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# THE SECOND WELFARE THEOREM WITH NONCONVEX PREFERENCES

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# Abstract

We prove several versions of the second theorem of welfare economics for exchange economies with nonconvex preferences.

# 1. Introduction

The Second Welfare Theorem asserts, under appropriate assumptions (chiefly convexity of preferences), that Pareto optimal allocations are Walrasian equilibria under some redistribution of income. If preferences are nonconvex, the theorem no longer holds. The purpose of this paper is to prove several versions of the Second Welfare Theorem in the case of nonconvex preferences.

It is useful to consider the interpretation commonly placed on the Second Welfare Theorem in undergraduate microeconomics courses. It is asserted that it would be better for government to redistribute income, and then allow the workings of the market to determine the allocation of commodities to individuals, rather than have the government establish subsidies for certain commodities or to allocate goods through non-market mechanisms. The argument is as follows: the outcome if the government undertakes non-market actions will probably not be Pareto optimal; in any case, we may find a Pareto optimal allocation $f$ which equals or Pareto dominates the non-market outcome. In the convex case, the Second Welfare Theorem asserts that the government can achieve $f$ merely through redistributing income; once redistribution has occurred, the workings of the market will yield the outcome $f$ without further intervention on the part of the government. To be more precise, let $A$ be the set of agents in an exchange economy, $e(a)$ the endowment of agent $a$. An income transfer is a function $t: A \to \mathbb{R}$ with $\sum_{a \in A} t(a) \leq 0$. The budget set of an individual $a$, relative to the transfer $t$ and price vector $p$, is $\{x\colon p \cdot x \leq p \cdot e + t(a)\}$. A Walrasian equilibrium relative to the transfer $t$ is $(f, p)$, where $f$ assigns consumption vectors to the agents, $p$ is a price vector, $f(a)$ maximizes the preference of $a$ over $a$'s budget set (relative to the transfer), and $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$. With convex preferences, the Second Welfare Theorem asserts that given any Pareto optimum $f$, there is an income transfer (with $\sum_{a \in A} t(a) = 0$) such that $f$ is a Walrasian equilibrium relative to the transfer $t$.

Three caveats are required, even in the convex case. First, the above story assumes that the operations of the market produce Walrasian equilibria as outcomes; this may not be the case if there are large agents, who have incentives not to act as price-takers. Second, a problem arises if Walrasian equilibrium is not unique. If the government is restricted to redistributing income, the income redistribution that makes $f$ a Walrasian equilibrium might also make some $f' \neq f$ Walrasian; $f'$ could be much more favorable to some individuals, and less favorable to others, than $f$. Of course, if $f$ is Pareto optimal, then under standard assumptions, it will be the unique Walrasian equilibrium for the economy with $f$ as the endowment map. However, allowing the government to redistribute goods to achieve $f$ as the initial allocation, and then noting that the market yields no further trade, violates the decentralized spirit of the interpretation of the Second Welfare Theorem described above. Third, the government might choose not to respect the preferences of individuals. For example, the government cannot provide an income transfer directly to young children; the transfer would have to be made to the parents of the children, and there may be a conflict between the interests of the children and the preferences of the parents. For this reason, the government might choose to provide certain specific goods to families with young children, rather than providing an income transfer to those families. We shall not, however, pursue these points here.

Farrell [10] gave an early discussion of welfare theory in large economies with nonconvexities. Hildenbrand [12] showed that the Second Welfare Theorem holds without convexity in economies with a measure space of agents. The first rigorous work on asymptotic versions of the Second Welfare Theorem for large finite economies with nonconvexities was done by Khan and Rashid [14], using Nonstandard Analysis. Mas-Colell ([16], Proposition 4.5.1) has proved an elementary version of the Second Welfare Theorem. He used the Shapley-Folkman Theorem (Starr [18]) to show that any Pareto optimal allocation $f$ can be approximately supported in the following sense: there is a price vector $p$ such that any $x \wp_a f(a)$ satisfies $p \cdot x$ is nearly as great as, or greater than, $p \cdot f(a)$. However, there is no reason to think that $f(a)$ is close to $a$'s demand set. Indeed, Anderson and Mas-Colell [7] give an example of a sequence of exchange economies with the number of agents going to infinity and a sequence of Pareto optimal allocations in which every agent is far from her demand set for every price and every income transfer.

The form of decentralization given in Mas-Colell's Theorem is not sufficient to justify the interpretation of the Second Welfare Theorem discussed above. Let us suppose the government has carried out the transfers needed to make $f(a)$ lie on the frontier of $a$'s budget set, with respect to the price $p$. We note first that there is no guarantee that there will be any price $q$ that clears the markets, since preferences are nonconvex; we are forced to consider prices that approximately clear the markets. Worse still, since $f(a)$ is not near $a$'s demand set with respect to the price $p$, there is no reason to think that $p$ will approximately clear the markets. Rather, as shown in Figure 1, it is possible to have a Pareto optimum $f$ so that the local supporting price $p$ at $f$ is *not* an equilibrium price, since agent I's demand will be at the point $x$, while II's demand will be at the point $y$. If one makes the income transfer necessary to make $f$ affordable for each agent at the price $p$, there will be a unique equilibrium price $q$

