Structures and Materials Research
Department of Civil Engineering
Division of Structural Engineering

Report No. 67-11

A NOTE ON THE ASYMPTOTIC PHASE VELOCITIES IN A TWO-LAYER PLATE

by

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May 1967

INTRODUCTION

In a recent paper[1], J.P. Jones discussed the propagation of waves in a two-layered plate. The plate is of infinite extent and is made of two plates, each with its own thickness and material properties, bonded at the interface. In the paper the frequency equation was developed which gives the relationship between the frequency and the wave length. This relationship was explored in depth for a large range of wave lengths and, whereas some study was made for waves with short wave length, the study for these waves was not complete and the conclusions regarding the phase velocities of these waves were left to speculation.

In this note we extend the findings of Jones by exploring the asymptotic velocities of waves as the wave length becomes extremely short. We find that in general the waves travel at the speeds of three classical waves: the Rayleigh waves of each of the two materials and the slower of the two possible shear waves. Waves will also travel along the interface of the two plates at the speed of Stoneley waves when the properties of the two plates admit them. For the particular plate studied by Jones, Stoneley waves do not exist, so we introduce a composite plate in which Stoneley waves are possible. To confirm the asymptotic character of the spectral lines we explore the corresponding displacement distributions.

Frequency Equation

In what follows we use the notation adopted by Jones and in the interests of brevity we will exploit his study and refer where possible to his paper. We assume, as did Jones, that shear waves travel in Plate 1 slower than they do in Plate 2. Each plate property is identified by its appropriate subscript.

We use a method that we have employed previously to study the asymptotic velocities in composite rods.[2]

The nature of the frequency equation (Eq. 9, Jones) depends on whether the arguments of the trignometric functions are real or imaginary: that is whether ε_1, ε_2, δ_1, and δ_2 (Eq. 5J) are real or imaginary. The asymptotic phase velocities are contained in only two of the five possible frequency equations represented by Eq. (9J) and will be the only two explored here. In both equations ε_1, ε_2, and δ_2 are real. In the first of the two δ_1 is real, in the second it is imaginary.

The first equation has been analyzed correctly by Jones. When the wave number is large, advantage can be taken of the relative magnitude of terms in the frequency equation, so that it becomes three uncoupled equations represented by Equations 11, 12, and 14 in Jones' paper. The phase velocities, one governed by each of these equations, are constant, that is they are independent of the frequency. They are the Rayleigh velocities in each of the two materials and a possible Stoneley velocity respectively. As has been pointed out by Sezewa and Kanai[(3)], Stoneley waves are only possible when the shear velocities in each of the two materials are close to one another. For the material properties of the plate chosen by Jones, Stoneley waves do not exist so we have chosen a plate capable of transmitting Stoneley waves. Our plate is described in Figure 2. We have determined the displacement distribution through our plate using a large wave number and the appropriate Stoneley velocity, and examination of Figure 4 will show that it is indeed the classical Stoneley distribution.

In addition to these velocities, waves can travel with an additional phase velocity which is contained in the second of the two frequency equations. When the wave number "k" is large we may neglect terms containing exp $(-\varepsilon_1 kh_1)$, exp $(-\varepsilon_2 kh_2)$ and exp $(-\delta_2 kh_2)$ and when we do the equation reduces to two uncoupled equations. The first of these equations is Equation (12) of Jones' paper which gives the velocity of Rayleigh waves in the "second" material. The fact that this Rayleigh equation is contained in both the frequency equations under study deserves an explanation. This Rayleigh equation can be written

$$(1 + \delta_2^2)^2 - 4\, \delta_2 \varepsilon_2 = 0 \, . \tag{1}$$

For both equations δ_2 and ε_2 are real, a necessary condition for the Rayleigh velocity to be a root. Whether this second Rayleigh velocity is actually a root of the first or second equation depends on the material properties of the plate. The "boundary" between the two equations occurs when

$$\delta_1 = 0, \tag{2}$$

which gives as a root the shear velocity in material one which by definition is the slower of the two shear velocities. If the plate is such that the second and faster Rayleigh velocity is less than this shear velocity this Rayleigh velocity will be a root of the first equation and it can be represented on a frequency spectrum by means of a single spectral line. If, as is more likely, this Rayleigh velocity is greater than the slower shear velocity it is a root of the second equation and is accommodated on frequency spectra by means of "terracing". This terracing was discussed by Jones and is shown for our rod in Figure 2.

