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Tambara Functors and Prisms

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ABSTRACT OF THE DISSERTATION

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by

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This thesis develops an emerging relationship between the number theoretic notion of prisms and the equivariant algebra of Tambara functors. We develop some of the theory of Weyl-invariant Tambara functors and present Witt vectors from this perspective. A new interpretation of divided power structures in terms of Tambara thickenings is also given. We reprove classical deformation-theoretic results and more recent work on presentations for the Witt vectors of perfect algebras. We then turn our attention to prisms, and show that there is an explicit recipe for constructing a Tambara functor from an oriented prism. Earlier results relating quotients of perfect prisms to Witt vectors are reproved from this perspective. We also offer a new Tambara-theoretic characterization of perfectoid rings, and introduce a candidate theory of global prisms with associated $\mathbb{Q}/\mathbb{Z}$-Tambara functors.

The dissertation of Alexander Nicholas Frederick is approved.

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CHAPTER 1

Introduction

This thesis charts an emerging relationship between genuine equivariant algebra and number theory. The key landmark is a mysterious functor from prisms to Tambara functors, inspired by work of Sulyma. Along the way we develop Witt vectors from the equivariant point of view as free Tambara functors, reproving specific presentation results of Cuntz-Deninger. We also show that the notion of divided power thickening has a natural interpretation in terms of Tambara algebra. We proceed to reformulate and reprove the main results of section 3 of [BMS19], showing perfect prisms are connected to Witt-Tambara functors. We also offer a new Tambara theoretic characterization of perfectoid rings. Finally we formulate and prove some results in the global direction, using the set-up of λ-rings and a new notion of "illustrious" element to sketch a possible candidate for a notion of "global prism". These are connected to the the theory of C_∞-Tambara functors in a way analogous to the connection between prisms and C_{p^∞}-Tambara functors.

The lines of inquiry pursued in this thesis - specifically, the connection between prisms and C_{p^∞}-Tambara functors, and more generally the applications of Tambara algebra to arithmetic - are inspired by several important developments.

1. Since at least [Bru03] it has been known that the p-typical Witt vectors admit an interpretation as free Tambara functors. Witt vectors are ubiquitous in commutative algebra and classical and modern number theory, but to the author's knowledge the literature is sparse on specific applications of the Tambara point of view to the theory and applications

of Witt vectors.

2. In the influential paper [BMS19], specifically in the third section “algebraic preliminaries on perfectoid rings”, the theory of perfectoid rings and Fontaine’s infinitisemal period rings is developed in a Witt-theoretic language with specific reference to the Frobenius maps. From the point of view of genuine equivariant algebra, the Frobenius maps are the restriction maps in the Witt-Tambara functor. Subsequent developments in prismatic cohomology ([BS22]) allow this section to be reformulated as a definite connection between perfect prisms, perfectoid rings, and Witt-Tambara functors.

3. In [Sul20], a recipe for associating a Green functor to an oriented prism is sketched, with the norm formulae also implicit. This was the most important development in suggesting a clear possibility of formulating aspects of prisms and prismatic cohomology in the language of Tambara algebra.

4. In the classical work of Illusie [Ill79], and more recent work of Langer-Zink [LZ04] and Hesselholdt [Hes15], the delicate structures of Witt complexes and the theory of the de Rham Witt complex have been laid out. The complicated data of a Witt complex admits a natural formalism in the language of Tambara algebra, and this immediately suggests the possibility of reconstructing the theory from this point of view, and also extending it to more general Tambara functors.

This thesis works through points (1) through (3); point (4), and further speculation on a Tambara-theoretic approach to the de Rham Witt complex is left for future work.

1.1 Equivariant Algebra

For any kind of mathematics, it has long been fashionable to consider G-mathematics for a given group of symmetries G. This opens the door to 'equivariant' mathematics. In algebra, this passage to G-mathematics is usually accomplished rigorously by the consideration of things like G-modules and G-rings, meaning abelian groups or associative rings equipped with an action of G that respects that structure. For the categorically minded, this corresponds to taking functors out of the category with one object and automorphism group G into the algebraic category of interest. For example, if one wanted to do equivariant commutative algebra over the group $C_2 = \langle\sigma\rangle$, one would consider rings such as $\mathbb{Z}[x]$ with the nontrivial action $\sigma(x) = -x$. As it turns out, various considerations in representation theory and especially algebraic topology lead one to consider more highly structured versions of G-mathematics. For a G-commutative ring R, it is natural to consider the collection of fixed points A^H for each subgroup $H \leq G$, and ask what kind of structure coheres in this collection.

For the example of $\mathbb{Z}[x]$ with nontrivial C_2-action, one has $\mathbb{Z}[x]^{C_2} = \mathbb{Z}[x^2]$. There is a natural inclusion, a homomorphism of rings, $\mathbb{Z}[x^2] \longrightarrow \mathbb{Z}[x]$. But there is also an additive transfer operation $T : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x^2]$ and multiplicative norm operation $N : \mathbb{Z}[x] \longrightarrow \mathbb{Z}[x^2]$ given by

$$Tf(x) = f(x) + \sigma \cdot f(x) = f(x) + f(-x) \quad Nf(x) = f(x)(\sigma \cdot f(x)) = f(x)f(-x)$$

Taken together, this collection of rings and structure maps form the data of a C_2-Tambara functor. The category of Tambara functors considerably broadens the horizons of equivariant algebra far beyond the more narrow idea of a G-ring. For example, the following gives a basic C_p-Tambara functor (the Burnside Tambara functor) that does not arise as the fixed points of a C_p-commutative ring. Associated to the subgroup C_p, one has the ring $\mathbb{Z}[\theta]/(\theta^2 - p\theta)$, while associated to the trivial subroup $e \subseteq C_p$ one puts $\mathbb{Z}$. These are connected by a restriction map $R : \mathbb{Z}[\theta] \longrightarrow \mathbb{Z}$, $R(\theta) = p$ as well as a transfer $T : \mathbb{Z} \longrightarrow \mathbb{Z}[\theta]$ and norm $N : \mathbb{Z} \longrightarrow \mathbb{Z}[\theta]$

given by

$$T(n) = n\theta \quad \text{and} \quad N(n) = n + \frac{n^p - n}{p}\theta$$

The world of Tambara functors is extensive and full of much strange phenomena (relative to the conventional setting of usual commutatie algebra). This thesis will focus on only a small slice of this world, a slice that has an unexpected and robust connection to recent developments in number theory. In fact, the basic example just given already hints at the connection this thesis will develop. The reader will notice that in the formula for the norm of the Burnside Tambara functor given above, the interesting operation

$$\delta_p : \mathbb{Z} \longrightarrow \mathbb{Z} \quad \delta_p(n) := \frac{n - n^p}{p}$$

occurs. This is the basic example of what is termed a δ-ring, which brings us to the next major thread of this thesis.

1.2 Prisms

Before continuing with the story of δ-rings, let us first give a brief overview of the kind of problems their theory was recently brought to bear on. In p-adic geometry there are a multitude of different cohomology theories: etale, crystalline, de Rham, etc. Each of these invariants can be thought of as arising from a process of probing a given space (or commutative ring) with specific kinds of neighborhoods. For example, in the case of crystalline cohomology, one looks at divided power thickenings, while for etale cohomology one looks at etale neighborhoods. Much of the utility of these different cohomology theories has come from powerful comparison isomorphisms that relate the various flavors of cohomology together. These comparison isomorphism tantalizingly suggest that each cohomology is itself just a particular avatar of a more fundamental cohomology theory that lies somewhere 'behind' or 'beneath' the others. This challenge of finding a kind of universal p-adic cohomology theory led Bhatt and Lurie to introduce the notion of prism and formulate prismatic cohomology, which accomplishes many of the goals of a 'universal' p-adic cohomology theory.

The basic notion underlying that of a prism is a δ-ring. When a commutative ring R is p-torsion free, the structure of a δ-ring is equivalent to that of a frobenius lift: a ring endomorphism $\varphi : R \longrightarrow R$ who's mod p reduction is the usual p-th power frobenius map. In other words, the ring homomorphism $\varphi : R \longrightarrow R$ is required to satisfy

$$\varphi(x) = x^p + p\delta x$$

for some operation $\delta : R \longrightarrow R$. If one unwinds in the strictest way exactly what δ must do in order for φ to be a ring homomorphism, one arrives at the notion of a δ-structure. In general, one can think of a δ-structure as being the data of a 'derived' Frobenius lift; as mentioned before, when there is no p-torsion, they correspond exactly to Frobenius lifts.

A prism is then defined to be a δ-ring A equipped with a special kind of element $d \in A$. The element d is required to be distinguished, which means that $\delta d \in A^{\times}$. It is equivalent to require that $(d, \varphi(d)) = (p)$; this is usually understood as expressing the geometric requirement that the vanishing loci $V(d)$ and $V(\varphi(d))$ meet only in characteristic p. Indeed, this is part of the basic motivation for the name, 'prism': one thinks of the ring A as being like light filtered through a prism (in the english sense) and refracted into its constituent pieces $A/d, A/\varphi(d), \cdots, A/\varphi^n(d), \cdots$. (For an illustration of this, see page 4 of [HN19]). Part of this thesis will consist in showing that this refraction process is, in a precise sense, similar to the way in which a C_{p^∞}-Tambara functor can be passed through the 'prism' of the total geometric fixed points functor and thereby refracted into its constituent pieces.

In [BS22], Bhatt and Scholze define prismatic cohomology by probing a given ring with prisms, and show that this cohomology can be used to recover de Rham, crystalline, and even etale cohomology (of the generic fiber). These comparison results, and indeed even the formulation of prismatic cohomology, lie outside the scope of this thesis. Instead we focus principally on the actual structure of prisms, and a surprising connection the theory of Tambara functors, first noticed in [Sul20] and subsequently developed in the recent article [Sul23].

1.3 Statement of Main Theorems

In this section we briefly summarize the main theorems of the text and indicate where the reader can find further elaboration.

The following description of the Witt vectors of perfect algebras was originally developed in [CD13]. We give essentially the same proof here, but from the Tambara point of view.

Theorem 1.3.1. *Cuntz-Deninger Presentation for Witt Vectors of Perfect $\mathbb{F}_p$-Algebras Let R be a perfect $\mathbb{F}_p$-algebra, and consider the monoid algebra $\mathbb{Z}[R]$ and augmentation map $\mathbb{Z}[R] \longrightarrow R$ with kernel I. Then for every $n \geq 0$ there is an isomorphism of rings*

$$\mathbb{Z}[R]/I^{n+1} \longrightarrow W_n(R)$$

This is developed in section 2.8.

In the p-local setting, we also offer a C_p-Tambara theoretic reformulation of divided power thickenings.

Theorem 1.3.2. *Tambara-Theoretic Reformulation of Divided Power Thickenings*

For a p-local ring R, the category of divided power thickenings of R embeds fully faithfully into the category of Φ^e-iso C_p-Tambara thickenings of $W_\bullet(R)$ as the subcategory of thickenings with injective transfer maps.

See section 2.6 for more details.

Perfectoid rings are now known to have a number of equivalent characterizations. In chapter 3, we show that they may be characterized naturally from the Tambara perspective.

Theorem 1.3.3. *Tambara-Theoretic Characterization of Perfectoid Rings*

Let R be a p-adically complete ring. Then R is perfectoid if and only if the C_p-Tambara functor $W_\bullet(R)$ satisfies the following properties:

1. *The restriction $F : W_1(R) \longrightarrow R$ is surjective.*

2. The element $\theta - p = T(1) - p$ lies in the ideal generated by p-th powers $(Ker F)^{[p]}$.

3. The kernel $Ker F$ is principal.

Moreover, a p-complete ring S satisfies conditions (1) and (2) if and only if it is semiperfectoid.

These results on perfectoid rings are developed in section 3.7.

The main result of this thesis is an explicit construction of the functor from orientable prisms to Tambara functors. Te recent article [Sul23] gives an equivalent construction, applicable to all prisms, from a slightly different point of view.

Theorem 1.3.4. *Prisms to C_{p^∞}-Tambara Functors*

There is a functor $\mathcal{S} : Prism^{or} \longrightarrow Tamb_{C_{p^\infty}}$ from orientable prisms to C_{p^∞}-Tambara functors satisfying the following properties:

(1) $\mathcal{S}(A, (d))(C_{p^\infty}/C_{p^n}) = A/d_n = A/d\varphi(d)\cdots\varphi^n(d)$

(2) The natural comparison map $W_\bullet(A/d) \longrightarrow \mathcal{S}(A, (d))$ is an isomorphism if and only if A is a perfect prism

(3) The total geometric fixed points of $\mathcal{S}(A, (d))$ corresponds to the collection of rings $A/\varphi^n(d)$ connected by the maps induced by the frobenius φ.

(4) There is an identification of Tambara functors $\mathcal{S}(A, d)_\bullet/(\theta_\bullet - p) = A/(d)^{\{\bullet\}}$ where the latter is the initial cohomological Tambara functor associated to $(A, (d))$

Finally, at the end of chapter 3, a potential candidate for 'global prisms' is developed. We use the language of λ-rings and introduce the notion of an illustrious element, which culminates in the following theorem constructing C_∞-Tambara functors.

Theorem 1.3.5. *Illustrious λ-Pairs and C_∞-Tambara Functors Let (A, ξ) be a λ-ring equipped with a globally illustrious element $\xi \in Jac(A)$. Suppose (A, ξ) is transversal, in the sense that A/ξ and $A/\psi^\ell(\xi)$, for every prime ℓ, is torsion free as an abelian group. Then there is* *an associated C_∞-Tambara functor $A_\bullet$ with $A_\bullet(C_\infty/C_n) = A_n = A/\psi^n(\xi)$. The associated Tambara functor is quasi-cohomological, in the sense that it is cohomological after taking the Green cohomologization.*

Remark 1.3.1. The transversal assumption in the above theorem can be removed if Conjecture 3.10.1 is solved in the affirmative.

We hope that the perspective and results taken in this thesis will inspire further development of this new linkage between number theory and equivariant algebra. Many open questions and directions for further inquiry arose in the process of writing this thesis; the interested reader is encouraged to look at the concluding section for a list of questions that might guide future developments.

CHAPTER 2

Witt Vectors and Equivariant Algebra

2.1 Tambara Functors

The setting of genuine equivariant algebra that this thesis concerns is not at all new. Mackey, Green, and Tambara functors have there roots in representation theory and even earlier group cohomology, and their formalism was set up several decades ago (papers of Mackey, Green, Tambara). This theory has seen renewed interest with recent significant advances in equivariant homotopy theory.

There are several ways to set up the theory of Tambara functors. One can directly define, via axioms and relations, Tambara functors as a kind of maximally structured generalization of the data of a G-commmutative ring. Alternatively, and this is the classical approach pursued by Tambara in [TNR], one may understand the theory of commutative semirings as being built out of functors from a category of bispans in finite sets; Tambara functors then arise as the analogous functors where the category of finite sets is replaced with the category of finite G-sets. See [Str12] for a nice development of this perspective. Finally, and perhaps at the current height of categorical abstraction, one may construct Mackey functors, the G-equivariant analogue of abelian groups, and show that this category has the structure not just of a symmetric monoidal category, but even of a G-symmetric monoidal category. This means that in addition to a usual symmetric monoidal product, the category is equipped with certain norm functors indexed by pairs of subgroups $K \leq H \leq G$. There is then a notion of a G-commutative monoid in a G-symmetric monoidal category, and this recovers once again the highly structured theory of Tambara functors. See [HM19] for more on this perspective.

All of these approaches are equivalent, and of course intersect in mutually enriching ways. However, as the work in this thesis hopes to be accessible to number theorists with less background in the equivariant theory, and moreover as the Tambara algebra in this thesis centers around various special, simplified cases (such as trivial Weyl actions, the restriction to [mostly] cyclic p-groups, etc), a bare-bones approach of axiomatics and relations will be taken. Those interested in other, especially more categorical approaches, may consult any of the articles mentioned in this introduction.

The data of a Weyl-invariant C_p-Tambara functor consists of a pair of commutative rings (A_1, A_0) equipped with three structure maps : $R : A_1 \longrightarrow A_0$ and $T, N : A_0 \longrightarrow A_1$. The restriction map R is required to be a ring homomorphism, while the transfer T is an additive map satisfying $T(Rx \cdot y) = x \cdot Ty$ (in other words, T is an A_1-module map from A_0 to A_1 where the former is considered as an A_1-module via R). One must also have that $RTx = px$. Finally the norm map N is a multiplicative map satisfying $RNx = x^p$, and interacts with addition through the formula

$$N(x + y) = Nx + Ny + TC_p(x, y)$$

Here $C_p(x, y)$ is the well-known polynomial with integer coefficients

$$C_p(x, y) = \frac{(x + y)^p - (x^p + y^p)}{p} = \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$$

Remark 2.1.1. The apparently coincidental appearance of the same polynomial $C_p(x, y)$ in both the behaviour of the norm map with respect to addition and that of a δ-structure is perhaps the fundamental observation that inspired much of the author's work on trying to connect the theory of δ-structures to that of Tambara functors.

In general, determining the explicit interaction between norms and transfers required in Tambara functors is quite difficult and delicate, as they depend on certain universal equivariant polynomials. However, in the case of Weyl-invariant Tambara functors, all of the equivariance collapses and leaves fairly digestible formulae. As an example, we go over the norm of the transfer in the C_{p^2}-situation, as well as later the $C_{p\ell}$-situation for $p \neq \ell$. As it turns out, this will be enough to deduce the requirements for C_{p^∞}-Tambara functors and even C_∞-Tambara functors. The reader interested only in the connection between C_{p^∞}-Tambara functors and prisms may of course skip the results for $\ell \neq p$; we include them only for reference and to support the more speculative, global work in section 3.10 connecting λ-rings to C_∞-Tambara functors.

Proposition 2.1.1. *In a Weyl-invariant C_{p^2}-Tambara functor, the interaction between the norm and transfer is given by*

$$N_1^2 T_0^1 x = NTx = p^{p-2} T^2 x^p = p^{p-2} T_0^2 x^p$$

Proof. To understand the required relation, we need to understand the structure of the set of sections of the projection $C_{p^2}/e \longrightarrow C_{p^2}/C_p$ as a C_{p^2}-set. Now we have $\mathrm{Sec}(C_{p^2}/e \longrightarrow C_{p^2}/C_p) \subseteq \mathrm{Map}(C_{p^2}/C_p, C_{p^2}/e)$. Now the map $C_{p^2}/e \longrightarrow C_{p^2}/C_p$ is p-to-1 and the codomain has size p, so we conclude that the cardinality $\#\mathrm{Sec}(C_{p^2}/e \longrightarrow C_{p^2}/C_p) = p^p$. Now these sections form a sub C_{p^2}-set of $\mathrm{Map}(C_{p^2}/C_p, C_{p^2}/e)$. The latter consists of only free orbits, since $\mathrm{Map}(\mathrm{Map}(C_{p^2}/C_p, C_{p^2}/e)^{C_p} = \mathrm{Map}_{C_p}(\mathrm{Map}(C_{p^2}/C_p, C_{p^2}/e) = \varnothing$, since a trivial C_p-set cannot map equivariantly into a nontrivial one. Thus the C_{p^2}-set of sections $\mathrm{Sec}(\mathrm{Map}(C_{p^2}/C_p, C_{p^2}/e))$ consists only of free C_{p^2} orbits; since its cardinality is p^p, there must be exactly p^{p-2} of these orbits.

In general, this would tell us that $NTx = T^2(\cdots)$ where the $\cdots$ indicate a sum of p^{p-2} C_p-equivariant polynomials in x all of weight p. In the Weyl-invariant situtation, these must all be equal to x^p, and so we can safely conclude that

$$NTx = T^2(p^{p-2}x^p) = p^{p-2}T^2x^p$$

□

Proposition 2.1.2. *Let $p \neq \ell$ be primes. Then the interaction between norms and transfers in a Weyl-invariant $C_{p\ell}$-Tambara functor is given by:*

$$N_\ell^{\ell p} T_1^\ell x = T_p^{p\ell} N_1^p x + \frac{\ell^p - \ell}{p\ell} T_1^{p\ell} x^p$$

Proof. In this case we must understand the decomposition of the $C_{p\ell}$-set of sections $\mathrm{Sec}(C_{p\ell}/e \longrightarrow C_{p\ell}/C_\ell) \subseteq \mathrm{Map}(C_{p\ell}/C_\ell, C_{p\ell}/e)$. Now the map $C_{p\ell}/e \longrightarrow C_{p\ell}/C_\ell$ is ℓ-to-1 and has codomain of size p. Thus the cardinality of the set of sections is given by ℓ^p.

Now evidently $\mathrm{Map}(C_{p\ell}/C_\ell, C_{p\ell}/e)$ has no C_ℓ-fixed points, since there are no C_ℓ-equivariant maps from a trivial C_ℓ-set to a free one. As $\mathrm{Sec}(C_{p\ell}/e \longrightarrow C_{p\ell}/C_\ell) \subseteq \mathrm{Map}(C_{p\ell}/C_\ell, C_{p\ell}/e)$, we can thus conclude that the set of sections decomposes only into free orbits and those of the form $C_{p\ell}/c_p$. To determine how many of the latter there are, we compute the size of $\mathrm{Sec}(C_{p\ell}/e \longrightarrow C_{p\ell}/C_\ell)^{C_p}$. This consists of the C_p-equivariant sections of projection $C_{p\ell}/e \longrightarrow C_{p\ell}/C_\ell$. But as C_p-sets, this projection is isomorphic to a projection

$$\coprod_\ell C_p/e \longrightarrow C_p/e$$

Thus there are exactly ℓ C_p-quivariant sections of this, one for each summand in the disjoint union. As $\#\mathrm{Sec}(C_{p\ell}/e \longrightarrow C_{p\ell}/C_\ell)^{C_p}) = \ell$, we conclude that there is exactly one $C_{p\ell}/C_p$-orbit and the rest are all free orbits. This gives the decomposition

$$\mathrm{Sec}(C_{p\ell}/e \longrightarrow C_{p\ell}/C_\ell)^{C_p}) \simeq C_{p\ell}/C_p \oplus \coprod_{\frac{\ell^p-\ell}{p\ell}} C_{p\ell}/e$$

The single $C_{p\ell}/C_\ell$-orbit gives the term $T_p^{p\ell} N_1^p x$, while the other orbits contribute a sum of transfers of weight p equivariant $C_{p\ell}$-monomials in x. In the Weyl-invariant sitaution, the only such monomial is x^p, so we arrive at the claimed formula:

$$N_\ell^{\ell p} T_1^\ell x = T_p^{p\ell} N_1^p x + \frac{\ell^p - \ell}{p\ell} T_1^{p\ell} x^p$$

∎

Remark 2.1.2. As before, this relation and the one above for $p = \ell$ is all we need to understand general C_∞-Tambara functors. This is because all of the 'atomic' structure maps happen one prime at a time, so all of the relations come from compositions of the two forms above. Finally we also are using that the same relatios hold when going from $C_{p\ell n}/C_n$ to $C_{p\ell n}/C_{p\ell n}$ as going from $C_{p\ell}/e$ to $C_{p\ell}/C_{p\ell}$ because the lattices of subgroups are identical (or equivalently there is an equivalence between the respective categories of G-sets at issue in each case).

2.2 Total Geometric Fixed Points, Ghost Maps, and Tombstone Tambara Functors

2.2.1 Geometric Fixed Points

Let $A_\bullet$ be a C_{p^∞}-Tambara functor. Associated to $A_\bullet$ is a direct system of commutative rings which we call the total geometric fixed points. For any $i = 0$ we set $\Phi^0 A_\bullet := A_0$ and for $i \geq 1$ we have $\Phi^i A_\bullet = A_i/TA_{i-1}$. The Frobenius relation $x \cdot Ty = T(Rx \cdot y)$ guarantees that TA_{i-1} is an ideal in A_i, so that each $\Phi^i A_\bullet$ is a commutative ring. Moreover, the norm maps $N_i : A_i \longrightarrow A_{i+1}$ descend to a ring homomorphism $A_i/TA_{i-1} \longrightarrow A_{i+1}/TA_i$ often denoted as $\varphi_i : \Phi^i A_\bullet \longrightarrow \Phi^{i+1} A_\bullet$.

