SMOOTHNESS OF HOROCYCLE FOLIATIONS

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1. Introduction

Let SM denote the unit tangent bundle of a compact C∞ Riemannian manifold M. Suppose that M has everywhere negative sectional curvature. In [1] Anosov proved that the geodesic flow ϕ on SM is of a certain type, called "Anosov" by later writers, and defined below.

Associated with any Anosov flow ϕ is a foliation by "strong stable manifolds"; this is called the horocycle foliation in the special case where ϕ is the geodesic flow on SM and M has negative curvature. The strong unstable manifold provides another isomorphic horocycle foliation.

The leaves of these foliations are as smooth as the Anosov flow ϕ, but Anosov showed that the foliations are not in general of class C∞, even when ϕ is real analytic. However, when M has dimension two or the curvature is j-pinned, we shall prove that the horocycle foliations (and even their tangent plane fields) are of class C1. In the course of the proof, the fact that "negative curvature > Anosov geodesic flow" fails out naturally. Our methods in §§ 5, 6 resemble those of Anosov and Sinai [2].

This smoothness result was suggested to us by an analogous situation we encountered in [8]; there, we showed that the strong stable manifold foliation of an Anosov diffeomorphism f is of class C1 provided that either the strong stable manifolds have codimension one in M or the spectrum of Tf is "bunched".

These cases are analogous to (i), (ii) below.

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2. The smoothness theorem

Let M be a C∞ compact boundaryless manifold with a C∞ Riemann structure dS. The geodesics of dS give rise to the geodesic flow ϕ on the tangent bundle T(M) of M:

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It is amusing that, to mean "generic", Russian mathematicians, such as Anosov, use a word translated from Russian to English as "rough". Here is an example where roughness is likely to be generic.
if \( v \) is tangent to \( TM \) and \( t \) is the unique \( \varphi \)-geodesic with 
\[
\varphi(0) = v, \quad \varphi(t) \in T_{x(t)} M.
\]

\( \varphi \) is a geodesic. The geodesic flow \( \varphi \) on \( SM \) is Anosov if there is a continuous splitting \( T(SM) = E^s \oplus E^c \oplus E^u \), invariant under the tangent flow \( T \varphi \) on \( T(SM) \), such that \( E^s \) and \( E^u \) are subbundles spanned by the geodesic spray \( T_x \varphi \), respectively expanding \( E^s \) and \( E^u \) exponentially shrink \( E^c \). This means that for some (hence any) Riemann structure or Finsler on \( T(SM) \), there are constants \( C, c > 0 \), \( \lambda > 1 \) such that
\[
|T \varphi(x)| \leq C|x| \quad \text{if} \quad x \in E^c \quad \text{and} \quad t \geq 0,
\]
\[
|T \varphi(x)| \geq C^t|x| \quad \text{if} \quad x \in E^u \quad \text{and} \quad t \geq 0.
\]

The subbundle \( E^s, E^u \) is known to be uniquely integrable. They are tangent to the horocycle foliations. Thus, to prove the horocycle foliations are of class \( C^c \), it suffices to prove \( E^s, E^u \) are of class \( C^c \).

The sectional curvature of \( \varphi \) at a 2-plane \( \Pi \subset T_x M \) is \( K_\varphi(\Pi) = \) the Gaussian curvature of \( \exp_\varphi(\Pi) \) relative to the inclusion-induced Riemann structure. If \( K_\varphi(\Pi) < 0 \) for all \( p \in M \) and all 2-planes \( \Pi \subset T_x M \), then \( \varphi \) is said to have negative curvature.

**Definition.** The curvature of \( \varphi \) is absolutely \( \alpha \)-pinched if
\[
\alpha < \inf \{ K_\varphi(\Pi)/K_\varphi(\Pi) \}.
\]

The inf is taken over all \( p, p' \in M \) and all 2-planes \( \Pi, \Pi' \) in \( T_p M, T_{p'} M \). The curvature of \( \varphi \) is relatively \( \alpha \)-pinched if
\[
\alpha < \inf \{ K_\varphi(\Pi)/K_\varphi(\Pi) \}.
\]

The inf is taken over all \( p \in M \) and all 2-planes \( \Pi, \Pi' \) in \( T_M M \).

**Smoothness Theorem.** Let \( \varphi \) be a Riemann structure on \( TM \). If either
(i) the curvature of \( \varphi \) is negative and \( M \) has dimension two or
(ii) the curvature of \( \varphi \) is negative and absolutely \( 1/2 \)-pinched, then the Anosov splitting \( T(SM) = E^s \oplus E^c \oplus E^u \) for the geodesic flow is of class \( C^c \). In particular, the horocycle foliations are of class \( C^c \). Under natural uniformity assumptions on the curvature, compactness of \( M \) can be relaxed to completeness.


**Question.** Is this theorem true for relative \( 1/2 \)-pinching? If it is, then it includes (i) and (ii) as special cases. For negative curvature on a 2-manifold is
always relatively π-pinched for all π < 1. Originally we were sure this world "follow easily" from the C-section theorem (see below), but now we doubt it. Also we conjecture that there are many cases when the horocycle foliation is not of class C. Even if the curvature is ρ-pinched, we expect the horocycle foliations are hardly ever of class C. Such results might follow from methods of R. Maţă who proved a converse to the C-section theorem (13). Anosov said the horocycle foliation is "obviously not smooth in general," (1, p. 12).

