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**Nonlinear stochastic response of marine vehicles**

**Duthoit, Christophe François, Ph.D.**

**University of California, Santa Barbara, 1987**

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UNIVERSITY OF CALIFORNIA  
Santa Barbara

**Nonlinear Stochastic Response of Marine Vehicles**

A Dissertation submitted in partial satisfaction  
of the requirements for the degree of

**Doctor of Philosophy**

in

**Mechanical & Environmental Engineering**

by

**Christophe Duthoit**

**Committee in charge:**

**Professor Jean-Louis Armand, Chairman**

**Professor Marshall P. Tulin**

**Professor Theodore Kokkinis**

**Professor Michel K. Ochi**

**August 1987**





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August 1987



**August 3, 1987**

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**v**



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## Publications

- "Nonlinear Dynamic Behavior of Moored Platforms Driven by Stochastic Seas," *Proceedings, Offshore Mechanics and Arctic Engineering Symposium, Houston, Texas, vol. 2, 1987, pp. 285-294 (with Pr. J.-L. Armand), also to appear in the Journal of Offshore Mechanics and Arctic Engineering, ASME.*
- "Nonlinear Response of Marine Vehicles to Stochastic Signals: a Review," *Technical Report 87-12, Ocean Engineering Laboratory, Dpt. of Mechanical & Environmental Engineering, University of California, Santa Barbara, Calif., 93106, 1987.*
- "The Principle of Maximum Entropy Applied to the Nonlinear Response of Marine Vehicles," submitted for publication in *Probabilistic Engineering Mechanics*, 1988 (with Pr. J.-L. Armand).



## **Abstract**

### **Nonlinear Stochastic Response of Marine Vehicles**

by

**Christophe Duthoit**

The dynamic behavior of marine vehicles in extreme sea states is a matter of great concern following some recent and dramatic accidents. The complex problem of its prediction can be approached through the study, yet of broader scope, of nonlinear dynamic systems driven by stochastic processes.

Nonlinear statistical dynamics is a relatively new and difficult field. Although the diversity of techniques now available may seem fostering, the achievement of a unified and general theory for nonlinear response to stochastic process appears as a quite remote event.

Second-order statistics contain the most important information to describe a random process. Both theoretical and empirical evidence showing the superiority of the method of equivalent linearization to predict second-order statistics are exhibited and exemplified. The rationale underlying the Wiener-Hermite functional model appears to

further support this affirmation.

However, higher-order statistics cannot be accurately predicted within the framework of this technique whenever deviation from normal behavior becomes significant. A new technique for predicting the response moments and cumulants of nonlinear systems is presented.

This technique relies upon the construction of a series of linear systems aimed at the prediction of the response statistics of a given order. Such linear systems are successively defined by linearizing the original nonlinear system and matching the Volterra functional model response statistics of the desired order. The linear system for predicting second-order statistics coincide with the one obtained using the method of equivalent linearization.

This technique is exemplified by a nonlinear system governed by the Duffing equation with linear plus cubic damping. Several innovative results related to the transfer functions and the response cumulant of Volterra series are exhibited and used in our model.

Response probability distributions can be constructed from knowledge of these statistical moments. Particular attention is devoted to the distribution of maximum entropy and its justification as a method of inference in such underdetermined moment problems.

Finally, several applications to the rigid body behavior of marine vehicles serve to assess the accuracy and the versatility of these techniques. Response distributions of maxima so predicted compare very well with exact solutions or time domain simulation estimates when no exact solution is available.

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## Nomenclature

$\exp$		exponential function
$E[ \ ]$		mathematical expectancy operator
$F(y)$		nonlinear, autonomous and causal function
$\mathcal{F}\{y(t)\}$		nonlinear, autonomous and causal functional
$g_n(\tau)$		n-th inverse Volterra kernel
$G_n(\omega)$		n-th order inverse Volterra transfer function
$\mathcal{G}_n\{y(t)\}$		n-th order inverse Volterra functional
$h_n(\tau)$		n-th order Volterra kernel or impulse response function
$H_n(\omega)$		n-th order Volterra transfer or harmonic response function
$He_n(u)$		Hermite polynomials
$he^{(n)}$		Hermite functional polynomials
$\mathcal{H}_n\{x(t)\}$		n-th order Volterra functional
$\mathcal{Im}\{ \ }$		imaginary part of
$k_n(\tau)$		n-th order Wiener-Hermite kernel
$K_n(\omega)$		n-th order Wiener Hermite transfer function

$\mathcal{K}_n\{x(t)\}$ .....	n-th order Wiener-Hermite functional
$\mathcal{L}\{y(t)\}$ .....	linear, autonomous and causal functional
$\ln$ .....	logarithmic function
$p(y)$ .....	probability density function of random process $y$
$R(\tau)$ .....	temporal autocorrelation function
$\mathcal{Re}\{ \}$ .....	real part of
$(S), (S_n)$ .....	original nonlinear system, "equivalent" linear systems
$S(\omega)$ .....	power spectral density
$\mathcal{S}\{p\}$ .....	entropy functional
$t, \tau, \sigma$ .....	time variables
$x(t)$ .....	excitation process
$y(t), y_n(t)$ .....	response process of system $(S), (S_n)$
$Z(u)$ .....	normal probability law
$\Gamma$ .....	gamma function
$\delta$ .....	Dirac distribution
$\delta_{ij}$ .....	Kronecker delta
$\varepsilon$ .....	expansion parameter
$\kappa_n, \mu_n, \beta_n$ .....	n-th order response cumulant, central moment, quasi-moment
$\lambda_n$ .....	Lagrange multipliers
$\xi$ .....	linear damping ratio
$\Xi$ .....	nonlinear damping coefficient
$\sigma_{y \text{ linear}}$ .....	linear response standard deviation
$\phi(it), \varphi(it)$ .....	moment, cumulant generating functions

## **Introduction**

Offshore technology has experienced a dramatic growth since 1937, when offshore drilling began in the Gulf of Mexico. Operating in ever increasing depths and amid an extremely hostile environment, ocean structures are characterized by their complexity as well as by the difficulty to design them.

Drawing heavily on past experience, the design of ocean structures, similar to that of ships, is for the most part based on empirical rules and codes. While such experience may be extensive in the case of ships, the design of which is the result of centuries of mostly careful and slow evolution, it is necessarily limited for ocean structures. Furthermore, ocean structures are built for specific purposes and designed to operate in particular environmental conditions. These may differ from one structure to another and from those of existing structures.

Extrapolation of available data can be a perilous exercise, as exemplified by various recent and fatal failures, partial or total, of ocean structures ("The Alexander Kielland Accident", 1981 and "Capsizing and Sinking of the ... Ocean Ranger ...", 1983)<sup>1</sup>. The

---

<sup>1</sup>Titles or authors followed by date refer to the bibliography.



need for a more rationally-based design procedure for ocean structures is now well recognized.

The awareness of the present lack of understanding of the mechanisms leading to large amplitude response and eventual capsizing of marine structures, such as semisubmersibles, has motivated extensive, worldwide research. However, the mathematical formulation of the problem represents a formidable challenge. Most of the various phenomena involved are not yet fully understood. Nevertheless, a useful insight into this complex problem can be gained from the consideration of simpler, though as yet unsolved, situations. Such mathematical models can be arrived at by isolating and idealizing the prevailing phenomena, while discarding the unessential.

The design process (figure 0.1) of ocean structures involves five successive steps. The environment interacts with the structure through a set of loading conditions. Mathematical modeling allows the idealization of such physical phenomena. The structure, in turn, responds to external loads in various ways (rigid-body motions, bending moments, stresses...). The derivation and study of methods of analysis to predict the response of an ocean structure to given external loads constitutes the essential scope of this work. Finally, a reliability and risk study can be drawn from the knowledge of the overall structure behavior. This is the design properly speaking.

The classical method of analysis does not adequately take into account the actual operating conditions of ocean structures. The paramount reason for such discrepancies stems from the diversity, as well as the complexity, of interacting effects governing the loading conditions. Ocean structures operate in an environment random in nature, which essentially governs their behavior. Therefore, rather than undertaking a deterministic approach, through which randomness of loading conditions is more or

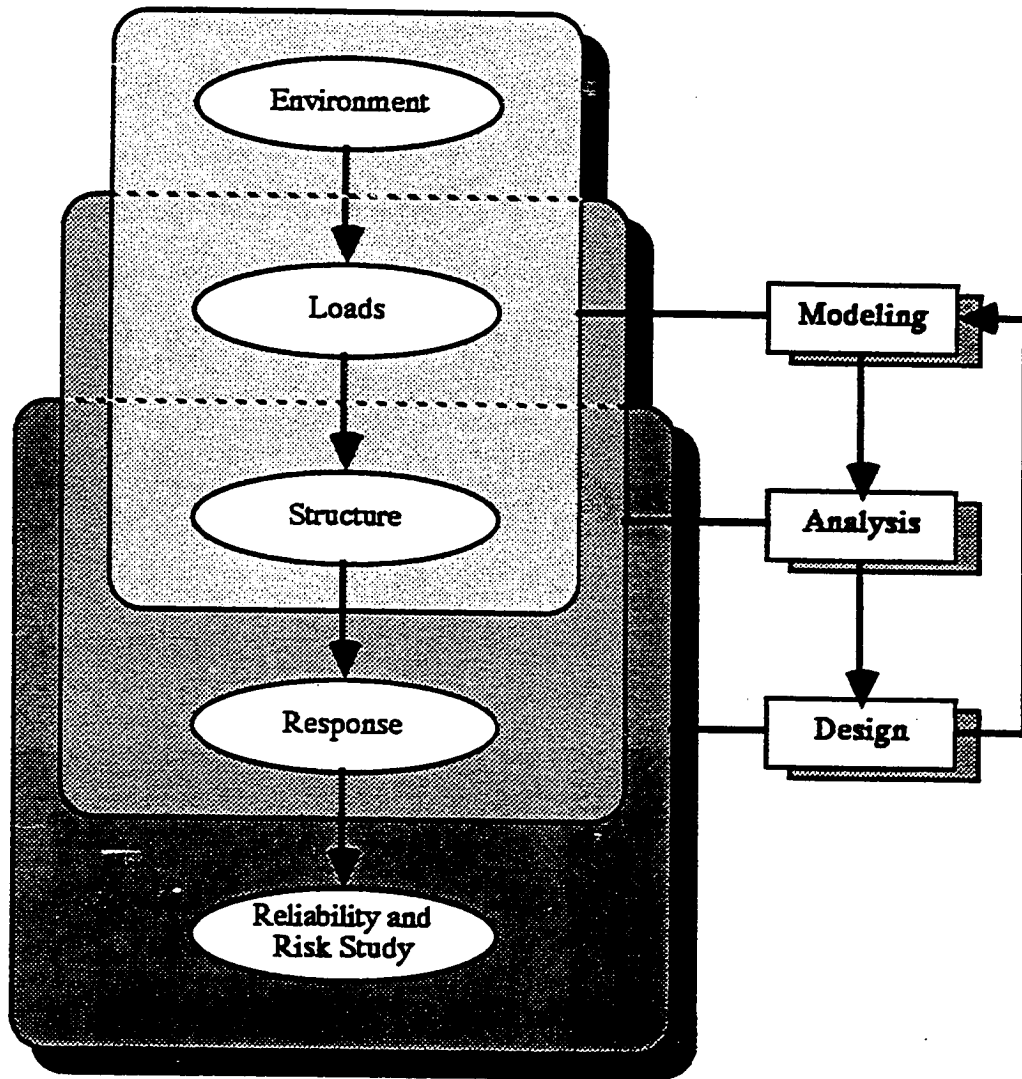


Figure 0.1. Structural Design Process.

less overcome by the introduction of "safety coefficients" into the design process, a *stochastic* approach seems more appropriate. Thus the essential notion of risk is quantified rather than simply eluded. Such a stochastic model may eventually lead to the notions of probability of failure and reliability, useful in engineering design.

The development and implementation of probabilistic models for predicting the loads acting on an ocean structure have already been the subject of comprehensive research. Yet to be thoroughly investigated is the stochastic prediction of an ocean structure's response to environmental loads which would provide the information essential for its rational design. The objective is to characterize the probabilistic structure governing the stochastic response of the system or the statistical properties exhibited by the response.

Linear system theory is a well developed body of knowledge, and its applications are relatively straightforward. It has, on the other hand, severe limitations now recognized in many situations involving ocean structures.

Nonlinearities play an important role in the design of moored floating structures, in particular. Also, the response to loads in unusual or extreme conditions, which constitutes a significant aspect of the design process, is essentially governed by *nonlinear* effects. This is for instance the case whenever the small amplitude motion and potential flow hypotheses, inherent to the linearized diffraction-radiation theory, do not apply. Furthermore, a nonlinear model alone can yield insight into the mechanisms leading to large amplitude response, and eventually to one or another mode of structural failure.

More often than not, the time scales describing the time-varying loads and the natural periods of ocean structures turn out to be of comparable magnitude. Thus, the

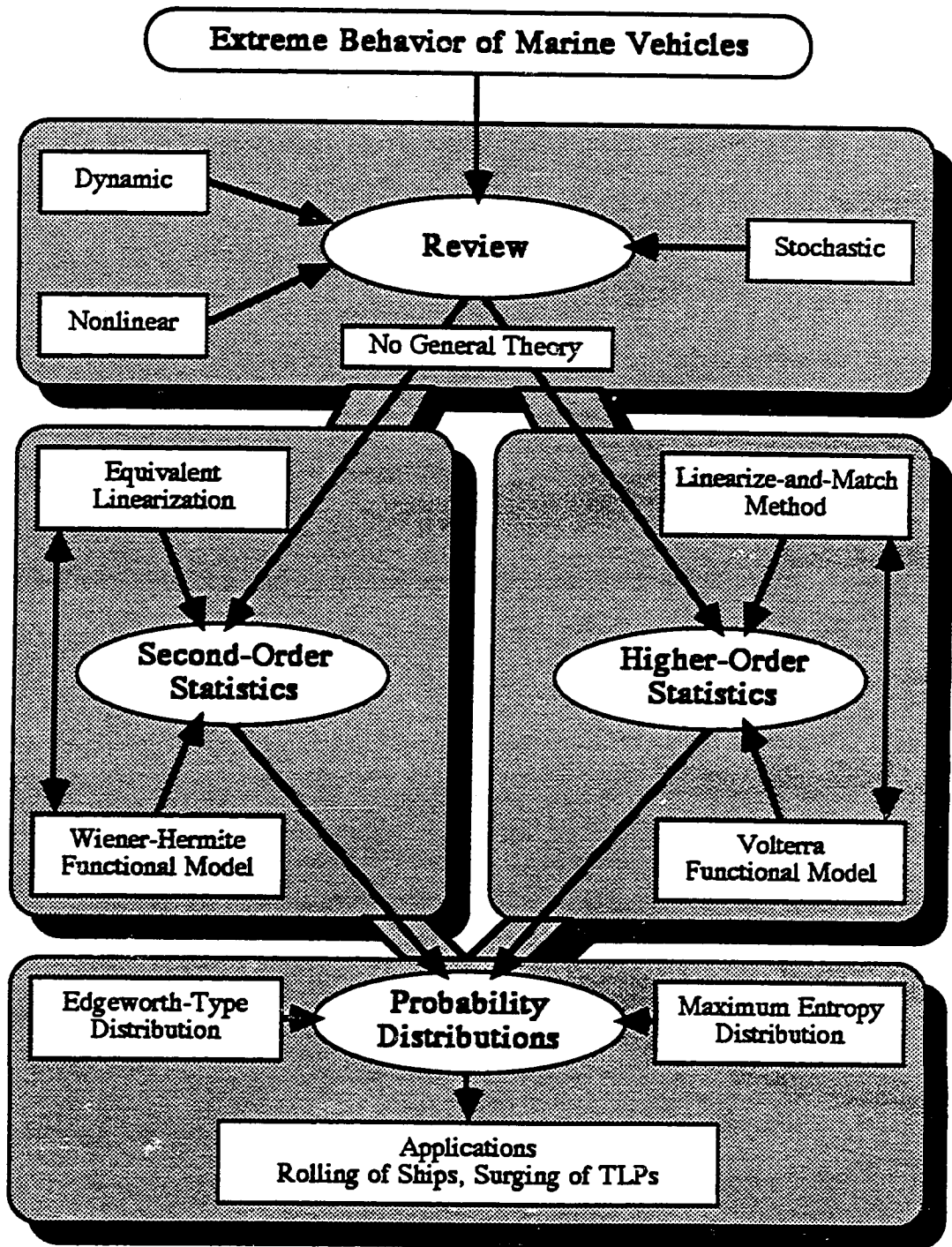


Figure 0.2. Dissertation Outline.

possibility for resonant behavior must, in general, be explored. These phenomena can only be successfully explained within the framework of a *dynamic* rather than a static model. Memory effects arising from the presence of the free surface must also be treated within a dynamic approach.

The prime objective of this work is to improve the methods of analysis for nonlinear, dynamic response to stochastic process (figure 0.1). Meanwhile, the necessary mathematical modeling is assumed given, whereas the reliability and risk study that follows is not discussed.

The following is an outline of the present work. The first chapter attempts to explore available methods. The emphasis is placed on ocean engineering applications. Statistical dynamics has inherited a considerable amount of knowledge from the work of physicists in the theory of Brownian motion, electrical engineers in the fields of information and control, and mathematicians in probability theory. In spite of a fruitful cross-fertilization between these different disciplines, a general theory for nonlinear response to stochastic process is not yet available.

The nonlinear features of a transformation represent such a difficulty that many techniques rely upon the consideration of one or several linear systems substituted to the original nonlinear one (equivalent linearization, Volterra functional model ...).

The proposed technique relies upon two essential observations. First, that linear modeling is a simple, versatile and appealing technique, then, that the method of equivalent linearization yields accurate estimates of second-order statistics (Chapter 2). If the method of equivalent linearization appears appropriate to predict second-order statistics, it may be possible to extrapolate this method to the construction of linear models to predict higher-order statistics (Chapter 3).

Finally, probability distributions can be constructed from knowledge of these statistics (Chapter 4). They provide the necessary information for the rational design of ocean structures. Figure 0.2 illustrates the relations among these different aspects of the resulting method of analysis. The connections with other available techniques are also exhibited.

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## CHAPTER 1

### Bibliographical Review

We have already discussed how useful insight into the behavior of ocean structures in unusual or extreme conditions can be gained from the study of nonlinear, dynamic systems driven by stochastic signals. Therefore, it is relevant to explore, in details, the field of nonlinear statistical dynamics. Meanwhile, the emphasis will be placed on ocean engineering applications.

The mathematical problem may, in general, be stated as follows. Let us consider a nonlinear dynamic system described by the functional input-output relationship<sup>1</sup>:

$$\mathcal{F}\{y(t)\} = x(t) \tag{1.1}$$

---

<sup>1</sup>Upper-case Roman symbols denote scalar functions and operators, lower-case Roman symbols denote scalar variables, the same notations in bold are used for vectorial and matrix quantities, whereas cursive Roman symbols represent functionals.



where  $x(t)$  and  $y(t)$  respectively denote the excitation and the response stochastic processes, and  $\mathcal{F}$  is a nonlinear autonomous (i.e. time-invariant) and generally causal functional.

The causality condition, usually satisfied by physical systems, may be mathematically written as:

$$\mathcal{F}\{y(t)\} = \mathcal{F}(y(\tau); \tau \leq t) = x(t) \quad (1.2)$$

It is required to derive some statistical or probabilistic properties of the response process  $y(t)$  when the deterministic system  $\mathcal{F}$  and stochastic excitation  $x(t)$  are given.

Such a general theory of the same scope as the linear theory, is not yet available, and thus progress toward a satisfactory stochastic theory of ship and platform motions has been rather slow.

Nonlinear stochastic modelling is a relatively new and difficult field, drawing on the latest advances in nonlinear system theory and stochastic processes. All the existing approaches are, in some way, limited in scope by assuming particular properties of both the excitation (Gaussian, harmonic, white noise, etc.) and the system (time-invariance, memoryless, analytic, etc.). Moreover, most of these techniques yield a limited description of the response (generally second-order statistical properties or first-order probabilistic structure).

We will examine some exact methods first. These techniques constitute the basis as well as most of the leading ideas governing other approximated methods which may apply whenever no exact one is possible.

## Exact Methods

Historically, the mathematical methods of analysis in the discipline of nonlinear statistical dynamics evolved quite independently from two fundamental theories.

The spectral analysis (or broadly speaking, the correlation methods) essentially appeared in connection with signal processing in the fields of communication and electrical engineering. Whereas the theory of Markov processes and the Fokker-Planck equation (methods of kinetic equations) are intimately linked to early works in the theory of Brownian motions and later to statistical physics.

Both theories are reviewed below. In addition, several fundamental theorems related to the behavior of memoryless (or static) systems are examined.

### Spectral Analysis Theory: Linear Systems

The theory governing the behavior of linear autonomous dynamic systems:

$$\mathcal{L}\{y(t)\} = x(t)$$

(1.3)

driven by stationary Gaussian signals is a well developed body of knowledge and we will not reproduce it here<sup>2</sup>

---

<sup>2</sup>Related literature and textbooks are abundants (Crandall, 1963b, Crandall and Mark, 1963, Lin, 1967, Bolotin, 1969, Papoulis, 1984, Newland, 1984 and Lin et al., 1986).

Let us simply state that the spectral analysis of linear systems originated during the first half of the century, and culminated with the much celebrated work of Rice (1944 & 1945).

The method is essentially based on the Wiener-Khintchine relations (Wiener, 1930) and the convolution theorem together with the assumptions of stationarity and ergodicity of the response process. They yield relatively straightforward frequency domain expressions (power spectral density) of the second-order statistics of linear autonomous transformations of stationary Gaussian processes.

For future reference it is essential to emphasize that the response of such transformations may be entirely described by its mean value and second-order statistics. These second-order statistics can be evaluated in the frequency domain through the power spectral density or in the time domain through the autocorrelation function.

Statistical moments of order 1 and 2, however, do not completely describe the behavior of random processes whenever deviation from normality is substantial, even if the transformation is linear. Furthermore, associated with this difficulty is the fact that nonlinear transformations, which are to be discussed below, do not preserve Gaussian character.

The case of non-Gaussian excitations is discussed in Kuznetsov, Stratonovitch and Tikhonov (1953).

The application of linear spectral theory to describe the motions of ships in irregular waves appeared with the pioneering work of St. Denis and Pierson (1953). Later, St. Denis (1973, 1974a, 1974b & 1975) and Yamanouchi (1974) discussed the severe limitations of linear models in various situations involving offshore platforms as well as ships.

## Nonlinear Static Systems

Although the theory of nonlinear system with memory remains our primary concern, it is instructive to examine the, quite extensive, literature on static transformations in order to gain further insight into the difficulties associated with these memory effects.

Nonlinear static systems can be understood as nonlinear transformations without memory, therefore the response of such systems only depends upon present or instantaneous values of the excitation process:

$$F(y(t)) = x(t) \tag{1.4}$$

In the case of a Gaussian excitation, Price's (1958 & 1964 and Baum, 1969) and Bussgang's (1952) theorems yield useful information on output second-order statistics. Whereas the case of a general excitation can be found in Abramson (1967). Generalization of Price's theorem to the functional version of (1.4) can be found in Gorman and Zaborszky (1968). While, the response probability distribution can be easily obtained by a simple transformation in most cases (Bendat, 1985).