Proposition 2.2.1. *The norm maps $N_i : A_i \longrightarrow A_{i+1}$ descend to ring homomorphisms $\varphi_i : A_i/TA_{i-1} \longrightarrow A_{i+1}/TA_i$.*

Proof. The norms themselves are already multiplicative, so it remains only to show that the norms are additive modulo transfers and that $N(TA_{i-1}) \subseteq TA_i$. The norms being additive modulo transfers is clear from the defining requirements of the norm. The norms taking transfer ideals into transfer ideals follows from the required relation in Proposition 2.1.1 above. □

Theorem 2.2.1. *Suppose $A_\bullet, B_\bullet$ are C_{p^n} ($n \in \mathbb{N} \cup \infty$) Tambara functors with injective transfer maps. Then if $f_\bullet : A_\bullet \longrightarrow B_\bullet$ is a map of Tambara functors inducing an isomorphism* *on total geometric fixed points, then it is an isomorphism of Tambara functors.*

Proof. The proof proceeds easily by induction. In level zero, geometric fixed points agree with the underlying, so one has an isomorphism $f_0 : A_0 \longrightarrow B_0$ by assumption. Now suppose it has been shown that $f_0, f_1, ..., f_{n-1}$ are isomorphisms. Then in level n one has the map of short exact sequences (exact on the left because of the crucial assumption that the transfers in $A_\bullet$ and $B_\bullet$ are injective:

$$0 \longrightarrow A_{n-1} \longrightarrow A_n \longrightarrow A_n/TA_{n-1} \longrightarrow 0$$

$$0 \longrightarrow B_{n-1} \longrightarrow B_n \longrightarrow B_n/TB_{n-1} \longrightarrow 0$$

The left-most vertical map is an isomorphism by the inductive hypothesis; the right-most map is an isomorphism by the assumption that $\Phi^*(f_\bullet)$ is an isomorphism. Therefore $f_n : A_n \longrightarrow B_n$ must also be an isomorphism, as desired. □

This basic result will be used many times in the rest of the thesis. In particular it is crucial to the new Tambara-theoretic proof of the result in [BMS19] that for a perfect prism $(A, (d))$, one has $A/d\varphi^(d) \cdots \varphi^n(d) \simeq W_n(A/d)$. See section XXX.

Remark 2.2.1. The condition that the transfer maps be injective cannot be done away with in full generality. For example, consider the map of C_p-Tambara functors $W_\bullet(\mathbb{F}_p) \longrightarrow \underline{\mathbb{F}_p}$. The map on geometric fixed points is evidently an isomorphism, but of course the map itself has kernel. In general, one can work out a criterion weaker than both Tambara functors having injective transfers: one needs something like $Tf(x) = 0 \implies Tx = 0$, i.e. $f^{-1}(\mathrm{Ker}T_B) \subseteq \mathrm{Ker}T_A$.

2.2.2 Tombstone Tambara Functors

This section is limited to Weyl-invariant C_{p^n}-Tambara functors (for $n \in \mathbb{N} \cup \{\infty\}$). The construction of tombstone Tambara functors can probably be extended to the case of nontrivial

Weyl group actions, but this is left for future work.

Given a direct system of commutative rings $R_0 \longrightarrow^{\varphi_0} R_1 \longrightarrow \cdots$ one can build a tombstone Tambara functor $\mathbf{T}(R_\bullet)$ that receives a map from any Tambara functor $A_\bullet$ equipped with a morphism of direct systems of rings from the geometric fixed points $\Phi^* A_\bullet \longrightarrow R_*$. In other words, the tombstone functor is right adjoint to the total geometric fixed points functor:

$$\mathrm{Hom}_{\mathrm{Tamb}}(A_\bullet, \mathbf{T}(R_*)) = \mathrm{Hom}(\Phi^* A_\bullet, R_*)$$

The name 'tombstone' Tambara functor is inspired by the fact that the natural map $A_\bullet \longrightarrow \mathbf{T}(\Phi^* A_\bullet)$ forms the universal phantom or ghost map.

First we approach the C_p-case in detail and then give the general construction after. Fix a diagram of rings $R_0 \longrightarrow^{n_0} R_1$. The tombstone Tambara functor associated to R_* has R_0 in level zero and $R_0 \times R_1$ in level one. The restriction map is simply the projection $R_0 \times R_1 \longrightarrow R_0$ onto the first factor. The transfer map is given by $T(x_0) = (px_0, 0)$ while the norm map is given by $N(x_0) = (x_0^p, n_0(x_0))$. It is immediate that $\mathbf{T}(R_*)$ is a Green functor; as the defined norm is evidently multiplicative and sends unity to unity, the only thing to check is that it interacts correctly with addition. This follows from computing

$$N(x+y) = ((x+y)^p, n(x+y)) = (x^p + pC_p(x,y) + y^p, n(x) + n(y))$$

$$= (x^p, n(x)) + (y^p, n(y)) + (pC_p(x,y), 0) = Nx + Ny + TC_p(x,y)$$

Definition 2.2.1. *The Tombstone Tambara Functor Fix $n \geq 0$ (including $n = \infty$). Given a direct system $R_* \in Fun([n], CRing)$, the tombstone tambara functor $\boldsymbol{T}(R_*)$ is defined as follows. In level k we have $\boldsymbol{T}(R_*)_k = R_0 \times R_1 \times \cdots \times R_k = \prod_{i=0}^k R_i$. The restriction maps are given by the canonical projections onto direct factors. The transfers are defined as*

$$T(x_0, ..., x_k) = (px_0, ..., px_k, 0)$$

while the norms are given by

$$N(x_0, ..., x_k) = (x_0^p, ..., x_k^p, n_k(x_k))$$

Remark 2.2.2. Note that only the norms take into account the morphisms in the direct system R_*. This reflects the fact that the Tombstone construction works just on the level of Green functors. The author speculates that it also works in the natural way for incomplete Tambara functors associated to any intermediate family of norms.

Proposition 2.2.2. *For any $G = C_{p^n}$ with $n \in \mathbb{N} \cup \{\infty\}$, the Tombstone Tambara functor construction $\boldsymbol{T} : Fun([n], CRing) \longrightarrow Tamb_{C_p^n}$ gives a right adjoint to the total geometric fixed points functor Φ^*.*

Proof. We construct a Hom-set bijection. Given a Tambara functor $A_\bullet$ and a direct system of rings R_*, the map $\mathrm{Hom}(\Phi^* A_\bullet, R_*) \longrightarrow \mathrm{Hom}_{\mathrm{Tamb}}(A_\bullet, \mathbf{T}_\bullet(R_*))$ takes a morphism $\{f_*\}$ to the morphism $\{F_\bullet\}$ given in level n by

$$F_n : A_n \longrightarrow \mathbf{T}_n(R_*) = R_0 \times R_1 \times \cdots \times R_n$$

$$F_n(a_n) = (f_0(R^n a_n), f_1(R^{n-1} a_n), \cdots, f_n(a_n))$$

To see that $\{_\bullet$ is indeed a morphism of Tambara functors, one can simply check that it commmutes with restrictions, transfers, and norms. For restrictions, one has

$$F_n(Ra_{n+1}) = (f_0(R^{n+1} a_{n+1}), f_1(R^n a_{n+1}), \cdots, f_n(Ra_{n+1}))$$

$$= R\left((f_0(R^{n+1} a_{n+1}), f_1(R^n a_{n+1}), \cdots, f_n(Ra_{n+1}), f_{n+1}(a_{n+1})\right) = RF_{n+1}(a_{n+1})$$

For transfers:

$$F_n(Ta_{n-1}) = (f_0(R^n Ta_{n-1}), f_1(R^{n-1} Ta_{n-1}), \cdots, f_n(Ta_{n-1}))$$

$$= (pf_0(R^{n-1} a_{n-1}), pf_1(R^{n-2} a_{n-1}), \cdots, 0) = TF_{n-1}(a_{n-1})$$

Here we have used that $A_\bullet$ is Weyl-invariant, so $RT = p$ always, and also that f_n is a morphism from the geometric fixed points A_n/TA_{n-1} to R_n, and therefore $f_n(Ta_{n-1}) = 0$.

Finally for norms, the check is similar:

$$F_n(Na_{n-1}) = (f_0(R^n Na_{n-1}), f_1(R^{n-1} Na_{n-1}), \cdots, f_n(Na_{n-1}))$$

$$= (f_0(R^{n-1}a_{n-1})^p, f_1(R^{n-2}a_{n-1})^p, \cdots, \varphi_n f_{n-1}(a_{n-1}))$$

$$= N(f_0(R^{n-1}a_{n-1}), f_1(R^{n-2}a_{n-1}), \cdots, f_{n-1}(a_{n-1})) = NF_{n-1}a_{n-1}$$

Here again we have crucially used the Weyl-invariance of $A_\bullet$ to get that $NRy = y^p$. In the last coordinate we use that $f_* : \Phi^* A_\bullet \longrightarrow R_*$ is a map of direct systems of rings, and so the f_* intertwine the φ_* with the maps on geometric fixed points induced by the norms of $A_\bullet$.

This shows that the association $f_* \mapsto F_\bullet$ is indeed a well defined map $\mathrm{Hom}(\Phi^* A_\bullet, R_*) \longrightarrow \mathrm{Hom}(A_\bullet, \mathbf{T}_\bullet(R_*))$. It remains only to check that it is a natural bijection. The naturality is evident from the construction. To see that it is a bijection, we construct a natural inverse. Given $G_\bullet : A_\bullet \longrightarrow \mathbf{T}_\bullet R_*$, the induced map on geometric fixed points gives a morphism of direct systems of rings $\Phi^* G_\bullet : \Phi^* A_\bullet \longrightarrow \Phi^* \mathbf{T}_\bullet(R_*)$. In level n this is a map

$$A_n/TA_{n-1} \longrightarrow (R_0 \times R_1 \times \cdots \times R_{n-1} \times R_n)/((p, p, \cdots, p, 0))$$

Projecting onto the last factor of the quotient gives the desired map $A_n/TA_{n-1} \longrightarrow R_n$. From the definitions, it is clear that this recovers the original f_* from the associated $F_\bullet$.

Going the other way requires us to verify that

$$G_n(a_n) = (g_0 R^n a_n, g_1 R^{n-1} a_{n-1, \cdots, g_n a_n}$$

The correctness of the last coordinate is evident from the definition, while the others follow from the fact that the $G_\bullet$ commute with the restriction maps on each side. □

It is natural to ask if the total geometric fixed points Φ^* admits a left adjoint. This is answered in the negative below, by specializing to the C_p-case and showing a certain functor isn't representable.

Proposition 2.2.3. *There is no left adjoint to the total geometric fixed points functor* Φ^*.

Proof. It suffices to show there is no left adjoint in the C_p-situation. Suppose to the contrary that there is a left adjoint $\mathcal{L}$. Then consider the direct system $\mathbb{Z} \longrightarrow \mathbb{Z}[x]$. On direct systems, it is clear that $\mathrm{Hom}(\mathbb{Z} \longrightarrow \mathbb{Z}[x], R_0 \longrightarrow R_1) \simeq R_1$. Thus one must have

$$\mathrm{Hom}_{\mathrm{Tamb}}(\mathcal{L}(\mathbb{Z} \longrightarrow \mathbb{Z}[x]), B_\bullet) \simeq \mathrm{Hom}(\mathbb{Z} \longrightarrow \mathbb{Z}[x], B_0 \longrightarrow B_1/TB_0) \simeq B_1/TB_0$$

Thus the functor $B_\bullet \mapsto B_1/TB_0$ needs to be representable. Now one can restrict to considering just constant Tambara functors $\underline{R}$ for commutative rings R. Then from the above, one must have

$$\mathrm{Hom}_{\mathrm{Tamb}}(\mathcal{L}(\mathbb{Z} \longrightarrow \mathbb{Z}[x]), \underline{R}) \simeq R/p$$

But on the other hand, maps into a constant Tambara functor are equivalent to maps of rings in bottom level (level zero). So one has

$$\mathrm{Hom}_{\mathrm{Tamb}}(\mathcal{L}(\mathbb{Z} \longrightarrow \mathbb{Z}[x]), \underline{R}) \simeq \mathrm{Hom}_{CRing}(L_0, R)$$

where L_0 is the commutative ring in level zero of $\mathcal{L}(\mathbb{Z} \longrightarrow \mathbb{Z}[x])$.

So within the category of rings, L_0 must represent the functor $R \mapsto R/p$. This is evidently impossible. A quick counterexample is afforded by looking at the injection $\mathbb{Z} \longrightarrow \mathbb{Q}$, which would give an injection of hom-sets $\mathrm{Hom}(L_0, \mathbb{Z}) \longrightarrow \mathrm{Hom}(L_0, \mathbb{Q})$. But the first hom set has cardinality $|\mathbb{Z}/p| = p$, while the second has cardinality $|\mathbb{Q}/p| = 1$, so there is a contradiction.

□

2.2.3 The Ghost Maps

In this section we develop the theory of ghost maps from a general point of view, later using it to recover well-known facts about the Witt vectors. Let then $A_\bullet$ be a Weyl-invariant C_{p^∞}-Tambara functor. Then adjoint to the identity map $\Phi^* A_\bullet \longrightarrow \Phi^* A_\bullet$ is the canonical morphism $A_\bullet \longrightarrow \mathbf{T}_\bullet \Phi^* A_\bullet$. This morphism is called the "ghost" or "total ghost" map, and in level n looks like a map

$$A_n \longrightarrow A_0 \times A_1/TA_0 \times \cdots \times A_n/TA_{n-1}$$

given by

$$a_n \mapsto (R^n a_n, \overline{R^{n-1} a_n}, \cdots, \overline{a_n})$$

First we analyze the injectivity of the total ghost map. Let $K_n \subseteq A_n$ denote the kernel of the n-th total ghost map. The K_i are difficult to get at it in full generality, but admit a workable inductive description. In level 0, we clearly have $K_0 = (0)$. Now for $x \in A_1$ to be in K_1, we need exactly that $Rx \in K_0$, and $x \in TA_0$. In other words, $K_1 = R^{-1}(K_0) \cap TA_0$. But if we take $Ty \in TA_0$, it is easy to check if $Ty \in R^{-1}(K_0)$, since this amounts simply to asking if $RTy = py \in K_0$. Thus in level 1, we have $K_1 = T(\frac{1}{p}K_0) = T(A_0[p])$. In general, the same analysis works, giving

$$K_{n+1} = T(\frac{1}{p}K_n)$$

where for an ideal I, $\frac{1}{p}I$ denotes the ideal of all x with $px \in I$.

Despite the general opacity of the kernels, one does have the following useful criterion for their vanishing (and equivalently, injectivity of the total ghost maps).

Proposition 2.2.4. *The total ghost maps are injective if for every n one has $A_n[p] \subseteq KerT$.*

Proof. This goes by induction. In level one, the kernel K_1 is just $TA_0[p]$, so it is clear that $K_1 = 0$ if and only if $TA_0[p] = 0$, that is, $A_0[p] \subseteq \mathrm{Ker}T$. Now suppose the claim is true up to level n. Then as $K_n = 0$, one has $\frac{1}{p}K_n = A_n[p]$ and thus $K_{n+1} = T(\frac{1}{p}K_n) = TA_n[p]$. So K_{n+1} vanishes if and only if $TA_n[p] = 0$, as desired. □

Corollary 2.2.1. *If $A_\bullet$ is p-torsion free, i.e. $A_n[p] = 0$ for all n, then the total ghost maps are injective.*

Proof. Follows immediately from the proposition above. □

In general, there does not seem to be a good description of the image of the total ghost map. For specific cases of the Witt-Tambara functors the classical Dwork's lemma provides an explicit description.

Also, when the total ghost map is injective, the geometric fixed points functor is faithful. When the total ghost map is an isomorphism (which is the case for example if p is invertible in every level), then the geometric fixed points functor is fully faithful and in fact gives an embedding of the category of Weyl-invaraint Tambara functors into direct systems of commutative rings.

Remark 2.2.3. Let $B_\bullet$ be a Tambara functor with injective total ghost maps, for example if $B_\bullet$ is p-torsion free. Then there is an injection of Hom-sets

$$\mathrm{Hom}_{\mathrm{Tamb}}(A_\bullet, B_\bullet) \longrightarrow \mathrm{Hom}(\Phi^* A_\bullet, \Phi^* B_\bullet)$$

Post-composition with the total ghost map $\mathrm{gh}_\bullet : B_\bullet \longrightarrow \mathbf{T}_\bullet \Phi^* B_\bullet$ is what induces the map

$$\mathrm{Hom}_{\mathrm{Tamb}}(A_\bullet, B_\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{Tamb}}(A_\bullet, \mathbf{T}_\bullet \Phi^* B_\bullet) \simeq \mathrm{Hom}(\Phi^* A_\bullet, \Phi^* B_\bullet)$$

So the total geometric fixed points functor is faithful on maps into Tambara functors with injective total ghost maps.

Remark 2.2.4. Let $B_\bullet$ be a Tambara functor such that the total ghost map is an isomorphism. Then the same reasoning in the remark above shows that Φ^* is fully faithful on maps into $B_\bullet$. If $B_\bullet$ has p invertible in every level, then the total ghost map is an isomorphism. This recovers the well-known fact that rational Tambara functors are equivalent to direct systems of rings via the total geometric fixed points functor.

2.3 Witt Vectors as Free Tambara Functors

The Tambara perspective taken on Witt vectors in this section is not new, and appears originally in the work of [Bru03]. In this work we approach Witt vectors from the perspective of equivariant algebra, which means that we will define them as certain kinds of free Tambara functors. Although we do not treat it specifically here, the connection with the more common classical definition using ghost polynomials follows almost immediately from this framework, once we explicate the nature of the total ghost map for free Tambara functors and identify it with the well-known ghost maps.

Let R be a commutative ring. The idea is to consider the C_{p^∞}-Tambara functor freely generated by R in level zero (that is, level C_{p^∞}/e), where it is considered as a commutative ring with trivial C_{p^∞}-action. There are rules for moving a norm past a transfer, and the restrictions of transfers and norms amount to multiplication by p and raising to the p-th power respectively, so the basic generating elements in level n will be of the form $T^iN^{n-i}a_i$ for some $0 \leq i \leq n$ and $a_i \in R$. In fact, as a set we have

$$W_n(R) = \{\sum_{i=0}^{n} T^iN^{n-i}a_i \;\mid\; a_i \in R\}$$

There is of course a natural bijection

$$R^{n+1} \longrightarrow W_n(R) \;\; (a_0, \cdots, a_n) \mapsto \sum_{i=0}^{n} T^iN^{n-i}a_i$$

and we will often write elements of $W_n(R)$ as ordered $n+1$-tuples.

Example 2.3.1. For $n = 1$, it is instructive to unwind the ring structure on $W_1(R)$ directly. We may compute the addition in $W_1(R)$ using the rules for adding norms in a C_p-Tambara functor. We have

$$(x_0, x_1) + (y_0, y_1) = Nx_0 + Tx_1 + Ny_0 + Ty_1 = Nx_0 + Ny_0 + T(x_1 + y_1)$$

$$= N(x_0 + y_0) - TC_p(x_0, y_0) + T(x_1 + y_1) = (x_0 + y_0, x_1 + y_1 - C_p(x_0, y_0))$$

Likewise the multiplication can be worked out:

$$(x_0, x_1) \cdot (y_0, y_1) = (Nx_0 + Tx_1)(Ny_0 + Ty_1) = Nx_0Ny_0 + Nx_0Ty_1 + Ny_0Tx_1 + Tx_1Ty_1$$

$$= Nx_0y_0 + T(x_0^py_1) + T(y_0^px_1) + T(px_1y_1) = (x_0y_0, x_0^py_1 + y_0^px_1 + px_1y_1)$$

Those familiar to the usual definition of Witt vectors in terms of various universal sum and product polynomials will recognize the first of those such polynomials emerging naturally from the Tambara point of view.

Now we turn to the determination of the geometric fixed points of the Witt-Tambara functor $W_\bullet(R)$. From the set-theoretic description of R, it is clear that the quotient $W_n(R)/TW_{n-1}(R)$ consists of the representatives

$$\{N^n a_0 + TW_{n-1}(R) \mid a_0 \in R\}$$

It is easy to see that these representatives multiply and add according to the multiplication and addition in the ring R, so that in fact there is an isomorphism

$$R \longrightarrow W_n(R)/TW_{n-1}(R) \quad a_0 \mapsto N^n a_0 + TW_{n-1}(R)$$

After making these identifications, the total geometric fixed points $\Phi^* W_\bullet(R)$ then becomes identified with the identity direct system of rings

$$\Phi^* W_\bullet(R) \simeq R \longrightarrow R \longrightarrow R \longrightarrow \cdots$$

With this understanding of $\Phi^* W_\bullet(R)$ in hand, one may determine the structure of the tombstone functor $\mathbf{T}_\bullet \Phi^* W_\bullet(R)$ and the total ghost map. Although we do not pursue it here, this is how the Tambara perspective on Witt vectors links up seamlessly with the usual constructions using ghost polynomials.

2.4 Perfect Interchange Phenomenon

In general, the Witt vectors of a polynomial algebra are related in only a delicate way to polynoimal algebras over the Witt vectors. However, in the (co)limit of adjoining all p-power roots of a generator, this difference evaporates. In particular, one has the following "perfect interchange phenomenon":

Theorem 2.4.1. *There is a canonical isomorphism of Tambara functors*

$$W_\bullet(R[t^{1/p^\infty}]) \longrightarrow W_\bullet(R)[\underline{x}^{1/p^\infty}]$$

The isomorphism is induced by adjunction from the isomorphism on bottom level $R[t^{1/p^\infty}] \longrightarrow R[x^{1/p^\infty}]$, $\quad t \mapsto x$ *and in level* n *the norm element* $N^n t$ *is sent to the generator* x.

Proof. The proof proceeds by showing that the two Tambara functors have naturally isomorphic representable functors. Given a Tambara functor $B_\bullet$, one has essentially by definition that

$$\mathrm{Hom}_{\mathrm{Tamb}}(W_\bullet(R[t^{1/p^\infty}]), B_\bullet) \simeq \mathrm{Hom}(R[t^{1/p^\infty}], B_0)$$

On the other hand, a morphism of Tambara functors $W_\bullet(R)[\underline{x}^{1/p^\infty}] \longrightarrow B_\bullet$ corresponds to a morphism of commutative rings $R \longrightarrow B_0$ together with compatible families $(x_n^{1/p^\infty}) \in B_n^\flat$. Here compatible means that for any $\alpha \in \mathbb{Z}[1/p]$ one has $Rx_{n+1}^\alpha = x_n^{p\alpha}$ and $Nx_n^\alpha = x_{n+1}^\alpha$. However, because $RNy = y^p$, the second compatibility condition implies the first, and moreover the second forces $x_n^\alpha = N^n(x_0^\alpha)$, so that everything is determined by the choice of $x_0^{1/p^\infty} \in B_0^\flat$. But together with the map $R \longrightarrow B_0$, this is exactly the data of a map of commutative rings $R[x^{1/p^\infty}] \longrightarrow B_0$.

□

Remark 2.4.1. The perfect interchange phenomenon extends in the C_{p^∞}-case to any uniquely p-divisble monoid M. That is, one has an isomorphism of Tambara functors $W_\bullet(R)[M] \longrightarrow W_\bullet(R[M])$. Even more generally, this should work in the setting of G-Tambara functors for any commutative monoid with $|G| : M \longrightarrow M$ an isomorphism. If G is infinite, one just means that the monoid is uniquely n-divisible for every n occurring as the order of a subgroup of G.

