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Author
Hernandez, E.S.

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E. S. Hernandez and B. Remaud

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QUANTAL FLUCTUATIONS AND INVARIANT OPERATORS FOR A GENERAL TIME-DEPENDENT HARMONIC OSCILLATOR

E. S. Hernandez* and B. Remaud**

Nuclear Science Division
Lawrence Berkeley Laboratory
Berkeley, California 94720

ABSTRACT

The connection between quantal fluctuations and invariant operators for a general time-dependent oscillator is discussed. The ground-state of the invariant operator is explicitly displayed. The use of this invariant is illustrated with a simple derivation of the generalized Wigner distribution function.

*Permanent address: Departamento de Fisica, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina.

**Permanent address: Institut de Physique, Universite de Nantes, 44000 Nantes, France.

Since the discovery by Lewis[1] of the exact invariants of the time-dependent harmonic oscillator, there has been a renewal of interest in this ancient problem. The studies of such invariants have two main orientations: i) the search of the solutions of the time-dependent quantal problem[2,3] and ii) the construction of generators of dynamical symmetry groups.[4]

The interpretation of such invariants is not obvious and several suggestions have been put forward (see for example Ref. 3.) In an earlier work, Symon[5] has shown that the invariant of the classical oscillator can be expressed through the amplitude of the motion; this has been generalized to the case of oscillations in presence of dissipation mechanisms.[6] In this paper, we want to concentrate ourselves on the quantal problem and display the relationship between the invariant operator and the quantal fluctuations; these quantities are the second moments of coordinate and momentum operators and do not possess classical equivalence. We shall restrict ourselves to a one-dimensional analysis, since generalization to several degrees of freedom is straightforward.

Let us assume a quantal harmonic oscillator whose position is described by the following general time-dependent Hamiltonian

$$H(t) = \frac{1}{2}m(t)\Omega^2(t)\, x^2 + \frac{1}{2m(t)}\, p^2 \quad , \tag{1}$$

where $m(t)$, $\Omega(t)$ are arbitrary, although differentiable, functions of time. The expectation values $x_o = \langle x \rangle, p_o = \langle p \rangle$ satisfy the Ehrenfest limit, namely

$$i\hbar\dot{x}_o = \langle [x,H] \rangle \tag{2a}$$

$$i\hbar\dot{p}_o = \langle [p,H] \rangle \tag{2b}$$

The second moments of the motion are defined as

$$\chi = \langle x^2 \rangle - x_o^2 \tag{3a}$$

$$\phi = \langle p^2 \rangle - p_o^2 \tag{3b}$$

$$\sigma = 1/2 \ \langle xp + px \rangle - x_o p_o \tag{3c}$$

Straightforward applications of the formula for the time-derivative of an operator lead to the following set of equations:

$$\dot{\chi} = 2\sigma/m \tag{4a}$$

$$\dot{\phi} = -2m \ \Omega^2 \sigma \tag{4b}$$

$$\dot{\sigma} = -m\Omega^2 \chi + \phi/m \tag{4c}$$

Let us postulate that there exists an invariant hermitian operator I that is quadratic in coordinate and momentum,

$$I = 1/2(\alpha(t)(p-p_o)^2 + \beta(t)(x-x_o)^2 + \gamma(t)\ \{p-p_o,q-q_o\}_+) \tag{5}$$

Here the symbol $\{\ \}_+$ is the usual anti-commutator and α,β,γ are real differentiable functions of time. The time derivative of I:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar}\ [I,H] \tag{6}$$

must vanish identically. This requirement leads to the following equation:

$$(\dot{\alpha} + \frac{2\gamma}{m})\ (p-p_o)^2 + (\dot{\beta} - 2m\Omega^2\gamma)(x - x_o)^2 \tag{7}$$

$$+ (\dot{\gamma} + \frac{\beta}{m} - m\Omega^2\alpha)\ \ \{p-p_o,x-x_o\}_+) = 0 \quad .$$

If we compare with Eqs. (4), we realize that α,β and $-\gamma$ respectively coincide with the second moments χ,ϕ,σ except for a common scaling factor. So the invariant I reads:[6]

$$I = \frac{1}{2}\ (\chi(p-p_o)^2 + \phi(x-x_o)^2 - \sigma\{p-p_o,x-x_o\}_+) \tag{8}$$

Its constant expectation value is:

$$\langle I\rangle = \chi\phi-\sigma^2 \tag{9}$$

This is to be interpreted as accounting for a definite, constant value, for the determinant of the covariance matrix of the classical phase-space distribution f(x,p,t) whose second moments are, precisely, χ, ϕ and σ (see Eq. (23b) below)

Notice that the Heisenberg uncertainty principle requires that $\langle I\rangle$ is always positive and verifies:

$$\langle I\rangle \geq \hbar^2/4 \tag{10}$$

If we introduce the reduced width of the quantal state whose first and second moments are given by Eqs. (2) and (3), respectively,

$$u^2 = m\langle I\rangle^{-1/2}\,\chi \tag{11}$$

we can decouple the set of Equations (4) and write a closed form equation for u:

$$\ddot{u} + \left(\Omega^2 - \frac{\ddot{m}}{2m} + \frac{1}{4}\left(\frac{\dot{m}}{m}\right)^2\right)\ u = \frac{1}{u^3} \tag{12}$$

We recognize the generalization to time-dependent mass of the auxiliary equation introduced by Lewis[1] in the classical case. It is worthwhile noticing that the principle of correspondence, when applied to the classical result of Symon,[5] entitles us to state that the amplitude of the x_o motion is solution of the Equation (12). We believe that the identification of the α,β,γ functions in Eq. 5 with the fluctuations of a quantal state with respect to the classical motion paves the way towards a better understanding of the invariant operator and its eigensolutions.