A number of the leading contributions to the theory of memoryless transformations of stochastic processes are reproduced in Haddad (1975).

## Fokker-Planck Equation: Nonlinear Dynamic Systems

The origins of the Fokker-Planck equation are intimately linked to the theory of Brownian motions, named after an English botanist Robert Brown who observed in 1827 that small particles suspended in fluids undergo erratic movements.

The very first satisfactory statistical theory of Brownian motions appeared with Einstein in 1905 through the diffusion equation in bringing together the Maxwell-Boltzmann theory and the random walk method (Fuller, 1969). However, it was not until a decade later that the combined works of Smoluchowski, Langevin, Fokker, Planck, and Ornstein among others led to considerable generalization of Einstein's pioneering work: the Fokker-Planck equation. Whereas, further mathematical aspects of the theory were examined by Wiener, Kolmogorov, and others (Uhlenbeck and Ornstein, 1930 and Ming Chen Wang and Uhlenbeck, 1945).

Essentially, the method relies on the fact that the response of a discrete dynamic system subjected to a Gaussian white noise signal behaves as a continuous multidimensional Markov process. Then, it is possible to show that Markov processes must satisfy a consistency equation; the Chapman-Kolmogorov or Smoluchowski equation:

$$p(y_3, t_3 | y_1, t_1) = \int_{-\infty}^{+\infty} p(y_3, t_3 | y_2, t_2) p(y_2, t_2 | y_1, t_1) dy_2 \quad t_1 \leq t_2 \leq t_3$$

(1.5)

The Smoluchowski equation, in turn, leads to the response transition probability density function  $p(z, s | y, t)$ , solution of a partial differential equation; the Fokker-

Planck-Kolmogorov equation (Ming Chen Wang and Uhlenbeck, 1945 and Caughey, 1971):

$$\frac{\partial p}{\partial t} = - \sum_i \frac{\partial [A_i(t,y)p]}{\partial y_i} + \frac{1}{2} \sum_i \sum_j \frac{\partial^2 [B_{ij}(t,y)p]}{\partial y_i \partial y_j} \quad (1.6)$$

The steady-state solution of equation (1.6) is the probability density function  $p(y)$  of a stationary response process  $y$ .

The Fokker-Planck equation has been subsequently applied to a variety of problems in electrical engineering, nonlinear control systems, nonlinear vibrations...

The appealing aspect of an approach based on the Fokker-Planck-Kolmogorov equation is that the derived solution is an exact one. However, the assumptions underlying the existence of an analytic stationary solution to equation (1.6) are quite restrictive: in general the nonlinearities are required to be of static nature only, and the excitation must be a Gaussian process the spectral density of which is that of a white noise (Caughey, 1963a & 1971).

A precise and comprehensive survey of the nonlinear systems which can be solved exactly by means of the Fokker-Planck equation can be found in Caughey (1971), Caughey and Ma (1982a & 1982b) and Ludwig (1975).

Concerning second-order properties of the response, virtually all the Markov processes the power spectrum of which can be evaluated exactly are the one which are Gaussian (Ming Chen Wang and Uhlenbeck, 1945). Therefore, the case of second-order statistics of nonlinear transformations remains unsolved at least from the theory of Markov processes.

Roberts (1981) derived the amplitude distribution of the slow drift oscillations of moored vessels from the Fokker-Planck equation.

## Approximate Methods

Drawing upon the two fundamental techniques described in the previous section, more general nonlinear transformations of random processes, for which no exact or closed-form solutions are known, can be discussed.

Again, two kinds of methods can be examined. Methods which relies on the theory of Markov processes and Itô stochastic calculus and those essentially based on spectral analysis and functional calculus. Both are addressed through this section.

Let us start this chapter on approximate methods with a rather systematic, yet cumbersome, technique which consists in a time domain simulation of the governing equations.

### Time Domain Simulation

Time domain digital simulation of the equations of motion remains the foremost way of predicting the response of a nonlinear system to some prescribed input.

Apart from its systematic aspect, such a method exhibits many well-known drawbacks, mostly linked to the prohibitive amount of calculations necessary. In particular, in the case of a stochastic excitation, a spectral or probabilistic description of

the response becomes rather cumbersome using a time domain simulation, the nature of which is essentially deterministic.

Clearly, pre- and post-processing of the time simulation of equations of motion generally include respectively simulation of the excitation power spectrum and spectral analysis of the response time series. Both of which must be handled with care and are quite demanding in computer capacity if conducted properly.

For these reasons, stochastic frequency domain techniques are generally preferred, whenever possible, to the more expensive time domain techniques.

Nevertheless, it has been applied to the description of ship motions. Dalzell (1971 & 1973) showed, through a time-stepping procedure, that for most of the practical dynamic domain, the distribution of roll maxima does not correspond to the distribution of the maxima of a random Gaussian process predicted by theory, i.e. the Cartwright and Longuet-Higgins (1956) distribution.

Pérez y Pérez (1974) modeled the motions of a steered ship in waves by a linear convolution integral and frequency independent nonlinearities considered to be part of the exciting forces, these nonlinearities being associated with the rudder forces, viscous roll damping and restoring forces and moments. When compared with experiments, the roll motion prediction proves to be not as accurate as the yaw and rudder motion predictions.

Besides digital computer simulations of the equations of motion, analog measurements of electronic or electro-mechanical circuits are possible and may prove to be useful when the system in hand can be successfully modeled in that manner (Broch, 1977).



## Equivalent Linearization

Linearizing the system comes next among the available techniques. Basically, an "equivalent" linear system:

$$\mathcal{L}_{\text{eq}}\{\tilde{y}(t)\} = x(t) \tag{1.7}$$

is substituted to the original nonlinear one; equations (1.1).

The price to be paid for such a drastic simplification of the nonlinear model lies in the choice of a linearization procedure, which does not follow any strict guidelines, as well as in an incomplete description of the nonlinear system in hand which ignores the specific features<sup>3</sup> of nonlinear systems. The system is thus essentially assumed to behave as a linear one.

In the case of a deterministic excitation, Kryloff, Bogoliuboff and Mitropolsky (1947 & 1961) invoke equivalent energy balance during one cycle.

Drawing upon Kryloff and Bogoliuboff work, the stochastic case is generally handled in replacing the original nonlinear equations describing the system, by equivalent linear equations which minimize the mean square error:

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<sup>3</sup>Nonlinearities usually have two different effects. The first of these leads to a response which differs only quantitatively from the linear response (amplitude modulation), while the second one induces phenomena which are not predictable within the framework of a linear approach, such as non-Gaussian response to Gaussian processes and certain types of dynamic instabilities, sub- or superharmonic responses (frequency modulation), bifurcations...

$$E[(\mathcal{F}\{y(t)\} - \mathcal{L}_{eq}\{y(t)\})^T (\mathcal{F}\{y(t)\} - \mathcal{L}_{eq}\{y(t)\})] \quad (1.8)$$

An application of functional calculus of variations (Volterra, 1930) leads to the following set of linear integral equations in terms of the kernels of equivalent system  $\mathcal{L}_{eq}$ :

$$R_{xy}(\tau) - \int_{-\infty}^{+\infty} h_{eq}(\sigma) R_{yy}(\tau - \sigma) d\sigma = 0 \quad \forall \tau \leq 0 \quad (1.9)$$

which are known, in linear mean square estimation (linear filtering), as the Wiener-Hopf equations (Wiener, 1949 and Papoulis, 1984).

The more common case of nonlinear differential equations of motion leads to linear algebraic equations in the equivalent linear system parameters. This technique was introduced by Booton (1953 & 1954) and Caughey (1959, 1960 & 1963b). The resulting system thus depends upon some statistics of the response<sup>4</sup>, which is not really inconvenient provided that stationarity is preserved.

Further theoretical foundations, like proofs of existence and uniqueness of the derived equivalent linear system, can be found in the works of Iwan (1973) and Spanos and Iwan (1978).

The class of nonlinear systems that can be handled within the framework of the method of equivalent linearization is quite large. In principle, the nonlinearities are not

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<sup>4</sup>Therefore the response probability law must be assumed. The various alternative procedures have been discussed by Crandall (1979), Beaman and Karl Hedrick (1979) and Tung (1979).

restricted to any term, but are generally assumed to be "weak" although the underlying reasons remain unclear.

Moreover, equivalent linearisation techniques may be applied to coupled sets of equations of motion (Caughey, 1963b, Kazakov, 1965 and Iwan and Yang, 1972) as well as asymmetric nonlinearities (Borgman, 1967 and Spanos, 1980).

Hysteretic behavior as well as non-stationary excitations are, at least in principle, amenable to equivalent linearization.

Nevertheless, it is yet unclear what kind of response statistics can be predicted within reasonable accuracy through the use of this method. In other words, does equivalence in the mean square sense preclude accurate prediction of statistics other than second order ones?

Moreover, the available literature seems to indicate that the method of equivalent linearization tends to underestimate the response statistics (Lin, et al., 1986) although no theoretical justification appears to support this observation.

This method has been applied to ship rolling in random waves by Kaplan (1966) (viscous damping) and Vassilopoulos (1971) (viscous damping and nonlinear restoring moment). A variation of this technique involves the use of a describing function in considering the ship as a feedback system (Flower & Mackerdichian, 1978).

Following the same idea, it is also possible to substitute an "equivalent" nonlinear system to the original system when, for example, the "equivalent" nonlinear system belongs to a class of problems which can be solved (exactly or not) (Caughey, 1984). A similar methodology has been proposed by Jaunet (1984) where a simplified nonlinear model (cascading systems: linear with memory + nonlinear without memory) of ship rolling in irregular seas is identified from experimental data .

## White Noise Spectrum Excitation

The mathematical idealization of white noise excitation<sup>5</sup> allows the use of techniques based on the Fokker-Planck equation as well as Itô calculus.

In this section nonlinear systems for which no exact solutions are known, are discussed. It will be shown how approximate response probability distributions or statistics can be evaluated through these techniques.

### Fokker-Planck Equation

The foremost and maybe most restrictive hypothesis related to the application of the Fokker-Planck equation is that the excitation power spectrum must be that of a white noise. In principle, this assumption may be removed if one recalls that colored spectrum may be whitened whenever they are factorizable<sup>6</sup> (Schetzen, 1980 and Papoulis, 1984). However, this possibility is more theoretical than it first appears since it leads to such an excessive complication of the associated Fokker-Planck equation that even numerical solutions are not easily obtained.

Therefore, one is left with approximate methods which may take more or less into account the shape of the excitation spectrum. One such method is a stochastic averaging procedure proposed by Stratonovitch (1963 & 1967) and applied with some

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<sup>5</sup>Such an idealization may be justified provided that the correlation time of the actual excitation process is small when compared with the relaxation time of the dynamic transformation (Lin, 1967, Lin et al., 1986).

<sup>6</sup>A procedure due to Wiener (1949).

success to the nonlinear broad-banded roll motion of ships by Roberts (1982a, 1982b, 1984 & 1985).

When the Fokker-Planck equation (1.6) associated with the nonlinear system in hand cannot be solved exactly, a number of approximate techniques exist. The method of equivalent nonlinearization can be invoked (Caughey, 1984 & 1986) and has been applied to the case of an oscillator with nonlinear damping by Kirk (1974). Haddara (1973 & 1974) used a perturbation method to solve the Fokker-Planck equation in the case of the roll motion of a ship.

The Fokker-Planck equation can also be solved numerically. The method of finite differences is used by Ochi (1984 & 1986), in the case of the Duffing oscillator with nonlinear damping driven by a Gaussian white-noise excitation which was assumed to model the surge motion of a tension leg platform in heavy seas.

A Galerkin method with Hermite polynomial expansion can also be used. Wen (1975) employed such a numerical scheme in the case of nonstationary excitations. While Taudin and Rocaboy (1986) proposed a similar technique in the case of multi-degree-of-freedom marine structures subjected to general wave excitations. Taudin and Rocaboy (1986) emphasized that this numerical scheme is about as consuming in computer time as a time domain simulation. Moreover numerical instabilities may arise and therefore such a versatile technique does not seem, at least for the time being, to be very useful in practical applications.

### **Cumulant Closure**

The statistical moments and cumulants of the response of nonlinear systems to white noise excitation can be computed through Itô stochastic calculus (Itô, 1951a &

1951b) from the governing equations. Successive coupled equations in the response power moments generally result. They form an infinite hierarchy. Therefore, an appropriate closure scheme must be used in order to obtain approximate closed-form solutions.

One such closure scheme relies upon neglecting cumulants, which represent higher and higher order measures of deviation from normal behavior, after some prescribed order (Crandall, 1980, Wu and Lin, 1984).

This closure scheme seems to yield accurate results (Crandall, 1980 and Wu and Lin, 1984) but has yield very few applications to ocean engineering problems so far (Kanegaonkar and Haldar, 1987).

### **Perturbation Techniques**

The general feature of perturbation methods is to substitute an infinite number of linear systems to a nonlinear one through an expansion in terms of a "small" parameter associated with the magnitude of the nonlinearities.

In this way, the difficulties related to the nonlinear features of the system temporarily disappear. On the other hand, the system response is now expressed in terms of a, generally infinite, series (Crandall, 1963a). The difficulty of evaluation of each term usually increases geometrically with its order in the series.

Two fundamental questions then arise; namely the convergence of the series, as well as the number of terms necessary to get an accurate description of the solution. The answer to the former is generally not easy although one may get a good approximation of the solution from the knowledge of the first few terms only, even

when the series diverges. Once again, the nonlinearities must remain weak in order to insure both convergence and accurate prediction of the solution with a limited number of terms.

Such an expansion procedure has been applied to the nonlinear rolling of ships in random seas, first in the case of viscous damping (Yamanouchi, 1964 & 1966) and later in the case of a static nonlinearity (Flower, 1976). Both papers discuss the influence of nonlinearities on the response power spectrum.

Among the perturbation techniques, multiple scale methods (spatial and/or temporal) may be applied to second order low frequency excitation problems (Triantafyllou, 1979 & 1982 and Agnon and Mei, 1983).

Basically, the general idea underlying the work of Triantafyllou (1979 & 1982) is that the motions of a floating body may be splitted into two components: a small amplitude, quickly varying motion and a large amplitude, slowly varying motion. The main hypothesis being to assume that the two motions may be treated separately. The solution may eventually be written as the sum of the solutions of the two linear problems.

### **Functional Series Representation Methods**

In most derivations of the equations of motion of floating bodies such as ships and offshore structures, the system is conveniently reduced to a set of second-order

differential equations with frequency-dependent coefficients<sup>7</sup>, whether linearity is assumed or not.

The simplicity of such a description is only apparent; it actually represents integral equations in the time domain, the physical interpretation of which lies in the fact that a structure freely floating in waves is actually a space-time system.

Various approximations and integrations are made to reduce it to a time only system (Tick, 1959 and Wehausen, 1971). The price to be paid to allow such a simplification resides in the memory effect which appears as we get rid of the space dimensions and is mathematically described by integral equations over the past history of the motion.

When the further assumption of linearity is made, the system is compactly described by its impulse response matrix in the time domain and by its harmonic response matrix in the frequency domain. The fundamental importance of these concepts in ship hydrodynamics have been introduced by Cummins (1962) and further emphasized by Bishop, Burcher and Price (1973).

In the nonlinear case, a perturbation technique would represent a natural generalization of this procedure to handle the non-linearities. It may be intuitively thought not as a regular expansion in power series, but rather as a power series "with memory", namely the Volterra functional series representation technique.

### **Volterra Functional Series**

The functional series representation of differential, integral and integro-differential equations originated with the work of Volterra by the end of last century (see e.g.

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<sup>7</sup>This dependency is omitted or neglected without much justification among virtually all the literature reviewed.



Volterra (1930) and Barrett (1980b) for a comprehensive bibliography). Essentially, the input-output relationship of a given analytic system is expanded in a functional power series which can be formally obtained, by analogy with Taylor series, through successive functional derivatives, as defined by Volterra (1930):

$$y(t) = \sum_{n=1}^{+\infty} \mathcal{H}_n\{x(t)\} \quad (1.10)$$

where the  $n$ -th order Volterra functional is defined by<sup>8</sup>:

$$\mathcal{H}_n\{x(t)\} = \iiint \dots \int h_n(\tau_1, \tau_2, \dots, \tau_n) x(t - \tau_1) x(t - \tau_2) \dots x(t - \tau_n) d\tau_1 d\tau_2 \dots d\tau_n \quad (1.11)$$

Such a power series is called a Volterra series, and the kernel  $h_n(\tau_1, \tau_2, \dots, \tau_n)$  is called the  $n$ -th order Volterra kernel. In the case of a causal (or physically realizable) system (equation (1.2)), kernels  $h_n$  must vanish whenever any of their arguments become negative.

It is a "power series with memory". That it is a power series can be readily seen recalling that the  $n$ -th order Volterra functional  $\mathcal{H}_n\{x(t)\}$  is a homogeneous functional of order  $n$ . It is a series with memory since  $\mathcal{H}_n\{x(t)\}$  is a  $n$ -fold convolution in time. Clearly, the generalized  $n$ -th order Volterra functional  $\mathcal{H}_n\{x_1(t), \dots, x_n(t)\}$  is  $n$ -linear, by analogy with multilinear function theory.

Wiener (1958) later applied Volterra's description of general functional relationships to nonlinear communication problems. The first systematic study of the

<sup>8</sup>Infinite bounds of integration will often be omitted.

application of the Volterra functional model to physical systems appeared with the work of Barrett (1963).

Vassilopoulos (1967) discussed the applicability of the Volterra and Wiener series to the motions of a ship in irregular seas modeled as a nonlinear autonomous system, together with particular applications to the cases of wave-induced ship resistance in random seas, as well as uncoupled nonlinear motions such as roll.

**Determination of the Volterra kernels.** The problem of determining the kernels  $h_n(\tau_1, \tau_2, \dots, \tau_n)$  or equivalently the transfer functions<sup>9</sup>  $H_n(\omega_1, \omega_2, \dots, \omega_n)$  can be understood differently depending on the problem in hand.

In the first case, the Volterra kernels are to be determined from knowledge of a general functional relationship of the type (1.1). This inversion problem admits a solution whenever the functional  $\mathcal{F}$  is analytic, and its linear part stable (therefore invertible (Barrett, 1963)). In this case, several methods are possible.

In the direct expansion method, the system equations are manipulated until they are brought into the form of the Volterra series expansion (1.10) (Bedrosian and Rice, 1971). In the case of nonlinear autonomous deterministic systems<sup>10</sup>, these identities lead to successive algebraic equations in terms of the transfer functions of any order of functionals  $\mathcal{H}_n$  and  $\mathcal{F}$  (Barrett, 1963 and Parente, 1970). Such an approach yields directly the transfer functions  $H_n$  via frequency association.

The harmonic input method relies on specific properties of the Volterra transfer functions. Clearly, they can be understood as harmonic response functions (Bedrosian and Rice, 1971).

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<sup>9</sup>The transfer functions are n-fold Fourier transforms of the kernels, and the Fourier transform is one-to-one.

<sup>10</sup>Assuming a deterministic system does not preclude the possibility of stochastic excitations.

In the second case, the Volterra series is determined from simultaneous measurements of the input and the output functions. This is the case of identification. Schetzen (1965b) proposed a method for measuring the Volterra kernels of nonlinear systems. Essentially, this approach is based on the n-linear properties of n-th order Volterra functional  $\mathcal{H}_n$ .

**Statistical and Probabilistic Properties of the Response.** Once the Volterra series is entirely determined, it is possible to evaluate the statistical properties of its response  $y(t)$ .

Most of the existing works deals with the prediction of second-order statistics<sup>11</sup> of nonlinear autonomous systems driven by ergodic random Gaussian processes. Essentially, the response autocorrelation function is evaluated first:

$$R_{yy}(\tau) = \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} E[\mathcal{H}_n\{x(t)\} \mathcal{H}_m\{x(t+\tau)\}] \quad (1.12)$$

The response power density spectrum may then be obtained by Fourier transform of the above equation (1.12). A direct approach leads to the result derived by Rudko and Weiner (1978). Clearly, if  $x(t)$  is of order  $\epsilon$ , the response power density spectrum  $S_{yy}(\omega)$  so obtained directly appears as a power series of  $\epsilon$ , keeping in mind the homogeneous property of functional  $\mathcal{H}_n$ . For example,  $E[\mathcal{H}_n\{x(t)\} \mathcal{H}_m\{x(t+\tau)\}]$  will be of order  $\epsilon^{m+n}$ .

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<sup>11</sup>Essentially because the Volterra and Wiener functional models were first applied in connection with signal processing and mean-square estimation in the field of communication.

In an alternative approach, proposed by Mircea & Sinnreich (1969), the terms in (1.12) are reordered, resulting in a single series which turns out to be closely related to the Wiener-Hermite expansion.

Dalzell (1976b & 1982) and Yamanouchi (1977) investigated the applicability of the Volterra series model to nonlinear ship rolling. Particular attention has been devoted to the third degree Volterra functional (Dalzell, 1982), where the transfer functions are evaluated using the so-called harmonic input method, while the roll spectrum is evaluated through the Mircea-Sinnreich series.

Dalzell (1976a) and Dalzell and Kim (1979), used this technique again in the problem of added ship resistance in irregular waves and lateral drift forces and moment (Kim and Dalzell, 1981 and Dalzell, 1986). Langley (1986) analyzed the low frequency motions of moored vessels.

Borgman (1982) described the general procedure to take into account various type of nonlinearities (wave theory, structure motion, drag force, mooring and free surface effects).

Finally, Bouché (1985) and Bouché and Olagnon (1985) analyzed the vibrations of a fixed circular cylinder in waves where the transfer functions were evaluated by a direct expansion procedure.

Although second order statistics are essential they only yield an incomplete description of the response of nonlinear systems, principally when deviation from normal behavior becomes significant. In particular, the accurate prediction of the probabilistic structure of the response necessarily involves higher order statistics.

Although there does not appear to exist any general formula, the first few output cumulants of general Volterra series can be computed from Bedrosian and Rice (1971),

which in turn, may lead to an Edgeworth-series type probability distribution. Yet, quite surprisingly, there does not seem to exist any application of these ideas.

In the case of second-order autonomous stochastic systems, the probabilistic description of the response is possible through the Kac-Siebert method (Kac and Siebert, 1947). Applications of this method to marine structures can be found in Neal (1974), Vinge (1983) and Naess (1985). Ultimately, extreme-value behavior of the marine structure response may be obtained (Naess, 1985).

### Wiener-Hermite Functional Series

Nonlinear system representation by Volterra functionals is but one technique among the functional representation techniques.

Two basic difficulties are associated with the practical application of the Volterra functional series. The first difficulty arises with the measurement (identification) of the Volterra kernels/transfer functions of a physical system, whereas the latter one is related to the question of convergence of the resulting series.

