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Bott Vanishing and Kawamata-Viehweg Vanishing for Algebraic Surfaces

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ABSTRACT OF THE DISSERTATION

Bott Vanishing and Kawamata-Viehweg Vanishing for Algebraic Surfaces

by

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Professor Burt Totaro, Chair

This dissertation has three parts. The first explores which del Pezzo surfaces with Gorenstein singularities satisfy Bott vanishing. The second expands on Bott vanishing that occurs for F-liftable varieties and toric varieties. The last provides a counterexample for Kawamata-Viehweg vanishing for a log Fano surface of characteristic five.

The dissertation of William Baker is approved.

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for my family

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Chapter 1

Introduction

Sheaf cohomology is one of the most powerful tools of modern algebraic geometry. Many algebraic varieties can be distinguished by sheaf cohomology, and many geometric properties can also be deduced. This dissertation concerns two types of sheaf cohomological vanishing. The first is Bott vanishing. Bott vanishing is an extremely powerful and rare form of vanishing which has gained recent interest due to its connection with Frobenius liftability. In this dissertation I produce many examples of singular surfaces which satisfy Bott vanishing. I also flesh out the connection between F liftability and Bott vanishing for singular varieties by extending the vanishing from Cartier divisors to Weil divisors. Kawamata-Viehweg vanishing is also very powerful, and it is a fundamental tool in the minimal model program. It is proven to always occur in characteristic zero, and here I find a counterexample for a log Fano surface in characteristic 5. This counterexample is connected to a literature of papers that establish for which log Fano surfaces does Kawamata-Viehweg vanishing hold.

1.1 Notation

Here I will recall a few basic notions used in the study of singular varieties.

Definition 1.1.1. Let X be a normal variety. A Weil divisor on X is a formal $\mathbb{Z}$-linear combination of closed codimension one subvarieties.

Weil divisors are often considered up to linear equivalence, where they are considered equivalent if their difference in principal. The group of Weil divisors up to linear equivalence is called the divisor class group of X, and is denoted $Cl(X)$.

Definition 1.1.2. Let X be a normal variety. A Cartier divisor is a global section of the sheaf $k(X)^*/O_X^*$, where * means taking the sheaf of units.

Again we can similarly define equivalence using principal Cartier divisors. Recall that in the smooth setting we have an isomorphism between $Pic(X)$, $Cl(X)$, and Cartier divisors up to equivalence. In the normal setting we still get this equivalence between line bundles and Cartier divisors, but Weil divisors are now more general. We still have that every line bundle/Cartier divisor corresponds to a Weil divisor by means of take zeroes and poles, but now for a Weil divisor, D, $O_X(D)$ the corresponding sheaf of functions with zeroes and poles prescribed by D is no longer necessarily a line bundle. We can at least say that $O_X(D)$ is a reflexive sheaf (see definition 3.2.3). We will call a Weil divisor $\mathbb{Q}$-Cartier if some multiple is Cartier.

I will also recall some types of particularly nice singularities. Note that for a normal variety we can always write $K_Y = \pi^*(K_X) + E$, where E is some divisor supported on the exceptional locus. One strategy for regulating singularities is then to have control over how antieffective E is (see [24, 3.5]). More precisely:

Definition 1.1.3. Let X be a normal, projective variety. Let Δ be an effective $\mathbb{Q}$-divisor whose coefficients are all less than 1, so that $K_X + \Delta$ is a $\mathbb{Q}$-Cartier divisor. Let $\pi : X' \to X$ be a resolution of singularities, and let Δ' be the proper transform of Δ. We have the following equation:

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta) + \Sigma a_i E_i$$

where E_i are the exceptional divisors. We call (X, Δ) klt if $a_i > -1$ for all i.

Note that requiring that $K_X + \Delta$ to be a $\mathbb{Q}$-Cartier divisor is necessary for the pullback to be well defined. Also this definition is independent of the choice of desingularization, although this is not obvious [24, 3.10]. In our applications we will refer to X as klt when it is klt for trivial boundary divisor. In the case of surfaces klt singularities are quotient singularities. Another important class of singularities are *canonical singularities*. These have the same definition except we insist that the Δ is trivial and that $a_i \geq 0$ for every i. For surfaces, where we can take a minimal desingularization, π, this is equivalent to insisting that $K_{X'} = \pi^*(K_X)$.

Chapter 2

Bott vanishing for del Pezzo surfaces

2.1 Introduction

A smooth projective variety X is said to satisfy Bott vanishing if for all ample line bundles L on X, $H^i(X, \Omega^j_X \otimes L) = 0$ for all $i > 0$, and all $j \geq 0$. Bott proved that $\mathbb{P}^n$ satisfies Bott vanishing. Later, it was discovered that all smooth projective toric varieties satisfy Bott vanishing. Even non-smooth toric varieties X satisfy a suitably adjusted Bott vanishing, by replacing Ω^j_X with reflexive differentials $\Omega^{[j]}_X$, and ample line bundles with ample $\mathbb{Q}$-Cartier $\mathbb{Z}$-Weil divisors [30]. F-liftable varieties (briefly, varieties whose Frobenius morphism lifts mod p^2, see 3.2.1) also satisfy Bott vanishing [8]. It seems that Bott vanishing is a rare property that some of the nicest varieties enjoy, but it is not completely clear if there is a geometrically meaningful necessary and sufficient condition for when a variety should satisfy it. Totaro answered a question by Achinger,Witaszek, and Zdanowicz [1, 2] by exhibiting K3 surfaces and one nontoric del Pezzo surface that satisfy Bott vanishing [37]. Torres recently studied when certain types of stable GIT quotients of $\mathbb{P}^n$ satisfy Bott vanishing [36].

Building on Totaro's classification of which smooth del Pezzo surfaces satisfy Bott vanishing, this chapter will study when del Pezzo surfaces with Gorenstein singularities over the complex numbers satisfy Bott vanishing. As in the non smooth toric case, we will use reflexive differentials. However, we insist that L is Cartier (I will call this Cartier Bott vanishing to avoid confusion). The main result is that a del Pezzo surface with Gorenstein singularities, X, satisfies Cartier Bott vanishing if and only if $H^1(X, \Omega^{[1]} \otimes K^*_X) = 0$. This is satisfying because $H^1(X, \Omega^{[1]} \otimes K^*_X) = 0$ can be interpreted geometrically as X not admitting any nontrivial locally trivial first order deformations [33, 1.2.9]. We can use this result to give a fairly explicit list of which of the del Pezzo surfaces with Gorenstein singularities satisfy Cartier Bott vanishing. All varieties are over the complex numbers.

2.2 Background

In this section I would like to state a few theorems that will be used in this paper. We will start by a special case of Kawamata-Viehweg vanishing.

Theorem 2.2.1 (Kawamata-Viehweg vanishing theorem). *[26, 2.70] Let X be a complex projective variety with canonical singularities, and let L be a big and nef line bundle on X. Then $H^i(X, L \otimes K_X) = 0$ for all $i > 0$.*

This is basically a strengthening of Kodaira vanishing. The statement given follows from picking 0 as the boundary divisor in the theorem cited.

Theorem 2.2.2. *[Reider's theorem] [32] Let X be a smooth complex projective surface, and let L be a nef line bundle on X. Suppose that $L^2 \geq 5$ and p is a base point of $L \otimes K_X$. Then there exists an effective divisor E passing through p such that either $L \cdot E = 0, E^2 = -1$ or $L \cdot E = 1, E^2 = 0$.*

We can sometimes use Reider's theorem to show that a nef divisor has no base points.

Theorem 2.2.3. *[Kodaira-Akizuki-Nakano (KAN) vanishing] [28, 4.2.3] Let X be a complex smooth projective variety, let L be an ample line bundle on X. Then $H^j(X, \Omega^i_X \otimes L) = 0$ for all $i + j > dim(X)$.*

KAN vanishing is quite useful for reducing the number of cohomological groups that must be checked for Bott vanishing.

2.3 Del Pezzo surfaces with Gorenstein singularities

In this section I will recall some background about del Pezzo surfaces and weak del Pezzo surfaces. We will start with their definitions. A good reference is [15].

Definition 2.3.1. We say a normal, projective surface X is a del Pezzo surface if it is klt and the anticanonical class $-K_X$ is ample.

This is the same thing as a two dimensional Fano variety.

Definition 2.3.2. We say a normal, projective surface X is a del Pezzo surface with Gorenstein singularities if it is a del Pezzo surface whose singularities are no worse than canonical.

The assumption of klt and Gorenstein implies that del Pezzo surface with Gorenstein singularities have canonical singularities. These singularities are also called du Val or ADE singularities. The singularities are classifed by the ADE Dynkin diagrams [5]

Definition 2.3.3. We say a smooth, projective surface X is a weak del Pezzo surface if K^*_X is big and nef.

