REPORT NO.
UCB/SESM-74/08

STRUCTURAL ENGINEERING
AND STRUCTURAL MECHANICS

FINITE ELEMENT FORMULATION AND SOLUTION OF CONTACT-IMPACT PROBLEMS IN CONTINUUM MECHANICS

by

THOMAS J. HUGHES
ROBERT L. TAYLOR
JEROME L. SACKMAN

MAY 1974

DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

Structures and Materials Research
Department of Civil Engineering

Report No. 74-8

FINITE ELEMENT FORMULATION AND SOLUTION OF CONTACT AND IMPACT PROBLEMS IN CONTINUUM MECHANICS

by

Thomas J. Hughes
Robert L. Taylor
Jerome L. Sackman

This report was prepared under subcontract number
N62399-73-C-0023

sponsored by

Naval Civil Engineering Laboratory
Port Hueneme, California

as part of NHTSA Interagency Agreement Number
DOT-HS-289-3-550 IA

Funded by National Highway Traffic
Safety Administration

Department of Transportation

May 1974

TABLE OF CONTENTS

Acknowledgment

This report was prepared under Subcontract number N62399-73-C-0023 sponsored by Naval Civil Engineering Laboratory, Port Hueneme, California as part of NHTSA interagency agreement number DOT-HS-289-3-550 IA which was funded by the National Highway Traffic Safety Administration, Department of Transportation.

The opinions, findings and conclusions expressed in this publication are those of the authors and not necessarily those of the Naval Civil Engineering Laboratory or the National Highway Traffic Safety Administration.

Introduction

In this report we consider the general problem of contact and impact between two bodies. The report is divided into three basic parts. These parts describe: (I) The general theory of contact-impact problems, (II) A numerical scheme for the analysis of contact-impact problems, and (III) The description of computer program FEAP 74 for the solution of contact-impact problems. In an appendix we include the program subroutines and general input description for FEAP 74.

In Sections 1 to 6, Part I, we deal with spatial aspects of the theory and in Section 7, Part I, we deal with temporal aspects. This splitting of the theory is motivated by the way we intend to numerically solve the equations, i.e., the finite element method spatially and a finite difference method temporally.

Part II considers a numerical implementation of the theory given in Part I. Section 9 deals with spatial notions of the numerical problem and Section 10 the temporal. The solution scheme for the resulting algebraic problem is discussed in Sections 11 and 12.

The computer program FEAP 74 was modified to incorporate the numerical contact-impact model. The program modifications and capabilities together with two numerical examples are contained in Part III.

Finally, in the appendix we give listings for the contact subroutines together with the data input instructions.

PART I

VARIATIONAL FORMULATION OF CONTACT-IMPACT PROBLEMS IN CONTINUUM MECHANICS

1. Preliminaries

Our conventions on indices are as follows:

Superscripts indicate to which body an entity pertains. Summation is to take place only when explicitly indicated.

Latin subscripts range over 1,2,3, while Greek subscripts range over 1,2. The summation convention is assumed to hold for both.

A body $\mathcal{B}$ is a nice connected region of $\mathbb{R}^3$ with a piecewise smooth boundary $\partial\mathcal{B}$. A contact* problem is a boundary value problem, or an initial-boundary value problem, in which two bodies, $\mathcal{B}^1$ and $\mathcal{B}^2$, interact according to the principles of mechanics. Thus the primary kinematic axiom of a contact problem is that configurations $\mathscr{b}^1$ and $\mathscr{b}^2$, of $\mathcal{B}^1$ and $\mathcal{B}^2$, respectively, do not penetrate each other, i.e.,

$$
\begin{aligned}
(\mathscr{b}^1)^\circ \cap \mathscr{b}^2 &= \emptyset, \\
\mathscr{b}^1 \cap (\mathscr{b}^2)^\circ &= \emptyset,
\end{aligned}
\tag{1}
$$

where $(\mathscr{b}^\alpha)^\circ$ denotes the interior of $\mathscr{b}^\alpha$, $\alpha = 1,2$.

On the other hand the unique conditon which characterizes contact problems is that material points on the boundaries of $\mathcal{B}^1$ and $\mathcal{B}^2$ may coalesce during the motion of the bodies. Thus we say $\mathcal{B}^1$ and $\mathcal{B}^2$ are in contact if $\partial\mathscr{b}^1 \cap \partial\mathscr{b}^2 \neq \emptyset$, and we define the contact surface $\mathscr{c}$ by

---

* It is usual for the term contact to have a static connotation while the term impact has a dynamic connotation. We shall use contact in the general sense to include static as well as dynamic phenomena.

$$\mathfrak{c} = \partial b^1 \cap \partial b^2 \quad . \tag{2}$$

If $\mathcal{B}^1$ and $\mathcal{B}^2$ are never in contact then $\mathfrak{c} = \emptyset$ for all configurations b^1 and b^2, and in this case an initial-boundary value problem for $\mathcal{B}^1$ and $\mathcal{B}^2$ reduces to one in which $\mathcal{B}^1$ and $\mathcal{B}^2$ may be treated separately. Thus a non-trivial contact problem is one in which $\mathfrak{c} \neq \emptyset$ for at least one instant during the motion of $\mathcal{B}^1$ and $\mathcal{B}^2$. The picture (Fig. 1) illustrates these notions.

Equation (1) implies that $\mathfrak{c}$ is a material surface with respect to both bodies, i.e., one which is not crossed by material particles. From this we may deduce the interface conditions on $\mathfrak{c}$.

Let $\underset{\sim}{x}$ be a persistent point of $\mathfrak{c}$ (one at which joining or releasing of the bodies is not instantaneously occurring) and $\underset{\sim}{v}$ be the velocity of $\underset{\sim}{x}$ ($\underset{\sim}{v} = \dot{\underset{\sim}{x}}$). Note that only the normal part of $\underset{\sim}{v}$ is independent of the parametrization of $\mathfrak{c}$. Let $\underset{\sim}{v}^1$ and $\underset{\sim}{v}^2$ be the velocities of the material particles located at the points $\underset{\sim}{x}^1$ and $\underset{\sim}{x}^2$, contained in ∂b^1 and ∂b^2 , respectively, such that $\underset{\sim}{x} = \underset{\sim}{x}^1 = \underset{\sim}{x}^2$ at the present instant. Then since $\mathfrak{c}$ is material and $\underset{\sim}{x}$ is persistant

$$\underset{\sim}{v} \cdot \underset{\sim}{n} = \underset{\sim}{v}^1 \cdot \underset{\sim}{n} = \underset{\sim}{v}^2 \cdot \underset{\sim}{n} \quad , \tag{3}$$

where $\underset{\sim}{n}$ is a unit normal vector to $\mathfrak{c}$ at $\underset{\sim}{x}$. From this it follows that a necessary condition for momentum to be balanced at $\underset{\sim}{x}$ is that

$$(\underset{\sim}{t}^1 + \underset{\sim}{t}^2) \cdot \underset{\sim}{n} = \underset{\sim}{0} \quad , \tag{4}$$

where $\underset{\sim}{t}^\alpha$ is the Cauchy traction vector with respect to ∂b^α .

In addition we assume that no tensile tractions can occur on $\mathfrak{c}$,

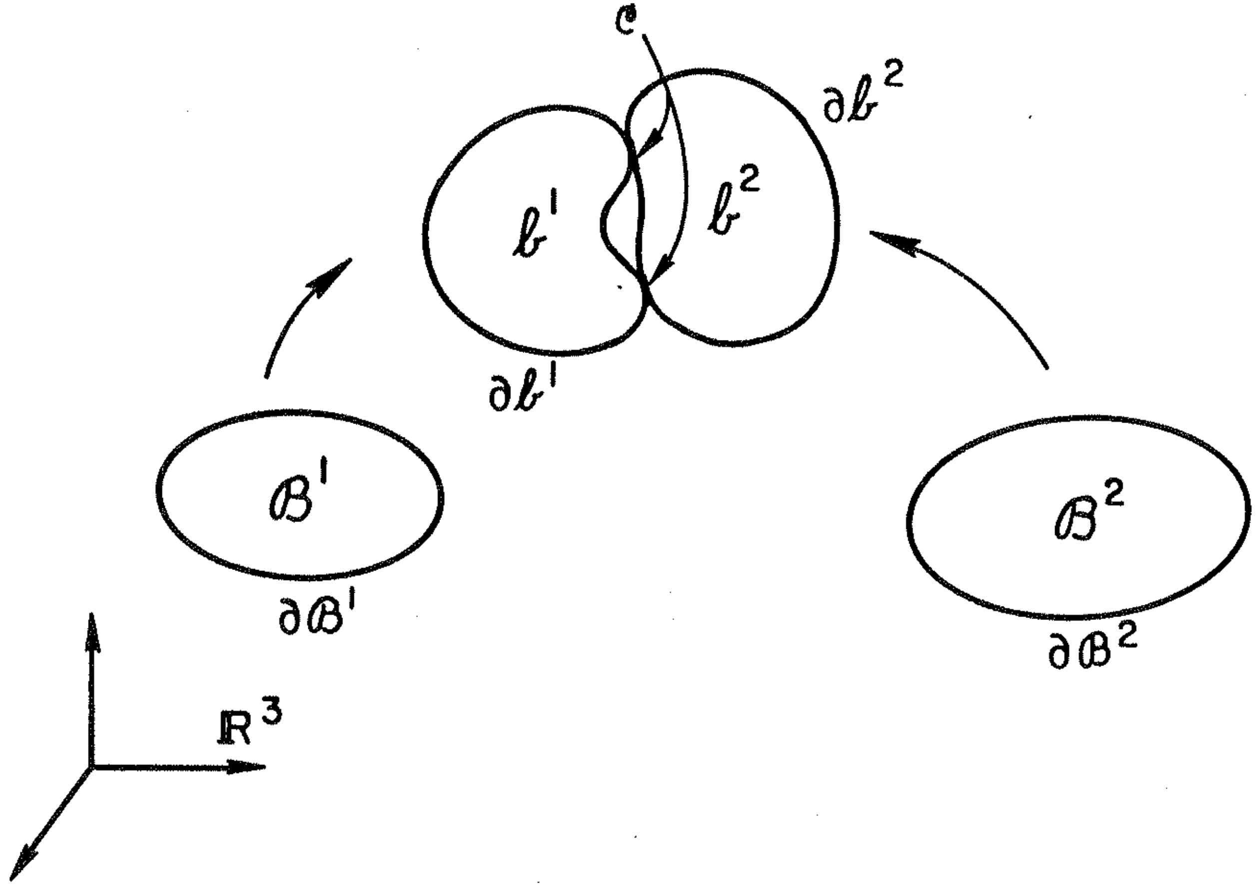

$\mathcal{C}$
∂b^2
b^1
b^2
∂b^1
$\mathcal{B}^1$
$\partial\mathcal{B}^1$
$\mathcal{B}^2$
$\partial\mathcal{B}^2$
$\mathbb{R}^3$

Figure 1

$$\underset{\sim}{t}^{\alpha} \cdot \underset{\sim}{n}^{\alpha} \leq 0 \ , \tag{5}$$

where $\underset{\sim}{n}^{\alpha}$ is the outward unit normal vector to $\partial \mathcal{B}^{\alpha}$. This condition excludes the possibility of the two bodies being glued together. Conditions (1-5) characterize our notion of a contact problem.

Note that thus far we have said nothing about the tangential parts of $\underset{\sim}{v}^{\alpha}$ and $\underset{\sim}{t}^{\alpha}$. These remaining conditions are determined by the frictional nature of the contact. We shall study two simple cases.

<u>Case I</u>: If we assume that points, once in contact, move with $\mathcal{C}$ until released, we have that

$$\underset{\sim}{v}^{1} = \underset{\sim}{v}^{2} \ , \tag{6}$$

and therefore

$$\underset{\sim}{t}^{1} + \underset{\sim}{t}^{2} = \underset{\sim}{0} \ . \tag{7}$$

For this model we say that a no-slip, or perfect friction, condition is achieved on $\mathcal{C}$. Thus condition (5) and equations (6) and (7) are the interface conditions for this case.

<u>Case II</u>: We may create the interface conditions for a frictionless, sliding contact by asserting that the tangential part of each $\underset{\sim}{t}^{\alpha}$ is identically zero,

$$\underset{\sim}{t}^{\alpha} - (\underset{\sim}{t}^{\alpha} \cdot \underset{\sim}{n}^{\alpha})\underset{\sim}{n}^{\alpha} = \underset{\sim}{0} \ . \tag{8}$$

Eq. (8), along with (3-5), are the interface conditions for this case.

2. Variational Theorems

We will formulate a variational theorem for the contact problem of finite elastodynamics. We point out, however, that our treatment is entirely general and could be used in conjunction with any field theory, as the only unique feature of the formulation involves the handling of interface conditions. At the same time finite elastodynamics, though lending itself to a clean and simple variational statement, is a case of wide practical interest.

We shall first obtain a variational theorem for the usual initial-boundary value problem of finite elastodynamics by a trivial generalization of some work done by S. Nemat-Nasser [1].

For notational simplicity let $\mathcal{A}$ denote $\partial\mathcal{B}$, and let $d\mathcal{A}$ and $d\mathcal{B}$ denote area and volume forms for $\mathcal{B}$ and $\mathcal{A}$, respectively. Let $\mathcal{A}_T \subset \mathcal{A}$ be that part of $\mathcal{A}$ where surface tractions are prescribed, and denote by $\bar{\underset{\sim}{T}}$ the Piola - Kirchhoff traction vector representing these prescribed tractions. Call ρ_o the density of $\mathcal{B}$ in the initial configuration, $\underset{\sim}{F}$ the extrinsic body force vector and let $\underset{\sim}{x} = \underset{\sim}{x}_t(\underset{\sim}{X})$ represent the position at time t of the material particle located at $\underset{\sim}{X}$ in the initial configuration. For convenience we take $\mathcal{B}$ to be the initial configuration. We denote by $\partial\underset{\sim}{x}/\partial\underset{\sim}{X}$ the deformation gradients and by $\Phi(\partial\underset{\sim}{x}/\partial\underset{\sim}{X})$ the strain energy density. Then if $\underset{\sim}{x}$ satisfies the kinematic boundary conditions

$$\underset{\sim}{x} = \bar{\underset{\sim}{x}} \tag{9}$$

on $\mathcal{A}_x \subset \mathcal{A}$, where

$$\mathcal{A}_x \cup \mathcal{A}_T = \mathcal{A},$$
$$\mathcal{A}_x \cap \mathcal{A}_T = \emptyset,$$

the functional Π defined by

$$\Pi(\underset{\sim}{x}) = \int_0^t \{ \int_{\mathcal{B}} (\Phi(\partial \underset{\sim}{x}/\partial \underset{\sim}{X}) - \rho_0 \dot{\underset{\sim}{x}} \cdot \dot{\underset{\sim}{x}}/2 - \rho_0 \underset{\sim}{F} \cdot \underset{\sim}{x}) d\mathcal{B} - \int_{\mathcal{A}_T} \underset{\sim}{x} \cdot \bar{\underset{\sim}{T}} \, d\mathcal{A} \} dt \, , \tag{10}$$

is stationary, i.e., its first variation vanishes

$$0 = \delta \Pi(\underset{\sim}{x}, \delta \underset{\sim}{x}) = \int_0^t \{ \int_{\mathcal{B}} (\rho_0 (\ddot{\underset{\sim}{x}} - \underset{\sim}{F}) - \mathrm{DIV}\, \underset{\sim}{P}) \cdot \delta \underset{\sim}{x} \, d\mathcal{B} + \int_{\mathcal{A}_T} (\underset{\sim}{T} - \bar{\underset{\sim}{T}}) \cdot \delta \underset{\sim}{x} \, d\mathcal{A} \} dt \, , \tag{11}$$

subject to the constraint on variations $\delta \underset{\sim}{x}_t = \delta \bar{\underset{\sim}{x}}_0 = \underset{\sim}{0}$, on $\mathcal{A}_x$ (12)

if and only if the equations of motion and traction boundary conditions are satisfied

$$\rho_0 (\ddot{\underset{\sim}{x}} - \underset{\sim}{F}) = \mathrm{DIV}\, \underset{\sim}{P} \, , \qquad \text{in } \mathcal{B} \, , \tag{13}$$

$$\underset{\sim}{T} = \bar{\underset{\sim}{T}} \, , \qquad \text{on } \mathcal{A}_T \, , \tag{14}$$

where $\underset{\sim}{P} = \partial \Phi / \partial(\partial \underset{\sim}{x}/\partial \underset{\sim}{X})$ is the first Piola - Kirchhoff stress tensor, $\underset{\sim}{T} = \underset{\sim}{N} \cdot \underset{\sim}{P}$ is the Piola - Kirchhoff traction vector, and $\underset{\sim}{N}$ is the outward unit normal vector to $\mathcal{A}$. The solution to the initial-boundary value problem must also satisfy the given initial conditions

$$\left. \begin{aligned} \underset{\sim}{x} &= \underset{\sim}{x}_0 \\ \underset{\sim}{v} &= \underset{\sim}{v}_0 \end{aligned} \right\} \quad \text{in } \mathcal{B} \, . \tag{15}$$

To interpret this variational theorem for two (non-interacting) bodies set

$$\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2 \, ,$$

$$\mathcal{A} = \mathcal{A}^1 \cup \mathcal{A}^2 \, , \quad \text{etc.} \, ,$$

and write

$$\mathbb{I}(\underset{\sim}{x}) = \mathbb{I}^1(\underset{\sim}{x}^1) + \mathbb{I}^2(\underset{\sim}{x}^2) .$$

The next step is to add to $\mathbb{I}$ terms manifesting the interface conditions on $\mathcal{C}$ and to stipulate the constraints under which the vanishing of the first variation of the appended functional corresponds to a solution of the contact problem. To do this we must consider further the kinematics and geometry of $\mathcal{C}$.

Define two piecewise smooth, invertible maps $\underset{\sim}{\chi}^1, \underset{\sim}{\chi}^2$ by the condition

$$(\underset{\sim}{\chi}^\alpha)^{-1} : \mathcal{C} \longrightarrow \mathcal{C}^\alpha \subset \partial\mathcal{B}^\alpha , \tag{16}$$

where each $\underset{\sim}{\chi}^\alpha$ identifies points on the boundary of the initial configuration $\mathcal{B}^\alpha$ which map into the contact surface $\mathcal{C}$ at each instant of time. If $\underset{\sim}{x} \in \mathcal{C}$, then $\underset{\sim}{X}^1 = (\underset{\sim}{\chi}^1)^{-1}(\underset{\sim}{x})$ and $\underset{\sim}{X}^2 = (\underset{\sim}{\chi}^2)^{-1}(\underset{\sim}{x})$ are the positions of particles in $\partial\mathcal{B}^1$ and $\partial\mathcal{B}^2$, respectively, which have coalesced at $\underset{\sim}{x} \in \mathcal{C}$. It is clear what the $\underset{\sim}{\chi}^\alpha$'s really are, viz., if $\underset{\sim}{x}^\alpha = \underset{\sim}{x}^\alpha_t(\underset{\sim}{X}^\alpha)$, for all $\underset{\sim}{X}^\alpha \in \mathcal{B}^\alpha$, represents the motion of body $\mathcal{B}^\alpha$ from the original configuration $\mathcal{B}^\alpha$ to the present one $\mathcal{b}^\alpha$, then $\underset{\sim}{\chi}^\alpha$ is the restriction of $\underset{\sim}{x}^\alpha$ to $\mathcal{C}^\alpha$,

$$\underset{\sim}{\chi}^\alpha(\underset{\sim}{X}^\alpha) = \underset{\sim}{x}^\alpha(\underset{\sim}{X}^\alpha) , \tag{17}$$

for each $\underset{\sim}{X}^\alpha \in \mathcal{C}^\alpha$, $\alpha = 1,2$. For the time being we consider the $\underset{\sim}{\chi}^\alpha$'s as maps defined independently of the $\underset{\sim}{x}^\alpha$'s and consider (17) a constraint on possible motions.

We are interested in to what extent the relation

$$\underset{\sim}{x} = \underset{\sim}{\chi}^{1}(\underset{\sim}{X}^{1}) = \underset{\sim}{\chi}^{2}(\underset{\sim}{X}^{2}) \quad , \tag{18}$$

is smooth in time and analogously under what circumstances the variations of the $\underset{\sim}{\chi}^{\alpha}$'s are equal. In general the $\underset{\sim}{\chi}^{\alpha}$'s will not even be continuous in time since contact surfaces can be instantaneously created or destroyed. If we eliminate such exceptional instants and consider only persistent points, the bodies still may slide with respect to each other, as depicted in Fig. 2. Thus tangential velocities are seen to be unequal in general. However, when $\underset{\sim}{x}$ is persistent, the impenetrability condition (1) forces the normal velocity components to be equal, and concomitantly the normal components of variations of the $\underset{\sim}{\chi}^{\alpha}$'s are also equal

$$\delta\underset{\sim}{\chi}^{1}\cdot\underset{\sim}{n} = \delta\underset{\sim}{\chi}^{2}\cdot\underset{\sim}{n} \tag{19}$$

For sliding contact (Case II), Eq. (19) characterizes the constraint on variations of the $\underset{\sim}{\chi}^{\alpha}$'s eqivalent to the velocity constraint (3).

For no-slip contact (Case I),

$$\delta\underset{\sim}{\chi}^{1} = \delta\underset{\sim}{\chi}^{2} \quad , \tag{20}$$

is easily seen to be the condition on variations equivalent to Eq. (6). We shall see that Eqs. (19) and (20) lead to the proper interface conditions in the variational theorems.

Introduce vector valued Lagrange multipliers $\underset{\sim}{\tau}^{\alpha}$, and add

$$\mathcal{K} = -\sum_{\alpha=1}^{2}\int_{0}^{t}\int_{\mathcal{C}^{\alpha}}\underset{\sim}{\tau}^{\alpha}\cdot(\underset{\sim}{x}^{\alpha}-\underset{\sim}{\chi}^{\alpha})\,d\mathcal{C}^{\alpha}dt \quad , \tag{21}$$

to the functional Π (Eq. 10). Note that when $\mathcal{C}\neq\emptyset$,

$$\mathcal{A}^{\alpha} = \mathcal{A}^{\alpha}_{x}\cup\mathcal{A}^{\alpha}_{T}\cup\mathcal{C}^{\alpha} \quad ,$$

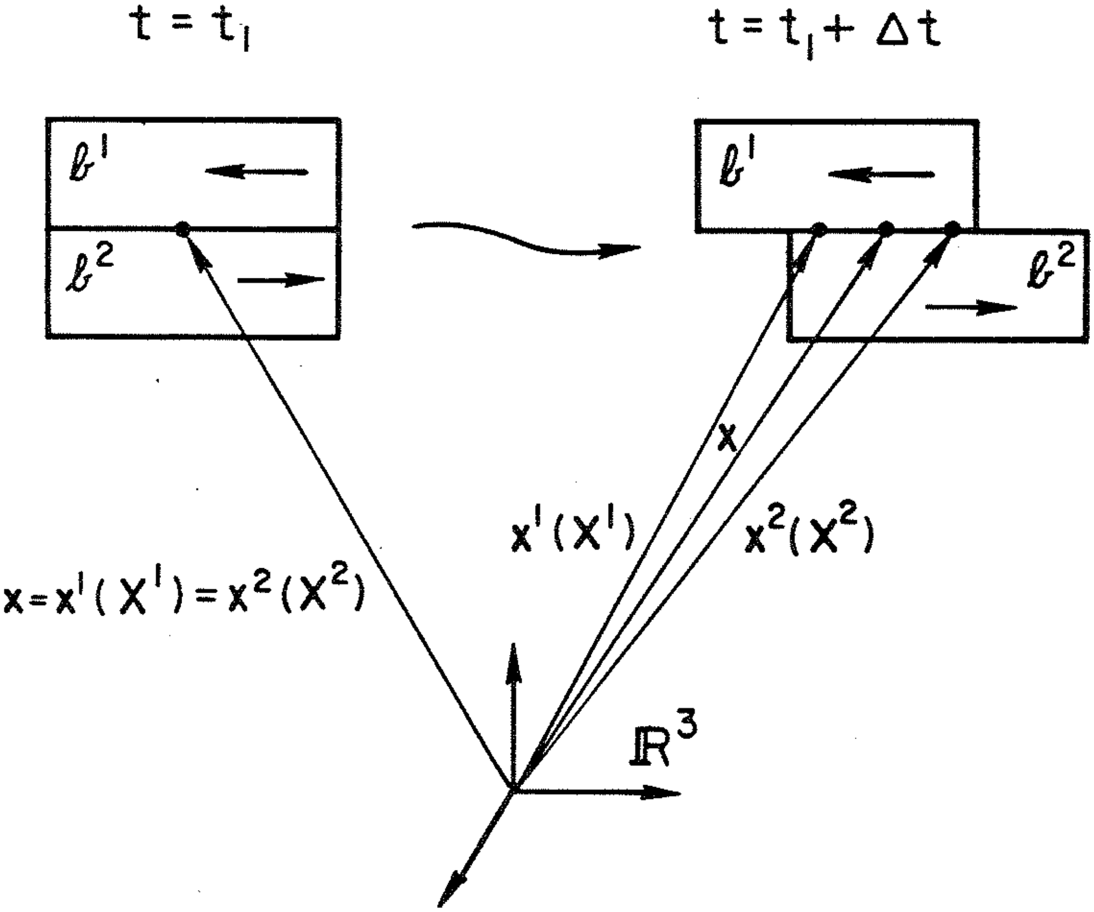


Figure 2

and assume for consistency's sake that

$$\mathcal{C}^{\alpha} \subset \mathcal{Q}^{\alpha}_{T} ,$$

$$\underset{\sim}{T} = \underset{\sim}{0} \qquad \text{on} \quad \mathcal{C}^{\alpha}. \tag{22}$$

This condition will preclude the ambiguous circumstance of non-zero tractions being specified on the contact area. Upon taking variations of $\mathcal{J} = \Pi + \mathcal{K}$ we get Eqs. (13), (14) and,

$$\begin{aligned} 0 = -\sum_{\alpha=1}^{2} \int_0^t \int_{\mathcal{C}^{\alpha}} \{ \delta \underset{\sim}{\tau}^{\alpha} \cdot (\underset{\sim}{x}^{\alpha} - \underset{\sim}{\chi}^{\alpha}) + \\ + \delta \underset{\sim}{x}^{\alpha} \cdot (\underset{\sim}{\tau}^{\alpha} - \underset{\sim}{T}^{\alpha}) - \delta \underset{\sim}{\chi}^{\alpha} \cdot \underset{\sim}{\tau}^{\alpha} \} d\mathcal{C}^{\alpha} dt \quad + \\ + \text{ transversality condition.} \end{aligned} \tag{23}$$

The transversality condition is the classical terminology for variations associated with the domain $\mathcal{C}^{\alpha}$.

The first summand of (23) gives us (17) which insures that the $\underset{\sim}{x}^{\alpha}$'s map into $\mathcal{E}$ properly. The second summand identifies $\underset{\sim}{\tau}^{\alpha}$ as the Piola - Kirchhoff traction vector $\underset{\sim}{T}^{\alpha}$ on $\mathcal{C}^{\alpha}$. Let us investigate the third summand.

Consider first Case I and define

$$\delta \underset{\sim}{\chi} = \delta \underset{\sim}{\chi}^{\alpha} \quad , \quad \alpha = 1,2 , \tag{24}$$

which makes sense because of Eq. (20). This condition is equivalent to insisting

$$\dot{\underset{\sim}{\chi}} = \dot{\underset{\sim}{\chi}}^{\alpha} \quad , \quad \alpha = 1,2 ,$$

thus the first summand of (23) also implies (6) holds whenever we have a persistent point. Let j^α denote the Jacobian determinant associated with χ^α,

$$da = j^\alpha dA^\alpha. \tag{25}$$

Notice then that since $\underset{\sim}{T}^\alpha$ is the Piola - Kirchhoff traction vector, $(1/j^\alpha)\underset{\sim}{T}^\alpha$ is the corresponding Cauchy traction vector. With these we have for the third summand,

$$0 = \sum_{\alpha=1}^{2} \int_{A^\alpha} \delta\underset{\sim}{\chi}^\alpha \cdot \underset{\sim}{T}^\alpha dA^\alpha = \int_{a} \delta\underset{\sim}{\chi} \cdot ((1/j^1)\underset{\sim}{T}^1 + (1/j^2)\underset{\sim}{T}^2)da \quad , \tag{26}$$

which in words means the Cauchy traction vectors are in equilibrium. Thus the momentum balance, Eq. (7), is satisfied on a.

