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A Coupled Pair of Luenberger Observers for Linear Systems to Improve Rate of Convergence and Robustness to Measurement Noise

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Abstract—Motivated by the need of observers that are both robust to disturbances and guarantee fast convergence to zero of the estimation error, we propose an observer for linear time-invariant systems that consists of the combination of two coupled Luenberger observers. The output of the proposed observer is defined as the average between the estimates of the individual ones. The convergence rate and the robustness to measurement noise of the proposed observer's output are characterized in terms of ISS estimates. Conditions guaranteeing that these estimates outperform those obtained with a standard Luenberger observer are given. The conditions are exercised in a stable scalar plant, for which a design procedure and numerical analysis are provided, and in a second order plant, numerically.

I. INTRODUCTION

We consider linear time-invariant systems of the form

\[ \dot{x} = Ax, \quad y = Cx + m(t), \]

where \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^p \), and \( t \mapsto m(t) \) denotes measurement noise, for which there exists a Luenberger observer

\[ \dot{\hat{x}}_0 = A\hat{x}_0 - \hat{K}_0(\hat{y}_0 - y), \quad \dot{\hat{y}}_0 = C\hat{x}_0 \]

leading to the exponentially stable estimation error system

\[ \dot{e}_0 = (A - \hat{K}_0C)e_0 + \hat{K}_0m(t) = \hat{A}_0e_0 + \hat{K}_0m(t) \]

with estimation error given by \( e_0 := \hat{x}_0 - x \). It is well-known that, under observability conditions of (1), the matrix gain \( \hat{K}_0 \) can be chosen to make the convergence rate of (2) arbitrarily fast. However, due to fast convergence speed requiring large gain, the price to pay is that the effect of measurement noise \( m \) is amplified. Indeed, the design of observers, such as those in the form (2), involves a trade off between convergence rate and robustness to measurement noise [1], [2].

Several observer architectures and design methods with the goal of conferring good performance and robustness to the error system have been proposed in the literature. In particular, \( H_\infty \) tools have been employed to formulate sets of Linear Matrix Inequalities (LMIs) that, when feasible, guarantee that the \( L_2 \) gain from disturbance to the estimation error is below a pre-established upper bound; see, e.g., [3], [4], [5], to just list a few. Following ideas from adaptive control [6], [7], observers with a gain that adapts to the plant measurements have been proposed in [8], [9], though the presence of measurement noise can lead to large values of the gains. Such issues also emerge in the design of high-gain observers, where the use of high gain can significantly amplify the effect of measurement noise. Indeed, in [1], [2], it is shown that measurement noise introduces an upper limit for the gain of a (constant) high-gain observer when good performance is desired.

More recently, observers using essentially two set of gains, one set optimized for convergence and the other for robustness, have been found successful in certain settings. Such approaches include the piecewise-linear gain approach in [10] for simultaneously satisfying steady-state and transient bounds, the high gain observer with nonlinear adaptive gain in [11], and the high gain observer with on-line gain tuning in [12].

In this paper, we propose a linear time-invariant observer and design conditions for both robustness to measurement noise and fast convergence of the estimation error. The proposed observer consists of two coupled Luenberger observers. We establish that, under certain conditions involving its parameters, and when compared to the Luenberger observer, the proposed observer improves the convergence rate and the effect of measurement noise. The main properties of the proposed observer, namely, convergence rate and the robustness to measurement noise, are characterized in terms of ISS estimates and compared with those of a standard Luenberger observer. A design procedure is formulated in terms of optimization problems. While general conditions for which this problem can be solved are not known at this time, a design procedure for the case of a stable scalar plant is provided. The design procedure is exercised in the scalar plant and, also for a second order plant, numerical results indicate improvement of performance and robustness.

The organization of the remainder of this paper is as follows. In Section II a motivational example is presented. Section III establishes the main results. Finally, Section IV shows a complete design for the motivational example and simulations. Complete proofs of presented results will be published elsewhere.