The second frequency equation gives, on degeneration resulting from using the large wave number, not only the Rayleigh equation but an additional equation

$$\Delta_2 = 0 \tag{3}$$

where

$$\Delta_2 = \begin{vmatrix} 0 & (1-\bar{\delta}_1^2) & -2\bar{\delta}_1 \operatorname{Sin} k\bar{\delta}_1 h_1 & 2\bar{\delta}_1 \operatorname{Cos} k\bar{\delta}_1 h_1 & 0 & 0 \\ 0 & 2\varepsilon_1 & (1-\bar{\delta}_1^2)\operatorname{Cos} k\bar{\delta}_1 h_1 & (1-\bar{\delta}_1^2)\operatorname{Sin} k\bar{\delta}_1 h_1 & 0 & 0 \\ 1 & 0 & 0 & \bar{\delta}_1 & -1 & -\delta_2 \\ \varepsilon_1 & 0 & -1 & 0 & \varepsilon_2 & 1 \\ \mu_1(1-\bar{\delta}_1^2) & 0 & 0 & 2\bar{\delta}_1\mu_1 & \mu_2(1+\delta_2^2) & -2\delta_2\mu_2 \\ -2\varepsilon_1\mu_1 & 0 & \mu_1(1-\bar{\delta}_1^2) & 0 & -2\varepsilon_2\mu_2 & -\mu_2(1+\delta_2^2) \end{vmatrix} \tag{4}$$

Eq. (3), after considerable manipulation, takes the form

$$\tan\,(k\bar{\delta}_1 h_1) = \frac{\Delta A}{\Delta B} \tag{5}$$

where

$$\delta_1 = i\bar{\delta}_1 \quad : \quad \bar{\delta}_1 = \left(\frac{c^2}{\beta_1^2} - 1\right)^{1/2} \tag{6}$$

$$\Delta A = \bar{\delta}_1 \begin{vmatrix} 1 & (1-\bar{\delta}_1^2)^2 & -1 & -\delta_2 \\ \varepsilon_1 & -4\varepsilon_1 & \varepsilon_2 & 1 \\ \mu_1(1-\bar{\delta}_1^2) & 2\mu_1(1-\bar{\delta}_1^2)^2 & -\mu_2(1+\delta_2^2) & -2\delta_2\mu_2 \\ -2\varepsilon_1\mu_1 & 4\varepsilon_1\mu_1(1-\bar{\delta}_1^2) & -2\varepsilon_2\mu_2 & -\mu_2(1+\delta_2^2) \end{vmatrix} \tag{7}$$

$$\Delta B = \begin{vmatrix} 1 & -4\varepsilon_1\overline{\delta}_1^2 & -1 & -\delta_2 \\ \varepsilon_1 & -(1-\overline{\delta}_1^2)^2 & \varepsilon_2 & 1 \\ \mu_1(1-\overline{\delta}_1^2) & -8\varepsilon_1\overline{\delta}_1^2\mu_1 & -\mu_2(1+\delta_2^2) & -2\delta_2\mu_2 \\ -2\varepsilon_1\mu_1 & \mu_1(1-\overline{\delta}_1^2)^3 & -2\varepsilon_2\mu_2 & -\mu_2(1+\delta_2^2) \end{vmatrix} \quad (8)$$

As β_1 is the shear velocity in the "first" material, a root $\overline{\delta}_1$ of Eq. (5) gives a corresponding phase velocity "c". We now analyze Eq. (5) in the same way as we did a similar equation in Reference 2.

The behavior of the spectral lines for large wave number will become apparent when we can find the roots for a particular, large k, say $\overline{k}$. On the left side of the equation the phase velocity "c" appears only in the $\overline{\delta}_1$. We now examine the plot of tan $(\overline{\delta}_1 \; kh_1)$ in Fig1 and establish the bounds of its argument as fixed by the bounds of validity of the second frequency equation. For the lower bound of validity, δ_1 is zero for which the argument is zero. The upper bound of validity occurs when either ε_1 or δ_2 is zero depending on the plate materials. For either, the phase velocity is a constant so we may say the argument corresponding to the upper bound is $(\overline{\delta}_1^* \; \overline{k}h_1)$. Tan $(\overline{\delta}_1 kh_1)$ is then sketched over this interval as "c" varies over the region of validity of this frequency equation or as $\overline{\delta}_1$ varies from zero to $\overline{\delta}_1^*$.