3. Background

In (9) we proved, with Mike Shub, a general theorem giving sufficient conditions for an invariant section of a bundle to be smooth. Let E be a C*-finite dimensional vector bundle over the compact C-manifold M. Assume E has a Findest Fairly continuous family of norms on the fibers). Let D be a disc subbundle of E.

Definition. The minimum norm (also called the concave norm) of an operator A is

\[ m(A) = \inf_{\|x\| \leq 1} \|Ax\| = \{A^t\}^{1/2}. \]

Definition. As r-fiber contraction is a C-fiber map F: D → D covering a C-fibration f: M → M such that for some Finslers on E and FM

\[ \sup_{\rho \in \mathbb{K}} k_{\rho, r} \rho^{\sigma_j} / \rho < 1, \quad 0 \leq j \leq r, \]

where \( k_{\rho, r} \) is the Lipschitz constant of F[D]. D is the D-fiber: at \( \rho \in M \), and \( \sigma_j = m(T_r f) \).

\( k_{\rho, r} \) is the fiber contraction rate; \( \sigma_j \) is the base contraction rate. The assumption \( \sup_{\rho \in \mathbb{K}} k_{\rho, r} \rho^{\sigma_j} / \rho < 1 \) implies F uniformly contracts the i-fibers (let \( j = 0 \)) and contracts \( D_r \) more sharply than i contracts the base at \( \rho \) (let \( j = 1 \)).

C-section theorem. If F is an r-fiber contraction of D, \( r \geq 0 \) then there is a unique F-invariant section \( s: M → D_r \). Besides, s is of class C.

This is a central result of (9).

A second concept we use from (8), (9) is that of the "graph-transform" \( F_{\rho} \). If \( F: D → D \) is a fiber map as above, then \( F \) induces a natural map \( F_{\rho}: \text{sec}(D) \to \text{sec}(D) \) on the sections of D defined by \( F_{\rho}(s) = F \circ s \circ f^{-1} \circ s \). This can be re-expressed as

\[ \text{image}(F_{\rho}) = F(\text{image} s) \].

Finally, we use the uniqueness of the hyperbolic splitting of a hyperbolic bundle automorphism. This result is part of (9, 2.9).

4. Proof of (i)

Let X be the geodesic spray generating the geodesic flow \( \phi \). Then \( X_0 \) preserves the subbundle of \( TM \) orthogonal to \( X \), and since the Anosov splitting is unique,
Since \( E \) is a smooth bundle, we can approximate \( E^a, E^b \) by smooth subbundles \( E^a, E^b \) of \( E \). Let \( \mathcal{F} \) be the smooth bundle over \( SM \) whose fiber at \( v \) is

\[
\mathcal{F}_v = \{ G \in L(E_b^a, E^b_1) : |G| \leq 1 \}.
\]

Put the "max Finsler" on \( T(SM) \) so that

\[
|z| = \max \{ |z_\alpha|, |z_\alpha| \}.
\]

where \( z = x \oplus v \oplus y \in E^b_1 \oplus \text{span} \{ y(\psi) \oplus E_b \} \), and \( |\cdot| \) is length respecting \( \mathcal{F} \).

This is a Finsler on the base-space of \( \mathcal{F} \).

Since \( T_{\mathcal{F}_1} \) preserves \( E^a \oplus E^b = E^b_1 \oplus E^b_2 \), the \( T_{\mathcal{F}_1} \)-graph transform \( (T_{\mathcal{F}_1})_p \) is a fiber map \( \mathcal{F} \rightarrow \mathcal{F} \) covering \( \gamma \), the time-one map of the geodesic flow. \( (T_{\mathcal{F}_1})_p \) is defined by

\[
(T_{\mathcal{F}_1})(\text{graph} G) = \text{graph}(T_{\mathcal{F}_1})_p(G), \quad G \in \mathcal{F}_v.
\]

where \( G = \{ x + (a)(y) \in E^a_1 \oplus E^b_1 \} \). Let \( T\mathcal{F}_1 = T\mathcal{F}_1^{E^a}, T\mathcal{F}_1 = T\mathcal{F}_1^{E^b} \). The fiber \( \mathcal{F}_v \) is contracted at a rate \( \Delta \approx \| T_{\mathcal{F}_1} \|^m(T_{\mathcal{F}_1})^{-1} \), and the base is contracted at the rate \( \Delta \approx m(T_{\mathcal{F}_1}) \). (To say this about the base-map we need the max Finsler.)

The hypothesis of the \( C^1 \) section theorem \((\nu = 1)\) is that (fiber contraction) \times (base contraction) \(-1 < 1 \), and we have shown this product to be

\[
\frac{\| T_{\mathcal{F}_1} \|^m(T_{\mathcal{F}_1})^{-1} \cdot m(T_{\mathcal{F}_1})^{-1}}{\nu} < 1.
\]

since \( E^b \) is one-dimensional. Hence the unique \( (T_{\mathcal{F}_1})_p \)-invariant section of \( \mathcal{F} \) is of class \( C^1 \). The section whose graphs give \( E^b \) is clearly invariant, since \( E^b \) is \( T_{\mathcal{F}_1} \)-invariant. Hence \( E^b \in C^1 \). Symmetrically. \( E^a \in C^1 \).