In Case II we only have that (19) holds, so define

$$\delta\chi(n) = \delta\underset{\sim}{\chi}^\alpha \cdot \underset{\sim}{n} \quad , \quad \alpha = 1,2 . \tag{27}$$

This requirement also insures that,

$$\dot{\underset{\sim}{\chi}}^1 \cdot \underset{\sim}{n} = \dot{\underset{\sim}{\chi}}^2 \cdot \underset{\sim}{n} \quad ,$$

thus the first summand of (23) implies (3). For this case the third summand takes the form,

$$\begin{aligned} 0 = \sum_{\alpha=1}^{2} \int_{A^\alpha} \delta\underset{\sim}{\chi}^\alpha \cdot \underset{\sim}{T}^\alpha dA^\alpha &= \int_{a} \delta\chi(n)((1/j^1)\underset{\sim}{T}^1\cdot\underset{\sim}{n} + (1/j^2)\underset{\sim}{T}^2\cdot\underset{\sim}{n})da \\ &+ \sum_{\alpha=1}^{2} \int_{A^\alpha} (\delta\underset{\sim}{\chi}^\alpha - \delta\chi(n)\underset{\sim}{n}) \cdot \underset{\sim}{T}^\alpha dA^\alpha . \end{aligned} \tag{28}$$

The integral over a gives us Eq. (4). The significance of the second integral hinges on the observation that $(\delta\underset{\sim}{\chi}^\alpha - \delta\chi(n)\underset{\sim}{n})$ is a tangent vector to a for each α. Thus the tangential part of each $\underset{\sim}{T}^\alpha$ is identically zero, which is equivalent to the shear free condition, Eq. (8), which we require for Case II.

A standard calculation enables us to write the transversality condition as,

$$0 = \sum_{\alpha=1}^{2} \int_{\partial C^\alpha} (\delta \chi^\alpha \cdot T^\alpha)(\tau^\alpha \cdot (x^\alpha - x'^\alpha))\, d(\partial C^\alpha) \quad , \tag{29}$$

where the transversal T^α is a unit vector field tangent to C^α, and perpendicular and pointing outward with respect to ∂C^α , Fig. 3. Thus (29) implies that

$$\tau^\alpha \cdot (x^\alpha - x'^\alpha) = 0 \quad \text{on} \quad \partial C^\alpha, \ \alpha = 1,2 \tag{30}$$

Assuming continuity of the integrands of (21) on the closure of C^α, condition (30) is already implied by the first summand of (23). This assumption precludes τ^α taking the form of a δ-distribution on ∂C^α.

Although this assumption is warranted here it may not be true when one employs certain approximate theories in mechanics. For instance consider the case where a Bernoulli-Euler beam is uniformly loaded and sits on a rigid parabolic surface (Fig. 4). At the contact points a, a', concentrated reactions must exist to balance shear forces. This example is actually from a completely different class of contact problems in that contact is made along a part of the interior rather than the boundary. Such problems as the contact of plates and shells also fall into this class. We could summarize such situations by the description -- m-dimensional contact of m-dimensional bodies, e.g., for the beam m=1, and for plates and shells m=2. The case under investigation in this paper (m=3) is an example of the (m-1)-dimensional contact of m-dimensional bodies.

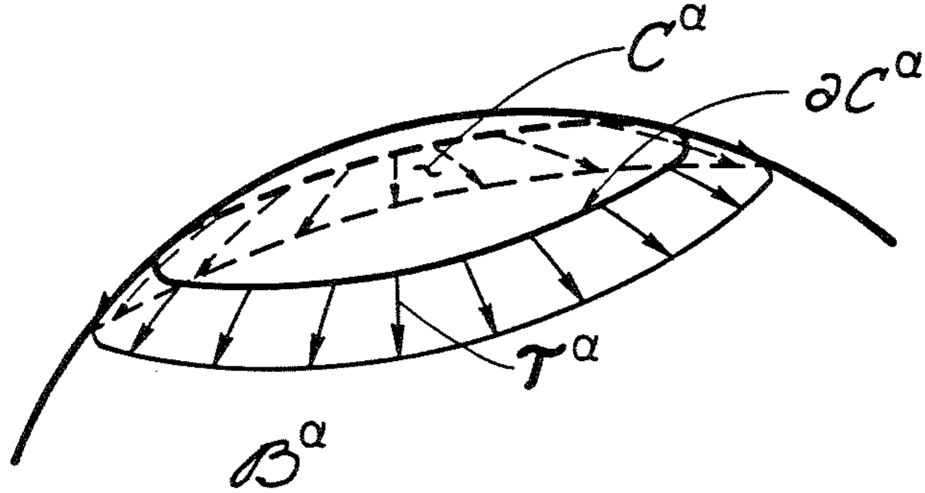


Figure 3

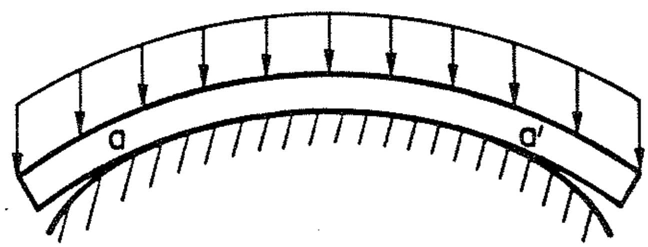


Figure 4

It is good to keep in mind cases such as that illustrated in Fig. 4 when considering specific boundary value problems.

A further point worth mentioning here is that the transversality condition will in general be an independent one in a numerical algorithm. For example, if the fields in the integrand of (21) are approximated by a family of trial functions, Eq. (23) only implies that some weighted integrals over the $\mathcal{C}^{\alpha}$'s vanish. The condition (29) requires that weighted integrals over the $\partial\mathcal{C}^{\alpha}$'s also vanish.

We now summarize our results in the following theorems:

Theorem I: Let (1), (2), (5), (9), (12), (15), and (20) hold. Then $\underset{\sim}{x}$ is a solution to the no-slip contact problem (Case I), that is, (6), (7), (13), (14) and (17) also hold if, and only if, $\delta\mathcal{G}=0$ for arbitrary variations of $\underset{\sim}{x}^{\alpha}, \underset{\sim}{\chi}^{\alpha}$ and $\underset{\sim}{\tau}^{\alpha}, \alpha=1,2$.

Theorem II: Let (1), (2), (5), (9), (12), (15), and (19) hold. Then $\underset{\sim}{x}$ is a solution to the sliding contact problem (Case II), that is, (3), (4), (8), (13), (14) and (17) also hold if, and only if, $\delta\mathcal{G}=0$ for arbitrary variations of $\underset{\sim}{x}^{\alpha}, \underset{\sim}{\chi}^{\alpha}$ and $\underset{\sim}{\tau}^{\alpha}, \alpha=1,2$.

3. <u>Consideration of Theorems I and II as Computational Tools</u>

Theorems I and II may be employed to generate numerical algorithms for the solution of contact problems. The basic idea is to represent $\underset{\sim}{x}^{\alpha}$, $\underset{\sim}{x}'^{\alpha}$ and $\underset{\sim}{\tau}^{\alpha}$ as the product of known functions on $\mathbb{R}^3$ with unknown parameters depending on time. Then Theorems I and II provide us with a method for generating an approximate system of equations (e.g., by the classical Ritz-Galerkin technique) in terms of these unknown parameters, which then can be solved incrementally and/or iteratively, subject to the side conditions of the theorems. The constraints (1) and (5) will both take the form of inequalities in actual computations, thus the ideas of optimization theory will probably be useful in the actual construction of a numerical algorithm.

The finite element method is a powerful technique for obtaining a system of approximate equations, and it is of interest to find out how amenable are Theorems I and II to a finite element formulation. Unfortunately the term $\mathcal{K}$ would result in a terrible mess if the integrand was represented by typical finite element functions. This is because the boundaries of the C^{α}'s are unknown and thus a parametric integration would bury the defining parameters of the C^{α}'s in the arguments of Heaviside functions representing the supports of the elements. Note that a classical Ritz-Galerkin approximation would not be subject to this pitfall, since the associated trial functions could be chosen to be real analytic and thus easily integrated parametrically to a relatively simple form. However, such a formulation is restricted to a geometrically simpler class of problems. Thus it is desirable to seek a generalization that will lend itself cleanly to a finite element formulation.

4. Variational Theorems Without Transversality Conditions

Let $\tilde{\mathcal{E}}^{\alpha}$ be a fixed part of $\mathcal{A}_{T}^{\alpha}$ such that

$$\tilde{\mathcal{E}}^{\alpha} \supset \mathcal{E}^{\alpha} \quad , \tag{31}$$

and

$$\underset{\sim}{T} = \underset{\sim}{0} \quad \text{on} \quad \tilde{\mathcal{E}}^{\alpha} \sim \mathcal{E}^{\alpha} . \tag{32}$$

Define a scalar valued function n^{α} on $\tilde{\mathcal{E}}^{\alpha}$ such that

$$n^{\alpha}(\underset{\sim}{X}^{\alpha}) = 0 \qquad \text{if} \quad \underset{\sim}{X}^{\alpha} \in \tilde{\mathcal{E}}^{\alpha} \sim \mathcal{E}^{\alpha} . \tag{33}$$

Let $\tilde{\varepsilon} \supset \varepsilon$, and define the maps $\underset{\sim}{\chi}^{\alpha}$ by the condition

$$(\underset{\sim}{\chi}^{\alpha})^{-1} : \tilde{\varepsilon} \longrightarrow \tilde{\mathcal{E}}^{\alpha} ,$$

where, as before, $\underset{\sim}{\chi}^{\alpha}$ represents $\underset{\sim}{x}^{\alpha}$ on $\mathcal{E}^{\alpha}$; but on $\tilde{\mathcal{E}}^{\alpha} \sim \mathcal{E}^{\alpha}$ we place no physical interpretation on $\underset{\sim}{\chi}^{\alpha}$. Thus on $\tilde{\mathcal{E}}^{\alpha}$ we will always have that,

$$n^{\alpha}(\underset{\sim}{x}^{\alpha} - \underset{\sim}{\chi}^{\alpha}) = \underset{\sim}{0} \quad , \tag{34}$$

since $\underset{\sim}{x}^{\alpha} = \underset{\sim}{\chi}^{\alpha}$ on $\mathcal{E}^{\alpha}$ and $n^{\alpha} = 0$ on the relative complement $\tilde{\mathcal{E}}^{\alpha} \sim \mathcal{E}^{\alpha}$.

Introduce vector valued Lagrange multipliers $\underset{\sim}{\sigma}^{\alpha}$ and let $\mathcal{L} = \mathrm{II} + \mathcal{M}$ where

$$\mathcal{M} = - \sum_{\alpha=1}^{2} \int_{0}^{t} \int_{\tilde{\mathcal{E}}^{\alpha}} \underset{\sim}{\sigma}^{\alpha} \cdot n^{\alpha} (\underset{\sim}{x}^{\alpha} - \underset{\sim}{\chi}^{\alpha}) \, d\tilde{\mathcal{E}}^{\alpha} dt . \tag{35}$$

We require that the variations of $\underset{\sim}{\chi}^{\alpha}$ satisfy the same conditions as before, but now for all $\tilde{\mathcal{E}}^{\alpha}$:

$$\left.\begin{array}{ll} \text{Case I:} & \delta\underset{\sim}{\chi} \overset{\text{def.}}{=} \delta\underset{\sim}{\chi}^{\alpha} \\ \text{Case II:} & \delta\chi(n) \overset{\text{def.}}{=} \delta\underset{\sim}{\chi}^{\alpha} \cdot \underset{\sim}{n} \end{array}\right\} \quad \text{on} \quad \tilde{\mathcal{E}}^{\alpha} \tag{36}$$

where $\underset{\sim}{n}$ is a unit normal vector to $\tilde{\mathcal{E}}$. Computing the first variation of $\mathcal{L}$ we have the usual conditions emanating from II and

$$
\begin{aligned}
0 = -\sum_{\alpha=1}^{2}\int_0^t\int_{\tilde{\mathcal{E}}^\alpha}\Big\{ & \delta\underset{\sim}{\sigma}^\alpha\cdot\left(\eta^\alpha(\underset{\sim}{x}^\alpha-\underset{\sim}{\chi}^\alpha)\right) \\
& +\delta\eta^\alpha\left(\underset{\sim}{\sigma}^\alpha\cdot(\underset{\sim}{x}^\alpha-\underset{\sim}{\chi}^\alpha)\right) \\
& +\delta\underset{\sim}{x}^\alpha\cdot\left(\eta^\alpha\underset{\sim}{\sigma}^\alpha-\underset{\sim}{T}^\alpha\right) \\
& -\delta\underset{\sim}{\chi}^\alpha\cdot\left(\eta^\alpha\underset{\sim}{\sigma}^\alpha\right)\Big\}\,d\tilde{\mathcal{E}}^\alpha\,dt .
\end{aligned}
\tag{37}
$$

The first summand gives us (34) and we define

$$
\mathcal{C}^\alpha = \{\, \underset{\sim}{X}^\alpha\in\tilde{\mathcal{E}}^\alpha : \underset{\sim}{x}^\alpha(\underset{\sim}{X}^\alpha)=\underset{\sim}{\chi}^\alpha(\underset{\sim}{X}^\alpha)\,\} . \tag{38}
$$

The third summand defines $\eta^\alpha\underset{\sim}{\sigma}^\alpha$ as the Piola - Kirchhoff traction vector. Note that this insures that $\underset{\sim}{T}=\underset{\sim}{0}$ on $\tilde{\mathcal{E}}^\alpha\sim\mathcal{C}^\alpha$ since $\eta^\alpha=0$ there. The fourth summand gives us the appropriate Cauchy traction condition across $\mathcal{C}$ for each case of (36). The second summand is identically satisfied on $\mathcal{C}^\alpha$ since $\underset{\sim}{x}^\alpha=\underset{\sim}{\chi}^\alpha$. On $\tilde{\mathcal{E}}^\alpha\sim\mathcal{C}^\alpha$ it tells us that $\underset{\sim}{\sigma}^\alpha$ is orthogonal to $\underset{\sim}{x}^\alpha-\underset{\sim}{\chi}^\alpha$, but this is of no physical interest.

Thus we can state the following theorems:

Theorem I': Let (1), (2), (5), (9), (12), (15), (31), (32) and $(36)_1$ hold. Then $\underset{\sim}{x}$ is a solution to the no-slip contact problem (Case I), that is, (6), (7), (13), (14) and (17) also hold where $\mathcal{C}^\alpha$ is defined by (38), if $\delta\mathcal{L}=0$ for arbitrary variations of $\underset{\sim}{x}^\alpha$, $\underset{\sim}{\chi}^\alpha$, η^α and $\underset{\sim}{\sigma}^\alpha$, $\alpha=1,2$.

Theorem II': Let (1), (2), (5), (9), (12), (15), (31), (32) and $(36)_2$ hold. Then $\underset{\sim}{x}$ is a solution to the sliding contact problem (Case II), that is, (3), (4), (8), (13), (14) and (17) hold where $\mathcal{C}^\alpha$ is defined by (38), if $\delta\mathcal{L}=0$ for arbitrary variations of $\underset{\sim}{x}^\alpha$, $\underset{\sim}{\chi}^\alpha$, η^α and $\underset{\sim}{\sigma}^\alpha$, $\alpha=1,2$.

The important feature of these theorems is that the regions $\tilde{\mathcal{E}}^{\alpha}$ are fixed. Thus transversality conditions are absent, and the theorems may be applied to finite element formulations. In fact one would naturally take $\tilde{\mathcal{E}}^{\alpha}$ to be a union of elements in $\mathcal{Q}^{\alpha}$, large enough to contain $\mathcal{E}^{\alpha}$ throughout the motion.

Thus far our considerations have been quite general and, in fact, more general than would be required for the solution of particular classes of contact problems. In the next section we illustrate the many simplifications which can be made in the application of the preceeding theorems to a class of problems of wide practical interest.

5. Hertzian Contact Problems

We wish to characterize contact problems in which the contact surface is approximately planar and the bodies have undergone small deformations in the neighborhood of the contact surface.

Assume the following:

(1) $\underset{\sim}{n} \overset{\text{def.}}{=} n_i \underset{\sim}{e}_i \approx \underset{\sim}{e}_3$ on $\mathcal{C}$, where the n_i indicate components with respect to the standard basis $\{\underset{\sim}{e}_i\}_1^3$ for $\mathbb{R}^3$, (see Fig. 5).

(2) $j^\alpha \approx 1$, $\alpha = 1,2$, thus $\underset{\sim}{t}^\alpha \approx \underset{\sim}{T}^\alpha$ on $\mathcal{C}^\alpha$.

Assumptions (1) and (2) together imply that,

$$t_3^\alpha \approx \underset{\sim}{t}^\alpha \cdot \underset{\sim}{n} \approx \underset{\sim}{T}^\alpha \cdot \underset{\sim}{n} \approx T_3^\alpha \quad ,$$

and that,

$$(t_1^\alpha, t_2^\alpha, 0) \approx \underset{\sim}{t}^\alpha - (\underset{\sim}{t}^\alpha \cdot \underset{\sim}{n})\underset{\sim}{n} \approx \underset{\sim}{T}^\alpha - (\underset{\sim}{T}^\alpha \cdot \underset{\sim}{n})\underset{\sim}{n} \approx (T_1^\alpha, T_2^\alpha, 0) .$$

(3) Material points which eventually contact have, to the first order, the same initial coordinates z_1 and z_2. Explicitly we manifest this idea by requiring that the $\underset{\sim}{\chi}^\alpha$'s satisfy

$$\underset{\sim}{\chi}^1(z_1, z_2, X_3^1(z_1, z_2)) = \underset{\sim}{\chi}^2(z_1, z_2, X_3^2(z_1, z_2)) \ . \qquad (39)$$

This is depicted in Fig. 6. Since X_3^α are given functions which define the surfaces $\mathcal{C}^\alpha$, it follows from (39) that,

$$\delta \underset{\sim}{\chi}^1 = \delta \underset{\sim}{\chi}^2 \ .$$

We term problems for which these assumptions hold Hertzian, since these assumptions are implicit in Hertz' classical theory [2] (see

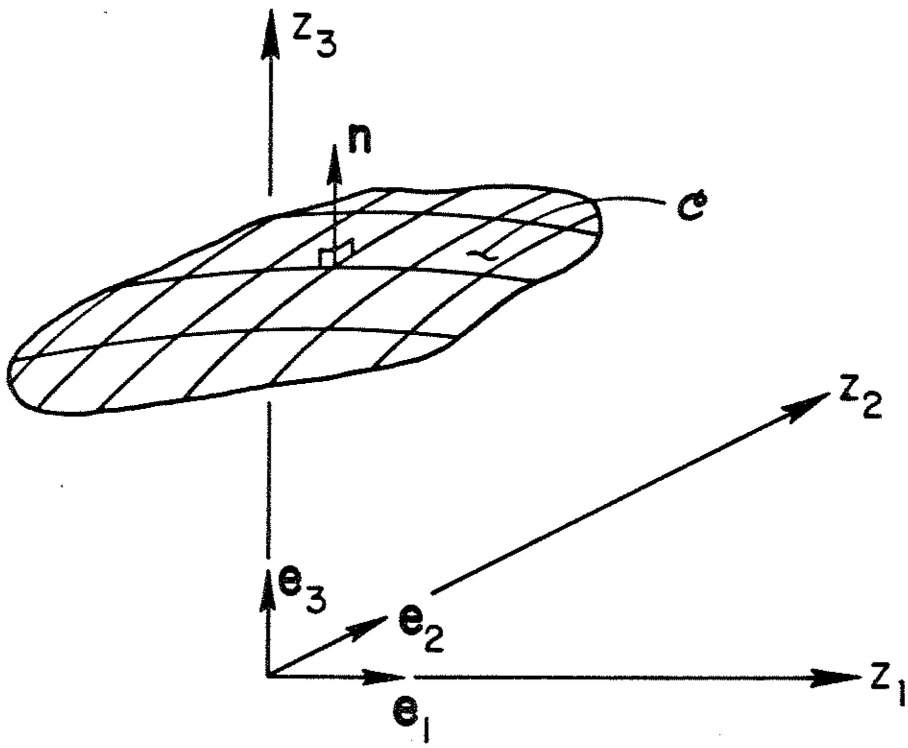


Figure 5

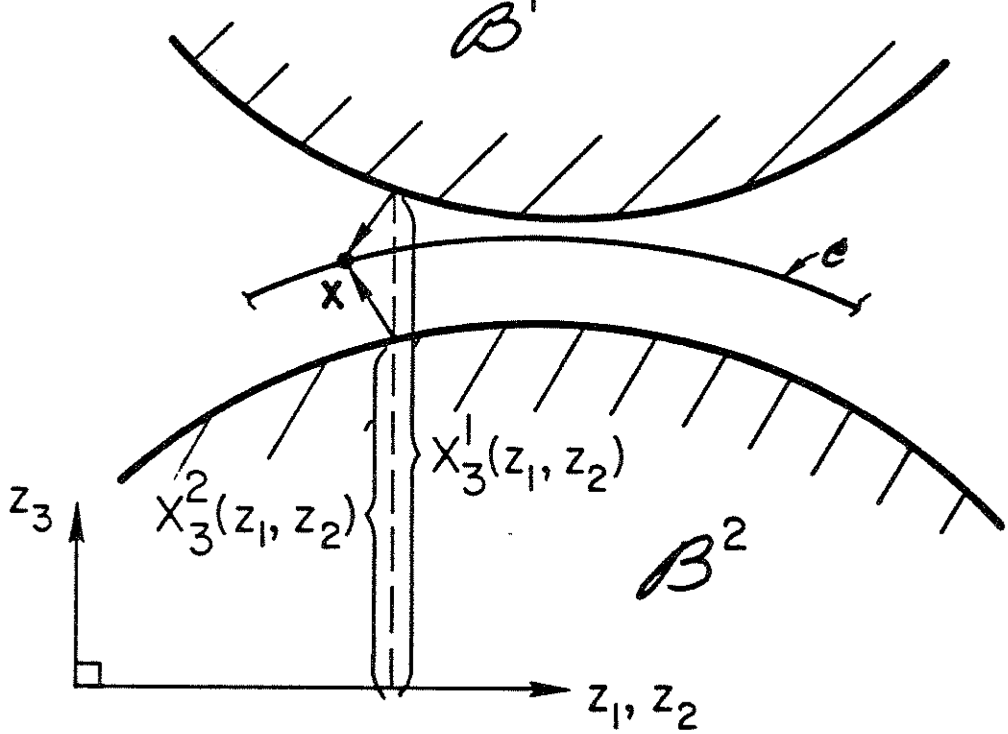


Figure 6

Goldsmith [3] for an excellent exposition of this work and also many applications of Hertz' theory to impact problems). It should be pointed out that the formulation we are about to give is still considerably more general than those to which Hertz' theory applies.

We now show how these assumptions allow us to make simplifications in the preceding theorems.

Theorems I and II:

Due to assumption (3) the term $\mathcal{K}$ can be replaced by an integral over a region in the z_1, z_2 -plane. This region, say c, is the projection of $\mathcal{C}$ onto the z_1, z_2 -plane, and due to assumption (2) it coincides, to the first order, with the projections of the $\mathcal{C}^\alpha$'s. Thus $\mathcal{K}$ can be written

$$\mathcal{K} = - \sum_{\alpha=1}^{2} \int_0^t \int_c \underset{\sim}{\tau}^\alpha \cdot (\underset{\sim}{x}^\alpha - \underset{\sim}{\chi}^\alpha) \, dc \, dt \; . \tag{40}$$

Since, for Case I, we know that the momentum balance on $\mathcal{C}$ requires that

$$\underset{\sim}{\tau}^1 + \underset{\sim}{\tau}^2 = \underset{\sim}{0} \; ,$$

we may make use of this relation immediately. Thus define

$$\underset{\sim}{\tau} = \underset{\sim}{\tau}^1 = -\underset{\sim}{\tau}^2 \; ,$$

and substitute into (40). Employing (39), the integrand simplifies to

$$\underset{\sim}{\tau} \cdot (\underset{\sim}{x}^2 - \underset{\sim}{x}^1) \, . \tag{41}$$

The analog of (23) becomes

$$0 = \int_0^t \int_c \Big\{ \delta \underset{\sim}{\tau} \cdot (\underset{\sim}{x}^2 - \underset{\sim}{x}^1) + \\ + \delta \underset{\sim}{x}^1 \cdot (\underset{\sim}{T}^1 - \underset{\sim}{\tau}) + \delta \underset{\sim}{x}^2 \cdot (\underset{\sim}{T}^2 + \underset{\sim}{\tau}) \Big\} \, dc \, dt \\ + \text{transversality condition}. \tag{42}$$

Thus the same conclusions of Theorem I can be drawn. However, from a numerical standpoint things are considerably different. First of all, since the $\underset{\sim}{\lambda}^\alpha$'s are absent in this formulation, we do not get a uniquely defined $\underset{\sim}{\mathfrak{b}}$; $\underset{\sim}{x}^1$ and $\underset{\sim}{x}^2$ will not in general be the same pointwise. If the graph of $\underset{\sim}{\mathfrak{b}}$ is important it could be constructed by averaging $\underset{\sim}{x}^1$ and $\underset{\sim}{x}^2$, which, if the solution is any good, should be reasonably close pointwise. On the other hand, the $\underset{\sim}{\lambda}^\alpha$'s being absent engenders a considerable saving in the number of equations to be solved and in their complexity.

The analogous case for Theorem II is constructed simply by setting

$$\tau_1 = \tau_2 = 0 \quad , \qquad \tau \overset{\text{def.}}{=} \tau_3 \ .$$

Then the integrand of $\mathcal{K}$ becomes

$$\tau\,(x_3^2 - x_3^1) \tag{43}$$

and (42) reduces to

$$0 = \int_0^t \int_c \Big\{ \delta\tau(x_3^2 - x_3^1) + \\ + \delta x_3^1 (T_3^1 - \tau) + \delta x_3^2 (T_3^2 + \tau) \\ + \delta x_\alpha^1 T_\alpha^1 \qquad + \delta x_\alpha^2 T_\alpha^2 \Big\} \, dc \, dt \\ + \text{transversality condition}. \tag{44}$$

Hence the conclusions of Theorem II hold.

Thus in the case of Hertzian contact we can add the simplifications manifested in (41) and (43) to the conditions of Theorems I and II, respectively, and still garner the same conclusions.