Examination of the right side of Eq. (5) shows that it is a function only of the phase velocity and the material properties, and that it is zero when $\overline{\delta}_1$ is zero. A possible plot of this function is shown in Fig 1. Whether the function is positive or negative, the lowest intersection is close to ¶.

If we set

$$\overline{\delta}_1 kh_1 = \P \tag{9}$$

We get

$$C_1 = \beta_1 (1 + \frac{\P^2}{h_1^2 k^2})^{1/2} \tag{10}$$

As the wave number k becomes large the phase velocity approaches the shear velocity β_1 from above.

In Fig. 1 the second intersection is close to, say $(\frac{5\P}{2})$ and if we set the argument of the angle equal to $(\frac{5\P}{2})$ we find the second velocity

$$C_2 = \beta_1 (1 + \frac{2\ 5\P^2}{4h_1^2 k^2}) \tag{11}$$

so we can conclude that a second spectral line approaches the same asymptote as the first as k goes to infinity, but approaches it more slowly.

The number of phase velocities for a particular $\overline{k}$ represents the number of spectral lines that intersect the line $k = \overline{k}$, derivable from the second frequency equation. If we take a new line say $k = 2\overline{k}$ the interval of the angle $(\overline{\delta}_1 kh_1)$ doubles so the number of roots doubles. The additional spectral lines in the interval $(2\overline{k} - \overline{k})$ come from the third frequency equation.

With the arguments just set forth, we may say that all spectral lines, except those derived from the first frequency equation, approach the line

$$\beta_1 = 0 \tag{12}$$

as the wave number approaches infinity. The phase velocity represented by this line is the slower of the two possible shear velocities. The asymptotic behaviour is demonstrated in Figure 2. In this Figure we choose to plot a normalized frequency versus a dimensionless wave number so that the phase velocity representing Eq. (12) is a sloping straight line passing through the origin.

As we identify the asymptotic velocities as those of the two Rayleigh waves, the Stoneley wave and the slower of the two shear waves, we validate this identification using displacement distributions through the thickness of the plate. The shear velocity does not have a distribution with which it can be characterized but the Rayleigh and Stoneley velocities do. Accordingly three points are chosen for a large propagation constant; these are shown as points 1, 2, and 3 in Figure 2 and are on spectral lines representing the Rayleigh velocity in Plate 1, the Stoneley velocity and the Rayleigh velocity in Plate 2 respectively. The displacement distributions are shown respectively in Figures 3, 4 and 5 and, in each of the three cases, the distribution has the classical form it should.