To circumvent these problems, Wiener constructed a new set of orthogonal functionals  $\mathcal{K}_n\{x(t)\}$ , orthogonal with respect to a Gaussian white noise input, determined from the Volterra functionals (Wiener, 1958, Barrett, 1963, Schetzen, 1980 and Rugh, 1981). The orthogonality relations are:

$$E[\mathcal{K}_n\{x(t)\} \mathcal{K}_m\{x(t + \tau)\}] = 0 \quad n \neq m \quad (1.13)$$

Orthogonalization of the Volterra series (1.10), through a Gram-schmidt orthogonalization procedure, leads to the Wiener series:

$$y(t) = \sum_{n=1}^{+\infty} \mathcal{K}_n \{x(t)\}$$

(1.14)

Because the convergence of an orthogonal series is a convergence in the mean, the class of nonlinear systems that can be described by the Wiener functionals is much broader than the class that can be described by a Volterra series (Schetzen, 1980).

Yet another and simpler derivation of the Wiener functionals expansion relative to a Gaussian white noise can be achieved through expansion in some set of orthogonal polynomials. A suitable choice of orthogonal functional polynomials are the Grad Hermite polynomials  $he^{(n)}$  (Barrett, 1963 & 1964):

$$\mathcal{K}_n \{x(t)\} = \iiint \dots \int k_n(\tau_1, \tau_2, \dots, \tau_n) he^{(n)} \{x; t - \tau_1, t - \tau_2, \dots, t - \tau_n\} d\tau_1 d\tau_2 \dots d\tau_n$$

(1.15)

where the Wiener kernels  $k_n(\tau_1, \tau_2, \dots, \tau_n)$  may be determined through the orthogonality condition (1.13):

$$n! S_0^n k_n(\tau_1, \tau_2, \dots, \tau_n) = E[y(t) he^{(n)} \{x; t - \tau_1, t - \tau_2, \dots, t - \tau_n\}]$$

(1.16)

The Wiener-Hermite expansion can be generalized to the case of a non-white Gaussian excitation process by redefining appropriate Hermite functionals (Barrett, 1980a & 1982).

Clearly, the Wiener-Hermite functional series can be understood as an orthogonal expansion with memory relative to a Gaussian input process.

It will be now shown how advantage can be gained from such an orthogonal expansion both in the measurement of the kernels and transfer functions and in the derivation of output second-order statistics.

**Determination of the Wiener-Hermite kernels.** The Wiener kernels  $k_n(\tau_1, \tau_2, \dots, \tau_n)$  and transfer functions  $K_n(\omega_1, \omega_2, \dots, \omega_n)$  can be determined from knowledge of the Volterra kernels and of the transfer functions (Barrett, 1980a & 1982 and Rugh, 1981)

However, one of the appealing aspects of these orthogonal expansions lies in the relative simplicity to identify the corresponding kernels and transfer functions:

$$\begin{aligned} \iint \dots \int k_n(\sigma_1, \sigma_2, \dots, \sigma_n) R_{xx}(\sigma_1 - \tau_1) R_{xx}(\sigma_2 - \tau_2) \dots R_{xx}(\sigma_n - \tau_n) d\sigma_1 d\sigma_2 \dots d\sigma_n \\ = \frac{1}{n!} R_{y \text{ he}^{(n)}(x)}(\tau_1, \tau_2, \dots, \tau_n) \end{aligned} \quad (1.17)$$

$$K_n(\omega_1, \omega_2, \dots, \omega_n) = \frac{S_{y \text{ he}^{(n)}(x)}(\omega_1, \omega_2, \dots, \omega_n)}{n! S_{xx}(\omega_1) S_{xx}(\omega_2) \dots S_{xx}(\omega_n)} \quad (1.18)$$

The use of the Wiener expansion in analog identification of nonlinear systems, driven by stationary Gaussian white noise, appeared with the work of Lee and Schetzen (1965) (equation (1.17)). Whereas French and Butz (1973) showed how the Wiener kernels may be measured in the frequency domain through equation (1.18). Extensive

applications to physiological systems can be found in Marmarelis and Marmarelis (1978) and Marmarelis (1979).

The application of identification techniques of linear autonomous systems to ship and offshore structure motions is well established. Essentially, cross-spectrum techniques (equation (1.18)) are used in order to evaluate the linear transfer function. The superiority of the cross-spectrum over the auto-spectrum technique is not only that it yields complete response characteristics, including both amplitude and phase relations, but also that cross-spectrum is free from the effect of any orthogonal<sup>12</sup> noise included in the response (Yamanouchi, 1974).

In this respect, such an approach, together with an auto-spectrum technique allows a convenient measure of the adequacy of the linear model through coherence functions (Bendat, 1982, 1983 & 1985).

The cross-spectrum approach (1.18) to the identification of nonlinear systems has been mentioned for some time in the ocean engineering literature, but without any reference to its mathematical foundation: the Wiener-Hermite expansion.

For example, as early as 1961 Tick (1961) and Hasselman (1966) showed that the nonlinear transfer functions can be obtained from high order statistical moments of the ship motions (equations (1.18)). This underlines the fact that when a nonlinear system is driven by a stationary Gaussian noise, the output is, in general non-Gaussian and therefore it cannot anymore be reasonably described by its first two moments alone.

Dalzell (1974) demonstrated, through identification, that a second order Volterra polynomial model is a reasonable representation of the added ship resistance produced by waves. He used identification techniques to evaluate the linear and quadratic kernels

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<sup>12</sup>Orthogonal in the sense of equation (1.13).



through the cross-spectrum and cross-bispectrum respectively. Assuming the Gaussian random wave model valid, Neal (1974) and Borresen (1978) treated the nonlinear response of a ship up to the second order, i.e. taking the low-frequency excitations into account.

A quite general and advanced review of the identification techniques of second and third order Volterra systems can be found in Bendat (1985), where particular emphasis is placed on square-law (with or without sign) and cubic systems.

A somewhat systematic approach to identify a general transformation giving the inline and transverse forces on a vertical cylinder element in random waves has been undertaken by Vugts and Bouquet (1985). The adequacy of the models to describe the relationship between input and output is evaluated by a total coherence function (Bendat, 1985). This work led to a revalidation of the Morison equation, together with the associated dilemma of an appropriate selection of the empirical coefficients. Moreover, among their conclusions was emphasized the coefficient dependency on the input conditions. Such a dependency is in contradiction with the "black box" concept.

**Statistical and Probabilistic Properties of the Response.** Keeping in mind the orthogonality of the Wiener-Hermite expansion (equation (1.12)), the response autocorrelation function can be written as the single sum:

$$R_{yy}(\tau) = \sum_{n=1}^{+\infty} E[\mathcal{K}_n\{x(t)\} \mathcal{K}_n\{x(t + \tau)\}] \quad (1.19)$$

The response power density spectrum may then be obtained by Fourier transform of the above equation (1.19) (Barrett, 1980a):

$$S_{yy}(\omega) = \iint \dots \int n! |K_n(\omega_1, \dots, \omega_n)|^2 S_{xx}(\omega_1) \dots S_{xx}(\omega_n) \delta(\omega - \omega_1 - \dots - \omega_n) d\omega_1 \dots d\omega_n \quad (1.20)$$

where  $\delta$  denotes the Dirac distribution.

This expression turns out to coincide with the response power density spectrum derived by Mircea and Sinnreich (1969) and Bedrosian and Rice (1971) and simply consists in a rearrangement of the terms in the Volterra series. It is interesting to mention that equation (1.20) provides a decomposition of the output power spectrum into its frequency components (Barrett, 1982).

In order to illustrate this frequency resolution, let us consider the simple and idealized case of a narrow-band excitation spectrum with modal frequency  $\omega_0$  and mean square  $S_0$ :  $S_{xx}(\omega) = S_0 \delta(\omega - \omega_0)$ . Upon substitution in equation (1.20) the response power density spectrum results:

$$S_{yy}(\omega) = \sum_{n=1}^{+\infty} n! |K_n(\omega_0, \dots, \omega_0)|^2 S_0^n \delta(\omega - n\omega_0) \quad (1.21)$$

It is clear from this example that nonlinear response occurs not only at the frequency of excitation but at superharmonic frequencies multiple of the fundamental one.

The output frequency spectrum  $S_{yy}(\omega)$  of analytic nonlinear systems can be evaluated either as a power series (by Fourier transform of equation (1.12)) or as the single sum (1.20). In practice such series must be truncated, thus (1.12) yields the

	Simplicity	Accuracy	Versatility
Simulation	Light Gray	Dark Gray	Dark Gray
Eq. Linearization	Dark Gray	Light Gray	Dark Gray
Fokker-Planck	Dark Gray	Dark Gray	Light Gray
Functionals	Light Gray	Dark Gray	Dark Gray

Table 1.1. Methods of Nonlinear Statistical Dynamics.

lowest order nonlinear terms whereas (1.20) yields the first few harmonics. Clearly, choice of either series should depend upon the problem in hand.

A general theory for nonlinear response to stochastic processes should satisfy three attributes: *simplicity* of implementation, *accuracy* of the resulting response statistics, and *versatility* of the method<sup>13</sup>.

Table 1.1 compares the different methods discussed with respect to these three attributes. The figure shows the darkest areas wherever the techniques perform best. Clearly, such a theory is not available, and one is left with the various techniques described and the associated dilemma of choosing the one which holds the most promises with regard to the problem in hand.

Furthermore, the singular lack of comprehensive full-scale as well as model measurements of situations involving marine vehicles seriously impairs the possibility of assessing the validity and adequacy of these techniques.

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<sup>13</sup>Versatile regarding both the class of nonlinear systems for which it applies and the resulting response statistics or probability distributions.



## CHAPTER 2

### Second-Order Statistics

Although second order statistics do not completely define nonlinear transformations of Gaussian random processes, they provide the necessary information related to the amount and repartition of energy so transmitted.

More precisely, the energy spectral density  $S(\omega)$  of a random process provides significant insight into the frequency content of such a process. This frequency repartition proves to be useful whenever significant dynamic effects, such as resonant frequencies and instabilities can be expected. Furthermore, the Bienaimé-Chebyshev's inequality provides an upper-bound of the probability distribution of any stochastic process, from knowledge of its mean value and variance alone.

Therefore, although the first- and second-order statistical properties do not completely characterize general stochastic processes, they still contain the most important information about that process.

Spectral analysis of linear systems essentially relies upon the correlation properties of such systems. They mainly consist in the convolution theorem and the Wiener-Khintchine relations. It is a well established body of knowledge thereby promoting

linear filtering as an effective mean to examine the statistical as well as the probabilistic content of the corresponding response process.

On the other hand, a careful review of the literature (Chapter 1) has revealed that there does not exist any unified and general theory aimed at predicting the response frequency spectrum of general nonlinear dynamic systems driven by stochastic Gaussian signals.

We will examine the connections among several approximated techniques, with particular reference to linear filtering, together with their applicability to practical problems related to the behavior of marine vehicles driven by stochastic seas.

## Linear Filtering

Let us consider a nonlinear, causal and autonomous (i.e. time invariant) system (S) described by equation (1.1). Probably the most appealing technique consists in linearizing the nonlinear system in hand. Clearly, such an input-output functional relationship can always be rewritten under the form:

$$\mathcal{L}\{y(t)\} + \mathcal{E}\{y(t)\} = x(t) \tag{2.1}$$

where,  $\mathcal{L}$  is a linear functional, and  $\mathcal{E}$  is a functional equal to the difference  $\mathcal{F} - \mathcal{L}$ .

The problem of linear estimation of the nonlinear system  $\mathcal{F}$  consists in constructing such a linear system  $\mathcal{L}$  through appropriate choice of the error functional  $\mathcal{E}$ . Therefore, the following linear approximations result:

$$\begin{aligned}\mathcal{L}\{\tilde{y}(t)\} &= x(t) \\ \mathcal{L}\{y(t)\} &= \tilde{x}(t)\end{aligned}\tag{2.2}$$

where tilde superscripts indicate estimated quantities resulting from the linear approximation.

The choice of such an "equivalent" linear functional is neither unique nor a priori evident. It usually results from the minimization of some measure of the error committed either on the excitation process  $x(t)$ , on the response process  $y(t)$  or alternatively on the error functional  $\mathcal{E}$ .

Therefore, a number of techniques are possible: they belong to the general problem of linear filtering which originated with the theory of communication.

### Mean Square Estimation

Linear mean square estimation of stochastic processes is a well established technique in the theory of communication, mainly in connection with signal analysis and processing (Wiener, 1949 and Makhoul, 1975). Essentially, it is desired to estimate the present value of a stochastic process  $y(t)$  in terms of the values  $x(\tau)$  of another process  $x(t)$  specified for every  $\tau$  in some time interval. When  $t \geq \tau$  this is the problem of linear filtering.

In the case of a nonlinear system  $\mathcal{F}\{y(t)\} = x(t)$ , it is possible to show how a linear estimate of the response process  $y(t)$  can be obtained, at least in principle, as a linear



transformation of past values of the excitation process  $x(t)$  from mean square estimation theory.

We may recall briefly the rationale underlying this technique. The linear mean square estimation problem reduces to find the linear transformation  $\mathcal{L}_{ms}$  which minimizes the output mean square error:

$$E[(y(t) - \bar{y}(t))^2] = R_{yy}(0) - 2 \int_{-\infty}^{+\infty} h_{ms}(\tau) R_{xy}(\tau) d\tau + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_{ms}(\tau) h_{ms}(\sigma) R_{xx}(\tau - \sigma) d\tau d\sigma \quad (2.3)$$

An application of functional calculus of variation (Volterra, 1930) leads to the following integral equation in the kernel  $h_{ms}$  of linear mean square system ( $S_{ms}$ ) known as the Wiener-Hopf equation (Wiener, 1949):

$$R_{xy}(\tau) d\tau = \int_{-\infty}^{+\infty} h_{ms}(\sigma) R_{xx}(\tau - \sigma) d\tau d\sigma \quad \tau \geq 0 \quad (2.4)$$

The solution  $h_{ms}(t)$  of the above equation is known as the Wiener filter. Generally, this equation must be solved numerically, and cannot be readily Fourier transformed to obtain frequency domain estimates rather than time domain ones because of the causality condition; equation (1.2), necessary to yield a physically realizable filter.

In order to overcome the difficulty related to the causality condition, Wiener and Hopf introduced a special technique called spectrum factorization. Note however that if

the poles of the resulting transfer function  $H_{ms}(\omega)$  all lies in the left half complex plane, then the causality condition is automatically satisfied, and the following relation holds:

$$H_{ms}(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} \quad (2.5)$$

Finally, the mean square error becomes:

$$E[\hat{y}^2] = E[(y - \tilde{y})^2] = E[y^2] - E[\tilde{y}^2] \geq 0 \quad (2.6)$$

In other words, the minimum value of the mean square error is equal to the difference of the mean squares of the exact process  $y(t)$  and its linear mean square estimate.

Clearly the method of linear mean square estimation invariably leads to an underestimated approximation, in the mean square sense, of the actual mean square of response process  $y(t)$  by an amount equal to the error (2.6). Thus, care must be exercised in interpreting such an unconservative estimation of the response mean square value in engineering design.

The transfer function of linear mean square system ( $S_{ms}$ ) is obtained through equation (2.5) as the ratio of the cross-spectrum  $S_{yx}(\omega)$  over the excitation spectrum  $S_{xx}(\omega)$ . Therefore, such a linear representation is most convenient whenever the system is to be identified through simultaneous measurements of the excitation and response processes.

In the case of a nonlinear system described through a functional relationship of the type (1.1) the cross-spectral density  $S_{yx}(\omega)$  is not known in general (at least not exactly). Nevertheless, we will show later how this technique can be related to the more conventional method of equivalent linearization in nonlinear statistical dynamics.

Let us now consider the integral version of Schwarz' inequality:

$$E[uv]^2 \leq E[u^2] E[v^2] \quad (2.7)$$

Substituting:

$$\begin{aligned} u &= y(t) \quad \text{and} \quad v = y(t+\tau) - \tilde{y}(t+\tau) \\ u &= y(t) - \tilde{y}(t) \quad \text{and} \quad v = \tilde{y}(t+\tau) \end{aligned} \quad (2.8)$$

in inequality (2.7), we respectively obtain the inequalities:

$$\begin{aligned} \{E[y(t)y(t+\tau)] - E[y(t)\tilde{y}(t+\tau)]\}^2 &\leq E[y^2] E[\tilde{y}^2] \\ \{E[y(t)\tilde{y}(t+\tau)] - E[\tilde{y}(t)\tilde{y}(t+\tau)]\}^2 &\leq E[\tilde{y}^2] E[\tilde{y}^2] \end{aligned} \quad (2.9)$$

Invoking the triangle inequality,  $|u + v| \leq |u| + |v|$ , it is possible to combine the two inequalities above to yield:

$$\{E[y(t)y(t+\tau)] - E[\tilde{y}(t)\tilde{y}(t+\tau)]\}^2 \leq E[\tilde{y}^2] \{\sqrt{E[y^2]} + \sqrt{E[\tilde{y}^2]}\}^2 \quad (2.10)$$

Finally, since the mean square is given by equation (2.6) and is necessarily positive, we obtain:

$$\frac{\{E[y(t)y(t+\tau)] - E[\tilde{y}(t)\tilde{y}(t+\tau)]\}^2}{E^2[y^2]} \leq \frac{4\{E[y^2] - E[\tilde{y}^2]\}}{E[y^2]} \quad (2.11)$$

Thus, the relative mean square error gives an upper bound of a measure of the error committed on general second-order statistics, the temporal autocorrelation function, keeping in mind the properties of stationarity and ergodicity of the excitation as well as the response processes.

Let us now assume that the actual process  $y$  and its linear mean square estimate are equal in the mean square sense, i.e. that the mean square error (2.6) is equal to zero. Then it follows from equation (2.11) that their temporal autocorrelation function as well as their spectral density (i.e. any second-order statistics) coincide.

Obviously, the relative mean square error:

$$\frac{E[y^2] - E[\tilde{y}^2]}{E[y^2]} \quad (2.12)$$

generally does not vanish, nevertheless it can be shown, through some examples, to remain quite reasonable and even bounded as departure from linear behavior becomes more and more significant (see Chapter 3, Figure 3.2). Note that this error is always positive and less than one.

All these results tend toward the same conclusion, namely that linear filtering using mean square estimation should yield reasonable prediction of second order statistics of the random process so estimated.

### The Method of Equivalent Linearization

In the method of equivalent linearization, an equivalent linear system ( $S_{eq}$ ) is substituted to the original nonlinear one ( $S$ ). The linear functional  $\mathcal{L}_{eq}$  describing the input-output relationship of system ( $S_{eq}$ ) is defined as the one which minimizes an "appropriate" measure of the corresponding error  $\mathcal{E}$ .

The choice of such a measure is not a priori straightforward and does not follow any specific guidelines. In the case of a stochastic excitation, the mean square error is generally minimized, mostly for computational convenience. This leads to the method of equivalent linearization as defined by Booton (1954) and Caughey (1959 & 1963b) (see Chapter 1).

The method of equivalent linearization as defined by Booton (1954) and Caughey (1959 & 1963b) was introduced essentially to deal with nonlinear sets of ordinary differential equations governing the behavior of electrical and mechanical systems, and therefore does not directly apply to more general nonlinear systems. Furthermore, the main idea underlying this technique relies upon the construction a priori of an equivalent linear system ( $S_{eq}$ ) in substituting linear terms to nonlinear terms in original system ( $S$ ), where the unknown coefficients of each linearized term are then determined in minimizing the mean square error  $E[\mathcal{E}^2]$ .

This technique has been proposed and applied to engineering problems for some time and with some success. However several fundamental, yet unanswered, questions can be raised:

What are we precisely doing in minimizing the mean square error  $\mathcal{E}$ ?

What kind of response statistics can be predicted (if any), within reasonable accuracy, through the method of equivalent linearization?<sup>1</sup>

Why does the method of equivalent linearization tends to underestimate the response variance?

When does it apply?

How does it compare to the Volterra (i.e. perturbation) and Wiener-Hermite functional models?

How is it related to linear mean square estimation?

All these questions will be considered, with various degree of completeness, through the following discussions.

In order to gain some further insight into the equivalent linearization method, let us consider the equivalent linear functional  $\mathcal{L}^{-1}$ , function of past values of the excitation process  $x(t)$ , with kernel  $h_{eq}(\tau)$ :

$$\mathcal{L}^{-1} \{x(t)\} = \int_{-\infty}^{+\infty} h_{eq}(t - \tau)x(\tau)d\tau$$

(2.13)

<sup>1</sup>Regarding this question, it is pertinent to quote Lyon (1960a & 1960b) discussing the method of equivalent linearization: "As a general matter it seems difficult to anticipate just which parameters will be estimated reasonably ... some caution would seem advisable in indiscriminate application of these techniques ..."

Clearly, the method of equivalent linearization minimizes the mean square error:

$$E[\mathcal{E}_{eq}^2\{y(t)\}] = R_{xx}(0) - 2 \int_{-\infty}^{+\infty} h_{eq}(\tau) R_{xy}(\tau) d\tau + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_{eq}(\tau_1) h_{eq}(\tau_2) R_{yy}(\tau_2 - \tau_1) d\tau_1 d\tau_2 \quad (2.14)$$

A procedure similar to the one used in the previous section i.e. linear mean square estimation leads to the transfer function  $H_{eq}(\omega)$  of equivalent linear system ( $S_{eq}$ ):

$$H_{eq}(\omega) = \frac{S_{yy}(\omega)}{S_{xy}(\omega)} \quad (2.15)$$

Whereas the mean square error becomes:

$$E[\hat{x}^2] = E[(x - \tilde{x})^2] = E[x^2] - E[\tilde{x}^2] \geq 0 \quad (2.16)$$

From this inequality and from linear equations (2.2), it is possible to conclude that:

$$E[y^2] - E[\tilde{y}^2] \geq 0 \quad (2.17)$$

if and only if  $\mathcal{L}_{eq}^{-1}$  is a monotonic decreasing functional in the mean square sense. In other words:

$$E[\tilde{x}^2] \geq E[y^2] \quad \text{or} \quad E[x^2] \geq E[\tilde{y}^2] \quad (2.18)$$

It is obvious that  $\mathcal{L}_{eq}^{-1}$  is monotonic since it is linear, though not formally because it depends upon some statistics of random process  $y(t)$ . In the case of mechanical systems the decreasing condition turns out to be met more often than not, since they are usually designed to act in such a way. In the increasing case, the response mean square is overestimated. Thus, the sign of the response mean square error can be predicted from consideration of the equivalent linear functional alone. Although such a general tendency to underestimate the response mean square value has been already observed (Lyon, 1960b, Iwan and Yang, 1972 and Lin, et al., 1986), it has never been thoroughly examined.

It is essential to emphasize that the rationale underlying the method of equivalent linearization is intimately related to the actual form of the functional relationship governing the behavior of nonlinear system (S). A nonlinear system described by an input-output relationship of the type (1.1) yields the equivalent linear transfer function (2.15), whereas the nonlinear system:

$$y(t) = \mathcal{F}^{-1} \{x(t)\} \quad (2.19)$$

leads to the same linear transfer function as the one obtained by linear mean square estimation, equation (2.5), although the original nonlinear systems are virtually identical.



When the nonlinear system is described through an explicit functional relationship of the type:  $y(t) = \mathcal{F}^{-1}\{x(t)\}$  the transfer functions obtained by each technique coincide.

Both methods yield a linearized system which necessarily depends upon some higher order statistics of the actual response process, therefore its probability law should be known. Since, in most cases this law is unknown, normal behavior is generally conveniently assumed.