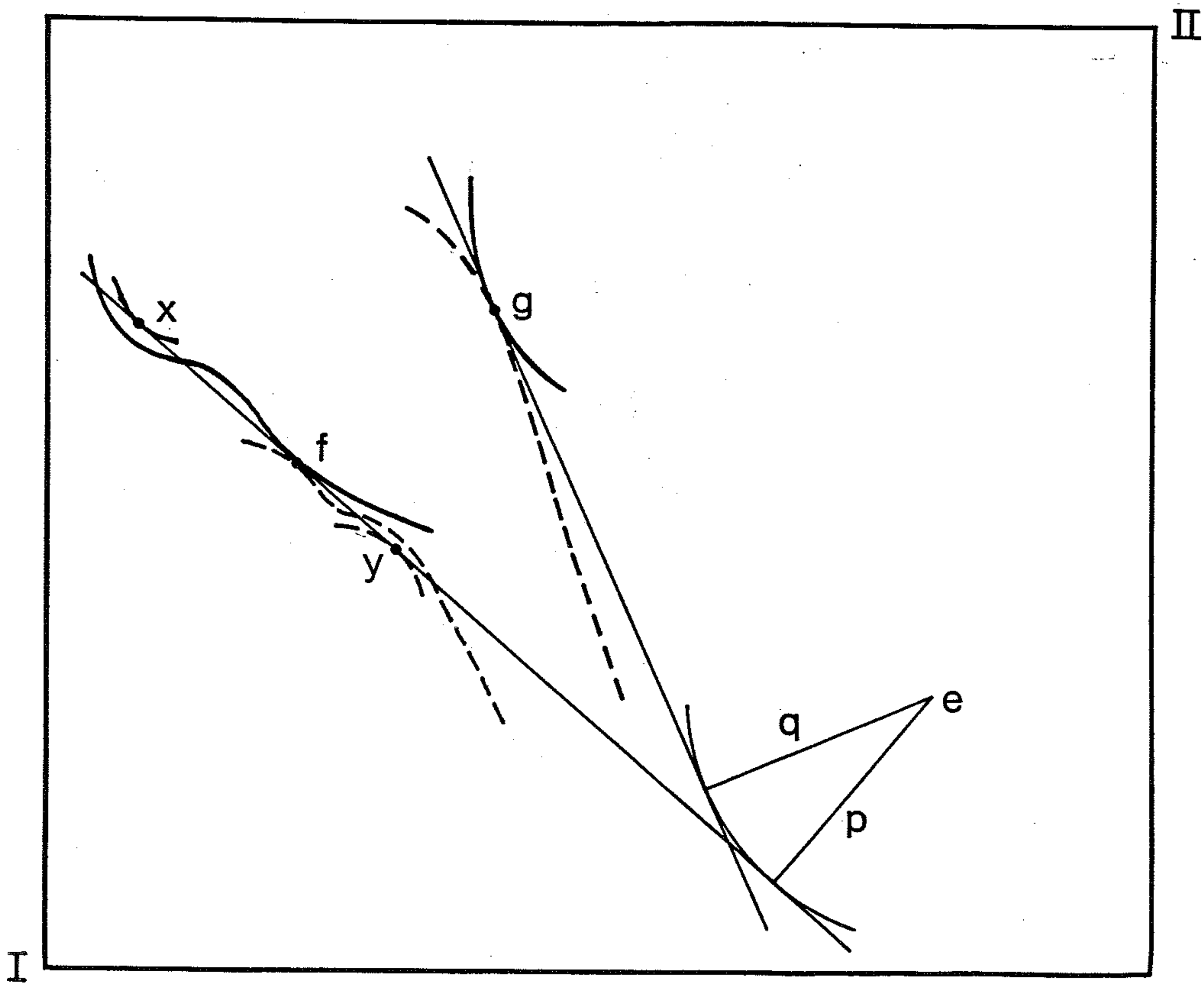


FIGURE 1

which yields an allocation $g$ far from $f$ and such that the utility levels of agents at $g$ are very different from the levels at $f$. Indeed, there is no income transfer $t$ such that there is a Walrasian equilibrium relative to $t$ which yields utility levels close to those of $f$ to both agents. Considering approximate Walrasian equilibria does not help either, since either agent I's demand will be near $x$ or II's demand will be near $y$, or both, so per capita excess demand is not small. In other words, the government cannot even achieve the desired utility levels of the agents through income transfers and a market mechanism. If one allows the government to dictate commodity transfers rather than income transfers, the government could specify $f$ as the initial endowment: in that case, there is *no* Walrasian equilibrium, nor even a price that makes per capita excess demand relatively small. Of course, if there were an approximate Walrasian allocation $g$, it would have the property that $f(a) \not\succ_a g(a)$ for all $a$ (observe that $f(a)$ is in the budget set). However, as in the convex case, allowing the government to dictate $f$ as the initial allocation destroys the interpretation of the Second Welfare Theorem as a story of decentralized allocation.

In Theorem 3.3, we show that the government can achieve the utility levels desired for all but $k$ agents, where $k$ is the dimension of the commodity space. In other words, the pathology illustrated in Figure 1 disappears (at least for most agents) provided that the number of agents is large relative to the number of commodities. The proof is elementary, relying primarily on the Shapley-Folkman Theorem. We focus on a particular choice of decentralizing price $\bar{p}$; this price is used by Mas-Colell in the proof of his theorem, and is closely related to the so-called gap-minimizing price studied in Anderson [6]; essentially, $\bar{p}$ is the price which minimizes the measure by which support fails in Mas-Colell's Theorem. Given any Pareto optimum $f$, there is an income transfer $t$ and a quasiequilibrium $\tilde{f}$ with respect to $t$ such that at most $k$ agents prefer $f$ to $\tilde{f}$. If preferences are monotone and a mild assumption on the distribution of goods at $f$ is satisfied, then we may show that $\bar{p}$ is strictly positive, and hence $\tilde{f}$ is a Walrasian equilibrium with respect to $t$. As an alternative, we can achieve an approximate equilibrium (i.e. total excess demand is bounded, independent of the number of agents) $\hat{f}$ such that *no* agent prefers $f$ to $\hat{f}$. It is worth emphasizing that Theorem 3.3 is a universal theorem, applying to all exchange economies, rather than a generic theorem. However, there is no guarantee that $\tilde{f}(a)$ is close to $f(a)$ for *any* $a$.

In Theorem 4.1, we show that, for most large exchange economies, all Pareto optima satisfying certain bounds are close in the commodity space to Walrasian equilibria with income transfers. Specifically, we assume that we are given a distribution $\mu$ of agents' preferences. We think of this as being the distribution of preferences of all possible people. We then form a sequence of exchange economies $\varepsilon_n$, with $A_n$ as the set of agents. Agents' endowments are assigned in an arbitrary way; their preferences are assigned by sampling from the given distribution. We show that, with probability one, the following conclusion holds: If $f_n(a)$ is a sequence of Pareto optimal allocations, bounded in an appropriate sense, there is a sequence of income transfers and Walrasian equilibria $(\tilde{f}_n, p_n)$ relative to those transfers such that $|f_n(a) - \tilde{f}_n(a)|$ converges to 0; the sense of convergence is either convergence in measure or mean, depending on the sense in which the sequence of Pareto optima is bounded.

The methodology used in proving Theorem 4.1 is similar to that used in Anderson [5] to prove a convergence theorem for the core in exchange economies with nonconvex preferences. We are grateful to Andreu Mas-Colell for suggesting that we apply that methodology to the set of Pareto optimal allocations. The proof uses Nonstandard Analysis (Robinson [17]). A general meta-theorem guarantees that a standard proof also exists, but it could be quite difficult to write it out. We do not know of a tractable standard proof of the result. Indeed, the general case of the result in Anderson [5] has stubbornly resisted the author's attempts to give a reasonable standard proof. Converting the proof of Theorem 4.1 to a standard proof poses additional problems not present in [5]. Specifically, the theorem says that, for a set of sequences of economies having probability one, the result applies to every Pareto optimal allocation satisfying certain bounds. Proving this requires interchanging the order of two quantifiers. Since there are an uncountable number of Pareto optima in each economy, there is no obvious way to interchange the quantifiers using standard measure-theoretic techniques. From the structure of the nonstandard proof, it appears that the best hope for obtaining a reasonable standard proof would be to consider finite sets of candidate supporting prices which fill out the price simplex as the number of agents grows.

One might suppose that Theorem 4.1 could be deduced from Anderson [5], using the fact that any Pareto optimal allocation $f$ is in the core of the economy with $f$ as the endowment map. However, the result in [5] applies only to economies in which the endowment maps are obtained by sampling from a distribution. Such economies have special properties. Our theorem applies to all Pareto optima within certain bounds; for many of them, the economies resulting from taking the optima as the new endowments do not satisfy the special properties.

Nor does it appear possible to deduce the result in [5] from Theorem 4.1. Any core allocation is Pareto optimal, so Theorem 4.1 tells us that core allocations are near demand sets after income transfers. However, [5] demonstrates that core allocations are near demand sets *without* income transfers.

As a corollary of the nonstandard proof, we are also able (in Theorem 4.4) to prove a version of the second welfare theorem for type sequences of exchange economies.

# 2. Preliminaries

We begin with some notation and definitions which will be used throughout. Suppose $x, y \in \mathbb{R}^k$, $B \subset \mathbb{R}^k$. $x^i$ denotes the $i^{th}$ component of $x$; $x \geq y$ means $x^i \geq y^i$ for all $i$; $x > y$ means $x \geq y$ and $x \neq y$; $x >> y$ means $x^i > y^i$ for all $i$; $x_+$ is defined by $(x_+)^i = \max\{x^i, 0\}$; $x_- = x_+ - x$; $||x||_\infty = \max_{1 \leq i \leq k} |x^i|$; $||x||_r = (\sum_{i=1}^{k} |x^i|^r)^{1/r}$; $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x \geq 0\}$ ; $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$; $\rho(x, B) = \inf\{||x - y||_\infty : y \in B\}$.

A preference is a binary relation $\wp$ on $\mathbb{R}^k_+$ satisfying the following conditions: (i) weak monotonicity: $x >> y \Rightarrow x \wp y$; and (ii) free disposal: $x >> y$, $y \wp z \Rightarrow x \wp z$. Let $P'$ denote the set of preferences. A preference $\wp$ is said to be (iii) continuous if $\{(x,y): y \wp x\}$ is relatively open in $\mathbb{R}^k_+ \times \mathbb{R}^k_+$; (iv) transitive if $x \wp y$, $y \wp z \Rightarrow x \wp z$; and (v) irreflexive if $x \not\wp x$. Let $P''$ denote the space of preferences satisfying (i)-(v). A preference is said to be (vi) monotone if $x > y \Rightarrow x \wp y$. Let $P$ denote the set of preferences satisfying (i)-(iii) and (vi).

An exchange economy is a map $\varepsilon: A \to P \times \mathbb{R}^k_+$, where $A$ is a finite set. For $a \in A$, let $\wp_a$ denote the preference of $a$ (i.e. the projection of $\varepsilon(a)$ onto $P$) and $e(a)$ the initial endowment of $a$ (i.e. the projection of $\varepsilon(a)$ onto $\mathbb{R}^k_+$). An allocation is a map $f: A \to \mathbb{R}^k_+$ such that $\sum_{a \in A} f(a) = \sum_{a \in A} e(a)$. Given an allocation $f$, we define $M_f = \max\{||\sum_{i=1}^{k} f(a_i)||_\infty : a_1, \ldots, a_k$ are distinct elements of $A\}$. Note that in the last definition, $k$ is the dimension of the commodity space $\mathbb{R}^k_+$. An allocation $f$ is said to be Pareto optimal if there does not exist an allocation $g$ such that $g(a) \wp_a f(a)$ for all $a \in A$. Let $\mathscr{P}(\varepsilon)$ denote the set of all Pareto optimal allocations for the economy $\varepsilon$. Observe that $\mathscr{P}(\varepsilon)$ depend only on the preferences and the social endowment $\sum_{a \in A} e(a)$, not on the individual endowments.