CAPTIONS FOR FIGURES

FIGURE 1 - GRAPHICAL SOLUTION OF THE SECOND REDUCED FREQUENCY EQUATION

FIGURE 2 - SPECTRUM SHOWING FREQUENCY VS REAL PROPAGATION CONSTANT

FIGURE 3 - DISPLACEMENT DISTRIBUTIONS CORRESPONDING TO THE POINT 1

FIGURE 4 - DISPLACEMENT DISTRIBUTIONS CORRESPONDING TO THE POINT 2

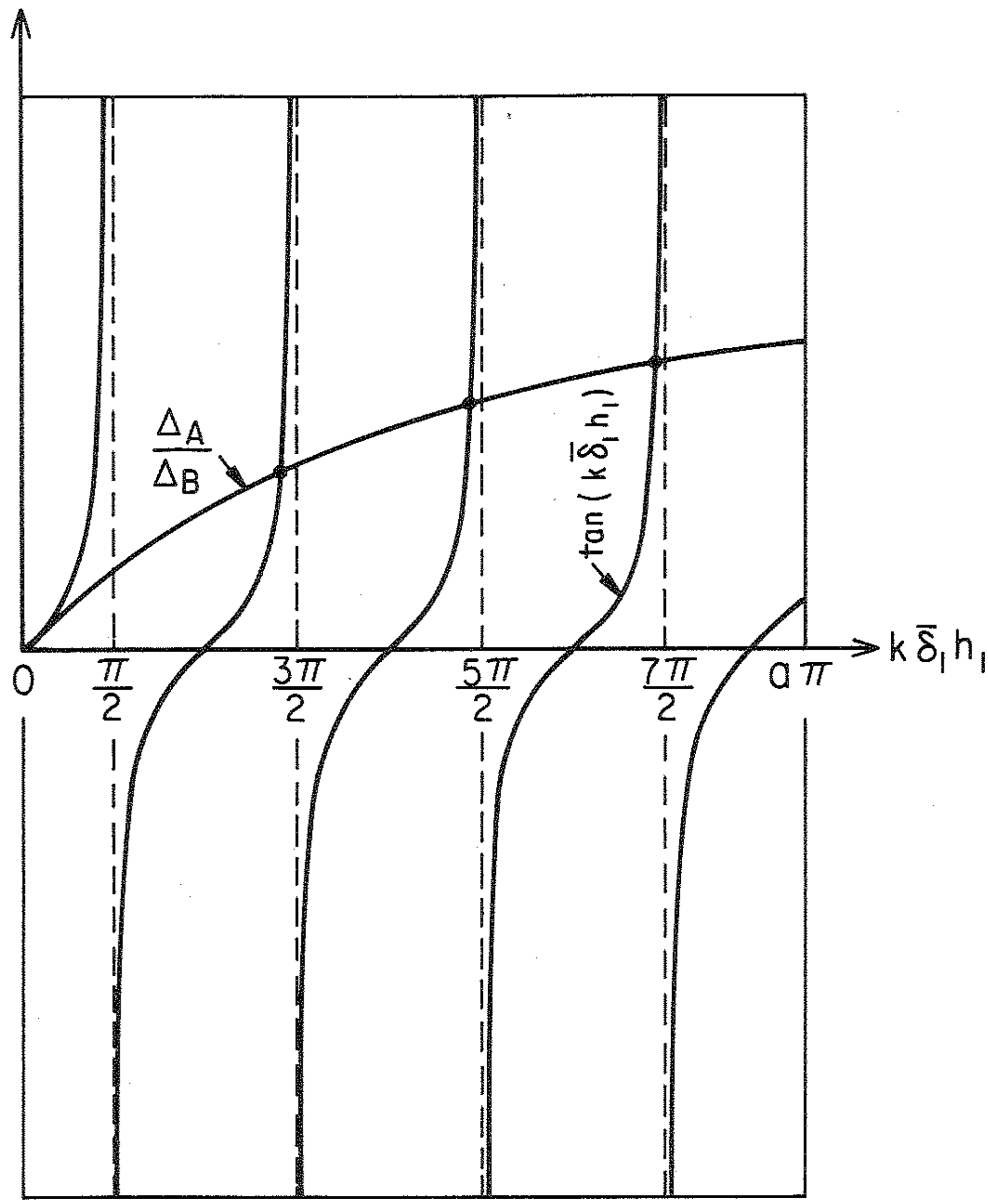
FIGURE 5 - DISPLACEMENT DISTRIBUTIONS CORRESPONDING TO THE POINT 3