Finally the connections between the methods of linear mean square estimation and equivalent linearization can be emphasized by their transfer functions respectively obtained from equations (2.5) and (2.15) in the case of an inverse nonlinear system of the type:  $\mathcal{F}\{y(t)\} = x(t)$ . In practice however, ratios (2.5) and (2.15) cannot be evaluated exactly since they both involve some statistics of unknown process  $y(t)$ . Assuming that the response process  $y(t)$  is respectively obtained from its mean square and equivalent linear approximations, then the two linear transfer functions coincide. This will be illustrated with the example below.

### The Duffing Equation

The fundamental nuances which characterize the method of equivalent linearization and linear mean square estimation of a nonlinear system described by an input-output relationship of the type  $\mathcal{F}\{y(t)\} = x(t)$  can be emphasized by considering the Duffing equation:

$$\ddot{y} + 2\xi\dot{y} + y + y^3 = x$$

(2.20)

Upon multiplying both sides by  $x(t+\tau)$  or  $y(t+\tau)$  and averaging with respect to the time variable, the following equations result:

$$R''_{yx}(\tau) + 2\xi R'_{yx}(\tau) + R_{yx}(\tau) + R_{yyyyx}(\tau) = R_{xx}(\tau) \quad (2.21)$$

$$R''_{yy}(\tau) + 2\xi R'_{yy}(\tau) + R_{yy}(\tau) + R_{yyyy}(\tau) = R_{xy}(\tau) \quad (2.22)$$

where the prime superscript denotes time differentiation.

By Fourier transform and substitution in equations (2.5) and (2.15), the linear mean square estimated and equivalent linear transfer functions can be obtained:

$$H_{ms}(\omega) = H_1(\omega) \left\{ 1 - \frac{S_{yyyyx}(\omega)}{S_{xx}(\omega)} \right\} \quad (2.23)$$

$$H_{eq}(\omega) = H_1(\omega) \left\{ 1 - \frac{S_{yyyy}(\omega)}{S_{xy}(\omega)} \right\} \quad (2.24)$$

Assuming now that deviation from normal behavior is light, we obtain:

$$R_{yyyyx}(\tau) = 3E[y^2]R_{yx}(\tau) \quad (2.25)$$

$$R_{yyyy}(\tau) = 3E[y^2]R_{yy}(\tau) \quad (2.26)$$

Fourier transforming and Substituting in equations (2.23) and (2.24), the optimum transfer function results:

$$H_{ms}(\omega) = H_{eq}(\omega) = \frac{1}{1 + 3E[y^2] - \omega^2 + 2i\xi\omega} \quad (2.27)$$

Thus the linear mean square estimated and equivalent linear systems are identical if normality is assumed. Although this equivalent system is essentially linear, it can still predict some nonlinear effects such as increased spectral bandwidth and natural frequency not revealed by the pure linear approximation.

### The Wiener-Hermite Functional Model

In order to gain further insight into the methods of equivalent linearization and mean square estimation, it is relevant to examine, in some details, the Wiener-Hermite functional model. Wiener constructed a new set of orthogonal functionals in order to overcome several difficulties associated with the convergence and the measurement of the functional series model originally developed by Volterra (Wiener, 1958).

However, one additional important feature of the truncated Wiener-Hermite functional series over the Volterra series stems from the resulting optimum nonlinear model (Barrett, 1963 and Schetzen, 1965c & 1980). In particular, the first term of the Wiener-Hermite functional expansion turns out to yield an optimum linear model closely related to the linear mean square model.

Wiener considered a complete set of functionals,  $\mathcal{K}_n\{x(t)\}$ , orthogonal with respect to a stationary Gaussian white noise excitation process  $x(t)$ . The generalization to a general Gaussian excitation process originated with the work of Barrett (1980a & 1982).

The further generalization to non-Gaussian input processes can be achieved using the gate functions at the expense of much greater complication (Schetzen, 1965a & 1980). Therefore, we will only consider Gaussian excitations. We can however emphasize that the assumption of a Gaussian random process can be overcome if it is possible to reconstruct the actual process through a nonlinear Wiener-Hermite transformation of a Gaussian one.

The hypothesis of normality justifies the choice of an expansion in a set of generalized Hermite polynomials  $h_n^{(n)}$ . The orthogonality relations (1.13) involving the Wiener-Hermite functionals result from the Hermite expansion. Where the Wiener-Hermite functionals are given by equation (1.15), and the corresponding Wiener-Hermite kernels  $k_n(\tau_1, \tau_2, \dots, \tau_n)$  are determined by the orthogonality conditions (equation (1.17)).

The Wiener-Hermite functional series is now completely defined, thus we are, at least in principle, able to derive any statistics describing the response process  $y(t)$ , defined by:

$$y(t) = \sum_{n=1}^{+\infty} \mathcal{K}_n\{x(t)\} \tag{2.28}$$

Taking advantage of the fact that the terms in the above series are all uncorrelated, the derivation of the cross-spectral density  $S_{yx}(\omega)$  between the excitation and the response processes is straightforward:

$$S_{yx}(\omega) = K_1(\omega)S_{xx}(\omega) \quad (2.29)$$

Similarly, the auto-spectral density is given by the Mircea-Sinnreich (1969) series:

$$S_{yy}(\omega) = \sum_{n=1}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} n! |K_n(\omega_1, \dots, \omega_n)|^2 S_{xx}(\omega_1) \dots S_{xx}(\omega_n) \delta(\omega - \omega_1 - \dots - \omega_n) d\omega_1 \dots d\omega_n \quad (2.30)$$

where  $K_n(\omega_1, \dots, \omega_n)$  is the  $n$ -th order Wiener-Hermite transfer function, Fourier transform of the  $n$ -th order Wiener-Hermite kernel  $k_n(\tau_1, \dots, \tau_n)$ .

When the response  $y(t)$  can be expanded as a Volterra functional series of the excitation process  $x(t)$ , the Wiener-Hermite transfer functions  $K_n(\omega_1, \dots, \omega_n)$  can be written in terms of the Volterra transfer functions  $H_n(\omega_1, \dots, \omega_n)$  (Barrett, 1980a & 1982):

$$n! K_n(\omega_1, \dots, \omega_n) = \sum_{m=0}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} 2^{-m} H_{2m+n}(\omega_1, \dots, \omega_n, \omega'_1, -\omega'_1, \dots, \omega'_m, -\omega'_m) S_{xx}(\omega'_1) \dots S_{xx}(\omega'_m) d\omega'_1 \dots d\omega'_m \quad (2.31)$$

Equations (2.5) and (2.15) together with (2.29) and (2.30) lead to the transfer functions of the linear mean square estimated and the equivalent linear systems:

$$H_{ms}(\omega) = K_1(\omega) \quad (2.32)$$

$$H_{eq}(\omega) = K_1(\omega) + \sum_{n=2}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} n! |K_n(\omega_1, \dots, \omega_n)|^2 S_{xx}(\omega_1) \dots S_{xx}(\omega_n) \delta(\omega - \omega_1 - \dots - \omega_n) d\omega_1 \dots d\omega_n / K_1^*(\omega) S_{xx}(\omega) \quad (2.33)$$

which, in turn, lead to the response spectrum predicted by the methods of equivalent linearization and mean square estimation:

$$S_{yyms}(\omega) = |K_1(\omega)|^2 S_{xx}(\omega) \quad (2.34)$$

$$S_{yyeq}(\omega) = |K_1(\omega)|^2 S_{xx}(\omega) + 2 \sum_{n=2}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} n! |K_n(\omega_1, \dots, \omega_n)|^2 S_{xx}(\omega_1) \dots S_{xx}(\omega_n) \delta(\omega - \omega_1 - \dots - \omega_n) d\omega_1 \dots d\omega_n + \left[ \sum_{n=2}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} n! |K_n(\omega_1, \dots, \omega_n)|^2 S_{xx}(\omega_1) \dots S_{xx}(\omega_n) \delta(\omega - \omega_1 - \dots - \omega_n) d\omega_1 \dots d\omega_n \right]^2 |K_1(\omega)|^{-2} S_{xx}^{-1}(\omega) \quad (2.35)$$

If we now assume that  $x(t)$  is of order  $\epsilon$ , then the excitation power spectrum will be of order  $\epsilon^2$ . Under the double further assumptions that the response process  $y(t)$  can be expanded in Volterra series and is an odd functional of the excitation process  $x(t)$ , it can be shown from equation (2.31) that the response power spectrum, as predicted by the method of equivalent linearization (equation (2.35)) as well as the method of mean square estimation (equation (2.34)) are exact up to the fifth order in  $\epsilon$  (note that the response power spectrum  $S_{xx}(\omega)$  is an even expansion in  $\epsilon$ ).

This confirms and demonstrates a property of the method of equivalent linearization already observed in some specific cases (Caughey, 1963b, Lin, 1967 and Payne, 1968).

Although, it is not obvious, it can be shown that the same property holds in the case of asymmetric Volterra series as well. It becomes clear recalling that the second order Volterra transfer function  $H_2(\omega_1, \omega_2)$  does not appear in the response centered power spectral density.

Such a property is very general and applies to any systems which can be expanded in Volterra series (i.e. analytic systems).

## Applications

The case of a nonlinear system which can be solved exactly is considered first. Then, particular attention is devoted to the case of the nonlinear roll motion of ships. Extensive time domain simulations of a nonlinear mathematical idealization of this problem by Dalzell (1971 & 1973) are used for the purpose of comparing and

characterizing various approximated analytical models leading to the roll spectral density.

### An Elementary Example

We consider the case of a nonlinear system, the response process of which, is exactly described by the two-term odd Volterra polynomial:

$$y(t) = \int_{-\infty}^{+\infty} h_1(\tau_1) x(t-\tau_1) d\tau_1 + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h_3(\tau_1, \tau_2, \tau_3) x(t-\tau_1) x(t-\tau_2) x(t-\tau_3) d\tau_1 d\tau_2 d\tau_3 \quad (2.36)$$

It is assumed that such a representation exactly described the response of a nonlinear system of the type:  $\mathcal{F}\{y(t)\} = x(t)$ . One such nonlinear system is the case of the adimensionalized differential equation:

$$\ddot{y} + 2\xi\dot{y} + y = x + x^3 \quad (2.37)$$

Clearly, it is possible to derive, from the functional representation (2.36), the response power spectral density resulting from various approximations, as well as the exact one:

(i) Linear approximation:



$$S_{yy}(\omega) = |H_1(\omega)|^2 S_{xx}(\omega) \quad (2.38)$$

(ii) Two-term perturbation (Volterra polynomial) approximation:

$$S_{yy}(\omega) = |H_1(\omega)|^2 S_{xx}(\omega) \{1 + 6\sigma_x^2\} \quad (2.39)$$

(iii) Mean square estimation, equivalent linearization approximations and first term of the Wiener functional model (equation (2.34)):

$$S_{yy}(\omega) = |H_1(\omega)|^2 S_{xx}(\omega) \{1 + 3\sigma_x^2\}^2 \quad (2.40)$$

(iv) Exact solution:

$$S_{yy}(\omega) = |H_1(\omega)|^2 S_{xx}(\omega) \{1 + 3\sigma_x^2\}^2 + 6 |H_1(\omega)|^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} S_{xx}(\omega_1) S_{xx}(\omega_2) S_{xx}(\omega - \omega_1 - \omega_2) d\omega_1 d\omega_2 \quad (2.41)$$

where the linear transfer function is given by:

$$H_1(\omega) = \frac{1}{-\omega^2 + 2i\xi\omega + 1} \quad (2.42)$$

The damping ratio is chosen equal to 0.1, while the adimensionalized natural frequency is equal to unity.

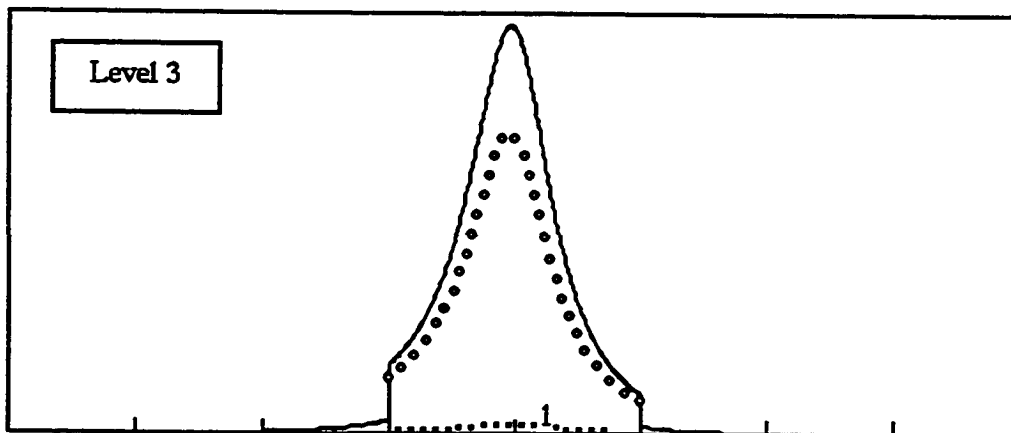
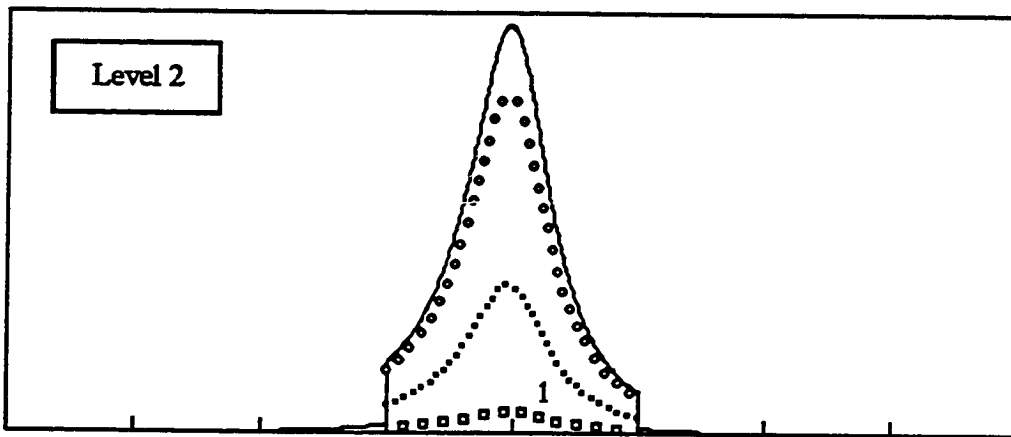
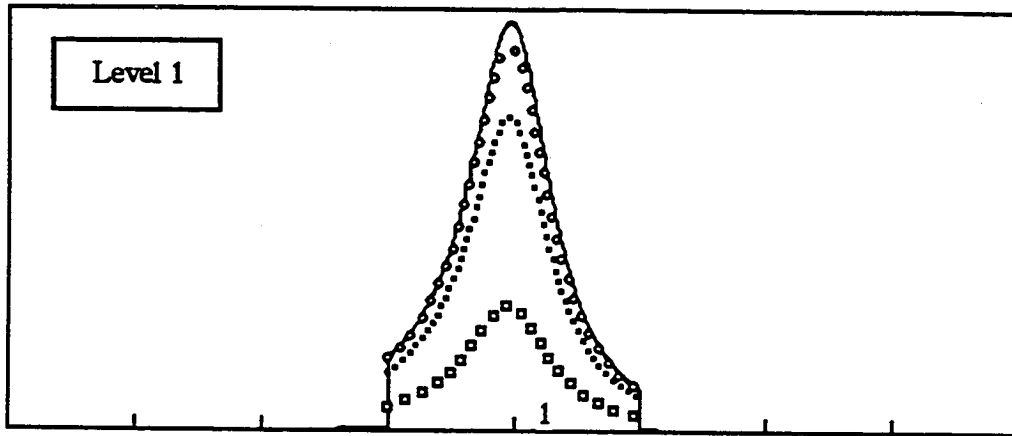
The case of an excitation process, the power spectrum of which is a band-limited white noise, serves to illustrate and compare the different approximations involved.

Figures 2.1 to 2.6 show the various approximations together with the exact response spectral density. The solid curve shows the exact spectrum (2.41), while the white squares, the black squares and the white dots respectively depict the linear (2.38), the 2-term Volterra (2.39) and the equivalent linear (2.40) approximations.

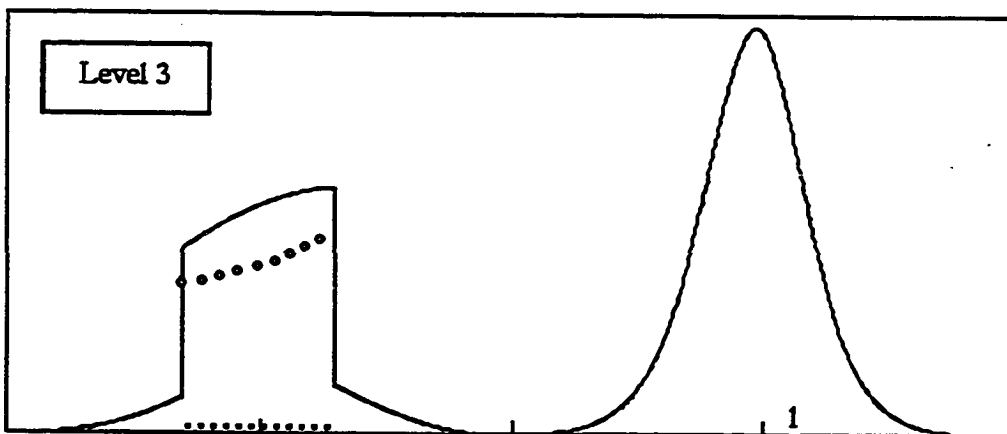
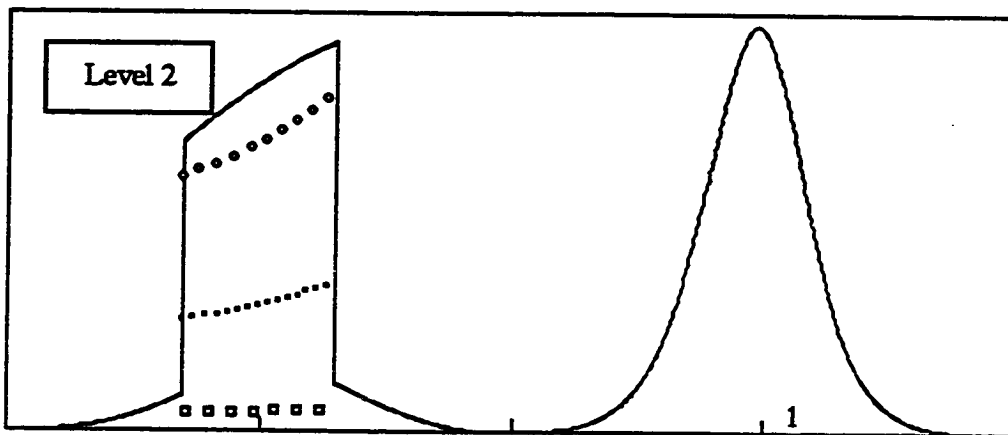
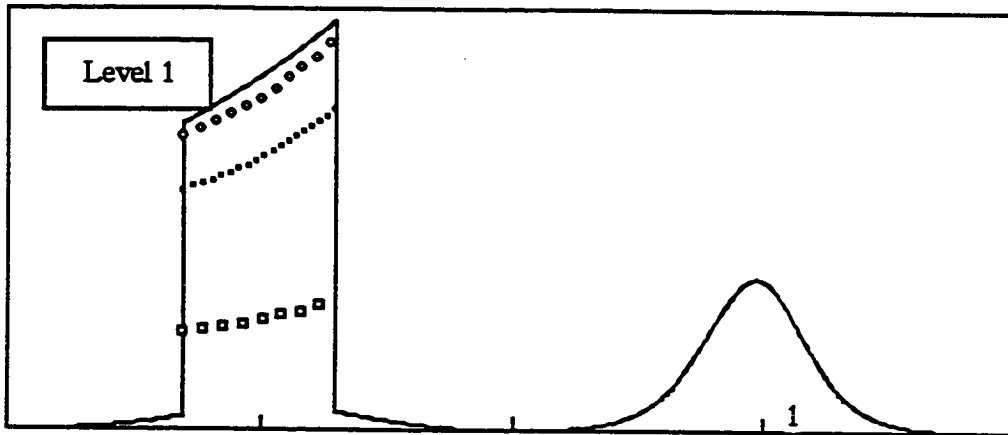
Figures 2.1, 2.2 and 2.3 show the response spectral density for increasing levels of nonlinearity (levels 1, 2 and 3 respectively correspond to excitation standard deviations equal to 0.5, 1 and 5), and excitation spectrum centered at the linear natural frequency of the system with a bandwidth of 0.5.

The equivalent linear approximation is, by far and consistently, the most accurate, whereas the 1- and 2-term Volterra approximations yield comparatively poor approximations. We may emphasize that the difference between the mean square and equivalent linear approximation (2.40) and the exact spectral density (2.41) is limited to the double convolution, the effect of which is to sensibly increase the bandwidth of the response process.

The same general pattern is observed when the excitation central frequency is a third of the natural one, Figures 2.4, 2.5 and 2.6, with excitation bandwidth 0.2. However no one of the approximations predict the apparition of a superharmonics at three times the mean excitation frequency as well as the significant features on both sides of the fundamental spectral response, resulting from the double convolution.



Figures 2.1-2.3. Spectral Density, Excitation 1.



Figures 2.4-2.6. Spectral Density, Excitation 2.

There probably lies one of the fundamental limitations of the methods of equivalent linearization and linear mean square estimation which cannot predict such typical qualitative nonlinear effects.

### A Mathematical Idealization of Ship Rolling

The mathematical modelling of the problem of the roll motion of ships among waves originated with the pioneering work of William Froude during the last century. When the roll motion of a ship can be decoupled with the other modes, the most widely agreed form of the equation governing the roll motion is:

$$\ddot{y} + a\dot{y} + b\dot{y}|y| + y - y^3 = x \quad (2.43)$$

and represents a slight truncation of Froude's model.

Dalzell (1971 & 1973) undertook a time domain simulation of this equation of motion, excited by a Gaussian sea state described by the 12th ITTC spectrum, in order to estimate the roll spectral density and probability distribution. We will use these simulated spectra in order to further characterize the various approximations already discussed. Among numerous configurations considered by Dalzell, we have chosen those for which departure from linear behavior is the most significant.

Figures 2.7-2.18 show the roll spectral density for four different configurations (Table 2.1), with varying linear damping ratio, nonlinear damping coefficient and excitation modal frequency. In each configuration three different sea states, by increasing level of intensity (the intensity of each sea state is defined by the

	a	b	$\omega_0$
Configuration 1	10.0%	1.00	0.55
Configuration 2	03.0%	1.00	0.90
Configuration 3	03.0%	3.00	0.90
Configuration 4	00.0%	1.00	0.90

Table 2.1. Roll Motion Configurations.

adimensionalized<sup>2</sup> wave slope standard deviation: 0.0039, 0.0117 and 0.0351) from sea state one to sea state three are considered.

On the figures, the white dots show Dalzell's time domain simulation estimates, while the solid curve labelled Linear, Equivalent Linear, Volterra and Wiener respectively correspond to the linear, the equivalent linear, the 2-term Volterra functional expansion (perturbation) and the 2-term Wiener-Hermite functional model (first two terms of equations (2.31) and (2.34)) spectral density approximations.