Theorems I' and II':

For these cases $\mathcal{M}$ can be written as an integral over $\tilde{C}$, the projection of $\mathcal{E}$:

$$\mathcal{M} = -\sum_{\alpha=1}^{2} \int_{0}^{t} \int_{\tilde{C}} \underset{\sim}{\sigma}^{\alpha} \cdot n^{\alpha} (\underset{\sim}{x}^{\alpha} - \underset{\sim}{\chi}^{\alpha})\, d\tilde{c}\, dt .$$

Due to the present geometric situation, it is appropriate to take

$$n^{1} = n^{2} ,$$

and thus define

$$n = n^{\alpha} , \qquad \alpha = 1,2 .$$

Analagous to the considerations for Theorems I and II, the momentum balance across $\mathcal{E}$ motivates the simplification

$$\underset{\sim}{\sigma} \overset{\text{def.}}{=} \underset{\sim}{\sigma}^{1} = -\underset{\sim}{\sigma}^{2} .$$

With these and (39), the integrand of $\mathcal{M}$ can be written

$$\underset{\sim}{\sigma} \cdot n (\underset{\sim}{x}^{2} - \underset{\sim}{x}^{1}) .$$

A further simplification can be made by setting*

$$\sigma_{3} = -n .$$

This eliminates one unknown function and, as we shall see, has the effect of satisfying (5) naturally. Thus the integrand of $\mathcal{M}$ becomes

---

* This is a standard ploy of optimization theory, see p. 82, [4].

$$\sigma_\alpha \eta (x_\alpha^2 - x_\alpha^1) - (\eta)^2 (x_3^2 - x_3^1) \,, \tag{45}$$

and the analog of (23) is

$$\begin{aligned} 0 = \int_0^t \int_{\tilde{c}} \Big\{ & \delta\eta\big(\sigma_\alpha(x_\alpha^2 - x_\alpha^1) - 2\eta(x_3^2 - x_3^1)\big) \\ & + \delta\sigma_\alpha\big(\eta(x_\alpha^2 - x_\alpha^1)\big) \\ & + \delta x_\alpha^1(T_\alpha^1 - \eta\sigma_\alpha) + \delta x_\alpha^2(T_\alpha^2 + \eta\sigma_\alpha) \\ & + \delta x_3^1(T_3^1 + (\eta)^2) + \delta x_3^2(T_3^2 - (\eta)^2)\Big\} d\tilde{c}\, dt . \end{aligned} \tag{46}$$

Summand two tells us that either $\eta = 0$ or $x_\alpha^1 = x_\alpha^2$, on $\tilde{c}$. Suppose $\eta \neq 0$, then $x_\alpha^1 = x_\alpha^2, \alpha = 1,2$. Summand one then gives us that $x_3^1 = x_3^2$ on $\tilde{c}$. Thus we have

$$\eta(\underset{\sim}{x}^2 - \underset{\sim}{x}^1) = \underset{\sim}{0} \,, \quad \text{on} \quad \tilde{c} \,,$$

as required, and c is defined as the subset of $\tilde{c}$ where $\underset{\sim}{x}^1 = \underset{\sim}{x}^2$.

The last four summands give the momentum balance conditions, as usual, and, in addition, the last two summands imply that the normal tractions are compressive (since $(\eta)^2 \geq 0$). Thus we have the conclusions of Theorem I' and condition (5).

The analogous set up for Theorem II' is accomplished by setting $\sigma_\alpha = 0$ in (45) yielding

$$-(\eta)^2 (x_3^2 - x_3^1) \tag{47}$$

for the integrand of $\mathcal{M}$. With this Eq. (46) becomes

$$0 = \int_0^t \int_{\bar{C}} \{ -2\,\delta\eta(\eta(x_s^2 - x_s^1)) + \delta x_\alpha^1 T_\alpha^1 + \delta x_\alpha^2 T_\alpha^2 + \delta x_s^1 (T_s^1 + (\eta)^2) + \delta x_s^2 (T_s^2 - (\eta)^2) \} \, d\bar{\tau}\, dt .$$

In this case we achieve the conclusion of Theorem II' and condition (5).

Thus to Theorems I' and II' we can delete condition (5), add the simplifications manifested in (45) and (47), and achieve the conclusions of Theorems I' and II', respectively, plus condition (5).

6. Contact Problems for One, Two and Three-dimensional Bodies

The previous work needs only trivial modification to be made applicable to contact problems involving bodies of different dimensions. There are many cases of considerable interest which fall into this category. For example, models consisting of a shell and a plate, or a solid and a plate, are useful for the study of head impact. The modifications necessary are essentially interpretative. An example illustrates this assertion.

Consider the frictionless Hertzian contact of a three-dimensional solid and a two-dimensional plate. Let $\mathcal{B}^1$ represent the solid and $\mathcal{B}^2$ the plate. In evaluating Π, the $\mathcal{B}^1$ part is as before while the $\mathcal{B}^2$ part would manifest the particular plate theory used. The contact term $\mathcal{K}$ (or $\mathcal{M}$) would be exactly as before. However note that c (or $\bar{c}$) is, in this case, also identifiable with part of the two-dimensional "volume" of the plate, rather than its boundary. Taking variations, everything is as before except that the term $\tau\,\delta x_3^2$ (or $-(\eta)^2\,\delta x_3^2$) contributes to the transverse momentum equation of the plate, rather than to its boundary conditions. The interpretation of τ (or $-(\eta)^2$) is thus two-fold, i.e., it is the normal component of the traction vector with respect to $\mathcal{B}^1$, as before, and it is also the equivalent normal "body force" with respect to $\mathcal{B}^2$, manifested by the interaction with $\mathcal{B}^1$ (Fig. 7).

This interpretation is general, namely, for one and two-dimensional bodies the contact force is an equivalent "body force" which contributes to the momentum equations, rather than the boundary conditions. With this interpretation in mind, the construction of variational theorems, analogous to the ones constructed in Sections 2, 4 and 5, for the class of one, two and three-dimensional contact problems, is just a formal deductive exercise involving only appropriate definitions for Π.

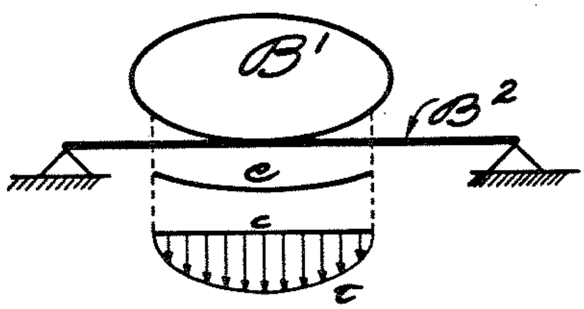


Figure 7

7. Impact

The previous sections deal with spatial aspects of contact problems. In this section we investigate temporal considerations, i.e., those phenomena which are unique to dynamic contact or impact. To manifest the problem encountered in such situations consider the following hypothetical situation. Assume that we are in the process of numerically solving some impact problem and suppose that it is discovered as we monitor the motion of the bodies that they impact somewhere in the time interval (t_1,t_2). At time t_1 we know the states of both bodies and we know that somewhere between t_1 and t_2 they have coalesced over a portion of their boundaries. Assume for the moment we know the geometry of the contact surface $\mathfrak{C}$. The question which arises then is what is the state of $\mathfrak{C}$ at time t_2, i.e., what are the velocity and traction vectors on $\mathfrak{C}$? It is necessary to know this information to carry forth the step forward time integration. The question though seems improperly posed without specifying considerable data about the nature of the impact. To get a handle on things, we will initially formulate a simple one-dimensional problem involving the impact of two elastic rods. Although this problem is trivial, it provides considerable insight into the general nature of impact of continuum bodies. Since we are interested in the state of $\mathfrak{C}$ (in this case a point) immediately after impact, whether the rods are finite or semi-infinite is immaterial.

Assume that the pre-impact states of the two bodies are given by the following data:

$$\mathcal{B}^{\alpha} : \quad v_-^{\alpha} , \quad (\partial x/\partial X)_-^{\alpha} , \quad P_-^{\alpha} \; ; \; \alpha = 1,2 . \tag{48}$$

At impact the rods coalesce at $\mathfrak{t}$, and for some finite time interval thereafter (at least) $x \in \mathfrak{t}$ is persistent. At the moment of impact shock waves begin to propagate in each body. The space-time picture is depicted in Fig. 8. As discussed in section 1, since $\mathfrak{t}$ is material and x is persistent, we have

$$v \overset{\text{def.}}{=} v_+^1 = v_+^2 \quad , \qquad P \overset{\text{def.}}{=} T_+^1 = -T_+^2 \quad , \tag{49}*$$

for the post-impact state (t_+). In addition to (49), the well known shock conditions must hold across the wave fronts:

$$\begin{aligned} [v^\alpha] + U^\alpha[(\partial x/\partial X)^\alpha] &= 0 \,, \\ \rho_0^\alpha U^\alpha [v^\alpha] &= [P^\alpha] \,, \end{aligned} \tag{50}$$

where U^α is the material velocity of the shock in $\mathcal{B}^\alpha$, and [ ] is the wave-front jump operator which assigns to a function the difference in its values behind and in front of the wave, i.e., $[f(X,t)] = f(X^-,t) - f(X^+,t)$ where X is a material point denoting the location of the wave-front. As can be deduced from Fig. 8, the states into which the shocks initially propagate are the pre-impact states given by (48), and the state at $\mathfrak{t}$, immediately after the shocks pass, is given by the post-impact state (49). These observations in conjunction with (50) yield,

$$\begin{aligned} v_-^\alpha - v + U^\alpha\{(\partial x/\partial X)_-^\alpha - (\partial x/\partial X)_+^\alpha\} &= 0 \,, \\ \rho_0^\alpha U^\alpha (v_-^\alpha - v) + P_-^\alpha - P &= 0 . \end{aligned} \tag{51}**$$

*For convenience we choose the initial state to be the pre-impact state, thus we need not distinguish between Cauchy and Piola tractions.

**A consistency condition for these equations is that $v_-^1 - v_-^2 > 0$. Otherwise the impact would not occur.

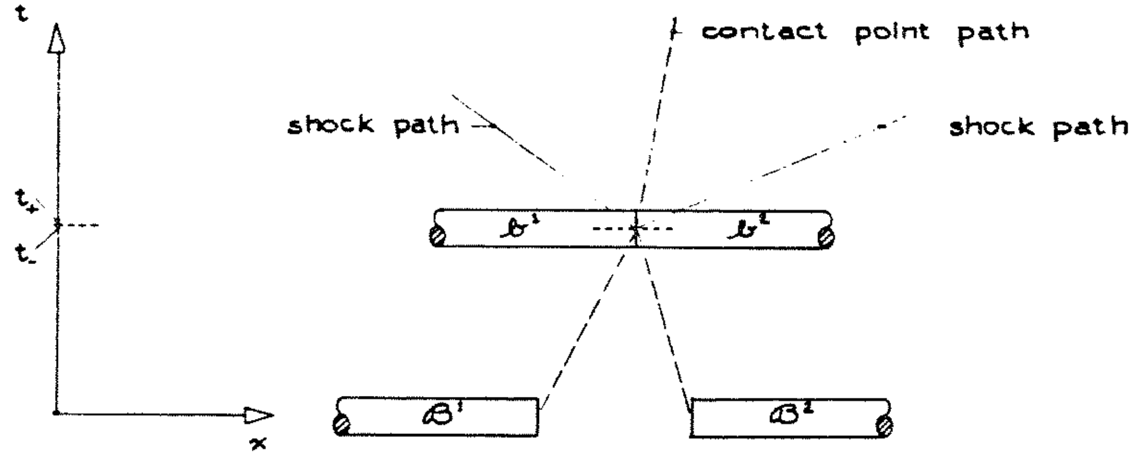


Figure 8

The four Eqs. (51) and constitutive equations relating P^α to $(\partial x/\partial X)^\alpha$ yield a formally deterministic system of six equations in the six unknowns $v, P, U^\alpha, (\partial x/\partial X)^\alpha$. Thus we see that the desired quantities v and P depend on the pre-impact states and material properties of both $\mathcal{B}^\alpha$. The precise form of this relationship depends upon the constitutive equations of the bodies. As a simple example, assume we have linear constitutive equations $P^\alpha = E^\alpha\{(\partial x/\partial X)^\alpha - 1\}$, E^α constant, and let the pre-impact state be given by

$$\begin{aligned} v_-^\alpha &= V^\alpha ,\\ (\partial x/\partial X)_-^\alpha &= 1 ,\\ P_-^\alpha &= 0 . \end{aligned} \tag{52}$$

These conditions, when inserted in Eqs. (51), lead to:

$$\begin{aligned} v &= \frac{\rho_0^2 U^2 V^2 - \rho_0^1 U^1 V^1}{\rho_0^2 U^2 - \rho_0^1 U^1} ,\\ P &= \frac{V^2 - V^1}{\left(\frac{U^2}{E^2} - \frac{U^1}{E^1}\right)} ,\\ (U^\alpha)^2 &= E^\alpha/\rho_0^\alpha . \end{aligned} \tag{53}$$

Note that the denominators in Eqs. $(53)_{1,2}$ present no problems since $\rho_0^\alpha > 0$, $E^\alpha > 0$ and $+1 = \operatorname{sgn} U^2 = -\operatorname{sgn} U^1$.
This result is also appropriate whenever the intensity of the impact is small enough such that the non-linear constitutive equation can be replaced by its linear approximation about the pre-impact state. In this case E^α is a tangent modulus evaluated at the pre-impact strain $\{(\partial x/\partial X)_-^\alpha - 1\} = 0$. To further simplify, consider the case when both rods have identical properties (i.e., $\rho_0 = \rho_0^\alpha$, $E = E^\alpha$, $\alpha = 1,2$). Then

$$v = \frac{V^1 + V^2}{2} \quad ,$$

$$P = \rho_o U (V^2 - V^1)/2 \quad ,$$

$$(U)^2 = E/\rho_o \, . \tag{54}$$

In Eqs. (54) U is positive, and since consistency requires $V^1 - V^2 > 0$, P is compressive.

Thus for the one-dimensional case at least the problem of computing the post-impact state is easily achieved. The solution of (51) for the fully non-linear case can be automated as part of a numerical algorithm. Although this problem is trivial, it serves to indicate that the post-impact problem, the solution of which is essential in a numerical algorithm, is one of wave propagation.

In the analysis of higher dimensional bodies the solution of the post-impact problem becomes greatly complicated due to the geometric variety of impact conditions. However, considerable simplifications can be taken advantage of if one keeps in mind the nature of the discrete problem. For instance, if a certain portion of the boundaries of two bodies have coalesced in e, each interior point of e, at which the tangent plane is well defined, may be treated, to the first order, as a point on the mating surface of two impacting half-spaces. As long as time steps are kept small enough, the local behavior is well represented. The post-impact problem for the general case, analogous to (51), can be automated as part of the numerical algorithm, and for many simple cases can be solved explicitly.

With these notions in mind, let us return to the case of main interest in this report, namely three-dimensional continuum bodies. We shall consider only the case of a frictionless contact surface (Case II), and leave the solution of the post-impact problem for the no-slip case (Case I), which is more difficult, for future work. With the proper interpretations, the one-dimensional rod formulation (Eqs. (48-54)) suffices to completely characterize this case. This is so because no tangential motions or stresses may be communicated across a frictionless surface, and thus we need only consider the configuration of normal incidence. In this case the requisite constitutive functionals in (51) would be those relating P^{α}, the normal Piola stress, to the normal component of strain, holding all other components of strain fixed at the pre-impact values. For example, in the linear isotropic case, E^{α} (Young's modulus) in Eqs. (53,54) would be replaced by $\lambda^{\alpha}+2\mu^{\alpha}$ ($\lambda^{\alpha}, \mu^{\alpha}$ are the Lamé and shear moduli, respectively) and the propagation velocity would be that of dilatational waves.

PART II

A NUMERICAL SCHEME FOR ANALYSIS OF CONTACT-IMPACT PROBLEMS

8. Numerical Solution of Contact-Impact Problems

In performing numerical computations based on the above described variational formulation for contact-impact problems we have employed three distinct levels of approximation: (1) a spatial discretization of the bodies and contact surfaces, (2) a temporal discretization to determine the response of the discretized bodies, and (3) a numerical solution for the resulting system of nonlinear algebraic equations.

In the following sections we shall restrict our attention to the Hertzian contact problem described in Section 5. Significant numerical difficulties are encountered in the solution of impact problems; to complicate the problem further by introducing the additional steps necessary to determine the contact surface maps for the full kinematically nonlinear case is left for a future study. While this is a simple impact problem in terms of determining the contact surface and the full power of the preceding theory is neither necessary nor exploited in its solution, many of the features of the general problem are employed here.

9. Spatial Discretization of the Bodies and Contact Surface

The bodies $\mathcal{B}^1$ and $\mathcal{B}^2$ are discretized using standard finite element methods, (e.g., see [5]). In order to facilitate the computation of a discrete Hertzian contact surface the nodes of $\mathcal{B}^1$ are arranged so that they align with the nodes of $\mathcal{B}^2$. This is consistent with the notions of condition 3 of Section 5 and ensures that during determination of the approximation to the contact surface contiguous nodes of the two bodies will meet. Thus, the simulation of the contact surface is trivial. The development of a numerical model for Hertzian contact problems is based upon the form of Theorem II' which uses (47) for the integrand of $\mathcal{M}$. For numerical computations we introduce the displacement vector $\underset{\sim}{u}$ such that

$$\underset{\sim}{x} = \underset{\sim}{X} + \underset{\sim}{u} \quad . \tag{55}$$

For a compatible finite element displacement field the integrand of $\mathcal{M}$ can be approximated by taking $\eta^2(\underset{\sim}{z},t)$ as the product of $\epsilon^2(t)$ and $\delta(\underset{\sim}{z}-\underset{\sim}{z}_i)$ (i.e., Dirac delta functions in space). This corresponds to taking η^2 as "concentrated nodal loads" which are the generalized forces of the contact pressure. With this discretization we can describe pseudo contact elements between each pair of candidate contact nodes. Let these nodes be denoted as $(\ )_i^\alpha$ and the generalized force as $(\epsilon_i)^2$; then

$$\mathcal{M} = \int_0^t \sum_i (\epsilon_i(t))^2 \left(u^2_{3i}(t) - u^1_{3i}(t) + X^2_{3i} - X^1_{3i} \right) dt \tag{56}$$

where $\{i\}$ are the set of candidate contact nodes which span $\mathcal{E}$; u^α_{3i} are the nodal displacements in the z_3 direction and X^α_{3i} are the nodal coordinates of the candidate contact nodes.*

*We assume here that 3 is the direction nominally normal to the contact surfaces, e.g., see Fig. 5.

Use of the finite element method in Theorem II' with $\mathcal{M}$ given by (56) produces a set of nonlinear second order ordinary differential equations which together with the impenetrability conditions define the discretized contact impact problem. These equations take the form:

$$\underset{\sim}{M}\,\ddot{\underset{\sim}{u}} + \underset{\sim}{K}(\underset{\sim}{u}) = \underset{\sim}{R} \quad , \tag{57}$$

where $\underset{\sim}{M}$ is the usual finite element mass matrix, $\underset{\sim}{K}$ represents the elastic stiffness forces together with the contact terms, $\underset{\sim}{R}$ is the set of generalized forces resulting from boundary tractions and $\underset{\sim}{u}$ is the set of time dependent nodal displacements (which also include the $(\varepsilon_i)^2$). For inelastic materials Theorem II' can be extended by treating the first variation as a Galerkin method (principle of virtual work) and replacing the elastic constitutive model by more general theories, e.g., viscoelastic, elasto-plastic, viscoplastic, etc. In this case

$$\underset{\sim}{K}(\underset{\sim}{u}) \longrightarrow \underset{\sim}{K}(\underset{\sim}{u}, \dot{\underset{\sim}{u}}) \tag{58}$$

in (57).

10. Temporal Discretization

A temporal discretization of the second order ordinary differential equations which result from a finite element spatial discretization of the contact-impact problem is accomplished herein by using the Newmark family of methods [6]. The Newmark family of methods is a one-step integration method with two free parameters which can be used to control stability and numerical damping. The method is essentially a difference method in time. The behavior of the method for linear elasto-dynamics problems is discussed in [6,7]. The algorithm is given by

$$\underset{\sim}{u}_{n+1} = \underset{\sim}{u}_n + \Delta t\, \dot{\underset{\sim}{u}}_n + (\tfrac{1}{2} - \beta)\Delta t^2 \ddot{\underset{\sim}{u}}_n + \beta \Delta t^2 \ddot{\underset{\sim}{u}}_{n+1} \quad , \tag{59}$$

and $\dot{\underset{\sim}{u}}_{n+1} = \dot{\underset{\sim}{u}}_n + (1-\gamma)\Delta t\, \ddot{\underset{\sim}{u}}_n + \gamma \Delta t\, \ddot{\underset{\sim}{u}}_{n+1}$,

where $\underset{\sim}{u}_n = \underset{\sim}{u}(t_n)$, $\Delta t = t_{n+1} - t_n$, and β, γ are the two parameters. For linear problems $\gamma = .5 + \delta = .5$ produces no artificial viscosity and $\beta \geq \frac{1}{4}(1+\delta)^2$ produces unconditional stability (i.e., the method is stiffly stable). Such generalization is not possible for nonlinear problems and during solution it may be necessary to monitor the solution for any signs of instability. In (59) $\beta = 0$ produces an explicit method for $\underset{\sim}{u}_{n+1}$ and if $\underset{\sim}{M}$ is diagonal (lumped mass) with $\underset{\sim}{K}$ and $\underset{\sim}{R}$ independent of $\dot{\underset{\sim}{u}}$ the solution can be advanced without solving a large set of simultaneous equations; for all other cases the method is implicit and equations must be solved. Solution of $(59)_1$ for $\ddot{\underset{\sim}{u}}_{n+1}$ in terms of the solution at t_n and $\underset{\sim}{u}_{n+1}$ gives

$$\ddot{\underset{\sim}{u}}_{n+1} = \frac{1}{\beta \Delta t^2}(\underset{\sim}{u}_{n+1} - \underset{\sim}{u}_n) - \frac{1}{\beta \Delta t}\dot{\underset{\sim}{u}}_n - \left(\frac{1-2\beta}{2\beta}\right)\ddot{\underset{\sim}{u}}_n \tag{60}$$

which can also be used in $(59)_2$ to express the velocity in terms of the solution at t_n and $\underset{\sim}{u}_{n+1}$. Since in this process we divide by β and Δt it is no longer possible to consider zero β or zero time steps.

11. Solution of the Nonlinear Algebraic Problem

Use of the Newmark method in (57) (including (58)) yields the set of nonlinear algebraic equations:

$$\frac{1}{\beta\Delta t^2} \underset{\sim}{M}\, \underset{\sim}{u}_{n+1} + \underset{\sim}{K}(\underset{\sim}{u}_{n+1}, \underset{\sim}{u}_n, \dot{\underset{\sim}{u}}_n, \ddot{\underset{\sim}{u}}_n) = \underset{\sim}{R}_{n+1} + \underset{\sim}{M}\, \underset{\sim}{A}_n \quad , \tag{61}$$

where

$$\underset{\sim}{A}_n = \frac{1}{\beta\Delta t^2} \underset{\sim}{u}_n + \frac{1}{\beta\Delta t} \dot{\underset{\sim}{u}}_n + \left(\frac{1-2\beta}{2\beta}\right) \ddot{\underset{\sim}{u}}_n \quad .$$

A Newton-Raphson iterative solution to this set of equations can formally be constructed, giving:

$$\left(\frac{1}{\beta\Delta t^2} \underset{\sim}{M} + \partial_u \underset{\sim}{K} - \partial_u \underset{\sim}{R}\right) \Delta \underset{\sim}{u}^{(i)} = \underset{\sim}{R} - \underset{\sim}{K}(\underset{\sim}{u}_{n+1}^{(i)}, \underset{\sim}{u}_n, \dot{\underset{\sim}{u}}_n, \ddot{\underset{\sim}{u}}_n) - \underset{\sim}{M}\, \ddot{\underset{\sim}{u}}_{n+1}^{(i)} \quad , \tag{62}$$

where $\partial_u \underset{\sim}{R}$ is the effect of loads varying with the deformation and

$$(\partial_u \underset{\sim}{K})_{ij} = \partial K_i / \partial u_j \quad , \tag{63}$$

is the tangent stiffness matrix. The coefficient to $\Delta \underset{\sim}{u}^{(i)}$ is generally called the Jacobian matrix of the Newton-Raphson iteration. The solution is advanced by taking

$$\underset{\sim}{u}_{n+1}^{(i+1)} = \underset{\sim}{u}_{n+1}^{(i)} + \Delta \underset{\sim}{u}^{(i)} , \tag{64}$$

and iterating until a norm of the solution satisfies

$$\| \Delta \underset{\sim}{u}^{(i)} \| \leq \epsilon \, \| \underset{\sim}{u}_{n+1}^{(i)} \| \quad , \tag{65}$$

where ϵ is some small positive error tolerance. In the work reported here the norm $\| \ \|$ is taken as the Euclidian norm

$$\| \underset{\sim}{x} \| = \left(\sum_i x_i^2 \right)^{1/2} \quad , \tag{66}$$

and the load vector $\underset{\sim}{R}$ is assumed to be independent of $\underset{\sim}{u}$. For stable elastic materials the resulting tangent stiffness is then symmetric and positive definite, consequently, standard direct solution methods normally employed in the solution of linear finite element problems can be used. For inelastic materials or deformation dependent loads the tangent stiffness may be asymmetric. In these cases some special methods may be necessary to effect a solution.

12. Discretized Impact Conditions

In the previous numerical development $\tilde{c}$ has been defined by discrete points which correspond to nodes along the boundaries of $\mathcal{B}^1$ and $\mathcal{B}^2$. When, during the course of advancing the solution in time, any one of these points violates the impenetrability condition a re-solution must be obtained in which the $(\mathcal{E}_i)^2$ are now non-zero and the u_3^* satisfy the impenetrability condition. Some control and monitoring are required to effect this in a computer program. In addition to satisfying these conditions, the impact relations denoted in Section 7 must be invoked. In the present study these conditions are applied to the solution at the end of a time step in which points first go into contact. Accordingly we compute from (50)*

$$\dot{u}_+ = \frac{\rho_o^2 U^2 \dot{u}_-^2 - \rho_o^1 U^1 \dot{u}_-^1}{\rho_o^2 U^2 - \rho_o^1 U^1} , \tag{67}$$

and assign this value to the appropriate node of $\mathcal{B}^1$ and $\mathcal{B}^2$.

To determine the solution vector $\underset{\sim}{u}$ at t_{n+1} we have solved the set of equations (61). As described above the shock conditions are then used to determine the value of the velocity at time t_{n+1} for all points which have come into contact during the time interval. In order to get a consistent solution at these points we must modify the accelerations and contact force to reflect the shock conditions. This is accomplished by re-solving the equilibrium conditions of $\mathcal{B}^1$ and $\mathcal{B}^2$ at point i. The expanded forms of the appropriate equations are:

---

*The $(\ )_-$ denotes a value which is computed before impact, whereas $(\ )_+$ denotes the value after impact.

$$M^1\ddot{u}^1_- + K^1(\underset{\sim}{u}) + (\varepsilon_i)^2_- = R^1 \, , \tag{68}$$

and

$$M^2\ddot{u}^2_- + K^2(\underset{\sim}{u}) + (\varepsilon_i)^2_- = R^2 \, .$$

For nodes which have come into contact we must enforce the condition on acceleration

$$\ddot{u}^1_+ = \ddot{u}^2_+ = \ddot{u}_+ \, , \tag{69}$$

and compute the contact force $(\varepsilon_i)^2_+$. The solution for these is obtained from

$$M^1\ddot{u}_+ + K^1(\underset{\sim}{u}) + (\varepsilon_i)^2_+ = R^1 \, ,$$

and

$$M^2\ddot{u}_+ + K^2(\underset{\sim}{u}) + (\varepsilon_i)^2_+ = R^2 \, .$$

These are two equations in two unknowns which can be solved for the $\ddot{u}_+$ and $(\varepsilon_i)^2_+$. If $K^\alpha(\underset{\sim}{u})$ is independent of velocity the stiffness forces and R^α will remain unchanged during the impact, hence we can solve the simpler problem

$$M^1\ddot{u}_+ + (\varepsilon_i)^2_+ = M^1\ddot{u}^1_- + (\varepsilon_i)^2_-$$

$$M^2\ddot{u}_+ - (\varepsilon_i)^2_- = M^2\ddot{u}^2_- - (\varepsilon_i)^2_-$$

whose solution is

$$\ddot{u}_+ = \frac{M^1\ddot{u}^1_- + M^2\ddot{u}^2_-}{M^1 + M^2} \, ,$$

and

$$2(\varepsilon_i)^2_+ = 2(\varepsilon_i)^2_- + M^1(\ddot{u}^1_- - \ddot{u}_+) - M^2(\ddot{u}^2_- - \ddot{u}_+) \, .$$

This completes the numerical specification of the solution at t_{n+1}; this solution process is now repeated for each of the succeeding time steps.