II. MOTIVATIONAL EXAMPLE

Consider the scalar plant

\[ \dot{x} = ax, \quad y = x + m, \]

where \( m \) denotes constant measurement noise (e.g., a bias) and \( a < 0 \). A standard Luenberger observer for this plant is

\[ \dot{x}_0 = a\hat{x}_0 - k_0(\hat{y}_0 - y) \quad \dot{\hat{y}}_0 = \hat{x}_0. \]
The estimation error system is given by (3) with \( \hat{A}_0 = a - k_0 \) while \( K_0 = k_0 \). Its convergence rate is \( a - k_0 \) and its steady-state error is \( e_0^* := \frac{k_0}{k_0 - a}m \). It is apparent that to get fast convergence, the constant \( k_0 \) needs to be positive and large. However, with \( k_0 \) large, the influence of measurement error is amplified as well. As argued in the introduction and suggested by Figure 1(a), a balance needs to be made between convergence rate and steady-state error induced by measurement noise.

\[
\begin{align*}
\dot{x}_1 &= a\hat{x}_1 - k_1(\hat{y}_1 - y) - \ell_1(\hat{y}_2 - y), \\
\dot{x}_2 &= a\hat{x}_2 - k_2(\hat{y}_2 - y) - \ell_2(\hat{y}_1 - y), \\
\hat{y}_i &= \hat{x}_i, \quad i \in \{1, 2\}, \quad \hat{x} = \frac{\hat{x}_1 + \hat{x}_2}{2}.
\end{align*}
\]

The coupling injection terms \( -\ell_1(\hat{y}_2 - y) \) and \( -\ell_2(\hat{y}_1 - y) \) define the innovation terms of the proposed observer. Compared to (5), with the proposed observer, a one-dimensional design problem on \( k_0 \) becomes a fourth dimensional design on \( k_1, k_2, \ell_1, \ell_2 \). The output \( \hat{x} \) of the coupled pair of observers defines the estimate of \( x \) as the average of the states \( \hat{x}_1 \) and \( \hat{x}_2 \) of the individual observers.

By defining the error variables \( e_i := \hat{x}_i - x \) for each \( i \in \{1, 2\} \), the error dynamics are captured by

\[
\dot{e}_1 = (a - k_1)e_1 - \ell_1e_2 + (k_1 + \ell_1)m, \\
\dot{e}_2 = -\ell_2e_1 + (a - k_2)e_2 + (k_2 + \ell_2)m,
\]

which can be written in matrix form as

\[
\dot{e} = \hat{A}e + \hat{K}m,
\]

where \( e = [e_1, e_2]^T \) and

\[
\hat{A} = \begin{bmatrix} a - k_1 & -\ell_1 \\ -\ell_2 & a - k_2 \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} k_1 + \ell_1 \\ k_2 + \ell_2 \end{bmatrix}.
\]

For \( i \in \{1, 2\} \), the steady-state error of (7) is given by

\[
e_i^* = \frac{k_1k_2 - \ell_1\ell_2 - k_1a - \ell_1a}{k_1k_2 - \ell_1\ell_2 - k_2a - k_1a + a^2}m.
\]

The estimation error of the proposed observer is given by the quantity

\[
\bar{e} := \hat{x} - x
\]

and has a steady-state value given by

\[
\bar{e}^* = \frac{k_1k_2 - \ell_1\ell_2 - (1/2)(k_1 + k_2 + \ell_1 + \ell_2)a}{k_1k_2 - \ell_1\ell_2 - k_2a - k_1a + a^2}m.
\]

Under the condition that all eigenvalues of the matrix \( \hat{A} \) are stable, they can be written in the general form \( \lambda_{1,2} = -\sigma \pm j\omega \), where \( \sigma \) is positive and \( \omega \in \mathbb{R} \). Then, by solving for the eigenvalues of \( \hat{A} \) and comparing with the rate of convergence of the observer (5), the following conditions guarantee a faster convergence rate of the proposed observer:

\[
-\sigma = \frac{-(k_1 - a) - (k_2 - a)}{2} < a - k_0 < 0, \quad (13)
\]

\[
\frac{((k_1 - a) + (k_2 - a))^2}{4} < 4 \det \hat{A}. \quad (14)
\]