Clearly, the method of equivalent linearization consistently yields the best overall agreement with simulation estimates, in 11 cases out of 12, when compared with the first few terms of the Volterra and Wiener-Hermite functional series.

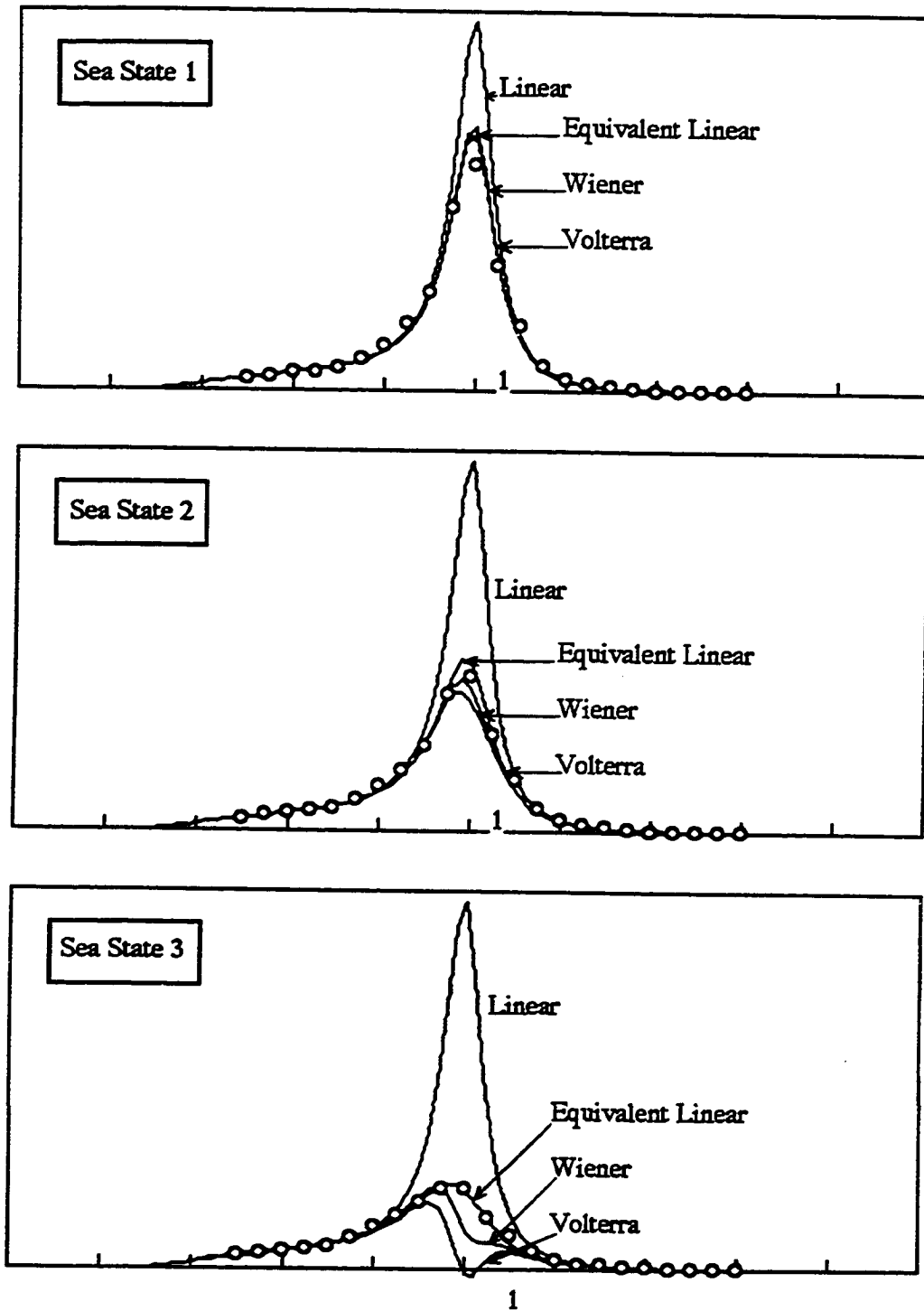
More terms in such expansions may result in an added accuracy, however the price to be paid would be significant, if not unachievable, in terms of computational costs. The configuration 2 was considered by Dalzell (1976a) in a comparison with a Wiener-Hermite functional model containing one more term at a significant expense (double convolution) without a much better accuracy.

We may nevertheless emphasize that the Wiener-Hermite model seems to attain a better level of accuracy than the Volterra model. Furthermore, the method of equivalent linearization and the Wiener functional model possess the advantage over the Volterra perturbation to guarantee the positivity of the resulting spectral density.

Increasing the level of nonlinearity results in larger spectral bandwidth because the damping does not increase linearly. This effect tends to overcome the effect of simultaneous nonlinear decrease in restoring force which acts in the opposite direction

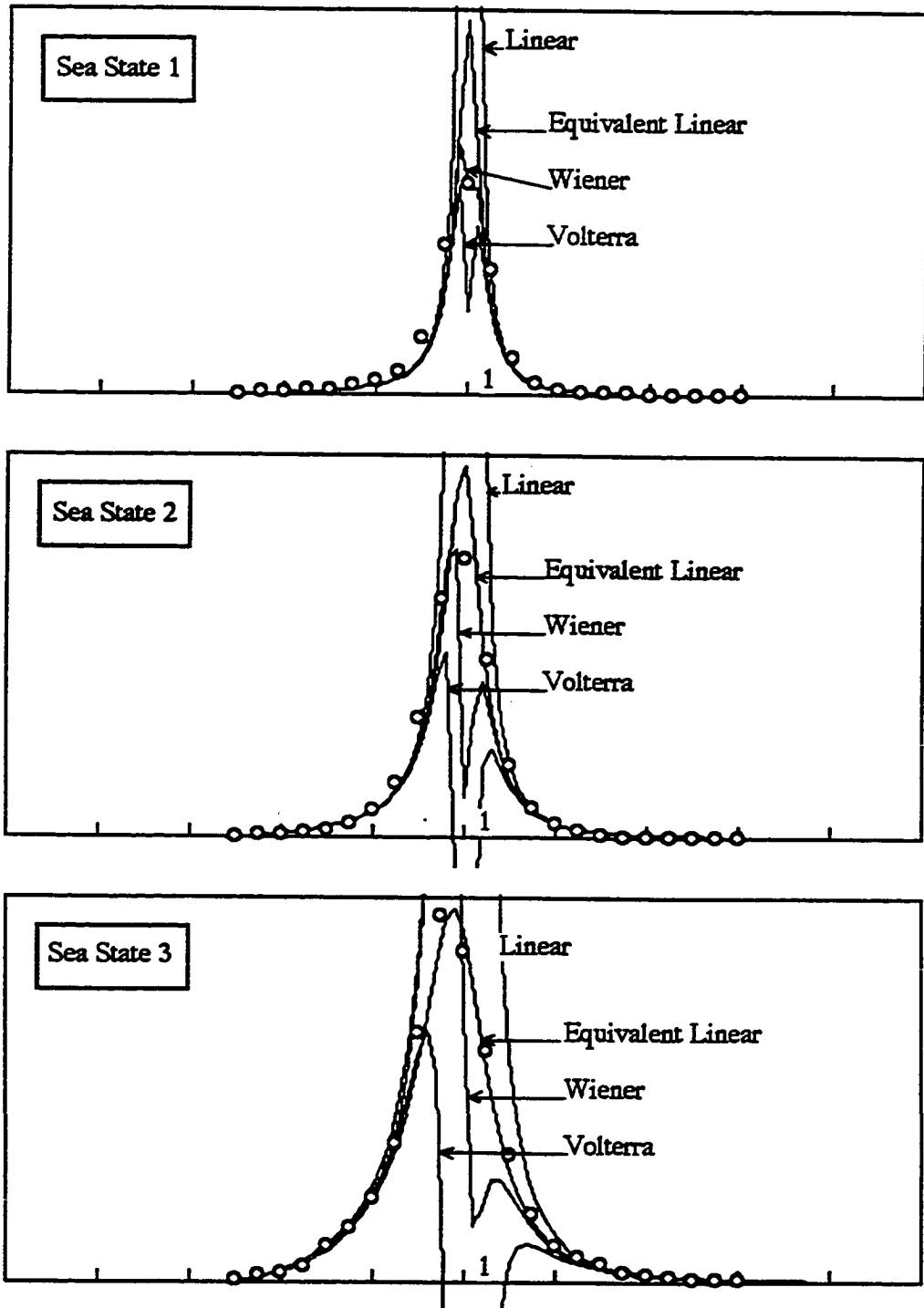
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<sup>2</sup>Relative to the half range of positive static stability of the ship.

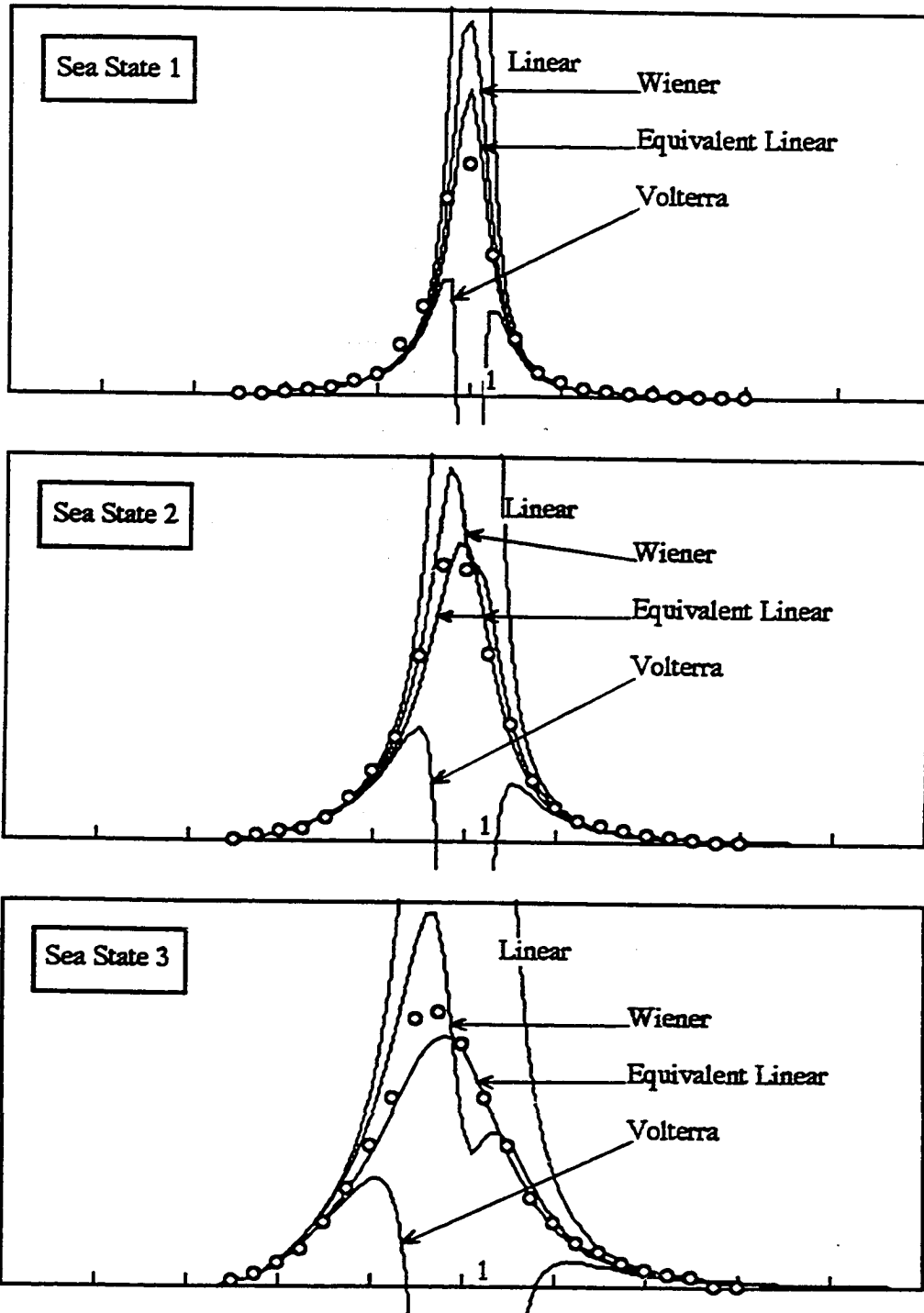


Figures 2.7-2.9. Roll Spectral Density, Configuration 1.

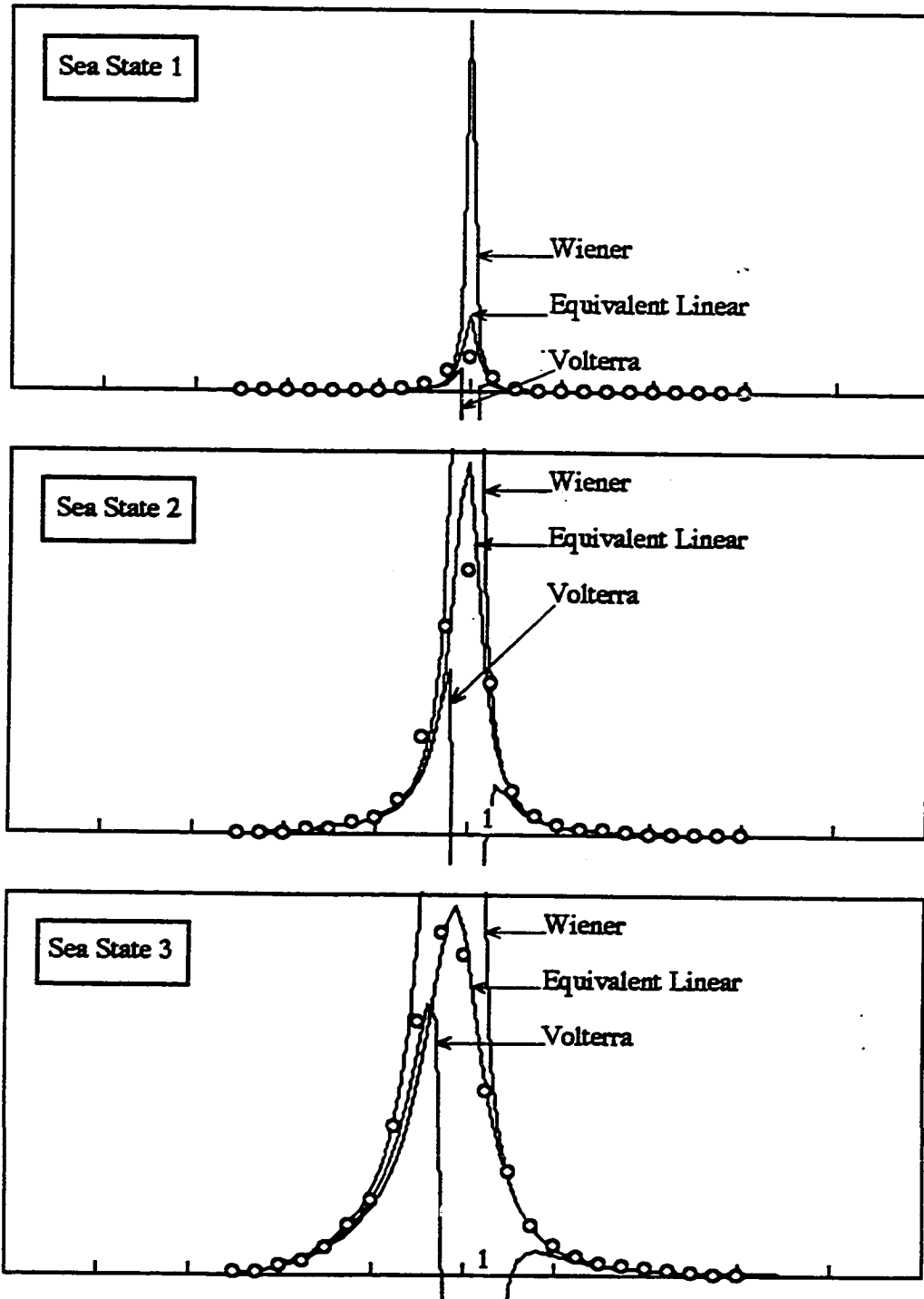




Figures 2.10-2.12. Roll Spectral Density, Configuration 2.



Figures 2.13-2.15. Roll Spectral Density, Configuration 3.



Figures 2.16-2.18. Roll Spectral Density, Configuration 4.

The method of equivalent linearization seems to yield the most accurate response spectral density whenever its bandwidth is the largest. This may be why the agreement tends to be better for higher levels of nonlinearity, although the average relative error of the roll motion standard deviation resulting from the equivalent linearization was found to increase with more severe sea states: 0.9% , 2.0% and 2.7% respectively, for sea states 1, 2 and 3. We can emphasize again that this error is consistently on the unconservative side.

Finally, we can state that the method of equivalent linearization should provide a rather accurate prediction of the roll motion spectral density, considering that such levels of nonlinearity correspond to very heavy sea states leading to some of the most extreme response a ship will encounter in its lifetime.

We have shown that the methods of equivalent linearization and mean square estimation are intimately related. In particular, they yield identical linear systems whenever the response process deviation from normal behavior can be neglected<sup>3</sup>. Moreover, the resulting linear model is the optimum linear approximation, in the mean square sense, of the actual nonlinear one.

We have also shown that the second order statistics resulting from this linear model are exact up to the 5th order in the expansion parameter  $\epsilon$ . Finally the corresponding linear transfer function is obtained analytically as the first term of the Wiener-Hermite functional series model (equations (2.31) & (2.34)) and alternatively through cross-spectrum identification (equations (2.5) & (2.15)).

This yields a simple and quite versatile technique for predicting the response second-order statistics of nonlinear dynamic systems. Several examples show the level

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<sup>3</sup>Note that it is neglected only for second-order statistics purposes.

of accuracy of the spectral density obtained when compared with exact and simulation results.

However, the main drawback of the technique of linear filtering probably lies in its intrinsic inability to predict specific nonlinear effects such as superharmonic response, non-Gaussian response to Gaussian random process... which result from the multiple convolutions (i.e. of order higher than one) appearing in the Wiener-Hermite form of the energy spectral density, equation (2.30).

Furthermore, prediction of statistics of order higher than two do not quite fit the methods of linear mean square estimation and equivalent linearization. Essentially, since linear behavior is assumed, higher order response statistics will be those of a Gaussian process if the excitation process is Gaussian. Therefore, we will now try to circumvent such drawbacks of the method of equivalent linearization in order to predict response higher-order statistics.

## **CHAPTER 3**

### **Higher-Order Statistics**

Throughout the previous chapter, we have exhibited both theoretical and empirical new evidences to promote the method of equivalent linearization (also known as statistical linearization) as one of the most useful procedure in nonlinear statistical dynamics to predict response second-order statistics.

However, higher-order statistics so predicted will necessarily be those of a Gaussian process. Furthermore, since nonlinear transformations do not preserve Gaussian character, response statistics of order higher than two must be considered.

Although most of the information pertaining to the physical description of stochastic processes, is contained in the first two moments, higher order statistics yield further information about deviation from normal behavior as well as extreme values. Extreme values constitute an essential body of information in probabilistic engineering design as well as in system reliability.

The main purpose of this chapter is to provide an innovative method for predicting such higher-order response statistics.

However, let us start this chapter by discussing and exposing some new results in connection with an already conventional technique: the Volterra functional model. These new results will serve both to emphasize some drawbacks of the Volterra functional model and thereafter to determine some unknowns in the proposed "linearize-and-match" method.

### The Volterra Functional Model

In general, a random variable can be described from the knowledge of its statistical moments  $\mu_n$  or alternatively of its statistical cumulants  $\kappa_n$  or its quasi-moments  $\beta_n$ . In the case of a stochastic process<sup>1</sup>, all the joint moments or cumulants of the noncountable infinity of random variables which specify the stochastic process must be considered. Such statistics are known as the moment functions  $\mu_n(\tau_1, \tau_2, \dots, \tau_n)$ , the cumulant functions<sup>2</sup>  $\kappa_n(\tau_1, \tau_2, \dots, \tau_n)$  and the quasi-moment functions  $\beta_n(\tau_1, \tau_2, \dots, \tau_n)$  (Kuznetsov, Stratonovitch and Tikhonov, 1951, Stratonovitch, 1963 and Lin, 1967).

It is essential to realize that the first N cumulants, moments and quasi-moments contain just as much information. If we confine ourselves to the first N such statistics, it is clear, from their definitions, that knowledge of any set of these leads to the remaining other two.

Owing to the complexity of the present problem, we choose to consider the moments and cumulants of the stationary response process. In other words, its

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<sup>1</sup>Also known as random process or random function.

<sup>2</sup>Also called correlation functions (Stratonovitch, 1963) or semi-invariants (Rice, 1944 & 1945).

moment and cumulant functions will be evaluated when their arguments are all equal:  
 $\tau_1 = \tau_2 = \dots = \tau_n$ .

A general expression for the response statistical cumulants of nonlinear systems described by a Volterra functional series is not yet available. Bedrosian and Rice's (1971) work appears the most advanced one regarding this problem.

Essentially, they derived the first four cumulants up to the 6th order in the small parameter  $\varepsilon$  defined below. They proposed 8th order terms, in cumulant  $\kappa_4$ , based on the conjecture, supported by the memoryless case, that only the terms that do not separate into products of integrals occur in the cumulants.

Their inability to push the expansion beyond that point was apparently motivated by the resulting excessive complexity of the higher order terms, as well as the lack of a general cumulant expression.

The following developments generalize and extend Longuet-Higgins' (1963) derivation of the cumulants of nonlinear algebraic transformations of independent random variables. His work was primarily motivated by the study of the effect of nonlinearities on statistical distributions of sea wave elevation. Let us consider the more general case of the output cumulants of a Volterra series (equations (1.10) and (1.11)) driven by a zero-mean stationary and ergodic Gaussian stochastic process  $x(t)$ :

$$y(t) = \varepsilon \int_{-\infty}^{+\infty} h_1(\tau_1) x(t-\tau_1) d\tau_1 + \varepsilon^2 \iint_{-\infty}^{+\infty} h_2(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) d\tau_1 d\tau_2 + \dots \quad (3.1)$$



where, without any loss of generality, the kernels  $h_k(\tau_1, \tau_2, \dots, \tau_k)$  can be assumed to be symmetric with respect to their arguments<sup>3</sup> (Bedrosian and Rice, 1971). For notational convenience, we choose to rewrite the Volterra series (3.1) as:

$$y(t) = h_i x_{t-i} + h_{ij} x_{t-i} x_{t-j} + h_{ijk} x_{t-i} x_{t-j} x_{t-k} + \dots \quad (3.2)$$

where repeated subscripts indicate integration in time<sup>4</sup>, and successive kernels are characterized, without ambiguity, by the number of their indices (our notation implicitly takes into account the successive power of parameter  $\epsilon$ ).