At this point it is important to compare the solution procedure for impact of a continuum discretized by a finite element method with the solution procedure for a physically discrete body, i.e., a body composed of mass points joined by massless elastic springs. Both problems may be described by algebraic equations of the form of (57). The impenetrability condition is also identical. The impact conditions, however, are different. For the discretized continuum the procedure is described above. The study of the impact of mass points is considered in elementary mechanics books, e.g. [8]. The impact of two mass points is described by impulsive motion such that at t_- the velocities of the two mass points are V_-^1 and V_-^2; after impact at time t_+, the two points have velocities V_+^1 and V_+^2. The two points will not in general stay in contact (i.e., $V_+^1 \neq V_+^2$) but will rebound. The conditions used to compute the V_+^1 and V_+^2 are: Balance of Momentum*

$$M^1 \{V^1\} + M^2 \{V^2\} = 0 , \tag{71}$$

and use of an equation involving the "coefficient of restitution", e :

$$\frac{V_+^2 - V_+^1}{V_-^1 - V_-^2} = e . \tag{72}$$

For $e=1$ energy is conserved whereas for $e=0$ the points "stick" and energy is dissipated. We must comment in passing that (72) is the energy

---

* $\{f(t)\} = f(t_+) - f(t_-)$.

equation in disguise. To see this we can write the jump conditions for energy as

$$\frac{1}{2} M^1 \{(V^1)^2\} + \frac{1}{2} M^2 \{(V^2)^2\} = \{\nu\} \quad . \tag{73}$$

The term $\{\nu\}$ can exist only if other energies are dissipated during the jump. We rewrite (73) by using

$$\frac{1}{2}\{(V^1)^2\} = [V^1]\langle V^1\rangle \quad ,$$

where

$$\langle V^1\rangle = \frac{1}{2}(V_+^1 + V_-^1) \quad . \tag{74}$$

Use of the momentum equation (71) then gives, after dividing by $M^1\{V^1\}$

$$\langle V^1\rangle - \langle V^2\rangle = \frac{\{\nu\}}{M^1\{V^1\}} \quad ,$$

or after recollecting terms and dividing by $(V_-^1 - V_-^2)$ we obtain:

$$\frac{V_+^2 - V_+^1}{V_-^1 - V_-^2} = 1 - \frac{\{\nu\}}{M^1\{V^1\}(V_-^1 - V_-^2)} \quad . \tag{75}$$

The significance of the coefficient of restitution then is associated with the right hand side of (75).

It is clear from the above developments that the numerical simulation of the discretized continuum and the physically discrete system involve two distinct methods for treating the impact conditions. It is imperative then to associate the correct method for the problem at hand. In the present study we are interested in the impact of continua, and in this case we shall employ the discrete shock condition to effect the solution. This a priori assumes that the response we are computing involves a time scale associated with wave propagation problems. Consequently, we cannot expect the computation procedure for advancing the solution in time to be accurate if we take time steps greatly in excess of transit times through each body. In this context it may be important to consider an "explicit" time integration procedure in future work. The stability restrictions may be too severe to make this feasible.

PART III

FEAP 74 - A COMPUTER PROGRAM FOR SOLUTION OF CONTACT-IMPACT PROBLEMS

13. Development of a Contact-Impact Model for FEAP

In order to incorporate an ability to compute solutions to contact-impact problems using a finite element method as described above it is necessary to have available a computer program which can solve the nonlinear equations of motion given by (61). The computer program FEAP is a general program to solve finite element problems. The program has a capability of solving both quasistatic and dynamic problems and can incorporate several types of elements simultaneously. The nonlinear capabilities required for the solution of contact-impact problems have been incorporated into FEAP and currently includes the user options (see Input Instructions):

(1) Selection of quasistatic or dynamic option: The dynamic option will integrate the equations of motion using the one-step Newmark method to advance the solution in time. Quasistatic analysis is accommodated by any one-step algorithm. The algorithm employed is incorporated into each element routine and thus id defined by the developer of each element. Impact problems require description of the contact surface and wave speeds.

(2) Selection of the nonlinear method to advance the solution: Options include:

  (a) No iterations in each time step. Unbalanced forces at each time are added to the next time step.

  (b) Iterations in each time step to achieve a balance of force within each time step. In this option the user can select to reform the Jacobian matrix for each iteration or only at the first iteration in each time step.

In the impact problems solved to date it has been necessary to use the general form of the Newton-Raphson algorithm. This includes a complete formating and factoring of the Jacobian matrix for each iteration of each time step in the analysis. If the method described herein is to become computationally effective improvements in the computer program are paramount. Undoubtedly the most important aspect in reducing computer times is to introduce a substructuring system so that the highly nonlinear equations in the vicinity of the contact surface can be isolated from the remainder of the bodies. This will normally involve only a small number of equations in the total system of (62). The solution of a large finite element problem will generally concentrate the computer solution time in the forming and factoring of the tangent stiffness matrix. The fewer times that it is necessary to perform this costly step the more efficient the solution algorithm. Substructuring can be used then to restrict the part of the equations which must be formed and factored often, and thus greatly reduce the computer costs in analyzing impact problems.

The version of FEAP which can currently be used to analyze contact-impact problems includes, in addition to the nonlinear Newton-Raphson iterative algorithm, a new special contact-impact element and a new subroutine to describe impact surfaces and the discrete shock conditions described in Section 12. These are described in the following sections.

14. Contact Element for Hertzian Contact

The contact-impact element which has been developed is called ELMT05 and can be used along any coordinate direction. As developed it cannot be used along normals which are in non-coordinate directions. The development of the contact element assumes that within the framework of linear elasticity theory a node on $\mathcal{B}^1$ will impact on a node of $\mathcal{B}^2$. In using this contact element we shall assume that the contact surface on $\mathcal{B}^2$ is located at larger coordinate values than the contact surface of $\mathcal{B}^1$. The contact element is described by three nodes. Node 1 is associated with $\mathcal{B}^1$, Node 3 is associated with $\mathcal{B}^2$, and Node 2 is used as storage for the contact force $(\mathcal{C})^2$. The user can select the direction of contact motion by specifying the degree of freedom of the nodal unknowns to which the contact is to be measured; this must agree with the physical direction of the element (see Fig. 9). The degree of freedom for the contact element is specified during the MATERIAL data input and consists of a single card in I5 format. The element nodes are described along with all other elements according to Section 4 of the Input Instructions. The Node order as shown in Fig. 9 must be observed.

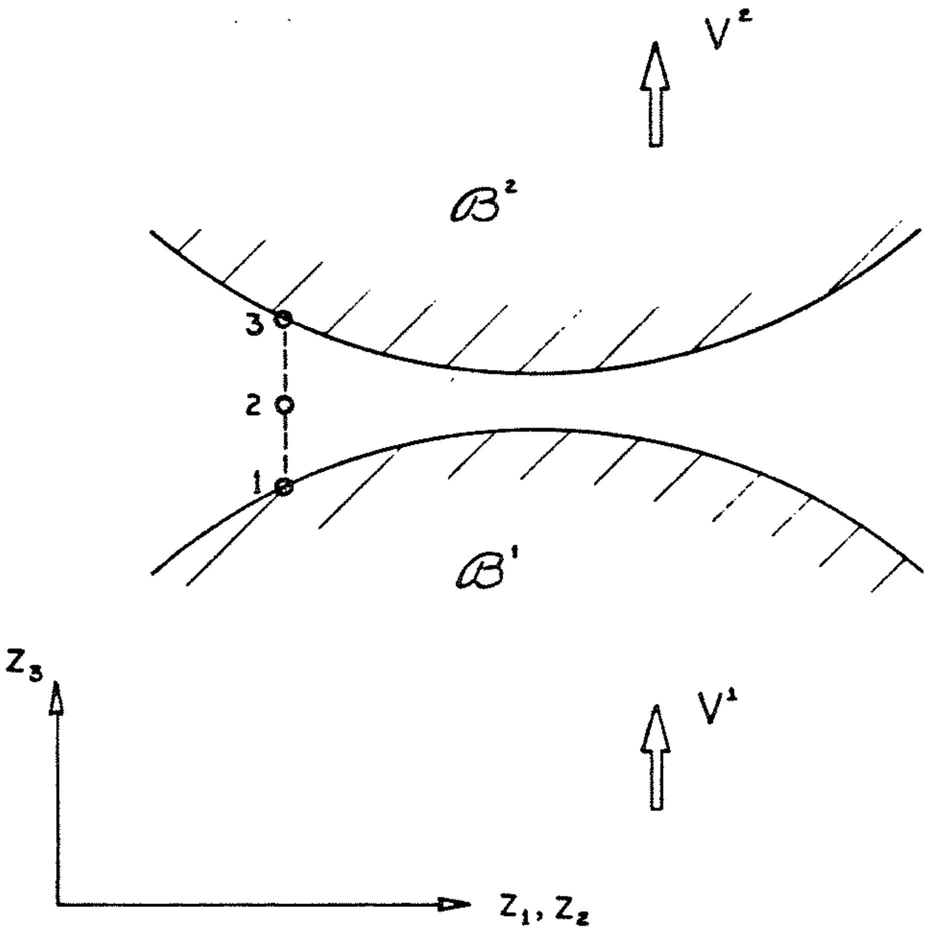


Figure 9

15. Impact Surface Description

The definition of the impact surface includes a list of all elements on the contact surface together with the degree of freedom describing the direction of contact motion (as described above). In addition, the product of mass density and wave velocity (always a positive number) for each body is input. This assumes, currently, that (1) each contact surface belongs to a linear material, and (2) the same material exists along all of the contact surface. This data need be prescribed only for impact problems, quasistatic contact problems do not require this data since no velocity or acceleration computations are performed in this class of problems. Data to be input for the impact surface is given in Table I.

Table I - Impact Surface Data

CARD 1) (6X,A6)		
COL. 7 to 12		Must contain CONTAC
CARD 2) (2F10.0)		
COL. 1 to 10	ρU	of body 1
COL. 11 to 20	ρU	of body 2
CARD 3) (I5)		
COL. 1 to 5		NLIST, number of elements on contact surface
CARD 4) (2I5)		
Repeat NLIST times		
COL. 1 to 5		Contact element number
COL. 6 to 10		Degree of freedom of this contact element

16. Example Problems

Two example problems are included to illustrate the characteristics of the methodology and the associated computer program described above for Hertzian contact problems. The first problem is a quasistatic contact problem which is used to demonstrate the ability of the computer program to compute an evolving contact surface. The second problem will demonstrate the ability of the program to properly model the temporal response of an impact problem.

To model a problem in which a contact surface will change under different load levels we consider two beams with an initial parabolic curvature. A symmetric configuration is analyzed and the resulting finite element model is shown in Fig. 10. Each element is nominally one unit by one unit. The gap at the load end is initially 0.5 units. The material properties used are $E = 500$ and $\nu = 0$. The load P is applied as shown and allowed to increase linearly in time. The problem then is to determine the contact surface at various load levels. In order to eliminate a singularity in the system of equations it was necessary to permanently attach the two nodes at the symmetry axis of the contact surface. All other nodes along the boundaries between the two bodies are assumed to be possible contact points and contact elements are assigned between vertical nodal pairs. The load was varied from 0.2 to 0.8 in increments of 0.1 and the computed contact surface and forces were computed. These are given in Table II. The deformed shape at a load of 0.4 is also shown in Fig. 10 as a dotted form. The attached node at the center has influenced the solution at loads above 0.3 since the contact pressure there is tensile (negative). The force is small and should not greatly affect the actual contact region computed. As the load increases the

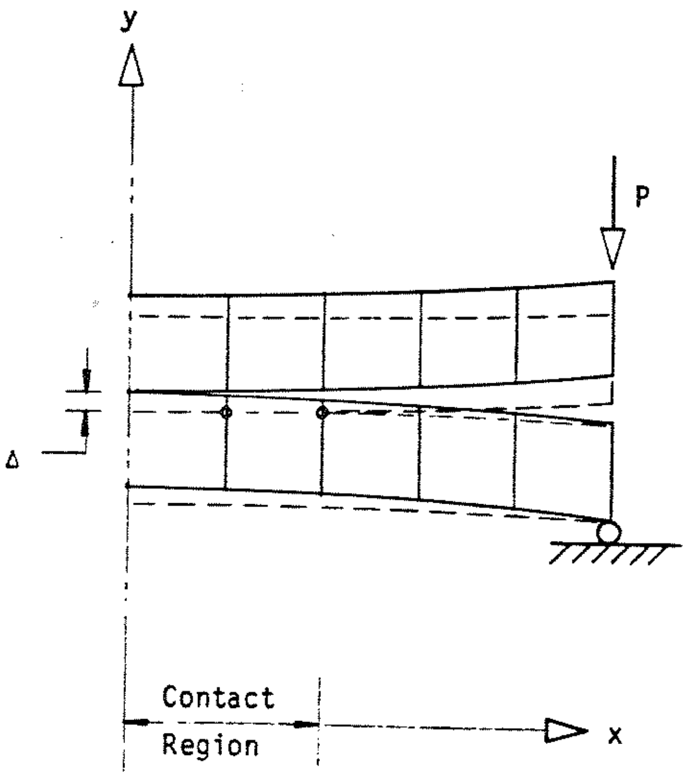


Figure 10

contact surface moves toward the load. This is conceptually correct since if the beams were modeled according to Euler-Bernoulli theory the contact force would be a point load which gradually moves from the center to the outer edge according to the relationship (using the above values for sizes and material properties)

$$X = 5\left(1 - \frac{1}{6P}\right) .$$

This relation predicts that the contact point will be non-zero only after P exceeds 1/6. The finite element model is in qualitative agreement with this beam theory, but since shear deformations are included the finite element solution gives a distributed load on the contact surface. It is interesting to also note that the contact force over the center of the beams is zero, just as in the beam theory.

Table II - Contact Forces

LOAD	X-COORDINATE 0	1	2	3	4	5	BEAM THEORY-X
0.2	0.2	-	-	-	-	-	0.83
0.3	.07	.23	-	-	-	-	2.22
0.4	-.01	.07	.34	-	-	-	2.92
0.5	-0.00	.02	.23	.25	-	-	3.33
0.6	-.01	-	.09	.51	.01	-	3.61
0.7	-.01	-	.07	.45	.19	-	3.81
0.8	-.01	-	.05	.38	.38	-	3.96

This problem demonstrates that the computer program can model the evolution of a contact surface. Of particular importance is to note that as the load increases the program can both attach and detach a contact point. This is an essential requirement for the analysis of impact problems as is shown in the next problem.

As a simple example we consider the impact against a rigid wall of a finite, linear elastic rod traveling at constant velocity. The rod has a modulus of elasticity E of 100, and a mass density ρ of 0.1. The arrival velocity is taken to be 0.1 (units may be assigned in any convenient system). The rod is taken to be 10 units long and is divided into 10 elements plus one contact element as shown in Fig. 11. At time zero the rod is just arriving at the wall. The exact solution predicts a contact duration of 0.2 time units. This corresponds to the time required for a wave to travel from the contact point to the left end and back to the contact point at which time the rod will part from the wall. The problem was analyzed using FEAP with time steps of 0.01 unit (transit time across an element) and the rod remains in contact until time 0.20 units and has rebounded at time 0.21. Thus the program can predict accurately the contact duration of the rod. The finite element solution obtained is compared with the exact solution in Fig. 12. The agreement of stresses and contact force is good. The largest discrepancy exists in defining the shock front, which is "smeared" by the finite element method and ordinary differential equation solution method used here. This is the same type of solutions which are commonly obtained with numerical solutions of this type even without impact. Solutions such as the impact shocks generated are probably one of the most difficult responses to accurately calculate by a finite element method.

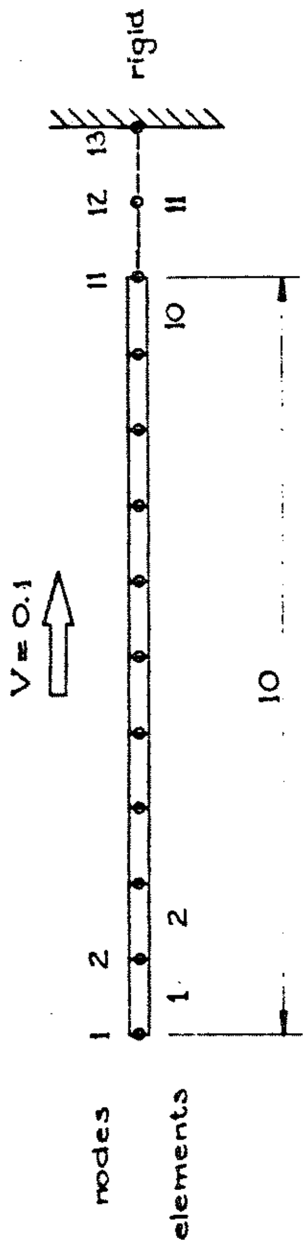
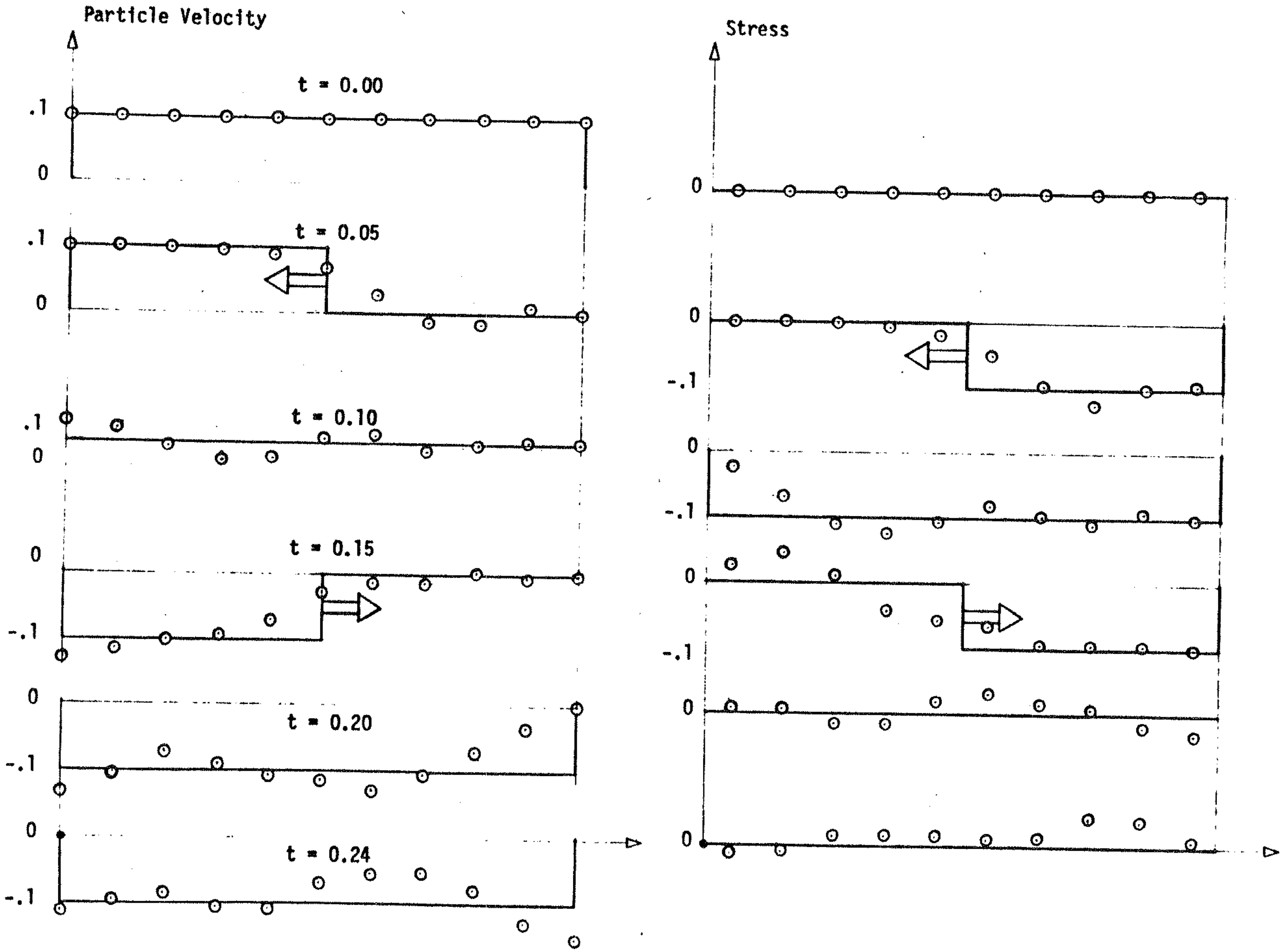


Figure 11

Comparison of Numerical Data with Exact Solution for Bar Impacting Rigid Wall



17. Closure and Recommendations for Future Work

In the preceding sections we have presented a theory for contact-impact problems together with the numerical development of a Hertzian contact-impact model. The computer program FEAP 74 has been modified to include the model and has successfully solved a contact problem and an impact problem. The work reported herein must be considered initiatory; the general theories and their numerical implementation have not been completed. The problem is of such a complicated nature and the literature existing prior to this study was so meager that we consider it fitting to document the work completed thus far.

We have attempted to qualify each stage of the development throughout the report, however, it may be fitting to reiterate future work which we consider to be essential for numerical models to be effective and efficient tools for predictive analyses.

(1) The restriction of Hertzian type contact must be removed. This involves the non-trivial task of finding appropriate numerical methods to handle the $\underset{\sim}{\chi}^{\alpha}$ maps.

(2) Improved methods for solving the set of nonlinear algebraic equations must be found. We have suggested two methods which should be considered: (a) Substructure the problem about the contact regime so that a more efficient forming and factoring of the tangent stiffness can be performed; and (b) Since the impact problem is a wave propagation problem an explicit time integration of the equations of motion should be explored. In complex situations the explicit integration method may have severe stability limitations which could make it unacceptable.

(3) Methods of utilizing the shock conditions need to be explored further. We have noted some peculiar anomalies when the bodies separate. These appear to be caused by a shock like separation phenomena.

(4) When the wave propagation property of the impact problem is ignored by taking time steps greatly in excess of the transit times in a body the computed response is meaningless. Under such situations the bodies rebound within a single time step. Currently the rebound velocity is much too large. When the shock conditions are used for a class of problems where the response desired is in the target instead of in the impactor, it may be expedient to take a large time step. Methods should be explored to accomplish this capability.

The above recommendations for future work should in no way minimize what has been accomplished by the present study. For the first time a contact-impact theory in the form of a variational problem has been presented in a general form. This formulation was motivated by the fact that numerical solutions would be obtained by a finite element method. In addition the necessary foundation for the numerical solution has been thought out and within this context a computer program has been developed for Hertzian contact-impact problems.

The implementations considered here have produced results which are hopeful signs for the eventual success of the more general impact problems.

References

1. S. Nemat-Nasser, On variational methods in finite and incremental elastic deformation problems with discontinuous fields, Quarterly Appl. Math., July 1972, 143.

2. H. Hertz, Über die Beruchrüng fester elastischer Körper, J. reine angew. Math. (Crelle), 92, 1881, 155.

3. W. Goldsmith, Impact, Edward Arnold, London, 1960.

4. Kowalik and Osborne, Methods for Unconstrained Optimization, Am. Elsevier, New York, 1968.

5. O. C. Zienkiewicz, The Finite Element Method in Engineering Science, McGraw-Hill, New York, 1971.

6. N. M. Newmark, A method of computation for structural dynamics, J. Engr. Mech. Division, ASCE, V. 85, No. EM3, July 1959, 67-94.

7. G. L. Goudreau and R. L. Taylor, Evaluation of numerical integration methods in elasto dynamics, Computer Methods in Appl. Mech. and Engng., 2, No. 1, 1972, 67-97.

8. J. L. Meriam, Dynamics, John Wiley & Son, Inc., New York, 1966.

APPENDICES

A. Input Instructions for Contact/Impact Problems

In order to analyze contact/impact problems in FEAP, users must prepare the data for a time dependent analysis. This will include the following Data Type Identification Cards (see Section 1, Appendix B):

FEAP 74
MATERIAL
NODAL
ELEMENT
CONTACT (for impact problems only)
loadings
INITIAL CONDITIONS (if non-zero)

and

VISCOE (for quasi-static contact problems)

or

IMPLICIT (for impact problems)

In performing the necessary solution to (62) the full Newton-Raphson method must be employed; this is controlled by the data in col. 76-80 of the first card following the VISCOE or IMPLICIT card, and consists of a negative number (negative uses full Newton-Raphson iteration with the absolute value of the number giving the number of iterations to be performed before going to the next time step). As an example of the required input data Table A shows the input data used for the impact problem reported in Section 16.

```
14.19.14 OUTPUT
     FEAP74 * * ONE DIMENSIONAL BAR CONTACT PROBLEM
    1        X
    1        U
    3
   13 NODAL
    1                       0.
   11                       10.
   13            111111     10.
   11 ELEMENTS
    1     1                1   1
   1    2
   11     2
  11   12  13
    3 INITIAL CONDITIONS
     DISPLACEMENT
     VELOCITY
     ACCELERATIONS
    1     1                 .1
   11     0                 .1


   2 MATERIALS
   1 ELM09
   100.      1.          .01
   2 ELM05
   1
     CONTACT
   1         1.0         1.0E+30
  11      1

   1 IMPLICIT
   1.0         5     1     1    13     1    11     0     0    0.25                      -3
  .01         25     1     1    13     1    11     0     0    0.25                      -3
     STOP
  .01         10     1     1    13     9    11     0     0    0.25                      -3
```

B. Input Instructions for FEAP 74

The input instructions for the description of a finite element mesh, together with the initial and boundary conditions, is described by subroutine MANUAL listed on the following pages. The input of the contact surface for impact problems is described in Table I of this report.