On the other hand, in order to assure an improvement on the effect of measurement noise, we want to guarantee that \( |\bar{e}^*| < |e_0^*| \), which leads to the following condition:

\[
\frac{k_1k_2 - \ell_1\ell_2 - (1/2)(k_1 + k_2 + \ell_1 + \ell_2)a}{k_1k_2 - \ell_1\ell_2 - k_2a - k_1a + a^2} < \frac{|k_0|}{|k_0 - a|}. \quad (15)
\]

It will be shown in Section IV-A that for any given \( k_0 \), there exist parameters \( k_1, k_2, \ell_1, \ell_2 \) of the proposed observer (6) such that conditions (13)-(15) hold. This observer leads to the improvement in rate of convergence and robustness suggested in Figure 2 where the dot dashed line denotes the state of a stable plant (3), dashed line denotes the estimate provided by the standard Luenberger observer (5), and the black line is the estimate from the proposed observer (6).

\[
\text{Fig. 2. Comparison between proposed observer (black, solid) and a standard Luenberger observer (blue, dashed). The plant solution is denoted in red, dash-dot.)}
\]

More generally, when \( m \) is bounded, Figure 3(a) shows the \( H_\infty \) norm from measurement noise \( m \) to estimation error \( e_0 \) for the nominal observer (3) as a function of \( k_0 \).

On the other hand, the rate of convergence, as shown in Figure 3(c) also increases when \( k_0 \) gets larger (\( \sigma^m \) and \( \sigma \) are defined as the absolute value of the real part of the dominant pole of closed-loop systems with the Luenberger observer and the coupled pair of observers, respectively). Such a tradeoff would become crucial when both rate of convergence and robustness are required. Figure 3(b) shows the \( H_\infty \) norm of the proposed observer as a function of \( \ell_1 \).\footnote{It should be noted that simply using two Luenberger observers without any coupling and taking the average of their estimates will not lead to both faster convergence rate and smaller steady state error.}
Fig. 4. Comparison of design regions for nominal observer in (3) and proposed observer in (6) with $a = -0.5$, $k_0^* = 2$, and particular choice of gain $k_1 = k_2 = k_0^*$. 

$G(t)$, norm $||G||_1$ is defined by $||G||_1 := \int_0^\infty ||G(t)||dt$, where $||G(t)|| = \sup\{||G(t)u|| : u \in \mathbb{R}^n \text{ and } |u| \leq 1\}$ for all $t \geq 0$. Given a bounded function $m_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_{\infty} := \sup_{t \geq 0} |m(t)|$. Give a bounded function $v_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $D^+ v(t) := \limsup_{h \to 0^+} \frac{v(t+h)-v(t)}{h}$. C defines the set of complex numbers. Given a symmetric matrix $P$, $\lambda_{\text{max}}(P) := \max\{\lambda : \lambda \in \text{eig}(P)\}$, $\lambda_{\text{min}}(P) := \min\{\lambda : \lambda \in \text{eig}(P)\}$.

B. Observer structure and basic properties

The proposed observer consists of a coupled pair of Luenberger observers with output given by the average between the states of the individual observers. The two coupled observers for system (1) can be formulated as

$$\dot{\hat{x}}_i = A\hat{x}_i - K_i(\hat{y}_i - y) - L_i(\hat{y}_i - y), \quad i \neq j, i, j \in \{1, 2\}$$

$$\hat{y}_i = C\hat{x}_i, \quad i \in \{1, 2\}, \quad \hat{x} = \frac{\hat{x}_1 + \hat{x}_2}{2},$$

where $K_1, K_2, L_1, L_2$ are constant matrix gains to be designed and $\hat{x}$ is the estimate of $x$. Defining the error vector $e = [e_1^T e_2^T]^T$, we obtain

$$\dot{e} = \hat{A}e + \hat{K}m,$$

where

$$\hat{A} = \begin{bmatrix} A - K_1C & -L_1C \\ -L_2C & A - K_2C \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} K_1 + L_1 \\ K_2 + L_2 \end{bmatrix}.$$  

Under a detectability condition, the following asymptotic stability property holds for the error system (17).