### Response Power Moments

The first moment  $\mu_1$  of stochastic process  $y(t)$  results upon time averaging equation (3.2) and taking advantage of the normality of excitation process  $x(t)$ :

$$\mu_1 = h_i \overline{x_i x_i} + 3h_{ijk} \overline{x_i x_j x_k} + 15h_{ijklmn} \overline{x_i x_j x_k x_l x_m x_n} + \dots \quad (3.3)$$

where, stationarity allows omission of the  $t$ -variable without any ambiguity. The first few terms in second and third moments  $\mu_2$  and  $\mu_3$  can be obtained similarly up to the 6th order in  $\epsilon$ :

<sup>3</sup>The Volterra kernels as well as the transfer functions are defined within any permutation of their arguments. Thus symmetry ensures uniqueness.

<sup>4</sup>By analogy with the summation convention in tensorial calculus. These quantities can actually be understood as tensors.

$$\begin{aligned} \mu_2 = & h_i h_j \overline{x_i x_j} + (h_{ij} h_{kl} + 2h_{ik} h_{jl} + 6h_i h_{jkl}) \overline{x_i x_j} \overline{x_k x_l} \\ & + (9h_{ijk} h_{lmn} + 6h_{ikm} h_{jln} + 30h_i h_{jklmn} + 6h_{ij} h_{klmn} + 24h_{ik} h_{jlmn}) \overline{x_i x_j} \overline{x_k x_l} \overline{x_m x_n} + \dots \end{aligned} \quad (3.4)$$

$$\begin{aligned} \mu_3 = & (3h_i h_j h_{kl} + 6h_{ik} h_{jl}) \overline{x_i x_j} \overline{x_k x_l} \\ & + (h_{ij} h_{kl} h_{mn} + 6h_{ij} h_{km} h_{ln} + 8h_{ik} h_{jm} h_{ln} + 18h_i h_{kl} h_{jmn} \\ & + 36h_i h_{jk} h_{lmn} + 36h_i h_{km} h_{jln} + 9h_i h_j h_{klmn} + 36h_i h_k h_{jlmn}) \overline{x_i x_j} \overline{x_k x_l} \overline{x_m x_n} + \dots \end{aligned} \quad (3.5)$$

Obviously, such a straightforward approach rapidly leads to an excessive amount of algebra. Instead, we will follow Longuet-Higgins' method.

Drawing upon Longuet-Higgins (1963) reasoning, it is possible to notice that some of the terms in  $\mu_2, \mu_3$  such as:

$$\begin{aligned} h_{ij} h_{kl} \overline{x_i x_j} \overline{x_k x_l} &= (h_{ij} \overline{x_i x_j}) (h_{kl} \overline{x_k x_l}) \\ 3h_i h_j h_{kl} \overline{x_i x_j} \overline{x_k x_l} &= 3 (h_i h_j \overline{x_i x_j}) (h_{kl} \overline{x_k x_l}) \end{aligned} \quad (3.6)$$

as well as in higher order power moments can be factorized whereas some others cannot. Longuet-Higgins calls such terms irreducible, and finds it convenient to introduce a compact notation for them. Let us denote by  $[H_p H_q \dots H_r]$  the sum of all the irreducible terms among the

$$\frac{1}{\sqrt{\pi}} 2^{\frac{p+q+\dots+t}{2}} \Gamma\left(\frac{p+q+\dots+t+1}{2}\right)$$

(3.7)

terms in the mean value:

$$\overline{h_{i_1 i_2 \dots i_p} x_{i_1} x_{i_2} \dots x_{i_p} h_{j_1 j_2 \dots j_q} x_{j_1} x_{j_2} \dots x_{j_q} \dots h_{m_1 m_2 \dots m_t} x_{m_1} x_{m_2} \dots x_{m_t}}$$

(3.8)

where  $\Gamma(p)$  denotes the usual gamma function (Abramowitz and Stegun, 1970). Note that  $p+q+\dots+t$  is always even since the excitation process is zero-mean and odd power moments of this process all vanish.

Clearly, the power moments of the random process  $y(t)$  can be written in terms of  $[H_p H_q \dots H_t]$ :

$$C(i, j, \dots, m) = \frac{1}{k!} C_n^i C_{n-i}^j \dots C_{n-i-j-\dots-1}^m = \frac{1}{k!} \frac{n!}{i! j! \dots m!}$$

(3.9)

where the outer sum extends over all the values of  $p, q, \dots, t$ , and the inner sum over all the groupings  $H_p, H_q, \dots, H_t$  into unordered sets such that  $i+j+\dots+m=n$ . Whereas the multinomial coefficient  $C(i, j, \dots, m)$  denotes the number of ways of choosing such sets, and  $k$  is the number of sets:

$$C(i, j, \dots, m) = \frac{1}{k!} C_n^i C_{n-i}^j \dots C_{n-i-j-\dots-1}^m = \frac{1}{k!} \frac{n!}{i! j! \dots m!}$$

(3.10)

where  $C_r^j$  denotes the usual binomial coefficient.

### Response Cumulants

Throughout the following, we choose to describe the response process  $y(t)$  through its cumulants rather than its power moments. Such a choice results from the double advantage that cumulants are simpler and are directly related to the deviation from normality of the process they describe<sup>5</sup>.

The moment  $\mu_n$  of a random process corresponds to the  $n$ -th derivative of the moment generating function  $\phi$  (Kendall and Stuart, 1958):

$$\phi(it) = \int_{-\infty}^{+\infty} p(\zeta) e^{it\zeta} d\zeta = \sum_{n=0}^{+\infty} \mu_n \frac{(it)^n}{n!} \quad (3.11)$$

While the cumulants  $\kappa_n$  are defined by the cumulant generating function  $\psi$ :

$$\psi(it) = \ln \phi(it) = \sum_{n=1}^{+\infty} \kappa_n \frac{(it)^n}{n!} \quad (3.12)$$

Equating the coefficients of same power in  $(it)^n$  in equations (3.11) and (3.12), general relations between moments and cumulants of any order are obtained:

<sup>5</sup>A process is said to be deterministic whenever all its cumulant functions vanish except the first one, and Gaussian whenever all its cumulant functions vanish except the first- and second-order ones.

$$\mu_n = n! \sum_{i_1+i_2+\dots+i_k=n} \frac{1}{k!} \frac{\kappa_{i_1}}{i_1!} \frac{\kappa_{i_2}}{i_2!} \dots \frac{\kappa_{i_k}}{i_k!} \quad (3.13)$$

$$\kappa_n = n! \sum_{i_1+i_2+\dots+i_k=n} (-1)^{k+1} \frac{1}{k} \frac{\mu_{i_1}}{i_1!} \frac{\mu_{i_2}}{i_2!} \dots \frac{\mu_{i_k}}{i_k!} \quad (3.14)$$

Upon identification of equation (3.9) with the moment - cumulant relationship, equation (3.13), the general cumulant expression results:

$$\kappa_n = \sum_i [H_{i_1} H_{i_2} \dots H_{i_n}] \quad (3.15)$$

The added simplicity of the cumulants over the power moments now appears obvious at the light of such a general expression. Only the so-called irreducible terms appear in the cumulant expression, whereas all the terms are needed to write the power moment of same order.

It appears clearly now that Bedrosian and Rice's (1971) intuitive idea, based on the memoryless case is correct, namely that only the terms that do not separate into products of integrals in the power moments contribute to the corresponding cumulants. To a lesser degree, Rice (1944 & 1945) also appeared to have explored an idea along similar lines.

In particular, retaining all the terms up to the 10th order in  $\epsilon$ , the following expressions are obtained for the first few cumulants:

$$\begin{aligned}
\kappa_1 &= [H_2] + [H_4] + [H_6] + [H_8] + [H_{10}] + \dots \\
\kappa_2 &= [H_1^2] + [H_2^2] + 2[H_1H_3] + [H_3^2] + 2[H_1H_5] + 2[H_2H_4] + [H_4^2] + 2[H_1H_7] \\
&+ 2[H_2H_6] + 2[H_3H_5] + [H_5^2] + 2[H_1H_9] + 2[H_2H_8] + 2[H_3H_7] + 2[H_4H_6] + \dots \\
\kappa_3 &= 3[H_1^2H_2] + [H_2^3] + 6[H_1H_2H_3] + 3[H_1^2H_4] + 3[H_2H_3^2] + 3[H_2^2H_4] + 6[H_1H_2H_5] \\
&+ 6[H_1H_3H_4] + 3[H_1^2H_6] + 3[H_3^2H_4] + 3[H_2^2H_6] + 3[H_2H_4^2] + 6[H_2H_3H_5] \\
&+ 6[H_1H_2H_7] + 6[H_1H_3H_6] + 6[H_1H_4H_5] + 3[H_1^2H_8] + \dots \\
\kappa_4 &= 3[H_1^4] + 6[H_1^2H_2^2] + 4[H_1^3H_3] + [H_2^4] + 12[H_1H_2^2H_3] + 6[H_1^2H_3^2] + 12[H_1^2H_2H_4] \\
&+ 4[H_1^3H_5] + 6[H_2^2H_3^2] + 4[H_2^3H_4] + 24[H_1H_2H_3H_4] + 12[H_1H_2^2H_5] + 4[H_1H_3^3] \\
&+ 6[H_1^2H_4^2] + 12[H_1^2H_2H_6] + 12[H_1^2H_3H_5] + 4[H_1^3H_7] + \dots \\
\kappa_5 &= 5[H_1^4H_2] + 12[H_1^2H_2^3] + 20[H_1^3H_2H_3] + 5[H_1^4H_4] + [H_2^5] + 20[H_1H_2^3H_3] \\
&+ 30[H_1^2H_2H_3^2] + 32[H_1^2H_2^2H_4] + 20[H_1^3H_3H_4] + 20[H_1^3H_2H_5] + 5[H_1^4H_6] + \dots \\
\kappa_6 &= [H_1^6] + 17[H_1^4H_2^2] + 6[H_1^5H_3] + 17[H_1^2H_2^4] + 64[H_1^3H_2^2H_3] + 15[H_1^4H_3^2] \\
&+ 38[H_1^4H_2H_4] + 6[H_1^5H_5] + \dots
\end{aligned}
\tag{3.16}$$

where it appears that:

$$[H_{i_1} H_{i_2} \dots H_{i_n}] = 0 \quad \text{for } i_1 + i_2 + \dots + i_n \leq 2n - 4 \quad (3.17)$$

Such a property can be generalized to any order, keeping in mind that such terms can always be factorized since  $i_k$  is always strictly less than  $n-2$  in these quantities (Longuet-Higgins, 1963). This, in turn, leads to the, quite remarkable, property of the response cumulants of a Volterra series (3.1), namely, that the non-vanishing lowest order term in  $n$ -th order cumulant  $\kappa_n$  is of order  $\epsilon^{2n-2}$  instead of  $\epsilon^n$  as we could have reasonably guessed:

$$\kappa_n = O(\epsilon^{2n-2}) \quad (3.18)$$

This cumulant property is quite noteworthy. It further justifies the description of the response process with a set of statistical cumulants rather than moments.

Furthermore, it is interesting to mention that the general cumulant expressions (equations (3.16)) could as well have been derived from the memoryless case without any influence of the memory effects.

Assuming now a Volterra series of the type (3.1) containing only odd-order terms, we have derived the general frequency domain expressions of all the cumulants up to the 10th order in  $\epsilon$ , through Fourier transform of original time domain integrals. In order to save space, we will use notations similar to Bedrosian and Rice's (1971).  $\epsilon^2 S_{xx}(\omega) d\omega$  and  $H_n(\omega_1, \omega_2, \dots, \omega_n)$  are simply respectively written as  $(S_{xx})$  and  $(1, 2, \dots, n)$ :

$$\begin{aligned}
\kappa_2 = & \int_{-\infty}^{+\infty} (1) (-1) (S_{xx}) (S_{xx}) \\
& + \iint_{-\infty}^{+\infty} 6 (1) (-1,2,-2) (S_{xx}) (S_{xx}) \\
& + \iiint_{-\infty}^{+\infty} [30 (1) (-1,2,-2,3,-3) + 9 (1,2,-2) (-1,3,-3) \\
& + 6 (1,2,3) (-1,-2,-3)] (S_{xx}) (S_{xx}) (S_{xx}) \\
& + \iiint_{-\infty}^{+\infty} [210 (1) (-1,2,-2,3,-3,4,-4) + 120 (1,2,3) (-1,-2,-3,4,-4) \\
& + 90 (1,-1,2) (-2,3,-3,4,-4)] (S_{xx}) (S_{xx}) (S_{xx}) (S_{xx}) \\
& + \iiint_{-\infty}^{+\infty} [1890 (1) (-1,2,-2,3,-3,4,-4,5,-5) \\
& + 1260 (1,2,3) (-1,-2,-3,4,-4,5,-5) + 630 (1,-1,2) (-2,3,-3,4,-4,5,-5) \\
& + 120 (1,2,3,4,5) (-1,-2,-3,-4,-5) + 600 (1,-1,2,3,4) (-2,-3,-4,5,-5) \\
& + 225 (1,-1,2,-2,3) (-3,4,-4,5,-5)] (S_{xx}) (S_{xx}) (S_{xx}) (S_{xx}) (S_{xx}) + \dots
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
\kappa_4 = & \iiint_{-\infty}^{+\infty} 24 (1) (2) (3) (-1,-2,-3) (S_{xx}) (S_{xx}) (S_{xx}) \\
& + \iiint_{-\infty}^{+\infty} [240 (1) (2) (3) (-1,-2,-3,4,-4) \\
& + 216 (1) (2) (-1,-2,3) (-3,4,-4) + 108 (1) (2) (-1,3,4) (-2,-3,-4) \\
& + 108 (1) (2) (-1,3,-3) (-2,4,-4)] (S_{xx}) (S_{xx}) (S_{xx}) (S_{xx})
\end{aligned}$$



$$\begin{aligned}
& + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [3240 (1) (-1,2,3) (-2,-3,4) (-4,5,-5) \\
& + 9000 (1) (2) (-1,-2,3) (-3,4,-4,5,-5) \\
& + 2520 (1,2,3) (-1,-2,-3,4,-4,5,-5)] (S_{xx}) (S_{xx}) (S_{xx}) (S_{xx}) (S_{xx}) + \dots
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\kappa_6 = & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [540 (1) (2) (3) (4) (-1,-2,-3) (-4,5,-5) \\
& + 720 (1) (2) (3) (4) (5) (-1,-2,-3,-4,-5)] (S_{xx}) (S_{xx}) (S_{xx}) (S_{xx}) (S_{xx}) + \dots
\end{aligned} \tag{3.21}$$

Bedrosian and Rice (1971) derived all the terms up to the sixth order in  $\epsilon$  in the cumulants above, as well as odd-order ones. It is believed that it is the farthest anyone carried out these expansions until these innovative results.

The particular case of a nonlinear system governed by the Duffing equation will serve to exemplify these new results in the last section of this chapter.

## The Linearize-and-Match Model

Although the general response cumulant expression of a Volterra functional system is now available, the usefulness of these expansions remains limited to the first few terms by the excessive complexity of both high order transfer functions and resulting multiple integrals. In most applications to the determination of the response of

nonlinear dynamic systems to stochastic excitation, perturbation and other general functional methods are not generally carried further than the first nonlinear term.

Moreover, it will be shown that the improvement provided by a two- or a three-term perturbation method only covers a region where the perturbation parameter is so "small" that the linear and the equivalent linear approximations are already quite accurate enough from a practical point of view (Morton and Corrsin, 1970 and Duthoit and Armand, 1987). This is believed to be true for most "inverse" nonlinear problems such as those described by a functional relationship of the type (1.1).

The main idea underlying the following developments is to provide an improved theory over the Volterra functional series model in deriving the response statistics (e.g. moments and cumulants) of autonomous nonlinear systems driven by Gaussian stochastic signals. Rather than expressing these statistics by a truncated series the convergence and accuracy of which remain questionable, closed-form approximate solutions are developed.

These statistics will eventually be used to evaluate response probability distributions (Chapter 4).

In order to illustrate this technique without involving any excessive amount of algebra, let us consider the particular case of the Duffing equation with linear plus cubic damping. Considerations regarding more general nonlinear systems are discussed in Appendix C.

## Linearizing

Let us consider the nonlinear system (S) described by the following differential equation, written in dimensionless form:

$$\ddot{y} + 2\xi\dot{y} + \Xi y^3 + y + y^3 = x \quad (3.22)$$

in which the excitation  $x(t)$  is assumed to be a zero-mean, stationary, ergodic, and Gaussian random process with double-sided frequency spectrum  $S_{xx}(\omega)$ .

The following "equivalent" linear systems ( $S_n$ ) are defined next in substituting terms proportional to their mathematical expectations to nonlinear terms in original system (S):

$$\ddot{y}_n + (2\xi + \Xi\beta_n E[y_n^2]) \dot{y}_n + (1 + \alpha_n E[y_n^2]) y_n = x(t) \quad (n \in N^*) \quad (3.23)$$

where successive parameters  $\alpha_n$  and  $\beta_n$  will be chosen in order to provide an accurate linear approximation to the output 2nth-order moment  $E[y^{2n}]$  of the original nonlinear system (S).

Clearly, for  $n = 1$ , ( $S_1$ ) corresponds to the equivalent linear system (equivalent in the mean square sense) as defined by Caughey (1963b).

The actual form of "equivalent" linear systems ( $S_n$ ) (equations (3.23)) is suggested by the form of the original differential equation describing the input-output relationship of system (S) (equation (3.22)).

Although the linearizing procedure is largely heuristic, and does not proceed from any formal, mathematical arguments, it is the simplest one which account for nonlinearities. Furthermore, choosing any other dimension preserving linearized quantity of the type  $E[y_n^{2k}]/E[y_n^{2k-2}]$  or any combination of these leads to identical linear systems  $(S_n)$  in this example, owing to the normal properties of stochastic process  $y_n$ .

It is important to emphasize that the proposed method is not restricted to systems of the type (3.22). In fact, a much more general class of differential equations and even integral or integro-differential equations can be handled within the framework of this technique (Appendix C).

Since each system  $(S_n)$  is linear, the corresponding output mean squares (or variances)  $E[y_n^2]$  and  $E[\dot{y}_n^2]$  are simply given by:

$$E[y_n^2] = \int_{-\infty}^{+\infty} |H_1^n(\omega)|^2 S_{xx}(\omega) d\omega \quad (n \in N^*) \quad (3.24)$$

$$E[\dot{y}_n^2] = \int_{-\infty}^{+\infty} \omega^2 |H_1^n(\omega)|^2 S_{xx}(\omega) d\omega \quad (n \in N^*) \quad (3.25)$$

where

$$H_1^n(\omega) = \frac{1}{-\omega^2 + i\omega(2\xi + \Xi\beta_n E[\dot{y}_n^2]) + (1 + \alpha_n E[y_n^2])} \quad (3.26)$$

is system ( $S_n$ ) transfer function. Clearly, (3.24) and (3.25) are coupled integral equations of system ( $S_n$ ) output variance and output time derivative variance.

Since system (S) only involves terms of odd order<sup>6</sup>, all output moments of odd order are identically zero, while output moments  $\mu_{2n}$  of even order are approximately obtained from the systems ( $S_n$ ) considered separately<sup>7</sup>:

$$\tilde{\mu}_{2n} = E[y_n^{2n}] \quad (n \in N^*) \quad (3.27)$$

Keeping in mind that  $y_n$  is the output of an autonomous linear system driven by a Gaussian, stationary and ergodic random process, it is possible to prove the following set of equalities:

$$E[y_n^{2n}] = \frac{1}{\sqrt{\pi}} 2^n \Gamma\left(n + \frac{1}{2}\right) E^n[y_n^2] \quad (3.28)$$

where  $\Gamma$  denotes the gamma function. The output moments  $\mu_{2n}$  are thus now given in terms of the output mean square of system ( $S_n$ ).

<sup>6</sup>The proposed technique is, in this form, restricted to analytic systems involving odd order terms only. The generalization to any analytic system is discussed in Appendix C.

<sup>7</sup>Tilde superscript indicates approximated moments or cumulants as predicted by the proposed technique.

Throughout the following, we will be more concerned with the cumulants  $\kappa_{2n}$  of the response of system (S), than with its moments  $\mu_{2n}$ . As already observed through the previous section, cumulants are simpler, are directly related to deviation from normality, and possess specific properties related to the Volterra functional model.

Thus, upon substitution of equations (3.27) and (3.28) into the cumulant-moment relationship (3.14), general approximate expressions for output cumulants result:

$$\tilde{\kappa}_{2n} = (2n)! \sum_{i_1+i_2+\dots+i_k=n} (-1)^{k+1} \frac{1}{k} \frac{E^{i_1}[y_1^2]}{(2^{i_1} i_1)!} \frac{E^{i_2}[y_2^2]}{(2^{i_2} i_2)!} \dots \frac{E^{i_k}[y_k^2]}{(2^{i_k} i_k)!} \quad (3.29)$$

where the sum extends over all the indicial combinations such that  $i_1+i_2+\dots+i_k=n$ . And the only unknowns are the  $\alpha_n$  and  $\beta_n$  coefficients which appear in  $E[y_n^2]$  through equations (3.24) and (3.25). For example, the first few cumulants are simply obtained by substituting  $n=1, 2, 3$  and  $4$ :

$$\begin{aligned} \tilde{\kappa}_2 &= E[y_1^2] \\ \tilde{\kappa}_4 &= 3E^2[y_2^2] - 3E^2[y_1^2] \\ \tilde{\kappa}_6 &= 15E^3[y_3^2] - 45E[y_1^2]E^2[y_2^2] + 30E^3[y_1^2] \\ \tilde{\kappa}_8 &= 105E^4[y_4^2] - 630E^4[y_1^2] - 315E^4[y_2^2] + 1260E^2[y_1^2]E^2[y_2^2] - 420E[y_1^2]E^3[y_3^2] \end{aligned} \quad (3.30)$$

## Matching

So far, expressions (3.27) and (3.29) for the output moments and cumulants would not provide a significant improvement over other existing techniques if we were unable to determine coefficients  $\alpha_n$  and  $\beta_n$ . Assuming that equations (3.27) and (3.29) represent uniformly valid approximate expressions of the output statistics of system (S) over some practical dynamic domain, some additional information, not provided by the linear theory, is needed in order to completely determine these approximate response statistics.

The main idea underlying the determination of the  $\alpha_n$  and  $\beta_n$  coefficients consists in matching the general cumulant expressions (3.29) with some "inner" solution obtained through an expansion procedure in terms of a small parameter  $\epsilon$  describing the magnitude of the response  $y(t)$ . Volterra functional series can be used (Volterra, 1930 and Wiener, 1958) to perform this expansion, though it is actually identical to a regular perturbation<sup>8</sup> "with memory".

In order to illustrate the matching procedure, an application to the case of a nonlinear system, the input-output relationship of which is described by equation (3.22), is provided. The response  $y(t)$  is assumed to be given by the Volterra series (Bedrosian and Rice, 1971):

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<sup>8</sup>The Volterra functional series model essentially consists in the Taylor series expansion, with memory, of an input-output relationship. Therefore, it does not provide anything more than what can otherwise be obtained through a regular expansion technique. The real advantage of the Volterra series model lies in the rather advanced body of results related to its application. Furthermore, the Volterra series approach has the virtue that many such problems can be treated by first computing the necessary transfer functions and then substituting those in the appropriate general formulas (e.g. Bedrosian and Rice, 1971, Chua and Ng, 1979, or in the present case equations (3.15)).

$$y(t) = \varepsilon \int_{-\infty}^{+\infty} h_1(t-\tau) x(\tau) d\tau + \varepsilon^3 \iiint_{-\infty}^{+\infty} h_3(t-\tau_1, t-\tau_2, t-\tau_3) x(\tau_1) x(\tau_2) x(\tau_3) d\tau_1 d\tau_2 d\tau_3 + \dots \quad (3.31)$$

Where the small parameter  $\varepsilon$  is introduced essentially to clarify the expansion and the matching procedures.