2.5 Example: Witt Vectors of $\mathbb{F}_p[t]$

As a fundamental example, this section develops the algebra structure of $W_n(R[t])$ as a $W_n(R)$-algebra. One can approach this directly and work out the algebra structure simply from a generators and relations perspective, as in the following proposition.

Proposition 2.5.1. *There is a presentation* $W_1(\mathbb{F}_p[t]) \simeq W_1(\mathbb{F}_p)[n]\langle t_1, ..., t_{p-1}\rangle$ *with relations given by*

$$t_i \cdot t_j = pt_{i+j} \text{ if } i + j \leq p - 1$$

$$t_i \cdot t_j = pnt_{i+j-p} \text{ if } i+j \geq p$$

Proof. Consider the map $W_1(\mathbb{F}_p)[n, t_1, ..., t_{p-1}] \longrightarrow W_1(\mathbb{F}_p[t])$ given by sending n to Nt, and t_i to Tt^i. Firstly one argues that this map is surjective. Note that $n^j t_i \mapsto (Nt)^j Tt^i = Tt^{pj+i}$. Therefore every Tt^a for $a \geq 0$ is hit, and as the powers of t additively generate $\mathbb{F}_p[t]$, the transfers of powers of t also additively generate the entire transfer ideal. On the other hand, for any $f(t) \in \mathbb{F}_p[t]$ one has $Nf(t) \equiv f^N(Nt)$ modulo transfers, and the latter is clearly hit by $f^N(n)$. So the map is indeed surjective.

The relations follow immediately from the usual relations in a Tambara functor. □

For $n > 1$, rather than reasoning directly via generators and relations, it is perhaps more convenient to embed the rings $W_n(\mathbb{F}_p[t])$ into more understandable rings. For this, consider the following important C_{p^n} Tambara functor, $W_\bullet(\mathbb{F}_p)[\underline{t}]$, the free $W_\bullet(\mathbb{F}_p)$-functor on a cohomological generator in top level. This means that in level n one has $W_n(\mathbb{F}_p)[t]$ with restriction given by $Rt = t$ and norm $Nt = t^p$ (the transfer is then implied by frobenius reciprocity, as $Tt = TRt = tT(1)$).

The identity map in bottom level induces a comparison map $W_n(\mathbb{F}_p[t]) \longrightarrow W_n(\mathbb{F}_p)[t]$. On geometric fixed points, this map is given by the iterated frobenius: $\mathbb{F}[t] \longrightarrow \mathbb{F}_p[t]$, $t = N^n(t) \mapsto N^n(t) = t^{p^n}$. These maps are evidently injective, and as both the Witt-Tambara functor and the target Tambara functor have injective transfers, one concludes that the entire comparison map is injective. This gives the following description of $W_n(\mathbb{F}_p[t])$.

Proposition 2.5.2. *There is an injection $W_n(\mathbb{F}_p[t]) \longrightarrow W_n(\mathbb{F}_p)[t]$. The image consists of the subring generated by the elements $t^{p^n}, pt^{p^{n-1}}, ..., p^n t$.*

Proof. The construction of the injection is detailed above. It remains only to check on the image. The algebra $W_n(\mathbb{F}_p[t])$ is generated by the elements $T^i N^{n-i} t$. Remembering that t is sent to a cohomological element, one immediately sees that $T^i N^{n-i} t$ is sent to $p^i t^{p^{n-i}}$. □

Remark 2.5.1. Consider the isomorphism $W_n(\mathbb{F}_p)[t] \longrightarrow W_n(\mathbb{F}_p)[s^{1/p^n}]$ with t corresponding to s^{1/p^n}. Under this isomorphism, the algebra $W_n(\mathbb{F}_p[t])$ embeds as the subring of $W_n(\mathbb{F}_p)[s^{1/p^n}]$ generated by the elements $s, ps^{1/p}, ..., p^n s^{1/p^n}$. This is precisely the subring of functions with integral differential form. The description of the Witt vectors of $\mathbb{F}_p[t]$ from the point of view of integral differential forms, originally due to Deligne, can be found in section I.2 of [Ill79].

A similar analysis can be conducted in the case of $W_n(R[x_1, .., x_n])$, leading to the same interpretation as functions with integral differentials.

2.6 Tambara Ideals of Witt Vectors

This section focuses on the C_p case and describes the various Tambara ideals of the C_p Witt Tambara functor $W_\bullet(R)$. In the p-local situation, the ideals turn out to be classified by data that is essentially equivalent to divided power structures. This connection will be explored more fully in section XXXXX.

Consider then a Tambara ideal $I_\bullet \subseteq W_\bullet(R)$. In the first case, note that the level zero elements $I_0 \subseteq R$ form an ideal of the underlying ring R, and one has $W_1(I_0) \subseteq I_1$. In this way I_1 corresponds exactly to an ideal in $W_1(R/I_0)$ which lies in the kernel of the Frobenius map (the restriction). Therefore it is enough to classify the usual ring-theoretic ideals of $W_1(R)$ contained in $\mathrm{Ker}F$ for general R.

Now suppose $J \subseteq \mathrm{Ker}F \subset W_1(R)$ is an ideal. Firstly one can consider the intersection of J with the transfer ideal $T \cdot R$ (usually written VR). One has $J \cap T \cdot R = T \cdot A$ for some additive subgroup $A \subseteq R$. Since $T \cdot A$ must be an ideal contained in $\mathrm{Ker}F$, the only options for A are subgroups of $R[p]$ closed under multiplication by p-th powers of elements of R. This is because one needs $FTa = pa = 0$ for any $a \in A$, and also that for any $r \in R$ the element $NrTa = T(r^p a)$ remains in TA. So $A \subseteq R[p]$ corresponds to a submodule of the Frobenius twisted module of p-torsion elements in R.

Now returning to the analysis of J, the image of J in the geometric fixed points $W_1(R)/T\cdot R \simeq R$ gives an ideal K of R. Since $J \cap T\cdot R = T\cdot A$, for any given $x \in K$ there is an element $\alpha(x) \in R$ well-defined up to A such that $Nx - T\alpha(x) \in J$. Note that since $Nx - T\alpha(x) \in J \subseteq \text{Ker}F$, one has $F(Nx - T\alpha(x)) = x^p - p\alpha(x) = 0$, or simply that $x^p = p\alpha(x)$. Now in order for the collection $J = \{Nx - T(\alpha(x)+a) \mid x \in K, a \in A\}$ to form an ideal of $W_1(R)$, the operation α must behave like a p-th divided power operator modulo A. In other words, one needs $\alpha : K \subseteq R \longrightarrow R$ to satisfy

$$\alpha(rx) - r^p\alpha(x) \in A$$

and

$$\alpha(x+y) - \alpha(x) - \alpha(y) - C_p(x,y) \in A$$

This analysis gives the following proposition describing all Tambara ideals of a C_p Witt Tambara functor $W_\bullet(R)$.

Proposition 2.6.1. *The ideals $J \subseteq KerF \subseteq W_1(R)$ correspond exactly to triples of data (I, V, α) where $I \subseteq R$ is an ideal, $V \subseteq R[p]$ is a submodule of the frobenius twisted p-torsion, and $\alpha : I \longrightarrow R/V$ is a divided p-th power operator satisfying*

$$\alpha(rx) = r^p\alpha(x)$$

$$\alpha(x+y) = \alpha(x) + \alpha(y) + C_p(x,y)$$

Proof. An ideal $J \subseteq \text{Ker}F$ corresponds to the triple $(\pi(J), T^{-1}(J \cap TR), \alpha)$ where $\pi : W_1(R) \longrightarrow R$ is the geometric fixed points projection and $\alpha(x)$ is determined by requiring $Nx - T\alpha(x) \in J$. On the other hand, a triple (I, V, α) corresponds to the ideal

$$\{Nx - T(\tilde{\alpha(x)} + v) \mid x \in I, v \in V\}$$

where $\tilde{\alpha(x)}$ denotes any lift of $\alpha(x) \in R/V$ to R. For more details see the discussion immediately preceding the proposition. □

2.6.1 Φ^e-iso Tambara Thickenings and PD-Thickenings

In this section we assume that every ring is a $\mathbb{Z}_{(p)}$-algebra.

First we must recall a description of the data of a divided power envelope in the case of $\mathbb{Z}_{(p)}$-algebras. Let (A, I, γ) be a pd-algebra, with A a $\mathbb{Z}_{(p)}$-algebra. Then we may define an operator $\alpha := \alpha_p : I \longrightarrow I$ by

$$\alpha(x) = (p-1)!\gamma_p(x)$$

The map α satisfies the following properties:

$$\alpha(x+y) = \alpha(x) + \alpha(y) + C_p(x, y)$$

$$\alpha(ax) = a^p\alpha(x)$$

$$p\alpha(x) = x^p$$

It turns out that the data of γ can be completely recovered from α. Essentially, because A is a $\mathbb{Z}_{(p)}$-algebra, the only issue in defining the divided powers is with the powers of p in $1/n!$. These are taken care of by repeatedly applying α (e.g. if $n = p^k m$ with m prime to p, we will have $\gamma_n(x) = \text{ unit } \cdot \alpha_p^k(x)$.

Now we introduce a particular kind of "Tambara thickening". Fix an algebra R. A Φ^e-iso C_p-Tambara thickening of R consists of the data of a Tambara functor surjection $\mathbf{A} \longrightarrow W.(R)$ with the requirement that the induced map on geometric fixed points $\Phi^e\mathbf{A} \longrightarrow \Phi^e W.(R) \simeq R$ be an isomorphism. (Alternately one can require just that $A_0 \longrightarrow R$ surjects and that there is an isomorphism on Φ^e; together these imply that A_1 surjects onto $W_1(R)$).

We claim that the data of a Φ^e-iso C_p-Tambara thickening with injective transfer is equivalent to that of a pd-thickening. The functor in one direction is easy to describe:

$$(A, I, \alpha) \mapsto W.(A)/(Nx - T\alpha(x) | x \in I)$$

On the other hand, if $\mathbf{A}$ is a Φ^e-iso C_p-Tambara thickening of R with injective transfer, we can define a pd structure on $I := \mathrm{Ker}(A_0 \longrightarrow R)$ by setting $\alpha(x) = y$ for the unique y such that $Ty = Nx$. (Here such a y must exist because of the Φ^e-iso property, and moreover y is unique since T has been assumed injective).

Clearly the composition of the functors is the identity, and this gives an embedding of the category. of pd-thickenings of R into the category of Φ^e-iso C_p-Tambara thickenings.

2.7 Localizations and Witt Vectors

Let R be a commutative ring and consider the C_{p^n}-Witt Tambara functor $(W_r(R))$. In this section we consider localizations of this Tambara functor at a subset of elements in bottom level, and show that taking Witt vectors commutes with localization in a precise sense.

Fix $S \subseteq R$ a multiplicative subset. Then clearly $(N^r S) \subseteq (W_r(R))$ forms the multiplicative subset (including being closed under norms) generated by S, and we may consider the corresponding localized Tambara functor $(N^r S)^{-1} W_r(S)$. Recall that in this Tambara functor the transfers of a fraction are defined as

$$T(\frac{f}{s}) = \frac{T(s^{p-1} f)}{Ns}$$

Now from the identify map $S^1 R \longrightarrow S^{-1} R$ we arrive at a family of comparison maps $W_r(S^{-1}R) \longrightarrow N^r(S)^{-1} W_r(R)$. We claim that these comparison maps are isomorphisms. Indeed, this follows immediately from identifying the universal properties satisifed by each Tambara functor. We have

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Tamb}}(W_\bullet(S^{-1}R), B_\bullet) &\simeq \mathrm{Hom}(S^{-1}R, B_0) \simeq \{f : R \longrightarrow B_0 \mid f(S) \subseteq B_0^\times\} \\
&\simeq \{g : W_\bullet(R) \longrightarrow B_\bullet \mid g_0(R) \subseteq B_0^\times\} \\
&= \{g : W_\bullet(R) \longrightarrow B_\bullet \mid g(N^\bullet(S)) \subseteq B_\bullet^\times\} \\
&= \mathrm{Hom}_{\mathrm{Tamb}}((N^\bullet S)^{-1} W_\bullet(R), B_\bullet)
\end{aligned}$$

2.8 Cuntz-Deninger Presentations for Witt Vectors

In “An Alternative to Witt Vectors” ([CD13]), Cuntz and Deninger show that the Witt vectors of a perfect characteristic p ring R admit a simple description in terms of the monoid algebra $\mathbb{Z}[(R,\cdot)]$ and the kernel I of the augmentation $\mathbb{Z}[(R,\cdot)] \longrightarrow R$. In this section we reconceptualize their results from the point of view of Tambara functors.

We start in full generality, and then gradually specialize to the case where we can recover exactly their results. Let R be a commutative ring, and consider the Witt-Tambara functor $W_\bullet(R)$. We begin with the identity map of multiplicative monoids $(R,\cdot) \longrightarrow (R,\cdot) = (W_\bullet(R),\cdot)_0$. This gives us a map of multiplicative semi-Mackey functors $\underline{(R,\cdot)}^\vee \longrightarrow (W_\bullet(R),\cdot)$. Applying another adjoint, the Tambarization of a semi-Mackey functor, we arrive at a map of Tambara functors

$$\underline{A}[\underline{(R,\cdot)}^\vee] \longrightarrow W_\bullet(R)$$

Now as this map is a surjection in level zero, and the Witt-Tambara functor is generated as a Tambara functor by level zero, we have evidently constructed a surjection. Moreover, by comparing the functors they represent, we may describe the kernel. As above let I be the kernel of the natural map $\mathbb{Z}[(R,\cdot)] \longrightarrow R$, so I is generated by elements of the form $[x+y]-[x]-[y]$. Consider the Tambara ideal $I_\bullet$ of $\underline{A}[\underline{(R,\cdot}^\vee]$ generated by $I = I_0$ in level zero. Then we have natural isomorphisms:

$$\mathrm{Hom}(\underline{A}[\underline{(R,\cdot)}^\vee]/I_\bullet, B_\bullet) = \mathrm{Hom}(\mathbb{Z}[(R,\cdot)]/I, B_0) \simeq \mathrm{Hom}(R, B_0) = \mathrm{Hom}(W_\bullet(R), B_\bullet)$$

This gives us our first result. For any commutative ring R, there is a canonical surjection $A_n[(R,\cdot)] \longrightarrow W_n(R)$ with kernel I_n. However, in this level of generality, an explicit description of I_n in terms of I seems hopelessly intractable. With a few more assumptions, however, the nature of I_n becomes clear.

So first we assume that R is of characteristic p. This means that $W_\bullet(R)$ is an algebra over $W_\bullet(\mathbb{F}_p)$, and the latter is known to be a $\underline{\mathbb{Z}}$-algebra. This is equivalent to saying the natural map from the Burnside functor $\underline{A}$ factors through $\underline{\mathbb{Z}}$, which is the same as saying that every transfer is equal to p. So in this case we may replace the Tambarization with the $\underline{\mathbb{Z}}$-Tambarization, and we arrive at surjections $\mathbb{Z}[(R, \cdot)] \longrightarrow W_n(R)$.

Now if we further assume that R is a perfect $\mathbb{F}_p$-algebra, then the semi-Mackey functor $\underline{(R, \cdot)}^\vee$ is naturally isomorphic to the semi-Mackey functor $\underline{(R, \cdot)}$. This allows us to replace the Tambara functor $\underline{\mathbb{Z}}[\underline{(R, \cdot)^\vee}]$ with $\underline{\mathbb{Z}}[\underline{(R, \cdot)}]$. Well this Tambara functor has the same algebra $\mathbb{Z}[(R, \cdot)]$ in each level, the change in its norm maps make the nature of the ideals I_n easier to identify.

Here it becomes clear that $I_{n+1} = I_n^{[p]} + pI_n$. We show by induction that $I_n = I^n$. Note that $p \in I$, so it is clear that

$$I_{n+1} = (I^n)^{[p]} + pI^n \subseteq I^{n+1}$$

So it only remains to show that $I^{n+1} \subseteq I_{n+1}$. For this, we use the lift of frobenius F, given on algebra generators $[r]$ by $F([r]) = [r^p]$. By perfection of R, we have $F(I) = I$ and thus $F(I^n) = I^n$. Now clearly

$$I^{n+1} = F(I^{n+1}) \subseteq (I^{n+1})^{[p]} + p\delta I^{n+1} \subseteq (I^n)^{[p]} + p\delta I^{n+1}$$

So it suffices to show that $\delta I^{n+1} \subseteq I^n$. This is also the key lemma used in [CD13].

Lemma 2.8.1. *Let A a δ-ring with φ the lift of frobenius. Suppose I is a φ-stable ideal. Then $\delta I^{n+1} \subseteq I^n$.*

Proof. We go by induction. For the base case $n = 1$, note for $xy \in I^2$

$$\delta(xy) = x^p \delta y + \varphi(y)\delta x \subseteq I^{[p]} + \varphi(I) \subseteq I$$

Moreover has $\delta(x + y) = \delta x + \delta y + C_p(x, y)$, and the last term is evidently in $I^p \subseteq I$. Since I^2 is additively generated by products xy, this gives the result in the base case.

Now suppose we have that $\delta I^{n+1} \subseteq I^n$. We will show that $\delta I^{n+2} \subseteq I^{n+1}$. Again like above, it is enough to show that $\delta(x_1 \cdots x_{n+2}) \in I^{n+1}$ for the collection of products $x_1 \cdots x_{n+2}$ which additively generate I^{n+2}. One has

$$\delta((x_1 \cdots x_{n+1})x_{n+2}) = \varphi(x_{n+2})\delta(x_1 \cdots x_{n+1}) + (x_1 \cdots x_{n+1})^p \delta x_{n+2} \in \varphi(I)I^n + I^{(n+1)p}$$

Now as I is φ-stable by assumption, the last ideal is evidently contained in I^{n+1} as desired. □

2.9 Applications to Deformation Theory

In this section we show that several deformation-theoretic results may be obtained from the Tambara perspective without recourse to the usual theory of the cotangent complex.

Let A and B be commutative rings with a map $A \longrightarrow B/p$. Then considering the crystalline Tambara functor $B/p^{\bullet+1}$, and using that $W_\bullet(A)$ is the free Tambara functor on A in level zero, we are immediately given comparison maps $W_r(A) \longrightarrow B/p^{r+1}$. Of course by functoriality these are determined by the case when $B = A/p$ and $A \longrightarrow A/p$ is the natural projection. So they depend only on the maps $W_r(A) \longrightarrow A/p^{r+1}$. In this case it is easy to check that the map on total geometric fixed points in level r corresponds to the r-th Frobenius $\varphi^r : A \longrightarrow A/p$.

It is well known that if A is a perfect $\mathbb{F}_p$-algebra, then it admits a unique deformation to a flat $W_r(k)$-algebra for all r. This is usually proven by showing that A has a vanishing cotangent complex and appealing to Illusie's deformation theory of commutative algebras in terms of the cotangent complex. In this section, we bypass the cotangent complex entirely and recover this now classical result from the Tambara-theoretic perspective.

Theorem 2.9.1. *If A is a perfect $\mathbb{F}_p$-algebra, then A admits a unique deformation to a flat $\mathbb{Z}/p^{n+1}$-algebra. This deformation is in fact given by $W_n(A)$.*

Proof. Let A be a perfect $\mathbb{F}_p$-algebra, and consider a flat deformation $\tilde{A}$, so that $\tilde{A}/p \simeq A$.

Then considering the crystalline tambara functor $\tilde{A}/p^{\bullet+1}$ and the isomorphism $A \longrightarrow \tilde{A}/p$, we are immediately afforded maps $W_r(A) \longrightarrow \tilde{A}/p^{r+1}$.

Since $\tilde{A}$ is flat as a $W_n(\mathbb{F}_p)$-algebra, the transfer maps in the crystalline Tambara functor $\tilde{A}/p^{\bullet+1}$ are injective, which is to say we have exact sequences

$$0 \longrightarrow A/p^r \longrightarrow^p A/p^{r+1} \longrightarrow A/p \longrightarrow 0$$

Since A is perfect, map on total geometric fixed points, which is given by powers of the frobenius in each level, is an isomorphism. A map of Tambara functors with injective transfers inducing an isomorphism on total geometric fixed points is automatically an isomorphism (see section 2.2), so we arrive at an isomorphism of Tambara functors $W_\bullet(A) \longrightarrow \tilde{A}/p^{\bullet+1}$ and in particular an isomorphism of rings $W_n(A) \longrightarrow \tilde{A}$. This shows that if a flat deformation exists, it is unique, and given by $W_n(A)$.

The fact that $W_n(A)$ provides a flat deformation of A follows from A being perfect. Since in particular A is semiperfect, the restriction maps (the frobenius maps) in the Witt-Tambara functor $W_\bullet(A)$ are surjective. This gives $(T(1)) = TW_{n-1}(A)$. Since A is an $\mathbb{F}_p$-algebra, one has $T(1) = p$. This shows that $W_n(A)/p \simeq A$. The fact that $W_n(A)$ is flat over $\mathbb{Z}_{p^{n+1}}$ follows from the injectivity of the transfer maps, which are given by multiplying a lift by p, in $W_\bullet(A) = \{\{W_n(A)/p^{\bullet+1}\}$. □

We can also record another deformation-theoretic result that follows quickly from the Tambara-theoretic perspective.

Theorem 2.9.2. *Suppose A, B are p-complete rings and that A is mod p semiperfect. Then the reduction map $Hom(A, B) \longrightarrow Hom(A/p, B/p)$ is an injection.*

Proof. Any map $A \longrightarrow B$ induces a map of cohomological Tambara functors $\{A/p^{\bullet+1}\} \longrightarrow \{B/p^{\bullet+1}\}$. Because A is mod p semiperfect, the natural morphism of Tambara functors $W_\bullet(A/p) \longrightarrow \{A/p^{\bullet+1}\}$ is surjective. (One may see this by checking that it is surjective on geometric fixed points, which follows immediately from A/p being semiperfect). Thus we have an injection of hom-sets

$$\mathrm{Hom}_{\mathrm{Tamb}}(\{A/p^{\bullet+1}\}, \{B/p^{\bullet+1}\}) \hookrightarrow \mathrm{Hom}_{\mathrm{Tamb}}(W_\bullet(A/p), \{B/p^{\bullet+1}\}) \simeq \mathrm{Hom}(A/p, B/p)$$

Thus we only need to check that the map

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}_{\mathrm{Tamb}}(\{A/p^{\bullet+1}\}, \{B/p^{\bullet+1}\})$$

is an injection. But this follows immediately from p-completeness, since this map has a section given by the inverse limit :

$$\lim : \mathrm{Hom}_{\mathrm{Tamb}}(\{A/p^{\bullet+1}\}, \{B/p^{\bullet+1}\}) \longrightarrow \mathrm{Hom}(\hat{A}_p, \hat{B}_p) \simeq \mathrm{Hom}(A, B)$$

□

CHAPTER 3

From Prisms to Tambara Functors

We use the basic set-up of prisms introduced in [BS22]. We quickly recall the definition of a δ-ring and distinguished elements, and then prisms.

Definition 3.0.1. *A δ-ring A is a commutative ring A equipped with an operation $\delta : A \longrightarrow A$ satisfying*

$$\delta(x+y) = \delta x + \delta y - C_p(x,y)$$

and

$$\delta(xy) = x^p \delta y + y^p \delta x + p\delta x \delta y$$

An element $d \in A$ is said to be distinguished if δd is a unit.

The notion of a δ-ring is a slight refinement of the notion of frobenius lift. Indeed, any δ-ring comes with a ring endomorphism $\varphi = \varphi_\delta : A \longrightarrow A$ lifting the frobenius map. One has

$$\varphi(x) = x^p + p\delta x$$

The axioms for a δ-structure can be deduced from finding out what δ must satisfy in order for φ to be a ring map, and then dividing by p when possible.