A price $p$ is an element of $\mathbb{R}_+^k$ with $||p||_1 = 1$. $\Delta$ denotes the set of prices, $\Delta^0 = \{p \in \Delta : p >> 0\}$. The demand set for $(\wp, e)$, with income augmented by $r \in \mathbb{R}$ is $D(p, (\wp, e), r) = \{x \in \mathbb{R}_+^k : p \cdot x \leq p \cdot e + r, \ \ y \wp x \Rightarrow p \cdot y > p \cdot e + r\}$. $D(p, (\wp, e), r)$ could be empty under the hypotheses we have placed on preferences. An income transfer is a function $t: A \rightarrow \mathbb{R}$ with $\sum_{t \in A} t(a) \leq 0$. By abuse of notation, we let $D(p, a, t) = D(p, (\wp_a, e(a)), t(a))$ if $a \in A$.

The quasidemand set for $(\wp, e)$, with income augmented by $r \in \mathbb{R}$ is $Q(p, (\wp, e), r) = \{x \in \mathbb{R}_+^k : p \cdot x \leq p \cdot e + r, \ \ y \wp x \Rightarrow p \cdot y \geq p \cdot e + r\}$. $Q(p, (\wp, e), r)$ could be empty under the hypotheses we have placed on preferences. By abuse of notation, we let $Q(p, a, t) = Q(p, (\wp_a, e(a)), t(a))$ if $a \in A$.

A Walrasian equilibrium for $\varepsilon$, relative to the income transfer $t$, is a pair $(f, p)$, where $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$, $p \in \Delta$, and $f(a) \in D(p, a, t)$ for all $a \in A$. Let $\mathcal{W}(\varepsilon, t)$ denote the set of Walrasian equilibria for $\varepsilon$, relative to the income transfer $t$.

A Walrasian quasiequilibrium for $\varepsilon$, relative to the income transfer $t$, is a pair $(f, p)$, where $\sum_{a \in A} f(a) \leq \sum_{a \in A} e(a)$, $p \in \Delta$, and $f(a) \in Q(p, a, t)$ for all $a \in A$. Let $\mathcal{Q}(\varepsilon, t)$ denote the set of Walrasian quasiequilibria for $\varepsilon$, relative to the income transfer $t$.

Given $x \in \mathbb{R}_+^k$, $\wp \in P$, and $p \in \Delta$, define

$$\phi(p, x, \wp) \quad = \quad |\inf\{p \cdot (y - x) : y \wp x\}|.$$

$\phi$ measures how far $x$ is from being demand-like. In particular, if $p >> 0$, then $\phi(p, x, \wp) = 0$ if and only if $x \in D(p, (\wp, x))$. By a slight abuse of notation, we let $\phi(p, f, a) = \phi(p, f(a), \wp_a)$ if $f$ is an allocation, and $\phi(p, x, a) = \phi(p, x, \wp_a)$ if $x \in \mathbb{R}_+^k$.

Next, we consider sequences of assignments of commodity bundles. Specifically, suppose we have sets $A_n$ with $|A_n| = n$. Define

$$\mathcal{G}_\gamma = \{\{f_n\} : f_n : A_n \rightarrow \mathbb{R}_+^k, \ \ |\{a \in A_n : f_n(a)^i > \gamma\}| / n > \gamma \text{ for each } i\},$$

and

$$\mathcal{G} = \bigcup_{\gamma \in \mathbb{R}_{++}} \mathcal{G}_\gamma.$$

$\mathscr{G}$ is the set of sequences of assignments with the property that, for each good, a positive fraction of the population possesses a positive amount of the good. Define

$$\mathscr{B}_{\gamma\eta n} = \{\{f_m\} \in \mathscr{G}_\gamma\colon\ M_{f_m}/m < \eta \text{ for } m \geq n\}$$

and

$$\mathscr{B} = \bigcup_{\gamma \in \mathbb{R}_{++}} \bigcap_{\eta \in \mathbb{R}_{++}} \bigcup_{n \in \mathbb{N}} \mathscr{B}_{\gamma\eta n} = \{\{f_n\} \in \mathscr{G}\colon\ M_{f_n}/n \to 0\}.$$

$\mathscr{B}$ is the subset of $\mathscr{G}$ consisting of sequences of assignments in which the largest bundle given to any agent becomes small in relation to the size of the economy. Define

$$\mathscr{U} = \{\{f_n\} \in \mathscr{G}\colon\ E_n \subset A_n, \frac{|E_n|}{n} \to 0 \Rightarrow \frac{1}{n}\|\sum_{a \in E_n} f_n(a)\|_\infty \to 0\}.$$

$\mathscr{U}$ is the subset of $\mathscr{G}$ consisting of assignments which are uniformly integrable; the economic interpretation is that no group consisting of a small proportion of the agents can possess a significant quantity, per capita, of the goods. Observe that $\mathscr{U} \subset \mathscr{B}$.

# 3. Two Consequences of the Shapley-Folkman Theorem

Mas-Colell ([16], Proposition 4.5.1) gave an elementary version of the Second Welfare Theorem, using the Shapley-Folkman Theorem (Starr [18]). We shall use Mas-Colell's theorem as the first step in the proof of our results. Since our normalization of prices differs from his, and our assumptions on preferences are a little weaker, we shall give a proof.

*Theorem 3.1* (Compare Mas-Colell [16], Proposition 4.5.1): Let $\varepsilon\colon A \to P' \times \mathbb{R}^k_+$ be an exchange economy. If $f \in \mathscr{P}(\varepsilon)$, there exists $p \in \Delta$ such that $\sum_{a \in A} \phi(p,f,a) \leq M_f$.

*Proof:* Suppose $f \in \mathscr{P}(\varepsilon)$. Let $\gamma(a) = \{y - f(a)\colon y \wp_a f(a)\}$ and $\Gamma = \sum_{a \in A} \gamma(a)$.

Suppose there exists $G \in \Gamma$ with $G << 0$. Then there exists $g\colon A \to \mathbb{R}^k_+$ with $g(a) \in \gamma(a)$ such that $\sum_{a \in A} g(a) = G$. Define $h(a) = g(a) + f(a) - G / |A|$. $h(a) >> (g(a) + f(a)) \wp_a f(a)$; since $\wp_a$ satisfies free disposal, $h(a) \wp_a f(a)$. But $\sum_{a \in A} h(a) = \sum_{a \in A} g(a) + \sum_{a \in A} f(a) - \sum_{a \in A} \frac{G}{|A|} = G + \sum_{a \in A} e(a) - G = \sum_{a \in A} e(a)$, which contradicts the Pareto optimality of $f$. Hence $G << 0 \Rightarrow G \notin \Gamma$.

Suppose $x \in \text{con}\ \Gamma$. By the Shapley-Folkman Theorem (Starr [18]), we can write $x$ in the form $x = \sum_{a \in A} g(a)$, where $g(a) \in \text{con}\ \gamma(a)$ for all $a \in A$ and $g(a) \in \gamma(a)$ for all but $m$ values of $a$, for some $m \leq k$. Let those values be $\{a_1, \ldots, a_m\}$. Let $\tilde{g}(a_i) = (\delta, \ldots, \delta)$ for some $\delta > 0$ and $\tilde{g}(a) = g(a)$ for $a \notin \{a_1, \ldots, a_m\}$. Since $\gamma(a_i) \geq -f(a_i)$, $\text{con}\ \gamma(a_i) \geq -f(a_i)$. Let $z = -(M_f, \ldots, M_f)$. Then $x = \sum_{a \in A} g(a)$ $= \sum_{a \in A} \tilde{g}(a) + \sum_{i=1}^{m} g(a_i) - m(\delta, \ldots, \delta) \geq \sum_{a \in A} \tilde{g}(a) - \sum_{i=1}^{m} f(a_i) - m(\delta, \ldots, \delta)$ $\geq \sum_{a \in A} \tilde{g}(a) + z - m(\delta, \ldots, \delta)$. Since $\tilde{g}(a) \in \gamma(a)$ for all $a \in A$, $\sum_{a \in A} \tilde{g}(a) \in \Gamma$, and hence we cannot have $x << z - m(\delta, \ldots, \delta)$. Since $\delta$ is arbitrary, we cannot have $x << z$.