These two classes of rational surfaces are intimately related, as seen by the following propositions.

Proposition 2.3.4. *Suppose X is a del Pezzo surface with Gorenstein singularities, let $\pi : Y \to X$ be its minimal desingularization. Then Y is a weak del Pezzo surface.*

Proof. Since all singularities of X are canonical, we have that $K_Y^* \backsimeq \pi^*(K_X^*)$; the proposition follows since the pullback of a big and nef divisor along a birational morphism is big and nef. □

Smooth weak del Pezzo surfaces have a nice classification.

Proposition 2.3.5. *X is a smooth weak del Pezzo surface if and only if X is a blow up of $\mathbb{P}^2$ at at most 8 points, such that the only curves with negative self intersection are (-1) or (-2) curves, or X is one of the Hirzebruch surfaces Σ_0 or Σ_2. Furthermore the connected components of (-2) curves have dual graphs corresponding to the ADE classification.*

More explicitly, a blow up of $\mathbb{P}^2$ at at most 8 points is a smooth weak del Pezzo surface if and only if the points are in so called *almost general position*, which means that no three points are on a line, no six points are on a conic, and no 8 points are on a singular cubic, with one point on the singular point [15, Corollary 8.1.24]. In fact, there is a 1-1 correspondence of smooth weak del Pezzo surfaces and del Pezzo surfaces with Gorenstein singularities.

Proposition 2.3.6. *Given any smooth weak del Pezzo surface, Y there exists a contraction $\pi : Y \to X$ of all (-2) curves of Y, where X is a del Pezzo surface with Gorenstein singularities. One can use a multiple of the anticanonical bundle for this contraction.*

Proof. These results can be found for example in [15, below 8.1.18, 8.2.27, 8.3.2,]. □

Much like smooth del Pezzo surfaces, it is often useful to distinguish our surfaces by their *degree.*

Definition 2.3.7. The degree of a weak del Pezzo surface, or del Pezzo surface with Gorenstein singularities Y, is the number K_Y^2.

Note that a smooth weak del Pezzo surface and its corresponding del Pezzo surface with Gorenstein singularities have the same degree. Our classification of smooth weak del Pezzo surfaces shows that the degree is a number between 9 and 1, exactly like for smooth del Pezzo surfaces. It is often more convenient to work with the associated smooth weak del Pezzo surface instead of the del Pezzo surface with Gorenstein singularities. We will use this approach in this paper.

2.4 Reflexive Differentials

In this section I will define and collect a few basic facts about reflexive differentials.

Definition 2.4.1. Given a normal variety, X, the sheaf of reflexive differentials on X, $\Omega_X^{[j]}$, is the double dual of the sheaf of Kähler differentials on X. Equivalently $\Omega_X^{[j]}$ is the pushforward of the sheaf of Kähler differentials on the smooth locus.

Greb, Kebekus and Peternell [17] proved that for mild enough singularities, we can view reflexive differentials as a pushforward of the sheaf of Kähler differentials on the desingularization. We will use this repeatedly.

Theorem 2.4.2. *Let X be a projective surface with canonical singularities, and let $\pi : Y \to X$ be a desingularization. Then $\Omega_X^{[j]} \backsimeq \pi_* \Omega_Y^j$.*

Proof. This is a very special case of [21]. □

I will define what this paper means by Cartier Bott vanishing here.

Definition 2.4.3. Given a normal projective variety X, we say that X satisfies Cartier Bott vanishing if for all ample line bundles L on X, $H^i(X, \Omega_X^{[j]} \otimes L) = 0$ for any $i > 0$, and for any $j \geq 0$.

2.5 A proposition on base points

We will start by obtaining a result controlling the base points for certain nef line bundles on smooth weak del Pezzo surfaces. This will be the key to our inductive step for our result.

Proposition 2.5.1. *Let Y' be a smooth weak del Pezzo surface of degree greater than 1, and let y be a point on Y', such that the blow up at y is still a smooth weak del Pezzo surface. Let $f : Y \to Y'$ be the blow up with exceptional divisor E. Suppose that M' is a nef divisor on Y' such that $(f^*M') \cdot C = 0$ for every (-2) curve, C, on Y. Then y is not a base point of M'.*

Proof. Before we begin let me review the genus formula. The genus formula tells us that for any smooth surface X, and smooth curve C, we have that $K_C \simeq (K_X \otimes O_X(C))|_C$. One common use is that we can infer how the canonical divisor intersects with a smooth rational curve from the self intersection on the rational curve. For example, if E is a (-1) curve, we have that $K_X \cdot E = -1$. By Riemann-Roch $\chi(M') = 1 + \frac{1}{2}(M'^2 - K_{Y'} \cdot M') > 0$, with the inequality following from M' being nef and $-K_{Y'}$ is effective. By Kawamata-Viehweg vanishing, we have that $\chi(M') = \chi(M' - K_{Y'} + K_{Y'}) = h^0(M')$. Thus we conclude that M' is effective. We will aim to use Reider's theorem, theorem 2.2.2. Suppose y is a base point of M', then E is in the base locus of f^*M'. Since therefore $f^*M' - E$ is effective, we have that $(-K_Y) \cdot (f^*M' - E) \geq 0$, so $(f^*M') \cdot (-K_Y) \geq 1$, by the genus formula and the nefness of $-K_Y$. Considering Riemann-Roch for surfaces we know that $\frac{1}{2}((f^*M')^2 - K_Y \cdot f^*M')$ is an integer. Since we also have by nefness that $(f^*M')^2 \geq 0$, we can conclude that $M'^2 - K_{Y'} \cdot M' = (f^*M')^2 - K_Y \cdot f^*M' \geq 2$. We can now compute $(M' - K_{Y'})^2 = ((f^*M')^2 - K_Y \cdot f^*M') - K_Y \cdot (f^*M') + K_{Y'}^2 \geq 2 + 1 + 2 = 5$ (note that we assumed that $K_{Y'}^2 > 1$). We can also verify that for $D = M' - K_{Y'}$ there does not exist an effective divisor G passing through y such that $G^2 = -1$ and $D \cdot G = 0$, or $G^2 = 0$ and $D \cdot G = 1$. Suppose $G^2 = -1$ and $D \cdot G = 0$, and G passes through y. Since M and $-K_{Y'}$ are both nef, we have that $(-K_{Y'}) \cdot G = 0$. Since for sufficiently large m, $-mK_Y$ induces the contraction of the (-2) curves, we have that G is supported on the (-2) curves, but by hypothesis such a divisor cannot pass through y, (since then Y would would have a (-3) curve), but this is a contradiction. Now suppose that $G^2 = 0$ and $D.G = 1$, and G passes through y. Similarly to our last argument, $(-K_{Y'}) \cdot G$ cannot be equal to 0. So $(-K_{Y'}) \cdot G = 1$. However, this is not possible, since then Riemann-Roch would give us a noninteger as the Euler characteristic for G. Thus we can invoke Reider's theorem 2.2.2, and we have a contradiction on y being a base point.

□

2.6 Rigidity and Cartier Bott vanishing

We will start by citing an analog of KAN (theorem 2.2.3) vanishing for reflexive differentials.

Proposition 2.6.1. *Let X be a projective surface with canonical singularities, let L be an ample line bundle on X. Then $H^j(X, \Omega_X^{[i]} \otimes L) = 0$ for $i + j > 2$*

Proof. This follows from the fact that canonical singularities for surfaces are quotient singularities, and [4, 17].

□

For X a del Pezzo surface with Gorenstein singularities, and L an ample line bundle, Cartier Bott vanishing is equivalent to checking that $H^1(X, \Omega_X^{[1]} \otimes L) = 0$. Note that $H^i(X, L) = 0$ for

$i > 0$ by Kawamata-Viehweg vanishing. We will continue by translating Cartier Bott vanishing into a statement about the desingularization of X.