We now turn to the definition of an (oriented) prism.

Definition 3.0.2. *An oriented prism (A, d) consists of a δ-ring A and a distinguished element $d \in A$. The element d is required to be a non-zero divisor, and A is required to be derived (p, d)-adically complete.*

Remark 3.0.1. Actually in [BS22] prisms are given by pairs (A, I) where I is an ideal locally generated by a distinguished non-zero divisor and A is derived (p, I)-adically complete. We do not concern ourselves with the non-orientable case, when I fails to be principal, in this thesis. Instead we give an explicit recipe for constructing a C_{p^∞}-Tambara functor associated to an oriented prism (A, d). Sulyma has given a more indirect construction for all prisms in [Sul23].

3.1 Special Units for Oriented Prisms

In this section we construct a family of special units $\{u_n\}$ associated to an oriented prism $(A, (d))$. We will use these units and their properties later to construct a C_{p^∞}-Tambara functor associated to the prism A and explicate some of its interesting properties.

We construct the family of units first in the universal case $A = \mathbb{Z}_p\{d, \delta d^{-1}\}$. The advantage of this is that A is p-torsion free, and so we may construct the u_n up to multiples of p, and then for any oriented prism $(B, (d))$ there is an automatic map $(A, (d)) \longrightarrow (B, (d))$ taking the universal family of units $\{u_n^A\}$ to the units $\{u_n^B\}$ for $(B, (d))$.

(The universal oriented prism is a localization (and completion) of free delta-ring, which is p-torsion free).

Proposition 3.1.1. *Let $(A, (d))$ denote the universal oriented prism as above. Then there is a unique family of units $\{u_n\}$ of A satisfying*

$$\varphi^n(d) = d^p \varphi(d)^{p-1} \cdots \varphi^{n-1}(d)^{p-1} + p u_n = d d_{n-1}^{p-1} + p u_n$$

Proof. As A is p-torsion free, it suffices to show that there exists units u_n satisfying the above equation. We go by induction. In the first case, we have

$$\varphi(d) = d^p + p\delta d = d^p + p u_1$$

where $u_1 = \delta d$ is a unit since d is distinguished. Now suppose we have constructed u_n, and have the equation

$$\varphi^n(d) = d^p\varphi(d)^{p-1}\cdots\varphi^{n-1}(d)^{p-1} + pu_n$$

Applying φ to each side, we see that

$$\varphi^{n+1}(d) = \varphi(d)^p\varphi^2(d)^{p-1}\cdots\varphi^n(d)^{p-1} + p\varphi(u_n)$$

$$= \varphi(d)\varphi(d)^{p-1}\cdots\varphi^n(d)^{p-1} + p\varphi(u_n)$$

$$= (d^p + pu_1)\varphi(d_{n-1})^{p-1} + p\varphi(u_n) = dd^{p-1}\varphi(d_{n-1}) + p(\varphi(u_n) + u_1\varphi(d_{n-1})^{p-1})$$

$$= dd_n^{p-1} + p(\varphi(u_n) + u_1\varphi(d_{n-1})^{p-1}$$

So now we set $u_{n+1} = \varphi(u_n) + u_1\varphi(d_{n-1})^{p-1}$, which is a unit because it is congruent to the unit $\varphi(u_n)$ modulo the topologically nilpotent ideal (p, d). This completes the proof of the construction of the u_n. □

Corollary 3.1.1. *In the course of the proof of the proposition, we deduced the following useful properties of the units, which we record here. We have the recursive relation*

$$u_{n+1} = \varphi(u_n) + u_1\varphi(d_{n-1})^{p-1}$$

and therefore the more explicit formula

$$u_n = \varphi^{n-1}(u_1) + \varphi^{n-2}(u_1)\varphi^{n-1}(d)^{p-1} + \varphi^{n-3}(u_1)\varphi^{n-2}(d_1)^{p-1} + \cdots + u_1\varphi(d_{n-2})^{p-1}$$

$$= \varphi^{n-1}(u_1) + \sum_{k=2}^{n} \varphi^{n-k}(u_1)\varphi^{n-k+1}(d_{k-2})^{p-1}$$

Finally, we record one other important property of the special units, which we will need to construct the norm maps in the Tambara functor we associate to the prism.

Proposition 3.1.2. *For any oriented prism $(A, (d))$, for every $n \geq 1$ the special unit u_n satisfies the following property:*

$$\delta d_n = u_{n+1}\varphi(d_{n-1})$$

Proof. Again we prove this in the universal case, where we may work up to multiples of p since the universal oriented prism is p-torsion free. So it is enough to show that

$$p\delta d_n = pu_{n+1}\varphi(d_{n-1})$$

Now the left hand side is evidently given by $\varphi(d_n)-d_n^p$. The right hand side, by the definition of the u_n, is given by

$$pu_{n+1}\varphi(d_{n-1}) = (\varphi^{n+1}(d) - dd_n^{p-1})\varphi(d_{n-1}) = \varphi(d_n) - dd_n^{p-1}\varphi(d_{n-1})$$

But now unwinding things using that $d_n = d\varphi(d)\cdots\varphi^n(d)$, we see that

$$\varphi(d_n) - dd_n^{p-1}\varphi(d_{n-1}) = \varphi(d_n) - d^p\varphi(d)^{p-1}\cdots\varphi^n(d)^{p-1}\varphi(d)\cdots\varphi^n(d) = \varphi(d_n) - d_n^p$$

This concludes the demonstration. □

3.2 Special Units and Change of Orientation

In this section we record how the special units react to a change of orientation. To allow us to freely work up to multiples of p and thereby simplify things, we work over the universal oriented prism with a choice of unit: $A\{v^{\pm}\}$. As this is a (completion) of a localization of a free delta-ring, it is safely p-torsion free.

Now consider the special units $\{u'_n\}$ associated to the orientation $d' = vd$. By definition, we have

$$\varphi^n(d') = d'{d'_{n-1}}^{p-1} + pu'_n = vv_{n-1}^{p-1}dd_{n-1}^{p-1} + pu'_n$$

But we also see that

$$\varphi^n(d') = \varphi^n(v)\varphi^n(d) = \varphi^n(v)dd_{n-1}^{p-1} + p\varphi^n(v)u_n$$

Now using the simple fact that $\varphi^n(v) \equiv v^{p^n} \equiv vv_{n-1}^{p-1} \pmod{p}$, we see that

$$u'_n \equiv \varphi^n(v)u_n \pmod{dd_{n-1}^{p-1}}$$

We will later use only the weaker congruence

$$u'_n \equiv \varphi^n(v)u_n \pmod{d_{n-1}}$$

to establish the independence of our constructed Tambara functor from the choice of orientation.

3.3 The Functor from Orientable Prisms to Tambara Functors

Now we may construct the principal object of investigation in this chapter: A functor from orientable prisms to C_{p^∞}-Tambara functors. Let now $(A,(d))$ be an arbitrary oriented prism. We will construct a Tambara functor $A_\bullet$ associated to $(A,(d))$, and in the next section show that it is independent of the choice of orientation.

In level n we set $A_n = A/d_n$, with the evident surjective restriction maps $A/d_{n+1} \longrightarrow A/d_n$. For the transfer in level n it is enough to define the transfer of 1, which we set as:

$$\theta_n := T_n^{n+1}(1) := u_{n+1}^{-1}\varphi^{n+1}(d)$$

We have

$$\theta_n = u_{n+1}^{-1}(dd_n^{p-1} + pu_{n+1}) = u_{n+1}^{-1}dd_n^{p-1} + p \equiv p \pmod{d_n}$$

so that $RT = p$ in every level, as required for a Weyl-invariant Tambara functor.

This gives the underlying Green functor, which interestingly only depends on A and the special units u_n, so only A, d, and the δ-iterates of d.

The construction of the norms is slightly more subtle, both in showing their well-definedness and checking the requisite properties. In level n, we define:

$$N_n(x) = \varphi(\tilde{x}) - \theta_n\delta\tilde{x} = \tilde{x}^p + (p - \theta_n)\delta\tilde{x}$$

where $\tilde{x}$ is any lift of x to A.

First we check that the norms are well-defined. Given a lift $\tilde{x}$ of $x \in A/d_n$ to A, any other lift is given by $\tilde{x}+ad_n$ for some $a \in A$. We check that the formula for the norm doesn't depend on a.

$$N_n(x) = \varphi(\tilde{x} + ad_n) - u_{n+1}^{-1}\varphi^{n+1}(d)\delta(\tilde{x} + ad_n)$$

$$= \varphi(\tilde{x}) + \varphi(a)\varphi(d_n) - u_{n+1}^{-1}\varphi^{n+1}(d)\left(\delta\tilde{x} + \delta(ad_n) + C_p(\tilde{x}, ad_n)\right)$$

Note that $C_p(\tilde{x}, ad_n) \in d_nA$, and as $d_{n+1} = d_n\varphi^{n+1}(d)$, we get

$$N_n(x) \equiv \varphi(\tilde{x}) + \varphi(a)\varphi(d_n) - u_{n+1}^{-1}\varphi^{n+1}(d)\left(\delta\tilde{x} + \delta(ad_n)\right) \pmod{d_{n+1}}$$

Now $\delta(ad_n) = \varphi(a)\delta d_n + d_n^p\delta a$, so working modulo d_{n+1} the norm expression simplifies once more to

$$\varphi(\tilde{x}) + \varphi(a)\varphi(d_n) - u_{n+1}^{-1}\varphi^{n+1}(d)\left(\delta(\tilde{x}) + \varphi(a)\delta d_n\right)$$

$$\equiv \varphi(\tilde{x}) - u_{n+1}^{-1}\varphi^{n+1}(d)\delta\tilde{x} + \varphi(a)\left(\varphi(d_n) - u_{n+1}^{-1}\varphi^{n+1}(d)\delta d_n\right) \pmod{d_{n+1}}$$

So it only remains to show that

$$\varphi(d_n) - u_{n+1}^{-1}\varphi^{n+1}(d) \equiv 0 \pmod{d_{n+1}}$$

But this is exactly Corollary 3.1.1 above. This completes the demonstration that the norms are well defined.

Restriction of the norms: It is easy to see that $RNx = x^p$ in every level, as we have

$$RNx = R(\varphi(\tilde{x}) - \theta_{n+1}\delta\tilde{x}) = R(\tilde{x}^p + (p - \theta_{n+1}\delta\tilde{x})$$

and of course $p - \theta_{n+1}$ is in the kernel of R, leaving us with

$$RNx = R(\tilde{x}^p) = R(\tilde{x})^p = x^p$$

as desired.

Multiplicativity of the Norms: Now we show that the norms are multiplicative. Let $x, y \in A_n = A/d_n$. Then we have

$$N_n(xy) = \varphi(\tilde{xy}) - \theta_{n+1}\delta\tilde{xy}$$

As we may use any lift of xy to compute the norm, we choose lifts $\tilde{x}$ and $\tilde{y}$ of x and y separately and then use the lift $\tilde{xy} = \tilde{x}\tilde{y}$. This gives

$$N_n(xy) = \varphi(\tilde{x})\varphi(\tilde{y}) - \theta_{n+1}\delta(\tilde{x}\tilde{y}) = \varphi(\tilde{x})\varphi(\tilde{y}) - \theta_{n+1}\left(\varphi(\tilde{x})\delta\tilde{y} + \varphi(\tilde{y})\delta\tilde{x} - p\delta x\delta y\right)$$

where here we have used the less common symmetric version of the Leibniz rule for a p-derivation δ:

$$\delta(ab) = \varphi(a)\delta b + \varphi(b)\delta a - p\delta a\delta b$$

Now on the other hand we may look at the product

$$N_n(x)N_n(y) = (\varphi(\tilde{x}) - \theta_{n+1}\delta\tilde{x})(\varphi(\tilde{y}) - \theta_{n+1}\delta\tilde{y})$$

$$= \varphi(\tilde{x})\varphi(\tilde{y}) - \theta_{n+1}\left(\varphi(\tilde{x})\delta\tilde{y} + \varphi(\tilde{y})\delta\tilde{x} - \theta_{n+1}\delta\tilde{x}\delta\tilde{y}\right)$$

But recall that in $A_{n+1} = A/d_{n+1}$ we have $\theta_{n+1}^2 = p\theta_{n+1}$. So the two expressions for $N_n(xy)$ and $N_n(x)N_n(y)$ agree, and we may conclude that the norms are multiplicative.

Interaction of Norms with Addition: Now we check that the norms behave appropriately with respect to addition. This follows quickly from the definition and the properties of the p-derivation δ. We have

$$N_n(x+y) = \varphi(\tilde{x}+\tilde{y}) - \theta_{n+1}\delta(\tilde{x}+\tilde{y})$$

$$= \varphi(\tilde{x}) + \varphi(\tilde{y}) - \theta_{n+1}\left(\delta\tilde{x} + \delta\tilde{y} - C_p(\tilde{x}, \tilde{y})\right)$$

$$= \varphi(\tilde{x}) - \theta_{n+1}\delta\tilde{x} + \varphi(\tilde{y}) - \theta_{n+1}\delta\tilde{y} + \theta_{n+1}C_p(\tilde{x}, \tilde{y})$$

$$= N_n(x) + N_n(y) + \theta_{n+1}C_p(\tilde{x}, \tilde{y}) = N_n(x) + N_n(y) + T_n(1)C_p(\tilde{x}, \tilde{y}) = N_nx + N_ny + T_nC_p(x, y)$$

This shows the norms interact appropriately with addition, and now it only remains to check that the norms cooperate with the transfers.

Interaction of Norms with Transfers: In general, checking that norms cooperate with transfers could be involved. However, as our prismatic Tambara functors have surjective restrictions, it is enough to check that the norms of the transfer elements θ_n do the appropriate thing. This amounts to checking for each $n \geq 1$ that

$$N_n T_{n-1}(1) = p^{p-2} T_n T_{n-1}(1)$$

So we must check that

$$N_n(u_n^{-1}\varphi^n(d)) = p^{p-2} u_{n+1}^{-1}\varphi^{n+1}(d) u_n^{-1}\varphi^n(d)$$

where the equality is happening in $A_{n+1} = A/d_{n+1}$.

For this, we first need a preparatory lemma.

Lemma 3.3.1. *Let $(A, (d))$ be an oriented prism. Then the special units u_n satisfy the following congruence:*

$$\varphi^{n+1}(d)\,(u_{n+1} - \varphi^n(u_1)) \equiv u_n^{p-1} p^{p-2} \varphi^n(d)\varphi^{n+1}(d) \pmod{d_{n+1}}$$

Proof. It is of course enough to prove this for the universal oriented prism and its special units $\{u_n\}$. This will be crucial because the universal oriented prism is transversal, and so each A/d_n is p-torsion free. We start with the equality

$$\varphi^n(d)^p + p\varphi^n(u_1) = \varphi^n(d^p + pu_1) = \varphi^{n+1}(d) = dd_n^{p-1} + pu_{n+1}$$

The last equality comes from the definition of the special units (see section 3.1). Now moving things around yields

$$p(u_{n+1} - \varphi^n(u_1) = \varphi^n(d)^p - dd_n^{p-1}$$

Now multiplying through by $\varphi^{n+1}(d)$ gives

$$p\varphi^{n+1}(d)(u_{n+1} - \varphi^n(u_1)) = \varphi^{n+1}(d)\varphi^n(d)^p - dd_n^{p-1}\varphi^{n+1}(d)$$

Now as we have $d_{n+1} = d_n\varphi^{n+1}(d)$, the last term vanishes when we go modulo d_{n+1}, giving the congruence

$$p\varphi^{n+1}(d)(u_{n+1} - \varphi^n(u_1)) \equiv \varphi^{n+1}(d)\varphi^n(d)^p \pmod{d_{n+1}}$$

Now we use that $\theta_n = u_n^{-1}\varphi^n(d)$ is a transfer element in A/d_n, so that $\theta_n^p \equiv p^{p-1}\theta_n$ in A/d_n. This gives

$$p\varphi^{n+1}(d)(u_{n+1} - \varphi^n(u_1)) \equiv \varphi^{n+1}(d)\varphi^n(d)^p \equiv p^{p-1}u_n^{p-1}\varphi^{n+1}(d)\varphi^n(d) \pmod{d_{n+1}}$$

Now dividing by p, which we may do in the universal case because it is transversal, gives the desired result. □

Now we return to proving the needed norm of transfer relation. As noted above, we must show

$$N_n(u_n^{-1}\varphi^n(d)) = p^{p-2}u_{n+1}^{-1}\varphi^{n+1}(d)u_n^{-1}\varphi^n(d)$$

where the equality is happening in $A_{n+1} = A/d_{n+1}$.

Using the definition of the norm, the left hand side becomes

$$u_n^{-p}\varphi^n(d)^p + (p - u_{n+1}^{-1}\varphi^{n+1}(d))\delta(u_n^{-1}\varphi^n(d))$$

$$\equiv u_n^{-p}\varphi^n(d)^p - u_{n+1}^{-1}dd_n^{p-1}\left(u_n^{-p}\delta\varphi^n(d) + \varphi^{n+1}(d)\delta u_n^{-1}\right)$$

$$\equiv u_n^{-p}\varphi^n(d)^p - u_{n+1}^{-1}dd_n^{p-1}u_n^{-p}\varphi^n(u_1)$$

$$\equiv u_n^{-p}u_{n+1}^{-1}\left(u_{n+1}\varphi^n(d)^p - dd_n^{p-1}\varphi^n(u_1)\right) \pmod{d_{n+1}}$$

Now using the basic relation $\varphi^n(d)^p = \varphi^{n+1}(d) - p\varphi^n(u_1)$ we arrive at

$$u_n^{-p}u_{n+1}^{-1}\left(u_{n+1}\varphi^{n+1}(d) - u_{n+1}p\varphi^n(u_1) - dd_n^{p-1}\varphi^n(u_1)\right)$$

$$= u_n^{-p} u_{n+1}^{-1} \left(u_{n+1} \varphi^{n+1}(d) - \varphi^{n+1}(d) \varphi^n(u_1) \right)$$

$$= u_n^{-p} u_{n+1}^{-1} \varphi^{n+1}(d) \left(u_{n+1} - \varphi^n(u_1) \right)$$

Now using the congruence from the lemma above, this becomes

$$u_n^{-p} u_{n+1}^{-1} \varphi^{n+1}(d) \left(u_{n+1} - \varphi^n(u_1) \right)$$

$$\equiv u_n^{-p} u_{n+1}^{-1} u_n^{p-1} p^{p-2} \varphi^n(d) \varphi^{n+1}(d)$$

$$\equiv p^{p-2} u_{n+1}^{-1} \varphi^{n+1}(d) u_n^{-1} \varphi^n(d) \pmod{d_{n+1}}$$

Thus the norm-of-transfer relations necessary for a Weyl-invariant, surjective restrictions Tambara functor are satisfied.

3.3.1 Independence of Choice of Orientation

Let $(A, (d))$ be an oriented prism. In this subsection we show that the associated Tambara functor $A_\bullet$ does not depend on the choice of the orientation. Any other orientation is given by vd for some unit $v \in A^\times$. Now it is immediately clear that the ideals (d_n) are not affected by a change in unit, since clearly

$$(vd)_n = v_n d_n$$

where $v_n = v\varphi(v) \cdots \varphi^n(v)$ is a product of units and therefore itself a unit.

Thus the rings $A_n = A/d_n$ and the restriction maps between them are independent of the choice of orientation. It only remains to show that the transfers and norms are unchanged. Now the transfers depend only on the transfer elements $T_n(1) = \theta_n = u_n^{-1}\varphi^n(d) \in A/d_n$, and in fact the norms also only depend on these, since they are defined by

$$N_n(x) = \varphi(\tilde{x}) - \theta_n \delta \tilde{x}$$

So it remains only to check that the special elements $u_n^{-1}\varphi^n(d) \in A/d_n$ do not depend on the choice of orientation. Above in section 3.2, we showed that the special units u_n' associated to vd are related to the units u_n associated to d in the following manner:

$$u'_n \equiv \varphi^n(v)u_n \pmod{d_{n-1}}$$

So we have $u'_n = \varphi^n(v) + ad_{n-1}$ for some $a \in A$. Then we check the special transfer elements θ'_n, we have:

$$\theta'_n = u_n'^{-1}\varphi^n(vd) = (\varphi^n(v)u_n + ad_n)^{-1}\varphi^n(v)\varphi^n(d)$$

Note that $(\varphi^n(v)u_n + ad_n)^{-1} \equiv \varphi^n(v)^{-1}u_n^{-1} \pmod{d_{n-1}}$. Multiplying this by $\varphi^n(d)$, and noting that $d_n = \varphi^n(d)d_{n-1}$, we see that

$$\theta'_n \equiv \varphi^n(v)^{-1}u_n^{-1}\varphi^n(v)\varphi^n(d) = u_n^{-1}\varphi^n(d) = \theta_n \pmod{d_n}$$

Thus the transfer elements are the same, giving the same transfers and norms as noted above. This concludes the demonstration that $A_\bullet$ depends only on the prism $(A,(d))$, and not on the choice of orientation.

3.3.2 Elementary Properties of The Functor

The first observation we make on the functor $(A,(d)) \mapsto A_\bullet$ is that, by adjunction, it gives us a natural comparison map $W_\bullet(A/d) \longrightarrow A_\bullet$. We will show later that this map is an isomorphism exactly when the prism A is perfect, and that it is surjective if A is semiperfect (in the sense that φ is surjective). Characterizing injectivity is more subtle, as the kernel of the comparison map is rather delicate. In general however, the further from being semiperfect a prism is, the smaller the image of the comparison map in $A_\bullet$. The Breuil-Kisin prism $(\mathbb{Z}_p[[u]],(u-p))$ provides an example where the image is quite small.

Remark 3.3.1. Molokov first constructed the comparison maps (as a morphism of inverse systems of rings) $W_\bullet(A/d) \longrightarrow A_\bullet$ ([Mol22]. Sulyma uses his transversal coordiantes in [Sul23] to show that the Tambara comparison map agrees with Molokov's comparison map.

In general, the comparison maps are difficult to express directly in terms of iterates of δ or even with Joyal's family of delta operations $\{\delta_n\}$. In [Sul23], another family of operations in a δ-ring are introduced to give a closed form expression of the comparison maps (see Definition 3.6 and Lemma 3.8 there).

The following theorem says that the functor provides a faithful embedding of the theory of prisms into Tambara algebra.

Theorem 3.3.1. *The functor* $(A,(d)) \mapsto A_\bullet$ *is faithful.*

Proof. Prisms are (p,d)-adically complete, and therefore complete with respect to the sequence of ideals (d_n). It follows that the inverse limit along the restrictions

$$\lim : \mathrm{Hom}_{\mathrm{Tamb}}(A_\bullet, B_\bullet) \longrightarrow \mathrm{Hom}_{\mathrm{CRing}}(A,B)$$

provides a section of the map $\mathrm{Hom}_{\Delta}(A,B) \longrightarrow \mathrm{Hom}_{\mathrm{Tamb}}(A_\bullet, B_\bullet)$. Thus the map on hom-sets induced by the functor is injective; in othe words, the functor is faithful. □

Remark 3.3.2. Unfortunately, the functor is not conservative. We will see in the crystalline examples in section 3.4 below that crystalline prisms entirely forget their δ-structures. So for example, for every δ-structure, the crystalline prisms $(\mathbb{Z}_p[x], \delta, p)$ are all sent to the same Tambara-functor. (Note that there are non-isomorphic δ-structures on $\mathbb{Z}_p[x]$, since there is more than one conjugacy class of frobenius lifts under the action of $\mathrm{Aut}(\mathbb{Z}_p[x])$).