\textbf{Remarks.} \( \text{If for any other reason bol}(T_{\mathcal{F}_1})^m(T_{\mathcal{F}_1})^{-1} < 1 \), then we get \( E^a \in C^1 \). By bol( ) we mean the "bolicity" which measures how nonconformal an isomorphism is:

\[
\text{bol}(T) = \frac{\| T \|}{m(T)} = \sup_{i \in [1, m]} \frac{|T_x|}{|T_y|} = \| T \| \cdot |T^{-1}|.
\]

\textbf{5. Second order linear differential equations}

To prove (ii) we need good norm-estimates on \( T_{\mathcal{F}_1}, T_{\mathcal{F}_1} \); the next lemma will provide them. By \( \mathcal{F}(R^n) = \mathcal{F} \) we mean symmetric linear endomorphisms of \( R^n \), i.e., self adjoint operators. By \( \mathcal{F}(R^n) \) we mean the convex cone of positive or negative definite maps.

\textbf{Lemma 1.} \( \text{Suppose } i \rightarrow P_i \) is a continuous map \( R \rightarrow \mathcal{F}(R^n) \), and \( \alpha, \beta \) are positive constants with
$\alpha < \inf \| P_x \|$, \quad $\sup \| P_x \| < \beta$.

Let $\Phi$ be the flow on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ generated by the artificially autonomous differential equation

$t := 1$, \quad $\dot{x} = y$, \quad $\dot{y} = P_x x$ ; \quad $t \in \mathbb{R}$, \quad $x, y \in \mathbb{R}^n$.

Then there exists a unique $\Phi$-invariant splitting $E_1 \oplus E_2 = \tau \times \mathbb{R}^n$ such that $E_2$, $E_1$ are graphs of uniformly bounded linear maps $\mathbb{R}^n \to \mathbb{R}^n$. Besides

$E_1 = \text{graph} G^*_2, \quad G^*_2 \subset \mathcal{F}^n(\mathbb{R}^n)$,

$\alpha < \langle G^*_2 x, x \rangle < \beta$,

$E_2 = \text{graph} G_2, \quad G_2 \subset \mathcal{F}^n(\mathbb{R}^n)$,

$\alpha < \langle -G_2 x, x \rangle < \beta$;

for all $x \in \mathbb{R}^n$ with $\langle x \rangle = 1$. This splitting $E^0 \oplus E^1$ of the product bundle $\mathbb{R} \times \mathbb{R}^n$ exhibits the hyperbolicity of $\Phi$. Norms on $E^0, E^1$ can be chosen, which are uniformly equivalent to the induced norms and make

$e^{\alpha t} < \| \Phi^t \| < e^{\beta t}$,

$x^{1+2\alpha} < \| x \phi^t \| < x^{1+2\beta}$

for all $t > 0$. If $P$ has period $\omega$, then so do $E^0$ and $E^1$.

Remark. A special case of this lemma is enlightening. Consider the autonomous constant coefficient linear differential equation:

$\dot{x} = y$, \quad $\dot{y} = px$, \quad $p > 0$

arising from the second order equation $x = g(t)$. This vector field on $\mathbb{R}^n$ generates the linear flow

$t \to \Phi_t = \begin{pmatrix} \cosh (pt) & \sinh (pt) \\ \beta \sinh (pt) & \cosh (pt) \end{pmatrix}$

which has the constant invariant splitting

$E^0 = \{ (s, 0) : s \in \mathbb{R} \}$, \quad $E^1 = \{ (s, -p) : s \in \mathbb{R} \}$.

It is a delightful coincidence that the hyperbolic trigonometric functions occur in a hyperbolic flow, and that this flow represents the tangent flow on the standard Poincaré hyperbolic plane (when $p = 1$).

Proof of Lemma 1. The flow $\Phi$ on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ naturally induces a (local) flow $\Phi_0$ on $\mathbb{R} \times \mathcal{O}_0(S)$ as follows. Fix $\tau \in \mathbb{R}$. For each $S \in \mathcal{O}_0(S)$ put $\Phi_0(\tau, S) = (\tau + t, S)$. Here $S$ is the unique linear map $\mathbb{R}^n \to \mathbb{R}^n$ such that

$(t + I) \times \text{graph}(S) = \Phi(\tau \times \text{graph} S)$.
When $S = S_i$ is fixed and $i$ is small, $S_i$ is well defined.

Fix $r$ and consider the solution $W_r = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix}$ of

$$\frac{\partial}{\partial t} W_r = \begin{bmatrix} 0 \\ P_{i \epsilon \tau} \end{bmatrix} W_r, \quad W_r(0) = I.$$

Thus $\Phi_{i \epsilon \tau} \times \mathbb{R}^d = W_r$. If $t > 0$ is small, then

$$S_i = (C_i + D_i) S_i (A_i + B_i S_i)^{-1}.$$

The tangent to the curve $S_i$ is

$$\frac{dS_i}{dt} = (C + DS_i (A + BS_i))^{-1} - (C + DS_i (A + BS_i))(A + BS_i)^{-1}.$$