Proposition 2.6.2. *Let X be a del Pezzo surface with Gorenstein singularities, let $\pi : Y \to X$ be its minimal desingularization. Let L be an ample line bundle on X, then we have a surjection $H^1(Y, \Omega^1_Y \otimes \pi^* L) \twoheadrightarrow H^0(X, R^1\pi_*\Omega^1_Y)$. It is an isomorphism if and only if $H^1(X, \Omega^{[1]}_X \otimes L) = 0$.*

Proof. We have that $\pi_*\Omega^1_Y \backsimeq \Omega^{[1]}_X$ by 2.4.2. The projection formula gives us that $R^1\pi_*(\Omega^1_Y \otimes \pi^*L) \backsimeq R^1\pi_*\Omega^1_Y \otimes L$. We can then consider the Leray spectral sequence $E_2^{pq} = H^p(X, R^q\pi_*(\Omega^1_Y \otimes \pi^*L)) \Rightarrow H^{p+q}(Y, \Omega^1_Y \otimes \pi^*L)$:

$$\begin{array}{ccc} 0 & 0 & 0 \\ H^0(X, R^1\pi_*\Omega^1_Y \otimes L) & H^1(X, R^1\pi_*\Omega^1_Y \otimes L) & H^2(X, R^1\pi_*\Omega^1_Y \otimes L) \\ H^0(X, \Omega^{[1]}_X \otimes L) & H^1(X, \Omega^{[1]}_X \otimes L) & H^2(X, \Omega^{[1]}_X \otimes L) \end{array}$$

We can see that the spectral sequence degenerates on the E_2 page, since $H^2(X, \Omega^{[1]}_X \otimes L) = 0$ by 2.6.1, and $R^2\pi_*\Omega^1 = 0$ since the exceptional locus is 1 dimensional. We thus have that $H^1(Y, \Omega^1_Y \otimes \pi^*L) \backsimeq H^0(X, R^1\pi_*\Omega^1_Y \otimes L) \oplus H^1(X, \Omega^{[1]}_X \otimes L)$. The theorem on formal functions [19, III.11.1] shows that $R^1\pi_*\Omega^1_Y$ is supported on the singularities of X, a zero dimensional set. L is isomorphic to the structure sheaf on stalks, so $R^1\pi_*\Omega^1_Y \otimes \pi^*L \backsimeq R^1\pi_*\Omega^1_Y$. The proposition then follows. □

Proposition 2.6.3. *With the same assumptions as 2.6.2, we have that $h^0(X, R^1\pi_*\Omega^1_Y) = n$ where n is the number of (-2) curves on Y.*

Proof. Because X has quotient singularities [35, 1.6] gives us that $H^2(X^{an}, \mathbb{C}) \backsimeq H^1(X, \Omega^{[1]}_X) \oplus H^2(X, O_X) \oplus H^0(X, K_X) = H^1(X, \Omega^{[1]}_X)$. The same identity holds for replacing X with Y. We can consider the Leray spectral sequence $E_2^{pq} = H^p(X, R^q\pi_*(\Omega^1_Y)) \Rightarrow H^{p+q}(Y, \Omega^1_Y)$:

$$\begin{array}{ccc} 0 & 0 & 0 \\ H^0(X, R^1\pi_*\Omega^1_Y) & H^1(X, R^1\pi_*\Omega^1_Y) & H^2(X, R^1\pi_*\Omega^1_Y) \\ H^0(X, \Omega^{[1]}_X) & H^1(X, \Omega^{[1]}_X) & H^2(X, \Omega^{[1]}_X) \end{array}$$

Note that $H^2(X, \Omega^{[1]}_X) = 0$. One way to see this is using [35, 1.6] and noting that $H^3(X^{an}, \mathbb{C}) = 0$. We can calculate this cohomology group by using the universal coefficient theorem, and noting that any resolution of singularities only changes the two skeleton. The Leray spectral sequence then gives us that

$$h^0(X, R^1\pi_*\Omega^1_Y) = dim(H^2(Y^{an}, \mathbb{C})) - dim(H^2(X^{an}, \mathbb{C})),$$

which is the number of (-2) curves of Y. □

Theorem 2.6.4. *Let X be a del Pezzo surface with Gorenstein singularities. X satisfies Cartier Bott vanishing iff $H^1(X, \Omega^{[1]} \otimes K_X^*) = 0$.*

Proof. Since K_X^* is ample this is a necessary condition. Suppose that $H^1(X, \Omega_X^{[1]} \otimes K_X^*) = 0$.

The strategy will progress similarly to Totaro's proof that the degree 5 smooth del Pezzo surface satisfies Bott vanishing [37]. Let $\pi : Y \to X$ be our minimal desingularization, and suppose that $\rho(Y) \leq 2$. We then have that Y is $\mathbb{P}^2$, or a Hirzebruch surface, Σ_n, with $n < 3$. All of these varieties are toric, so Y and X are both toric and satisfy Bott vanishing. We can assume that $\rho(Y) > 2$ from here on out.

I will now give a description of the cone of curves $\overline{NE}(X)$ of X. First, each irreducible negative self intersection curve is an extremal ray of $\overline{NE}(X)$. These (-2) curves account for all extremal rays that are not K_X-negative, since they are precisely the curves that $-K_X$ has zero intersection with. Let C be a K_X-negative extremal ray by [23, III.2.1.7] there exists a contraction of C, which is in fact a blow up map from another surface. This proves that C is a (-1) curve. Thus the extremal rays of $\overline{NE}(X)$ are precisely the (-1) and (-2) curves. We also know that X has at least one (-1) curve. This is because since $-K_X$ is big, ample divisors must be K_X-negative, but then at least one extremal ray must be K_X-negative. Also, there are only finitely many extremal rays for example see [15, 8.2.25, 8.2.34].

Given an ample line bundle A on X, we have that $L = \pi^*A$ has zero intersection number with all (-2) curves. Let m be the minimum intersection number of L with the finitely many (-1) curves on Y. Let $M = L + mK_Y$; we see that this divisor has non negative intersection number with all (-1) and (-2) curves (using the genus formula in the case of (-1) curves). Since these curves generate the cone of curves, we see that M is nef. Since M intersects some (-1) curve trivially, there exists a $f : Y \to Y'$ obtainable by contracting the (-1) curve, and a line bundle M', so that $M = f^*M'$.

We see that M intersects all (-2) curves trivially, since L and K_Y are pullbacks of line bundles on X. By viewing f as a blow up of Y' at y, we are in the situation of proposition 2.5.1, and can conclude that y is not a base point of M'.

By proposition 2.6.2 and 2.6.3, we can conclude that $h^1(Y, \Omega^1_Y \otimes K^*_Y) = n$, where n is the number of (-2) curves on Y and the theorem will follow if we can conclude that $h^1(Y, \Omega^1_Y \otimes L) \leq n$. We get this inequality from the following lemma.

□

Lemma 2.6.5. *Let Y be a smooth weak del Pezzo surface, and let M and L be as before. We then have that $h^1(Y, \Omega^1_Y \otimes L) \leq h^1(Y, \Omega^1_Y \otimes M \otimes K^*_Y) \leq h^1(Y, \Omega^1_Y \otimes K^*_Y) = h^1(Y, T_Y)$.*

Proof. The last equality follows from $\Omega^1_Y \otimes K^*_Y \simeq T_Y$, which is true for any smooth surface.

Given Y a weak del Pezzo surface, we can find a smooth divisor in the linear system $|K^*_Y|$ [14, page 40, Corollaire]. The first inequality follows since we can choose a smooth effective divisor D corresponding to K^*_Y. After tensoring the short exact sequence $0 \to O_Y(-D) \to O_Y \to O_D \to 0$ with $\Omega^1_Y \otimes M \otimes K^{(-n)}_Y$, for $n > 1$ we have the long exact sequence:

$$... \to H^1(Y, \Omega^1_Y \otimes M \otimes K^{(1-n)}_Y) \to H^1(Y, \Omega^1_Y \otimes M \otimes K^{(-n)}_Y)$$

$$\to H^1(D, \Omega^1_Y|_D \otimes M|_D \otimes K^{(-n)}_Y|_D) \to ...$$

The last term is zero though, since we have the short exact sequence on $D : 0 \to O_D(-D) \to \Omega^1_Y|_D \to K_D \to 0$. We can tensor through by $(M \otimes K^{(-n)}_Y|_D)$. Since K_Y has positive self intersection, $n > 1$, and D is a genus 1 curve we have that the two line bundles in the short exact sequence are of positive degree, which means (because D is genus 1 and Riemann-Roch) that the line bundles have trivial higher cohomology. We can thus conclude that the middle term has trivial higher cohomology as desired.

The middle inequality will be done by induction on $\rho(Y)$. Suppose that Y is a weak del Pezzo surface, and let N be a nef divisor that has zero intersection with all (-2) curves. We will show that $h^1(Y, \Omega^1_Y \otimes N \otimes K^*_Y) \leq h^1(Y, \Omega^1_Y \otimes K^*_Y)$.