Proposition 3.1: (Asymptotic stability): For the case of $m \equiv 0$, if the pair $(C, A)$ of the plant defined in (1) is detectable, then there exist gains $K_1, K_2, L_1, L_2$ such that the origin of the error dynamics for coupled pair observers as in (17) is asymptotically stable.

C. Conditions for improving rate of convergence and robustness

The performance and measurement noise effect of the observers are characterized in terms of input-to-state-stability-like bounds. More precisely, given an observer with

2Note that in Figure 3(b) the $H_\infty$ grows unbounded at points on the $\ell_1 - \ell_2$ plane corresponding to the case of purely imaginary poles. This can be seen from Figure 3(d) where, at such points, the rate of convergence is zero.

3More general linear combinations are possible.
estimation error $e$, we are interested in bounds of the form
\[ |e(t)| \leq \beta(|e(0)|, t) + \gamma(|m|_\infty) \quad \forall t \geq 0, \]
where $\beta$ is a class-$\mathcal{KL}$ function and $\gamma$ is a class-$\mathcal{K}_\infty$ function.
For the particular Luenberger observer in (2), it is well known that, when $A_0$ is Hurwitz with distinct eigenvalues and $A_0$ is decomposed as $A_0 = \dot{X}_0 J_0 \dot{X}_0^{-1}$, the estimation error $e_0$ satisfies [13]
\[ |e_0(t)| \leq \beta_0(|e_0(0)|, t) + \gamma_0(|m|_\infty) \quad \forall t \geq 0, \]
with, for example, for all $s \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}_{\geq 0}$,
\[ \beta_0(s, t) = \kappa(A_0) \exp(\alpha(A_0)t)s, \quad \gamma_0(s) = \frac{|K_0|}{\mu(A_0)}. \quad (18) \]
To establish and compare this property with that of the proposed observer, the next result guarantees that the upper bounds on the rate of convergence and the steady-state error due to the proposed coupled pair of observers outperform those due to a standard Luenberger observer.

**Lemma 3.2:** Consider the plant (1), the Luenberger observer (2) with estimation error (3), and the coupled pair of observers (16) with error dynamics (17). Suppose that
\begin{itemize}
  \item [a)] $A_0$ is dissipative, i.e., for some $\overline{\alpha} > 0$
  \[ \dot{A}_0^T + A_0 \leq -2\overline{\alpha}_0 I; \quad (19) \]
  \item [b)] $\exists K_1, K_2, L_1, L_2$ such that $\dot{\hat{A}}$ is dissipative, i.e., for some $\overline{\alpha} > 0$
  \[ \hat{A}^T + \hat{A} \leq -2\overline{\alpha} I; \quad (20) \]
  \item [c)] each of $A_0$ and $\dot{\hat{A}}$ has distinct eigenvalues, $\alpha($\dot{\hat{A}}$) < \alpha(A_0)$;
  \item [d)] $\frac{|K|}{\overline{\alpha}} \leq \frac{\hat{K}_0}{\overline{\alpha}_0}$. \quad (22)
\end{itemize}
Then, there exists a class-$\mathcal{KL}$ function $\beta$ and a class-$\mathcal{K}_\infty$ function $\gamma$ such that the error $e$ in (17) satisfies the following:
\begin{itemize}
  \item [a)] $|\hat{e}(t)| \leq \beta(|e(0)|, t) + \gamma(|m|_\infty) \quad \forall t \geq 0$;
  \item [b)] Given nonzero $e(0)$ and $e_0(0)$, $\exists \overline{t} \geq 0$ such that $\beta(|e(0)|, t) \leq \beta_0(|e_0(0)|, t) \forall t \geq \overline{t}$;
  \item [c)] $\gamma(s) < \gamma_0(s)$, for all $s \neq 0$ and $s \in \mathbb{R}_{\geq 0}$.
\end{itemize}
A Lyapunov-based set of conditions for rate of convergence and robustness improvement is given next.