At $t = 0$ this reduces to $P_\tau = S_i$ since

$$\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} C & D \\ PA & PB \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Thus the flow $\Phi_{i \epsilon \tau}$ is tangent to the vector field (on $\mathbb{R} \times \text{GL}(n)$) given by $(r, S) \mapsto (1, P_\tau = S_i)$. Note that its integral curves are solutions to the Riccati equation $\dot{S} = P - S^2$. Since this vector field is tangent to $\mathbb{R} \times \mathcal{F}(\mathbb{R}^n)$ by inspection, the flow $\Phi_{i \epsilon \tau}$ leaves $\mathbb{R} \times \mathcal{F}(\mathbb{R}^n)$ invariant. We claim that all points of the boundary $\partial (\mathbb{R} \times \mathcal{F}(\mathbb{R}^n))$ are strict ingress points for $\Phi_{i \epsilon \tau}$ where

$$\mathcal{F}(\mathbb{R}^n) = \{ s \in \mathcal{F} : c\in (-\infty, \langle s, x \rangle \leq \beta \leq \infty \text{ for all } x \in \mathbb{R}^n, |x| = 1 \}.$$

A boundary point $p$ of a region $U$ is a strict ingress point for a local flow $\varphi$ if $\varphi_{t=0} \in \text{Int}(U)$ for all small $r > 0$. This is an idea due to Ważewski. For $x \in \mathbb{R}^n$ and $S \in \mathcal{F}$ we have

$$s_1 = y_1, \quad x_1 = x \in \mathbb{R}^n,$$

$$s_2 = P_{x_1} x_1, \quad y_2 = S_{x_1} x_2,$$

and compute

$$\frac{d}{dt} \begin{bmatrix} s_{x_1} \cr x_{x_1} \end{bmatrix} = \begin{bmatrix} s_{x_1} \cr x_{x_1} \end{bmatrix} + \begin{bmatrix} s_{x_1} \cr x_{x_1} \end{bmatrix} + \begin{bmatrix} s_{x_1} \cr x_{x_1} \end{bmatrix} = \begin{bmatrix} s_{x_1} \cr x_{x_1} \end{bmatrix} = \begin{bmatrix} s_{x_1} \cr x_{x_1} \end{bmatrix} + \begin{bmatrix} s_{x_1} \cr x_{x_1} \end{bmatrix} = 2(\langle s_{x_1}, x_1 \rangle + \langle s_{x_1}, x_1 \rangle).$$
For small \( t, x \mapsto x_t \) defines an embedding of the unit sphere \( S^{\alpha - 1} \) of \( \mathbb{R}^\beta \) which is near the identity; thus the mapping \( S^{\alpha - 1} \ni x \mapsto x_t/(x_t, x)^{\alpha/2} \)

is near the identity; therefore it is subjective. This implies that

\[
\inf_{i \geq 1} \left\langle S_{x, t} \right\rangle = \inf_{i = 1} \left\langle \left( x_t, x^2 \right) \right\rangle
\]

for small \( t \).

Choose \( \alpha, \alpha_1, \beta_1, \beta_2 \) such that

\[
\alpha < \alpha_1 < \alpha_2 \leq \inf \{ \mu(P) \} , \quad \sup \{ \beta_1 \} \leq \beta_2 < \beta_3 ,
\]

\[
\alpha - \alpha_0 < \alpha_1 = \beta - \beta_0 < \beta_2 < \beta_3.
\]

Since \( P_1 \) is symmetric, \( \left\langle P, x, x \right\rangle \geq \alpha_0 |x|^2 \).

Suppose \( S \in \mathcal{A}_0 \) and consider the sets

\[
X_{1}(S) = \{ x \in S^{\alpha - 1} : \omega^{\alpha} \leq \left\langle S_{x, x} \right\rangle \leq \alpha_2 \} ,
\]

\[
X_{2}(S) = \{ x \in S^{\alpha - 1} : \beta_3 \leq \left\langle S_{x, x} \right\rangle \leq \beta_2 \} .
\]

For each \( x \in X_{1}(S) \) we have from (1)

\[
\frac{d}{dt} \left\langle S_{x, t} \right\rangle = \left\langle P_{x, x} \right\rangle + \left\langle S_{x, x} \right\rangle - 2\left\langle S_{x, x} \right\rangle.
\]

It follows from (2) that if \( x \in X_{2}(S) \), then

\[
\left\langle S_{x, x} \right\rangle > \alpha_2^{\alpha/2} \quad \text{for all small } t > 0.
\]

But if \( x \in S^{\alpha - 1} - X_{2}(S) \) and \( t \) is small, then

\[
\left\langle S_{x, x} \right\rangle \geq \left\langle S_{x, x} \right\rangle > \alpha_1^{\alpha/2}
\]

by continuity. Thus (3) holds for all \( x \in S^{\alpha - 1} - X_{2}(S) \), that is,

\[
\inf_{i \geq 1} \left\langle S_{x, t} \right\rangle > \alpha_3^{\alpha/2} \quad \text{for all small } t > 0.
\]

The same reasoning proves that also

\[
\sup_{i \geq 1} \left\langle S_{x, t} \right\rangle < \beta_1^{\alpha/2} \quad \text{for all small } t > 0.
\]

This shows that \( r \times S \) is a strict ingress point of \( \mathcal{A} \) for the local flow \( \Phi_t \).
The set $\mathcal{F}$ is a compact convex subset of the (finite dimensional) linear space $\mathcal{F}^r$. All the points of its boundary were shown to be strict ingress points. Since $R \times \mathcal{F}$ is not a retrofit of $\mathcal{F}$, Wazewski’s Principle [6, p. 279] says there must be a trajectory of $\Phi_t$ remaining in $R \times \mathcal{F}$ for all time. Let $r \to \tau \times G_t$ be such a trajectory, and set $F^r_t = \text{graph } G_t^r \subset R$. Clearly $G_t^r$ is interior to $\mathcal{F}$, and $\Phi_t(F_t^r) = F_t^r$.