The description of material properties for the contact element is described in Section 14. For material properties for other elements in FEAP special input instructions must be supplied.

```
      SUBROUTINE MANUAL                                                 MAN  1C
C                                                                       MAN  2C
C**** USER INSTRUCTIONS AND INPUT FORMATS FOR FEAP-74  ***************** MAN  3C
C                                                                       MAN  4C
C                                                                       MAN  5C
C     FINITE ELEMENT ANALYSIS PROGRAM                                   MAN  6C
C                                                                       MAN  7C
C      ****** I N D E X ******                                          MAN  8C
C                                                                       MAN  9C
C  SECTION    WORD      C O N T E N T                                   MAN 10C
C                                                                       MAN 11C
C    1.0                DATA DIRECTORY WORDS                            MAN 12C
C    2.0      FEAP74    PROBLEM INITIATION AND CONTROL CARDS            MAN 13C
C    2.1      REMARK    REMARK CARD                                     MAN 14C
C    2.2      TITLE     TITLE CHANGE                                    MAN 15C
C    2.3      STOP      EXECUTION TERMINATION                           MAN 16C
C    3.0      MATERI    MATERIAL CHARACTERIZATION                       MAN 17C
C    4.0      NODAL     SEQUENTIAL NODAL GENERATION                     MAN 18C
C    4.1      GENERA    NON-SEQUENTIAL NODAL GENERATION                 MAN 19C
C    4.2      BOUNDA    BOUNDARY CODE INPUT                             MAN 20C
C    4.3      POLAR     POLAR TO CARTESIAN CONVERSION                   MAN 21C
C    5.0      ELEMEN    ELEMENT GENERATION                              MAN 22C
C    5.1      BLOCK     BLOCK MESH GENERATION                           MAN 23C
C    6.0      VECTOR    USER VECTOR INPUT                               MAN 24C
C    6.1      INITIA    INITIAL CONDITION                               MAN 25C
C    7.0      FORCE     GENERALIZED NODAL FORCE                         MAN 26C
C    7.1      BLOADS    SURFACE LOADS                                   MAN 27C
C    7.2      ELOADS    ELEMENT LOADS                                   MAN 28C
C    7.3                PROPORTIONAL LOADS                              MAN 29C
C    8.0      SOLVE     INITIATION OF STATIC SOLUTION                   MAN 30C
C             RESOLV    STATIC SOLUTION                                 MAN 31C
C    8.1      EXPLIC    EXPLICIT DYNAMIC INTEGRATION                    MAN 32C
C    8.2      IMPLIC    IMPLICIT INTEGRATION                            MAN 33C
C             VISCOE    IMPLICIT TIME INTEGRATION                       MAN 34C
C    8.3      MESH      MESH CHECK                                      MAN 34C
C             PLOT      PLOT MESH                                       MAN 34C
C    8.4      FOURIE    FOURIER SERIES HARMONICS                        MAN 35C
C    9.0                OUTPUT CONTROL                                  MAN 35C
C                                                                       MAN 36C
C        FEAP74 IS A GENERAL (F)INITE (E)LEMENT (A)NALYSIS (P)ROGRAM    MAN 37C
C        WHICH FURNISHES TO THE USER MESH INPUT/OUTPUT, ELEMENT ASSEMBLY MAN 38C
C        AND SOLUTION OF EQUATIONS (LINEAR, IMPLICIT AND EXPLICIT TIME  MAN 39C
C        DEPENDENT, NONLINEAR), PRESCRIBED GENERALIZED NODAL FORCES,    MAN 40C
C        PRESCRIBED NODAL AND ELEMENT DATA, AND OUTPUT OF THE           MAN 41C
C        GENERALIZED DISPLACEMENTS AND FORCES.  ELEMENT MATRICES FOR TWO MAN 42C
C        AND THREE DIMENSIONAL LINEAR ELASTICITY, SHELLS, PLATES, AND   MAN 43C
C        FIELD (LAPLACE EQUATION) PROBLEMS ARE AVAILABLE.  ALTERNATIVELY MAN 44C
C        USERS MAY SUPPLY THEIR OWN ELEMENT LIBRARY BY PROVIDING A      MAN 45C
C        SUBROUTINE CALLED ELMTNN,  WHERE NN IS A TWO DIGIT NUMBER      MAN 46C
C        (01-10), IDENTIFYING THE ELEMENT SUBROUTINE. EACH ELEMENT      MAN 47C
C        SUBROUTINE HAS AT LEAST FOUR BASIC FUNCTIONS WHICH ARE         MAN 48C
C        DELINEATED BY A SWITCHING PARAMETER, ISW, IN THE SUBROUTINE.   MAN 49C
C                                                                       MAN 50C
```

```
C        ELMTNN(N,MA,NDIM,NDF,NEL,NEL1,NSTF,NSIZV,NVEC,MCT,DM,D,XYZ,
C         IX,F,FORCE,ESTIF,U,VECT,ISW)
C
C                       N IS ELEMENT NUMBER.
C                       MA IS THE MATERIAL NUMBER.
C                       NDIM IS SPATIAL DIMENSION, 1,2,OR 3.
C                       NDF IS NUMBER OF DEGREES OF FREEDOM PER NODE.
C                       NEL IS THE NUMBER OF EXTERNAL NODES PER ELEMENT
C                       NEL1 IS DIMENSION OF ELEMENT PROPERTY ARRAY.
C                       NSTF IS THE SIZE OF THE ELEMENT STIFFNESS.
C                       NSIZV IS THE SIZE OF UTILITY VECTORS.
C                       NVEC IS THE NUMBER OF UTILITY VECTORS.
C                       MCT IS A PRINTER LINE COUNTER.
C                       DM IS A PARAMETER FOR MATERIAL IDENTIFICATION.
C                       D(L,1) IS MATERIAL PROPERTY MATRIX (63 CELLS).
C                       XYZ(NDIM,1) ARE NODAL COORDINATES.
C                       IX(NEL1,1) ARE ELEMENT  PROPERTIES, NODES, ETC.
C                       F(NDF,1) ARE NODAL GENERALIZED FORCES.
C                       FORCE(NSTF,2) IS ELEMENT FORCE VECTOR TO BE
C                       COMPUTED. COLUMN 2 IS LUMPED MASS
C                       ESTIF(NSTF,NSTF) IS ELEMENT MATRIX TO BE
C                       COMPUTED.
C                       VECT(NSIZV,1) ARE PRESCRIBED NODAL OR ELEMENT
C                       QUANTITIES. TEMPERATURES ETC.
C                       U(NDF,1) IS SOLUTION VECTOR.
C                       ISW IS SWITCHING PARAMETER,
C
C                           ISW=1, ** MATERIAL CHARACTERIZATION**
C                           ISW=2, ** CHECK ELEMENT FOR POSITIVE AREA *
C                           ISW=3, ** ELEMENT STIFFNESS AND
C                                     LOAD COMPUTATION**
C                           ISW=4, ** ELEMENT STRESSES AND PRINTOUT**
C                           ISW=5, ** ELEMENT LOAD COMPUTATION ONLY**
C                           ISW=6, ** NONLINEAR GENERALIZED FORCES**
C                           OTHER ISW MAY BE USED FOR SPECIAL PURPOSES.
C
C        USERS CAN GENERATE SURFACE LOADINGS BY PROVIDING SLDNN
C        SUBROUTINES (WHERE NN IS A TWO DIGIT NUMBER BETWEEN 01 AND 05)
C        THAT SPECIFY THE LOAD ROUTINE.  THE SUBROUTINE IS ACCESSED BY
C        THE CALL TO
C
C        SLDNN(NDIM,NDF,NDP,NPRES,IPRES,PR,XYZ,FS)
C
C                       WHERE IN ADDITION TO QUANTITIES DEFINED ABOVE
C                       FOR ELMTNN,
C                       NDP IS THE DIMENSION OF LOADED SURFACE
C                       NPRES IS NUMBER OF LOADED NODES (MAX 8)
C                       IPRES(8) ARE NODE NUMBERS OF LOADED NODES.
C                       PR(8) ARE LOAD VALUES AT CORRESPONDING IPRES
C                           NODES.
C                       FS(6,8) ARE THE COMPUTED GENERALIZED (NODAL)
C                           FORCES FOR EACH DEGREE OF FREEDOM AT EACH
C                           IPRES NODE.
C
```

```
C        SEE SECTION 7.1 FOR DATA INPUT DETAILS.                              MAN104C
C                                                                             MAN105C
C                                                                             MAN106C
C        INTEGRATION TABLE IS ACCESSED BY THE CALL                            MAN107C
C                                                                             MAN108C
C          CALL  INTEGL(LIM,NC1,NDIM,LINT,STUW)                               MAN109C
C                                                                             MAN110C
C                    STUW(4,M)  INTEGRATION POINTS AND WEIGHTS.               MAN111C
C                    **NOTE** M MUST BE SET EXPLICITLY AND BE LARGER          MAN112C
C                    THAN OR EQUAL TO LINT.                                   MAN113C
C                    LINT - RETURNS WITH NUMBER INTEGRATION POINTS.           MAN114C
C                    NC1 = 0  RETURNS GAUSS POINTS AND WEIGHTS IN             MAN115C
C                        STUW.                                                MAN116C
C                        LIM = 1 TO 5 IS NUMBER OF GAUSS POINTS/DIR-          MAN117C
C                        ECTION.                                              MAN118C
C                    NC1 = 1  RETURNS A SPECIAL 3-D GAUSS FORMULA.            MAN119C
C                        SET LIM = 1 FOR 6 PT. CUBIC ACCURACY                 MAN120C
C                        SET LIM = 2 FOR 14 PT. QUINTIC ACCURACY.             MAN121C
C                    NC1 = 2  RETURNS TRIANGULAR INTEGRATION FORMULA          MAN122C
C                        SET LIM = 1 FOR 1 PT. LINEAR ACCURACY.               MAN123C
C                        SET LIM = 2 FOR 3 PT. QUADRATIC ACCURACY.            MAN124C
C                        SET LIM = 3 FOR 7 PT. QUARTIC ACCURACY.              MAN125C
C                                                                             MAN126C
C                                                                             MAN127C
C      1.) DATA TYPE IDENTIFICATION CARDS (I5,1X,12A6).                       MAN128C
C                                                                             MAN129C
C        EACH DATA SEGMENT IS PREDEDED BY A CARD WHICH IDENTIFIES THE         MAN130C
C        TYPE OF DATA AND LIMITS ON THE AMOUNT OF DATA WHICH IMMEDIATELY      MAN131C
C        FOLLOWS THE CARD.  EXCEPT AS NOTED THE DATA SEGMENTS MAY APPEAR      MAN132C
C        IN ANY ORDER.  THE IDENTITY CARDS MAY ALSO AID THE USER IN           MAN133C
C        INTERPRETTING THE INPUT DATA CARDS.  AS SUPPLIED THERE ARE           MAN134C
C        TWENTY-FIVE DIFFERENT DATA IDENTIFICATION CARDS.  THESE ARE          MAN135C
C                                                                             MAN136C
C        COL  7 TO 12     IDENTITY(RESTRICTIONS)                              MAN137C
C                                                                             MAN138C
C        FEAP74           START OF EACH PROBLEM (MUST PRECEDE ALL OTHER       MAN139C
C                         DATA).                                              MAN140C
C        TITLE            CHANGE OUTPUT PAGE HEADINGS                         MAN141C
C        REMARK           COMMENTS ON OUTPUT                                  MAN142C
C        MATERI           MATERIAL CHARACTERIZATION.                          MAN143C
C        NODAL            NODAL CARDS                                         MAN144C
C        POLAR            POLAR CONVERSION. (PRECEDE BY NODAL,GENERA, OR      MAN145C
C                         BLOCK )                                             MAN146C
C        ELEMEN           ELEMENT CONNECTION CARDS.                           MAN147C
C        GENERA           GENERATE NODES IN A LINEAR PATH BY ANY INCREMNT     MAN148C
C        BLOCK            GENERATE ALL MESH DATA (BOTH NODAL AND ELEMENT)     MAN149C
C                         FOR A 2,(OR 3)DIMENSIONAL REGION WHOSE BOUNDARY     MAN150C
C                         MAY BE DEFINED BY 4(8)OR 8(20)COLLOCATED POINTS     MAN151C
C        BOUNDA           BOUNDARY CODE PRESCRIPTION (PRECEDE BY NODAL OR     MAN152C
C                         GENERATE OR BLOCK)                                  MAN153C
C        VECTOR           PRESCRIBED NODAL OR ELEMENT DATA (PRECEDE BY        MAN154C
C                         NODAL, OR GENERA AND ELEMEN, OR BLOCK)              MAN155C
C        FORCE            NODAL GENERALIZED FORCES (PRECEDE BY NODAL OR       MAN156C
C                         GENERA OR BLOCK).                                   MAN157C
```

```
C       BLOADS          SURFACE LOADINGS (SAME AS FORCE).                    MAN158C
C       ELOADS          ELEMENT LOADINGS (SAME AS FORCE).                    MAN159C
C       MESH            CHECK CONSISTENCY OF MESH ONLY (SAME AS SOLVE)       MAN160C
C       PLOT            PLOT MESH (SAME AS SOLVE)                            MAN161C
C       INITIA          INITIAL CONDITION PRESCRIPTION FOR DYNAMIC           MAN162C
C                       ANALYSIS (PRECEDE BY NODAL, GENERA OR BLOCK)         MAN163C
C       SOLVE           COMPLETE FORMULATION AND SOLUTION FROM ELEMENTS      MAN164C
C                       (PRECEDE BY MATERI,NODAL OR GENERA, AND ELEMEN       MAN165C
C                       OR PRECEDE BY MATERI AND BLOCK )                     MAN166C
C       RESOLV          USE PREVIOUS PROBLEM DESCRIPTION WITH NEW LOAD       MAN167C
C                       ONLY (PRECEDE BY SOLVE AND NEW LOADING CARDS).       MAN168C
C       EXPLIC          DYNAMIC SOLUTION BY EXPLICIT INTEGRATION, (SAME      MAN169C
C                       AS SOLVE)                                            MAN170C
C       IMPLIC          IMPLICIT INTEGRATION OF DYNAMIC PROBLEMS             MAN171C
C                       (PRECEDE BY SAME DATA AS FOR SOLVE)                  MAN172C
C       VISCOE          QUASI-STATIC LINEAR VISCOELASTIC INTEGRATION         MAN173C
C                       (PRECEDE BY SAME DATA AS FOR SOLVE)                  MAN174C
C       FOURIE          FOURIER COMPOSITION (SAME AS SOLVE)                  MAN175C
C       ADDUP           ACCUMULATE FOURIER SOLUTION (AFTER FOURIE)           MAN176C
C       STOP            NORMAL EXIT (MUST FOLLOW ALL DATA)                   MAN177C
C                                                                            MAN178C
C       ***NOTE*** EACH IDENTIFIER IS PUNCHED STARTING IN COL 7 (LEFT        MAN179C
C           JUSTIFIED).                                                      MAN180C
C                                                                            MAN181C
C     EXCESS CARDS MAY EXIST BETWEEN EACH SECTION OF DATA, HOWEVER,          MAN182C
C     THE DATA TO BE USED MUST IMMEDIATELY FOLLOW THE TYPE CARD AND          MAN183C
C     MUST BE IN PROPER ORDER.  NO PARTICULAR ORDER OF THE TYPE              MAN184C
C     CARDS IS NECESSARY EXCEPT THAT THE FEAP74 CARD MUST ALWAYS BE          MAN185C
C     THE FIRST CARD IN EACH SET OF DATA, AND RESTRICTIONS MUST BE           MAN186C
C     OBSERVED.                                                              MAN187C
C                                                                            MAN188C
C                                                                            MAN189C
C     2.) PROBLEM INITIATION AND CONTROL CARDS                               MAN190C
C                                                                            MAN191C
C       CARD 1. (6X,12A6)                                                    MAN192C
C                                                                            MAN193C
C       COL  7 TO 12    MUST CONTAIN WORD FEAP74                             MAN194C
C       COL 13 TO 78    OUTPUT PAGE HEADER                                   MAN195C
C                                                                            MAN196C
C       CARD 2. (I5,1X,3A6)                                                  MAN197C
C                                                                            MAN198C
C       COL  1 TO 5     NDIM - SPATIAL DIMENSION OF PROBLEM (1 TO 3)         MAN199C
C       COL  7 TO 12    NAMES TO BE PRINTED AS OUTPUT HEADERS TO             MAN200C
C       COL 13 TO 18    COORDINATES - IF BLANK SET TO 1,2,3 AS NEEDED.       MAN201C
C       COL 19 TO 24                                                         MAN202C
C                                                                            MAN203C
C       CARD 3. (I5,1X,6A6)                                                  MAN204C
C                                                                            MAN205C
C       COL  1 TO  5    NDF - NUMBER OF UNKNOWNS PER NODE (1 TO 6)           MAN206C
C       COL  7 TO 12    NAMES TO BE PRINTED AS OUTPUT HEADERS OF THE         MAN207C
C       COL 13 TO 18    GENERALIZED DISPLACEMENTS AND FORCES - IF            MAN208C
C           ........    BLANK SET TO 1,2,3,4,5,6 AS NECESSARY                MAN209C
C       COL 37 TO 42                                                         MAN210C
C                                                                            MAN211C
```

```
C        CARD 4. (6I5,5F10.0)                                                        MAN212C
C                                                                                    MAN213C
C        COL  1 TO 5     NEN - MAXIMUM NUMBER OF NODES CONNECTED TO ANY              MAN214C
C                        ELEMENT (1 TO 20).                                          MAN215C
C        COL  6 TO 10    NEXTRA - INCREASES ELEMENT MATRIX SIZE FROM                 MAN216C
C                        NDF*NEN TO NDF*NEN + NEXTRA                                 MAN217C
C        COL 11 TO 15    IREC - COMPUTE GENERALIZED FORCE CHECK IF                   MAN218C
C                        NONZERO (FOR TIME INVARIANT ANALYSIS ONLY)                  MAN219C
C        COL 16 TO 20    MBAN - MAXIMUM EXPECTED BANDWIDTH. DEFAULT IS               MAN220C
C                        SET TO 100. USED AS AN ERROR CHECK TO PREVENT               MAN221C
C                        RUNNING WITH AN OBVIOUS ERROR.                              MAN222C
C        COL 21 TO 25    IBUF - BUFFER SIZE FOR STORAGE OF HISTORY                   MAN223C
C                        EFFECTS IN TIME DEPENDENT ANALYSIS. DEFAULT IS              MAN224C
C                        IBUF = ISZDT/20                                             MAN225C
C        COL 26 TO 30    NC1  - USER INTEGER CONSTANT                                MAN226C
C        COL 31 TO 40    CON1 - USER DEFINED CONSTANT                                MAN227C
C        COL 41 TO 50    CON2 - USER DEFINED CONSTANT                                MAN228C
C        COL 51 TO 60    CON3(1) - USER DEFINED CONSTANT                             MAN229C
C        COL 61 TO 70    CON3(2) - USER DEFINED CONSTANT                             MAN230C
C        COL 71 TO 80    CON3(3) - USER DEFINED CONSTANT                             MAN231C
C                                                                                    MAN232C
C      2.1)  *REMARK* USER COMMENTS ON OUTPUT. (6X,12A6)                             MAN233C
C                                                                                    MAN234C
C        SUBSEQUENT CARDS                                                            MAN235C
C                                                                                    MAN236C
C        COL  7 TO 12    MUST CONTAIN REMARK                                         MAN237C
C        COL 13 TO 78    STATEMENTS TO BE OUTPUT , USE AS MANY REMARK                MAN238C
C                        CARDS AS DESIRED. INSERT BEFORE ANY TYPE CARD.              MAN239C
C                                                                                    MAN240C
C      2.2) TITLE CHANGE ON OUTPUT (6X,12A6)                                         MAN241C
C                                                                                    MAN242C
C        COL  7 TO 12    MUST CONTAIN TITLE                                          MAN243C
C        COL 13 TO 78    NEW TITLE DESCRIPTOR                                        MAN244C
C                                                                                    MAN245C
C      2.3) EXECUTION TERMINATION (6X,A4)                                            MAN246C
C                                                                                    MAN247C
C        COL  7 TO 10    MUST CONTAIN STOP. INSERT AFTER LAST PROBLEM.               MAN248C
C                                                                                    MAN249C
C                                                                                    MAN250C
C     3.) MATERIAL CHARACTERIZATION (I5,1X,12A6)                                     MAN251C
C                                                                                    MAN252C
C        COL  1 TO  5    NUMMAT - NUMBER OF DIFFERENT MATERIAL CHARACT-              MAN253C
C                        ERIZATIONS TO FOLLOW.                                       MAN254C
C        COL  7 TO 12    MUST CONTAIN WORD MATERI                                    MAN255C
C                                                                                    MAN256C
C        THE FOLLOWING CARDS ARE SUPPLIED FOR EACH MATERIAL TO BE CHARAC             MAN257C
C        TERIZED (MUST BE EXACTLY NUMMAT SETS OF CARDS)                              MAN258C
C                                                                                    MAN259C
C        CARD 1.) ELEMENT SELECTOR CARD (I5,1X,A5,11A6)                              MAN260C
C                                                                                    MAN261C
C        COL  1 TO  5    MATERIAL NUMBER (1 TO NUMMAT)                               MAN262C
C        COL  7 TO 11    ELMNN - WHERE NN IS NUMBER OF ELEMENT CLASS (01             MAN263C
C                        TO 10) TO WHICH THE CHARACTERIZATION BELONGS.               MAN264C
C        COL 12 TO 77    ALPHANUMERIC INFORMATION TO BE OUTPUT.                      MAN265C
```

```
C                                                                      MAN266C
C        CARD 2.), ETC. ** USER DEFINED FOR EACH ELEMENT TYPE PROVIDED. MAN267C
C                                                                      MAN267C
C        EXCESS BLANK CARDS ARE PERMISSIBLE BETWEEN EACH MATERIAL SET.  MAN267C
C                                                                      MAN268C
C     4.) NODAL CARDS (I5,1X,A6)                                       MAN269C
C                                                                      MAN270C
C        COL  1 TO  5    NUMNP - NUMBER OF NODAL POINTS                MAN271C
C        COL  7 TO 12    MUST CONTAIN NODAL                            MAN272C
C                                                                      MAN273C
C        SUBSEQUENT CARDS  LAST NODAL CARD MUST NOT BE GENERATED.      MAN274C
C        (I5,I15,3F10.0)                                               MAN275C
C                                                                      MAN276C
C        COL  1 TO  5    NODE NUMBER                                   MAN277C
C        COL 15          1 IF 1 DISPLACEMENT IS SPECIFIED              MAN278C
C        COL 16          1 IF 2 DISPLACEMENT IS SPECIFIED              MAN279C
C        COL 17          1 IF 3 DISPLACEMENT IS SPECIFIED              MAN280C
C        COL 18          1 IF 4 DISPLACEMENT IS SPECIFIED              MAN281C
C        COL 19          1 IF 5 DISPLACEMENT IS SPECIFIED              MAN282C
C        COL 20          1 IF 6 DISPLACEMENT IS SPECIFIED              MAN283C
C        COL 21 TO 30    1 COORDINATE VALUE                            MAN284C
C        COL 31 TO 40    2 COORDINATE VALUE * AS REQUIRED              MAN285C
C        COL 41 TO 50    3 COORDINATE VALUE                            MAN286C
C                                                                      MAN287C
C     NODAL CARDS MUST BE IN ORDER. MISSING NODES ARE INTERPOLATED     MAN288C
C     LINEARLY FROM INPUT NODES. IF SUCCEEDING CARDS HAVE IDENTICAL    MAN289C
C     BOUNDARY CODES, THIS BOUNDARY CODE WILL BE ASSIGNED TO THE INTER- MAN290C
C     VENING NODES. IN ALL OTHER CASES THE BOUNDARY CODE IS SET TO ZERO MAN291C
C     *TERMINATE ON NODE NUMNP OR A BLANK CARD*                        MAN292C
C                                                                      MAN293C
C     4.1) NON SEQUENTIAL NODAL GENERATOR OPTION. (I5,1X,A6)           MAN294C
C                                                                      MAN295C
C        COL  1 TO  5    NUMBER OF NODAL POINTS                        MAN296C
C        COL  7 TO 12    MUST CONTAIN GENERA                           MAN297C
C                                                                      MAN298C
C        SUBSEQUENT CARDS (2I5,I10,3F10.0)                             MAN299C
C                                                                      MAN300C
C        COL  1 TO  5    NODE-NUMBER                                   MAN301C
C        COL  6 TO 10    NODE-NUMBER-INCREMENT WHICH WILL BE SUCESSIVELY MAN302C
C                        ADDED TO NODE-NUMBER UNTIL SUM IS GREATER THAN MAN303C
C                        NODE-NUMBER ON FOLLOWING CARD (ALGEBRAIC).    MAN304C
C        COL 15 TO 20    BOUNDARY CODE. SAME AS INPUT FOR NODAL.       MAN305C
C                        IF SUCCEEDING CARDS HAVE IDENTICAL BOUNDARY   MAN306C
C                        CODES, THIS BOUNDARY CODE WILL BE ASSIGNED TO MAN307C
C                        THE INTERVENING NODES. IN ALL OTHER CASES THE MAN308C
C                        BOUNDARY CODE IS SET TO ZERO.                 MAN309C
C        COL 21 TO 30    1 COORDINATE VALUE  *                         MAN310C
C        COL 31 TO 40    2 COORDINATE VALUE  * AS REQUIRED *           MAN311C
C        COL 41 TO 50    3 COORDINATE VALUE  *                         MAN312C
C                                                                      MAN313C
C        *TERMINATE WITH BLANK CARD *                                  MAN314C
C                                                                      MAN315C
C     4.2) BOUNDARY CODE PATCH-UP OPTION. (6X,A6)                      MAN316C
C                                                                      MAN317C
```

```
C       COL  7 TO 12    MUST CONTAIN BOUNDA                                          MAN318C
C                                                                                    MAN319C
C       SUBSEQUENT CARDS. (8I5)                                                      MAN320C
C                                                                                    MAN321C
C       COL  1 TO  5    N, NODE NUMBER TO HAVE REDEFINED BOUNDARY CODE.              MAN322C
C       COL  6 TO 10    NX, GENERATOR INCREMENT TO BE ADDED ALGEBRAI-                MAN323C
C                       CALLY TO N, UNTIL SUM EXCEEDS (MAX OR MIN) THE               MAN324C
C                       N OF THE FOLLOWING CARD.                                     MAN325C
C       COL 11 TO 15    IBC(I), (I=1,2...NDF) CODE FOR SPECIFYING FORCE              MAN326C
C       COL 16 TO 20    OR DISPLACEMENT BOUNDARY CONDITIONS,                         MAN327C
C       COL  ...                                                                     MAN328C
C                           IBC(I) .EQ. 0, FORCE SPECIFIED.                          MAN329C
C                           IBC(I) .GT. 0, DISPLACEMENT SPECIFIED, NO                MAN330C
C                           INTERVENING GENERATION.                                  MAN331C
C                           IBC(I) .LT. 0, DISPLACEMENT SPECIFIED,                   MAN332C
C                           GENERATE BETWEEN MISSING NODES IN ALGEBRAIC              MAN333C
C                           INCREMENTS OF NX.                                        MAN334C
C                                                                                    MAN335C
C       * TERMINATE WITH A BLANK CARD. *                                             MAN336C
C                                                                                    MAN337C
C     4.3) POLAR OR CYLINDRICAL COORDINATE CONVERSION TO CARTESIAN                   MAN338C
C          COORDINATES (6X,A6)                                                       MAN339C
C                                                                                    MAN340C
C       COL  7 TO 12    MUST CONTAIN POLAR    (LEFT JUSTIFIED)                       MAN341C
C                                                                                    MAN342C
C       CARD 1.  (3I5,5X,2F10.0)                                                     MAN343C
C                                                                                    MAN344C
C       COL  1 TO  5    N1, FIRST NODE TO BE CONVERTED                               MAN345C
C       COL  6 TO 10    N2, LAST  NODE TO BE CONVERTED                               MAN346C
C       COL 11 TO 15    N3, INCREMENT ADDED (ALGEBRAICALLY), N1 TO N2                MAN347C
C       COL 21 TO 30    X0, ORIGIN OF POLAR X-COORDINATE                             MAN348C
C       COL 31 TO 40    Y0, ORIGIN OF POLAR Y-COORDINATE                             MAN349C
C                                                                                    MAN350C
C     5.) ELEMENT CARDS (I5,1X,A6)                                                   MAN351C
C                                                                                    MAN352C
C       COL  1 TO  5    NUMEL - NUMBER OF ELEMENTS                                   MAN353C
C       COL  7 TO 12    MUST CONTAIN ELEMEN                                          MAN354C
C                                                                                    MAN355C
C       SUBSEQUENT CARDS (4I5,20I3/20I4)                                             MAN356C
C                                                                                    MAN357C
C       CARD 1.                                                                      MAN358C
C                                                                                    MAN359C
C       COL  1 TO  5    ELEMENT NUMBER                                               MAN360C
C       COL  6 TO 10    MATERIAL NUMBER                                              MAN361C
C       COL 11 TO 15    NUMBER OF SUBSEQUENT ELEMENTS USING SAME                     MAN362C
C                       STIFFNESS MATRIX * SAVES RECOMPUTATION OF                    MAN363C
C                       SIMILAR MATRICES.  ELEMENT MUST ALSO HAVE                    MAN364C
C                       SAME ELEMENT FORCE VECTOR * IF THESE ARE                     MAN365C
C                       IN THE STIFFNESS SUBROUTINE *                                MAN366C
C       COL 16 TO 20    PRINT ELEMENT MATRIX IF NONZERO.                             MAN367C
C       COL 21 TO 23    IXD(1) ELEMENT INCREMENT ARRAY ON NODE 1.                    MAN368C
C       COL 24 TO 26    IXD(2)  * IF NOT INPUT IS SET AUTOMATICALLY                  MAN369C
C                       UP TO                                                        MAN370C
C       COL 79 TO 80    IXD(20)   FOR SERENDIPITY ELEMENTS * SEE REPORT              MAN371C
```

```
C                                                                      MAN372C
C       CARD 2.                                                        MAN373C
C                                                                      MAN374C
C       COL  1 TO  4    NODE 1                                         MAN375C
C       COL  5 TO  8    NODE 2                                         MAN376C
C       COL  9 TO 12    NODE 3         IX(NEL1,1) ARRAY                MAN377C
C                           CONTINUE IN I4 FORMAT TO A MAXIMUM         MAN378C
C       COL 77 TO 80    NODE 20                                        MAN379C
C                                                                      MAN380C
C       ELEMENT CARDS MUST BE IN ORDER. MISSING ELEMENTS ARE GENERATED  MAN382C
C       BY INCREMENTING NODES.                                         MAN383C
C       LAST ELEMENT CARD MUST NOT BE GENERATED.                       MAN384C