**Lemma 3.3:** Consider the plant (1), the Luenberger observer (2) with estimation error (3), and a coupled pair of observers (16) with error dynamics (17). Suppose that
\begin{itemize}
  \item [a)] the measurement noise $m$ is bounded;
  \item [b)] $\exists P_0^T = P_0 > 0$ such that for some $\overline{\alpha} > 0$
  \[ \dot{A}_0^T + P_0 \dot{A}_0 \leq -2\overline{\alpha}_0 P_0; \quad (23) \]
  \item [c)] $\exists K_1, K_2, L_1, L_2$ and $P^T = P > 0$ such that for some $\overline{\alpha} > 0$
  \[ \hat{A}^T P + P \hat{A} \leq -2\overline{\alpha} P; \quad (24) \]
  \item [d)] $\frac{\lambda_{\min}(P_0)}{\lambda_{\max}(P_0)} < \frac{\overline{\alpha}}{\overline{\alpha}_0} \frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}$. \quad (25)
\end{itemize}
where $A_0 = \dot{X}_0 J_0 \dot{X}_0^{-1}$, $A_0$ is Hurwitz with distinct eigenvalues, and $A_0$ is decomposed as $A_0 = \dot{X}_0 J_0 \dot{X}_0^{-1}$.
\[ e(0) = (\lambda_{\max}(P)^2 |\hat{K}|)^{1/2} < (\lambda_{\max}(P_0)^2 |\hat{K}_0|)^{1/2}. \quad (26) \]
Then, there exists a class-$\mathcal{KL}$ function $\beta$ and a class-$\mathcal{K}_\infty$ function $\gamma$ such that the error $e$ in (17) resulting from the coupled pair of observers satisfies the following:
\begin{itemize}
  \item [a)] $|\hat{e}(t)| \leq \beta(|e(0)|, t) + \gamma(|m|_\infty) \quad \forall t \geq 0$;
  \item [b)] $|\hat{e}(t)| \leq \beta(|e(0)|, t) + \gamma(|m|_\infty) \quad \forall t \geq 0$;
  \item [c)] Given nonzero $e(0)$ and $e_0(0)$, $\exists \overline{t} \geq 0$ such that $\beta(|e(0)|, t) \leq \beta_0(|e_0(0)|, t) \forall t \geq \overline{t}$;
  \item [d)] $\gamma(s) < \gamma_0(s)$, for all $s \neq 0$ and $s \in \mathbb{R}_{\geq 0}$.
\end{itemize}

**Remark 3.4:** Lemma 3.2 is a special case of Lemma 3.3 with matrices $P = I$ and $P_0 = I$.

Lemmas 3.2 to 3.3 establish boundedness properties from noise $m$ to error $e$ for the proposed observer. A property from $m$ to the estimation error $\overline{\sigma}$ is established next.

**Theorem 3.5:** For the plant (1) with the Luenberger observer (2) and the coupled pair of observers (16), suppose the measurement noise $m$ is bounded. If there exist matrices $\hat{K}_0, K_1, K_2, L_1, L_2$ such that (19), (22), or (23), (26), then there exists a class-$\mathcal{K}_\infty$ function $\beta$ and a class-$\mathcal{KL}$ function $\gamma$ such that the error $e$ resulting from the coupled pair of observers satisfies the following:
\begin{itemize}
  \item [a)] $|\hat{e}(t)| \leq \overline{\sigma}(|e(0)|, t) + \gamma(|m|_\infty) \quad \forall t \geq 0$;
  \item [b)] Given nonzero $e(0)$ and $e_0(0)$, $\exists \overline{t} \geq 0$ such that $\overline{\sigma}(|e(0)|, t) \leq \beta_0(|e_0(0)|, t) \forall t \geq \overline{t}$;
  \item [c)] $\gamma(s) < \gamma_0(s)$, for all $s \neq 0$ and $s \in \mathbb{R}_{\geq 0}$.
\end{itemize}