The base case is $\mathbb{P}^2$, which is obvious. Suppose $\rho(Y) = m$. I will reduce to the case that N intersects at least one (-1) curve trivially. If it does not, then treat N as L in the preceding argument and use the inequality $h^1(Y, \Omega^1_Y \otimes L) \leq h^1(Y, \Omega^1_Y \otimes M \otimes K^*_Y)$. Now we can use M in that argument as our new N. We can form Y' as before, so we have a nef divisor N' on Y' who pulls back to N. We have the ideal sheaf short exact sequence on Y': $0 \to I_y \to O_{Y'} \to O_y \to 0$. After tensoring through by $T_{Y'} \otimes N'$, and using that $I_y \otimes T_{Y'} = f_* T_Y = Rf_* T_Y$, we get the long exact sequence

$$0 \to H^0(Y, T_Y \otimes N) \to H^0(Y', T_{Y'} \otimes N') \to H^0(y, T_y \otimes N|_y) \to H^1(Y, T_Y \otimes N)$$

$$\to H^1(Y', T_{Y'} \otimes N') \to 0$$

The last term is 0 because O_y is supported on y, a zero dimensional set. Inductively we have that $h^1(Y', T_{Y'} \otimes N') \leq h^1(Y', T_{Y'})$. We can construct a similar long exact sequence by replacing N with the trivial bundle, yielding:

$$0 \to H^0(Y, T_Y) \to H^0(Y', T_{Y'}) \to H^0(y, T_y) \to H^1(Y, T_Y) \to H^1(Y', T_{Y'}) \to 0$$

Since y is not a base point of N', we have that tensoring through by N' does not reduce the dimension of the image of $H^0(Y', T_{Y'} \otimes N') \to H^0(y, T_{Y'} \otimes N'|_y) = H^0(y, T_y)$. Thus by exactness we have that

$$h^1(Y, \Omega^1_Y \otimes N \otimes K^*_Y) =$$
$$h^0(y, T_y) - dim(im(H^0(Y', T_{Y'} \otimes N') \to H^0(y, T_{Y'} \otimes N'|_y))) + h^1(Y', T_{Y'} \otimes N')$$
$$\leq h^0(y, T_y) - dim(im(H^0(Y', T_{Y'}) \to H^0(y, T_{Y'}))) + h^1(Y', T_{Y'}) = h^1(Y, T_Y).$$

□

2.7 Examples

I will rephrase the main theorem in terms of global vector fields, allowing us to use the work of Martin and Stadlmayr to get a classification. I will start with computing $\chi(T_Y)$, for Y a smooth weak del Pezzo surface.

Proposition 2.7.1. *Let Y be a smooth rational surface, then $\chi(T_Y) = 10 - 2\rho(Y)$*

Proof. Suppose that $f : Y \to Y'$ is a blow up map of smooth rational surfaces, at a point y in Y'. The exact sequence $0 \to I_y \to O_{Y'} \to O_y \to 0$, gives us $\chi(T_Y) + \chi(T_{Y'}|_y) = \chi(T_{Y'})$ after tensoring with $T_{Y'}$. Finally since $T_{Y'}|_y$ is a rank 2 vector bundle on a point, we see that $\chi(T_{Y'}|_y) = h^0(T_{Y'}|_y) = 2$. The proposition follows after noticing that $\chi(T_{\mathbb{P}^2}) = h^0(T_{\mathbb{P}^2}) = 8$. One can alternatively do a simple Hirzebruch Riemann-Roch computation to get the proposition. □

We have that $\rho(Y) = n + \rho(X)$, where n is the number of (-2) curves on Y, and X is the del Pezzo surface with Gorenstein singularities corresponding to Y. Note that $H^2(Y, T_Y) = 0$ since by Serre duality we have that $h^2(Y, T_Y) = h^2(Y, \Omega^1_Y \otimes K^*_Y) = h^0(Y, K^2_Y \otimes T_Y) = h^0(Y, K_Y \otimes \Omega^1_Y) = 0$, with the last equality following from the fact that K_Y is anti effective, and since Y is rationally connected $H^0(Y, \Omega^1_Y) = 0$. We can compute that $h^0(Y, T_Y) - h^1(Y, T_Y) = 10 - 2n - 2\rho(X)$, using 2.7.1 and we know that Cartier Bott vanishing holds for X if and only if $h^1(Y, T_Y) = n$ by theorem 2.6.4, or in other words if and only if $h^0(Y, T_Y) = 10 - n - 2\rho(X)$. Using that $h^0(Y, T_Y) \geq 0$, we thus trivially get that lots of del Pezzo surfaces with Gorenstein singularities fail Bott vanishing (for example if $\rho(X)$ is greater than 5).

Given a del Pezzo surface with Gorenstein singularities, X, of degree $K^2_X = d$, and rank $\rho(X) = m$, we see that its desingulariation Y has rank $\rho(Y) = 10 - d$, using that smooth blow ups lower the degree by one. Also we know that $h^2(Y, T_Y) = 0$, so we have that $h^0(Y, T_Y) - h^1(Y, T_Y) = 2d - 10$. Thus X satisfies Cartier Bott vanishing if and only if $h^0(Y, T_Y) = 2d - 10 + n$. In particular if $h^0(Y, T_Y) = 0$, then X satisfies Cartier Bott vanishing if and only if $n + 2d = 10$. [29] supplies tables of smooth weak del Pezzo surfaces with the necessary information to read off which ones satisfy Cartier Bott vanishing, when $h^0(Y, T_Y) \neq 0$. I will use the notation of the paper, but include a second table converting their notation into singularity type. I recommend checking out their paper for more complete information about each type. I include a third table that classifies which del Pezzo surfaces with $h^0(Y, T_Y) = 0$ satisfy Cartier Bott vanishing sorted by singularity type; a list of possible singularity types can be found in [15]. I will separate the case of degree 1 del Pezzo surfaces with Gorenstein singularities, since so many of them do not satisfy Cartier Bott vanishing.

Table 2.1: del Pezzo surfaces with nontrivial vector fields

degree	Satisfies Bott Vanishing	Fails Bott Vanishing
9	9A	none
8	8A, $\mathbb{P}^1 \times \mathbb{P}^1, \Sigma_2$	none
7	7A,7B	none
6	6A,6B,6C,6D,6E,6F	none
5	5A,5B,5C,5D,5E,5F	none
4	4B,4C,4D,4E,4F,4G,4H,4I,4J,4K, 4L	4A
3	3C,3D,3E,3F,3G,3H,3I,3J	3A,3B
2	2D,2E,2F,2G,2H,2I,	2A,2B,2C

Table 2.2: del Pezzo surfaces with nontrivial vector fields by singularity type

degree	Satisfies Bott Vanishing	Fails Bott Vanishing
9	$\mathbb{P}^2$	none
8	$\emptyset, \mathbb{P}^1 \times \mathbb{P}^1, \Sigma_2$	none
7	$\emptyset, A_1$	none
6	$\emptyset, A_1, A_1, 2A_1, A_2, A_2 + A_1$	none
5	$A_1, 2A_1, A_2, A_2 + A_1, A_3, A_4$	none
4	$3A_1, A_2 + A_1, A_3, A_3, 4A_1, A_2 + 2A_1,$	$2A_1$
4	$A_3 + A_1, A_4, D_4, A_3 + 2A_1, D_5$	
3	$2A_2 + A_1, A_3 + 2A_1, A_4 + A_1, A_5, D_5, 3A_2, A_5 + A_1, E_6$	$2A_2, D_4$
2	$2A_3 + A_1, D_4 + 3A_1, A_5 + A_2, D_6 + A_1, A_7, E_7$	$2A_3, D_5 + A_1, E_6$

Table 2.3: del Pezzo surfaces with no nonvanishing vector fields

degree	Satisfies Bott vanishing	Fails Bott vanishing
5	$(\emptyset)$	none
4	$2A_1, A_2$	$\emptyset, A_1$
3	$A_4, A_3 + A_1, 4A_1$	$\emptyset, A_1, A_2, A_3, A_2 + A_1, 3A_1$
2	$A_4 + A_2, A_6, A_3 + A_2 + A_1, 2A_3$	$D_4 + A_1, D_5, A_5, A_3 + A_2, 2A_2 + A_1, A_4 + A_1, A_3 + 2A_1, A_2 + 3A_1$
2	$D_4 + 2A_1, 3A_2, 6A_1, A_5 + A_1, D_6$	$5A_1, D_4, 2A_2, A_4, A_3 + A_1, A_2 + 2A_1, 4A_1, A_3, A_2 + A_1, 3A_1, A_2, 2A_1, A_1, \emptyset$

Proposition 2.7.2. *Let X be a degree 1 del Pezzo surface with Gorenstein singularities, if X satisfies Cartier Bott vanishing, then the corresponding weak del Pezzo surface, Y, must have 8 (-2) curves. Further, there are 16 types of such del Pezzo surfaces, 12 of which satisfy Cartier Bott vanishing.*

Proof. Since $n = \rho(Y) - \rho(X)$, and Y is the blow up of $\mathbb{P}^2$ at at most 8 points, we know that $n \leq 8$. If X satisfies Cartier Bott vanishing, then $h^0(Y, T_Y) = 2d - 10 + n$, thus n must be 8 if $d = 1$. Further if n is equal to 8, we just need to check if $h^0(Y, T_Y) = 0$. We can see from [29] this is not true exactly in the cases 1A, 1B, 1C, or 1D. Alternatively, Alexeev and Nikulin described all 16 classes, and studied which satisfy $h^0(Y, T_Y) = 0$. [3] The singularities of the del Pezzo surfaces with Gorenstein singularities that do satisfy Cartier Bott vanishing are given as follows: $E_8, A_8, A_7+A_1, A_5+A_2+A_1, 2A_4, D_8, D_5+A_3, E_6+A_2, E_7+A_1, D_6+2A_1, 2A_3+2A_1, 4A_2$. $\square$

It is interesting to note that there are 16 types of Gorenstein toric del Pezzo surfaces. This is well known, but one reference is [22]. 5 types are smooth, and the remaining 11 types have singularities as follows $A_1, A_1, A_1, 2A_1, 2A_1, A_2 + A_1, A_2 + A_1, 4A_1, A_2 + 2A_1, 2A_2 + A_1, A_3 + 2A_1, 3A_2$. These automatically satisfy Bott vanishing since they are toric. The previous tables give many more examples of del Pezzo surfaces with Gorenstein singularities that satisfy Cartier Bott vanishing.