C       * TERMINATE ON ELEMENT NUMEL OR A BLANK CARD *                 MAN385C
C                                                                      MAN386C
C     5.1) BLOCK GENERATOR, GENERATES ALL MESH DATA.(6X,A6)            MAN387C
C                                                                      MAN388C
C       COL  1 TO  5    NUMBER OF NODAL POINTS TO BE GENERATED.        MAN389C
C       COL  7 TO 12    MUST CONTAIN BLOCK                             MAN390C
C                                                                      MAN391C
C       SUBSEQUENT CARDS (10I5/6(4X,I6)/(10X,3F10.0))                  MAN392C
C                                                                      MAN393C
C       CARD 1.                                                        MAN394C
C                                                                      MAN395C
C       COL  1 TO  5    NN, NUMBER OF POINTS REQUIRED TO DEFINE        MAN396C
C                       BOUNDARY OF REGION. FOR 2-DIM., NN=4 OR 8.     MAN397C
C                       FOR 3-DIM., NN= 8 OR 20.                       MAN398C
C       COL  6 TO 10    NUMBER OF ELEMENTS IN R-DIRECTION.             MAN399C
C       COL 11 TO 15    NUMBER OF ELEMENTS IN S-DIRECTION.             MAN400C
C       COL 16 TO 20    NUMBER OF ELEMENTS IN T-DIRECTION.             MAN401C
C       COL 21 TO 25    INITIAL NODE NUMBER, DEFAULT = 1.              MAN402C
C       COL 26 TO 30    INITIAL ELEMENT NUMBER, DEFAULT = 1.           MAN403C
C       COL 31 TO 35    MATERIAL NUMBER OVER REGION, DEFAULT = 1       MAN404C
C       COL 36 TO 40    BOUNDARY CODE SKIP. A NON-ZERO ENTRY WILL OMIT MAN405C
C                       SETTING ALL INTERIOR BOUNDARY CODES TO ZERO.   MAN406C
C       COL 41 TO 45    IREUSE - REUSE ELEMENT STIFFNESS OPTION - USES MAN407C
C                       EACH ELEMENT STIFFNESS IREUSE TIMES BEFORE     MAN408C
C                       GENERATING A NEW ELEMENT STIFFNESS MATRIX.     MAN409C
C       COL 46 TO 50    ELEMENT STIFFNESS-PRINT. A NON-ZERO ENTRY  WILL MAN410C
C                       CAUSE PRINT-OUT OF FIRST ELEMENT.              MAN411C
C       COL 51 TO 55    IMSH - IF NONZERO PRINT COMPUTED NODES.        MAN412C
C       COL 56 TO 60    IELM - IF NONZERO PRINT COMPUTED ELEMENTS.     MAN413C
C                                                                      MAN414C
C       CARD 2.  (BOUNDARY CODE AS DEFINED IN NODAL CARD.)             MAN415C
C                                                                      MAN416C
C       COL  5 TO 10    BOUNDARY CODE OVER FACE -R.                    MAN417C
C       COL 15 TO 20    BOUNDARY CODE OVER FACE +R.                    MAN418C
C       COL 25 TO 30    BOUNDARY CODE OVER FACE -S.                    MAN419C
C       COL 35 TO 40    BOUNDARY CODE OVER FACE +S.                    MAN420C
C       COL 45 TO 50    BOUNDARY CODE OVER FACE -T.                    MAN421C
C       COL 55 TO 60    BOUNDARY CODE OVER FACE +T.                    MAN422C
C                                                                      MAN423C
C       CARD 3.  (REPEAT NN TIMES.)                                    MAN424C
C                                                                      MAN425C
C       COL 11 TO 20    1-COORDINATE OF BOUNDARY-DEFINING-POINT.       MAN426C
```

```
C         COL 21 TO 30     2-COORDINATE OF BOUNDARY-DEFINING-POINT.      MAN427C
C         COL 31 TO 40     3-COORDINATE OF BOUNDARY-DEFINING-POINT.      MAN428C
C                                                                        MAN429C
C         NOTES, 1. BLOCK GENERATES ONLY 4 PT. QUADRILATERALS OR 8 PT.   MAN430C
C                   BRICKS.                                              MAN431C
C                2. INPUT OF CARDS 3.) FOLLOW ORDER RULES FOR ELEMENT    MAN432C
C                   INPUT (SEE SECTION 5.)).                             MAN433C
C                3. R-S-T ARE LOCAL COORDINATES,                         MAN433C
C                   I.E. (-1 .LE. R,S,T .LE. 1), WHERE R IS DIRECTED     MAN433C
C                   FROM NODE 1 TO 2, S IS IN PLANE OF FIRST THREE NODES MAN433C
C                   AND T IS NORMAL TO R-S PLANE.                        MAN433C
C                                                                        MAN435C
C      6.)  VECTOR CARDS, I.E. USER DEFINED INPUT (I5,1X,A6)             MAN436C
C                                                                        MAN437C
C         COL  1 TO  5    NVEC, NUMBER OF DIFFERENT VECTORS (7 MAX)      MAN438C
C         COL  7 TO 12    MUST CONTAIN VECTOR                            MAN439C
C                                                                        MAN440C
C         SUBSEQUENT CARDS                                               MAN441C
C                                                                        MAN442C
C         CARD 1.  (2I5)                                                 MAN443C
C                                                                        MAN444C
C         COL  1 TO  5    NSIZV, VECTOR LENGTH,COMMON TO ALL NVEC VECTORS MAN445C
C         COL  6 TO 10    IPICK, CODED PARAMETER,                        MAN446C
C                                                                        MAN447C
C                         IPICK = 0, VECTORS ASSOCIATED WITH NODES       MAN448C
C                         IPICK = 1, VECTORS ASSOCIATED WITH DEG.FREEDOM MAN449C
C                         IPICK = 2, VECTORS ASSOCIATED WITH ELEMENTS    MAN450C
C                                                                        MAN451C
C         CARD 2.  (6X, 2A6) REPEAT NVEC TIMES                           MAN452C
C                                                                        MAN453C
C         COL  7 TO 18     DESCRIPTIVE TITLE FOR VECTOR                  MAN454C
C                                                                        MAN455C
C         CARD 3.  (2I5,7F10.0)                                          MAN456C
C                                                                        MAN457C
C         COL  1 TO  5    POSITION NUMBER OF VECTOR ELEMENT, 1 TO NSIZV  MAN458C
C         COL  6 TO 10    GENERATOR INCREMENT                            MAN459C
C         COL 11 TO 20    VECTOR ELEMENT VALUE OF VECTOR 1               MAN460C
C         COL 21 TO 30    VECTOR ELEMENT VALUE OF VECTOR 2               MAN461C
C         COL ........    AS REQUIRED FOR NVEC VECTORS                   MAN462C
C                                                                        MAN463C
C         LINEAR INTERPOLATION IS PERFORMED ON ALL VECTORS BETWEEN       MAN464C
C         NON-CONSECUTIVE POSITION NUMBERS SPECIFIED IN COL 1 TO 5       MAN465C
C         IF INCREMENT IS NONZERO.                                       MAN466C
C         IF DESCRIPTIVE TITLES OF ALL VECTORS ARE BLANK CARDS,          MAN467C
C         PRINTING OF THE VECTOR VALUES IS SUPRESSED.                    MAN468C
C                                                                        MAN469C
C         * TERMINATE ON BLANK CARD *                                    MAN470C
C                                                                        MAN471C
C      6.1)  INITIAL CONDITIONS FOR TIME DEPENDENT ANALYSIS.             MAN472C
C                                                                        MAN473C
C         COL  1 TO  5    NICD, NUMBER OF INITIAL CONDITION VECTORS      MAN474C
C         COL  7 TO 12    MUST CONTAIN INITIA                            MAN475C
C                                                                        MAN476C
C         SUBSEQUENT CARDS                                               MAN477C
```

```
C                                                                      MAN478C
C     CARD 1. (6X,2A6) REPEAT NICD TIMES                               MAN479C
C                                                                      MAN480C
C     COL  7 TO 18    DESCRIPTIVE TITLE FOR INITIAL CONDITIONS         MAN481C
C                                                                      MAN482C
C     CARD 2.  (2I5,7F10.0)                                            MAN483C
C                                                                      MAN484C
C     COL  1 TO  5    POSITION NUMBER, AS IN VECTOR CARDS FOR IPICK=1  MAN485C
C     COL  6 TO 10    GENERATOR INCREMENT                              MAN486C
C     COL 11 TO 20    INITIAL CONDITION 1                              MAN487C
C     COL 21 TO 30    INITIAL CONDITION 2                              MAN488C
C     COL ........    AS REQUIRED FOR NICD INITIAL CONDITIONS          MAN489C
C                                                                      MAN490C
C     INTERPOLATION BETWEEN INPUT VALUES AS DESCRIBED IN VECTOR INPUT  MAN491C
C     **** NOTE **** IF MISSING THE INITIAL CONDITIONS ARE SET ZERO    MAN492C
C                                                                      MAN493C
C   7.) FORCE CARDS (I5,1X,A6)                                         MAN494C
C                                                                      MAN495C
C     COL  1 TO  5    LAST NODE TO WHICH A FORCE IS TO BE SPECIFIED    MAN496C
C     COL  7 TO 12    MUST CONTAIN FORCE                               MAN497C
C                                                                      MAN498C
C     SUBSEQUENT CARDS (I5,5X,6F10.0)                                  MAN499C
C                                                                      MAN500C
C     THE FOLLOWING VALUES ARE EACH INTERPRETTED AS FORCES IF THE      MAN501C
C     CORRESPONDING BOUNDARY CODE IS A 0 *ZERO* AND AS A DISPLACEMENT  MAN502C
C     IF THE CORRESPONDING BOUNDARY CODE IS 1 *ONE*.                   MAN503C
C                                                                      MAN504C
C     COL  1 TO  5    NODE TO WHICH FORCE OR DISPLACEMENT IS APPLIED   MAN505C
C     COL 11 TO 20    VALUE OF 1 FORCE/DISPLACEMENT                    MAN506C
C     COL 21 TO 30    VALUE OF 2 FORCE/DISPLACEMENT    * AS *          MAN507C
C     COL 31 TO 40    VALUE OF 3 FORCE/DISPLACEMENT * REQUIRED *       MAN508C
C     COL 41 TO 50    VALUE OF 4 FORCE/DISPLACEMENT                    MAN509C
C     COL 51 TO 60    VALUE OF 5 FORCE/DISPLACEMENT                    MAN510C
C     COL 61 TO 70    VALUE OF 6 FORCE/DISPLACEMENT                    MAN511C
C                                                                      MAN512C
C                                                                      MAN513C
C   7.1) SURFACE LOAD CARDS (I5,1X,A6)                                 MAN514C
C                                                                      MAN515C
C     COL  1 TO  5    NUMBER OF LOADED FACE CARDS                      MAN516C
C     COL  7 TO 12    MUST CONTAIN BLOADS                              MAN517C
C                                                                      MAN518C
C     CARD 1. (I5,1X,A5,I4,8I5,8I3)                                    MAN519C
C                                                                      MAN520C
C     COL  1 TO  5    DIMENSION OF LOADING SURFACE, (1 OR 2).          MAN521C
C     COL  7 TO 11    SLD(NN), ALPHA-NUMERIC NAME OF SURFACE LOADING   MAN522C
C                     SUBROUTINE (NN IS BETWEEN 1 AND 5)               MAN523C
C     COL 12 TO 15    NRT,NUMCER OF ADDITIONAL ELEMENT LOAD            MAN524C
C                     SURFACES TO BE GENERATED FROM CURRENT MODEL.     MAN525C
C     COL 16 TO 20    IPRES(N), NODE NUMBERS DEFINING LOADING SURFACE  MAN526C
C     COL 21 TO 25    OF CURRENT ELEMENT.                              MAN527C
C     COL ...         (IDENTIFY FROM 2 TO 8 AS REQUIRED)               MAN528C
C     COL 51 TO 55                                                     MAN529C
C     COL 56 TO 58    INC(N), INCREMENT VALUE ADDED TO IPRES(N) TO     MAN530C
C     COL 59 TO 61    IDENTIFY NODE NUMBERS OF A GENERATED SEQUENCE.   MAN531C
```

```
C
C       COL ...        (IDENTIFY FROM 2 TO 8 AS REQUIRED)
C
C       CARD 2. (8F10.0)
C
C       COL  1 TO 80   LOAD AT NODES GIVEN ON PREVIOUS CARD *
C                      MUST CORRESPOND IN SEQUENCE TO THE NODE NUMBERS
C
C
C
C     7.2) ELEMENT LOAD CARDS (I5,1X,A6 )
C
C       COL  1 TO  5   NLD, NUMBER OF ELEMFNT LOAD CARDS.
C       COL  7 TO 11   MUST CONTAIN ELOADS
C
C       SUBSEQUENT CARDS (I5,1X,A5,I4,I5,6F10.0)
C
C       COL  1 TO  5   IEL, INITIAL ELEMENT OF A GENERATED SEQUENCE.
C       COL  7 TO 11   ELM(NN), ALPHA-NUMERIC NAME OF ELEMENT
C                      SUBROUTINE WHERE ELEMENT LOADS ARE COMPUTED.
C                      USED AS CHECK TO INSURE IEL, ETC. ARE PROPER
C                      ELEMENTS.
C       COL 12 TO 15   INC, INCREMENT NUMBER IN A GENERATED SEQUENCE,
C                      (DEFAULT = 1).
C       COL 16 TO 20   JEL, TERMINAL ELEMENT NUMBER IN A GENERATED
C                      SEQUENCE, IF JEL = 0, ONLY IEL IS COUNTED.
C       COL 21 TO 80   PR-USER DEFINED VALUES FOR DETERMINING BODY
C                      LOADS IN THE ISW=5 PORTION OF ELM(NN).
C
C       NOTE, USER MUST PROVIDE COMPUTATION OF LOADS IN ELMTNN.
C       PR IS TRANSFERED TO SUBROUTINE ELMTNN IN THE U VECTOR,
C       WHEN ISW =5, ONLY.
C
C     7.3) PROPORTIONAL LOADS FOR TIME DEPENDENT ANALYSIS
C
C       TRANSFER TO THIS OPTION OCCURS ONLY FOR TIME ANALYSES.
C
C       ONE CARD FOR EACH PROPORTIONAL LOAD REQUIRED
C
C       COL  1 TO  5   PROPORTIONAL LOAD TYPE, 1,2 OR 3
C       COL  6 TO 10   K, TABLE CONSTANT
C       COL 11 TO 20   TMIN, SMALLEST TIME LOADING IS VALID
C       COL 21 TO 30   TMAX, LARGEST TIME LOADING IS VALID
C       COL 31 TO 40   A0
C       COL 41 TO 50   A1
C       COL 51 TO 60   A2
C       COL 61 TO 70   A3
C       COL 71 TO 80   A4
C
C       LOAD TYPE 1.  T = TIME
C
C           PROP = A0 + A1*T + A2*T*T + A3*T*T*T + A4*T*T*T*T
C
C       LOAD TYPE 2.
C
```

```
C             PROP = A0*(SIN(A1*T))**K + A2*(COS(A3*T))**K + A5          MAN586C
C        LOAD TYPE 3.                                                     MAN587C
C                                                                         MAN588C
C             PROP = USER DEFINED FUNCTION FROM SUBROUTINE                MAN589C
C                    EXPRLD(PROP,T,A)  WHERE A IS AN ARRAY WITH           MAN590C
C                    INFORMATION FOR COLUMNS 6-80 OF DATA CARDS.          MAN591C
C                                                                         MAN592C
C        **NOTE** PROPORTIONAL LOADS CAN BE ACCUMULATED FROM DIFFERENT    MAN593C
C                 TYPES AT THE SAME TIME.                                 MAN594C
C                                                                         MAN595C
C     8.) INITIATION OF TIME INDEPENDENT SOLUTION (I5,1X,A6)              MAN596C
C                                                                         MAN597C
C       COL  1 TO  5    IOUT, OUTPUT CONTROL CODE.                        MAN598C
C                                                                         MAN599C
C                       IOUT .EQ. 0, ALL STRESSES AND DISP. PRINTED       MAN600C
C                       IOUT .NE. 0, SELECTED PRINTOUT, MORE DATA INPUT   MAN601C
C                       SEE SECTION 9 FOR DATA PREPARATION.               MAN602C
C                                                                         MAN603C
C       COL  7 TO 12    MUST CONTAIN  SOLVE  *INDICATES ALL DATA INPUT*   MAN604C
C                       COMPLETE FORMULATION AND SOLUTION OF EQUATIONS.   MAN605C
C       COL  7 TO 12    MUST CONTAIN  RESOLV TO OBTAIN SUBSEQUENT         MAN606C
C                       SOLUTIONS WHERE BOUNDARY CODES DO NOT CHANGE      MAN607C
C                       AND ALL PRESCRIBED DISPLACEMENTS ARE ZERO.        MAN608C
C                                                                         MAN609C
C     8.1) INITIATION OF DYNAMIC SOLUTION BY EXPLICIT INTEGRATION.        MAN610C
C                                                                         MAN611C
C       COL  1 TO  5    IPRT.  OUTPUT CONTROL FOR DISPLACEMENT AND        MAN612C
C                       STRESS PRINTOUT. SEE SECT. 9 FOR DATA INPUT.      MAN613C
C                       IOUT = 1 - MIN(1,IPRT)                            MAN614C
C                       IF IOUT .NE. 0, THE SPATIAL CONTROL DATA          MAN614C
C                       COMES AT THE END OF THE DYNAMIC SEGMENT.          MAN615C
C       COL  7 TO 12    MUST CONTAIN EXPLIC                               MAN616C
C                                                                         MAN617C
C       SUBSEQUENT CARDS (2I5,2F10.0,2I5)                                 MAN618C
C                                                                         MAN619C
C       COL  1 TO  5    NUMBER OF TIME STEPS                              MAN620C
C       COL  6 TO 10    PRINT INTERVAL                                    MAN621C
C       COL 11 TO 20    TIME INCREMENT                                    MAN622C
C       COL 21 TO 30    NEWMARK DELTA-DAMPING TERM (GAMMA - .5)           MAN623C
C       COL 31 TO 35    NUMBER OF TIME EVOLUTION ELEMENT VARIABLE PLOTS   MAN624C
C       COL 36 TO 40    NPROP, NUMBER OF PROPORTIONAL LOADS TO BE INPUT   MAN625C
C       COL 41 TO 45    NFORC, LAST NODE ON WHICH A FORCE IS CHANGED      MAN626C
C                       DURING EACH TIME STEP.                            MAN627C
C       COL 46 TO 50    KKK, STABILITY CHECK OVERRIDE ** CAUTION USE      MAN628C
C                       ONLY WHEN A BETTER ESTIMATE OF THE STABLE TIME    MAN629C
C                       STEP IS AVAILABLE THAN CAN BE PERFORMED BY CODE   MAN630C
C                       KKK ZERO, USES INTERNAL STABILITY CHECK.          MAN631C
C                       KKK NONZERO, DISREGARDS STABILITY CHECK.          MAN632C
C       COL 51 TO 55    NONL,  0 FOR LINEAR                               MAN633C
C                              1 FOR NON LINEAR                           MAN633
C       COL 56 TO 60    LPR,   0 PRINTS LOADS                             MAN633C
C                              1 SUPPRESS PRINT                           MAN633C
C                                                                         MAN633C
C                                                                         MAN634C
```

```
C        SUBSEQUENT CARDS (3I5) ONE FOR EACH STRESS PLOT.                MAN635C
C                                                                        MAN636C
C        COL  1 TO  5    ELEMENT NUMBER CONTAINING STRESS TO BE PLOTTED. MAN637C
C        COL  6 TO 10    LOCAL COORDINATE POINT CODE, 1 TO 9, AS         MAN638C
C                        PATTERNED AFTER, COL 11 TO COL 19, IN SECT. 9.  MAN639C
C        COL 11 TO 15    PLOT COMPONENT CODE, 1 TO 6 FOR SIGMA(I,J),     MAN640C
C                        I.E., SIGMA(1,1)=1, SIGMA(1,2)=2, SIGMA(1,3)=3, MAN641C
C                        SIGMA(2,2) = 4, SIGMA(2,3) = 5, SIGMA(3,3) = 6. MAN642C
C                                                                        MAN643C
C        IF(NPROP.NE.0) READ PROPORTIONAL LOAD CARDS, SEE SECT. 7.3      MAN644C
C                                                                        MAN645C
C        IF(NFORC.NE.0) READ FORCE CARDS AT EACH TIME STEP. IF OUTPUT IS MAN646C
C        LIMITED BY IOUT NONZERO, THE FIRST FORCE CARD SET PRECEDES      MAN647C
C        OUTPUT CARDS AND THE REMAINDER FOLLOW THE OUTPUT CARDS NO BLANK MAN648C
C        CARDS MAY BE USED BETWEEN SETS OF CARDS OTHER THAN THE USUAL    MAN649C
C        BLANK TERMINATOR CARD FOR FORCE INPUT CARDS.                    MAN650C
C                                                                        MAN651C
C        IF(IOUT.NE.0)   DATA FOR SPATIAL PRINTOUT CONTROL, SEE SECT.9.  MAN652C
C                                                                        MAN653C
C                                                                        MAN654C
C       SPECIAL COMMENTS FOR DYNAMIC OPTION                              MAN655C
C                                                                        MAN656C
C       (1) ONLY COLUMNS 1 TO 66 ARE AVAILABLE FOR PAGE HEADING.         MAN657C
C       (2) MAXIMUM ADVANTAGE OF ELEMENT REUSE OPTION SHOULD BE TAKEN.   MAN658C
C       (3) INITIAL CONDITIONS FOR DISPLACEMENT AND VELOCITY VECTORS,    MAN659C
C           AS WELL AS STORAGE FOR ACCELERATION VECTOR, MAY BE MADE      MAN660C
C           THROUGH INPUT OF AN INITIAL CONDITION CARD SET. WITHOUT      MAN661C
C           SPECIFIED INITIAL CONDITIONS THEY ARE AUTOMATICALLY SET ZERO.MAN662C
C       (4) SPATIAL LOADING IS INPUT THROUGH FORCE OR BOUNDARY           MAN663C
C           PRESSURE CARDS.                                              MAN664C
C       (5) EXTREME CAUTION ON ORDER OF DATA CARDS MUST BE OBSERVED.  NO MAN665C
C           EXTRA CARDS ARE PERMITTED AND STRICT COUNTS ARE OBSERVED     MAN666C
C           EXCEPT FOR THE NUMBER OF FORCE CARDS USED IN EACH TIME STEP. MAN667C
C                                                                        MAN668C
C       8.2) INITIATION OF IMPLICIT TIME INTEGRATIONS (I5,1X,A6)         MAN669C
C                                                                        MAN670C
C         COL  1 TO  5    NSEQ, NUMBER OF TIME SEQUENCES                 MAN671C
C         COL  7 TO 12    MUST CONTAIN VISCOE FOR LINEAR VISCOELASTIC    MAN672C
C                         QUASI-STATIC PROBLEMS (ONE INITIAL CONDITION   MAN673C
C                         ONLY MUST BE USED)                             MAN674C
C         COL  7 TO 12    MUST CONTAIN IMPLIC FOR DYNAMIC IMPLICIT       MAN675C
C                         INTEGRATIONS (THREE INITIAL CONDITIONS ARE     MAN676C
C                         REQUIRED, MORE CAN BE SPECIFIED WITHOUT ERROR) MAN677C
C                                                                        MAN688C
C         SUBSEQUENT CARDS, ONE SET FOR EACH TIME SEQUENCE               MAN689C
C                                                                        MAN690C
C         CARD 1. (F10.0,8I5,2F10.0,2I5)                                 MAN691C
C                                                                        MAN692C
C         COL  1 TO 10    DT, TIME INCREMENT (NONZERO FOR IMPLIC)        MAN693C
C         COL 11 TO 15    NTS, NUMBER OF TIME STEPS IN SEQUENCE          MAN694C
C         COL 16 TO 20    INT, PRINT INTERVAL (DEFAULT 1)                MAN695C
C         COL 21 TO 25    NNI, FIRST NODE PRINTED                        MAN696C
C         COL 26 TO 30    NNE, LAST  NODE PRINTED                        MAN697C
C         COL 31 TO 35    NEI, FIRST ELEMENT STRESS TO BE PRINTED        MAN698C
```

```
C        COL 36 TO 40    NEF, LAST  ELEMENT STRESS TO BE PRINTED                 MAN699C
C        COL 41 TO 45    NPROP, NUMBER OF PROPORTIONAL LOADS IN SEQUENCE         MAN700C
C        COL 46 TO 50    NFORC, LAST NODE FOR GENERALIZED FORCES TO BE           MAN701C
C                        INPUT FOR EACH TIME IN SEQUENCE (SEE SECT.7.            MAN702C
C                        FOR DATA PREPARATION FORMATS)                           MAN703C
C        COL 51 TO 60    BETA, NEWMARK INTEGRATION PARAMETER (IMPLIC)            MAN704C
C                        DEFAULT VALUE IS 0.25                                   MAN705C
C        COL 61 TO 70    DEL = GAMMA - 0.5, NEWMARK INTEGRATION                  MAN706C
C                        PARAMETER (IMPLIC)                                      MAN707C
C        COL 71 TO 75    NUMPLT, NUMBER OF STRESS PLOTS REQUESTED                MAN708C
C                        (INPUT ON FIRST TIME SEQUENCE ONLY)                     MAN709C
C        COL 76 TO 80    NITS, NUMBER OF ITERATIONS PER TIME STEP                MAN710C
C                        (TO ACHIEVE EQUILIBRIUM IN NON LINEAR PROBLEMS)         MAN711C
C                        ZERO OR BLANK FOR LINEAR PROBLEMS.                      MAN712C
C                        POSITIVE - REFORM THE NEWTON JACOBIAN MATRIX            MAN713C
C                                   AT START OF EACH TIME STEP ONLY.             MAN714C
C                        NEGATIVE - REFORM THE NEWTON JACOBIAN MATRIX            MAN715C
C                                   AT EACH ITERATION IN EACH TIME STEP.         MAN716C
C                        IF CONVERGENCE OCCURS BEFORE ABS(NITS)                  MAN717C
C                        ITERATIONS, THE PROGRAM WILL PROCEED TO THE             MAN718C
C                        NEXT TIME STEP.                                         MAN719C
C                        NOTE * * BECAUSE OF THE CONVERGENCE TEST USED,          MAN720C
C                                 A MINIMUM OF TWO ITERATIONS MUST BE            MAN721C
C                                 PERFORMED BEFORE THE CHECK CAN BE              MAN722C
C                                 IMPOSED.  THUS, USE OF NITS CAN BE             MAN723C
C                                 INEFFICIENT IF LINEAR PROBLEMS ARE             MAN724C
C                                 BEING SOLVED.                                  MAN725C
C                                                                                MAN726C
C        CARD 2. (2I5,7F10.0)                                                    MAN727C
C                                                                                MAN728C
C        ONE FOR EACH NPROP. SEE SECT.7.3 FOR DATA PREPARATION                   MAN729C
C                                                                                MAN730C
C                                                                                MAN731C
C        SUBSEQUENT CARDS FOR EACH TIME STEP IN THE SEQUENCE                     MAN732C
C                                                                                MAN733C
C        FORCE CARDS.  SEE SECTION 7. FOR DATA PREPARATION FORMATS.              MAN734C
C                                                                                MAN735C
C      8.3) MESH CHECKS AND PLOTS                                                MAN736C
C                                                                                MAN736C
C        COL  7 TO 10    MUST CONTAIN MESH.                                      MAN736C
C                        PROGRAM WILL CHECK THE PREVIOUSLY INPUTTED MESH         MAN736C
C                        FOR ERRORS, BUT DOES NOT PROCEED TO SOLVE.              MAN736C
C        COL  7 TO 10    MUST CONTAIN PLOT.                                      MAN736C
C                        PERFORMS SAME CHECKS AS FOR MESH AND PLOTS THE          MAN736C
C                        MESH. (GDS OR NOVA PLOT ROUTINES AVAILABLE).            MAN736C
C                                                                                MAN736C
C     8.4) INPUT OF FOURIER SERIES HARMONICS                                     MAN736C
C                                                                                MAN736C
C        COL  1 TO  5    IOUT. (SEE 8.)                                          MAN736C
C        COL  7 TO 12    FOURIE                                                  MAN736C
C                                                                                MAN736C
C        CARD 1.  (I5.5X,6F10.0)                                                 MAN736C
C                                                                                MAN736C
C        COL  1 TO 10    HARMONIC NUMBER                                         MAN736C
```

```
C        COL 11 TO 20     1 - FORCE/DISPL. MULTIPLIER                   MAN736C
C        COL 21 TO 30     2 - FORCE/DISPL. MULTIPLIER                   MAN736C
C        COL 31 TO 40     3 - FORCE/DISPL. MULTIPLIER                   MAN736C
C        COL 41 TO 50     4 - FORCE USER MULTIPLIER                     MAN736C
C        COL 51 TO 60     5 - FORCE USER MULTIPLIER                     MAN736C
C        COL 61 TO 70     6 - FORCE USER MULTIPLIER                     MAN736C
C        COL 71 TO 80     USER CONSTANT.                                MAN736C
C                                                                       MAN736C
C                                                                       MAN736C
C     9.) OUTPUT CONTROL FOR LIMITED PRINTS                             MAN737C
C                                                                       MAN738C
C        DISPLACEMENT OUTPUT CONTROL, IF IOUT .NE. 0.                   MAN739C
C                                                                       MAN740C
C        CARD 1.  (I5)                                                  MAN741C
C                                                                       MAN742C
C        COL  1 TO  5     NUMDIS - NUMBER OF DISPLACEMENT PRINT CARDS   MAN743C
C                                                                       MAN744C
C        SUBSEQUENT CARDS (2I5) SKIP IF NUMDIS = 0                      MAN745C
C                                                                       MAN746C
C        COL  1 TO  5     NODAL NUMBER TO BE OUTPUT.                    MAN747C
C        COL  6 TO 10     HIGHER NODE NUMBER OF A GENERATED SEQUENCE,   MAN748C
C                         IF ZERO JUST FIRST NODE IS COUNTED.           MAN749C
C        COL 11 TO 15     INCREMENT TO GENERATOR, DEFAULT = 1           MAN750C
C                         *** REPEAT UNTIL,NUMDIS CARDS HAVE BEEN READ  MAN751C
C                                                                       MAN752C
C        STRESS OUTPUT CONTROL, IF IOUT .NE. 0.                         MAN453C
C                                                                       MAN754C
C        CARD 1.  (I5,5X,9I1)                                           MAN755C
C                                                                       MAN756C
C        COL  1 TO  5     NUMSTR - NUMBER OF STRESS OUTPUT CARDS        MAN757C
C        COL 11 TO 19     NSIG(9) - PRINT PATTERN WITHIN AN ELEMENT.    MAN758C
C                         LOCAL POINTS OF EACH ELEMENT CAN BE           MAN759C
C                         SUPRESSED BY NON-ZERO ENTRIES AS FOLLOWS,     MAN760C
C                                                         E.G.          MAN761C
C              COL  11    SUPRESS PRINT AT LOCAL POINT 1, ( 0, 0, 0)    MAN762C
C              COL  12    SUPRESS PRINT AT LOCAL POINT 2, (-1, 0, 0)    MAN763C
C              COL  13    SUPRESS PRINT AT LOCAL POINT 3, ( 1, 0, 0)    MAN764C
C              COL  14    SUPRESS PRINT AT LOCAL POINT 4, ( 0,-1, 0)    MAN765C
C              COL  15    SUPRESS PRINT AT LOCAL POINT 5, ( 0, 1, 0)    MAN766C
C              COL  16    SUPRESS PRINT AT LOCAL POINT 6, ( 0, 0,-1)    MAN767C
C              COL  17    SUPRESS PRINT AT LOCAL POINT 7, ( 0, 0, 1)    MAN768C
C              COL  18    SUPRESS PRINT AT LOCAL POINT 8                MAN769C
C              COL  19    SUPRESS PRINT AT LOCAL POINT 9                MAN770C
C                                                                       MAN771C
C        SUBSEQUENT CARDS (2I5) SKIP IF NUMSTR = 0                      MAN772C
C                                                                       MAN773C
C        COL  1 TO  5     ELEMENT NUMBER TO BE PRINTED.                 MAN774C
C        COL  6 TO 10     HIGHER ELEMENT NUMBER OF A GENERATED SEQUENCE,MAN775C
C                         IF ZERO ONLY FIRST ELEMENT IS COUNTED.        MAN776C
C        COL 11 TO 15     INCREMENT TO GENERATOR, DEFAULT = 1           MAN777C
C                         *** REPEAT UNTIL,NUMSTR CARDS HAVE BEEN READ  MAN778C
C                                                                       MAN779C
C                                                                       MAN780C
C*********************************************************************  MAN781C
```