Let $\mathcal{F}^r = \{x \in \mathcal{F}; a^{i/2} \leq \langle -S(x), x \rangle \leq b^{i/2} \text{ for all } x \in \mathcal{F}, \langle x \rangle = 1 \}$. Then all points of $3(R \times \mathcal{F})^r$ are strict egress points. This can be seen by some reasoning similar to the above. Again by Wazewski’s Principle, there is a $\Phi_t^r$ trajectory remaining in $\mathcal{F}^r$ for all time. This gives $G_t^r, F_t^r$ as claimed and completes the existence part of Lemma 1.

Uniqueness of $E_t, E^r_t$ follows from hyperbolicity of $\Phi_t$ and Hirsch-Pugh-Shub [9, 2.9]. To prove hyperbolicity and the asserted estimates on its strength, we introduce the new inner product in $R^m \times R^m$ by setting

$$
\langle \xi', \xi'' \rangle_e = \langle x', x'' \rangle; \quad x' = (x^1, \ldots, x^m), \quad j = 1, 2.
$$

By restriction we get new inner products on each $E_t, E^r_t (t \in \mathcal{F})$. This makes $x \to (x, G^\alpha x), x \to (x, G^\alpha x)$ isometries of $R^m$ onto $E_t, E^r_t$.

Denote $\Phi_t(x, \xi)$ by $(r + t, z_t)$ and put $z_t = (x_t, \xi_t) \in R^m \times R^m$. Thus

$$
x_t = x_t, \quad \xi_t = F_t \cdot x_t,
$$

and so

$$
\frac{d}{dt} \langle x_t, z_t \rangle_e = \frac{d}{dt} \langle x_t, \xi_t \rangle = 2 \langle \xi_t, G^*_t (\xi_t) \rangle\]

$$
= 2 \langle \xi_t, G^*_t (\xi_t) \rangle = 2 \langle \xi_t, G^*_t (\xi_t) \rangle
$$

by invariance of $E_t$. Since $G_t \in \mathcal{F}^r$, this last quantity lies between $2a^{\mu_0}$ and $2b^{\mu_0}$. Hence $\langle z_t, z_t \rangle_e$ satisfies the differential inequality

$$
2a^{\mu_0} \frac{d}{dt} \langle z_t, z_t \rangle_e < 2b^{\mu_0}, \quad t > 0,
$$

while

$$
\langle z_t, z_t \rangle_e = \langle z_t, z_t \rangle_e, \quad 0 \leq \tau \in E^r_t.
$$

From Hartman [6, p. 24] we conclude that

$$
e^{\mu_0 t} \langle z_t, z_t \rangle_e \leq \langle z_0, z_0 \rangle_e < e^{\mu_0 t} \langle z_0, z_0 \rangle_e
$$

for all $t > 0$. Taking square roots gives the growth estimate on $\Phi_t^r$ in Lemma 1. Similarly, if $z \in E_t^r$ then
\[ \frac{d}{dt} \langle z_t, z_t \rangle = 2 \langle x_t, G_{z_t}(z_t) \rangle, \]

which lies between \(-2\alpha^{(2)}\) and \(-2\beta^{(3)}\) since \(G_t \in \mathcal{F}_{z_t} \). This gives the growth estimate on \(\Phi_7\) in Lemma 1.

As remarked before, hyperbolicity of \(\Phi\) implies the uniqueness of \(F^+, F^\circ\). Suppose \(\Phi\) has period \(\omega\). Set \(F^\circ = \Sigma_{\omega}, F^+ = E^{\omega}\). Then \(F^+ \cap F^\circ\) is a \(\Phi\)-invariant splitting of \(R \times R^\omega\) since \(\Phi(x + \omega) = \Phi(x, \omega) + (0, \omega)\). Clearly \(F^+ \cap F^\circ\) also exhibits the hyperbolicity of \(\Phi\) so by [9, 2.9] \(E^\circ = F^+, E^\circ = F^, \) and \(\omega\)-periodicity of \(E^+, E^\circ\) is proved. This completes the proof of Lemma 1.

**Remark.** An alternative proof that \(E^+, E^\circ\) exist can be devised by showing that the flow \(\Phi_t\) contracts \(\mathcal{F}_{z_t}\), instead of using Wazewski's principle. Contractiveness of \(\Phi_t\) on \(\mathcal{F}_{z_t}\) follows from considering the first variation equation of \(S = F - S\), along a \(\Phi\)-trajectory \(S_t\), namely, \(\dot{S} = -(S, S^\perp)\). While \(S_t\) is in \(\mathcal{F}_{z_t}\), it is a positive operator so the above \(F\) is "negative," showing that \(\Phi_t\) contracts infinitesimally, \(t > 0\). Contractiveness of \(\Phi_t\) in the large follows by the mean value theorem since \(\mathcal{F}_{z_t}\) is convex. The details of this argument involve use of the inner product

\[ \langle A, B \rangle = \text{trace}(A^t B) \]

on \(L(R^n, R^n)\) and its corresponding norm. This is not the operator norm on \(L(R^n, R^n)\), and it does not have an analogue for an infinite dimensional real Hilbert space \(E\). The estimates in the proof of Lemma 1 remain valid for \(E\), but Wazewski's Principle fails because \(\mathcal{F}_{z_t}\) probably is a retract of \(\mathcal{F}_{z_{w+t}}\); compare Kric [11]. Thus the generalization of Lemma 1 to Hilbert space remains unproved by us.