Total Geometric Fixed Points: This will be used importantly in following section on perfect prisms (3.6). Consider the total geometric fixed point functor applied to a prismatic Tambara functor. We have $\Phi^n A_\bullet = A_n/TA_{n-1} = A_n/u_n^{-1}\phi^n(d) = A/\varphi^n(d)$, and the norm homomorphisms $N_n : \Phi^n A_\bullet \longrightarrow \Phi^{n+1} A_\bullet$ are given by $\varphi : A/\varphi^n(d) \longrightarrow A/\varphi^{n+1}(d)$, since we have $N_n(x) = \varphi(\tilde{x}) - \theta_n \delta \tilde{x} \equiv \varphi(x) \pmod{\theta_n}$. In particular, if A is a perfect prism, then the total geometric fixed points of $A_\bullet$ is given by the string of isomorphisms

$$A/d \xrightarrow{\sim} A/\varphi(d) \xrightarrow{\sim} \cdots \xrightarrow{\sim} A/\varphi^n(d) \xrightarrow{\sim} \cdots$$

.

We record this description of the total geometric fixed points in the next theorem.

Theorem 3.3.2. *Let $(A, (d))$ be a prism with associated Tambara functor $A_\bullet$. Then the total geometric fixed points of $A)_\bullet$ consist of the diagram*

$$A/d \longrightarrow A/\varphi(d) \longrightarrow \cdots$$

where the connecting maps are induced by the frobenius : $\varphi : A/\varphi^n(d) \longrightarrow A/\varphi^{n+1}(d)$.

Proof. See the discussion immediately preceding the statement of the theorem. □

In [HN19], the choice of the name 'prism' is explained through a picture that visualizes Spec$A$ as a collection of colored rays consisting of the Spec$A/\varphi^n(d)$ for each n, emanating from their mutual intesection that represents SpecA/p. Indeed, one way of understanding the distinguished condition for d is in demanding that SpecA/d and Spec$A/\varphi(d)$ meet only in characteristic p. The description of the total geometric fixed points of the Tambara functor $A_\bullet$ presents another way of understanding this picture.

Remark 3.3.3. The results of this chapter actually do not depend on all the assumptions of a prism. In particular, the construction of an associated C_{p^∞}-Tambara functors extends to δ-rings equipped with a distinguished element lying in the Jacobson ideal. In particular, the usual condition in a prism that $d \in A$ be a non-zero divisor (in other symbols, $A[d] = 0$) is not used. The exact significance of this condition remains to be unearthed. The author believes that transversal prisms (in particular (p, d) is a regular sequence) coincide exactly with the δ-pairs having associated C_p^∞-Tambara fuctor flat (as a Mackey functor, or equivalently over the Burnisde functor $W_\bullet(\mathbb{Z})$. On the other hand crystalline prisms should amount to the softer condition that the associated Tambara functor, in this case cohomological, be flat over $\underline{\mathbb{Z}}$.

Remark 3.3.4. The approach taken in [Sul23], based on lemma 2.2.8 in [BL22], shows that the functor can be extended to nonorientable prisms. The lemma in [BL22] supplies the needed transfer elements, but it specifies them only in the quotient rings A/d_n. The approach taken here for orientable prisms shows that a choice of orientation also provides one with specific lifts of the transfer elements to A (in this case given by $\theta_n = u_n^{-1}\varphi^n(d)$).

3.4 Examples

(1) Crystalline Prisms

Let $(A, (p))$ be a crystalline prism. This means that A is a (derived) p-adically complete, p-torsion free ring equipped with a δ-structure. Since A is p-torsion free, the delta structure is equivalent to its associated lift of Frobenius φ. Clearly in the crystalline case one has $d = p$ and as $\varphi^k(p) = p$ for all k, one has $d_n = d\varphi(d)\cdots\varphi^n(d) = p^{n+1}$.

As for the special units u_n, note that $u_1 = \delta p = 1 - p^{p-1}$. The general formula

$$\varphi^n(d) = dd_{n-1}^{p-1} + pu_n$$

shows that

$$p = pp^{n(p-1} + pu_n$$

Dividing by p, which produces no ambiguities because A is p-torsion free, yields

$$1 = p^{n(p-1)} + u_n$$

or simply

$$u_n = 1 - p^{n(p-1)}$$

This leads to transfer elements $\theta_n = u_n^{-1}\varphi^n(d) = (1 - p^{n(p-1)})^{-1}p \equiv p \pmod{p}^{n(p-1)+1}$. Now in all cases one has $n(p-1) + 1 \geq n + 1$, so in fact the transfer element $\theta_n \in A/d_n = A/p^{n+1}$ is simply given by p.

Finally as the $\theta_n - p$ terms therefore all vanish, it is the case that the associated norm formula simply gives $Nx = \tilde{x}^p$, i.e. raising a lift of x to the p-th power. So the Tambara functor associated to the crystalline prism $(A,(p))$ is the cohomological Tambara functor $\{A/p^{\bullet+1}\}$.

Note that in the crystalline case the associated Tambara functor appears to completely lose all information about the δ-structue (equivalently Frobenius lift). Heuristically, this may be an acceptable situation, as the crystalline comparison theorem of [BS22] says that prismatic cohomology (up to a Frobenius twist) over a crystalline prism recovers crystalline cohomology, which does not depend at all on the underlying δ-structure of A. So the Tambara functor associated to a crystalline prism may yet retain enough information to compute prismatic cohomology, despite losing the δ-structure entirely.

As any crystalline prism $(A,(p))$ gives rise to a cohomological Tambara functor, it is natural to ask if the converse is true. Indeed, this is conjectured to be the case (and proven for $p=2$ in [Sul23]. We cannot improve on the investigation of this issue found there. We only add the following interesting example, which shows that the prism condition (specifically $d \in A$ being a nonzero divisor) cannot be relaxed in the formulation of the conjecture.

Example 3.4.1. Consider the ring of Witt vectors $W(\mathbb{Z}_p)$ with its natural δ-structure as a co-free δ-ring. Then the element $V(1) \in W(\mathbb{Z}_p)$ is distinguished. One can see this by noting

$$FV(1) = p = V(1)^p + p(1 - \frac{V(1)^p}{p}) = V(1)^p + p(1 - p^{p-2}V(1))$$

so

$$\delta V(1) = 1 - p^{p-2}V(1) = 1 - V(p^{p-2})$$

This is always a unit because $W(\mathbb{Z}_p)$ is always $V(1)$-adically complete. Note that the pair $(W(\mathbb{Z}_p), V(1))$ does not give a prism, because $V(1)$ is not a zero divisor, for example:

$$V(1)(V(1)-p) = V(p) - pV(1) = V(p-p) = V(0) = 0$$

However, it is a δ-ring equipped with a distinguished element, so the machinery of this section still applies to yield a C_{p^∞}-Tambara functor associated to it. It is not hard to see that this Tambara functor is cohomological. On the other hand, the element $V(1)$ is not assoicate to

p in $W(\mathbb{Z}_p)$. Thus in Sulyma's conjecture, one cannot relax from prisms to general δ-rings with distinguished element.

(2) The q-crystalline prism

The q-crystalline prism $(\mathbb{Z}_p[[q-1]], ([p]_q))$ has δ-structure determined by $\delta q = 0$ and distinguished element $[p]_q = \frac{q^p-1}{q-1} = 1 + q + \cdots + q^{p-1}$, the q-analogue of p. As the frobenius acts by $\varphi(q) = q^p$, one sees immediately that $\varphi^k([p]_q) = [p]_{q^{p^n}} = \frac{q^{p^{n+1}}-1}{q^{p^n}-1}$. Thus the elements d_n are given by the q-analogues of p^{n+1}:

$$d_n = d \cdots \varphi^n(d) = [p]_q \cdots [p]_{q^{p^n}} = \frac{q^p-1}{q-1}\frac{q^{p^2}-1}{q^p-1} \cdots \frac{q^{p^{n+1}}-1}{q^{p^n}-1} = \frac{q^{p^{n+1}}-1}{q-1} = [p^{n+1}]_q$$

Since the q-crystalline prism is p-torsion free, the special units u_n can be deduced from the general formula $\varphi^n(d) = dd_{n-1}^{p-1} + pu_n$. For $d = [p]_q$ this reads as

$$[p]_{q^{p^n}} = [p]_q[p^n]_q^{p-1} + pu_n$$

Now the transfer element $\theta_n = u_n^{-1}\varphi^n(d) \in A/d_n$ depends only on the unit u_n modulo d_{n-1}, or in this case u_n modulo $[p^n]_q$. Going mod $[p^n]_q$ gives

$$[p]_{q^{p^n}} \equiv pu_n \pmod{[p^n]_q}$$

Now modulo $[p^n]_q$ one has $q^{p^n} = 1$, so the left hand side is simply p. Finally, $\mathbb{Z}_p[[q-1]]/[p^n]_q$ is p-torsion free as it injects into a product of p-torsion free rings, so in fact $u_n \equiv 1 \pmod{[p^n]_q}$. Thus it follows that the transfer elements θ_n are always given simply by $[p]_{q^{p^n}} \in A/[p^{n+1}]_q$.

As for the norms, one evidently has $N_n(q) = q^p$ since $\delta q = 0$. Since in each level the element q generates as an algebra, this completely determines the norm structure. One arrives at the formula

$$N_n(f(q)) = f(q^p) + [p]_{q^{p^n}}(\frac{f(q)^p - f(q^p)}{p})$$

which of course could also be written down immediately from the definition, where

$$N_n(f(q)) = \varphi(\tilde{f(q)}) - \theta_n \delta \tilde{f(q)}$$

(3) Nonreduced Prisms

This section gives an explicit example of distinct non-reduced prisms having the same associated Tambara functor. This shows explicitly that the functor from prisms to Tambara functors fails to be conservative. It would be interesting to know if there exists a pair of non-crystalline prisms that the functor fails to distinguish as well.

Consider the p-complete, p-torsion free ring $\mathbb{Z}_p[x]/x^2$. This may be made into a crystalline prism by equipping it with a δ-structure, or, as it is p-torsion free, equivalently a lift of Frobenius. For any $c \in \mathbb{Z}_p$, one may equip it with the lift of Frobenius $\varphi(x) = pcx$ (corresponding to $\delta x = cx$). For different c these are all nonisomorphic prisms (see lemma below). However, the associated Tambara functor is in all cases the cohomological Tambara functor $\mathbb{Z}/p^{\bullet+1}[x]/x^2$.

Lemma 3.4.1. *The p-torsion free ring $\mathbb{Z}_p[x]/x^2$ may be equipped with a delta structure δ_c satisfying $\delta_c x = pcx$ for any $c \in \mathbb{Z}_p$. For distinct c, these are all non-isomorphic delta structures.*

Proof. Because the ring in question is p-torsion free, it is enough to show that the endomorphisms φ_c are all nonisomorphic. Every ring automorphism $\sigma_v : \mathbb{Z}_p[x]/x^2 \longrightarrow \mathbb{Z}_p[x]/x^2$ is determined by sending x to vx for some p-adic unit $v \in \mathbb{Z}_p^\times$. For φ_c and $\varphi_{c'}$ to be equivalent, one needs

$$\sigma_v \varphi_c = \varphi_{c'} \sigma_v$$

evaluating on x means one would need

$$\sigma_v(x^p + cpx) = \varphi_{c'}(vx)$$

or

$$(vx)^p + cvpx = (vx)^p + p\delta_{c'}(vx) = (vx)^p + p\varphi_{c'}(v)\delta_{c'}x = (vx)^p + pvc'x$$

This of course amounts to requiring exactly that $c = c'$. □

(4) Perfect and Semiperfect Prisms In section 3.6 below, we will show that for any perfect prism the comparions map $W_\bullet(A/d) \longrightarrow A_\bullet$ is an isomorphism.

If A is semiperfect, then A surjects onto it's colimit perfection A_{perf}, and in fact $A_{\text{perf}} \simeq A/\text{Ker}\varphi^\infty$ where $\text{Ker}\varphi^\infty$ consists of the (p-adic closure) of the union of the kernels of the φ^n for all n. In this case it ought to be possible to describe the associated Tambara functor $A_\bullet$ as an explicit nilpotent thickening of the Witt-Tambara functor $W_\bullet(A_{\text{perf}}/d)$. We do not yet have an explicit description of this thickening.

3.5 Quasi-Cohomological Tambara Functors

The Tambara functors constructed from prisms in this chapter satisfy a peculiar property that makes them not so far from functors cohomological Tambara. From the definition of the norms, we see that in these Tambara functors there is the congruence for every n

$$NRx \equiv x^p \pmod{\theta_n - p}$$

We call these quasi-cohomological. In fact, more generally we define quasi-cohomological elements in a Tambara functor.

Definition 3.5.1. *Let $A_\bullet$ be a Tambara functor and $x_n \in A_n$ an element in level n. Then x_n is said to be a quasi-cohomological element if $NRx \equiv x^p \pmod{\theta_n - p}$ The Tambara functor $A_\bullet$ is said to be quasi-cohomological in level n if every element in level n is quasi-cohomogoical. The Tambara functor $A_\bullet$ is quasi-cohomological if it is quasi-cohomological in level n for every n.*

This may look like an unusual condition at first, but we will quickly see that it satisfies several natural properties and that in fact all Witt-Tambara functors are quasicohomological.

Lemma 3.5.1. *The quasi-cohomological elements of level n form a subring.*

Proof. Consider the operation $\mu(x)$ defined by

$$\mu(x) = NRx - x^p$$

Of course really $\mu = \mu_n$, but we suppress the dependence on n. Then it is easy to check from the properties of N that

$$\mu(x + y) = \mu(x) + \mu(y) + (\theta - p)C_p(x, y)$$

and

$$\mu(xy) = \mu(x)y^p + \mu(y)x^p + \mu(x)\mu(y)$$

Thus if x, y satisfy $\mu(x), \mu(y) \in (\theta - p)$, so do their sum and product. One sees immediately that 0 and 1 are quasi-cohomological as well. Given that quasi-cohomological elements are closed under products, it remains only to check that -1 is quasicohomological. We have

$$\mu(-1) = NR(-1) - (-1)^p = N(-1) - (-1)^p$$

When p is odd, $N(-1) = -1$ and -1 is actually cohomological. When $p = 2$, one has $N(-1) = -1 + \theta$, so that $\mu(-1) = -1 + \theta - 1 = \theta - 2$, so that -1 is indeed quasi-cohomological. □

Lemma 3.5.2. *Elements that arise as norms or transfers are quasi-cohomological.*

Proof. For a norm Nx this is immediate, since

$$NR(Nx) = N(x^p) = (Nx)^p$$

That is, norms are actually always cohomological, not just quasi-cohomological. For a transfer Ty, we have

$$NR(Ty) = N(py) = NpNy$$

while on the other hand

$$(Ty)^p = p^{p-1}T(y^p) = p^{p-1}T(1)Ny$$

The identity $Np = p + T(p^{p-1} - 1)$ holds at every level, so we see that modulo $p - T(1)$ one has $Np = p^{p-1}T(1)$ This shows that transfers are also quasi-cohomological. □

Warning In spite of the previous two lemmas, in general there is no sub-functor of quasi-cohomological elements. This is because the restriction of a quasi-cohomological element is not necessarily quasicohomological. For an example, consider the free C_{p^2}-Tambara functor generated by a polynomial generator x in level 2. Then the norm NRx is quasi-cohomological but its restriction, $RNRx = (Rx)^p$, fails to be quasi-cohomological.

Corollary 3.5.1. *Any Witt Tambara functor is quasi-cohomological.*

Proof. Every element in a Witt-Tambara functor is a sum of norms and transfers, and therefore quasi-cohomological. □

Remark 3.5.1. There is a close relationship between quasi-cohomological Tambara functors and δ-structures. If A is quasi-cohomological, then the map $\mu_n : A_n \longrightarrow A_n$ lands in the principal ideal $(\theta_n - p)$. Dividing by $\theta_n - p$ yields a map

$$\delta_n : A_n/\mathrm{Ann}(\theta_n - p)$$

One can check from the properties of μ_n that δ_n is a p-derivation subordinate to the projection $A_n \longrightarrow A_n/\mathrm{Ann}(\theta_n - p)$. In future work we hope to use this relationship to more closely connect the theory of prisms to epic, quasi-cohomological Tambara functors and characterize prismatic Tambara functors completely.

3.6 Perfect Prisms and Witt-Tambara Functors

In this section we investigate the Tambara functors associated to perfect prisms, and use it to recover the uniqueness of maps out of perfect prisms. Much of the results and descriptions in this section can be found in a different language in section 3 of [BMS19], but we develop the theory from the Tambara perspective here.

There are two main ingredients in characterizing (many) perfect prismatic Tambara functors - first, the description of the total geometric fixed points described in the previous section, and secondly, the fact that in two major cases the transfers are injective.

Proposition Suppose $(A, (d))$ is a prism with the property that for every $n \geq 1$, $\varphi^n(d)$ is a non-zerodivisor. Then the transfers in $A_\bullet$ are injective.

Proof. Consider the transfer in level $n-1$, $T : A/d_{n-1} \longrightarrow A/d_n$. Then one has

$$Tx = u_n^{-1}\varphi^n(d)\tilde{x}$$

for any choice of lift $\tilde{x}$. As u_n is a unit, one has $Tx = 0$ if and only if $\varphi^n(d)\tilde{x} = 0$ in A/d_n. This gives

$$\varphi^n(d)\tilde{x} = ad_n = a\varphi^n(d)d_{n-1}$$

for some $a \in A$. Since $\varphi^n(d)$ is a non-zero divisor, one may conclude that $\tilde{x} = ad_{n-1}$, whence in fact $x = 0 \in A/d_{n-1}$, so the transfers are injective in this case. □

There are three major cases in which the $\varphi^n(d)$ are guaranteed to be non-zerodivisors: 1. If A is crystalline, then all $\varphi^n(d) = d = p$ and p is a nonzerodivisor by assumption 2. If A is perfect, then d being a nonzero-divisor carries over to every $\varphi^n(d)$ being a non zero divisor and 3. if A is transversal. Only the last case requires some work, which we include in the lemma below.

Lemma 3.6.1. *Suppose $(A, (d))$ is a transversal prism. Then every frobenius iterate $\varphi^n(d)$ is also a non-zero divisor.*

Proof. Suppose $(A, (d))$ is transversal and that $x\varphi(d) = 0$. Then one has

$$\varphi(d)x = (d^p + pu_1)x = 0$$

so that $x \in A/d[p]$. As this is zero by transversality, it follows that $x = dy$ for some y. Then we have

$$\varphi(d)dy = 0$$

But d is a nonzero-divisor, so again $\varphi(d)y = 0$. From this one inductively infers that $x \in (d)^\infty$, and the latter ideal is zero because A is derived (p, d)-adically complete. Therefore $\varphi(d)$ is a non-zero-divisor, and in fact $(A, \varphi(d))$ forms a transversal prism. It is transversal because $(\varphi(d), p)$ being a regular sequence follows from (d, p) being regular. Arguing inductively shows that every $\varphi^n(d)$ is a non-zero divisor. □

Theorem 3.6.1. *Suppose $(A, (d))$ is a perfect prism. Then the natural comparison map $W_\bullet(A/d) \longrightarrow A_\bullet$ is an isomorphism.*

Proof. Note that by adjunction the identity $A/d \longrightarrow A/d$ prolongs to a natural map of Tambara functors $\eta : W_\bullet(A/d) \longrightarrow A_\bullet$. Now both of these Tambara functors have injective transfer maps, so to show that η is an isomorphism it suffices to show that the map on total geometric fixed points $\Phi^\bullet(\eta)$ is an isomorphism. Note that the total geometric fixed points of the Witt Tambara functor $W_\bullet(A/d)$ are given (essentially) by the diagram of identity maps

$$A/d \longrightarrow A/d \longrightarrow A/d \longrightarrow \cdots$$

Meanwhile the diagram of total geometric fixed points of $A_\bullet$ is the diagram with transition maps isomorphisms induced by ϕ:

$$A/d \longrightarrow A/\varphi(d) \longrightarrow A/\varphi^2(d) \longrightarrow \cdots$$

As the map in level zero, $\Phi^0(\eta) : A/d \longrightarrow A/d$, is the identity and so an isomorphism, and all transition maps in each system are isomorphisms, the total map $\Phi^\bullet(\eta)$ is indeed an isomorphism. As mentioned earlier, together with the fact that both Tambara functors have injective transfer maps, this shows that the comparison $W_\bullet(A/d) \longrightarrow A_\bullet$ is indeed an isomorphism. (See theorem 2.2.1 for this Tambara-theoretic result). □

In fact, as observed in [Sul23], the converse is also true. If the comparison map $W_\bullet(A/d) \longrightarrow A_\bullet$ is an isomorphism, then upon applying geometric fixed points one sees that $\varphi : A/d \longrightarrow A/\varphi(d)$ is an isomorphism. Because d (and $\varphi(d)$) are non-zero divisors, and because A is (p,d)-adically complete, this is enough to guarantee that $\varphi : A \longrightarrow A$ itself is an isomorphism. See Theorem 3.19 of loc.cit. for the complete argument.

Corollary 3.6.1. *Let $(A,(d))$ be a perfect prism and $(B,(d))$ another prism with a map of prisms $A \longrightarrow B$. Then this map is uniquely determined by the map $A/d \longrightarrow B/d$.*

Proof. The reduction map of hom-sets

$$\mathrm{Hom}_{\Delta}(A,B) \longrightarrow \mathrm{Hom}(A/d, B/d)$$

factors as

$$\mathrm{Hom}_{\Delta}(A,B) \longrightarrow \mathrm{Hom}_{\mathrm{Tamb}}(A_\bullet, B_\bullet) \longrightarrow \mathrm{Hom}(A/d, B/d)$$

When A is perfect, we have $A_\bullet \simeq W_\bullet(A/d)$ and so the last map is an isomorphism by adjunction. The first map is an injection because of each prism being complete with respect to the sequence of ideals (d_n). □

Remark 3.6.1. In [BS22] it is shown that for a perfect prism (A,I) and any prism (B,J) the reduction map

$$\mathrm{Hom}_{\Delta}((A,I),(B,J)) \longrightarrow \mathrm{Hom}(A/I, B/J)$$

is an isomorphism. In other words, a morphism out of a perfect prism is determined by its residual map, and any residual map lifts to a morphism of prisms. Using the result of this section, that $A_\bullet \simeq W_\bullet(A/I)$ when A is perfect, one can directly lift any map $A/I \longrightarrow B/J$ to a map of rings $A \longrightarrow B$ taking I into J. However, to show that it is a map of prisms, one needs to also check that it respects the δ-structures. We cannot offer a different proof of this than the one given in loc. cit.

3.7 Perfect Prisms / Perfectoid Rings May Be Detected at Cp level

The notion of perfectoid ring now has several equivalent definitions. Here we use the description in [BMS19] to show there is yet another equivalent, Tambara-theoretic characterization.

Definition 3.7.1. *A C_p-Green Functor $A_\bullet$ is said to be of perfectoid type if*

(1) The restriction $A_1 \longrightarrow A_0$ is surjective

(2) The element $\theta - p$ is in $(KerR)^{[p]}$

(3) The ideal KerR is principal

First a proposition.