Chapter 3

Weil Bott vanishing and F-liftability

3.1 Introduction

In this section I will show that F-liftable varieties satisfy Weil Bott vanishing. It was proven that they satisfy Cartier Bott vanishing in [8]. I will follow their proof with modifications to account for the Weil divisors. I will start by recalling some facts and definitions. As an application I will give an alternate proof that toric varieties satisfy Weil Bott vanishing, which requires a deformation argument for toric varieties in characteristic 0.

3.2 Weil Bott vanishing and reflexive sheaves

We will start by recalling the definition of F-liftability and making Weil Bott vanishing precise. Note that $E^* := \mathcal{H}om(E, O_X)$ will denote the dual of a sheaf E for this chapter.

Definition 3.2.1. Let X be a k-scheme, for a perfect field k of characteristic p. Let F be the k relative Frobenius morphism. We say that X is F-liftable if there exists a lifting of X mod p^2 (that is, to the second Witt vectors $W_2(k)$) together with a lifting of F.

We will call a Weil divisor ample, if for some positive integer e, eD is an ample Cartier divisor.

Definition 3.2.2. Let X be a normal projective variety. We say that X satisfies Weil Bott vanishing if for all ample Weil divisors, D, on X, we have that $H^i(X, ((\Omega_X^{[r]} \otimes O_X(D))^{**}) = 0$, for $i > 0$ and $r \geq 0$

The double dual here ensures that the sheaf in question is reflexive, this is quite standard when working with reflexive sheaves. I will now recall some facts and definitions about reflexive sheaves. These properties are well known and can be found in [20].

Definition 3.2.3. Let X be a variety, and G a sheaf on X. We say that G is reflexive if the natural map $\phi : G \to G^{**}$ is an isomorphism.

Proposition 3.2.4. *Let X be a normal projective variety, and suppose that D is a Weil divisor on X. Then $O_X(D)$ is a reflexive sheaf.*

Proposition 3.2.5. *Let X be a normal projective variety with smooth locus U. Let G be a reflexive sheaf on X, and j the inclusion map of U to X, then the natural map $G \to j_* j^* G$ is an isomorphism.*

Proof. This is a restatement of [34, Tag 0AVT]]. □

Proposition 3.2.6. *Let X be a normal projective variety with smooth locus U. Suppose that F_1 and F_2 are reflexive sheaves on X, then F_1 and F_2 are isomorphic if and only if their restrictions to U are isomorphic.*

Proof. For any reflexive sheaf, G, G is isomorphic to j_*j^*G where j is the inclusion map of U. Thus the proposition follows by functoriality of j^* and j_*. □

Proposition 3.2.7. *Let X be a normal projective variety, and suppose that G is a reflexive sheaf, and V is a locally free sheaf. Then $V \otimes G$ is reflexive.*

Proof. We can check that the natural map $\phi : V \otimes G \to (V \otimes G)^{**}$ is an isomorphism locally. Thus we can assume that V is free, and the proposition then easily follows from the reflexivity of G. □

Proposition 3.2.8. *Let X be a normal projective variety, and suppose that G is a coherent sheaf. Then G^* is a reflexive sheaf.*

Proof. See Theorem 2.8 from [8]. □

I will state generic flatness here, which will be used when we deform toric varieties to characteristic p.

Proposition 3.2.9. *Let $f : X \to Y$ be a finite type morphism of schemes. Assume that Y is integral. Let G be a coherent sheaf on X, then there is a nonempty open subset U of Y such that $G|_{f^{-1}(U)}$ is flat over U.*

One reference is [34, Tag 052A] We will use this flatness to allow us to commute duals with pullbacks by restricting to an open set of our base.

3.3 F-liftability and Weil Bott vanishing

I will now give the main proof in this chapter.

Theorem 3.3.1. *Let X be a normal projective variety that is F-liftable. Then X satisfies Weil Bott vanishing.*

Proof. First let D be an ample Weil divisor with Cartier index e. Since eD is an ample Cartier divisor we can, by Serre vanishing, for $0 \leq i \leq e-1$, find numbers N_i such that $H^j(X, (\Omega_X^{[k]} \otimes O_X(iD)^{**} \otimes O_X(neD)) = 0$ for $n \geq N_i$, and $j > 0$. Now take N to be the max of the N_i, and choose l such that $p^l \geq e(N+1)$. Write $p^l = me + r$, with $0 \leq r \leq e-1$, and $m \geq N$.

We will show that $(\Omega_X^{[k]} \otimes O_X(rD))^{**} \otimes O_X(meD) \simeq (\Omega_X^{[k]} \otimes O_X(rD) \otimes O_X(meD))^{**}$. Note that both sheaves are reflexive, since taking the double dual of a coherent sheaf is reflexive by 3.2.8, and tensoring with a locally free sheaf preserves reflexivity by 3.2.7. It is then sufficient to check on the smooth locus by 3.2.6, U, of X since X is normal. We see that both sheaves are the same vector bundle on U.

Using the above isomorphism, we get that $H^j(X, (\Omega_X^{[k]} \otimes O_X(p^l D))^{**}) = 0$. Thus to show Weil Bott vanishing, it is sufficient to show that $H^j(X, (\Omega_X^{[k]} \otimes O_X(D))^{**})$ injects into $H^j(X, (\Omega_X^{[k]} \otimes O_X(pD))^{**})$, the latter being 0 by descending induction. Because X is F-liftable, U is also F-liftable, and we have that $\Omega_U^k \to F_*(\Omega_U^k)$ is a split monomorphism (here F is the Frobenius morphism) [8].The split mono is defined in a slightly complicated way, but it is related to the Cartier operator. We can tensor through by $O_U(D)$, which is a line bundle on U. We then have the split monomorphism,

$$\Omega_U^k \otimes O_U(D) \to F_*(\Omega_U^k) \otimes O_U(D)$$

By noting that $F^* O_U(D) \simeq O_U(pD)$, and using the projection formula, we have the split monomorphism

$$\Omega_U^k \otimes O_U(D) \to F_*((\Omega_U^k) \otimes O_U(pD))$$

Now we will push forward along the inclusion $j : U \to X$, and use the fact that the Frobenius commutes with j :

$$j_*(\Omega_U^k \otimes O_U(D)) \to F_* j_*((\Omega_U^k) \otimes O_U(pD)))$$

is a split monomorphism. Now since $((\Omega_U^k) \otimes O_U(D)$ is a vector bundle on U, we have that $j_*((\Omega_U^k) \otimes O_U(D)))$ is a relexive sheaf. I claim that $j_*(\Omega_U^k \otimes O_U(D)) \simeq (j_*\Omega_U^k \otimes j_* O_U(D))^{**} = (\Omega_U^{[k]} \otimes O_X(D))^{**}$. We can check this isomorphism on U since both sides are reflexive. Using that pulling back along open inclusions commutes with tensor products and duals, we have reduced to checking that $j^* j_*(\Omega_U^k \otimes O_U(D) \simeq (j^* j_* \Omega_U^k \otimes j^* j_* O_U(D))^{**}$. This isomorphism is true since $j^* j_*$ is the acts as the identity, and $O_U(D)$ is free. Similarly we can show that $j_*(\Omega_U^k \otimes O_U(pD) \simeq (j_* \Omega_U^k \otimes j_* O_U(pD))^{**} = ((\Omega_U^{[k]} \otimes O_X(pD))^*)^*$. Using that cohomology is an additive functor, we have the desired result, that $H^j(X, (\Omega_X^{[k]} \otimes O_X(D))^{**})$ injects into $H^j(X, (\Omega_X^{[k]} \otimes O_X(pD))^{**})$.