Hence, $\text{con}\ \Gamma$ does not intersect $\{w \in \mathbb{R}^k\colon w << z\}$. Therefore, there exists $p \in \Delta$ with $\inf p \cdot \Gamma \geq \sup p \cdot \{w\colon w << z\} = -M_f$. Therefore, $\sum_{a \in A} \phi(p, f, a) \leq M_f$.

*Remark 3.2:* For certain applications, it may be useful to use a different normalization of prices. If $(1/r) + (1/s) = 1$ (with the usual convention that $1/\infty = 0$), and prices are normalized so that $||p||_s = 1$, then the theorem remains true with $M_f = k^{1/r} \max || \sum_{i=1}^{k} f(a_i) ||_r$. The reader is warned that there was an error in this remark in the working paper version of this paper: the factor $k^{1/r}$ was omitted.

*Theorem 3.3:* Suppose $\varepsilon\colon A \to P'' \times \mathbb{R}^k_+$ is an exchange economy. If $f \in \mathscr{P}(\varepsilon)$, then there exists an income transfer $t$ with $-2M_f \leq \sum_{a \in A} t(a) \leq 0$ and $(\tilde{f}, \bar{p}) \in \mathscr{Q}(\varepsilon, t)$ such that, for all but $k$ agents $a \in A$, $f(a) \not\wp_a \tilde{f}(a)$. Alternatively, we may find an income transfer $\hat{t}$ with $-M_f \leq \sum_{a \in A} \hat{t}(a) \leq 0$ and $\hat{f}(a) \in Q(\bar{p}, a, \hat{t})$ such that, for *all* $a \in A$, $f(a) \not\succ_a \hat{f}(a)$ and

$$\bar{p} \cdot \left[ \left( \sum_{a \in A} (\hat{f}(a) - e(a)) \right)_+ + \left( \sum_{a \in A} (\hat{f}(a) - e(a)) \right)_- \right] \leq 3M_f.$$

If in addition we assume that $\succ_a \in P \cap P''$ for all $a$ and

$$\sum_{a \in S} f(a) < \frac{1}{2} \sum_{a \in A} f(a) \text{ whenever } |S| \leq k \qquad (*),$$

then we can take $\bar{p} >> 0$, $(\tilde{f}, \bar{p}) \in \mathcal{W}(\varepsilon, t)$, and $\hat{f}(a) \in D(\bar{p}, a, \hat{t})$ for all $a \in A$.

*Remark 3.4:* Since preferences may be nonconvex, $\mathcal{W}(\varepsilon, t)$ may be empty. The conclusion that it is not empty is less surprising that it might at first appear. Since we only require that $\sum_{a \in A} \tilde{f}(a) \leq \sum_{a \in A} e(a)$, and we may have $\sum_{a \in A} t(a) < 0$, the government ends up with some quantity of goods. It is as if the government had a linear preference relation with indifference curves perpendicular to $p$; this provides the necessary freedom to obtain a Walrasian equilibrium. The alternative formulation involving $\hat{f}$ is a notion of approximate Walrasian equilibrium. The theorem indicates that the market value of the absolute value (taken componentwise) of the excess demand is bounded. This result is obtained essentially by combining the formulation involving $\tilde{f}$ with the argument in Anderson [3]. Finally, note that the statement of the Second Welfare Theorem in Debreu's *Theory of Value* [9] involves the notion of equilibrium relative to a price system. An equilibrium relative to a price system is immediately seen to be a Walrasian equilibrium after income transfers in the sense that we use here.

*Lemma 3.5:* Suppose $\succ$ satisfies free disposal. If $x \in \mathbb{R}^k_+$, then con $\overline{\{z: z \succ x\}}$ is closed.

*Proof:* Let $B = \{z\colon z \succsim x\}$. If $y \in \overline{B}$ and $w > y$, then we may find $w_m \to w$ and $y_m \to y$ such that $w_m >> y_m$ and $y_m \succsim x$. By free disposal, $w_m \succsim x$, so $w \in \overline{B}$. Thus, $\overline{B} + I\!R^k_+ \subset \overline{B} \subset I\!R^k_+$. Hence, con $\overline{B}$ is closed (Mas-Colell [16], p. 27, F.1.2.).

*Proof of Theorem 3.3:* By Theorem 3.1, there exist $p \in \Delta$ such that $\inf p \cdot \Gamma \geq -M_f$. If $p \in \Delta$, then $\inf p \cdot \Gamma \leq 0$. Hence, $\alpha = -\sup_{p \in \Delta} \inf p \cdot \Gamma$ exists. Suppose $p_n$ is such that $\inf p_n \cdot \Gamma \to -\alpha$. By taking a convergent subsequence, we may assume that $p_n \to \bar{p}$ for some $\bar{p} \in \Delta$. Then $-\alpha \geq \inf \bar{p} \cdot \Gamma$; on the other hand, if $\inf \bar{p} \cdot \Gamma < -\alpha$, then there exists $G \in \Gamma$ and $\delta > 0$ such that $\bar{p} \cdot G < -\alpha - \delta \leq \inf p_n \cdot \Gamma - \delta/2$ (for $n$ sufficiently large) $\leq p_n \cdot G - \delta/2$, a contradiction since $p_n \to \bar{p}$. Hence, $\inf \bar{p} \cdot \Gamma = -\alpha$. The price $\bar{p}$ is used in the proof of Mas-Colell [16], Proposition 4.5.1. We call $\bar{p}$ the gap-minimizing price because it minimizes the "decentralization gap" for the Pareto optimum $f$; see Anderson [6] for a related construction for core allocations.

Let $z = (-\alpha, \ldots, -\alpha)$. We claim that $z \in \overline{\text{con}\ \Gamma}$; if not, there exists $q \neq 0$, $q \cdot z < \inf q \cdot \overline{\text{con}\ \Gamma}$. Observe that for any $i$ and any $t > 0$, $(0, \ldots, 0, t, 0 \ldots, 0) \in \overline{\Gamma}$, where the $t$ occurs in the $i$'th place, by weak monotonicity. Hence, if $q^i < 0$ for some $i$, $\inf q \cdot \overline{\text{con}\ \Gamma} = -\infty$, a contradiction. Hence, $q > 0$, so we may assume $q \in \Delta$. Thus, $\inf q \cdot \Gamma = \inf q \cdot \overline{\text{con}\ \Gamma} > q \cdot z = -\alpha$, contradicting the definition of $\alpha$. Hence, $z \in \overline{\text{con}\ \Gamma} = \overline{\text{con} \sum_{a \in A} \gamma(a)} = \overline{\sum_{a \in A} \text{con}\ \gamma(a)} = \sum_{a \in A} \overline{\text{con}\ \gamma(a)}$ (since $\gamma(a)$ is bounded below by $-f(a)$) $= \sum_{a \in A} \left(\overline{\text{con}\ \{z\colon z \succsim_a f(a)\}} - f(a)\right) = \sum_{a \in A} \left(\text{con}\ \overline{\{z\colon z \succsim_a f(a)\}} - f(a)\right)$ (by Lemma 3.5) $= \sum_{a \in A} \text{con}\ \overline{\gamma(a)}$. Hence, we may write $z = \sum_{a \in A} g(a)$, where $g(a) \in \text{con}\ \overline{\gamma(a)}$. By the Shapley-Folkman Theorem, there is a set $\{a_1, \ldots, a_k\} \subset A$ so that we may choose $g(a) \in \overline{\gamma(a)}$ for all $a \notin \{a_1, \ldots, a_k\} \subset A$. Since $\sum_{a \in A} \bar{p} \cdot g(a) = \bar{p} \cdot z = -\alpha = \inf \bar{p} \cdot \Gamma = \sum_{a \in A} \inf \bar{p} \cdot \gamma(a)$, $\bar{p} \cdot g(a) = \inf \bar{p} \cdot \gamma(a)$ for all $a \in A$.