C. Listing of the Contact/Impact Subroutines Added to FEAP 74

The listings for the subroutines which are added to FEAP 74 for the contact/impact theory described herein are given below.

```
                SUBROUTINE CONTAC(ISW,IX,NEL1,NDF,NUMNP,DU,U,UD,UDD)
          C****************** CONTAC ***** 04/29/74 ******************************
000014          LOGICAL CFLAG,FLAGC
000014          DIMENSION IX(NEL1,NUMNP),DU(NDF,NUMNP),U(NDF,NUMNP),UD(NDF,NUMNP),
               C         UDD(NDF,NUMNP)
000014          DIMENSION RMP(10)
000014          COMMON/CNTACT/FLAGC,CFLAG,LIST,ICLIST(10),ICDEG(10),RU1,RU2,
               C     RM(2,10)
000014          COMMON/LABELS/HEAD(12),O,IPG,XHED(3),UHED(6),XH,FH,UH,NSTR,FLAG(7)
000014          COMMON/TAPES/ ITP5,ITP6
000014          DATA CFLAG,FLAGC/.FALSE.,.FALSE./
000014          GO TO (1,2,3),ISW
000022    1     READ(ITP5,1000) LIST,RU1,RU2
000034          RP = RU1+RU2
000036          IF(RP.EQ.0.0) RP = 1.
000044          READ(ITP5,1001) (ICLIST(L),ICDEG(L),L=1,LIST)
000070          WRITE(ITP6,2000) O,HEAD,IPG,RU1,RU2,(ICLIST(L),ICDEG(L),L=1,LIST)
000131          IPG = IPG + 1
000133          FLAGC = .TRUE.
000134          RETURN
000134    2      DO 200 L = 1,LIST
000141          N = ICLIST(L)
000143          IDEG = ICDEG(L)
000144          I = IX(1,N)
000150          RM(1,L) = DU(IDEG,I)
000155          I = IX(3,N)
000161    200   RM(2,L) = DU(IDEG,I)
000171          WRITE(ITP6,2002)
000175          DO 220 L = 1,LIST
000202          N = ICLIST(L)
000204          RMP(L) = RM(1,L) + RM(2,L)
000210          IF(RM(1,L).EQ.0.0.OR.RM(2,L).EQ.0.0) RMP(L) = RMP(L)*1.E+30
000222    220   WRITE(ITP6,2001) N,(RM(I,L),I=1,2),RMP(L)
000256          DO 210 N = 1,NUMNP
000260          DO 210 I = 1,NDF
000261    210   DU(I,N) = 0.0
000272          RETURN
000272    3     DO 300 L = 1,LIST
000274          N=ICLIST(L)
000276          IDEG = ICDEG(L)
000277          K = IX(2,N)
000303          UD(IDEG,K) = 0.0
000310          UDD(IDEG,K) = 0.0
000313          IF(U(IDEG,K)) 300,300,310
000317    310   I = IX(1,N)
000323          J = IX(3,N)
000326          CU =(RU1*UD(IDEG,I)+RU2*UD(IDEG,J))/RP
000342          UD(IDEG,I) = CU
000347          UD(IDEG,J) = CU
000352          CU=(RM(1,L)*UDD(IDEG,I)+RM(2,L)*UDD(IDEG,J))/RMP(L)
000375          IF(RM(1,L).GT.RM(2,L))U(IDEG,K)=U(IDEG,K)-RM(2,L)*(UDD(IDEG,J)-CU)
000420          IF(RM(1,L).LE.RM(2,L))U(IDEG,K)=U(IDEG,K)+RM(1,L)*(UDD(IDEG,J)-CU)
000443          UDD(IDEG,I) = CU
```

```
RUN FORTRAN COMPILER VERSION 2.3 B.3                                    06 MAY 74 14:09:30  PAGE NO. 2

000451          UDD(IDEG,J) = CU
000454    300   CONTINUE

000457          RETURN

000457   1000   FORMAT(I5,5X,2F10.0)
000457   1001   FORMAT(2I5)
000457   2000   FORMAT(A1,12A6,30X,4HPAGE,I4//5X,25HCONTACT SHEET DESCRIPTION//
               C      10X,13HBODY 1 RO*U =,E13.5/
               C      10X,13HBODY 2 RO*U =,E13.5//
               X   11X,7HELEMENT,1X,9HDIRECTION/(I15,I10))
000457   2001   FORMAT(I10,6E13.4)
000457   2002   FORMAT(/10H   ELEMENT,13H  BODY 1 MASS,13H  BODY 2 MASS)
000457          END
```

```
                SUBROUTINE ELMT05(N,MA,NDIM,NDF,NEL,NEL1,NSTF,NSIZV,NVEC,MCT,DM,D,        ELM  1C
               X XYZ,IX,F,FORCE,ESTIF,U,VECT,ISW)
          C****************** ELMT05 ***** 04/29/74 ******************************
000027          LOGICAL NPR
000027          DIMENSION ESTIF(NSTF,NSTF),FORCE(NSTF,2)
000027          DIMENSION U(NDF,1),IX(NEL1,1),PLOT(8),THED(8)
000027          COMMON/LOCALS/ DUL(6,20),UL(6,20),UDL(6,20),UDDL(6,20)
000027          COMMON/PRTPLT/ NSIG(9),NPLT(9,2),NT,NSTEP,NUMPLT,NEDATA(20,3),NPR
000027          COMMON/SHAP/ XJAC,SHAPE(4,20),SG(3,3),SK(3,3),X(3,20)
000027          COMMON/TAPES/ ITP5,ITP6
000027          COMMON/TIMHIS/ TIME,DT,DTP,NH,ISZH,C0,C1,C2,C3,C4,C5,C6,NCT
000027          EQUIVALENCE (PLOT,TAU),(PLOT(2),ETA)
000027          DATA THED/3HTAU,3HETA,2HU1,3HUD1,4HUDD1,2HU3,3HUD3,4HUDD3/
000027          GO TO (1,2,3,3,5,3), ISW
000041    1     READ(ITP5,1000) IDEG,NPS
000051          IDEG = MAX0(IDEG,1)
000055          WRITE(ITP6,2000) IDEG,NPS
000064          TOL = 1.E-06
000066          I = NDF  + IDEG
000073          J = I + NDF
000074          RI = 0.5
000075          RJ = 0.5
000076          C6 = 0.
000077          NV = NPS
000100          NA = NPS + NPS
000101          RETURN
000102    2     DM = X(IDEG,3) - X(IDEG,1)
000106          DM = DM + 1.E-25
000110          RETURN
000110    3     TAU = UL(IDEG,2)
000112          DD = (X(IDEG,3)-X(IDEG,1)) + (UL(IDEG,3)-UL(IDEG,1))
000120          ETA = 0.
000121          IF(DD.LT.TOL) ETA = 1.
000125          IF(TAU.LT.0.0  ) ETA = 0.0
000130          IF(ISW.GT.3) GO TO 4
000134          ESTIF(IDEG,I) = ETA
000142          ESTIF(I,IDEG) = ETA
000145          ESTIF(I,J) = -ETA
000151          ESTIF(J,I) = -ETA
000155          ESTIF(I,I) = 1.0 - ETA
000163          RETURN
000163    4      IF(ISW.EQ.4) GO TO 6
000166          FORCE(I,1) = ETA*DD
000174          IF(ETA.EQ.0.0) FORCE(I,1) = - TAU
000203          TAU = AMAX1(0.0,TAU)
000207          FORCE(IDEG,1) = -TAU
000215          FORCE(J,1) = TAU
000220    6     IF(.NOT.NPR) WRITE(ITP6,2001) N,DM,MA,TAU,ETA
000243          IF(NUMPLT.LE.0) RETURN
000246          PLOT(3) = UL(IDEG,1)
000250          PLOT(4) = UDL(IDEG,1)
000252          PLOT(5) = UDDL(IDEG,1)
000254          PLOT(6) = UL(IDEG,3)
```

```
 RUN FORTRAN COMPILER VERSION 2.3 B.3                                      06 MAY 74 14:09:30  PAGE NO. 2

000256           PLOT(7) = UDL(IDEG,3)
000260           PLOT(8) = UDDL(IDEG,3)
000262           DO 60 K = 1,NUMPLT
000263           KK = NPLT(K,2)
000265           IF(NPLT(K,1).GT.0) CALL PLDATA(NDIM,NPLT(K,1),THED(KK),X(1,2),
               C    PLOT(KK))
000306    60     CONTINUE
000311    5      RETURN
000312    1000   FORMAT(2I5)
000312    2000   FORMAT(5X,21HPOINT CONTACT ELEMENT/
               C           5X,27HCONTACT DEGREE OF FREEDOM = I3,5X,5HNPS = I5)
000312    2001   FORMAT(5X,7HELEMENT,I5,5X,A5,5X,8HMATERIAL,I3,5X,5HTAU =,E15.5,5X,
               X    5HETA =,F3.1)
000312           END
```

```
                 SUBROUTINE ELMT09(N,MA,NDIM,NDF,NELM,NEL1,NSTF,NSIZV,NVEC,MCT,DM,
                X D,XYZ,IX,F,FORCE,ESTIF,U,VECT,ISW)                                  ELM  1C
000027           LOGICAL NOPRNT,NPR,NPL                                               ELM  2C
000027           DIMENSION ESTIF(NSTF,NSTF),D(3,21,1),IX(NEL1,1),U(NDF,1),
                X V(3,20),DU(3),XX(3),FORCE(NSTF,2)                                   ELM  4C
000027           COMMON/GAUS/ LIM,SGAUSS(5,5),WGAUSS(5,5)                             ELM  5C
000027           COMMON/LABELS/HEAD(12),O,IPG,XHED(3),UHED(6),XH,FH,UH,NSTR,FLAG(7)   ELM  8C
000027           COMMON/LOCALS/ DUL(6,20),UL(6,20),UDL(6,20),UDDL(6,20)
000027           COMMON/PRTPLT/ NSIG(9),NPLT(9,2),NT,NSTEP,NUMPLT,NEDATA(20,3),NPR
000027           COMMON/SHAP/ XJAC,SHAPE(4,20),SG(3,3),SK(3,3),X(3,20)
000027           COMMON/TAPES/ ITP5,ITP6                                              ELM 10C
000027           COMMON/TIMHIS/ TIME,DT,DTP,NH,ISZH,C0,C1,C2,C3,C4,C5,C6,NCT
000027           DATA SH,EH,BL/6HSTRESS,6HSTRAIN,6H      /                            ELM 11C
000027           GO TO (1,2,3,4,5,4) , ISW
000041    1      CONTINUE                                                             ELM 13C
          C....  BAR STIFFNESS CHARACTERIZATION                                       ELM 14C
000041           READ(ITP5,1000) E,AA,RO
000053           WRITE(ITP6,2100) E,AA,RO
000065           D(1,1,MA) = E
000075           D(1,2,MA)=AA                                                         ELM 17C
000100           D(1,3,MA)=RO
000103           C6 = 0.
000104           RETURN
000104    2      CONTINUE                                                             ELM 20C
000104           RETURN                                                               ELM 21C
          C....  COMPUTATION OF BAR STIFFNESS NO BENDING * ISOPARAMETRIC 1ST AND 2N   ELM 22C
000105    3      CONTINUE                                                             ELM 23C
          C....  CONSTRUCT STIFFNESS AT INTEGRATION POINTS                            ELM 24C
000105           EA = D(1,1,MA)*D(1,2,MA)                                             ELM 25C
000113           RA = D(1,3,MA)*D(1,2,MA)                                             ELM 26C
000116           DO 260 II = 1,NELM                                                   ELM 27C
000120           SS = SGAUSS(II,NELM)                                                 ELM 28C
000123           CALL LINE(SS,NDIM,NELM)                                              ELM 29C
000125           WW=WGAUSS(II,NELM)                                                   ELM 30C
000134           DVOL=EA*WW/(XJAC*XJAC*XJAC)
          C....  COMPUTE A LUMPED MASS MATRIX
000137           IU = 0                                                               ELM 32C
000140           RAA=RA*XJAC*WW                                                       ELM 33C
000142           DO 217 I = 1,NELM
000143           AA=RAA*SHAPE(2,I)                                                    ELM 34C
000146           DO 215 KK = 1,NDIM
000150    215    FORCE(IU+KK,2) = FORCE(IU+KK,2) + AA                                 ELM 35C
000162    217    IU = IU + NDF                                                        ELM 37C
000166           DO 250 KK = 1,NDIM                                                   ELM 38C
000167           DO 250 LL = KK,NDIM                                                  ELM 39C
000170           SGD=SG(KK,LL)*DVOL                                                   ELM 40C
000174           I1 = KK
000175           DO 240 I = 1,NELM                                                    ELM 41C
000177           SHI=SHAPE(1,I)*SGD                                                   ELM 42C
000202           L = 1
000203           IF(KK.EQ.LL) L = I                                                   ELM 43C
000206           J1 = LL + NDF*(L-1)                                                  ELM 44C
000213           DO 230 J = L,NELM                                                    ELM 45C
                                                                                      ELM 46C
```

```
000214          ESTIF(I1,J1)=ESTIF(I1,J1)+SHI*SHAPE(1,J)
000225   230    J1 = J1 + NDF
000231   240    I1 = I1 + NDF                                   ELM 48C
000234   250    CONTINUE                                        ELM 49C
000241   260    CONTINUE                                        ELM 50C
         C....  CONSTRUCT SYMMETRIC PART OF STIFFNESS MATRIX    ELM 51C
000243          KK = NELM*NDF                                   ELM 52C
000245          LL = KK - 1                                     ELM 53C
000247          DO 360 I=1,LL                                   ELM 54C
000250          K = I + 1                                       ELM 55C
000252          DO 360 J = K,KK                                 ELM 56C
000254          ESTIF(I,J) = ESTIF(I,J) + ESTIF(J,I)            ELM 57C
000266   360    ESTIF(J,I) = ESTIF(I,J)                         ELM 58C
000302          IF(C6.EQ.0.0) RETURN                            ELM 59C
000304          DO 370 I = 1,NSTF
000306   370    ESTIF(I,I) = ESTIF(I,I) + C6*FORCE(I,2)
000324          RETURN
000325   4      CONTINUE                                        ELM 60C
000325          E = D(1,1,MA)                                   ELM 61C
000331          NELP = NELM                                     ELM 64C
000332          DO 400 NN = 1,NELP
000334          NOPRNT=NPR
000336          IF(NSIG(NN).GT.0) NOPRNT=.TRUE.
000341          NPL=.TRUE.
000342          IF(NPLT(NN,1).GT.0) NPL = .FALSE.
000345          SS = SGAUSS(NN,NELP)
000351          CALL LINE(SS,NDIM,NELM)
         C....  COMPUTE STRESS AND STRAIN                       ELM 71C
000353          DO 100 KK = 1,NDIM                              ELM 72C
000360          XX(KK) = 0.0                                    ELM 73C
000361          DU(KK)= 0.                                      ELM 74C
000362          DO 100 LL = 1,NELM                              ELM 75C
000364          XX(KK) = XX(KK) + X(KK,LL)*SHAPE(2,LL)          ELM 76C
000372   100    DU(KK) = DU(KK) +UL(KK,LL)*SHAPE(1,LL)          ELM 77C
000404          EPS = 0.
000405          DO 200 KK = 1,NDIM                              ELM 79C
000406   200    EPS = EPS + SK(KK,1)*DU(KK)/XJAC/XJAC           ELM 80C
000416          SIG = E*EPS                                     ELM 81C
000420          IF(NOPRNT) GO TO 350                            ELM 82C
000421          MCT = MCT - 1
000423          IF(MCT.GT.0) GO TO 300                          ELM 83C
000425          MCT=50                                          ELM 84C
000426          WRITE(ITP6,2000) HEAD,TIME,IPG,( XHED(I),XH,I=1,NDIM),BL,SH,BL,EH
000465          IPG = IPG + 1
000467    300   WRITE(ITP6,2001) N,MA,DM,(XX(I),I=1,NDIM),SIG,EPS   ELM 87C
000523   350    IF(.NOT.NPL) CALL PLDATA(NDIM,NPLT(NN,1),5HAXIAL,XX,SIG)   ELM 88C
000542          IF(ISW.EQ.4) GO TO 400
000545          WT = WGAUSS(NN,NELM)*D(1,2,MA)
000553          WW = WT*D(1,3,MA)*XJAC
000557          IU = 0
000560          DO 600 I = 1,NELM
000562          AF = WW*SHAPE(2,I)
000565          BB = SHAPE(1,I)*SIG*WT/XJAC
000572          DO 610 KK = 1,NDIM
```

```
000573   615   FORCE(KK+IU,1) = FORCE(KK+IU,1) - AA*UDDL(KK,I) - BB*SK(KK,I)
000615   600   IU = IU + NDF
000621   400   CONTINUE                                                        ELM 91C
000624   5     RETURN
000625   1000  FORMAT(3F10.0)                                                  ELM 93C
000625   2000  FORMAT(1H1,12A6,E13.5,17X,4HPAGE,I3//10H   ELEMENT,4X,8HMATERIAL,
              X 3X,5(2A6))
000625   2001  FORMAT(I10,I5,5X,A5,5E12.4)                                     ELM 96C
000625   2100  FORMAT(37H0LINEAR ELASTIC MATERIAL, BAR ELEMENT//
              X       5X,3HE =,E15.5,5X,6HAREA =,E15.5,5X,9HDENSITY =E15.5/)   ELM 97C
000625   4000  FORMAT(A5,I5,1P3E12.5,24X,I10)                                  ELM 98C
000625         END                                                             ELM 99C
```

```
RUN FORTRAN COMPILER VERSION 2.3 B.3                                  06 MAY 74 14:09:30  PAGE NO. 1

               SUBROUTINE LINE(S,NDIM,NEL)                                                   LIN  1C
          C***** LINE         ****** 7/02/73 ***************************************
          C.... SHAPE FUNCTION ROUTINE FOR 1ST AND 2ND ORDER ISOPARAMETRIC LINES             LIN  2C
000006         COMMON/SHAP/ XJAC,SHAPE(4,20),SG(3,3),SK(3,3),X(3,20)
          C.... FORM SHAPE FUNCTIONS                                                         LIN  4C
000006         SHAPE(1,1) = -0.5                                                             LIN  5C
000007         SHAPE(2,1) = 0.5*(1.0-S)                                                      LIN  6C
000012         SHAPE(1,NEL) = 0.5                                                            LIN  7C
000014         SHAPE(2,NEL) = 0.5*(1.0+S)                                                    LIN  8C
000016         IF(NEL.EQ.2) GO TO 350                                                        LIN  9C
000020         SHAPE(1,2) = -2.0*S                                                           LIN 10C
000021         SHAPE(2,2) = 1.0-S*S                                                          LIN 11C
000024         K = NEL-1                                                                     LIN 12C
000026         DO 100 I = 1,NEL,K                                                            LIN 13C
000027         SHAPE(1,I) = SHAPE(1,I) - 0.5*SHAPE(1,2)                                      LIN 14C
000033    100  SHAPE(2,I) = SHAPE(2,I) - 0.5*SHAPE(2,2)                                      LIN 15C
          C.... FORM JACOBIAN MATRIX AND DETERMINANT                                         LIN 16C
000040    350  DO 360 I = 1,3                                                                LIN 17C
000042    360  SK(I,1) = 0.0                                                                 LIN 18C
000046         DO 400 I = 1,NDIM                                                             LIN 19C
000047         DO 400 J = 1,NEL                                                              LIN 20C
000050    400  SK(I,1) = SK(I,1) + X(I,J)*SHAPE(1,J)                                         LIN 21C
000063         XJAC = 0.0                                                                    LIN 22C
000064         DO 500 I = 1,NDIM                                                             LIN 23C
000065         XJAC = XJAC + SK(I,1)*SK(I,1)                                                 LIN 24C
000070         DO 500 J = 1,NDIM                                                             LIN 25C
000071    500  SG(I,J) = SK(I,1)*SK(J,1)                                                     LIN 26C
000103         XJAC = SQRT(XJAC)                                                             LIN 27C
000105         RETURN                                                                        LIN 28C
000105         END                                                                           LIN 29C
```

```
              SUBROUTINE NLNORM(ISW,MDEG,NN,U,DU,DF,F,PROP,NCT,NTT,IDEST,IBLK)      NLN  1C
        C***** NLNORM    ****** 12/14/73   ********************************************  NLN  2C
000017        DIMENSION U(1),DU(MDEG),DF(1),F(MDEG),IDEST(1)                        NLN  3C
000017        COMMON/NORMS/ UNORM,DNORM,ANORM,CS,CSP,DP,DNP,IFL,XFLAG,ERR           NLN  4C
000017        GO TO (100,200,200,100), ISW                                          NLN  5C
000026    100 CS=1.0                                                                NLN  6C
000027        CSP=1.0                                                               NLN  7C
000030        DP=0.75                                                               NLN  8C
000032        IFL=0                                                                 NLN  9C
000032        XFLAG=.FALSE.                                                         NLN 10C
000033        ERR=1.0E-3                                                            NLN 11C
000035        RETURN                                                                NLN 12C
000036    200 UNORM = 0.                                                            NLN 13C
000037        DNORM = 0.                                                            NLN 14C
000037        ANORM=0.                                                              NLN 15C
000041        DO 500 N = 1,MDEG                                                     NLN 16C
000042        DUN = DU(N)                                                           NLN 17C
000044        UN = U(N)                                                             NLN 18C
000046        IF(ISW.EQ.3) UN = UN + DF(N) + DUN                                    NLN 19C
000053        ANORM=ANORM+UN*DUN                                                    NLN 20C
000056        UNORM=UNORM+UN*UN                                                     NLN 21C
000060    500 DNORM=DNORM+DUN*DUN                                                   NLN 22C
000065        UNORM=SQRT(UNORM)                                                     NLN 23C
000067        DNORM=SQRT(DNORM)                                                     NLN 24C
000070        IF(NN.EQ.1) DNP=UNORM                                                 NLN 25C
000077        CS=1.0                                                                NLN 26C
000100        IF(UNORM*DNORM.NE.0.) CS=ANORM/(UNORM*DNORM)                          NLN 27C
000105        IF(IFL.EQ.1) DP=0.75                                                  NLN 28C
000111        IF(IFL.EQ.1.AND.XFLAG) CSP=CS                                         NLN 29C
000120        IFL=0                                                                 NLN 30C
000121        IF(DNORM.LE.0.5*UNORM) GO TO 550                                      NLN 31C
000124        IFL=1                                                                 NLN 32C
000125        DP=0.5*UNORM/DNORM                                                    NLN 33C
000127    550 IF(DP*DNORM.GT.DNP) DP=DNP/DNORM                                      NLN 34C
000134        IF(DP.EQ.0.0) DP = 1.0                                                NLN 35C
000136        IF(CS*CSP.LT.0.) DP=DP/2.                                             NLN 36C
000142        IF(CS*CSP.GT.0.) DP=1.25*DP                                           NLN 37C
000146        DNP=DP*DNORM                                                          NLN 38C
000150        CSP=CS                                                                NLN 39C
000150        IF(ISW.EQ.2) RETURN                                                   NLN 40C
000153        DO 600 N = 1,MDEG                                                     NLN 41C
000155        DF(N) = DF(N) + DU(N)                                                 NLN 42C
000160        IF(NCT.EQ.0) DF(N)= DU(N)                                             NLN 43C
000164  600   DU(N) = DF(N)                                                         NLN 44C
000171        RETURN                                                                NLN 45C
000171        END
```

```
               SUBROUTINE TSOLVE(NUMNP,NUMEL,NUMMAT,NDIM,NDF,NEN,NEL1,NSTF,NVEC,        TSO  1C
             1   NSIZV,NICD,NSICD,IBLK,ISZA,NEQB,MAXBAN,NDEG,ID1,M8,IBUF,DS,            TSO  2C
             2  TYPE,D,ICOD,XYZ,F,IX,IDEST,VECT,FORCE,ESTIF,LD,A,DU,MAXB,H,U,DF,        TSO  3C
             3   NSEQ,TYME)                                                             TSO  4C
       C***** TSOLVE      ****** 03/06/74   ***********************************          TSO  5C
000053         LOGICAL CFLAG,FLAGC
000053         LOGICAL NPR,VFLAG,DFLAG                                                  TSO  6C
000053         REAL LABL(12)
000053         DIMENSION TYPE(1),ICOD(1),XYZ(NDIM,1),F(NDF,1),IX(NEL1,1),               TSO  8C
             1   IDEST(NDF,1),VECT(NSIZV,1),FORCE(NSTF,2),ESTIF(NSTF,NSTF),             TSO  9C
             2   A(NEQB,1),DU(NDF,1),H(IBUF),U(NDF,1),DF(1),DS(M8),TYME(7),LD(1)        TSO 10C
000053         COMMON/CNTACT/FLAGC,CFLAG,LIST,ICLIST(10),ICDEG(10),RU1,RU2,
             C       RM(2,10)
000053         COMMON/LABELS/HEAD(12),O,IPG,XHED(3),UHED(6),XH,FH,UH,NSTR,FLAG(7)       TSO 11C
000053         COMMON/LOCALS/ DUL(6,20),UL(6,20),UDL(6,20),UDDL(6,20)                   TSO 12C
000053         COMMON/NORMS/ UNORM,DNORM,ANORM,CS,CSP,DP,DNP IFL,XFLAG,ERR              TSO 13C
000053         COMMON/PRTPLT/ NSIG(9),NPLT(9,2),NT,NSTEP,NUMPLT,NEDATA(20,3),NPR        TSO 14C
000053         COMMON/SHAP/ XJAC,SHAPE(4,20),SG(3,3),SK(3,3),X(3,20)                    TSO 15C
000053         COMMON/TAPES/ ITP5,ITP6                                                  TSO 16C
000053         COMMON/TIMHIS/ TIME,DT,DTP,NH,ISZH,C0,C1,C2,C3,C4,C5,C6,NCT              TSO 17C
000053         COMMON/VTAPE/ ITP13,ITP14,ITPD,ITWR                                      TSO 18C
000053         DATA AWORD1,AWORD2/8H  F13.4,,8H  E13.4,/,AWORD3/8H  F12.2,/
000053         DATA VH,AH,AWORD4,AWORD5/6H VEL. ,6H ACCEL ,6H(/12X,,8H9E13.4))/
000053         DATA LABL(1)/8H (I12,  /                                                 TSO 20C
000053         DATA CFLAG,FLAGC/.FALSE.,.FALSE./
       C....   SUBROUTINE TO PERFORM IMPLICIT INTEGRATION WITH 1 OR 3 INIT. COND.       TSO 21C
000053         DO 2 N=1,NEN                                                             TSO 22C
000054         DO 2 J=1,NDF                                                             TSO 23C
000055         DUL(J,N)=0.                                                              TSO 24C