6. **Fermi coordinates**

The next lemma concerns a special coordinate system along a geodesic, called a "Fermi chart." For the geodesic flow, the bundle-chart over a Fermi chart serves the same purpose as a flowbox does for a flow. Let \(\mathcal{M}\) be a smooth Riemannian structure on \(\mathcal{M}\), and let \(\gamma \in \mathcal{M}\). Let \(X\) be the geodesic spray on \(\mathcal{M}\). Let \(x, x_0, \ldots, x_n\) be an orthonormal basis for \(T_{x_0}\mathcal{M}\) with \(x = x_0\), and let \(\gamma\) be the geodesic initially tangent to \(x\). Parallel translation down \(\gamma\) gives smooth orthonormal vector fields \(e_0(t), \ldots, e_n(t)\) on \(\gamma\) such that \(e_0(t) = f(t)\). Since exp is tangent to the identity,

\[ f(s, e_0) = \exp_{x(t)} \left( \sum_{i=1}^n s_i e_i(0) \right) \]

defines an immersion \(f_t\), called the Fermi chart associated with \(x\) and \(v \in S_{x_0}\mathcal{M}\). The domain of \(f_t\) includes
\( \mathcal{D}_c = \{ vv + v' \in T_\mathcal{M} : v' \perp v, v |v'| \leq c, v \in \mathcal{H} \} \),

where \( c \) is a positive constant. \( \iota_c \) sends span \((v)\) isometrically onto \( \mathcal{H} \). Since \( f_c \) is an immersion, \( \mathcal{H} \) pulls back \( \omega \) a Riemann structure \( \mathcal{H} \mathcal{F} \) on \( T\mathcal{D}_c = \mathcal{D}_c \times T\mathcal{M} \). Thus \( \mathcal{F} \mathcal{F} \mathcal{F} \) is \( \mathcal{H} \) expressed in the \( f_c \)-chart. Let \( g_{ab}, \Gamma^c_{ab} \) and \( R^c_{ab} \) be the components of \( \mathcal{F} \mathcal{F} \mathcal{F} \), its Christoffel symbols and its Riemannian curvature tensor in the \( f_c \)-chart.

**Lemma 2.** The Fermi chart \( f_c \) has the following properties at all points of span \((v)\):

- \((0,0)\text{-th order}\)
  \[ g_{ab} = \delta_{ab}, \]

- \((1,1)\text{-order}\)
  \[ \Gamma^c_{ab} = 0, \]

- \((2,0)\text{-order}\)
  \[ R^c_{ab} = \frac{1}{2} \frac{\partial \mathbf{q}_{ab}}{\partial x^i} = \frac{\partial \Gamma^c_{ab}}{\partial x^i} \]

**Proof.** The 0th and 1st order assertions are proved in Gromov, Klingenberg-Maver [5]. In any chart

\[ \Gamma^c_{ab} = \frac{1}{2} \sum_{g} g^{cd} (\partial \mathbf{q}_{bd} + \partial \mathbf{q}_{da} - \partial \mathbf{q}_{ab}), \]

where \( (g^{cd}) \) is the matrix inverse to \( (g_{cd}) \). By \( \partial \), etc., we mean \( \partial / \partial x^r \) where \( x^1, \ldots, x^n \) are the coordinates in the chart. Juggling indices and summing as in Weatherburn [15] we get

\[ \partial_{(a} \mathbf{q}_{b)} = 0, \quad 1 \leq a,b \leq m \]

at any point of a chart where \( \mathbf{r} = 0 \) and \( (g_{ab}) = (\delta_{ab}) \). This means the map

\[ x \rightarrow (g_{ab}(x)) \in \text{real} \times m \times m \text{ matrices} \]

has zero derivative at all points of span \((v)\) in the Fermi chart. By the chain rule the same is true of

\[ x \rightarrow (g^{ab}(x))^{-1} = (g^{cd}(x))^{-1} \]

Thus all first partials of \( g_{ab} \) and \( g^{cd} \) vanish along span \((v)\). From this constancy we conclude \( \partial_{(a} \mathbf{q}_{b)} = \partial_{(a} \mathbf{q}_{b)} = 0 \) along span \((v) = x^n \mathbf{q}_{ab} \).

In any chart the components \( R^c_{ab} \) are related to the \( \mathbf{r} \) by

\[ R^c_{ab} = \partial_{(a} \mathbf{r}_{b)} - \partial_{(a} \mathbf{r}_{b)} + \sum_{g} \left( \frac{\partial \mathbf{r}_{ab}}{\partial x^g} \right) - \left( \Gamma^c_{ab} \mathbf{r}_c - \Gamma^c_{ab} \mathbf{r}_c ight) \]

(see Hicks [7]), so in the Fermi chart along span \((v)\)
\[ R^I_{\mu\nu} = \frac{\partial}{\partial a^\mu} (\frac{\partial}{\partial a^\nu} \partial_{\nu a} - \partial_{\nu a} \partial_{\nu a} ) \]
\[ + \frac{1}{2} \sum \partial \partial^*(\partial \partial_{\nu a} + \partial a_{\nu} - \partial a_{\nu}) \]
\[ - \frac{1}{2} \sum \partial^* \partial \partial_{\nu a} + \partial a_{\nu} - \partial a_{\nu} ) \]
\[ - \frac{1}{2} \sum \partial^* \partial \partial_{\nu a} + \partial a_{\nu} - \partial a_{\nu} ) \]
\[ = - \frac{1}{2} \partial \partial_{\nu a} + \partial a_{\nu} - \partial a_{\nu} ) \]
\[ = - \frac{1}{2} \frac{\partial}{\partial a^*} \partial^* + \partial a_{\nu} - \partial a_{\nu} ) \]