□

3.4 The example of Toric Varieties

I will recall some facts about toric varieties. Two good references are *The geometry of Toric Varieties* by V.I Danilov [12] and *Introduction to Toric Varieties* by William Fulton [16]. Let k be a field, and let X be a normal variety over k of dimension n. We say that X is a toric variety if it has an algebraic torus, G_m^n as a dense open set whose action extends to all of X. I will summarize the equivalent combinatorial description of toric varieties that is often easier to work with and extends more readily to more general bases. Let R be an integrally closed domain. A fan is a set of rational strongly convex polyhedral cones in $\mathbb{R}^n$ such that each face of a cone in the set is itself a cone in the set, and each intersection of cones in the set is a face of each. Given a fan Δ, we can associate to each cone σ the semigroup $S_\sigma = \sigma^* \cup (\mathbb{Z}^n)^*$. We can form an affine toric variety over R by associating to each cone the affine variety $\mathrm{Spec}(R[S_\sigma])$. The intersections of cones in the fan are faces which - also corresponding to affine toric varieties - can be used as gluing data, which gives rise to a toric variety corresponding to the fan Δ and the ring R, which we will denote $X^R(\Delta)$. Toric varieties are sometimes defined to be schemes that arise in this way. Given a ring homomorphism $f : R \to R'$, we can pull back $X^R(\Delta)$ along f^* to get the toric variety $X^{R'}(\Delta)$. This is clear if $X^R(\Delta)$ is affine, since $R'[S_\sigma] = R' \otimes R[S_\sigma]$, and since the gluing data is all compatible, it is true for all toric varieties. If given a cone τ in fan Δ, we can associate a closed subvariety, $V^R(\tau)$, by taking the closure of the orbit O_τ in $X^R(\Delta)$ [16, 3.1]. The ideal corresponding to this closed subvariety is $\bigoplus R\chi^u$, where the sum is over $u \in S_\sigma$ such that $\langle u, v\rangle > 0$ for v in the relative interior of τ. [16, page 54] We see that these ideals are compatible with base change, that is that $f^*(O_{X^R(\Delta)}(-V^R(V_\tau)) \simeq O_{X^{R'}(\Delta)}(-V^{R'}(\tau))$. Every Weil divisor on a toric variety $X^R(\Delta)$ is linearly equivalent to a T-Weil divisor [16, 3.4], that is, a divisor that is generated by divisors of the form $V^R(\tau_i)$, for rays τ_i in Δ. We can also describe reflexive differentials in terms of Δ. We can describe reflexive differentials on open sets of the form $A = \mathrm{Spec}(R[S_\sigma])$ as the R-module $\bigoplus_{m\in\sigma\cup M} \Lambda^r(\bigcap_{m\in\theta} V_\theta)x^m$, where θ are the codimension one faces of σ, and V_θ is the lattice spanned by $M \cup \theta$ [12, I.4.3].

Many properties of a toric variety can be read off of Δ, for example, if Δ is complete (the union of cones is the whole vectorspace), then $X^R(\Delta)$ is proper over $\mathrm{Spec}(R)$. Also whether a toric divisor is ample just depends on the combinatorics τ and Δ [16, page 99]. These facts allow us to switch base fields while preserving ampleness and other key properties.

We will use generic flatness to understand reflexive sheaves in families.

Lemma 3.4.1. *(i) Let $f : R \to R'$ be a faithfully flat map of rings. Let X be an integral scheme of finite type over R, and let $g : X' \to X$ be the base change. Let G be a coherent sheaf on X, then*

$$g^*(G^*) \simeq g^*(G)^*.$$

(ii) Let X be an integral scheme of finite type over a Noetherian integral domain R. Let G be a sheaf on X, then there exists a nonempty open set of $\mathrm{Spec}(R)$, $\mathrm{Spec}(R[\frac{1}{f}])$, such that for every point $p \in \mathrm{Spec}(R[\frac{1}{f}])$, we have that $G^*|_p \simeq (G|_p)^*$.

Proof. These are proven similarly. I will prove (ii). There exists a natural homomorphism $\phi : F^*|_p \to (F|_p)^*$. We can check that this is an isomorphism locally, so replace X with $\mathrm{Spec}(A)$, and let F correspond to the finitely presented A module M. We need to show that the natural map $\phi : Hom_A(M, A) \otimes R/p \to Hom_{A/p}(M/pM, A/p)$, is an isomorphism for some principal open set of values of p. Note that since M is finitely presented, $Hom_{A_f}(M_f, A_f) = Hom_A(M, A)_f$, so we can replace A with A_f, and M with M_f as it suits us. Since M is finitely presented, we can write the exact sequence

$$A^m \to A^n \to M \to 0$$

We can now take duals, and since Hom is contravariantly left exact in the second variable, we have the exact sequence

$$0 \to M^* \to (A^n)^* \to (A^m)^*$$

I want to now argue that after shrinking $\mathrm{Spec}(R)$, that tensoring with R/p yields an exact sequence. We can cut our above exact sequence into two short exact sequences:

$$0 \to M^* \to (A^n)^* \to N \to 0$$

$$0 \to N \to (A^m)^* \to N' \to 0$$

Using generic flatness, we can shrink $\mathrm{spec}(R)$ so that N and N' are flat. However, because of properties of the tor functors, we can tensor these short exact sequences with R/p to yield the exact sequences:

$$0 \to M^* \otimes R/p \to (A^n)^* \otimes R/p \to N \otimes R/p \to 0$$

$$0 \to N \otimes R/p \to (A^m)^* \otimes R/p \to N' \otimes R/p \to 0$$

We can now stitch these sequences back to get the exact sequence

$$0 \to M^* \otimes R/p \to (A^n)^* \otimes R/p \to (A^m)^* \otimes R/p$$

Similarly applying generic flatness before dualizing we can get the exact sequence

$$A^m \otimes R/p \to A^n \otimes R/p \to M \otimes R/p \to 0$$

We can now dualize to get the exact sequence

$$0 \to (M \otimes R/p)^* \to (A^n \otimes R/p)^* \to (A^m \otimes R/p)^*$$

We further have natural maps giving us the diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M^* \otimes R/p & \longrightarrow & (A^n)^* \otimes R/p & \longrightarrow & (A^m)^* \otimes R/p \\
\downarrow & & \downarrow \phi & & \downarrow & & \downarrow \\
0 & \longrightarrow & (M \otimes R/p)^* & \longrightarrow & (A^n \otimes R/p)^* & \longrightarrow & (A^m \otimes R/p)^*
\end{array}$$

By augmenting this diagram with another column of zeros on the left, we can use five lemma to conclude that ϕ is an isomorphism as desired.

(i) We can prove this similarly, by reducing locally and working with our presentation. The major modification is to use the flatness of R' instead of generic flatness. □

We will now show that projective toric varieties satisfy Weil Bott vanishing.

Proposition 3.4.2. *Let X be a projective toric variety over a field k. X satisfies Weil Bott vanishing.*

Proof. If $\text{char}(k) = p$, then X is F-liftable [8] and projective so we can use the previous result 3.3.1. The main idea behind showing that X is F-liftable is to use the pth power endomorphism as a lifting. This endomorphism is induced locally by the semigroup morphism $e \mapsto pe$ on S_σ. These maps clearly glue together to create a globally defined endomorphism on X. Now suppose $\text{char}(k) = 0$; we will do a deformation argument to finite characteristic. First let us fix that

$X = X^k(\Delta)$ for some fan Δ. Since every Weil divisor is linearly equivalent to a toric divisor, and the cohomology groups only depend on $O_X(D)$, for ample Weil divisors D, we can assume D is toric. We will first reduce to the case that $k = \mathbb{Q}$. Let $j : X \to X^{\mathbb{Q}}(\Delta)$ be the base change map. Note that $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{Q})$ is faithfully flat.