Define $\tilde{f}(a) = g(a) + f(a)$ if $a \notin \{a_1, \ldots, a_k\}$ and $\tilde{f}(a_i) = 0$ $(1 \leq i \leq k)$. $\sum_{a \in A} \tilde{f}(a) = \sum_{a \notin \{a_1, \ldots, a_k\}} (g(a) + f(a)) \leq \sum_{a \in A} (g(a) + f(a)) = z + \sum_{a \in A} f(a) \leq \sum_{a \in A} f(a) = \sum_{a \in A} e(a)$.

Define $t(a) = \bar{p} \cdot (\tilde{f}(a) - e(a))$. $\bar{p} \cdot \tilde{f}(a) = \bar{p} \cdot e(a) + t(a)$. $\sum_{a \in A} t(a) = \sum_{a \in A} \bar{p} \cdot (\tilde{f}(a) - f(a))$ $= \sum_{a \in A} \bar{p} \cdot g(a) - \sum_{i=1}^{k} \bar{p} \cdot f(a_i) = \bar{p} \cdot (z - \sum_{i=1}^{k} f(a_i))$, so $-2M_f \leq \sum_{a \in A} t(a) \leq 0$. We will show that $\tilde{f}(a) \in Q(\bar{p}, a, t)$ for all $a$. Suppose first $a \notin \{a_1, \ldots, a_k\}$. Suppose $x \wp_a \tilde{f}(a)$. By continuity there exists $y \in \gamma(a)$ such that $x \wp_a (y + f(a))$. By the definition of $\gamma(a)$, $(y + f(a)) \wp_a f(a)$; by transitivity, $x \wp_a f(a)$. Hence, $\bar{p} \cdot x \geq \inf \bar{p} \cdot \gamma(a) + \bar{p} \cdot f(a) = \bar{p} \cdot \tilde{f}(a)$ $= \bar{p} \cdot e(a) + t(a)$. Thus, $\tilde{f}(a) \in Q(\bar{p}, a, t)$. If $a \in \{a_1, \ldots, a_k\}$, $\bar{p} \cdot \tilde{f}(a) = \bar{p} \cdot e(a) + t(a)$ $= 0$, so it is trivial that $\tilde{f}(a) \in Q(\bar{p}, a, t)$. Hence, $(\tilde{f}, \bar{p}) \in \mathcal{Q}(\varepsilon, t)$.

Suppose $a \notin \{a_1, \ldots, a_k\}$. If $f(a) \wp_a \tilde{f}(a)$, then by continuity we may find $y \in \gamma(a)$ such that $f(a) \wp_a (y + f(a))$. By the definition of $\gamma(a)$, $(y + f(a)) \wp_a f(a)$; by transitivity, $f(a) \wp_a f(a)$, contradicting irreflexivity. Hence, for all $a \notin \{a_1, \ldots, a_k\}$, $f(a) \not\wp_a \tilde{f}(a)$.

Define $\hat{f}(a) = g(a) + f(a)$ if $a \notin \{a_1, \ldots, a_k\}$ and let $\hat{f}(a_i) = \hat{g}(a_i) + f(a_i)$, where $\hat{g}(a_i)$ is chosen arbitrarily from $\{x \in \overline{\gamma(a)} : \bar{p} \cdot x = \inf \bar{p} \cdot \gamma(a)\}$. Let $\hat{t}(a) = \bar{p} \cdot (\hat{f}(a) - e(a))$. $\sum_{a \in A} \hat{t}(a) = \sum_{a \in A} \bar{p} \cdot g(a) = \bar{p} \cdot z$, so $-M_f \leq \sum_{a \in A} \hat{t}(a) \leq 0$. By the same arguments as in the last two paragraphs, it follows for all $a \in A$ that $\hat{f}(a) \in Q(\bar{p}, a, t)$ and $f(a) \not\wp_a \hat{f}(a)$.

Note that

$$\sum_{a \in A} \left( \hat{f}(a) - e(a) \right) = \sum_{a \in A} \left( \hat{f}(a) - f(a) \right) = z + \sum_{i=1}^{k} \left( \hat{g}(a_i) - g(a_i) \right)$$

$$= z + \sum_{i=1}^{k} \left[ \left( \hat{g}(a_i) + f(a_i) \right) - \left( g(a_i) + f(a_i) \right) \right],$$

and so

$$z - \sum_{i=1}^{k} \left(g(a_i) + f(a_i)\right) \leq \sum_{a \in A} \left(\hat{f}(a) - e(a)\right) \leq \sum_{i=1}^{k} \left(\hat{g}(a_i) + f(a_i)\right).$$

Therefore

$$\bar{p} \cdot \left[ \left( \sum_{a \in A} (\hat{f}(a) - e(a)) \right)_+ + \left( \sum_{a \in A} (\hat{f}(a) - e(a)) \right)_- \right]$$

$$\leq -\bar{p} \cdot z + \bar{p} \cdot \sum_{i=1}^{k} \left(g(a_i) + f(a_i)\right) + \bar{p} \cdot \sum_{i=1}^{k} \left(\hat{g}(a_i) + f(a_i)\right) \leq M_f + 2\bar{p} \cdot \sum_{i=1}^{k} f(a_i) \leq 3M_f.$$

Now suppose that preferences are monotone and the condition (*) holds. If we can show that $\bar{p} >> 0$, then it follows by standard arguments that $\mathcal{Q}(\varepsilon, t) = \mathcal{W}(\varepsilon, t)$, so we will be done. Suppose $p^i = 0$ for some $i$. Suppose $a \notin \{a_1, \dots, a_k\}$. If $\bar{p} \cdot \tilde{f}(a) > 0$, then by monotonicity and continuity, there exists $y \wp_a \tilde{f}(a)$, $\bar{p} \cdot y < \bar{p} \cdot \tilde{f}(a)$. By continuity we may find $x$ such that $y \wp_a x$ and $x - f(a) \in \gamma(a)$. By the definition of $\gamma(a)$, $x \wp_a f(a)$, and so $y \wp_a f(a)$ by transitivity, a contradiction. Therefore, $\bar{p} \cdot \tilde{f}(a) = 0$ for $a \notin \{a_1, \dots, a_k\}$.

By changing the units in which the commodities are measured, we can assume without loss of generality that

$$\left( \sum_{a \in A} f(a) \right)^1 = \dots = \left( \sum_{a \in A} f(a) \right)^k.$$

Then, choosing $i$ to minimize

$$\left(\sum_{a \notin \{a_1,\dots,a_k\}} f(a)\right)^i$$

and $j$ to maximize

$$\max_{|S| \le k}\left(\sum_{a \in S} f(a)\right)^j,$$

$$\sum_{a \in A} \bar{p} \cdot z = \sum_{a \in A} \bar{p} \cdot g(a) = \sum_{a \notin \{a_1,\dots,a_k\}} \bar{p} \cdot (\tilde{f}(a) - f(a)) + \sum_{m=1}^{k} \bar{p} \cdot g(a_m)$$

$$= - \sum_{a \notin \{a_1,\dots,a_k\}} \bar{p} \cdot f(a) + \sum_{m=1}^{k} \bar{p} \cdot g(a_m) \le - \sum_{a \notin \{a_1,\dots,a_k\}} \bar{p} \cdot f(a)$$

$$\le -\left(\sum_{a \notin \{a_1,\dots,a_k\}} f(a)\right)^i < -\frac{1}{2}\left(\sum_{a \in A} f(a)\right)^i \quad \text{(by (*))}$$

$$= -\frac{1}{2}\left(\sum_{a \in A} f(a)\right)^j < -\max_{|S| \le k}\left(\sum_{a \in S} f(a)\right)^j \quad \text{(by (*))} = -M_f.$$

Hence, inf $\bar{p} \cdot \Gamma < -M_f$, contradicting the definition of $\bar{p}$ and the last line of the proof of Theorem 3.1. Thus, $\bar{p} >> 0$, completing the proof.

# 4. Random Sequences of Economies

The purpose of this section is to show that a version of the Second Welfare Theorem stronger than the results of section 3 holds for almost all sequences of economies drawn at random: agents' consumptions are close to their consumptions at a Walrasian equilibrium with income transfers. The key observation in the proof is that sequences of economies drawn at random from a given distribution of agents' characteristics converge in a stronger sense than weak convergence. The proof is modelled after the proof of a strong core convergence theorem with nonconvex preferences in Anderson [5]. As a corollary of the proof, we also prove a theorem (Theorem 4.4) for type sequences of exchange economies.