As an illustration of the appearance of the invariant operator in actual applications, we shall focus on the generalization of the Wigner distribution function[7] for a general-time dependent oscillator.

Our starting point is the observation that the Schrödinger's equation for the Hamiltonian (1) admits a non-stationary, gaussian wave packet solution of the form[8,9]

$$\psi(x,t) = \left(\frac{\mathrm{Re}\,\alpha^{-1}}{\pi}\right)^{1/4} \exp\left\{-\frac{(x-x_o)^2}{2\alpha} + \frac{i}{\hbar}\left[p_o(x-x_o) + \theta\right]\right\} \qquad (13)$$

provided that the complex width α and the real phase θ are solutions of the following equations of motion,

$$\dot{\alpha} = \frac{i\hbar}{m} - \frac{i}{\hbar} m\Omega^2\alpha^2 \qquad (14a)$$

$$\dot{\theta} = L(x_o,p_o) - \frac{\hbar^2}{2m} \mathrm{Re}(\alpha^{-1}) \qquad (14b)$$

In Eq. (14b), $L(x_o,p_o)$ is the classical Lagrangian for the oscillator under consideration. It is also straightforward to establish the relationship between the above defined fluctuations χ, ϕ and σ (Eqs. 3) and the complex width α, that reads,

$$\chi = \frac{1}{2\text{Re}\alpha^{-1}} \tag{15a}$$

$$\phi = \frac{\hbar^2}{2\text{Re}\alpha} \tag{15b}$$

$$\sigma = \frac{\hbar}{2}\,\frac{\text{Im }\alpha}{\text{Re }\alpha} \tag{15c}$$

An important consequence of Eqs. (15) is the fact that these three quantities are not independent, but fulfill the condition

$$\chi\phi - \sigma^2 = \hbar^2/4 \tag{16}$$

This corresponds to the absolute minimum of the constant expectation value $\langle I \rangle$. Furthermore, it can be easily checked that the wave packet (13) is the ground-state eigenfunction of the I-operator, namely,

$$I\psi(x,t) = \hbar^2/4\ \psi(x,t) \tag{17}$$

The quantal state represented by the wave packet (13) can be described through a density operator whose matrix elements in configuration space read,

$$\rho(x',x'',t) = \psi(x',t)\ \psi^*(x'',t) \tag{18}$$

This means,

$$\rho(x', x'', t) = \left(\frac{\mathrm{Re}\,\alpha^{-1}}{\pi}\right)^{1/2} \exp\left\{-\frac{(x'-x_o)^2}{2\alpha} - \frac{(x''-x_o)^2}{2\alpha^*} + \frac{i}{\hbar} P_o(x'-x'')\right\} \tag{19}$$

The evaluation of the Wigner distribution function[7,10] associated with this density operator is straightforward. We just resort to its definition[7],

$$d_w(x,p,t) = \frac{1}{2\pi\hbar}\int dz \; \exp(ipz/\hbar)\,\rho(x-\frac{z}{2}\;,\; x+\frac{z}{2}\;,\; t) \tag{20}$$

With the help of Eqs. (16) and (17), the value of the integral can be cast into the form,

$$d_w(x,p,t) = \frac{1}{\pi\hbar} \exp\left\{-\frac{1}{\hbar^2}\left[\phi(x-x_o)^2 + \chi(p-p_o)^2 - 2\sigma(x-x_o)(p-p_o)\right]\right\} \tag{21}$$

We recognize a gaussian distribution function that can be expressed, in matrix notation, in the compact way,

$$d_w(x,p,t) = \frac{1}{\pi\hbar} \exp\left(-\frac{1}{2} V^t M^{-1} V\right) \tag{22}$$

where $$V = \begin{pmatrix} x-x_o \\ p-p_o \end{pmatrix} \tag{23a}$$

and $$M = \begin{pmatrix} \chi & \sigma \\ \sigma & \phi \end{pmatrix} \tag{23b}$$

is the covariance matrix of this distribution. The argument of the exponential is just the semiclassical version of the invariant derived in Eq. (8). The appealing fact is that Eq. (20) provides a generalization of the Wigner distribution function for an ordinary harmonic oscillator[11] (in the zero-temperature limit), namely,

$$d_{w_o}(x,p) = \frac{1}{\pi\hbar} \exp\left(\frac{-2H_o}{\hbar\Omega_o}\right) \tag{24}$$

Here H_o is given by Eq. (1), but the mass and frequency have been taken as constants m_o, Ω_o, respectively and $x_o = p_o = 0$. Finally, Eq. (23) reads,

$$d'_{w_o} = \frac{1}{\pi\hbar} \exp\left\{-\frac{2}{\hbar^2}\left[\frac{\hbar m_o\Omega_o}{2} x^2 + \frac{\hbar}{2m_o\Omega_o} p^2\right]\right\} \tag{25}$$

The quantities accompanying the coordinate and momentum factor are readily recognized as the fluctuations of momentum and coordinate in the ground-state of the oscillator, respectively. In this sense, Eq. (25) provides the correct limit of the more general Eq. (21).

As a final remark, we believe that the connection between quantal fluctuations and invariant operators can provide a useful tool for the description of a class of processes (nuclear fission, relaxation of collective degrees of freedom in heavy ion reactions[6]) in which the variable mass reaches a singularity.

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