Proposition 3.7.1. *The element $\theta - p \in W_1(R)$ lies in the ideal generated by p-th powers $(KerF)^{[p]}$ if and only if the ideal generated by the second coordinates of elements of KerF gives the unit ideal in $R/R[p]$.*

Proof. The element $p - \theta = (p, -p^{p-1})$ lies in $(\mathrm{Ker}F)^{[p]}$ if and only if there exists elements $(x_i, y_i) \in \mathrm{Ker}F$ and $(a_i, b_i) \in W_1(R)$ such that

$$(p, -p^{p-1}) = \sum (a_i, b_i)(x_i, y_i)^p$$

This expression can be simplified several ways. Using that $TR \cdot \mathrm{Ker}F = 0$, one has for $(x, y) \in \mathrm{Ker}F$ that

$$(x, y)^p = (x, y)(x, y)^{p-1} = (x, y)(x^{p-1}, 0)$$

$$= (x^p, x^{p(p-1)}y) = (-py, (-p)^{p-1}y^p) = (-p, (-p)^{p-1})(y, 0)$$

Thus the sum above becomes

$$(-p, (-p)^{p-1}) = \sum (a_i, 0)(y_i, 0)(-p, (-p)^{p-1})$$

This equation holds if and only if

$$p = p(\sum a_i y_i)$$

and

$$p^{p-1} = p^{p-1}(\sum a_i y_i)^p$$

The first implies the second, so this can happen if and only if one has

$$p(\sum a_i y_i - 1) = 0$$

or in other words, $\sum a_i y_i \in 1 + R[p]$. As the y_i consist of elements that appear as the second coordinates of elements of Ker$F$ and the a_i may be freely chosen, this gives the proposition. □

Corollary 3.7.1. *If Ker$F$ is principal and $R[p] \subseteq Jac(R)$, then $\theta - p \in (KerF)^{[p]}$ if and only if there exists an element $\pi \in R$ with π^p associate to p.*

Proof. Suppose Ker$F$ is generated by an element (a, b). From the proposition above, we have that $\theta - p \in (\text{Ker}F)^{[p]}$ if and only if b generates the unit ideal in $R/R[p]$. Since we have assumed $R[p] \subseteq \text{Jac}(R)$, this implies b is a unit in R itself. But then as $(a, b) \in \text{Ker}F$ one has

$$F(a, b) = 0 = a^p + pb$$

since $b \in R^\times$, $\pi = a$ gives an element with p-th power associate to p.

□

Theorem 3.7.1. *Suppose R is p-adically complete. Then the C_p-Tambara functor $W_\bullet(R)$ is of perfectoid type if and only if R is a perfectoid ring.*

Proof. Let R be a perfectoid ring. Then there exists a non-zero divisor $\pi \in R$ with $\pi^p = pu$ for some unit $u \in R^\times$. Therefore the element $(\pi, -u)$ is in the kernel of the restriction / Frobenius $F : W_1(R) \longrightarrow R$. We claim that in fact Ker$F$ is principal and generated by $(\pi, -u)$. Let $(a, b) \in \text{Ker}F$. Then $a^p + pb = 0$. Now since R is perfectoid, the p-th power map $R/\pi \longrightarrow R/p$ is an isomorphism. It follows that $a \in \pi R$. Let $a = \pi c$. Then we have $0 = a^p + pb = \pi^p(c^p + u^{-1}b)$. As π is a non-zero divisor, we have $b = -uc^p$. Thus we see that $(a, b) = (\pi c, -uc^p) = (c, 0)(\pi, -u)$, so $\mathrm{Ker}F = ((\pi, -u))$. Thus condition (3) is satisfied.

Condition (1) is immediate, since $F : W_1(R) \longrightarrow R$ is surjective if and only if $\varphi : R \longrightarrow R/p$ is surjective, and this is the case for a perfectoid ring.

For condition (2), we note that $\theta - p = (0, 1) - (p, 1 - p^{p-1}) = (p, -p^{p-1})$. Now $(\pi, -u)^p = (\pi, -u)(\pi^{p-1}, 0) = (\pi^p, -u\pi^{p(p-1)}) = (-pu, -u(-pu)^{p-1}) = (-pu, (-u)^p p^{p-1}) = (p, -p^{p-1})(-u, 0)$. So in fact $(\mathrm{Ker}F)^{[p]} = (\theta - p)$.

Now for the forward direction, assume that $W_\bullet(R)$ is of perfect prismatic type. From condition (3), we have $\mathrm{Ker}F = ((x, y))$ for some element $(x, y) \in W_1(R)$. Using condition (2), We will show that there exists a distinguished element $(\pi, u) \in \mathrm{Ker}F$ with $u \in R^\times$, from which it follows that $((\pi, u)) = ((x, y)) = \mathrm{Ker}F$. The principality of $\mathrm{Ker}F$ will force $\phi : R/\pi \longrightarrow R/p$ injective and $R[\pi] = 0$; then condition (1) gives surjectivity. Lemma 3.10 , condition (ii), of [BMS19] says that this is the same as perfectoid.

□

This characterization of perfectoid rings also makes it easy to prove they are reduced.

Corollary 3.7.2. *A perfectoid ring R is necessarily reduced.*

Proof. Suppose $x \in R$ satisfies $x^p = 0$. Then $(x, 0) \in \mathrm{Ker}F = ((\pi, u))$ where (π, u) is a generator of this principal ideal, and necessarily $u \in R^\times$. It follows that there exists $a \in R$ with

$$(x, 0) = (a\pi, a^p u)$$

The first coordinate shows $x \in (\pi)$, while the second coordinate forces $a^p = 0$. Repeating the argument inductively shows $x \in (\pi)^\infty = (p)^\infty = 0$ with the last equality because R is p-adically separated. □

Corollary 3.7.3. *A p-adically complete ring S is semiperfectoid (that is, can be written as a quotient of a perfectoid ring) if and only if the C_p Witt-Tambara functor $W_\bullet(S)$ satisfies conditions (1) and (2) of the definition of prismatic type.*

Proof. The forward direction is direct. Suppose $S = R/J$ for R a perfectoid ring. Then by theorem 3.7.1 above, $F : W_1(R) \longrightarrow R$ is surjective and $\theta - p \in (\mathrm{Ker}F)^{[p]}$. Thus $F_{R/J} : W_1(R/J) \longrightarrow R/J$ is surjective and $\theta - p \in (\mathrm{Ker}F_{R/J})^{[p]}$ as well.

For the converse, assume S satisfies conditions (1) and (2). Consider the Fontaine ring $\mathbf{A}(S) := \lim_F W_r(S)$. Then $\mathbf{A}(S)$ is a perfect δ-ring with a surjection $\mathbf{A}(S) \longrightarrow S$ (as a consequence of (1) and theorem 3.2 of [DK14]). Choose an element $\xi \in \mathbf{A}(S)$ such that the component of ξ in $W_1(S)$ is given by (π, u). Then ξ is distinguished and $(\mathbf{A}(S), \xi)$ forms a perfect prism with a surjection $\mathbf{A}(S)/\xi \longrightarrow S$. The quotient $\mathbf{A}(S)/\xi$ is then known to be perfectoid, so this shows that S is semiperfectoid. □

Remark For a general ring R one has that $(\mathrm{Ker}F)^{[p]} \subseteq (\theta - p)$. Indeed, let $(x, y) \in \mathrm{Ker}F$. Then, using that $\mathrm{Ker}F \cdot VR = 0$, one has

$$(x, y)^p = (x, y)(x^{p-1}, 0) = (x^p, x^{p(p-1)}y)$$

But as $(x, y) \in \mathrm{Ker}F$, one has that $x^p + py = 0$, or $x^p = -py$. This gives

$$(x, y)^p = (-py, (-py)^{p-1}y) = (-py, (-p)^{p-1}y^p) = (y, 0)(-p, (-p)^{p-1})$$

Now if p is odd, one has $(-p, (-p)^{p-1}) = (-p, p^{p-1}) = p - \theta$. On the other hand, if $p = 2$, one still has $(-p, (-p)^{p-1}) = (-2, -2)$ while $2 - \theta = (2, -2) = (-1, 0)(-2, -2)$. So in every case one sees that $(\mathrm{Ker}F)^{[p]} \subseteq (\theta - p)$.

Thus one way of describing semiperfectoid rings S is as p-complete rings that are mod-p semiperfect (condition (1) of Definition above) and have $(\mathrm{Ker}F)^{[p]}$ as large as possible (this corresponds to condition (2) together with the above remark).

On the other hand, we can also use theorem 3.7.1 to give a condition for when quotients of a perfectoid ring remain perfectoid.

Corollary 3.7.4. *Let R be a perfectoid ring and $J \subseteq R$ a p-adically closed ideal. Then the quotient R/J is perfectoid if and only if the ideal J is Witt-closed, meaning $J = J^{[p]} + pJ$.*

Proof. The quotient R/J is perfectoid if and only if the C_p-Tambara functor $W_\bullet(R/J)$ is prismatic. Conditions (1) and (2) are guaranteed; one only needs to check the preservation of condition (3), the principality of $\mathrm{Ker}F$. From the pair of long exact sequences

$$0 \longrightarrow W_1(J) \longrightarrow W_1(R) \longrightarrow W_1(R/J) \longrightarrow 0$$

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

the snake lemma gives an exact sequence

$$0 \longrightarrow \mathrm{Ker}(F : W_1(J) \to J) \longrightarrow \mathrm{Ker}F_R \longrightarrow \mathrm{Ker}F_{R/J} \longrightarrow \mathrm{Cok}(F : W_1(J) \to J) \longrightarrow 0$$

Now from the proof of Theorem XXX above, one knows that $\mathrm{Ker}F_{R/J}$ is principal if and only if it is generated by an element of type (π, u). Now the image of a generating $(\pi, u) \in \mathrm{Ker}F$ generates $\mathrm{Ker}F_{R/J}$ if and only if the term $\mathrm{Cok}(F_J)$ vanishes. But the latter is by inspection given by $J/(J^{[p]} + pJ)$, and so vanishes if and only if J is Witt-closed in the sense that $J = J^{[p]} + pJ$. □

Remark 3.7.1. If $(A, (d))$ is an orientable prism, then the associated C_p-Tambara functor satisfies conditions (1), (2), and (3) of definition 3.7.1. If (A, I) is a not necessarily orientable prism, Sulyma has shown in [Sul23] that one can still associate a C_{p^∞}-Tambara functor. The C_p-Tambara functor underlying this will still satisfy conditions (1) and (2), i.e. be of semiperfectoid type.

It is natural to ask if one can give conditions for a C_p-Tambara functor to arise from a prism (orientable or not). Unfortunately conditions (1) and (2) are not sufficient, since the

C_p-Tambara Witt functor of a semiperfectoid ring satisfies these conditions, but semiperfectoid rings are not in general lenses (that is, quotients A/I for (A, I) a prism). The author does not know if conditions (1), (2), and (3) together are enough to guarantee a C_p-Tambara functor arises from an orientable prism. For more discussion of this open question, see Question XXX of the conclusion.

3.7.1 Examples Involving the Prismatic Conditions for Witt Tambara Functors

In this short section we give some examples of rings satisfying different pieces of the requirements for perfectoid rings. Recall that a p-complete ring R is perfectoid if and only if its C_p Witt-Tambara functor satisfies the following three properties (this is Theorem 3.7.1 above):

(1) $F : W_1(R) \longrightarrow R$ is surjective (2) The element $\theta - p$ lies in the ideal generated by p-th powers $(\mathrm{Ker} F)^{[p]}$ (3) The ideal $\mathrm{Ker} F$ is principal

First we consider the case that R is an $\mathbb{F}_p$-algebra. In this case, condition (2) is vacuously satisfied, as $\theta = p$. Condition (1) is visibly seen to correspond exactly to R being semiperfect, while condition (3) is actually equivalent to R being perfect, and thus implies condition (1).

Proposition 3.7.2. *An $\mathbb{F}_p$-algebra R satisfies condition (3) above if and only if it is perfect.*

Proof. Let R a $\mathbb{F}_p$-algebra and suppose the kernel of the restriction/ frobenius $F : W_1(R) \longrightarrow R$ is principal. Let $(x, y) \in \mathrm{Ker} F$ be a generator. As $(0, 1) \in \mathrm{Ker} F$, we must have

$$(0, 1) = (x, y)(a, b) = (ax, a^p y)$$

for some $(a, b) \in W_1(R)$. Thus $a^p y = 1$, so both a and y are units, and so the equation $ax = 0$ forces $x = 0$. This already suffices to see that R must be reduced, as if $z \in R$ has $z^p = 0$, then $(z, 0) \in KerF = ((0, y))$, which forces $z = 0$. Now on the other hand the ideal generated by $(0, y)$ consists of the set $\{(0, a^p y)\}_{a \in R}$. But of course since R is a $\mathbb{F}_p$-algebra, $\{(0, c)\}_{c \in R} \subseteq \mathrm{Ker} F$. It follows that every element of R can be written as a p-th power, and thus R is also semiperfect. But of course reduced and semiperfect are equivalent to perfect. $\square$

In characteristic zero, any combination of the conditions can be satisfied; we provide a list of basic examples.

None: $\mathbb{Z}_p[\varepsilon]/\varepsilon^2$ Condition (1) fails because $\mathbb{F}_p[\varepsilon]/\varepsilon^2$ is visibly not semiperfect (the element ε is not a p-th power). Condition (2) fails as it is equivalent to the existence of an element π with π^p associate to p. Finally condition (3) fails because the element $(\varepsilon, 0)$ lies in $\mathrm{Ker}F$, as does $(p, -p^{p-1})$. These can have no common factor since even ε and p have no common factor.

Just (1), but not (2) or (3): $\mathbb{Z}_p[x^{1/p^\infty}]/x$ Condition (1) is satisfied since A/p is visibly semiperfect. Condition (2) fails becuase there is no π with π^p associate to p, while (3) fails for similar reasons to the previous example, because of the element $(x^{1/p}, 0) \in \mathrm{Ker}F$.

Just (2), but not (1) or (3): $\mathbb{Z}_p[\zeta_{p^2}][\varepsilon]/\varepsilon^2$ Condition (1) fails for several reasons, but in particular the element ε is not a p-th power modulo p. Condition (2) succeeds because the ramification index of $\mathbb{Z}_p[\zeta_{p^2}]$ is $p(p-1)$, so there is an element of p-adic valuation $\frac{1}{p}$. Condition (3) fails for reasons similar to the previous examples; consider the element $(\varepsilon, 0) \in \mathrm{Ker}F$.

Just (3), but not (1) or (2): $\mathbb{Z}_p[x]$ Condition (1) fails since $\mathbb{F}_p[x]$ is not semiperfect. Condition (2) fails as there is no element π with π^p associate to p. Condition (3) holds as the kernel is generated by $(p, -p^{p-1})$.

(1) and (2), but not (3): $\mathbb{Z}_p[\zeta_{p^\infty}][x^{1/p^\infty}]/x$

(2) and (3), but not (1): $\mathbb{Z}_p[\zeta_{p^2}]$

(1) and (3), but not (2:) $\mathbb{Z}_p$

Finally we also give an example of a ring R satisfying (2) but not having an element π with p-th power associate to p. Via the recipe of Sulyma (depending on a result of Bhatt-Lurie), any non-orientable prism would also give a C_p-Tambara functor satisfying (1) and

(2) but failing (3) for reasons more interesting than failing to be reduced.

Take $R = \mathbb{Z}_p[x, y, (px)^{1/p}, (py)^{1/p}]/(p(1-x-y))$. Then clearly the elements $((px)^{1/p}, -x)$ and $((py)^{1/p}, -y)$ lie in Ker$F$. Thus the element $-(x+y) = 1+(1-x-y) \in 1+R[p]$ lies in the ideal generated by the second coordinates of elements of Ker$F$. From Proposition 3.7.1 above, this means $\theta - p \in (\mathrm{Ker}F)^{[p]}$. However, Ker$F$ is not principal (the elements $(px)^{1/p}$ and $(py)^{1/p}$ have no common factor, and so there is no element $\pi \in R$ with π^p associate to p.

A similar example, this time p-torsion free, is given by $R = \mathbb{Z}_p[x, y, (px)^{1/p}, (py)^{1/p}][\frac{1}{x+y}]$. Again this satisfies condition (2) without having Ker$F$ principal, even though it is reduced.

3.8 Description of the Additive Cohomologization of Prismatic Tambara Functors

In this subsection, we examine two interesting and interconnected properties of a prismatic Tambara functor. We show that the Green cohomologization agrees with the Tambara cohomologization, and that both are the "Witty" cohomological Tambara functor associated to $(A, (d))$. The Witty Tambara functor associated to a pair (A, I) is essentially the free cohomological Tambara functor with $A_\infty = A$ and $I \subseteq \mathrm{Ker}\theta_0$.

In the first place we investigate the Green ideal generated by all the $\theta_n - p$, for $n \geq 1$.

Proposition 3.8.1. *Let $A_\bullet$ denote a prismatic Tambara functor associated to $(A, (d))$. Let $I_\bullet$ denote the Green ideal of $A_\bullet$ generated by the $\theta_n - p$ for every $n \geq 1$. Then in level n, for $n \geq 1$, we have*

$$A_n/I_n = A/(d^{p^n}, pd^{p^{n-1}}, ..., p^n d) = A/\omega_n(d)$$

Proof. We go by induction. In the base case $n = 1$, we have

$$I_1 = (\theta_1 - p) = (u_1^{-1}\varphi(d) - p) = (u_1^{-1}d^p) = (d^p)$$

Thus

$$A_1/I_1 = A/(d\varphi(d), d^p) = A/(d^{p+1} + pu_1 d, d^p) = A/(d^p, pd)$$

as desired. Now set $A_n/I_n = A/J_n$. As $A_\bullet/I_\bullet$ is a cohomological Green functor, we must have $pJ_n \subseteq J_{n+1}$. In fact, since I_{n+1} is generated by $\theta_{n+1} - p$ and transfers from I_n, we have $J_{n+1} = (\theta_{n+1} - p, pJ_n)$. So it only remains to show that $(\theta_{n+1} - p, pJ_n) = (d^{p^{n+1}}, pJ_n)$.

Next observe that

$$\theta_{n+1} - p = u_{n+1}^{-1}\varphi^{n+1}(d) - p = u_{n+1}^{-1} d^p \varphi(d)^{p-1} \cdots \varphi^n(d)^{p-1}$$

So it is enough to show that

$$dd_n^{p-1} \equiv d^{p^{n+1}} \pmod{pd^{p^n}, p^2 d^{p^{n-1}}, \cdots, p^{n+1} d}$$

Now we use the basic fact that for every $k \geq 1$

$$\varphi^k(d) \equiv d^{p^k} \pmod p$$

Thus

$$dd_n^{p-1} = d^p\varphi(d)^{p-1} \cdots \varphi^n(d)^{p-1} = d^p(d^p + px_1)^{p-1}(d^{p^2} + px_2)^{p-1} \cdots (d^{p^n} + px_n)^{p-1}$$

$$= d^p(d^{p(p-1)} + py_1)(d^{p^2(p-1)} + py_2) \cdots (d^{p^n(p-1)} + py_n)$$

for some elements $x_1, ..., x_n$ and $y_1, ..., y_n$ whose exact nature are not important. We see that this product consists of the first term, $d^p(d^p)^{p-1} \cdots (d^{p^n})^{p-1} = d^{p^{n+1}}$ as well as various cross terms involving binomials in p and d. From inspection, we see that the lowest power of d occurring on all cross terms divisible by p^k (but not p^{k+1} is given by

$$d^p(d^p)^{p-1} \cdots (d^{p^{n-k}})^{p-1} = d^{p^{n-k+1}}$$

(Any cross-term involves a choice of k terms divisible by p, and $n - k$ of the powers of d present. The lowest possible power of d occurring is then the product of the lowest $n - k$

powers of d present). So we have that dd_n^{p-1} is equal to $d^{p^{n+1}}$ plus cross terms all divisible by one of $pd^{p^n}, ..., p^{n+1}d$. But this gives exactly what we set out to show:

$$dd_n^{p-1} \equiv d^{p^{n+1}} \pmod{pd^{p^n}, ..., p^{n+1}d}$$

$\square$

3.9 Decontraction of the q-crystalline Tambara Functor

The category of prisms has a natural endofunctor given by applying the Frobenius: $(A, I) \mapsto (A, \varphi(I))$. On the Tambara side of things, this operation corresponds to the 'contraction' of a C_{p^∞}-Tambara functor, whereby one kills the bottom level and then shifts everything down. In this section we will show that the q-crystalline Tambara functor admits a natural decontraction that curiously does not arise from any prism.

The decontraction is given by $A_n = \mathbb{Z}_p[q]/(q^{p^n} - 1)$, with transfers in level n given by $\theta_n = [p^n]_q$ and norms determined by the fact that q is cohomological in every level, that is, $N_n(q) = q^p$. In order to justify that this is well defined, we first note that clearly

$$\theta_n = [p^n]_q = 1 + q^{p^{n-1}} + \cdots + q^{p^{n-1}(p-1)} \equiv p \pmod{q^{p^{n-1}} - 1}$$

So the only difficult part is showing that the norms are well defined. For this, we first define the norm on the level of $\mathbb{Z}_p[q]$, by setting $N_n(q) = q^p$ and then extending via the required norm relations. As $\mathbb{Z}_p[q]$ is a free algebra, this can be done, and ends up giving us a map $N_n : \mathbb{Z}_p[q] \longrightarrow \mathbb{Z}_p[q]/(q^{p^n} - 1)$. Now we just need to show that N_n actually descends to the quotient $\mathbb{Z}_p[q]/(q^{p^{n-1}} - 1)$. Consider applying N_n to an element $a + b(q^{p^{n-1}} - 1) \in Z_p[q]$. We see that

$$N_n(a + b(q^{p^{n-1}} - 1)) = N_n(a) + N_n(b)N_n(q^{p^{n-1}} - 1) + [p^n]_q C_p(a, b(q^{p^{n-1}} - 1))$$

Now $q^{p^{n-1}} - 1$ divides $C_p(a, b(q^{p^{n-1}})$, so the last term is divisible by $[p^n]_q(q^{p^{n-1}} - 1) = q^{p^n} - 1$

as desired. So we are reduced to showing that $N_n(q^{p^{n-1}} - 1)$ is divisible by $q^{p^n} - 1$. We have:

$$N_n(q^{p^{n-1}} - 1) = N_n(q^{p^{n-1}}) + N_n(-1) + [p^n]_q C_p(q^{p^{n-1}}, -1)$$

For simplicity take $p > 2$ odd; otherwise a slightly different argument is needed. When p is odd, $N_n(-1) = -1$, so we get

$$q^{p^n} - 1 + [p^n]_q C_p(q^{p^{n-1},-1)}$$

and finally because p is odd, we have $C_p(1, -1) = 0$, so we see that $(q^{p^{n-1}} - 1)|C_p(q^{p^{n-1}}, -1)$, and thus the last term is again divisible by $[p^n]_q(q^{p^{n-1}} - 1) = (q^{p^n} - 1)$ as desired.

Now the contraction of this functor is evidently the q-crystalline Tambara functor, as moding out by the bottom level leaves $\mathbb{Z}_p[q]/([p]_q \cdots [p^n]_q)$ in each level and the norms match up since q remains cohomological in every level, and the transfers are evidently the same as those in the q-crystalline functor.

What is very curious is that this decontraction $\{A_n\} = \{\mathbb{Z}_p[q]/(q^{p^n} - 1)\}$ does not arise from a prism. To see this, note that $A_\infty = \mathbb{Z}_p[[q-1]]$. Now for any prism structure $(\mathbb{Z}_p[[q-1]], \delta, (f(q)))$ on this ring, the associated Tambara functors will have specific $\mathbb{Z}_p$-ranks in each level. Specifically, $\deg \varphi^n(f(q)) = p^n \deg f$, so $\deg(f \cdots \varphi^n(f)) = \frac{p^{n+1}-1}{p-1} \deg f$. But we have $\operatorname{rank}_{\mathbb{Z}_p} A_n = p^n$, which clearly cannot have the indicated form. So this decontraction is not prismatic, although in many other respects it behaves as if it were a prismatic Tambara functor.

A natural question is whether or not other prismatic Tambara functors admit 'non-prismatic' decontractions. It is easy to see that the Tambara functors arising from crystalline prisms do. If $(A, (p))$ is a crystalline prism, then the associated Tambara functor is $A/p^{\bullet+1}$. This is visibly obtained from the cohomological constant Tambara functor $\underline{A}$ by contraction. We end this section with a definite question in this direction.

Question Do the Tambara functors associated to Breuil-Kisin prisms ever admit non-prismatic decontractions?

3.10 Global Analogues via λ-Rings

This section is of a preliminary nature, and merely opens up a host of interesting questions as to how far the results in this chapter can be taken in a global (that is, simultaneously at every prime or simply multiple primes at once) direction. The key example is the q-crystalline prism, which visibly extends to an 'integral model'.