**D. Design of coupled pair of observers**

The design of the proposed observer can be described as an optimization problem, particularly, under the constraints of pole placement and of minimizing the $H_\infty$ gain of the transfer function from noise $m$ to the output $\overline{\sigma}$ of the system (16). To formulate such an optimization problem following [14], the error dynamics for (16) can be rewritten as
\[ \dot{e} = A_c e + B_c u, \quad y_c = C_c e + D_c m, \quad z_\infty = C_c e, \quad (27) \]
where
\[ A_c = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad B_c = I_{2n \times 2n}, \quad C_c = -\begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}, \]
\[ D_c = \begin{bmatrix} I_{p \times p} & I_{p \times p} \end{bmatrix}^T, \quad C_\infty = \frac{1}{2} \begin{bmatrix} I_{n \times n} & 0 \\ 0 & I_{n \times n} \end{bmatrix} \]
and the “input” $u$ is assigned via $u = M_u y_c$ with
\[ M_u = \begin{bmatrix} K_1 & L_1 \\ L_2 & K_2 \end{bmatrix}. \]
Moreover, $z_\infty$ denotes the estimation error of the proposed observer, i.e., $z_\infty = \overline{\sigma}$. In frequency domain, the transfer function from $m$ to $z_\infty$ for (27) can be written as
\[ T_{mz_\infty}(s) = C_c (s I - A_c)^{-1} B_c + D_c, \quad (28) \]
where $A_{cl} = A_c + M_u C_c$, $B_{cl} = M_u D_c$, $C_{cl} = C_c$, $D_{cl} = 0$.

Within this setting, the optimization problem for the proposed observer are formulated in the following two subsections.
1) Rate of convergence as an inequality constraint: To guarantee a rate of convergence requirement, we are interested in placing the poles in a particular region such as that one achieved by a Luenberger observer, i.e., all poles located at left of vertical line $-\sigma^*$ in the complex plane. Following [15], the system (27) has all poles located at left of $-\sigma^*$ in the complex plane if and only if there exists a symmetric positive definite matrix $P_D$ such that

$$A^T_D P_D + P_D A_D + 2\sigma^* P_D < 0. \tag{29}$$

It is worth to note that, for system (27), the above inequality constraint is nonlinear because of the appearance of the cross term $P_D M_u$. The following theorem provides an equivalent linear formulation and a sufficient condition for (29).

**Proposition 3.6:** The inequality (29) is satisfied if

a) and only if there exist $P_D$ and $M_p$ such that

$$A^T_D P_D + P_D A_D + C^T_c M_p + M_p C_c + 2\sigma^* P_D < 0,$$

in which case $M_u = P_D^{-1} M_p$.

b) there exists $h_1, h_2 \in \mathbb{R}$ such that the following hold:

b.1) $h_1 + h_2 \geq \sigma^*$;

b.2) $P_i = P_i^T > 0$, for each $i \in \{1, 2\}$

b.3) $(A - K_c C)^T P_i + P_i (A - K_c C) + 2h_1 P_i < 0$ for each $i \in \{1, 2\}$

b.4) 

$$2h_2 P_i - (L_c C)^T P_i - P_i L_c C < 0.$$ 

2) Bound of $H_\infty$ gain as an inequality constraint: We are interested in minimizing the bound of the transfer function $T_{mz_\infty}$, i.e., finding the minimum $\gamma \geq 0$ such that $|T_{mz_\infty}(j\omega)| < \gamma$ for all $\omega \in \mathbb{R}$.