For along span (v): \( \partial \partial_{\nu a} + \partial a_{\nu} - \partial a_{\nu} ) \) vanishes, \( \partial a_{\nu} - \partial a_{\nu} ) \) vanishes, and \( \partial a_{\nu} - \partial a_{\nu} ) \) vanishes, and \( \partial^* = \partial^* \). For the same reasons

\[ \frac{\partial^*}{\partial a^*} = \frac{1}{2} \sum \partial \partial^* (\partial_a \partial_{\nu a} + \partial a_{\nu} - \partial a_{\nu} ) \]
\[ + \frac{1}{2} \sum \partial^* \partial \partial_{\nu a} + \partial a_{\nu} - \partial a_{\nu} ) \]
\[ = - \frac{1}{2} \partial \partial_{\nu a} + \partial a_{\nu} - \partial a_{\nu} ) \]

along span (v). This completes the proof of Lemma 2.

7. Proof of (ii)

Let \( \partial^* \) be the given Riemann structure on \( TM \). Let \( \nu \in S \), \( \rho \in M \), and choose an orthonormal basis of \( T \nu \rho M, e_1 , \ldots , e_n \) with \( e_i = \nu \). Let \( f_i \) be the Frenet chart determined by \( e_1 , \ldots , e_n \), and let \( f_i \) be the bundle chart of \( TM \) tangent to \( f_i \).

\[ \partial \in \text{dom} \partial \]. The geodesic spray \( X \) is represented in any \( TM \)-bundle-chart for \( TM \) as the first order ordinary differential equation

\[ (1) \]

\[ [ \partial \partial_{\mu} ] = [ \partial \partial_{\mu} ] \]

where \( \partial \partial_{\mu} \): \( T \nu \rho M \times T \nu \rho M \rightarrow T \nu \rho M \) is the symmetric bilinear map such that

\[ \Gamma_{\lambda}(e_i , e_j) = \frac{1}{2} \partial \partial_{\mu} \partial_{\mu} e_i = e_i , \]  

\[ \in \partial \partial_{\mu} \). The \( f_{\mu} \) are the Christoffel symbols of \( \partial^* \) expressed in the \( f_i \) chart.

The geodesic \( \text{flow} \) \( \phi \) of \( \partial \partial_{\mu} \), represented in the \( f_{\mu} \)-chart, is the solution of (1). The assertion of the smoothness theorem concerns the tangent flow \( T \partial \partial_{\mu} \) on
\( T(TM) \). When represented in the \( TF_p \)-chart, \( T_p \) is the solution of the first variation equation of (1):

\[
W = D(F^2X)_{\gamma}(\xi), \quad W(0) = I
\]

for \( w_1 = F^1_{\gamma}(\varphi) \circ F(x), \) \( w \in \mathcal{D} \times T_pM. \) By \( F^2X \) we mean the vector field \( X \circ F^1_{\gamma} \) on \( \mathcal{D} \times T_pM. \) At \( F^1_{\gamma}(q_0, \xi) \) we calculate

\[
D(F^2X)_{\gamma}(\xi, \xi) = \begin{bmatrix}
\xi & 0 \\
-\Gamma(x)(\xi, \xi) & -2\Gamma(x)(\xi, \xi)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 0 \\
\frac{\partial \Gamma(x)}{\partial x} & 0
\end{bmatrix}
\]

by Lemma 2 since

\[
\begin{bmatrix}
\frac{\partial \Gamma(x)}{\partial x} \\
\frac{\partial \Gamma(x)}{\partial y} + \frac{\partial \Gamma(x)}{\partial z} + \frac{\partial \Gamma(x)}{\partial v}
\end{bmatrix}
\]

by Lemma 2 since

\[
\text{(The } R^i_{\gamma} \text{ are the components of the curvature tensor in the } F \text{-chart.) Thus, along } F^1_{\gamma}(q_0, \xi), \text{ (2) becomes}
\]

\[
W = \begin{bmatrix}
0 & 0 \\
-\Gamma_{\xi}(v_0) & 0
\end{bmatrix} W, \quad W(0) = I.
\]

In general, \( R^{i}_{\gamma} \) is skew-symmetric in \( \beta \) and \( R^{i}_{\gamma} = 0 \), so we see that

\[
(R^i_{\gamma}) = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}, \quad 2 \leq k, l \leq m.
\]

These extra zeros indicate that \( T_p \) preserves \( X^1 \) (as does any tangent flow) and that \( T_p \) preserves \( X^1 \) (as does any tangent geodesic flow). Let \( E = X^1 \cap T(TM) \). Then \( T_p \) preserves \( E \) and \( \phi_t = T_p \circ E \cdot t \), expressed in the \( F \)-chart, solves

\[
\phi_t = \begin{bmatrix}
0 & 0 \\
F_t & 0
\end{bmatrix}, \quad \phi_0 = I,
\]

where

\[
F_t = [-R^i_{\gamma}(v_0)]_{2 \leq k, l \leq m}.
\]