A 3rd order Volterra functional series yields cumulants  $\kappa_{2n}$ ,  $n \leq 2$ , which, in turn, lead to the determination of unknown coefficients  $\alpha_1$ ,  $\alpha_2$  and  $\beta_1$ ,  $\beta_2$ .

Clearly, for a Gaussian excitation the response 2nd order cumulant  $\kappa_2$  is obtained by integrating the output power density spectrum at this order (see equation (3.19) and the transfer functions calculated in Appendix A):

$$\kappa_2 = \left\{ \varepsilon^2 - 6\varepsilon^4 \int_{-\infty}^{+\infty} [\operatorname{Re}\{H_1(\omega)\} - \Xi\omega\mathcal{L}\{H_1(\omega)\}] |H_1(\omega)|^2 S_{xx}(\omega) d\omega \right. \\ \left. \int_{-\infty}^{+\infty} |H_1(\omega)|^2 S_{xx}(\omega) d\omega + \dots \right\} \quad (3.32)$$

where  $H_1(\omega)$  is the linear frequency response function of system (S), and

$$\sigma_{y \text{ linear}}^2 = \varepsilon^2 \int_{-\infty}^{+\infty} |H_1(\omega)|^2 S_{xx}(\omega) d\omega \quad (3.33)$$

is the linear response variance.



We now turn to the 4th order cumulant evaluated through equation (3.20) together with the transfer functions evaluated in Appendix A:

$$\kappa_4 = -24\varepsilon^6 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\Re\{H_1(\omega_1 + \omega_2 + \omega_3)\} + \Xi\omega_1\omega_2\omega_3 \Im\{H_1(\omega_1 + \omega_2 + \omega_3)\}] |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2 S_{xx}(\omega_1) S_{xx}(\omega_2) S_{xx}(\omega_3) d\omega_1 d\omega_2 d\omega_3 + \dots \quad (3.34)$$

Coefficients  $\alpha_n$  and  $\beta_n$ ,  $n=1,2$  are obtained identifying terms of same order in  $\varepsilon$  in equations (3.29) with equations (3.32) and (3.34):

$$\alpha_1 = 3 \quad \beta_1 = 3$$

$$\alpha_2 = 3 + 2 \frac{J_1}{\sigma_{y \text{ linear}}^4 I_1} \quad \beta_2 = 3 + 2 \frac{J_2}{\sigma_{y \text{ linear}}^2 \sigma_{y \text{ linear}}^2 I_2} \quad (3.35)$$

where  $I_1$ ,  $I_2$ ,  $J_1$  and  $J_2$  are defined in Appendix B.

One reason to describe the response through its cumulants rather than its moments lies in the fact that the lowest order terms of the output cumulant  $\kappa_n$  of a Volterra series is of order  $\varepsilon^{2n-2}$ ,  $n \geq 2$  (see equation (3.18)). Taking advantage of this property, all coefficients  $\alpha_n$  and  $\beta_n$ ,  $n \geq 3$ , can be determined in eliminating terms of order lower than  $\varepsilon^{2(2n-1)}$  in cumulant  $\kappa_{2n}$  (equation (3.29)):

$$\alpha_n = 3 + 2(n-1) \frac{J_1}{\sigma_{y \text{ linear}}^4 I_1} \quad \beta_n = 3 + 2(n-1) \frac{J_2}{\sigma_{y \text{ linear}}^2 \sigma_{y \text{ linear}}^2 I_2} \quad (3.36)$$

Finally, the first few output moments (equations (3.27)) and cumulants (equations (3.29)) may be obtained by substituting above coefficients  $\alpha_n$  and  $\beta_n$  in equations (3.24) and (3.25), and formally solving these integral equations. It is pertinent to mention that these integral equations are apparently well-behaved. They can be solved through a rather simple iteration scheme involving the linear values as initial conditions. Convergence is consistently achieved after a few iterations.

Although we have considered the particular case of a nonlinear system described by the Duffing equation, the linearize-and-match method is applicable to a quite more general variety of nonlinear transformations. Essentially, the method applies to any type of analytic<sup>9</sup> system driven by general stochastic processes. The case of asymmetric nonlinear systems the excitation of which do not possess a zero mean is exposed in Appendix C. Coupled multiple degree of freedom systems can similarly be treated within the general framework of this technique provided that they can be expanded in Volterra series.

Non-Gaussian excitation processes can as well be treated within the general framework of this method, provided that such processes can be reasonably approximated as the response of an auxiliary Volterra system driven by a Gaussian process. Original system (S) is thereby enlarged by such an auxiliary system.

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<sup>9</sup>I.e. nonlinear systems which can be expanded in Volterra functional series.

Two particular cases involving the Duffing equation will serve to assess the accuracy and the possibilities of the method.

### **The Duffing Equation**

The response cumulants resulting from the linearize-and-match method can be compared with the exact ones as well as with other approximations available, whenever exact methods are possible.

In the case  $\Xi = 0$  (linear damping only), equation (3.22) coincides exactly with the so-called Duffing equation (equation (2.20)). There exist at least two situations where the response moments and cumulants of the Duffing equation can be determined exactly.

The first of these is the case of the idealized white noise excitation, and the latter is the case of a narrow-banded excitation centered at the origin (static case). In both cases, the validity and the possibilities of our method has been assessed through comparison with the exact solution and some other methods.

#### **White Noise Excitation**

In the case of the Duffing oscillator excited by a Gaussian white noise, the response probability density function is obtained as the stationary solution of an appropriate Fokker-Planck-Kolmogorov equation (Caughey, 1963a):

$$p(y) = \exp\left(-\lambda_0 - \frac{y^2}{2\sigma_{y \text{ linear}}^2} - \frac{y^4}{4\sigma_{y \text{ linear}}^2}\right) \quad (3.37)$$

where  $\lambda_0$  is a normalizing constant.

The response power moments as well as cumulants can easily be obtained exactly from this distribution. On the other hand, the Volterra functional model can lead to a series expansion in terms of the "small" parameter  $\sigma_{y \text{ linear}}$  of these cumulants from the knowledge of the transfer functions (Appendix A) and residue integration in the frequency domain of equations (3.19), (3.20) and (3.21). Keeping all the terms up to the 10th order in this parameter, we obtain:

$$\begin{aligned} \kappa_2 &= \sigma_{y \text{ linear}}^2 - 3\sigma_{y \text{ linear}}^4 + 24\sigma_{y \text{ linear}}^6 - 297\sigma_{y \text{ linear}}^8 + 4896\sigma_{y \text{ linear}}^{10} + \dots \\ \kappa_4 &= -6\sigma_{y \text{ linear}}^6 + 126\sigma_{y \text{ linear}}^8 - 2682\sigma_{y \text{ linear}}^{10} + \dots \\ \kappa_6 &= 360\sigma_{y \text{ linear}}^{10} + \dots \end{aligned} \quad (3.38)$$

We should however mention frankly that only the terms up to the 6th order could actually be derived from the Volterra functional model<sup>10</sup>, because of the excessive complexity resulting from higher order transfer functions (Appendix A). The remaining terms can be obtained directly from the exact probability distribution

<sup>10</sup>The 6-th order term in  $\kappa_2$  is in contradiction with Smith's (1977 & 1980) assertion. Namely that, motivated by the fact that some integrals cannot be evaluated analytically, for most cases this 6-th order term is approximately equal to  $18\sigma_{y \text{ linear}}^6$  instead of its exact analytical value  $24\sigma_{y \text{ linear}}^6$ !

(equation (3.37)), through an appropriate change of variable and Taylor series expansion in terms of small parameter  $\sigma_{y \text{ linear}}$

Finally, the linearize-and-match method, as described throughout the previous section, yields the following approximated expressions for the first few even order response cumulants:

$$\begin{aligned}\tilde{\kappa}_2 &= \frac{1}{6} (\sqrt{1 + 12\sigma_{y \text{ linear}}^2} - 1) \\ \tilde{\kappa}_4 &= \frac{3}{64} (\sqrt{1 + 16\sigma_{y \text{ linear}}^2} - 1)^2 - \frac{1}{12} (\sqrt{1 + 12\sigma_{y \text{ linear}}^2} - 1)^2 \\ \tilde{\kappa}_6 &= \frac{3}{200} (\sqrt{1 + 20\sigma_{y \text{ linear}}^2} - 1)^3 + \frac{5}{36} (\sqrt{1 + 12\sigma_{y \text{ linear}}^2} - 1)^3 \\ &\quad - \frac{45}{384} (\sqrt{1 + 16\sigma_{y \text{ linear}}^2} - 1)^2 (\sqrt{1 + 12\sigma_{y \text{ linear}}^2} - 1)\end{aligned}\tag{3.39}$$

Expanding in terms of the small parameter  $\sigma_{y \text{ linear}}$  we obtain:

$$\begin{aligned}\tilde{\kappa}_2 &= \sigma_{y \text{ linear}}^2 - 3\sigma_{y \text{ linear}}^4 + 18\sigma_{y \text{ linear}}^6 - 135\sigma_{y \text{ linear}}^8 + 1134\sigma_{y \text{ linear}}^{10} + \dots \\ \tilde{\kappa}_4 &= -6\sigma_{y \text{ linear}}^6 + 105\sigma_{y \text{ linear}}^8 - 1554\sigma_{y \text{ linear}}^{10} + \dots \\ \tilde{\kappa}_6 &= 330\sigma_{y \text{ linear}}^{10} + \dots\end{aligned}\tag{3.40}$$

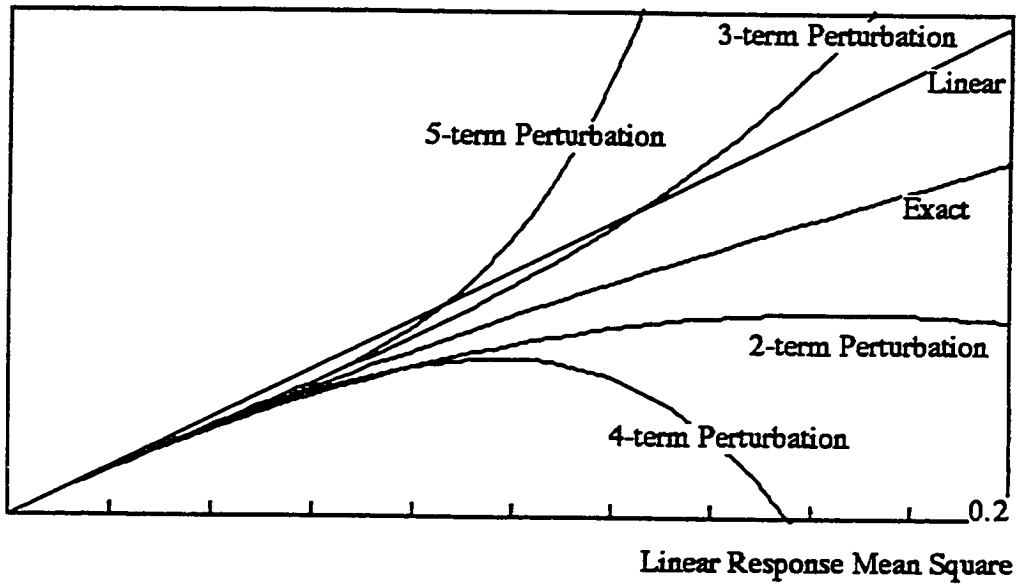


Figure 3.1. Response 2nd Order Cumulant, Kappa-2.

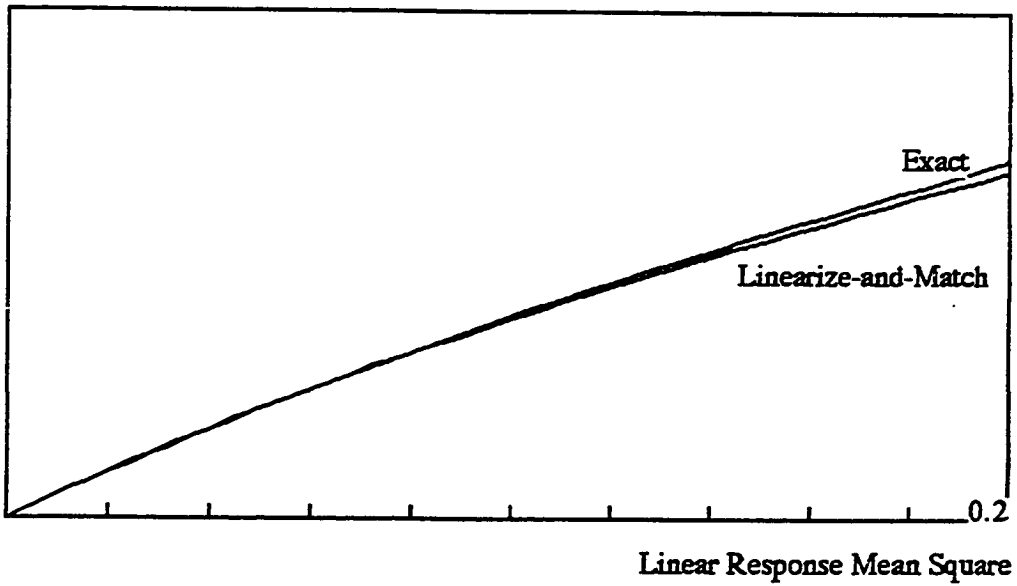


Figure 3.2. Response 2nd Order Cumulant, Kappa-2.

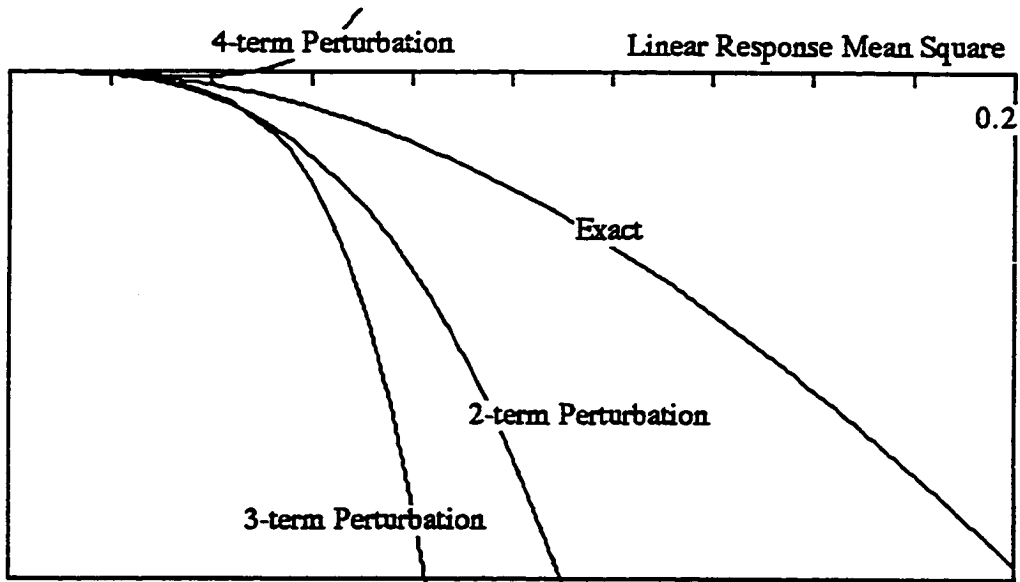


Figure 3.3. Response 4th Order Cumulant, Kappa-4.

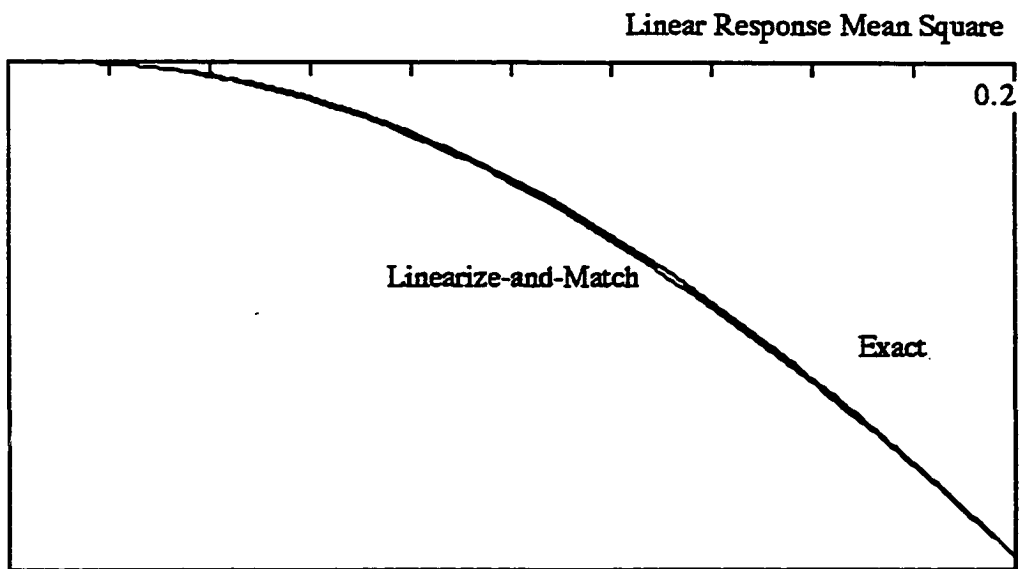


Figure 3.4. Response 4th Order Cumulant, Kappa-4.

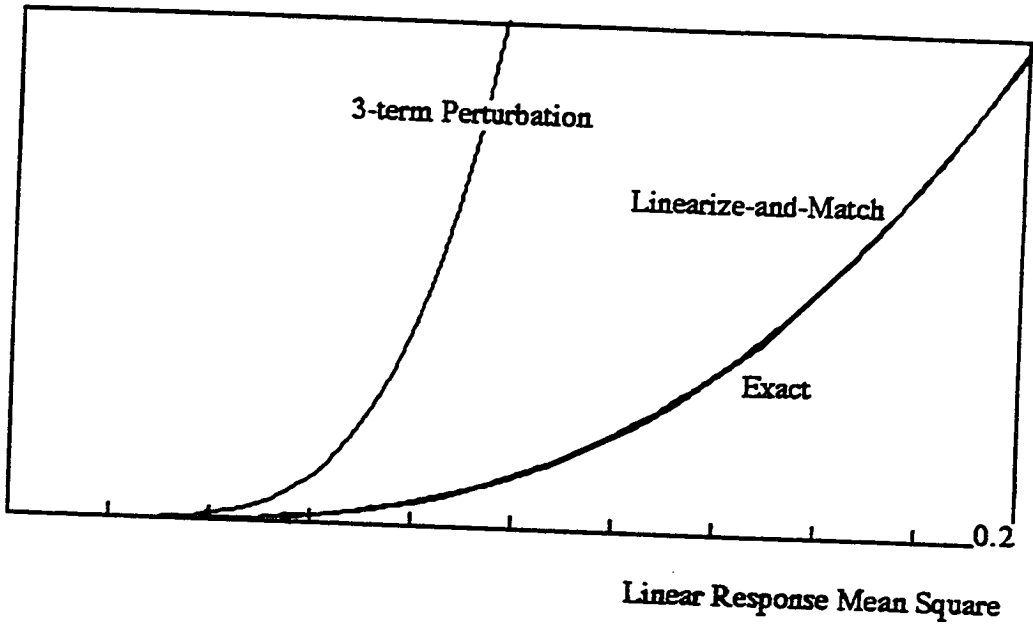


Figure 3.5. Response 6th Order Cumulant, Kappa-6.

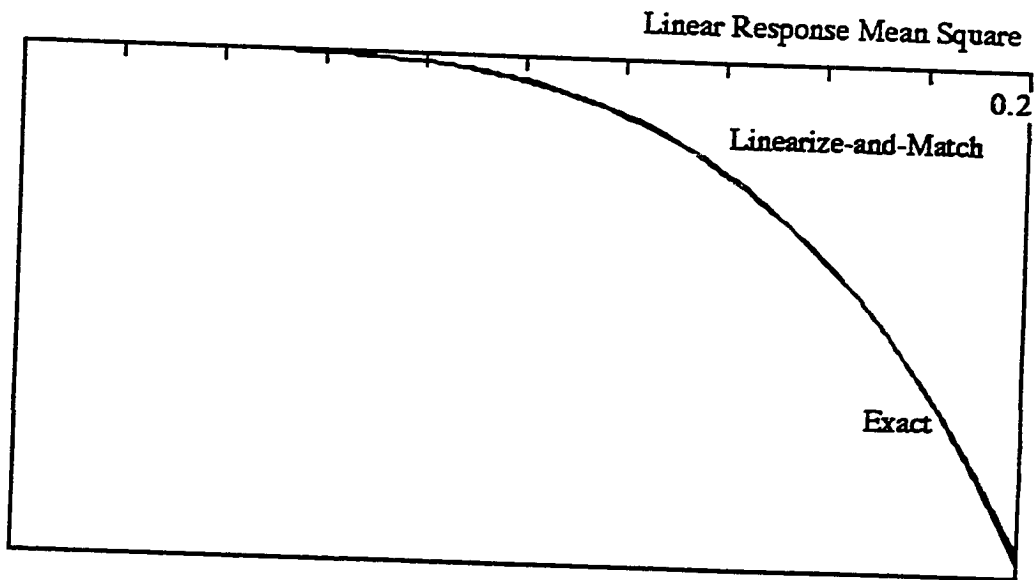


Figure 3.6. Response 8th Order Cumulant, Kappa-8.



So that the predicted response cumulants  $\kappa_{2n}$  are exact up to the  $2(n+1)$ th order in this parameter, as expected from the matching procedure.

Figures 3.1 to 3.6 show the various approximations of response cumulants  $\kappa_2$ ,  $\kappa_4$ ,  $\kappa_6$  and  $\kappa_8$  as function of parameter  $\sigma_{y \text{ linear}}$  together with the exact one, as given by equation (3.37).

Clearly, the linearize-and-match method consistently yields, by far, the best agreement with the exact cumulants. Whereas the Volterra functional model leads to comparatively poor estimates for the levels of nonlinearity shown. Furthermore, the Volterra functional model results in such an amount of computational effort that evaluating terms beyond the 6th order in  $\sigma_{y \text{ linear}}$  would be quite cumbersome if not practically impossible. This difficulty is related to both higher order transfer functions and multiple integral computation. Finally, the linear and equivalent linear approximations lead to vanishing 4th and higher order cumulants, simply because they necessarily predict Gaussian response processes.

Furthermore, the linearize-and-match does provide a relatively simple and accurate technique for predicting such response statistics.