Consider the "global" version of the q-crystalline prism given by $\mathbb{Z}[q-1] = \mathbb{Z}[q]$. The 'decontraction' discussed in section XX above actually can be extended to a full C_∞-Tambara functor as follows. One sets $\underline{A}(C_\infty/C_n) := \mathbb{Z}[q]/(q^n-1)$. The restriction maps are clearly all surjective, so the transfers are determined by transfer elements. These transfer elements are given by $T_n^{np}(1) = [p]_{q^n}$. Finally the norms make use of the fact that $\mathbb{Z}[q]$ carries compatible (in a sense to be made precise below) δ-structures for every prime p. Writing δ_p for the δ-structure at a prime p, which is determined by $\delta_p q = 0$, one has again the formula

$$\delta_p f(q) = \frac{f(q^p) - f(q)^p}{p}$$

The norms are determined by the usual formula

$$N_n^{pn} f(q) = f(q)^p + (p - T_n^{pn})\delta_p f(q)$$

$$= f(q)^p + (p - [p]_{q^n}\delta_p f(q) = f(q^p) - [p]_{q^n}\frac{f(q^p) - f(q)^p}{p}$$

Most of the properties needed for this to assemble into a C_∞-Tambara functor are fairly straightforward. For example it is clear that $[p]_{q^n} \equiv p \pmod{q^n - 1}$, so that the restriction of a transfer is multiplication by the appropriate number, and thus also the restriction of a norm is evidently the right power. The only major thing to check is that the norm maps and transfer maps from C_n to C_m when $n|m$ are actually well defined. This amounts to checking that the "atomic" transfers and norms (those given by going from C_n to C_{np} for any prime p) have compositions that do not depend on the order. So for example we need to check that for any two primes p, ℓ we have

$$T_{n\ell}^{n\ell p} T_n^{n\ell} = T_{np}^{n\ell p} T_n^{np}$$

This amounts to checking the equality

$$[\ell]_{q^n} [p]_{q^{n\ell}} = [p]_{q^n} [\ell]_{q^{np}}$$

This follows quickly from the definition of these q-analogues:

$$\frac{q^{n\ell}-1}{q^n-1} \frac{q^{pn\ell}-1}{q^{n\ell-1}} = \frac{q^{np\ell}-1}{q^n-1} = \frac{q^{np}-1}{q^n-1} \frac{q^{np\ell}-1}{q^{np}-1}$$

Checking that the norms behave is considerably more delicate, but can be done directly. However, we defer the proof for later in this section where it will be conducted in greater generality, so we separate this demonstration into its own result.

Proposition 3.10.1. *The atomic norm maps defined above commute in the sense that*

$$N_{n\ell}^{n\ell p} N_n^{n\ell} x = N_{np}^{n\ell p} N_n^{np} x$$

Proof. See Theorem XXX below. Apply in the case that the λ-ring is $\mathbb{Z}[q]$ with $\psi^p(q) = q^p$ for all primes p, and the globally illustrious element is $q-1$. • □

As mentioned above, the ring $\mathbb{Z}[q]$ carries the structure of a δ_p-ring for every prime p, and moreover they are compatible in the sense that all of the associated frobenius operations $\{\psi^p\}$ commute. This amounts to what is known as the structure of a λ-ring. Historically the notion of λ-ring emerged from Grothendieck's study of K-theory and was expressed in terms of operations $\lambda^i : R \longrightarrow R$ for $i \geq 0$ mimicking the behavior of exterior powers on K-theoretic classes of vector bundles or projective modules. However, since the work of Wilkerson it has been known that a λ-ring structure is equivalent to having suitably compatible δ_p operations for every prime p ([Wil82]). In the torsion-free case, the compatibility is given exactly by requiring that the Frobenii commute; in general, there is a further division of this condition that must hold for general rings. This point of view is taken in [Bor15] and one can find great exposition and development there.

For a λ-ring R we write the commuting Frobenii as $\{\psi^p\}$ and for composite $n = p_1^{e_1} \cdots p_k^{e_k}$ we write $\psi^n = \psi_{p_1}^{e_1} \circ \cdots \circ \psi_{p_k}^{e_k}$. In the language of λ-rings, the $\{\psi^n\}$ are the Adams operations.

We can in fact mimic the construction of a C_∞ Tambara functor from $\mathbb{Z}[q]$ in the case of a general λ-ring R equipped with a special kind of element $\xi \in R$. To see what we want from this element ξ, we must reinspect the nature of the transfer elements $[p]_{q^n}$. Note that the Adams operation ψ^n on $\mathbb{Z}[q]$ is determined by $\psi^n(q) = q^n$. So we have

$$[p]_{q^n} = \psi^n([p]_q)$$

and it is enough to think about what is going on with the elements $[p]_q$. But in fact we have

$$[p]_q = \frac{q^p - 1}{q - 1} = \frac{\psi^p(q-1)}{q-1}$$

This suggests then that the fundamental element is $\xi := q - 1$. One then wants to be able to mold a transfer element out of the fraction $\frac{\psi^p(\xi)}{\xi}$. This leads to the following crucial definition.

Definition 3.10.1. *Let A be a λ-ring. A non zero-divisor $\xi \in A$ is said to be illustrious at a prime p if $\xi | \psi^p(\xi)$ and one has a congruence*

$$\frac{\psi^p(\xi)}{\xi} \equiv u_p p \pmod{\xi}$$

for some unit $u_p \in A$. The element ξ is said to be globally illustrious (or simply just illustrious) if it is illustrious at every prime p.

We can connect this notion to the p-local notion of distinguished that is used in the definition of prisms.

Remark 3.10.1. If $\xi \in \mathrm{Jac}(A)$ is illustrious at p, then the element $[p]_\xi := \frac{\psi^p(\xi)}{\xi}$ is δ_p-distinguished in the p-adic completion $\hat{A}_p$. Indeed, since ξ is illustrious at p we have

$$\frac{\psi^p(\xi)}{\xi} = \xi^{p-1} + pu_p$$

for a unit $u_p = \frac{\delta_p \xi}{\xi}$. Applying δ_p and using the usual addition formula gives

$$\delta_p[p]_\xi = \delta_p(\xi^{p-1} + pu_p) = \delta_p(\xi^{p-1}) + \delta_p(pu_p) - C_p(\xi^{p-1}, pu_p)$$

As ξ is illustrious, one has $\xi|\psi^p(\xi)$ and in fact $\xi|\delta_p\xi$. This forces $\xi|\delta_p(\xi^{p-1})$ as well. The last term is also evidently divisible by ξ. Finally the middle term $\delta_p(pu_p)$ is a unit because u_p is a unit. Since we have assumed $\xi \in \mathrm{Jac}(A)$, it follows that $\delta_p[p]_\xi$ is itself a unit and so $[p]_\xi$ is distinguished.

Warning In general, we regard the pair $(\hat{A}_p, [p]_\xi)$ only as a distinguished δ-pair; there is no guarantee that it will be a prism without further assumptions about completeness and $[p]_\xi$ being a non-zerodivisor. However, the construction of a C_{p^∞}-Tambara functor given earlier in this chapter works for most distinguished δ-pairs, not just prisms, so we may continue to think about an associated Tambara functor.

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Example 3.10.1. The main example of an illustrious element comes from the q-crystalline theory. Let A be a λ-ring and $q \in A$ an element of rank one, meaning that $\delta_p q = 0$ for all primes p. In the torsion free case, this is equivalent to the condition that $\psi^n(q) = q^n$ for every Adams operation ψ^n. Then $\xi := q - 1$ is a globally illustrious element. Indeed, one has $\psi^p(\xi) = \psi^p(q-1) = q^p - 1$, so it is clear that $\xi|\psi^p(\xi)$. One also has

$$\frac{\psi^p(\xi)}{\xi} = \frac{q^p - 1}{q - 1} = 1 + q + \cdots + q^{p-1} \equiv p \pmod{\xi = q - 1}$$

A fundamental open question is finding more examples of globally illustrious elements, especially ones not arising from rank one elements. We know of only one such example, which is given in subsection 3.10.1 below.

Suppose now that (A, ξ) is a λ-ring equipped with a globally illustrious element ξ. We assume that (A, ξ) is transversal in the sense that A/ξ and every $A/\psi^\ell(\xi)$ is torsion-free (as an abelian group). We do not know if A/ξ being torsion free implies that $A/\psi^\ell(\xi)$ is torsion free for every ℓ. The construction of a C_∞-Tambara functor associated to (A, ξ) will be done under the assumption that (A, ξ) is transversal; however, this assumption can be removed if the following conjecture is true.

Conjecture 3.10.1. *Consider the free λ-ring A on a globally illustrious element ξ. As a λ-ring, this has the presentation (in the category of λ-rings)*

$$A = \mathbb{Z}\{\xi, \left(\frac{\delta_p \xi}{\xi}\right) \pm \ |p \text{ a prime}\}$$

Then (A, ξ) is transversal, meaning A/ξ is torsion-free as an abelian group and so is $A/\psi^\ell(\xi)$ for every prime ℓ.

If this conjecture is true, then the following construction for transversal lambda-rings with globally illustrious elements can be applied to all lambda-rings with globally illustrious elements simply by specializing the universal case; this is the same approach taken earlier in this chapter for the construction of C_{p^∞}-Tambara functors associated to orientable prisms.

Before heading to the main construction of this section, we include an important lemma about the behaviour of illustrious elements under the Adams operations.

Lemma 3.10.1. *Let (A, ξ) be a λ-ring with globally illustrious element ξ. Then if $d|n$, one has $\psi^d(\xi)|\psi^n(\xi)$, or equivalently the containment of ideals $(\psi^n(\xi)) \subseteq (\psi^d(\xi))$.*

Proof. From the definition of illustrious at p, one has that $\xi|\psi^p(\xi)$. Applying the adams operation ψ^k to this for any k leads to the divisibility $\psi^k(\xi)|\psi^{pk}(\xi)$. If $d|n$, then we may write $n = dp_1 \cdots p_i$ for some sequence of (not necessarily unique) primes p_i. One has the divisibilities $\psi^d(\xi)|\psi^{dp_1}(\xi)$, $\psi^{dp_1}(\xi)|\psi^{dp_1p_2}(\xi)$ ..., $\psi^{dp_1\cdots p_{i-1}}(\xi)|\psi^n(\xi)$. As divisibility is transitive, this gives the desired result that $\psi^d(\xi)|\psi^n(\xi)$ whenever $d|n$. ☐

The main theorem of this section is the following construction of a C_∞-Tambara functor.

Theorem 3.10.1. *Let (A,ξ) be a λ-ring with a globally illustrious element ξ. Assume that ξ is a non-zero divisor and $\xi \in Jac(A)$, and that (A,ξ) is transversal. Then there is a Weyl-invariant C_∞ Tambara functor $\underline{A} = A_\bullet$ with the following structure. The rings are given in level n (i.e. level C_∞/C_n) by*

$$\underline{A}(C_\infty/C_n) = A_n := A/\psi^n(\xi)$$

The atomic transfer maps from n to np are given by

$$T_n^{np}x = \theta_{n\to np}\tilde{x} := \psi^n\left(u_p^{-1}\frac{\psi^p(\xi)}{\xi}\right)\tilde{x}$$

where $u_p := \frac{\delta_p\xi}{\xi}$, which exists and is a unit because ξ is illustrious at p. Finally the atomic norms from n to np are given by

$$N_n^{np}(x) = \tilde{x}^p + (p-\theta_{n\to np})\delta_p\tilde{x} = \psi^p(\tilde{x}) - \theta_{n\to np}\delta_p\tilde{x}$$

We break the proof of this theorem up into the main pieces. First we check that the transfers are well-defined, additive, and satisfy the necessary relations with restrictions. Next we check that the norms are well-defined, multiplicative, and satisfy the necessary relations with the restrictions and addition. Both of these steps are straightforward and almost identical to the proofs earlier in this chapter for the single prime, prism case.

The interesting and more involved steps involve the interaction between atomic norms and transfers from different primes. We must show that the atomic transfers for different primes commute, so that there is a well defined norm from level d to level n whenever $d|n$. This must also be shown for the norms. Finally we must check that the norms and transfers interact in the right way; if both the atomic norm and transfer are associated to a single prime p, this is again straightforward and follows the argument in the single prime, prism case. When the atomic norm and transfer in question come from distinct primes, this is a wholly new argument.

1. Transfers well defined, additive, interact with restrictions correctly.

It is clear from the definition that the transfers are additive, we need only check that they are well-defined and interact with restrictions correctly. We have defined $T_n^{np} : A/\psi^n(\xi) \longrightarrow A/\psi^{np}(\xi)$ by $T_n^{np}x = \theta_{n\to np}\tilde{x}$ for any lift of $\tilde{x}$. To show that it is well defined, we simply need to show that in $A/\psi^{np}(\xi)$

$$\theta_{n\to np}\psi^n(\xi) = 0$$

But from the definition we have

$$\theta_{n\to np}\psi^n(\xi) = \psi^n(u_p^{-1}\frac{\psi^p(\xi)}{\xi})\psi^n(\xi) = \psi^n(u_p^{-1})\psi^{np}(\xi)$$

so the transfers are well defined.

To see that $R_{np}^n T_n^{np} x = px$, it suffices to show

$$\theta_{n\to np} \equiv p \pmod{\psi})^n(\xi)$$

Since $\theta_{n\to np}$ is given by $\psi^n(\theta_{1\to p}$, it suffices to show this for $n = 1$, as then the general case follows by applying the Adams operation ψ^n. But of course we have

$$\theta_{1\to p} = u_p^{-1}(\frac{\psi^p(\xi)}{\xi}) = u_p^{-1}(\xi^{p-1} + pu_p) \equiv p \pmod{\xi}$$

Finally one can easily see that Frobenius reciprocity is immediate, as

$$T(Rx \cdot y) = \theta\tilde{Rxy} = \theta x\tilde{y} = xTy$$

2. Norms are well-defined, multiplicative, and interact correctly with addition and restriction

This follows from the fact that the δ_p are p-derivations and the general theory of quasi-cohomological Tambara functors (see the end of section 3.5 for a brief discussion).

3. Atomic transfers commute

Let $p \neq \ell$ be distinct primes. We need to show that transferring from level n to level $np\ell$ is well defined, so in other words we need to show

$$T_{np}^{np\ell} T_n^{np} x = T_{n\ell}^{np\ell} T_n^{n\ell} x$$

For this it is enough to show that we have the following identity in $A/\psi^{np\ell}(\xi)$

$$\theta_{np \to np\ell} \theta_{n \to np} = \theta_{n\ell \to np\ell} \theta_{n \to n\ell}$$

This means we need to show the congruence

$$\psi^{np}\left(u_\ell^{-1} \frac{\psi^\ell(\xi)}{\xi}\right) \psi^n \left(u_p^{-1} \frac{\psi^p(\xi)}{\xi}\right) \equiv \psi^{n\ell}\left(u_p^{-1} \frac{\psi^p(\xi)}{\xi}\right) \psi^n \left(u_\ell^{-1} \frac{\psi^\ell(\xi)}{\xi}\right) \pmod{\psi^{np\ell}(\xi)}$$

Now it is enough to show this congruence in the case that $n = 1$, as then applying the Adams operation ψ^n yields the general result. So we are reduced to showing

$$\psi^p \left(u_\ell^{-1} \frac{\psi^\ell(\xi)}{\xi}\right) \left(u_p^{-1} \frac{\psi^p(\xi)}{\xi}\right) \equiv \psi^\ell \left(u_p^{-1} \frac{\psi^p(\xi)}{\xi}\right) \left(u_\ell^{-1} \frac{\psi^\ell(\xi)}{\xi}\right) \pmod{\psi^{p\ell}(\xi)}$$

This becomes

$$\psi^p(u_\ell^{-1}) u_p^{-1} \frac{\psi^{p\ell}(\xi)}{\xi} \equiv \psi^\ell(u_p^{-1}) u_\ell^{-1} \frac{\psi^{p\ell}(\xi)}{\xi} \pmod{\psi^{p\ell}(\xi)}$$

From here we see that it is enough to show the congruence

$$\psi^p(u_\ell^{-1}) u_p^{-1} \equiv \psi^\ell(u_p^{-1}) u_\ell^{-1} \pmod{\xi}$$

Inverting each side yields the more palatable form

$$\psi^p(u_\ell) u_p \equiv \psi^\ell(u_p) u_\ell \pmod{\xi}$$

This congruence can be seen from the fact that ψ^p and ψ^ℓ commute. One has

$$\psi^p(\psi^\ell(\xi)) = \psi^{p\ell}(\xi) = \psi^\ell(\psi^p(\xi))$$

giving

$$\psi^p(\xi^\ell + \ell u_\ell \xi) = \psi^\ell(\xi^p + p u_p \xi)$$

$$\psi^p(\xi)^\ell + \ell\psi^p(u_\ell)\psi^p(\xi) = \psi^\ell(\xi)^p + p\psi^\ell(u_p)\psi^\ell(\xi)$$

Now dividing by ξ and going modulo ξ gives the congruence

$$\ell\psi^p(u_\ell)pu_p \equiv p\psi^\ell(u_p)\ell u_\ell \pmod{\xi}$$

As we have assumed that (A, ξ) is transversal, the quotient A/ξ is $p\ell$-torsion free and so we may divide this congruence by $p\ell$ and arrive at the desired result.

Remark 3.10.2. Here we have crucially used the transversal assumption to give a direct proof of the result. However, it shouldn't be necessary. The author expects that one can prove the wanted congruence between u_p and u_ℓ directly using the how δ_p and δ_ℓ must interact in a λ-ring, but this is likely far more complicated than the argument in the transversal case.

4. Atomic norms commute

Showing the atomic norms commute is a somewhat arduous task in symbol manipulation: we will need several preparatory lemmas.

Lemma 3.10.2. *For distinct primes $p \neq \ell$ we have the equality in $A_{np\ell} = A/\psi^{np\ell}(\xi)$*

$$\theta_{n\ell\to np\ell}\theta_{np\to np\ell} = \theta_{n\ell\to np\ell}\theta_{n\to n\ell}$$

Proof. We want to show the congruence

$$\psi^{n\ell}(u_p^{-1}\frac{\psi^p(\xi)}{\xi})\psi^{np}(u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}) \equiv \psi^{n\ell}(u_p^{-1}\frac{\psi^p(\xi)}{\xi}\psi^n(u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}) \pmod{\psi^{np\ell}(\xi)}$$

It is clear that it suffices to show the congruence for $n = 1$, as the general case follows from applying the Adams operation ψ^n to this. So we want to show the congruence

$$\psi^{\ell}(u_p^{-1}\frac{\psi^p(\xi)}{\xi})\psi^{p}(u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}) \equiv \psi^{\ell}(u_p^{-1}\frac{\psi^p(\xi)}{\xi}\psi^n(u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}) \pmod{\psi^{p\ell}(\xi)}$$

Canceling the unit factor of $\psi^\ell(u_p^{-1})$ from both sides and then moving things to one side and factoring out $\psi^\ell(\frac{\psi^p(\xi)}{\xi})$ gives

$$\psi^\ell(\frac{\psi^p(\xi)}{\xi})\left(\psi^p(u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}) - u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}\right) \equiv 0 \pmod{\psi^{p\ell}(\xi)}$$

Now it suffices to show that the bracketed term is zero modulo $\psi^\ell(\xi)$ because of the factor of $\psi^\ell(\frac{\psi^p(\xi)}{\xi}) = \frac{\psi^{p\ell}(\xi)}{\psi^\ell(\xi)}$ out front. This reduces us to showing

$$\psi^p(u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}) - u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi} \equiv 0 \pmod{\psi^\ell(\xi)}$$

or more explicitly

$$\psi^p(u_\ell^{-1}\xi^{\ell-1} + \ell) - (u_\ell^{-1}\xi^{\ell-1} + \ell) \equiv 0 \pmod{\psi^\ell(\xi)}$$

Of course ψ^p is a ring homomorphism and ℓ is an integer, so $\psi^p(\ell) = \ell$ and those terms cancel, leaving just

$$\psi^p(u_\ell^{-1}\xi^{\ell-1}) - u_\ell^{-1}\xi^{\ell-1} \equiv 0 \pmod{\psi^\ell(\xi)}$$

Now we will show that this term is zero modulo ξ and modulo $\frac{\psi^\ell(\xi)}{\xi}$. This is enough because in the transversal case we have $(\psi^\ell(\xi)) = (\xi) \cap (\frac{\psi^\ell(\xi)}{\xi}) = (\xi) \cap (\xi^{\ell-1} + \ell u_\ell)$. Of course it is always true that $\psi^\ell(\xi) = \xi(\xi^{\ell-1} + \ell u_\ell)$ so that the principal ideal $(\psi^\ell(\xi))$ is the product ideal $(\xi) \cdot (\xi^{\ell-1} + \ell u_\ell)$. Transversality implies that ξ, ℓ form a regular sequence, from which it follows that $\xi, \xi^{\ell-1} + \ell u_\ell$ form a regular sequence and therefore the product of the two ideals is the intersection. For the reader's convenience a proof of this piece of commutative algebra is given just below.

The expression

$$\psi^p(u_\ell^{-1}\xi^{\ell-1}) - u_\ell^{-1}\xi^{\ell-1}$$

is evidently zero modulo ξ, since $\xi | \psi^p(\xi)$. On the other hand, returning to the expression written as

$$\psi^p(u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}) - u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi}$$

we see that showing it is zero modulo $\frac{\psi^\ell(\xi)}{\xi}$ amounts to showing that the term $\psi^p(\frac{\psi^\ell(\xi)}{\xi})$ is divisible by $\frac{\psi^\ell(\xi)}{\xi}$.

This divisibility follows from the divisibility

$$\frac{\psi^\ell(\xi)}{\xi} | \frac{\psi^{p\ell}(\xi)}{\psi^p(\xi)} \frac{\psi^p(\xi)}{\xi}$$

together with the fact that $\frac{\psi^\ell(\xi)}{\xi}$ and $\frac{\psi^p(\xi)}{\xi}$ are coprime. One can see this by going modulo ξ, where the ideal generated by both of them becomes $(\ell, p) = (1)$. Since $\xi \in \mathrm{Jac}(A)$ by assumption, it follows that the ideal generated by both of them is actually the unit ideal in A. This finishes the demonstration.

□

Lemma 3.10.3. *Suppose R is a commutative ring with regular sequence x, y (that is, $R/x[y] = 0$. Then one has an equality of ideals $(xy) = (x) \cap (y)$.*

Proof. The containment $(xy) \subseteq (x) \cap (y)$ is immediate. For the reverse containment, suppose that $a \in (x) \cap (y)$. Then one can write $a = bx = cy$ for some $b, c \in R$. Therefore in R/x one has $cy = bx = 0$. Now $R/x[y] = 0$ forces $c \in (x)$, and therefore $a \in (xy)$. □

Proposition 3.10.2. *The atomic norms for $p \neq \ell$ commute.*

Proof. We want to show that there is an equality of compositions of norm operators from $A/\psi^n(\xi)$ to $A/\psi^{np\ell}$

$$N_{n\ell}^{np\ell} N_n^{n\ell} = N_{np}^{n\ell p} N_n^{np}$$

To reduce clutter, in what follows we do not distinguish between x and a chosen lift $\tilde{x}$ of x to any other level. Consider the expression

$$N_{n\ell}^{np\ell} N_n^{n\ell} x = N_{n\ell}^{n\ell p} \left(\psi^\ell(x) - \theta_{n\to n\ell}\delta_\ell x\right)$$

$$= \psi^p \left(\psi^\ell(x) - \theta_{n\to n\ell}\delta_\ell x\right) - \theta_{n\ell\to n\ell p}\delta_p \left(\psi^\ell(x) - \theta_{n\to n\ell}\delta_\ell x\right)$$

$$= \psi^{p\ell}(x) - \psi^p(\theta_{n\to n\ell})\psi^p(\delta_\ell x)$$

$$-\theta_{n\ell\to n\ell p} \left[\delta_p\psi^\ell(x) + \delta_p(\theta_{n\to n\ell}\delta_\ell x) - C_p(\psi^\ell(x), -\theta_{n\to n\ell}\delta_\ell x)\right]$$

Now it is enough to show that this expression is invariant upon switching p and ℓ.