000060         UL(J,N) = 0.0                                                            TSO 25C
000063         UDL(J,N)=0.                                                              TSO 26C
000065   2     UDDL(J,N) = 0.0                                                          TSO 27C
000073         DO 3 I=2,7                                                               TSO 28C
000074   3     NSIG(I)=1                                                                TSO 29C
000077         IF(IBLK.EQ.0) GO TO 4                                                    TSO 30C
000101         KU = NUMNP*NICD                                                          TSO 31C
000104         REWIND 7                                                                 TSO 32C
000106         WRITE(7) DS                                                              TSO 33C
       C....   SET THE OUTPUT LABELS FOR DISPLACEMENT PRINTS                            TSO 34C
000117   4     HEAD(12) = 6H TIME=                                                      TSO 35C
000124         IFLG =-NSICD                                                             TSO 36C
000126         MDEG = NSICD                                                             TSO 37C
000127         MDEG2 = MDEG + MDEG                                                      TSO 38C
000130         I = 2                                                                    TSO 39C
000131         DO 5 N = 1,NDIM                                                          TSO 40C
000132         LABL(I) = AWORD1                                                         TSO 41C
000134   5     I = I + 1                                                                TSO 42C
000137         L = NDIM + 1
000140         DO 7 N = L,9
000141         LABL(I) = AWORD2
000143   7     I = I + 1
000146         LABL(I) = AWORD4
000150         LABL(I+1) = AWORD5
```

```
000152           DTP = 0.                                                    TSO 44C
000153           TIME = 0.                                                   TSO 45C
000154           MB = MAXBAN + 1                                             TSO 46C
000156           NE1 = NEN + 1                                               TSO 47C
        C.... INITIALIZE INCREMENTAL FORCE VECTOR                            TSO 48C
000157           DO 6 N = 1,NSICD                                            TSO 49C
000160   6       DF(N) = 0.0                                                 TSO 50C
000165           DO 10 N = 1,NUMEL                                           TSO 51C
000166   10      IX(NE1,N) = 0                                               TSO 52C
        C.... PUT INITIAL DATA ON TAPE                                       TSO 53C
000177           ISZH = IBUF                                                 TSO 54C
000201           ITRD = ITP13                                                TSO 55C
000202           ITWR = ITP14                                                TSO 56C
000204           REWIND ITRD                                                 TSO 57C
000206           IF(IBLK.GT.0) WRITE(ITRD) ((U(I,J),I=1,NDF),J=1,KU)         TSO 58C
000241           DO 11 N = 1,ISZH                                            TSO 59C
000243   11      H(N) = 0.                                                   TSO 60C
000247           NTB= (NH+ISZH-1)/ISZH                                       TSO 61C
000253           WRITE(ITRD) H                                               TSO 62C
000264           IF(NTB.LE.1) GO TO 13                                       TSO 63C
000272           NT = NTB*2                                                  TSO 64C
000273           DO 12 N = 1,NT                                              TSO 65C
000275   12      WRITE(ITRD) H                                               TSO 66C
000314   13        REWIND ITRD                                               TSO 67C
000316           IF(IBLK.GT.0) READ(ITRD) DM                                 TSO 68C
000331           NEP = NUMNP + NUMNP                                         TSO 69C
000332           C6 = 0.0                                                    TSO 70C
000333           NSTEP = 0                                                   TSO 71C
000334           NST = 0                                                     TSO 72C
000335           NUMPLT=0                                                    TSO 73C
000336           PROP = 0.                                                   TSO 74C
000337           CFLAG = .FALSE.                                             TSO 75C
000340           DO 900 M = 1,NSEQ                                           TSO 76C
000341           READ(ITP5,1000) DT,NTS,INT,NNI,NNF,NEI,NEF,NPROP,NFORC,C0,DM,NT,NL   TSO 77C
000376           IF(NICD. GT.1) C6= DM                                       TSO 78C
000406           IF(NL.NE.0) NST= NL                                         TSO 79C
000410           NTT = IABS(NST)                                             TSO 80C
000412           DFLAG = .FALSE.                                             TSO 81C
000413           IF (NTT. NE. 0 )       DFLAG = .TRUE.                       TSO 82C
000415           NSTEP=NSTEP+NTS                                             TSO 83C
000417           IF(M.EQ.1) NUMPLT=NT                                        TSO 84C
000422           IF(INT.LE.0) INT = 1                                        TSO 85C
000425           WRITE(ITP6,2000) O,HEAD,TIME,IPG,DT,NTS,INT,NNI,NNF,NEI,NEF TSO 86C
000457           IF(DFLAG) WRITE(ITP6,2032) NST                              TSO 87C
000476           IPG = IPG + 1                                               TSO 88C
000500           IF(NICD.EQ.1) GO TO 1                                       TSO 89C
000501           IF(DT.EQ.0.0) GO TO 901                                     TSO 90C
000502           CALL UPDATE(3)                                              TSO 91C
000504   1       IF(NPROP.GT.0) PROP = PROPLD(TIME,NPROP)                    TSO 92C
000517           IF(NFORC.GT.6.AND.NPROP.GT.0) WRITE(ITP6,4001)              TSO 93C
        C.... READ STRESS PLOT INFORMATION                                   TSO 94C
000536           IF(NUMPLT.LE.0.OR.M.GT.1) GO TO 50C                         TSO 95C
000547           REWIND 12                                                   TSO 96C
000551           WRITE(ITP6,2005)
```

```
000555          DO 501 N=1,NUMPLT
000562          READ(ITP5,1006) (NEDATA(N ,I),I=1,3)                                      TS0 97C
000600   501    WRITE(ITP6,2006) N,(NEDATA(N ,I),I=1,3)                                   TS0 98C
000632          CALL PLZERO                                                               TS0 99C
000633   500    VFLAG = .FALSC.                                                           TS0100C
000634          DO 600 NT = 1,NTS                                                         TS0101C
000641          PPROP = PROP                                                              TS0102C
000643          PROP = 1.0                                                                TS0103C
000644          IF(DFLAG)CALL NLNORM(4,MDEG,NT,U,DF,DU,F,PPROP,NCT,NTT,IDEST,IBLK)        TS0104C
000667          IF(NPROP.GT.0) PROP = PROPLD(TIME+DT,0)                                   TS0105C
000701          IF(NFORC.GT.0) CALL RESET(-NFORC,NUMNP,NDF,F)                             TS0106C
000713          NCT = 0                                                                   TS0107C
000714   14     REWIND ITWR                                                               TS0103C
000716          IF(IBLK.EQ.0) GO TO 15                                                    TS0109C
000720          IF(.NOT.VFLAG) GO TO 20                                                   TS0110C
000721          DO 21 I = 1,NDEG                                                          TS0111C
000723   21     DF(I) = 0.                                                                TS0112C
000730          GO TO 20                                                                  TS0113C
000730   15     DO 19 N = 1,NUMNP                                                         TS0114C
000732          DO 18 K = 1,NDF                                                           TS0115C
000733          J = IDEST(K,N)                                                            TS0116C
000740          IF(J.EQ.0) GO TO 18                                                       TS0117C
000741          IF(VFLAG) GO TO 17                                                        TS0118C
000743          DO 16 I = 1,MAXBAN                                                        TS0119C
000744   16     A(J,I) = 0.0                                                              TS0120C
000756   17     DF(J) = F(K,N)*PROP                                                       TS0121C
000766   18     CONTINUE                                                                  TS0122C
000771   19     CONTINUE                                                                  TS0123C
000773   20     NH = 1                                                                    TS0124C
000774          IF(NTB.GT.1) READ(ITRD) H                                                 TS0125C
001614          IF(NCT.GT.0) GO TO 44                                                     TS0126C
        C.... OUTPUT THE SOLUTION VECTOR FOR THE CURRENT TIME                             TS0127C
001017          IF(NNI.EQ.0.OR.((NT-1)/INT)*INT.NE.NT-1) GO TO 40                         TS0128C
001032          MCT = 0                                                                   TS0129C
001033          DO 30 N = NNI,NNF                                                         TS0130C
001034          MCT = MCT - 1                                                             TS0131C
001036          IF(MCT.GT.0) GO TO 31                                                     TS0132C
001040          IF(NICD.EQ.1) WRITE(ITP6,2001) O,HEAD,TIME,IPG,PPROP,M,NT,
               X   (XHED(I),XH,I=1,NDIM),(UHED(I),UH,I=1,NDF)
001121          IF(NICD.NE.1) WRITE(ITP6,2001) O,HEAD,TIME,IPG,PPROP,M,NT,
               X   (XHED(I),XH,I=1,NDIM),(UHED(I),UH,I=1,NDF),(UHED(I),VH,I=1,NDF)
               X  ,(UHED(I),AH,I=1,NDF)
001235          IPG = IPG + 1                                                             TS0136C
001237          MCT = 50                                                                  TS0.37C
001240   31     IF(NICD.EQ.1)WRITE(ITP6,LABL)N,(XYZ(I,N),I=1,NDIM),(U(I,N)
               X   ,I=1,NDF)
001311          IF(NICD.NE.1)WRITE(ITP6,LABL)N,(XYZ(I,N),I=1,NDIM),(U(I,N),I=1,NDF
               X ) ,(U(I,NUMNP+N),I=1,NDF),(U(I,NEP+N),I=1,NDF)
001420   30     CONTINUE
        C.... UPDATE (1) * * FOR DYNAMIC SOLUTIONS ONLY                                   TS0139C
001423   40     IF(NICD.GT.1) CALL UPDATE(1,MDEG,NICD,U,U(1,NUMNP+1),U(1,NEP+1),          TS0140C
               X   DU,F,DF,IDEST,PROP)                                                    TS0141C
001457          IF(IBLK.GT.0) WRITE(ITWR) ((U(I,J),I=1,NDF), J=1,KU )                     TS0142C
001512   44     MP = 0                                                                    TS0143C
```

```
001513             TEMP = 0.                                                      TS0144C
001514             DO 400 N = 1,NUMEL                                             TS0145C
001515             NPR = .TRUE.                                                   TS0146C
001516             DO 43 I = 1,9                                                  TS0147C
001520             NPLT(I,1) = 0                                                  TS0148C
001521      43     NPLT(I,2) = 0                                                  TS0149C
001524             IF(NCT.GT.0) GO TO 46                                          TS0150C
001526             IF(N.GE.NEI.AND.N.LE.NEF) NPR = .FALSE.                        TS0151C
001540             IF(NUMPLT.LE.0) GO TO 46                                       TS0152C
001542             DO 45 I = 1,NUMPLT                                             TS0153C
001543             IF(NEDATA(I,1).NE.N) GO TO 45                                  TS0154C
001546             J = NEDATA(I,2)                                                TS0155C
001547             NPLT(J,1) = I                                                  TS0156C
001551             NPLT(J,2) = NEDATA(I,3)                                        TS0157C
001553      45     CONTINUE                                                       TS0158C
001556      46     MA = MOD(IX(NEL1,N),100)                                       TS0159C
001567             IF(MR.LE.0) MRR= IX(NEL1,N)/1000                               TS0160C
001600             IF(MR.LE.0) MR = MRR                                           TS0161C
001603             DO 60 I = 1,NSTF                                               TS0162C
001605             FORCE(I,1) = 0.                                                TS0163C
001612             FORCE(I,2)=0.                                                  TS0164C
001616             LD(I) = 0                                                      TS0165C
001620             IF(MR.NE.MRR.OR.VFLAG) GO TO 60                                TS0166C
001625             DO 50 J = 1,NSTF                                               TS0167C
001627      50     ESTIF(I,J) = 0.0                                               TS0168C
001640      60     CONTINUE                                                       TS0169C
001643             L = 0                                                          TS0170C
001644             DO 110 I = 1,NEN                                               TS0171C
001645             K = IX(I,N)                                                    TS0172C
001653             DO 90 J = 1,NDIM                                               TS0173C
001654      90     X(J,I) = 0.                                                    TS0174C
001661             IF(K.EQ.0)  GO TO 120                                          TS0175C
001662             NEL = I                                                        TS0176C
001663             DO 100 J = 1,NDIM                                              TS0177C
001665      100    X(J,I) = XYZ(J,K)                                              TS0178C
001701             DO 110 J = 1,NDF                                               TS0179C
001702             L = L + 1                                                      TS0180C
001704             DUL(J,I) = DU(J,K)                                             TS0181C
001713             UL(J,I) = U(J,K)                                               TS0182C
001722             IF(NCT.GT. 0) UL(J,I) =UL(J,I) +DUL(J,I)                       TS0183C
001731             IF(NICD.EQ.1) GO TO 110                                        TS0184C
001734             UDL(J,I) = U(J,K+NUMNP)                                        TS0185C
001744             UDDL(J,I) = U(J,K+NEP)                                         TS0186C
001754             IF(NCT.GT.0) UDDL(J,I)= UDDL(J,I) +C6*DUL(J,I)                 TS0187C
001765      110    LD(L) = IDEST(J,K)                                             TS0188C
002001      120    DM = TYPE(MA)                                                  TS0189C
002004             IF(DM.NE.TEMP) MCT = 0                                         TS0190C
002007             TEMP = DM                                                      TS0191C
002010             CALL TICTOC(TYME,6)                                            TS0192C
            C.... COMPUTE ELEMENT STRESSES AND UPDATE FORCES                      TS0193C
002013             CALL ELMLIB(N,MA,NDIM,NDF,NEL,NEL1,NSTF,NSIZV,NVEC,MCT,DM,D,XYZ,  TS0194C
                  X IX,H,FORCE,ESTIF,U,VECT,6)                                    TS0195C
002047             CALL TICTOC(TYME,5)                                            TS0196C
            C.... FORM STIFFNESSIF NEEDED FOR THE NEXT TIME STREP                 TS0197C
```

```
002052         IF((.NOT.VFLAG.AND.MR.EQ.MRR).OR.(VFLAG.AND.IX(NE1,N).EQ.1))
              XCALL ELMLIB(N,MA,NDIM,NDF,NEL,NEL1,NSTF,NSIZV,NVEC,MCT,DM,D,XYZ,
              X IX,H,FORCE,ESTIF,U,VECT,3)
002131         IF(MOD(IX(NEL1,N),1000)/100.GT.0.AND..NOT.VFLAG)
              X CALL PRTMAT(HEAD,IPG,N,NSTF,ESTIF,FORCE,LD,NSTF,0)
         C.... MODIFY FOR THE DISPLACEMENT B.C.
002165         CALL MODIFY(NDF,NEL,NEL1,NE1,IBLK,NSTF,PROP,IX,ICOD,F,FORCE,ESTIF,
              X   N)
002205         IF(VFLAG) GO TO 300
002212         IF(IBLK.EQ.0) CALL COREIN(A,DF,NEQB,ESTIF,FORCE,LD,NSTF)
002231         IF(IBLK.GT.0) WRITE(7) ESTIF,FORCE,LD
002263         IF(.NOT.FLAGC.OR.CFLAG) GO TO 400
002266         L = 1
002267         DO 130 I = 1,NEL
002270         K = IX(I,N)
002276         DO 130 J = 1,NDF
002277         DU(J,K) = DU(J,K) + FORCE(L,2)
002311   130   L = L + 1
002317         GO TO 400
002320   300   CONTINUE
         C.... ADD THE FORCE TO THE SOLUTION FOR A RESOLVE
002320         DO 310 K = 1,NSTF
002322         J = LD(K)
002325         IF(J.GT.0) DF(J) = DF(J) + FORCE(K,1)
002335   310   CONTINUE
002340   400   CALL TICTOC(TYME,3)
002351         IF(FLAGC.AND..NOT.CFLAG) CALL CONTAC(2,IX,NEL1,NDF,NUMNP,DU)
002364         CFLAG = .TRUE.
002365         IF(NTB.GT.1) WRITE(ITWR) H
002405         IF(IBLK.GT.0) WRITE(ITWR) (IX(NE1,N),N=1,NUMEL)
002436         IF(.NOT.VFLAG) CALL SOLVEQ(NUMNP,NUMEL,NDF,ID1,M8,MAXBAN,9,NSTF,
              1   ISZA,NEQB,IBLK,A,DU,DF,IDEST,FORCE,ESTIF,LD,MAXB,NDEG)
002476         IF(VFLAG) CALL RESVEQ(NUMNP,NDF,M8,MAXBAN,ISZA,NEQB,IBLK,A,DF,DF,
              1   IDEST,IDEST,MAXB,IFLG)
002524         CALL TICTOC(TYME,4)
         C.... UPDATE THE SOLUTION
002527         IF(NT.EQ.NTS.AND.M.EQ.NSEQ) GO TO 900
002542         IF(NCT.GT.0) GO TO 410
002544         DTP = DT
002545         I = ITRD
002546         ITRD = ITWR
002547         ITWR = I
002550   410   IF(IBLK.GT.0) BACKSPACE ITRD
002554         IF(IBLK.GT.0) READ(ITRD) (IX(NE1,N),N=1,NUMEL)
002605         REWIND ITRD
002607         IF(IBLK.GT.0) READ(ITRD) ((U(I,J),I=1,NDF),J=1,KU)
002642         IF(DFLAG)CALL NLNORM(3,MDEG,NT,U,DF,DU,F,PPROP,NCT,NTT,IDEST,IBLK)
002666         VFLAG = .TRUE.
002667         IF(NST.LT.0) VFLAG = .FALSE.
002672         NCT = NCT + 1
002674         IF(.NOT.DFLAG) GO TO 700
002675         WRITE(6,6000) NCT,UNORM,DNORM,ANORM
002711         IF(NCT.LT.NTT.AND.DNORM.GT.UNORM*EPP) GO TO 14
002730   700   IF(DFLAG) VFLAG=.FALSE.
```

```
         C.... UPDATE (2)                                                         TS0245C
002733         CALL UPDATE(2,MDEG,NICD,U,U(1,NUMNP+1),U(1,NEP+1),DU,F,DF,IDEST,  TS0246C
              X   PROP)                                                           TS0247C
002761         IF(FLAGC) CALL CONTAC(3,IX,NEL1,NDF,NUMNP,DU,U,U(1,NUMNP+1),
              X          U(1,NEP+1))
003016         MR = MR - 1                                                        TS0248C
003020   800   TIME = TIME + DT                                                   TS0249C
003024   900   CONTINUE                                                           TS0250C
003027         IF(NUMPLT.GT.0) CALL PLOTGO(NDIM,NEL,DM,U)                         TS0251C
003040         CALL TICTOC (TYME,6)                                               TS0252C
003043         WRITE(ITP6,2033) TYME                                              TS0253C
003052         RETURN                                                             TS0254C
003053   901   WRITE(ITP6,2030)                                                   TS0255C
003057         IPG = 0                                                            TS0256C
003060         RETURN                                                             TS0257C
         C.... FORMATS                                                            TS0258C
003061  1000   FORMAT(F10.0,8I5,2F10.0,2I5)                                       TS0259C
003061  1006   FORMAT(3I5)                                                        TS0260C
003061  2000   FORMAT(A1,12A6,E13.5,17X,4HPAGE,I4//5X,23HTIME DEPENDENT SOLUTION  TS0261C
              X//10X,17HTIME INCREMENT  = 1PE15.4/                                TS0262C
              X  10X,17HNUMBER OF STEPS = I5/                                     TS0263C
              X  10X,17HPRINT INTERVAL  = I5/                                     TS0264C
              X  10X,14HPRINT NODES    ,I5,3H TO,I5/                              TS0265C
              X  10X,14HPRINT ELEMENTS,I5,3H TO,I5)                               TS0266C
003061  2001   FORMAT(A1,12A6,E13.5,17X,4HPAGE,I4//5X,12HNODAL VALUES,5X,         TS0267C
              X    27HPROPORTIONAL LOAD FACTOR = ,F8.3,5X, 8HSEQUENCE,I4,5X,
              X   9HTIME STEP,I4//12H NODAL POINT,9(1X,2A6),(12X,9(1X,2A6)) )
003061  2002   FORMAT(10X,17HITERATION LIMIT =,I5/1X)                             TS0270C
003061  2005   FORMAT(4X,50H DESCRIPTION OF STRESS EVOLUTION PLOTS TO BE MADE.  / TS0271C
              X  5X,10H PLOT NO.  5X,8H ELEMENT ,7X, 9H XYZ-CODE,6X,9H SIG-CODE/) TS0272C
003061  2006   FORMAT(4(7X,I5,3X))                                                TS0273C
003061  2030   FORMAT(5X,53H**FATAL ERROR 33** TIME STEF ZERO FOR DYNAMIC PROBLEM TS0274C
              X/1X)                                                               TS0275C
003061  2033   FORMAT (    22X,12HELAPSED TIME///10X,25HINPUT PROPERTIES AND MESH TS0276C
              ,F10.3/10X,25HCHECK AND PLOT INPUT DATA,F10.3/10X,14HFORM STIFFNESS TS0277C
              X,F21.3/10X,21HSOLUTION OF EQUATIONS,F14.3/10X,15HOUTPUT STRESSES,  TS0278C
              XF20.3/10X,27HIMPLICIT ALGORITHM + DISPL.,F8.3/10X,19HTOTAL TIME/IM TS0279C
              XPLICIT,F16.3/1X)                                                   TS0280C
003061  4001   FORMAT(5X,89H**WARNING  3**  BOTH THE PROPORTIONAL LOADING AND FOR TS0281C
              XCE ARE BEING RESET ON EACH TIME STEP )                             TS0282C
003061  6000   FORMAT(5X*NCT*I5,*  UNORM*E15.5*  DNORM*E15.5*  ANORM*E15.5)       TS0242C
003061         END
```

```
14.11.57 OUTPUT
  RUN FORTRAN COMPILER VERSION 2.3 B.3                                    06 MAY 74 14:09:30  PAGE NO. 1

                SUBROUTINE UPDATE(ISW,NDEG,NICD,U,UD,UDD,DU,F,DF,IDEST,PROP)
          C***** UPDATE     ****** 12/14/73  ******************************************
000016          DIMENSION U(1),DU(1),F(1),DF(1),IDEST(1),UD(1),UDD(1)
000016          COMMON/TAPES/ ITP5,ITP6
000016          COMMON/TIMHIS/ TIME,DT,DTP,NH,ISZH,C0,C1,C2,C3,C4,C5,C6,NCT
000016          GO TO (100,300,500), ISW
          C.... UPDATE (1) * * PREUPDATE OF ACCELERATIONS FOR DYNAMIC SOLUTIONS.
000024    100   DO 200 N = 1,NDEG
000026    200   UDD(N) = C4*UDD(N) - C5*UD(N)
000035          RETURN
          C.... UPDATE (2) * * UPDATE THE SOLUTION FOR GENL. DISPL, VEL, ACCL.
000035    300   DO 400 N = 1,NDEG
000037          DU(N) = DF(N)
000043          IF(IDEST(N).EQ.0) DU(N) = F(N)*PROP - U(N)
000053          TEMP = DU(N)
000056          IF(NICD.EQ.1) GO TO 400
000060          P = UDD(N)
000061          UD(N) = C1*UD(N) + C2*P + C3*TEMP
000067          UDD(N) = P + C6*TEMP
000073    400   U(N) = U(N) + TEMP
000100          RETURN
          C.... UPDATE(3) * * SET INTEGRATION CONSTANTS
000100    500   C6 = C6 + 0.5
000102          IF(C0.EQ.0.0) C0 = 0.25
000104          WRITE(ITP6,2002) C0,C6
000114          C5 = 1./C0/DT
000116          C4 = 1. - 0.5/C0
000121          C3 = C6*C5
000123          C2 = (1. - C6/C0/2.)*DT/C4
000127          C1 = 1. - C6/C0 + C2*C5
000134          C6 = C5/DT
000137          RETURN
000137    2002  FORMAT(10X,30HNEWMARK INTEGRATION PARAMETERS/
               C10X17HBETA VALUE      = F6 3/
               C10X17HGAMMA VALUE     = F6.3)
000137          END
```