\( \phi \) is a linear flow on \( \text{span } (v) \times H_v, \) \( v \circ \mathcal{D} \times \mathcal{R}^{n-1} \times \mathcal{R}^{n-1} \) where \( H_v = \{ (x, 0) \in T_pM \times T_pM, x \perp v \}, \) \( V = \{(0, \xi) \in T_pM \times T_pM : x \perp v \}. \)
In any chart at a point where the coordinates are orthonormal, the sectional curvature of a pair of vectors $Y, Z \in T_pM$ is

$$K_p(Y, Z) = \langle R(Y, Z)Y, Z \rangle, \quad Y = \sum_{i,j} y^i e_i, \quad Z = \sum_{i,j} z^i e_i,$$

and thus finally using the negative curvature hypothesis we have

$$\langle P(Z, Z) = -\sum_{i,j} P_{i,j}z^i e_i z^j = -K e_i, Z \rangle > 0,$$

where $P_{i,j} = R^{i,j}(e_i)$. Choose constants $K > k > 0$ such that every sectional curvature lies strictly between $-K$ and $-k$. By (4), in applying Lemma 1 we can take $k = \tilde{k}, \quad \tilde{k} = K$.

By Lemma 1, $\Phi$ is hyperbolic and the strength of its hyperbolicity can be estimated. Using the $F_r$-chart we get a well defined $T_pE$-invariant splitting $E_p \oplus E_p$ of $E$ over the $\varphi$-orbit of $v$. (If $t \in \mathbb{R}$, $v$ is periodic in $t$, then $P_t$ is periodic and, by Lemma 1, so is the $\Phi$-invariant-splitting. Hence $E_p \oplus E_p$ is well defined.) Choose one $v$ on each $\varphi$-orbit and make the preceding construction. This gives a well defined $T_pE$-invariant splitting of $E$ over all $SM$.

Since the Finsler on span $(v) \times J_\pm \times V_p$ adapted to $\Phi$ is uniformly equivalent to the standard Finsler, and since $J_\pm$ is a Fermi-chart, we use the estimates

$$e^{t_k} < m(\delta^i) \leq \|\delta^i\| < e^{t_k}, \quad e^{-t_k} < m(\Phi^i) \leq \|\Phi^i\| < e^{-t_k},$$

which are valid for all $t > 0$ when the adapted Finsler is used—imply

$$e^{t_k} < m(T_\partial \Phi) \leq \|T_\partial \Phi\| < e^{t_k}, \quad e^{-t_k} < m(T_\partial \Phi) \leq \|T_\partial \Phi\| < e^{-t_k}$$

respecting the $\mathbb{R}$-norms for all large $t$. By $T_\partial \Phi, T_\partial E$ we mean $T_{\partial \Phi}(E), T_{\partial \Phi}(E)$. Thus, respecting the fixed $\mathbb{R}$-norms, $T_{\partial \Phi}E$ is a linear uniformly hyperbolic flow and so, by [9, 2.9], $E_p$ and $E_p'$ are automatically continuous and independent of which $v$ was chosen on each $\varphi$-orbit. Hence $\Phi$ is Anosov.

By (5) we get

$$\text{bol}(T_\partial \Phi) < e^{-t_k}, \quad m(T_\partial \Phi) > e^{t_k}, \quad \|T_\partial \Phi\| < e^{t_k}$$

for all large $t$. Now return to the proof of (ii). Since $E$ is a smooth bundle we can approximate $E_p, E_p'$ by smooth subbundles $E_p, E_p'$ of $E$. Then we can consider, for a large $t \in \mathbb{N}$, the $\varphi$-map $(T_\partial \Phi): \varphi \to \varphi$, where $\varphi = \{G \in L(\mathbb{E}_p, E_p') : \|G\| \leq 1\}$. As in the proof of (i), $(T_\partial \Phi)$ is a fiber contraction with

- (fiber contraction): the contraction of $e^{t_k}$
Since the curvature is \( \frac{1}{2}\)-pinched, we have \( K - 2k < 0 \) and the hypothesis of the \( C^1 \) section theorem is satisfied; therefore the unique \( (T_0)_{\xi} \)-invariant section of \( \xi \) is of class \( C^1 \). Since \( \xi \) gives such a section, \( \xi \) is of class \( C^1 \). Working with the reverse flow and \( \xi' := \xi + \xi' \), \( \xi' \) is \( C^1 \), \( \xi' \) gives the same result for \( \xi \). This completes the proof of (ii).

Remarks on the smoothness of \( \xi \). For simplicity, we assumed the Riemannian structure \( \xi \) was \( C^1 \). However, the above constructions work equally naturally when \( \xi \) is \( C^2 \), the smoothness theorem holds when \( \xi \) is \( C^2 \), and \( \xi \) is Anosov when \( \xi \) is \( C^1 \) with negative curvature. This can be seen by \( C^1 \)-approximating \( \xi \) by a \( C^2 \) Riemannian structure \( \xi' \) and using the uniformities in the hyperbolicity estimate. Alternatively, the Fermi chart could be smoothed as were flow boxes in Pugh-Robinson [14].

Standard question. If the geodesic flow of \( \xi \) is Anosov, then does \( M \) admit a Riemann structure \( \xi' \) with negative curvature? Wilhelm Klingenberg showed in [12], [16] that all known topological properties of \( M \) which are implied by negative curvature are equally implied by \( \phi \) being Anosov.

References


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