$P$ can be made into a Borel subset of a compact metrizable space (Hildenbrand [13], Grodal [11]); the topology this metric generates is called the topology of closed convergence. Let $\mathcal{M}$ be the space of Borel probability measures $\mu$ on $P$.

Suppose $\mu \in \mathcal{M}$. We may think of $\mu$ as describing the underlying distribution of preferences of "all possible people," and construct sequences of finite economies by sampling from $\mu$. Specifically, we take $\Omega$ to be the countable product $P^{\mathbb{N}}$, with the countable product measure $\mu^{\mathbb{N}}$. Any $\omega \in \Omega$ is a sequence $\{\omega_1, \omega_2, ...\}$ $\omega_i \in P$, of preferences. Let $A_n = \{1, ..., n\}$. Given such an $\omega$ and an arbitrary sequences of endowment maps $e_n$: $A_n \to \mathbb{R}^k_+$, we form a sequence of economies $\varepsilon^\omega_n : A_n \to P \times \mathbb{R}^k_+$, where $\varepsilon^\omega_n(i) = (\omega_i, e_n(i))$. In other words, $\varepsilon^\omega_n$ is the economy whose agents have characteristics $(\omega_1, e_n(1)), ..., (\omega_n, e_n(n))$. The construction of a sequence of economies by sampling in this way is due to Hildenbrand; see [13], page 138.

*Theorem 4.1:* Suppose $\mu \in \mathcal{M}$. There is a set $\overline{\Omega} \subset \Omega$ with $\mu^{\mathbb{N}}(\overline{\Omega}) = 1$ with the following property: if $\omega \in \overline{\Omega}$, then for every sequence of endowments $e_n$ satisfying

$$0 << \underline{\lim}_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e_n(i) \text{ and } \overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e_n(i) << \infty$$

and every sequence $\{f_n\} \in \mathcal{B}$, $f_n \in \mathcal{P}(\varepsilon^\omega_n)$, there exist (for sufficiently large $n$) income transfer functions $t_n$ with $\sum_{a \in A_n} t_n(a) \leq 0$ and $(\tilde{f}_n, p_n) \in \mathcal{W}(\varepsilon^\omega_n, t_n)$ such that, for all $\delta > 0$,

$$\frac{1}{|A_n|} \, | \{a \in A_n : \; || f_n(a) - \tilde{f}_n(a) ||_\infty > \delta\} | \to 0.$$

Furthermore, if the sequence $\{f_n\} \in \mathcal{U}$, then we may choose $t_n$ and $\tilde{f}_n$ such that

$$\frac{1}{|A_n|} \sum_{a \in A_n} t_n(a) \to 0 \quad \text{and} \quad \frac{1}{|A_n|} \sum_{a \in A_n} \| f_n(a) - \tilde{f}_n(a) \|_\infty \to 0.$$

*Remark 4.2:* The comments in Remark 3.4 apply here also. In addition, note that our assumptions on preferences are not sufficient to guarantee that demand sets are not empty. However, one consequence of Theorem 4.1 is that the existence of Pareto optima implies that demand sets are nonempty.

We shall first prove the following result for Nonstandard exchange economies, which were introduced by Brown and Robinson [8].

*Theorem 4.3:* Let $\varepsilon: A \to {}^*(P \times \mathbb{R}^k_+)$ be an internal exchange economy. Define the counting measure $\lambda(B) = |B| \,/\, |A|$ for $B$ an internal subset of $A$. Suppose $f \in {}^*\mathscr{P}(\varepsilon)$, and

(i) $|A| \in {}^*\mathbb{N} - \mathbb{N}$.

(ii) $\frac{M_f}{|A|} \simeq 0$ and, for each $i$, $L(\lambda)(\{a \in A\colon 0 < {}^\circ(f(a))^i\}) > 0$,

where $L(\lambda)$ is the Loeb measure generated by $\lambda$ [15], and

(iii) $0 << {}^\circ(\frac{1}{|A|} \sum_{a \in A} e(a)) << \infty$.

(iv) the induced measure $\nu(B) = \frac{|\{a \in A\colon \wp_a \in B\}|}{|A|}$ on ${}^*P$

is standardly distributed (see section 8 of Anderson [4]).

Then there exists an internal income transfer $t\colon A \to {}^*\mathbb{R}$ with $\sum_{a \in A} t(a) \le 0$ and $(\tilde{f}, p) \in {}^*\mathscr{W}(\varepsilon, t)$ such that $\tilde{f}(a) \simeq f(a)$ for $L(\lambda)$-almost all $a \in A$.

If, instead of (ii), we substitute the stronger assumption

(ii′) $f$ is S-integrable (i.e. $\frac{|E|}{|A|} \simeq 0 \Rightarrow \frac{1}{|A|} \sum_{a \in A} e(a) \simeq 0$),

then we may choose $t$ and $\tilde{f}$ such that

$$\frac{1}{|A|}\sum_{a\in A} t(a) \simeq 0 \quad \text{and} \quad \frac{1}{|A|}\sum_{a\in A} \| f(a) - \tilde{f}(a) \|_\infty \simeq 0.$$

*Proof:* Let $\mu(B) = L(\nu)(st^{-1}(B))$ for Borel $B \subset P$. By Anderson [4] (Proposition 8.4(ii)), $\mu$ is a Radon Probability measure. In particular, $\wp_a$ is near-standard for $\mu$-almost all $a$.

Given $p \in \Delta^\circ$ and $\varepsilon,\ \delta \in I\!R_{++}$, define

$$B_{p\varepsilon\delta} = \{\wp \in P\colon \ \|x\|_\infty \leq 1/\varepsilon,\ \ \phi(p,x,\wp) < \delta \Rightarrow \exists y\ \rho(x, D(p,(\wp,y))) < \varepsilon\}.$$

Fix $\varepsilon \in I\!R_+$, $p \in \Delta^\circ$, $x \in I\!R^k_+$ and $\wp \in P$. If $\phi(p,x,\wp) = 0$, then

$$y \wp x \Rightarrow p \cdot y \geq p \cdot x. \qquad (*)$$

Since $p >> 0$, $x \in D(p,(\wp,x))$ if $x = 0$. If $x \neq 0$, $y\wp x$, and $p \cdot y = p \cdot x$, then we can find $y'$ arbitrarily close to $y$ with $p \cdot y' < p \cdot x$. Since $\wp$ is continuous, we can choose such a $y'$ satisfying $y' \wp x$, contradicting (*). Hence, $\phi(p,x,\wp) = 0 \Rightarrow x \in D(p,(\wp,x))$. Now, let $x$ vary, subject to the constraint $\|x\| \leq 1/\varepsilon$. We claim that there exists $\delta$ such that $\wp \in B_{p\varepsilon\delta}$. If not, then we may find a sequence $x_n$ with $\phi(p,x_n,\wp) \to 0$ but $\rho(x_n, D(p,(\wp,y))) > \varepsilon$ for every $y$. By taking a convergent subsequence, we may assume without loss of generality that $x_n \to x$ for some $x$. $\phi(p,x,\wp) = 0$, and so $x \in D(p,(\wp,x))$. Letting $y = x$, we arrive at a contradiction. Thus, for fixed $p \in \Delta^\circ$ and $\varepsilon \in I\!R_{++}$,

$$\bigcup_{\delta>0} B_{p\varepsilon\delta} = P.$$

Since $\mu$ is countably additive, given $\varepsilon \in I\!R_{++}$, there exists $\delta \in I\!R_{++}$ such that $\mu(B_{p\varepsilon\delta}) > 1 - \varepsilon$. Since $\nu$ is standardly distributed, $\nu({}^*B_{p\varepsilon\delta}) \simeq {}^*(\mu(B_{p\varepsilon\delta})) = \mu(B_{p\varepsilon\delta})$, so $\nu({}^*B_{p\varepsilon\delta}) > 1 - 2\varepsilon$.

Suppose $f \in {}^*\mathscr{P}(\varepsilon)$. Transferring Theorem 3.1, we see that there exists $p \in {}^*\Delta$ such that $\frac{1}{|A|}\sum_{a\in A}\phi(p,f,a) \leq M_f/|A| \simeq 0$. Therefore $\phi(p,f,a) \simeq 0$, $L(\lambda)$-almost surely.