REFERENCES

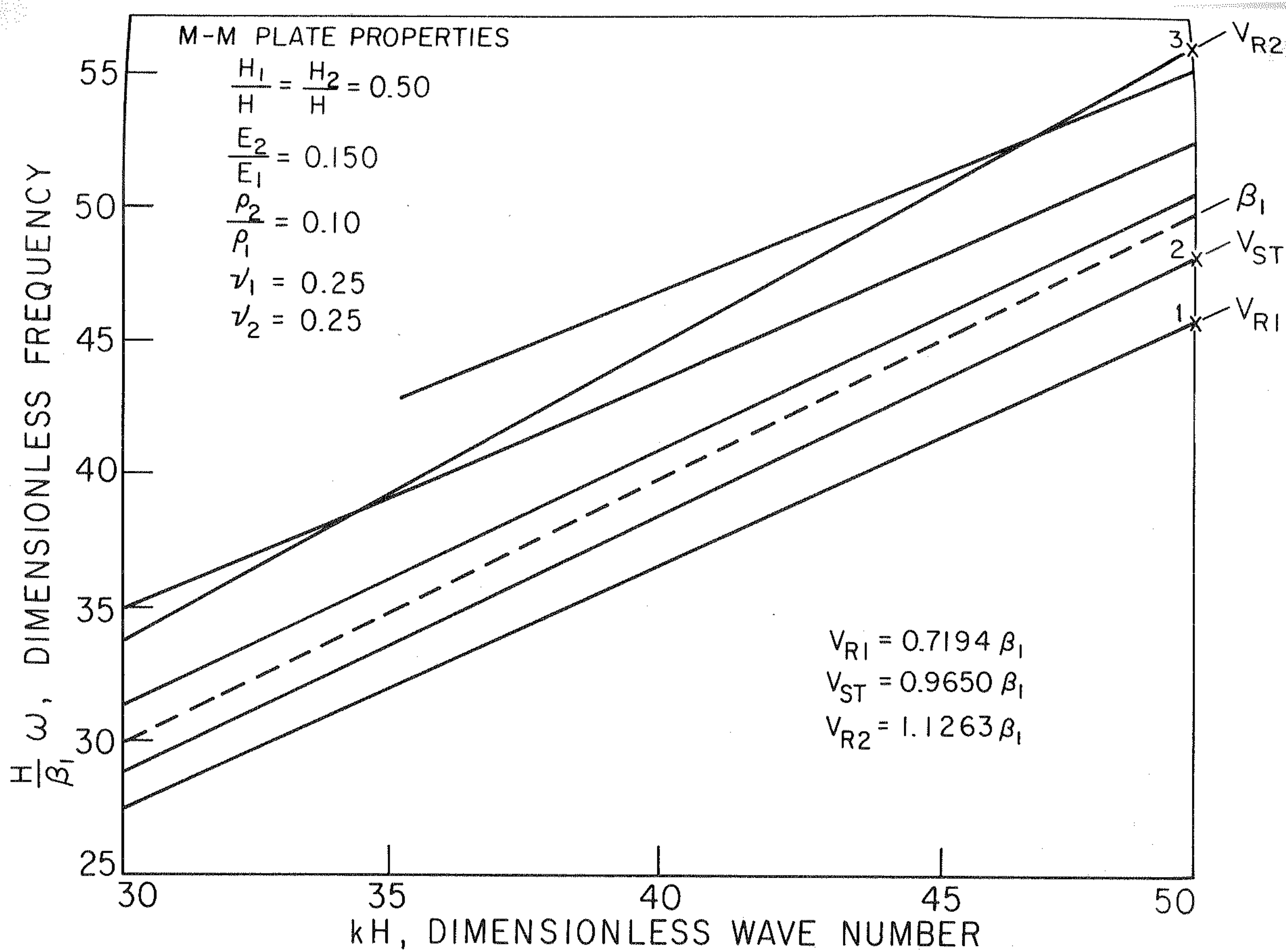
1 J.P. Jones, "Wave Propagation in a Two-Layered Medium," Journal of Applied Mechanics, Transactions of the ASME, Series E, pp. 213-222, June 1964.

2 Y. Mengi and H.D. McNiven, "Asymptotic Phase Velocities of Axisymmetric Waves in Composite Rods", The Journal of the Acoustical Society of America, Vol. 41, No. 7, pp. July 1967.