Now evidently the first term $\psi^{p\ell}(x)$ is invariant upon switching p and ℓ, so we may disregard it. Similarly, the collected terms

$$-\psi^p(\theta_{n\to n\ell})\psi^p(\delta_\ell x) - \theta_{n\ell\to n\ell p}\delta_p\psi^\ell(x)$$

are also invariant upon exchanging p and ℓ, since $\psi^p(\theta_{n\to n\ell}) = \theta_{np\to np\ell}$ and ψ^p and δ_ℓ commute. This leaves only the somewhat gruesome expression

$$\theta_{n\ell\to n\ell p}\left[\delta_p(\theta_{n\to n\ell}\delta_\ell x) - C_p(\psi^\ell(x), -\theta_{n\to n\ell}\delta_\ell x)\right]$$

Using the formula for δ_p applied to the product $\theta_{n\to n\ell}\delta_\ell x$ gives

$$\theta_{n\ell\to n\ell p}\left[\theta_{np\to np\ell}\delta_p\delta_\ell x - (\delta_\ell x)^p\delta_p(-\theta_{n\to n\ell}) - C_p(\psi^\ell(x), -\theta_{n\to n\ell}\delta_\ell x)\right]$$

Now using the fact that $\theta^2_{n\to n\ell} \equiv \ell\theta_{n\to n\ell}$ modulo $\psi^{n\ell}(\xi)$, we may rewrite the term with the polynomial $C_p(a,b)$, giving

$$\theta_{n\ell\to n\ell p}\left[\theta_{np\to np\ell}\delta_p\delta_\ell x - (\delta_\ell x)^p\delta_p(-\theta_{n\to n\ell}) - \theta_{n\to n\ell}\frac{1}{\ell}C_p(\psi^\ell(x), -\ell\delta_\ell x)\right]$$

Now the polynomial C_p satisfies the general identity (actually a cocycle condition)

$$C_p(a+b,c) + C_p(a,b) = C_p(a,b+c) + C_p(b,c)$$

Using this for the case at hand shows

$$C_p(\psi^\ell(x), -\ell\delta_\ell x) = C_p(x^\ell + \ell\delta_\ell x, -\ell\delta_\ell x)$$

$$= C_p(x^\ell, 0) + C_p(\ell\delta_\ell x, -\ell\delta_\ell x) - C_p(x^\ell, \ell\delta_\ell x)$$

One always has $C_p(a,0) = 0$, while for $p > 2$ one has $C_p(a,-a) = 0$. So for $p > 2$ this leaves us with

$$C_p(\psi^\ell(x), -\ell\delta_\ell x) = -C_p(x^\ell, \ell\delta_\ell x)$$

Plugging this back into the original expression yields (for $p > 2$)

$$\theta_{n\ell\to n\ell p}\left[\theta_{np\to np\ell}\delta_p\delta_\ell x - (\delta_\ell x)^p\delta_p(-\theta_{n\to n\ell}) + \theta_{n\to n\ell}\frac{1}{\ell}C_p(x^\ell, \ell\delta_\ell x)\right]$$

Now when $p = 2$ a slightly different argument is required, but in all cases one arrives at the expression

$$\theta_{n\ell\to n\ell p}\left[\theta_{np\to np\ell}\delta_p\delta_\ell x - (\delta_\ell x)^p\delta_p(-\theta_{n\to n\ell}) + \theta_{n\to n\ell}\frac{1}{\ell}C_p(x^\ell, \ell\delta_\ell x)\right]$$

So this is the expression that we must show is invariant in ℓ and p, or in other words we want to show the explicit equality

$$\theta_{n\ell\to n\ell p}\left[\theta_{np\to np\ell}\delta_p\delta_\ell x - (\delta_\ell x)^p\delta_p(-\theta_{n\to n\ell}) + \theta_{n\to n\ell}\frac{1}{\ell}C_p(x^\ell, \ell\delta_\ell x)\right]$$

$$= \theta_{np\to n\ell p}\left[\theta_{n\ell\to np\ell}\delta_\ell\delta_p x - (\delta_p x)^\ell\delta_\ell(-\theta_{n\to np}) + \theta_{n\to np}\frac{1}{p}C_\ell(x^p, p\delta_p x)\right]$$

For this we must use the relation between the compositions $\delta_p\delta_\ell$ and $\delta_\ell\delta_p$ in a λ-ring. This can be found in [Bor15], see equation (1.19.1) there. The equation in our notation reads

$$\delta_p\delta_\ell x = \delta_\ell\delta_p x + \frac{1}{p}C_\ell(x^p, p\delta_p x) - \frac{1}{\ell}C_p(x^\ell, \delta_\ell x) - \frac{\delta_p\ell}{\ell}(\delta_\ell x)^p + \frac{\delta_\ell p}{p}(\delta_p x)^\ell$$

Plugging this in to the above and canceling terms using lemma 3.10.2 (which gives the congruences $\theta_{n\ell\to n\ell p}\theta_{np\to np\ell} = \theta_{n\ell\to n\ell p}\theta_{n\to n\ell} = \theta_{np\to n\ell p}\theta_{n\to np}$ leaves us only with the need to show

$$\theta_{n\ell\to n\ell p}\left[\theta_{np\to np\ell}(-\frac{\delta_p\ell}{\ell}(\delta_\ell x)^p + \frac{\delta_\ell p}{p}(\delta_p x)^\ell) - (\delta_\ell x)^p\delta_p(-\theta_{n\to n\ell})\right]$$
$$= \theta_{np\to np\ell}\left[-(\delta_p x)^\ell\delta_\ell(-\theta_{n\to np})\right]$$

We see now that it suffices to show the equality (in $A/\psi^{n\ell p}(\xi)$)

$$\theta_{n\ell\to n\ell p}\theta_{np\to np\ell}\frac{\delta_p\ell}{\ell}(\delta_\ell x)^p = -\theta_{n\ell\to n\ell p}(\delta_\ell x)^p(\delta_p(-\theta_{n\to n\ell})$$

The proof of this is given below. Granting this result, then applying it to the above cancels one pair of terms, while the other pair of terms is canceled by using the same result but exchanging the role of ℓ and p. This concludes the arduous demonstration.

□

Lemma 3.10.4. *We have the congruence*

$$\theta_{n\to np}\frac{\delta_p \ell}{\ell} \equiv \delta_p(\theta_{n\to n\ell}) \pmod{\psi)^{n\ell}(\xi}$$

Proof. We see that it suffices to show the result for $n = 1$, as the general case then follows from applying Adams operation ψ^n to the congruence for $n = 1$. Thus we want to show

$$\theta_{1\to p}\frac{\delta_p \ell}{\ell} \equiv \delta_p(\theta_{1\to \ell}) \pmod{\psi^\ell(\xi)}$$

Now we use that (A, ξ) is assumed transversal, so that $A/\psi^\ell(\xi)[p] = 0$. So it is enough to show the equation obtained by multiplying both sides by p. This gives

$$\theta_{1\to p}(1 - \ell^{p-1}) \equiv \psi^p(\theta_{1\to\ell}) - \theta^p_{1\to\ell}) \pmod{\psi^\ell(\xi)}$$

Now because $\theta_{1\to\ell}$ is a transfer element, we have $\theta^p_{1\to\ell} = p^{p-1}\theta_{1\to\ell}$. That reduces us to showing

$$\theta_{1\to p} \equiv \psi^p(\theta_{1\to\ell}) \pmod{\psi^\ell(\xi)}$$

But this congruence was obtained at the end of the proof of Lemma 3.10.2 above. □

Remark 3.10.3. Here is where we use the crucial general efinition of transversal, that even $A/\psi^\ell(\xi)$ be torsion-free.

5. Norms of Transfers for $p = \ell$

This again follows from the same argument as in the p-local case for prisms.

6. Norms of Transfers for $p \neq \ell$

From Proposition 2.1.2 in chapter 1, the required relation for norms of transfers at different primes is given by

$$N^{n\ell p}_{n\ell} T^{\ell n}_n x = T^{np\ell}_{pn} N^{pn}_n x + \frac{\ell^p - \ell}{p\ell} T^{np\ell}_n x^p$$

This amounts to the claim that in $A/\psi^{np\ell}(\xi)$ one has

$$N^{n\ell p}_{n\ell}(\theta_{n\to n\ell} x) = \theta_{np\to np\ell} N^{pn}_n x + \frac{\delta_p \ell}{\ell}\theta_{n\to np\ell} x^p$$

which using one of the equivalent definitions of the norm operators gives

$$\psi^p(\theta_{n\to n\ell}x) - \theta_{n\ell\to n\ell p}\delta_p(\theta_{n\to n\ell}x)$$
$$= \theta_{np\to np\ell}\left(\psi^p(x) - \theta_{n\to np}\delta_p x\right) + \frac{\delta_p \ell}{\ell}\theta_{n\to np\ell}x^p$$

Now one has $\psi^p(\theta_{n\to n\ell}) = \theta_{np\to np\ell}$, so the first terms of each side are identical. This leaves us with showing

$$-\theta_{n\ell\to n\ell p}\delta_p(\theta_{n\to n\ell}x) = -\theta_{np\to np\ell}\theta_{n\to np}\delta_p x + \frac{\delta_p \ell}{\ell}\theta_{n\to np\ell}x^p$$

Now we may rewrite the left hand side using the fact that $\delta_p(ab) = \psi^p(a)\delta_p b + b^p\delta_p a$. This gives

$$-\theta_{n\ell\to n\ell p}\left(\theta_{np\to np\ell}\delta_p x + x^p\delta_p\theta_{n\to n\ell}\right)$$

Now once again applying Lemma 3.10.2, which says the products $\theta_{np\to np\ell}\theta_{n\to np}$ and $\theta_{n\ell\to n\ell p}\theta_{np\to np\ell}$ are identical in $A/\psi^{np\ell}(\xi)$, reduces us finally to showing the equation

$$-\theta_{n\ell\to n\ell p}\delta_p(\theta_{n\to n\ell})x^p = \frac{\delta_p \ell}{\ell}\theta_{n\to np\ell}x^p$$

This follows from applying ψ^n to the congruence in the lemma below.

Lemma 3.10.5. *There is a congruence*

$$\theta_{\ell\to\ell p}\delta_p(\theta_{1\to\ell}) \equiv \frac{\ell - \ell^p}{p\ell}\theta_{1\to p\ell} \pmod{\psi^{p\ell}(\xi)}$$

Proof. Recalling the formula $\frac{\ell-\ell^p}{p\ell} = \frac{\delta_p\ell}{\ell}$ and noting that $\theta_{1\to p\ell} = \theta_{1\to\ell}\theta_{\ell\to p\ell}$, we may rewrite the congruence as

$$\theta_{\ell\to p\ell}\left(\theta_{1\to\ell}\frac{\delta_p\ell}{\ell} - \delta_p(\theta_{1\to\ell})\right) \equiv 0 \pmod{\psi^{p\ell}(\xi)}$$

Now the term $\theta_{\ell\to\ell p}$ is associate to $\frac{\psi^{p\ell}(\xi)}{\psi^{\ell}(x)}$, so it is enough to show that the parenthesized term is divisible by $\psi^\ell(\xi)$. We thus want to show

$$\theta_{1\to\ell}\frac{\delta_p\ell}{\ell} - \delta_p(\theta_{1\to\ell}) \equiv 0 \pmod{\psi^{\ell}(\xi)}$$

As in the proof of Lemma 3.10.2 above, it suffices to show that this expression is divisible by ξ and by $\frac{\psi^\ell(\xi)}{\xi}$. Now because ξ is illustrious, the principal ideal (ξ) is actually a λ-ideal and so we can go modulo ξ and safely commute the δ_p operator around it. The expression becomes, modulo ξ,

$$\ell\frac{\delta_p\ell}{\ell} - \delta_p\ell$$

so we indeed have the divisibility by ξ.

Now for the other part, showing it is zero modulo $\frac{\psi^\ell(\xi)}{\xi}$, it is clearly enough to show that

$$\frac{\psi^\ell(\xi)}{\xi}|\delta_p(u_\ell^{-1}\frac{\psi^\ell(\xi)}{\xi})$$

Now in Lemma 3.10.4 above we showed that ψ^p takes the ideal $(\frac{\psi^\ell(\xi)}{\xi})$ into itself. Transversality allows us to conclude that $A/(\frac{\psi^\ell(\xi)}{\xi})$ is p-torsion free, from which it follows that δ_p takes this ideal into itself as well. Thus we have the desired divisibility. This completes the demonstration.

□

3.10.1 Further Examples of Illustrious Elements

Interestingly, in the global case there is no apparent notion of "crystalline", as there is no obvious integer candidate for a globally illustrious element in a λ-ring, unlike in the prism case where the distinguished element p is always present. If one is willing to soften the requirements of globally illustrious to being illustrious only at cofinitely many primes, one does arrive at more examples, including the crystalline-like first example in the remark below.

Remark 3.10.4. S-illustrious elements Let S denote a finite set of primes. It is possible to produce examples of S-illustrious elements, i.e. elements that are illustrious at every prime outside of p. For example, let n be an integer and take S to be the set of primes dividing n.

Then for every $p \notin S$, one has

$$\frac{\psi^p(n)}{n} = \frac{n}{n} = 1 \equiv u_p p \pmod{n}$$

for some unit u_p, since p is a unit modulo n. This allows the production of not quite a C_∞-Tambara functor, but a $C_{\infty,S}$-Tambara functor associated to $(A, (n))$. In this case the Tambara functor in question will be the constant Tambara functor $\underline{A/n}$.

This significance of a theory of S-illustrious elements and their associated $C_{\infty,S}$-Tambara functors remains unclear and unexplored.

We have found one example of a globally illustrious element that does not arise from an element of rank one. The λ-ring in question is the free divided power algebra on one generator. This example may be related to a kind of global crystalline theory, as its associated Tambara functor is cohomological, and its associated prisms are crystalline for every prime p.

Example 3.10.2. Here is an example of a globally illustrious element not of the form $q-1$ for q of rank one. Take $A = \mathbb{Z}[x, \frac{x^2}{2}, \cdots, \frac{x^n}{n!}, \cdots]$, the free divided power algebra on one generator. This is evidently a torsion-free ring, so from Wilkerson's result, a λ-ring structure on A is equivalent to giving commuting Frobenius lifts. For the Adams operators we define

$$\psi^k(x) = kx$$

These are Frobenius lifts, as

$$\psi^p(\frac{x^n}{n!}) = p^n\frac{x^n}{n!} = (\frac{x^n}{n!})^p + p\left(p^{n-1}\frac{x^n}{n!} - \frac{(np)!}{(n!)^p p}\frac{x^{np}}{(np)!}\right)$$

Using the well known bound $v_p(n!) < \frac{n}{p-1}$, one may check that the term $\frac{(np)!}{(n!)^p p}$ is indeed integral (see lemma below for more details).

Finally, it is clear that $x \in A$ is a globally illustrious element, since

$$\frac{\psi^p(x)}{x} = \frac{px}{x} = p$$

This λ-ring with illustrious element x can be thought of as an avatar of a global cristalline prism, as each of the x-analogues $[p]_x = \frac{\psi^p(x)}{x} = p$ recovers the local crystalline setting.

Lemma 3.10.6. *Let p be a prime and n a positive integer. The fraction $\frac{(pn)!}{(n!)^p p} \in \mathbb{Q}$ is integral.*

Proof. Let ℓ denote an arbitrary prime. Legendre's formula gives the ℓ-adic valuation of a factorial as

$$v_\ell(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{\ell^k} \rfloor$$

Thus we see that

$$v_\ell(\frac{(pn)!}{(n!)^p}) = v_\ell((np)!) - pv_\ell(n!) = \sum_{k=1}^{\infty} \lfloor \frac{pn}{\ell^k} \rfloor - p\lfloor \frac{n}{\ell^k} \rfloor$$

Now each term in the sum is ≥ 0 since $\lfloor px \rfloor \geq p\lfloor x \rfloor$ for any x. This shows that $\frac{(pn)!}{(n!)^p}$ is ℓ-integral for every prime ℓ. Now if $p = 1$, one can go further, as then

$$v_p(\frac{(pn)!}{(n!)^p}) = \sum_{k=1}^{\infty} \lfloor \frac{pn}{p^k} \rfloor - p\lfloor \frac{n}{p^k} \rfloor$$

$$= n - (p-1)\sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor > n - (p-1)\frac{n}{p-1} = 0$$

Since the p-adic valuation is greater than zero, it is at least one, and therefore the integer $\frac{(pn)!}{(n!)^p}$ is also divisible by p, as desired.

□

The example mentioned above, after being completed at a prime p, leads to the prism $(\Gamma_{\mathbb{Z}_p}(x), (p))$. This prism represents an interesting functor on the category of prisms:

$$\mathrm{Hom}_\Delta(\Gamma_{\mathbb{Z}_p}(x), A) = \begin{cases} \varnothing \text{ if } A \text{ is not crystalline} \\ A^{\varphi=p} \text{ if } A \text{ is crystalline} \end{cases}$$

The significance of this remains unexplored.

3.10.2 The Habiro Ring and Habiro Completeness

It is not immediately clear what kind of completeness properties one should require for this candidate for global prisms; inspecting this case helps shed some light on what might be required.

Take then $A = \mathbb{Z}[q]$ the λ-ring with q of rank 1. Then the associated C_∞-Tambara functor has in level n the ring $\mathbb{Z}[q]/(q^n - 1) \simeq \mathbb{Z}[C_n]$. Now the inverse limit over the restriction maps of this Tambara functor gives the ring

$$A_\infty := \lim_n Z[q]/(q^n - 1)$$

This is a strange ring at first glance, but it has appeared before in the literature in another guise. The inverse system $\{Z[q]/(q^n - 1)\}$ of rings is indexed by the poset of positive integers under divisibility. We may replace it with an equivalent inverse system indexed by the usual poset of natural numbers by considering

$$\cdots \longrightarrow \mathbb{Z}[q]/(q^n - 1)(q^{n-1} - 1) \cdots (q - 1) \longrightarrow \cdots \longrightarrow \cdots \mathbb{Z}[q]/(q^2 - 1)(q - 1) \longrightarrow \mathbb{Z}[q]/(q - 1)$$

The inverse limit of this system, which is isomorphic to A_∞, consists of infinite series of the form

$$\sum a_n(q)(q^n - 1)(q^{n-1} - 1) \cdots (q^2 - 1)(q - 1)$$

where $a_n(q) \in \mathbb{Z}[q]$ is a polynomial of degree $\leq n$. This is known as the Habiro ring and was introduced in [Hab02]. Interestingly, any element of the Habiro ring may be evaluated at any root of unity, because the infinite tail will vanish beyond the order of the root of unity. In [Man09], the Habiro ring is viewed as a candidate for the ring of analytic functions on the (algebraic closure) of the field with one element. It is quite interesting that this ring appears naturally in this Tambara theoretic framework, and seems to lend some heuristic weight to the idea that λ-rings with illustrious elements might be a reasonable candidate for "global prisms".

The association of a C_∞-Tambara functor to a λ-ring with illustrious element ξ having $A/\psi^n(\xi)$ in level n suggests the following notion of completeness, which we expect to be relevant to using λ-rings as model for "global prisms".

Definition 3.10.2. *Let A be a λ-ring and ξ an illustrious element, so that $\psi^d(\xi)|\psi^n(\xi)$ whenever $d|n$. Then A is said to be Habiro complete at ξ if the natural morphism*

$$A \longrightarrow \lim_n A/\psi^n(\xi)$$

is an isomorphism. Here the inverse limit is over the poset of natural numbers ordered by divisibility.

Setting $(n)_\xi := \psi^n(\xi)$ and $(n)_\xi! := (n)_\xi(n-1)_\xi \cdots (2)_\xi(1)_\xi$, one may contemplate generalized Habiro series of the form

$$\sum a_n (n)_\xi!$$

These of course will be convergent and well-defined in the case that the λ-ring A is Habiro-complete at ξ. We leave further investigation of these to future work.

3.10.3 Global Comparison Map

In this section for a commutative ring R we use $W_\bullet(R)$ to refer to the C_∞ Witt-Tambara functor. As this is the free Tambara functor on R in level 1 (meaning level C_∞/e), for any illustrious λ-pair (A, ξ) there is a natural comparison map

$$W_\bullet(A/\xi) \longrightarrow A/\psi^\bullet(\xi)$$

It is natural to wonder if this is ever an isomorphism, and in particular if it matches up with a notion of perfection as in the prism case. Now a λ-ring A is said to be perfect if every Frobenius lift ψ^p is an isomorphism. One might hope for the comparison map to be an isomorphism in this case, but this is not quite true. Philosophically, this has something to do with this notion of global prism lying 'behind' that of local prisms.

CHAPTER 4

Future Directions

This thesis has charted a surprising relationship between prisms and the theory of Tambara functors. Many open questions and directions for future study remain; we list just a few important ones here.

Andre's Lemma and Perfect Interchange

Andre's Lemma is an important result in the theory of perfectoid rings that says given a perfectoid ring R and a monic polynomial $f(x) \in R[x]$, there is a faithfully flat morphism of perfectoid rings $R \longrightarrow S$ such that S contains a root of $f(x)$. Early proofs involved a delicate approach via perfectoid theory, while since [BMS19] a new approach involving using prismatic cohomology and the notion of perfectoidization is available. One would like to know if there is another proof from the Tambara point of view; in particular the perfect interchange results of section 2.4 should be relevant.

Cartier-Witt Stack The Cartier-Witt stack is studied in Bhatt-Lurie ([BL22]) and used to provide a stacky formulation of absolute prismatic cohomology. The connections between this stack and Green and Tambara geometry remain opaque, but in the basic case of CW_1, one can show that the stack is equivalent to a stack of 'prismatic Green functors'. Extending these results and probing this connection remains open.

Crystalline Cohomology via Tambara Functors In section 2.6, the Tambara ideals of the C_p-Tambara Witt functor were classified with a striking connection to the notion of divided power structures. It would be very favorable to extend this classification of Tambara ideals to higher levels (the C_{p^n} for $n \geq 2$), and to see if one can develop the theory of crystalline cohomology from the point of view of Tambara thickenings. The higher level structures may be related to the higher level pd-structures of Berthelot.

Equivariant Directions So far the work in this thesis has focused on the case of Weyl-invariant Tambara functors, where there are no nontrivial group actions. It is natural to ask to extend many different parts of the theory in a more equivariant direction, where nontrivial group actions occur. For example, the Witt C_p-Tambara functor has an extension to all C_p-rings, but its behavior is far more delicate in the presence of a nontrivial action. Can the results of sections XXX be extended to yield a workable notion of equivariant divided power structure (one might call this 'divided norms')? Are there rings with nontrivial C_p-action who's C_p-Tambara functors enjoy some of the properties of perfectoid Tambara functors? Is there a notion of C_{p^∞}-ring equipped with some extra structure that becomes the familiar notion of δ-ring when the group action is trivial, and allows for the extension of the prism-to-Tambara construction developed in this thesis to the case with nontrivial Weyl actions?

Tambara Cohomology, C_{p^n}-Prisms, and Finite Level Prismatic Cohomology A crucial goal is to characterize the essential image of the functor from prisms to Tambara functors. If this is done, one can attempt to set up theories of finite level prismatic cohomology by probing a ring with prismatic C_{p^n}-Tambara functors. These can also be compared to more general notions of Tambara cohomology obtained by probing a ring with more general kinds of Tambara thickenings. All of this will be taken up in future work.

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