We show ${}^\circ p >> 0$. If not, we can assume without loss of generality that ${}^\circ p^1 > 0$, ${}^\circ p^2 = 0$. $L(\lambda)(\{a\colon \phi(p,f,a) \simeq 0, \wp_a \text{ near-standard}, {}^\circ f(a) << \infty\}) = 1$. By assumption (ii), we can find $a$ such that $\phi(p,f,a) \simeq 0$, $\wp_a$ is near-standard, ${}^\circ f(a) << \infty$, and

${}^\circ(f(a)^1) > 0$. Let $y = f(a) + (-2\frac{\phi(p,f,a)+p_2}{p_1},1,0,\dots,0)$. $y \in {}^*\mathbb{R}^k_+$, ${}^\circ y = {}^\circ f(a) + (0,1,0,\dots,0)$, so ${}^\circ y \; {}^\circ\wp_a \; {}^\circ f(a)$. Hence, $y \wp_a f(a)$. But $p \cdot y = p \cdot f(a) - 2\phi(p,f,a) - 2p_2 + p_2 < p \cdot f(a) - \phi(p,f,a)$, a contradiction. Thus, ${}^\circ p >> 0$.

Thus, given $\varepsilon \in \mathbb{R}_{++}$, we may find $\delta \in \mathbb{R}_{++}$ such that $\nu({}^*B_{{}^\circ p \varepsilon\delta}) > 1 - 2\varepsilon$. $\phi({}^\circ p,f,a) \leq \phi(p,f,a) + 2\,||p - {}^\circ p||_1 \, ||f(a)||_\infty / \min_i p^i \simeq 0$ $L(\lambda)$-almost surely. Hence, $\lambda(\{a \in A\colon \phi({}^\circ p,f,a) < \delta\}) \simeq 1$ for $\delta \in \mathbb{R}_{++}$. Hence there is some $\bar{\delta} \simeq 0$ such that $\lambda(\{a \in A\colon \phi({}^\circ p,f,a) < \bar{\delta}\}) \simeq 1$. For all $\varepsilon \in \mathbb{R}_{++}$, $\nu({}^*B_{{}^\circ p \varepsilon\bar{\delta}}) > 1 - 2\varepsilon$. Hence, there is some $\bar{\varepsilon} \simeq 0$ such that $\nu(B_{{}^\circ p \bar{\varepsilon}\bar{\delta}}) > 1 - 2\bar{\varepsilon} \simeq 1$.

For $L(\lambda)$-almost all $a \in A$, the following conditions hold: (i) $\phi({}^\circ p,f,a) < \bar{\delta}$, (ii) $\wp_a \in B_{{}^\circ p \bar{\varepsilon}\bar{\delta}}$, and (iii) $||f(a)||_\infty < 1/\bar{\varepsilon}$. For such $a$'s, there exists $y$ such that $\rho(f(a), D(p,(\wp_a,y))) < \bar{\varepsilon} \simeq 0$. For all $a \in A$, choose $g(a)$ to minimize $||g(a) - f(a)||_\infty$ subject to $g(a) \in {}^*D({}^\circ p,(\wp_a,y))$ for some $y$; if ${}^*D({}^\circ p,(\wp_a,y))$ is empty for all $y$, we set $g(a) = 0$. Then define $h(a) = g(a)$ if $||g(a) - f(a)||_\infty \leq 1$, and $h(a) = 0$ otherwise. $h$ is internal, and $h(a) \simeq f(a)$ $L(\lambda)$-almost surely.

If we defined $\tilde{f} = h$ and $t(a) = {}^\circ p \cdot (\tilde{f}(a) - e(a))$, we would satisfy all the conclusions of the theorem except possibly $\sum_{a \in A} \tilde{f}(a) \leq \sum_{a \in A} e(a)$ and $\sum_{a \in A} t(a) \leq 0$. We shall show how to modify $h$ in order to obtain these last properties. Note that $h(a)^i \leq f(a)^i + 1$ for each $a$ and $i$. Further, since $h(a) \simeq f(a)$ $L(\lambda)$-almost surely, there exists $\eta \simeq 0$ such that

$$\frac{1}{|A|}\sum_{a \in A} h(a)^i \leq \frac{1}{|A|}\sum_{a \in A} f(a)^i + \eta = \frac{1}{|A|}\sum_{a \in A} e(a)^i + \eta$$

for each $i$. By assumption (ii), there is some $\beta \in \mathbb{R}_{++}$ such that

$$\frac{1}{|A|} |\{a \in A: f(a)^i > \beta\}| > 2\beta.$$

Since $h(a) \simeq f(a)$ $L(\lambda)$-almost surely,

$$\frac{1}{|A|} |\{a \in A: h(a)^i > \beta\}| > \beta.$$

Hence, we can find $A^i \subset A$ with $h(a)^i > \beta$ for $a \in A^i$ and $0 \simeq \frac{|A^i|}{|A|} > \frac{\eta}{\beta}$. Define $\tilde{f}(a) = 0$ for $a \in A^1 \cup ... \cup A^k$, and $\tilde{f}(a) = h(a)$ otherwise. Define $t(a) = {}^\circ p \cdot (\tilde{f}(a) - e(a))$ for all $a$. For each $i$, $\sum_{a\in A} \tilde{f}(a)^i \leq \sum_{a\in A} h(a)^i - \beta\frac{\eta}{\beta}|A| = \sum_{a\in A} h(a)^i - \eta|A| \leq \sum_{a\in A} f(a)^i = \sum_{a\in A} e(a)^i$. $\sum_{a\in A} t(a) = {}^\circ p \cdot \sum_{a\in A} (\tilde{f}(a) - e(a)) \leq 0$. Then $\tilde{f}(a) \in {}^*D({}^\circ p, a, t)$ for all $a$, so $(\tilde{f}, {}^\circ p) \in {}^*\mathscr{W}(\varepsilon, t)$.

Now suppose assumption (iii′) holds. $||\tilde{f}(a) - f(a)||_\infty \leq 1$, unless $\tilde{f}(a) = 0$. Since $f(a)$ is S-integrable, so is $\tilde{f}$. By Anderson [1] (Theorem 9), $\frac{1}{|A|} \sum_{a\in A} ||f(a) - \tilde{f}(a)||_\infty \simeq \int_A {}^\circ||f(a) - \tilde{f}(a)||_\infty \, dL(\lambda) = 0$, since ${}^\circ||f(a) - \tilde{f}(a)||_\infty = 0$ $L(\lambda)$ –almost everywhere. $\frac{1}{|A|} \sum_{a\in A} t(a) = \frac{1}{|A|} {}^\circ p \cdot (f(a) - \tilde{f}(a)) \leq \frac{1}{|A|} \sum_{a\in A} ||f(a) - \tilde{f}(a)||_\infty \simeq 0$. This completes the proof.

A type sequence of economies is a sequence of economies $\varepsilon_n: A_n \to T$, where $T$ is a finite subset of $P \times I\!R^k_+$. The elements of $T$ are called types; two individuals $a$, $b \in A_n$ with $\varepsilon_n(a) = \varepsilon_n(b)$ are said to be of the same type, in that they have identical characteristics. Note however that allocations (including Pareto optimal allocations) may give different consumption vectors in $I\!R^k_+$ to $a$ and $b$. Let $M_T$ be the largest $||\ .\ ||_\infty$-norm of the endowments in $T$.