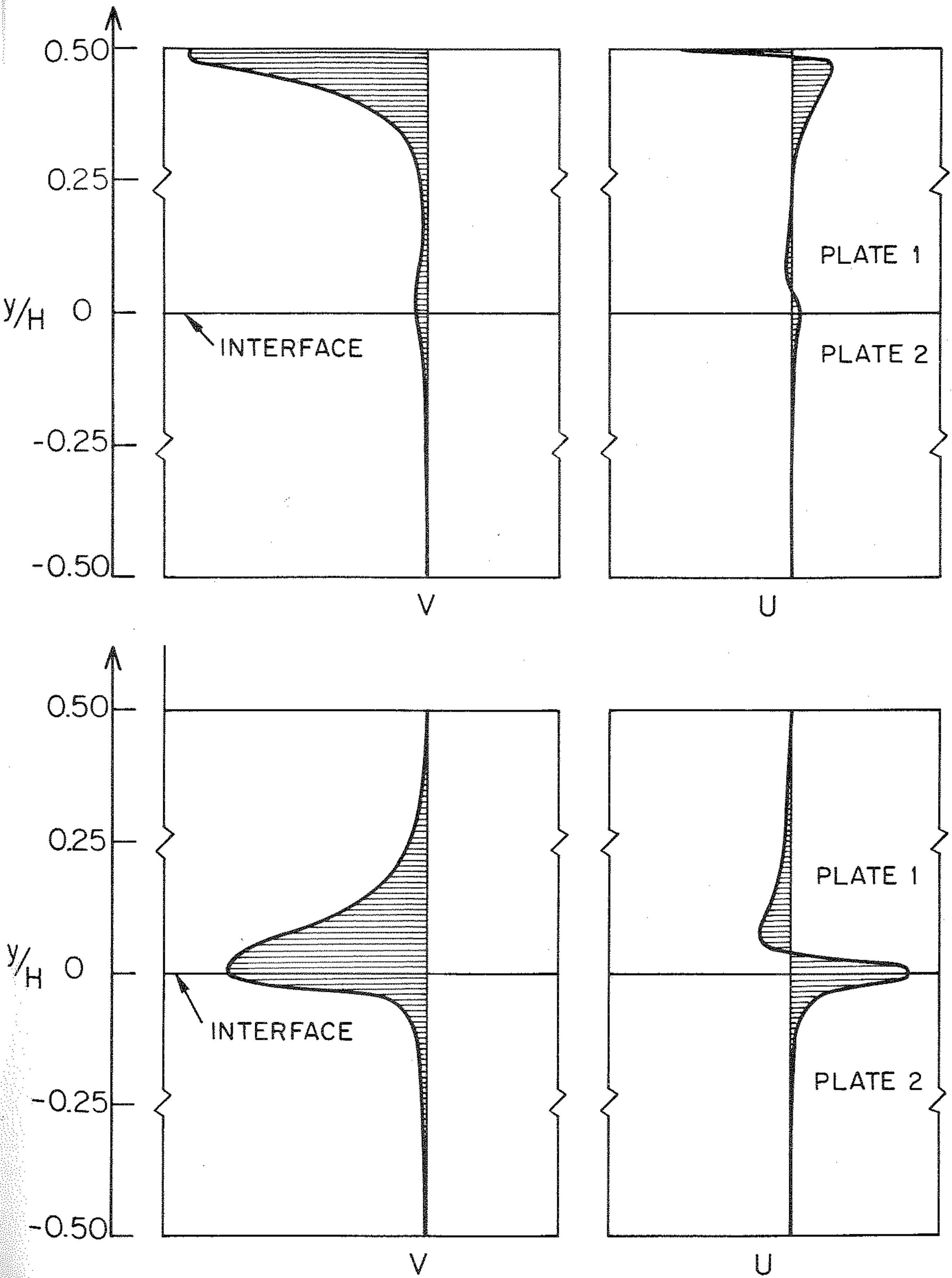
3 Sezewa, K., and K. Kanai, "The Range of Possible Existence of Stoneley Waves and Some Related Problems", Bull Earthquake Research Institute (Tokyo), Vol. 17, pp. 1-8, 1939.