Since D is toric, we can write $D = \Sigma a_i V^k(\tau_i)$. We see that

$$j^*((\Omega^{[r]}_{X^{\mathbb{Q}}(\Delta)} \otimes O_{X^{\mathbb{Q}}(\Delta)}(D))^{**})$$

$$\simeq (j^*(\Omega^{[r]}_{X^{\mathbb{Q}}(\Delta)} \otimes O_{X^{\mathbb{Q}}(\Delta)}(D)))^{**}$$

by 3.4.1. We can check that $j^*(\Omega^{[r]}_{X^{\mathbb{Q}}(\Delta)}) \simeq \Omega^{[r]}_X$, since reflexive differentials are the double dual of kahler differentials, which do behave nicely with respect to base change. Finally we have that

$$O_{X^{\mathbb{Q}}(\Delta)}(D) \simeq X^{\mathbb{Q}(\Delta)}(\Sigma a_i V^{\mathbb{Q}}(\tau_i)) \simeq (\bigotimes((O_X^{\mathbb{Q}(\Delta)}(-V^{\mathbb{Q}}(\tau_i)))^{(\otimes a_i)})^*)^{**}.$$

Thus using lemma 3.4.1 and commuting/distributing the pullback across the duals and tensor products, we get that

$$j^*((\Omega^{[r]}_{X^{\mathbb{Q}}(\Delta)} \otimes O_{X^{\mathbb{Q}}(\Delta)}(D))^{**}) \simeq (\Omega^{[r]}_X \otimes O_X(D))^{**}.$$

We can use faithfully flat base change to get the desired reduction:

$$H^i(X^{\mathbb{Q}}, (\Omega^{[r]}_{X^{\mathbb{Q}}} \otimes O_{X^{\mathbb{Q}}(\Delta)}(D)))^{**}) = H^i(X, (\Omega^{[r]}_X \otimes O_X(D))^{**}).$$

We can now view $X^{\mathbb{Z}}(\Delta)$ as a family in mixed characteristic. Using a similar argument to our previous reduction to $\mathbb{Q}$, after localizing away from some amount of small primes, by the lemma we have that $(\Omega^{[r]}_{X^{\mathbb{Z}}(\Delta)} \otimes O_{X^{\mathbb{Z}}(\Delta)}(D)))^{**})|_{\mathbb{F}_{|}} \simeq (\Omega^{[r]}_{X^{\mathbb{F}_{|}}(\Delta)} \otimes O_{X^{\mathbb{F}_{|}}(\Delta)}(D)))^{**}$. Note that $X^{\mathbb{F}_{||}}(\Delta)$ is proper, since Δ is complete, and $O_{X^{\mathbb{F}_{||}}(\Delta)}(D)$ is ample. By F-liftability, $H^i(X^{\mathbb{F}_{||}}, (\Omega^{[r]}_{X^{\mathbb{F}_{|}}} \otimes O_{X^{\mathbb{F}_{|}}(\Delta)}(D)))^{**}) = 0$ for $i > 0$, and since $X^{\mathbb{Z}}$ is proper, we can apply the semicontinuity [19, III.12.8] to get that $H^i(X^{\mathbb{Q}}(\Delta), (\Omega^{[r]}_{X^{\mathbb{Q}}(\Delta)} \otimes O_{X^{\mathbb{Q}}(\Delta)}(D)))^{**}) = 0$ as desired.

Lemma 3.4.3. *Let X be an integral normal scheme of finite type over* $\mathrm{Spec}(\mathbb{Z})$. *Let D be an ample Weil Divisor on X, and let U be a nonempty open subset of X as in lemma 3.4.1 for $O_X(D)$. Then for p in U, we have that $O_X(D)|F_p$ is a reflexive sheaf associated to an ample Weil divisor on $X|_{F_p}$*

Proof. First we will show that $O_X(D)|F_p$ is a reflexive sheaf. This follows by taking the double dual and applying lemma 3.4.1 to show that $O_X(D)|F_p{}^{**} \simeq (O_X(D))^{**}|F_p \simeq O_X(D)|F_p$. Now that we know it is reflexive, observe that we can check that whether it arises from a Weil divisor by restricting to the smooth locus (using that X is normal). We have that its restriction to the smooth locus is clearly isomorphic to the restriction of the cartier divisor on the smooth locus of X corresponding to D. To check ampleness, just note that $(O_X(D)|F_p{}^{\otimes n})^{**} \simeq O_X(nD)|F_p$ for any integer n. □

We can simplify 3.4.2 using 3.4.3. Also we can extend our deformation argument to quotients of toric varieties by finite groups using that these varieties are also F-liftable [1].

□

Chapter 4

Counterexample for Kawamata-Viehweg vanishing

4.1 Introduction

The Kawamata-Viehweg vanishing theorem is a generalization of the Kodaira vanishing theorem which serves as a powerful tool in the minimal model program over $\mathbb{C}$. Cascini, Tanaka and Witaszek showed that Kawamata-Viehweg vanishing holds for log del Pezzo surfaces over an algebraically closed field of sufficiently high characteristic [11]. Lacini recently made this result effective by giving $p \geq 7$ as a lower bound [27]. His method was to show that all log del Pezzo surfaces over an algebraically closed field with characteristic greater than 5 lift to characteristic 0 over a smooth base. Alternatively, Cascini and Tanaka produced an example in characteristic 2 of a log del Pezzo surface which fails Kawamata-Viehweg vanishing [9]. Then, Bernasconi produced a similar example in characteristic 3 [7]. Thus only in characteristic 5 was it unknown if Kawamata-Viehweg vanishing holds for log del Pezzo surfaces over an algebraically closed field. In this note I will use the methods in [7] to show that the surface given by Lacini in [27] as an example of failure for log del Pezzo surfaces in characteristic 5 to lift to characteristic 0 over a smooth base is also an example of Kawamata-Viehweg vanishing theorem failing.

This paper gives the same example as the one given in [6]. I did my work independently of theirs. The proofs differ in how we establish the vanishing of the higher pushforwards.

4.2 Preliminaries

The actual cohomology computation will not use very much special machinery. I would like to give some background information about how the specific Weil divisor was selected to find the counterexample. First let me review Kawamata-Viehweg vanishing for characteristic zero, which I will state with trivial boundary divisor.

Theorem 4.2.1 (Kawamata-Viehweg vanishing theorem)**.** *[26, 2.70] Let X be a klt projective variety over $\mathbb{C}$. Let L be a big and nef $\mathbb{Q}$-Cartier Weil divisor on X. Then $H^i(X, O_X(K_X+L)) = 0$ for $i > 0$.*

Proof. The only thing to note is that I gave the dual version, following from Serre duality. □

Note that even in our singular characteristic p setting, we get Serre duality in its traditional form.

Theorem 4.2.2. *[13, beginning of section 2] Let X be a normal projective surface. Let D be a Weil divisor on X. Then $H^i(X, O_X(D)) \simeq H^{2-i}(X, O_X(K_X - D))^*$ for all $i \geq 0$.*

Kawamata-Viehweg vanishing extends at least partially to log Fano surfaces in positive characteristic, p. As discussed in the introduction, the same statement is true if $p > 5$. Also the statement is true if we can assume that L is effective [10], or if L is Cartier and $p > 3$ [31]. On a surface of Picard rank 1, the L can only be a counterexample if it is a (non Cartier) Weil divisor which is numerically effective, but not linearly equivalent to an effective divisor. This significantly narrowed down my search for a counterexample. I will close this section by stating birational Kawamata-Viehweg vanishing which is a powerful tool for studying surfaces in characteristic p. I again state the result with trivial boundary to simplify the statement.

Lemma 4.2.3. *[25, 10.4] Let X be a regular surface and $f : X \to Y$ be a proper birational morphism with exceptional curves C_i such that the union is connected. Let L be a line bundle on X, and assume that $(L - K_X) \cdot C_i \geq 0$ for every i. Then $R^1 f_* L = 0$.*

4.3 The example and the proof

The example is taken from [27]. We will abuse notation by identifying every divisor with its proper transforms. Let k be an algebraically closed field of characteristic 5. Take four points in general position on $\mathbb{P}^2, p_1, p_2, p_3, p_4$, and call the six lines going through p_i and p_j for various i and j, L_{ij}. Blow up at each of the four points, to form the degree 5 del Pezzo surface X, with exceptional divisors above p_i which we will call B_i. Now blow up at the intersections of B_1 and L_{13}, B_2 and L_{24}, B_3 and L_{23}, and B_4, and L_{14} forming a degree 1 weak del Pezzo surface we will call $\tilde{X}$. Call these exceptional divisors G_1, G_2, G_3, and G_4 respectively. In our situation, we have that there exists a cuspidal curve in the anticanonical linear system $|-K_{\tilde{X}}|$ (this phenomenon is unique to characteristic 5 for our blow up configuration). Now we blow up three more times to resolve the cusp and to make the exceptional divisor have simple normal crossings, creating a new surface $\tilde{Y}$. I will call the (-3), and (-2) exceptional divisors E_3 and E_2 respectively. I will call the proper transform of the cuspidal curve, a (-5) curve, E_5. I will call the last exceptional divisor, a (-1) curve, C.