*Theorem 4.4:* Let $\varepsilon_n : A_n \to T$ be a type sequence of exchange economies and $f_n \in \mathscr{P}(\varepsilon_n)$ satisfying (i) $|A_n| \to \infty$, (ii) $\{f_n\} \in \mathscr{B}$, (iii) $\inf_n |\varepsilon_n^{-1}(t)| / |A_n| > 0$ for each $t \in T$, and $\sum_{(\wp,e)\in T} e >> 0$. Then for sufficiently large $n$, there exist income transfers $t_n$ with $\sum_{a\in A_n} t_n(a) \le 0$ and $(\tilde{f}_n, p_n) \in \mathscr{W}(\varepsilon_n, t_n)$ such that, for all $\delta > 0$,

$$\frac{1}{|A_n|} |\{a \in A_n : \, ||f_n(a) - \tilde{f}_n(a)||_\infty > \delta\}| \to 0.$$

Furthermore, if the sequence $\{f_n\} \in \mathscr{U}$, then we may choose $t_n$ and $\tilde{f}_n$ such that

$$\frac{1}{|A_n|} \sum_{a\in A_n} t_n(a) \to 0 \quad \text{and} \quad \frac{1}{|A_n|} \sum_{a\in A_n} ||f_n(a) - \tilde{f}_n(a)||_\infty \to 0.$$

*Proof of Theorem 4.4.* Consider the nonstandard extension of the sequences $A_n, f_n$, etc. and choose $n \in {}^*\mathbb{N} - \mathbb{N}$. We shall apply Theorem 4.3 to the economy $\varepsilon_n$. Assumptions (i) - (iii) of Theorem 4.3 follow from the corresponding assumptions in Theorem 4.4. Assumption (iv) follows from the fact that the sequence is a type sequence (Anderson [4], Example 8.2). Hence, there is an internal income transfer $t_n$ and $(\tilde{f}_n, p_n) \in {}^*\mathscr{W}(\varepsilon_n, t_n)$ such that $\tilde{f}_n(a) \simeq f_n(a)$ for $L(\lambda)$ – almost all $a \in A_n$. Hence, $\frac{1}{|A_n|} |\{a \in A_n : \, ||\tilde{f}_n(a) - f_n(a)||_\infty > \delta\}| \simeq 0$. Hence, $\exists \bar{n} \in \mathbb{N}$ such that, for $n > \bar{n}$, we may find $t_n$ and $(\tilde{f}_n, p_n) \in \mathscr{W}(\varepsilon_n, t_n)$ such that

$$\frac{1}{|A_n|} |\{a \in A_n : \, ||\tilde{f}_n(a) - f_n(a)||_\infty > \delta\}| \to 0.$$

If the sequence $f_n$ is uniformly integrable, then for infinite $n$, $f_n$ is S-integrable (Anderson [4], Theorem 6.5). By Theorem 4.3, $\frac{1}{|A_n|} \sum_{a\in A_n} t_n(a) \simeq 0$ and

$\frac{1}{|A_n|}\sum_{a\in A_n} ||f_n(a)-\tilde{f}_n(a)||_\infty \simeq 0$. It follows that, for $n \in I\!N$, $\frac{1}{|A_n|}\sum_{a\in A_n} t_n(a) \to 0$ and $\frac{1}{|A_n|}\sum_{a\in A_n} ||f_n(a)-\tilde{f}_n(a)||_\infty \to 0$. This completes the proof.

*Proof of Theorem 4.1:* Let $n \in {}^*I\!N - I\!N$. By Theorem 8.7(i) of Anderson [4], there exists an internal $\Omega''_n \subset {}^*\Omega$ such that ${}^*\mu^{I\!N}(\Omega''_n) \geq 1 - 2ne^{-\sqrt{n}/4}$ and $\nu_n^\omega$ is standardly distributed for all $\omega \in \Omega_n'$. Letting $\Omega'''_n = \bigcap_{m\geq n} \Omega''_n$, we find $\Omega'''_n$ with ${}^*\mu^{I\!N}(\Omega'''_n) \simeq 1$ such that $\nu_m^\omega$ is standardly distributed for all $\omega \in \Omega'''_n$ and all $m \geq n$.

Suppose $\omega \in \Omega_n$. We shall apply Theorem 4.3 to the economy $\varepsilon_m^\omega$, where $m \geq n$. Assumption (i) holds trivially, since $|A_m| = m$. Assumption (iii) follows from the assumption on the endowments $e_n$. Assumption (iv), that $\nu_m^\omega$ is standardly distributed, follows from the definition of $\Omega_n$.

Let

$$\Omega_{\beta\gamma\delta\eta n} = \{\omega\colon\ m \geq n, f_m \in \mathscr{P}(\varepsilon_m), \{f_m\} \in \mathscr{B}_{\gamma\eta n}$$

$$\Rightarrow\ \exists t_m \exists (\tilde{f}_m, p_m) \in \mathscr{W}(\varepsilon_m, t_m)\ \ \lambda(\{a\colon\ ||f(a) - \tilde{f}(a)||_\infty > \delta\}) < \beta\}.$$

Fix $\alpha, \beta, \gamma, \delta \in I\!R_{++}$. Suppose $\eta \simeq 0$, and $n \in {}^*I\!N - I\!N$. By Theorem 4.3,

$${}^*\mu^{I\!N}({}^*\Omega_{\beta\gamma\delta\eta n}) \geq {}^*\mu^{I\!N}(\Omega_n) > 1 - \alpha.$$

Since this statement holds for all $n \in {}^*I\!N - I\!N$ and all $\eta \simeq 0$, it also holds for some $n \in I\!N$ and some $\eta \in I\!R_{++}$. Thus, by the Transfer Principle,

$$\mu^{I\!N}(\Omega_{\beta\gamma\delta\eta n}) > 1 - \alpha.$$

In other words, letting

$$\Omega_{\beta\gamma\delta} = \bigcup_{\eta,n} \Omega_{\beta\gamma\delta\eta n},$$

$\mu(\Omega_{\beta\gamma\delta}) = 1.$

Let $\overline{\Omega} = \bigcap_{\beta,\gamma,\delta} \Omega_{\beta\gamma\delta}$. $\mu^{\mathbb{N}}(\overline{\Omega}) = 1$, since the intersection can be taken over a countable number of $\beta$'s, $\gamma$'s and $\delta$'s. Suppose $\omega \in \overline{\Omega}$, $\{f_m\} \in \mathscr{B}$, $f_m \in \mathscr{P}(\varepsilon_m)$. Fix $\beta, \delta \in \mathbb{R}_{++}$. Since $\{f_m\} \in \mathscr{B}$,

$$\{f_m\} \in \bigcup_{\gamma} \bigcap_{\eta} \bigcup_{n} \mathscr{B}_{\gamma\eta n},$$

so

$$\{f_m\} \in \bigcap_{\eta} \bigcup_{n} \mathscr{B}_{\overline{\gamma}\eta n}$$

for some $\overline{\gamma} \in \mathbb{R}_{++}$. $\omega \in \overline{\Omega} \Rightarrow \omega \in \Omega_{\beta\overline{\gamma}\delta}$. Thus, $\omega \in \Omega_{\beta\overline{\gamma}\delta\overline{\eta}\overline{n}}$ for some $\overline{\eta} \in \mathbb{R}_{++}$ and some $\overline{n} \in \mathbb{N}$. $\{f_m\} \in \bigcup_{n} \mathscr{B}_{\overline{\gamma}\overline{\eta}n}$, and so $\{f_m\} \in \mathscr{B}_{\overline{\gamma}\overline{\eta}\overline{n}}$ for some $\overline{n} \in \mathbb{N}$.

Since $\omega \in \Omega_{\beta\overline{\gamma}\delta\overline{\eta}\overline{n}}$, $f_m \in \mathscr{P}(\varepsilon_m)$, and $\{f_m\} \in \mathscr{B}_{\overline{\gamma}\overline{\eta}\overline{n}}$,

$$\exists t_m \exists (\tilde{f}_m, p_m) \in \mathscr{W}(\varepsilon_m, t_m) \quad \lambda(\{a: \ ||f(a) - \tilde{f}(a)||_\infty > \delta\}) < \beta$$

for $m > \overline{n}$. Since $\beta$ is arbitrary,

$$\frac{1}{|A_n|} |\{a \in A_n: \ ||f(a) - \tilde{f}(a)||_\infty > \delta\}| \to 0,$$

as required.

Now suppose in addition that $\{f_n\} \in \mathscr{U}$. Since $||\tilde{f}_n(a) - f_n(a)||_\infty \leq 1$ or $\tilde{f}_n(a) = 0$, $\tilde{f}_n(a)$ is uniformly integrable. Since $||f_n(a) - \tilde{f}_n(a)||_\infty$ converges to 0 in measure and is uniformly integrable, it converges to 0 in mean, i.e.

$$\frac{1}{|A_n|} \sum_{a \in A_n} ||f_n(a) - \tilde{f}_n(a)||_\infty \to 0.$$

Then

$$\left|\frac{1}{|A_n|}\sum_{a\in A_n} t_n(a)\right| = \left|\frac{1}{|A_n|}p_n \cdot \sum_{a\in A_n}(e_n - \tilde{f}_n(a))\right|$$

$$\leq \frac{1}{|A_n|}\left|\left|\sum_{a\in A_n} f_n(a) - \tilde{f}_n(a)\right|\right|_\infty \leq \frac{1}{|A_n|}\sum_{a\in A_n}\left|\left|f_n(a) - \tilde{f}_n(a)\right|\right|_\infty \to 0.$$

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