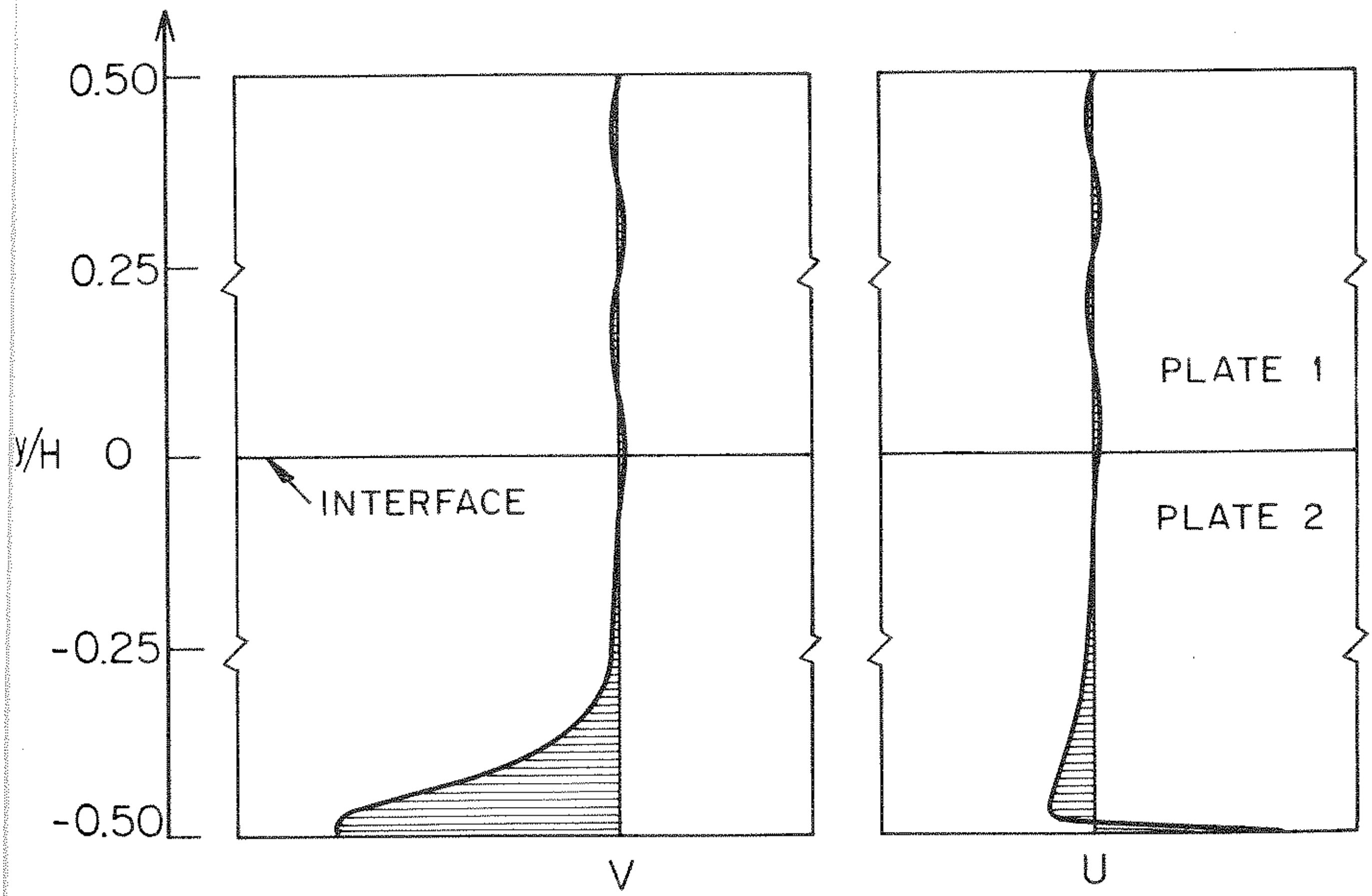
$\frac{\Delta_A}{\Delta_B}$
$\tan(k\bar{\delta}_1 h_1)$
$k\bar{\delta}_1 h_1$
0
$\frac{\pi}{2}$
$\frac{3\pi}{2}$
$\frac{5\pi}{2}$
$\frac{7\pi}{2}$
$a\pi$



M-M PLATE PROPERTIES
$\frac{H_1}{H} = \frac{H_2}{H} = 0.50$
$\frac{E_2}{E_1} = 0.150$
$\frac{\rho_2}{\rho_1} = 0.10$
$\nu_1 = 0.25$
$\nu_2 = 0.25$
$V_{R1} = 0.7194\,\beta_1$
$V_{ST} = 0.9650\,\beta_1$
$V_{R2} = 1.1263\,\beta_1$
V_{R2}
β_1
V_{ST}
V_{R1}
1
2
3
$\frac{H}{\beta_1}\omega$, DIMENSIONLESS FREQUENCY
kH, DIMENSIONLESS WAVE NUMBER
25
30
35
40
45
50
55



0.50
0.25
y/H 0
-0.25
-0.50
INTERFACE
PLATE 1
PLATE 2
V
U
0.50
0.25
y/H 0
-0.25
-0.50
INTERFACE
PLATE 1
PLATE 2
V
U



0.50
0.25
y/H 0
-0.25
-0.50
INTERFACE
V
PLATE 1
PLATE 2
U