**Lemma 3.7:** For the system (28) defined by $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$, the following statements are equivalent.

a) The system is stable and the $H_\infty$ gain of the system is less than $\gamma$ for some $\gamma > 0$, i.e., $||T_{mz_\infty}||_\infty < \gamma$.

b) There exists $P_H = P_H^T > 0$ such that

$$\begin{bmatrix} A^T_D & P_H & B_{cl} & C^T_{cl} \\ B_{cl}^T & -\gamma I & D_{cl} \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0. \tag{30}$$

**Remark 3.8:** The condition in b) is the so-called Bounded Real Lemma condition; see, e.g., [17], [18].

3) Minimization of the $H_\infty$ norm under a rate of convergence constraint: Using the formulations in terms of inequality constraints and LMIs in Section II-D.1 and Section III-D.2, we formulate optimization problems to minimize the $H_\infty$ norm from $m$ to $\bar{u}$ under constraints imposing an specific rate of convergence.

**Theorem 3.9:** Given $\sigma^* > 0$, the poles of system (27) are located in the region $D = \{ s \in \mathbb{C} : \text{Re}(s) \leq -\sigma^* \}$, and the $H_\infty$ gain is less or equal than $\gamma$ if and only if there exist $M_u$, $P_D$, and $P_H$ such that the following optimization problem is feasible:

$$\begin{align*}
\min \ & \gamma \\
\text{s.t.:} \ & A^T_D P_D + P_D A_D + 2\sigma^* P_D \leq 0, \\
& A^T_D P_H + P_H A_D + P_H B_{cl} C^T_{cl} < 0, \\
& P_H = P_H^T > 0, \quad P_D = P_D^T > 0.
\end{align*} \tag{31}$$

Note that the optimization problem (31) is not jointly convex over the variables $(P_D, P_H, M_u)$. Moreover, it is nonlinear because of the existence of cross terms $P_H M_u$ and $P_D M_u$. In order to remove the nonlinearities and make the two constraints jointly convex, following [14], we reformulate the problem by seeking common solutions of $P_D$ and $P_H$, and changing variables to $M_p := P_M u$.

**Theorem 3.10:** Given $\sigma^* > 0$, the poles of system (27) are located in region $D = \{ s \in \mathbb{C} : \text{Re}(s) \leq -\sigma^* \}$, and the $H_\infty$ gain is less or equal than $\gamma$ if there exists $M_p$ and $P$ such that the following optimization problem (LMI) is feasible:

$$\begin{align*}
\min \ & \gamma \\
\text{s.t.:} \ & A^T_c P + P A_c + C^T_c M_p + M_p C_c + 2\sigma^* P \leq 0, \\
& A^T_c P + P A_c + C^T_c M_p + M_p C_c + 2\sigma^* P \leq 0, \\
& \begin{bmatrix} A^T_c P + P A_c + C^T_c M_p + M_p C_c & C^T_{cl} \\ D^T_{cl} M_p & -\gamma I & 0 \\ 0 & -\gamma I \end{bmatrix} < 0, \\
& P = P^T > 0.
\end{align*} \tag{31}$$

**Remark 3.11:** The resulting observer gain matrix from Theorem 3.10 is given by $M_u = P^{-1} M_p$. By making the optimization problem linear and convex, a global optimizer is guaranteed. However, asking for $P_H = P_D$ may eliminate a better feasible solution to the original optimization in (31).

The following result assures that the performance and robustness of the proposed observer are no worse than those of a Luenberger observer.