I will now summarize some facts about $\tilde{Y}$. Since the cuspidal curve is in the anticanonical system of $\tilde{X}$, one can calculate that $-K_{\tilde{Y}} = E_5 + E_3 + E_2 + 2C$. $\tilde{Y}$ has two disjoint A_4

configurations of (-2) curves; B_1, L_{14}, L_{23}, and B_2 is the first configuration, and B_3, L_{13}, L_{24}, and B_4 is the second configuration. A basis for the divisor class group of $\tilde{Y}$ is $L, B_1, B_2, B_3, B_4, G_1, G_2, G_3, G_4, E_3, E_2, C$; where L is the proper transform of a generic line in $\mathbb{P}^2$. The intersection matrix relative to this ordered basis is:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

We also have some relations (with respect to linear equivalence) that will be useful:

$$L_{12} = L - B_1 - B_2 - G_1 - G_2$$

$$L_{13} = L - B_1 - B_3 - 2G_1 - G_3$$

$$L_{14} = L - B_1 - B_4 - G_1 - 2G_4$$

$$L_{23} = L - B_2 - B_3 - G_2 - 2G_3$$

$$L_{24} = L - B_2 - B_4 - 2G_2 - G_4$$

$$L_{34} = L - B_3 - B_4 - G_3 - G_4$$

$$E_5 = L - 2E_3 - 3E_2 - 6C$$

We can now form Y by contracting the 11 rational curves with self intersection less than

(−1). I will call the contraction $\pi : \tilde{Y} \to Y$. This variety has a Picard rank of 1, and has 5 singularities, three of which are canonical, together denoted as $2A_4 + A_1$, there are also two klt singularities resulting from contracting E_3 and E_5. We can use the previous relations and some Gaussian elimination to compute a presentation of the divisor class group of Y. We have generators C, G_1, and $(G_1 - G_2)$ with relations:

$$6C = 3G_1 - 4(G_1 - G_2)$$

$$5(G_1 - G_2) = 0$$

We can also compute the canonical divisor K_Y, by taking the cycle pushforward of $K_{\tilde{Y}}$, which works out to be $-2C$. Using that the Picard rank is 1, and that $-K_{\tilde{Y}}$ is effective, we see that Y is log Fano. We can identify $Pic(Y)$ inside the class group $Div(Y)$ by seeing which divisors are integral after pulling back to $\tilde{Y}$.

$$\pi^*(C) = C + \frac{1}{2}E_2 + \frac{1}{3}E_3 + \frac{1}{5}E_5$$

$$\pi^*(G_1) = G_1 + (\frac{4}{5}B_1 + \frac{3}{5}L_{14} + \frac{2}{5}L_{23} + \frac{1}{5}B_2) + (\frac{3}{5}B_3 + \frac{6}{5}L_{13} + \frac{4}{5}L_{24} + \frac{2}{5}B_4)$$

$$\pi^*(G_2) = G_2 + (\frac{4}{5}B_2 + \frac{3}{5}L_{23} + \frac{2}{5}L_{14} + \frac{1}{5}B_1) + (\frac{3}{5}B_4 + \frac{6}{5}L_{24} + \frac{4}{5}L_{13} + \frac{2}{5}B_3)$$

Thus we have that $30C$ and $5G_1$ are Cartier, and one can see that these together generate $Pic(Y)$. I will note that $G_1 - G_2 - C - K_Y = G_1 - G_2 + C$, is big and nef since it is numerically effective on a variety with Picard rank 1.

Proposition 4.3.1. $H^1(Y, O_Y(G_1 - G_2 - C)) \neq 0$. *Thus Kawamata-Viehweg vanishing fails, since $G_1 - G_2 - C - K_Y$ is nef and big, and Y is klt.*

Proof. Our strategy will be to pullback our divisor to $\tilde{Y}$ and use Rieman-Roch. We compute that

$$\pi^*(G_1 - G_2 - C) =$$

$$G_1 + (\frac{4}{5}B_1 + \frac{3}{5}L_{14} + \frac{2}{5}L_{23} + \frac{1}{5}B_2) + (\frac{3}{5}B_3 + \frac{6}{5}L_{13} + \frac{4}{5}L_{24} + \frac{2}{5}B_4)$$

$$-G_2 - (\frac{4}{5}B_2 + \frac{3}{5}L_{23} + \frac{2}{5}L_{14} + \frac{1}{5}B_1) - (\frac{3}{5}B_4 + \frac{6}{5}L_{24} + \frac{4}{5}L_{13} + \frac{2}{5}B_3)$$

$$-C - (\frac{1}{5}E_5 + \frac{1}{3}E_3 + \frac{1}{2}E_2)$$

Thus we can compute the round down $\lfloor(\pi^*(G_1 - G_2 - C))\rfloor = G_1 - G_2 - L_{23} - B_2 - L_{24} - B_4 - C - E_5 - E_1 - E_2$. I will name this divisor D. We can compute $\chi(D) = 1 + \frac{1}{2}(D^2 - K_{\tilde{Y}} \cdot D) = -1$ by Riemann-Roch. Thus $h^1(\tilde{Y}, O_{\tilde{Y}}(D)) > 0$. Now we must use this calculation to get our result. First note that $\pi_*(O_{\tilde{Y}}(D)) = O_S(G_1 - G_2 - C)$ [13]. Thus if we can show that the higher push forwards of $O_{\tilde{Y}}(D)$ are zero, then we can finish the proof using that the Leray spectral sequence gives us

$$H^1(\tilde{Y}, O_{\tilde{Y}}(D)) \simeq H^1(Y, O_Y(G_1 - G_2 - C)).$$

Birational relative Kawamata-Viehweg vanishing (see theorem 4.2.3) is sufficient for checking this for all of the contractions except one; unfortunately,

$$(D - K_{\tilde{Y}}) \cdot L_{14} = -1,$$

so we do not have nefness on all of components of the exceptional fibers. This amounts to us needing to show by hand that $R^1\pi_*O_{\tilde{Y}}(D))_p = 0$, where p is the point below the fiber containing L_{14}. We will show this using the theorem on formal functions in the next paragraph [19, III.11.1]. First I will list the intersection numbers of $D - K_{\tilde{Y}}$ with each of the exceptional curves.

$$(D - K_{\tilde{Y}}) \cdot E_5 = 1$$

$$(D - K_{\tilde{Y}}) \cdot E_3 = 1$$

$$(D - K_{\tilde{Y}}) \cdot E_2 = 1$$

$$(D - K_{\tilde{Y}}) \cdot B_1 = 1$$

$$(D - K_{\tilde{Y}}) \cdot L_{14} = -1$$

$$(D - K_{\tilde{Y}}) \cdot L_{23} = 1$$

$$(D - K_{\tilde{Y}}) \cdot B_2 = 0$$

$$(D - K_{\tilde{Y}}) \cdot B_3 = 0$$

$$(D - K_{\tilde{Y}})) \cdot L_{13} = 0$$

$$(D - K_{\tilde{Y}}) \cdot L_{24} = 0$$

$$(D - K_{\tilde{Y}}) \cdot B_4 = 1$$

Let E be the exceptional fiber of p. E is a chain of (-2) curves, and, though it is singular at the intersection points, we can still calculate the dualizing sheaf of E, ω_E. Briefly, sections of this sheaf can be described as differential forms on each component with at most simple poles on the nodes, with the condition on each node being that the two differential forms have residue that add to 0 [18, 3.3]. Differential forms on $\mathbb{P}^1$ have two poles, and we can choose which points these are on. By choosing the poles to be on the nodes, and scaling by constants appropriately to match residues, we can construct a section of ω_E whose only nonnodal poles are on B_1 and B_2. Thus we have that $\omega_E = -P - Q$ where P is a general point of B_1, and Q is a general point of B_2. Alternatively, one may use [19, II.7.11] and compute the dualizing sheaf directly from the intersection theory of $\tilde{Y}$. Now I want to show that $R^1\pi_* O_{\tilde{Y}}(D))_p = 0$. Using the theorem on formal functions this follows from showing that for all natural numbers n, $H^1(nE, O_{nE}(D)) = 0$ [19, III.11.1]. For n a nonnegative integer, we have that $H^1(E, O_E(D - nE)) \simeq H^0(E, O_E(\omega_E - D + nE))^*$, using Serre duality. We can see that $\omega_E + nE - D$ has negative degree on all divisors except for L_{14}, where it is degree 1. Any section of $H^0(E, O_E(\omega_E - D + nE))$ must be zero on all components except for L_{14}, where it must have a zero on the intersections with B_1 and L_{23}. The only global section of a degree 1 line bundle on P^1 that has two zeroes is the zero section, thus

$H^1(E, O_E(D - nE)) = 0$. I will now use the exact sequence $0 \to O_E(D - nE) \to O_{(n+1)E}(D) \to O_{nE}(D) \to 0$. The long exact sequence on cohomology yields that $H^1((n+1)E, O_{(n+1)E}(D)) \simeq H^1(nE, O_{nE}(D))$. We will proceed in proving that $H^1(nE, O_{nE}(D)) = 0$ by induction on n. When $n = 1$, $H^1(E, O_E(D)) \simeq H^0(E, O_E(K - D))^* = 0$. The inductive step obviously follows from $H^1((n + 1)E, O_{(n+1)E}(D)) \simeq H^1(nE, O_{nE}(D))$.

□

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