**Theorem 3.12:** Given $\sigma^* > 0$, the poles of error dynamics of the Luenberger observer (3) for the plant (1) are located in the region $D = \{ s \in \mathbb{C} : \text{Re}(s) \leq -\sigma^* \}$, and the $H_\infty$ gain from $m$ to $e_0$ is less or equal than $\gamma^* \geq 0$ if and only if there exist $K_0$, $X_D$, and $X_H$ such that the following optimization problem is feasible:

$$\begin{align*}
\min \ & \gamma^* \\
\text{s.t.:} \ & \bar{A}^T D X_D + X_D \bar{A}_0 + 2\sigma^* X_D \leq 0, \\
& \bar{A}^T D X_H + X_H \bar{A}_0 + X_H K_0 I \leq 0, \\
& K_0 X_H - \gamma^* I \leq 0, \\
& X_H = X_H^T > 0, \quad X_D = X_D^T > 0.
\end{align*} \tag{32}$$

Moreover, if such $K_0$, $X_D$, and $X_H$ exist, then the optimization problem in Theorem 3.9 on $P_D$, $P_H$, and $M_u$ is feasible, and its solution $\gamma$ has the property $\gamma \leq \gamma^*$.

For simplicity, we do not linearize (32) but that is possible following the approach in Theorem 3.10.
IV. EXAMPLES

A. Numerical results for first order plant

To illustrate the main feature of the proposed coupled pair of observers, we revisit the motivational example. Consider the plant in (4) with $a = -0.5$. The Luenberger observer is designed following (5) with $k_0 = 2$. The proposed observer is designed following (6) with error dynamics (7). Conditions (19)-(22) of Theorem 3.4 can be rewritten as

$$\alpha(A) \leq a - k_0,$$

$$\frac{\sqrt{2} \sqrt{(k_1 + \ell_1)^2 + (k_2 + \ell_2)^2}}{|\mu(A)|} < \frac{a}{a - k_0}. \quad (33)$$

By solving (33), we pick parameters $k_1 = 1.7896$, $k_2 = 2.2278$, $\ell_1 = 0.0538$, $\ell_2 = -1.1633$. It can be verified that the eigenvalue of $\hat{A}$ according to this set of parameters are $-2.5087 \pm 0.1208i$. Moreover, $\mu(\hat{A}) = -1.9123$. With initial conditions $x(0) = 3$, $\pi_1(0) = \pi_2(0) = \pi_3(0) = 5$, a simulation for $m(t) \equiv 0.3$ is shown in Figure 5. It is worth to note that there is an improvement of the steady-state error by the proposed observer, $\pi^* = 0.2272$, while the Luenberger observer gives $e^*_0 = 0.2400$. As shown in Figure 5(b), we obtain $t^* = 2s$, and $\pi$ becomes closer to 0 than $\pi_0$ thereafter.

Based on Theorem 3.4, we are able to find better parameters by using the solver PENBMI [19]. For values $k_1 \approx 3.5198$, $k_2 \approx 0.4802$, $\ell_1 \approx -8.0142$, $\ell_2 \approx 0.2883$, the resulting $H_\infty$ gain is $\approx 0.4953$, which is $\approx 38.09\%$ smaller than that of Luenberger observer ($\gamma_0 = 0.8$) with $k_0 = 2$. The simulation in Figure 2 was obtained using these parameters.

B. Numerical results for second-order plant

Consider the second-order plant given as in (1) with

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}. $$

For a given Luenberger observer with $\hat{K}_0 = \begin{bmatrix} 2 & 5 \end{bmatrix}^T$, its rate of convergence is $-1$ and its $H_\infty$ norm from measurement noise $m$ to estimation error $e_0$ is 0.4859. By formulating the problem according to Theorem 3.12 with $\sigma^* = 1$, we obtain $\gamma^* \approx 0.2850$, which is a great improvement from the non-optimized Luenberger $H_\infty$ norm of 0.4859, with $\hat{K}_0 = \begin{bmatrix} 0.2852 & 0.3574 \end{bmatrix}^T$. However, Theorem 3.12 gives $\gamma \approx 0.0594$ for the proposed observer with

$$M_u^T = \begin{bmatrix} 0.2790 & 0.2160 & 0.1400 & -0.1149 \\ -0.0367 & -0.9901 & 0.3470 & 0.7500 \end{bmatrix}^T.$$