### **Behavior at Infinity**

Since we can expect the worst accuracy of our method whenever departure from linear behavior is the most significant, it is interesting to examine the predicted power moments when nonlinear effects should be the most significant i.e. whenever:

$$\sigma_{y \text{ linear}} \rightarrow \infty$$

(3.41)

The first term of the asymptotic expansion of the exact power moments as obtained from the response probability density, equation (3.37) is:

$$\mu_{2n} \approx 2^n \sigma_{y \text{ linear}}^n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

(3.42)

Similarly, the first term of the asymptotic expansion of the predicted response power moments is:

$$\tilde{\mu}_{2n} \approx 2^n \sigma_{y \text{ linear}}^n \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi} (n+2)^{\frac{n}{2}}}$$

(3.43)

It follows immediately from these expressions that the exact and predicted power moments are of the same order of magnitude in  $\sigma_{y \text{ linear}}$ , which guarantees the fact that the resulting relative error:

	$\kappa_2$	$\kappa_4$	$\kappa_6$	$\kappa_8$
Linear	$\sigma^2$	0	0	0
Eq. Linearization	$0.52 \sigma$	0	0	0
Linearize-Match	$0.52 \sigma$	$-0.25 \sigma^2$	$1.77 \sigma^3$	$-22.1 \sigma^4$
Exact Solution	$0.68 \sigma$	$-0.37 \sigma^2$	$1.15 \sigma^3$	$-8.01 \sigma^4$

Table 3.1. Behavior at infinity.

$$\frac{\mu_{2n} - \tilde{\mu}_{2n}}{\mu_{2n}} \approx 1 - 2^{-n} \frac{\Gamma(\frac{1}{4}) \Gamma(n + \frac{1}{2})}{\pi (n + 2)^{\frac{n}{2}}} \quad (3.44)$$

will always remain bounded.

The behavior of the first few cumulants in the limit as  $\sigma_{y \text{ linear}}$  tends to infinity can be obtained from equations (3.42) and (3.43). Table 3.1 ( $\sigma$  denotes the perturbation parameter  $\sigma_{y \text{ linear}}$ ) compares the different approximations with the exact solution. The cumulants resulting from the linearize-and-match approximation are clearly the most accurate.

### Static Approximation

The second application is the case of the Duffing equation excited by an idealized Gaussian narrow-band process centered at the origin. This case corresponds to a pure static configuration.

The response probability density function is exactly:

$$p(y) = \frac{1}{\sqrt{2\pi} \sigma_{y \text{ linear}}} (1 + 3y^2) \exp\left(-\frac{(y + y^3)^2}{2\sigma_{y \text{ linear}}^2}\right) \quad (3.45)$$

Since this nonlinear transformation is static, the response statistical cumulants can be expanded in power series directly from the equation of motion, without difficulty:

$$\begin{aligned}
\kappa_2 &= \sigma_{y \text{ linear}}^2 - 6\sigma_{y \text{ linear}}^4 + 105\sigma_{y \text{ linear}}^6 - 3150\sigma_{y \text{ linear}}^8 + 135135\sigma_{y \text{ linear}}^{10} + \dots \\
\kappa_4 &= -24\sigma_{y \text{ linear}}^6 + 1152\sigma_{y \text{ linear}}^8 - 60480\sigma_{y \text{ linear}}^{10} + \dots \\
\kappa_6 &= 6750\sigma_{y \text{ linear}}^{10} + \dots
\end{aligned}
\tag{3.46}$$

The linearize-and-match method yields the response power moments as solution of a polynomial, and the  $\alpha$ -coefficients are:

$$\alpha_n = 1 + 2n
\tag{3.47}$$

The behavior of cumulants  $\kappa_2$ ,  $\kappa_4$ ,  $\kappa_6$  and  $\kappa_8$ , as functions of the parameter describing the magnitude of the nonlinearity are not shown. But observations similar to the white noise case can be drawn. Again, the linearize-and-match method yields the most accurate estimates.

## CHAPTER 4

### Probability Distributions

We have shown, through Chapters 2 and 3, how the response statistics of a broad variety of nonlinear systems can be predicted rather accurately. Although such cumulants and moments represent significant statistical quantities to describe stochastic processes, their probability distributions<sup>1</sup> constitute an essential body of information in engineering design. They, in turn, may lead to the useful notions of system reliability and probability of failure.

Whereas the methods based on functional representation techniques yield response statistics, the prediction of response probability distributions belongs, for the most part, to the realm of methods drawing upon the theory of Markov processes. These methods are however limited in scope. Instead, we will explore another avenue consisting in the construction of these distributions from appropriate response statistics.

A stochastic process  $y(t)$  can be described, in increasing order of completeness, by the set of probability density functions (Ming Chen Wang and Uhlenbeck, 1945):

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<sup>1</sup>As it is customary, probability distributions of a stochastic process denote, loosely speaking, both density functions and distributions, as well as any other set of functions related to the probabilistic description of such a process.

$$\begin{aligned}
p(y_1, \tau_1) dy_1 &= \Pr\{y_1 \leq y(\tau_1) \leq y_1 + dy_1\} \\
p(y_1, \tau_1; y_2, \tau_2) dy_1 dy_2 &= \Pr\{y_1 \leq y(\tau_1) \leq y_1 + dy_1 \text{ and } y_2 \leq y(\tau_2) \leq y_2 + dy_2\} \\
&\dots \\
p(y_1, \tau_1; \dots; y_n, \tau_n) dy_1 \dots dy_n &= \Pr\{y_1 \leq y(\tau_1) \leq y_1 + dy_1 \dots y_n \leq y(\tau_n) \leq y_n + dy_n\} \\
&\dots
\end{aligned}
\tag{4.1}$$

These distributions must fulfill some obvious conditions. They must be positive, normalized and satisfy the compatibility conditions:

$$p(y_1, \tau_1; \dots; y_n, \tau_n) = \iint \dots \int p(y_1, \tau_1; \dots; y_k, \tau_k) dy_{n+1} \dots dy_k \quad n < k
\tag{4.2}$$

We will show how such probability distributions can be constructed from knowledge of their statistical moments or cumulants, even when a finite number of those are known.

Although a complete set of such distributions can be constructed similarly, we will only consider the first order one  $p(y)$  in the case of a stationary stochastic process  $y$ . Several techniques for reconstructing these distributions will be described and compared through various mathematical idealizations of the rigid body behavior of marine vehicles.

## The Underdetermined Moment Problem

In the classical problem of moments, a probability density function  $p(y)$  is to be determined from knowledge of its power moments:

$$\mu_n = \int_{-\infty}^{+\infty} y^n p(y) dy \quad (4.3)$$

The extent to which the density  $p(y)$  may be uniquely determined from its moments has been discussed through the literature. Briefly, expanding the moment generating function of the probability density  $p$  near the origin (equation (3.11)), it can be shown that the density of a random variable is uniquely determined by its power moments if all moments are finite and the series converges absolutely in some neighborhood of the origin (Papoulis, 1984).

In practice, only a finite number  $N$  of these moments are known<sup>2</sup>. Clearly then there exists an infinite number of density  $p(y)$ , the moments of which coincide with the  $N$  prior power moments.

The construction of a probability density from knowledge of a finite number of its power moments can thus be understood as an underdetermined moment problem. Nevertheless, a number of approximation procedures exists which aim at reconstructing such a function  $p(y)$ .

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<sup>2</sup>The normalizing moment  $\mu_0=1$  is implicitly included in these  $N$  moments.



## Edgeworth Series Distribution

One such technique involves the expansion of  $p(y)$  in a set of orthogonal polynomials, the first  $N$  power moments of which are required to be correct. This implies the solution of an  $N \times N$  linear system. In practice, the choice of a suitable set of orthogonal polynomials is difficult, so the resulting sequences may exhibit oscillating approximations to  $p(y)$  which are further impaired by lack of positivity of the truncated series (e.g. Ochi, 1986).

When Hermite polynomials are used,  $p(y)$  is expressed as the usual Gram-Charlier series in the case of moment constraints (Cramer, 1945, Kendall and Stuart, 1958). If cumulants are used instead of moments, the Edgeworth series results (Edgeworth, 1904).

Although any set of statistical moments and cumulants up to the same order contain precisely the same amount of information, the Gram-Charlier and Edgeworth series do not coincide. In other words, different ways of taking the same information into account lead to different distributions. This is yet another drawback of the truncation procedure.

One important property of the Edgeworth series is that it represents a non-linear perturbation of a linear problem the probability density of which is Gaussian. Clearly, unknown cumulants, which are directly related to the deviation of  $p(y)$  from normality, are automatically set equal to zero. Such probability distributions have been used, for some time, in connection with ocean engineering problems (e.g. Longuet-Higgins, 1963 and Ochi, 1986).

Edgeworth (1904) developed an asymptotic series probability density function for the sum of random variables related to the law of error. In practice, the series is truncated assuming slightly non Gaussian random variables. The first few terms of the Edgeworth series distribution of a random variable are given by<sup>3</sup> (see e.g. Abramowitz and Stegun, 1970):

$$\begin{aligned}
 p(u) = & Z(u) - \left[\frac{\gamma_1}{3!} Z^{(3)}(u)\right] + \left[\frac{\gamma_2}{4!} Z^{(4)}(u) + \frac{\gamma_1^2}{72} Z^{(6)}(u)\right] \\
 & - \left[\frac{\gamma_3}{5!} Z^{(5)}(u) + \frac{\gamma_1\gamma_2}{144} Z^{(7)}(u) + \frac{\gamma_1^3}{1296} Z^{(9)}(u)\right] \\
 & + \left[\frac{\gamma_4}{6!} Z^{(6)}(u) + \left(\frac{\gamma_2^2}{1152} + \frac{\gamma_1\gamma_3}{720}\right) Z^{(8)}(u) + \frac{\gamma_1^2\gamma_2}{1728} Z^{(10)}(u) + \frac{\gamma_1^4}{31104} Z^{(12)}(u)\right] + \dots
 \end{aligned}
 \tag{4.4}$$

where:

$$\begin{aligned}
 u &= \frac{y}{\sqrt{\kappa_2}} & \gamma_{n-1} &= \frac{\kappa_n}{\sqrt{\kappa_2^n}} \\
 Z(u) &= \frac{1}{\sqrt{2\pi\kappa_2}} \exp\left(-\frac{u^2}{2}\right) & Z^{(n)}(u) &= \frac{dZ(u)}{du} = He_n(u) Z(u)
 \end{aligned}
 \tag{4.5}$$

and  $He_n(u)$  denotes the usual Hermite polynomial of order  $n$ .

<sup>3</sup>Note that the advantage of the quasi-moments is that they directly yield the coefficients of the Hermite polynomial expansion (Kuznetsov, Stratonovitch and Tikhonov, 1951).

## Distribution of Maximum Entropy

The maximum entropy approach offers yet another procedure for the construction of a positive density  $p(y)$ .

### Background

Although the distribution of maximum entropy constitutes a relatively recent and still unused model, when compared with the more conventional expansions in a set of orthogonal polynomials, its scientific origins are quite old.

**Entropy and Thermodynamics.** The physical concept of entropy originated with the work of Clausius in 1850, and is intimately linked to the essence of the second law of thermodynamics.

While the first law of thermodynamics does not place any restriction on the direction of the process involved, the second law draws its historical importance as a law of nature allowing energy transfer to occur spontaneously only in certain preferred directions. Mathematically, this limitation is expressed as an inequality<sup>4</sup> stating that the internal entropy production (a state function related to the heat transfer) is never negative.

Clearly, entropy is a quantity that remains constant in any reversible process, while it must increase in any irreversible process.

**Entropy and Probability.** The probabilistic interpretation of entropy, in the context of statistical mechanics, is attributed to Boltzmann (1877). However, the

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<sup>4</sup>The Clausius-Duhem inequality.

explicit relationship between entropy and probability did not appear before 1906 with Planck.

In statistical mechanics, the entropy of a state is directly related to the probability of occurrence of that state among all the possible states that could occur. And it is found that changes of state are more likely to occur in the direction of greater disorder when a system is left to itself. Thus, increasing entropy is associated with increasing disorder. The second law of thermodynamics implies an almost natural preference for situations of disorder.

In other words, entropy is a measure of randomness of the organisation of a system, and this measure can only increase as the system moves into state space regions of greater probability.

The much celebrated work of Shannon and Weaver (1949) laid the foundations of modern information theory and led to Jaynes's principle of maximum entropy (1957a & 1957b) and Kullback's principle of minimum cross-entropy (1959).

The method of maximum entropy has been applied to a variety of problems involving the determination of probability distributions from incomplete data (Jaynes, 1957a & 1957b, Agmon, Alhassid and Levine, 1979, Levine, 1980 and Mead and Papanicolaou, 1984).

Historically, the entropy functional  $S\{p\}$  was derived from a number of postulates based on our heuristic understanding of uncertainty and information (Shannon and Weaver, 1949 and Papoulis, 1984). It can be shown that the discrete version of the sum:

$$\mathcal{S}\{p\} = - \int_{-\infty}^{+\infty} p(y) \ln p(y) dy$$

(4.6)

satisfies these postulates and is unique within a constant factor (see e.g. Papoulis, 1984).

### Rationale

Until now we have not exhibited any "hard" evidence that the principle of maximum entropy would actually provide any worse or better an approximation of an unknown probability function than any other available technique.

Intuitively, the distributions of higher entropy represent more "disorder," they are "smoother," "more probable," and they "assume less" (Jaynes, 1982). Nevertheless, one may wonder what exactly are we doing in maximizing entropy. Keeping in mind that it is generally possible to get an accurate prediction of any density provided that a sufficient number of its power moments are known.

At least two kinds of justifications can be provided, the former is based on the entropy concentration theorem (Jaynes, 1982), while the latter relies on an axiomatic derivation of the principle of maximum entropy (Shore and Johnson, 1980).

**Entropy Concentration theorem.** Let us consider a random experiment which has  $m$  possible results for each of the  $M$  trials. Each of the  $n^M$  outcomes yields a set of frequencies  $\{p_i, i \in [1,m]\}$ . The quantity  $\Delta\mathcal{S}$  is defined next:

$$\mathcal{S}_{\max} - \Delta\mathcal{S} \leq \mathcal{S}(p_1, p_2, \dots, p_m) \leq \mathcal{S}_{\max} \quad (4.7)$$

Then, it is possible to show, using combinatorial analysis, that  $2M\Delta\mathcal{S}$  is asymptotically distributed, over the class of all possible outcomes that could be observed in  $M$  trials, as chi-squared with  $k=M-N-1$  degrees of freedom, independently of the actual nature of the constraints (Jaynes, 1982).

Invoking the entropy concentration theorem, Jaynes (1982) concludes by stating that given incomplete information, the distribution of maximum entropy is not only the one that can be realized in the greatest number of ways; in fact, for large  $M$  the overwhelming majority of all possible distributions compatible with our information have entropy very close to the maximum.

In other words, distributions of entropy away from the maximum are improbable relative to those allowed by the constraints.

Drawing upon some elementary examples Jaynes finally concludes: "... it is prudent to adopt for purposes of inference that distribution which has maximum entropy subject to the data we have. It is prudent, not for any vague, mystical reason; but for the very clear and pragmatic reason that the maxent predictions are the most reliable ones that can be made on the given information. It is a combinatorial theorem that to choose any other estimate would amount to ignoring the vast majority of all the possibilities allowed by the data..."

**Axiomatic Justification.** As already mentioned, the entropy functional  $\mathcal{S}\{p\}$  was derived from a number of postulates based upon desired properties of information measures (Shannon and Weaver, 1949).

Yet, another axiomatic derivation, proposed by Shore and Johnson (1980) and based upon self-consistent properties of inference methods, provides new evidence toward choosing distributions of maximum entropy.

The principle of maximum entropy can be illustrated by an elementary, but typical example (van Rooeckeghem, 1986). Let us assume the probability of having a black court card is to be estimated. For that purpose, the following events are defined:

$x_1$ : get a black court card

$x_2$ : get a red court card

$x_3$ : get another black

$x_4$ : get another red

It is further assumed that we are given the probability of getting a court card and the probability of getting a black card:

$$p(x_1) + p(x_2) = \frac{3}{13} \quad p(x_1) + p(x_3) = \frac{1}{2} \quad (4.8)$$

Clearly, there is an infinite number of probability assignments that satisfy those constraints. Maximizing the entropy:

$$\mathcal{S}\{p\} = - \sum_{n=1}^4 p(x_n) \ln p(x_n) \quad (4.9)$$

subject to the constraints (4.3) leads to the probabilities:

$$p(x_1) = p(x_2) = \frac{3}{26} \quad p(x_3) = p(x_4) = \frac{10}{26} \quad (4.10)$$

There is no correlation between the color and the value of the cards induced from the maximization procedure. This should be expected from any reasonable solution. Maximizing any function other than entropy would have induced undesirable correlations.

This is one of the arguments that led to the axiomatic derivation of the principle of maximum entropy (Shore and Johnson, 1980).

Essentially, their approach postulates that reasonable methods of inference should lead to consistent results when there are several ways of taking the same constraints into account (e.g. different sets of coordinates).

They show that there is a unique distribution obtained by maximization that satisfies the constraints assuming that the technique used verifies the postulates (or consistency axioms); this distribution can be obtained by maximizing the entropy.

### **The Maximum Entropy Probability Distribution**

The entropy  $\mathcal{S}\{p\}$  is maximized under the constraints that the first  $N$  power moments be equal to the true moments  $\mu_n$ . Defining appropriate Lagrange multipliers  $\lambda_n$ , one aims at maximizing the entropy functional redefined from (Mead and Papanicolaou, 1984):



$$\mathcal{S}\{p\} = - \int_{-\infty}^{+\infty} [p(y) \ln p(y) - p(y)] dy - \sum_{n=0}^N \lambda_n \left( \int_{-\infty}^{+\infty} y^n p(y) dy - \mu_n \right) \quad (4.11)$$

where the sum extends over all the prior moment constraints (equation (4.3)).

Upon maximization, the following density function  $p(y)$  results:

$$p_N(y) = \exp\left(- \sum_{n=0}^N \lambda_n y^n\right) \quad (4.12)$$

where the Lagrange multipliers are determined from the  $N$  moment constraints (4.3), and  $\lambda_0$  is a normalizing constant. Several peculiar features of the probability density of maximum entropy (4.12) can be emphasized:

First it is interesting to mention that for  $N=2$ , the normal probability density is obtained.

The density of maximum entropy  $p_N(y)$  remains always positive, at each finite stage of the iteration, unlike densities derived from sets of orthogonal polynomial.

Using cumulant or quasi-moment constraints instead of moments leads to the same distribution, provided that these constraints are of the same order. In other words, different ways of expressing the same information lead to identical distributions<sup>5</sup>.

For numerical purposes, a potential is introduced by an appropriate Legendre transformation. It can be shown to be everywhere convex through explicit construction

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<sup>5</sup>This is not the case of orthogonal polynomial expansions, e.g. Edgeworth and Gram-Charlier distributions.

of the corresponding Hessian. Such a property ensures that a local stationary point actually maximizes entropy (Mead and Papanicolaou, 1984).

Given a set of moment constraints (4.3), the maximum entropy distribution can be evaluated through the Raphson-Newton algorithm. At each step of the iteration procedure, the first  $2N$  power moments must be computed:

$$\mu_n = \int_{-\infty}^{+\infty} y^n \exp\left(-\sum_{i=0}^N \lambda_i y^i\right) dy \quad (4.13)$$

Integrating by parts, the general expressions are obtained:

$$\mu_n = \frac{1}{N\lambda_N} \left( (n - N + 1)\mu_{n-N} - \sum_{i=1}^{N-1} i \lambda_i \mu_{n-N+i} \right) \quad n \geq N \quad (4.14)$$

Thus, only the first  $N$  moments actually need to be evaluated including  $\mu_0=1$ . The remaining power moments  $\mu_n, n \geq N$  are given in terms of the lower order ones by the above recurrence relationship.

Such original relationships are very useful in the construction of the Hessian at each iterative step of the Raphson-Newton algorithm.

Referring to the two examples already discussed (Chapter 3), empirical evidences leading to the preference of the maximum entropy distributions over other distributions can be exhibited.

In the case of the Duffing oscillator excited by a Gaussian white noise, the exact response probability density is obtained from equation (3.37). Such a probability

density coincides with the maximum entropy distribution as soon as the 2nd- and 4th-order moments (or cumulants) belong to any set of moment constraints of the type (4.3).

Let us consider now the pure static case of the Duffing oscillator excited by a Gaussian narrow-banded signal at the frequency  $\omega_0 = 0$ . The response probability function is exactly given by equation (3.45). Clearly, such a probability density cannot generally coincide with the distribution of maximum entropy given any finite set of moment constraints of the type (4.3). Nevertheless we show, starting with a set of exact moments, that the distribution of maximum entropy is in far better agreement with the exact distribution than the corresponding Edgeworth series-type distributions (Figures 4.1 and 4.2).

These observations agree with other works on the distribution of maximum entropy in the problem of moments (Levine and Tribus, 1979, Agmon, Alhassid and Levine, 1979, Levine, 1980 and Mead and Papanicolaou, 1984).

It is essential to keep in mind that the maximum entropy approach, just as any other approximation, should not be regarded as the panacea for the solution of all moment problems. After all, a polynomial expansion would be perfect if the exact density happened to be a finite polynomial, while maximum entropy would be ideal if the density were the exponential of a finite degree polynomial (Mead and Papanicolaou, 1984).

However, in the absence of any other information than our moment constraints (4.3), the distribution of maximum entropy "is uniquely determined as the one which is maximally noncommittal with regard to missing information" (Jaynes, 1957a & 1957b).

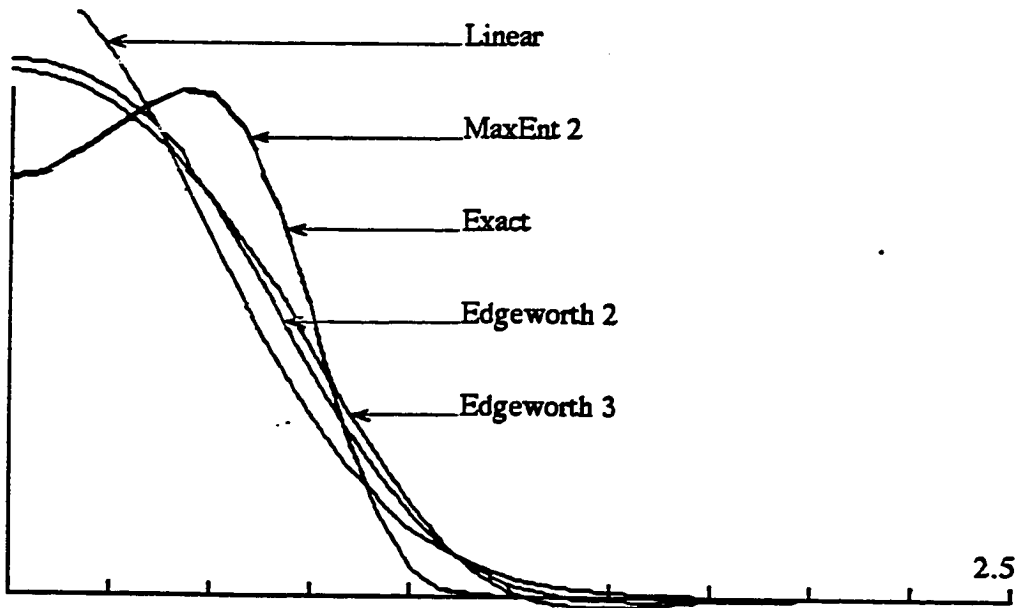


Figure 4.1. Response Probability Density Function, Static Case.  
 $\sigma_{y \text{ linear}}^2 = 0.5$

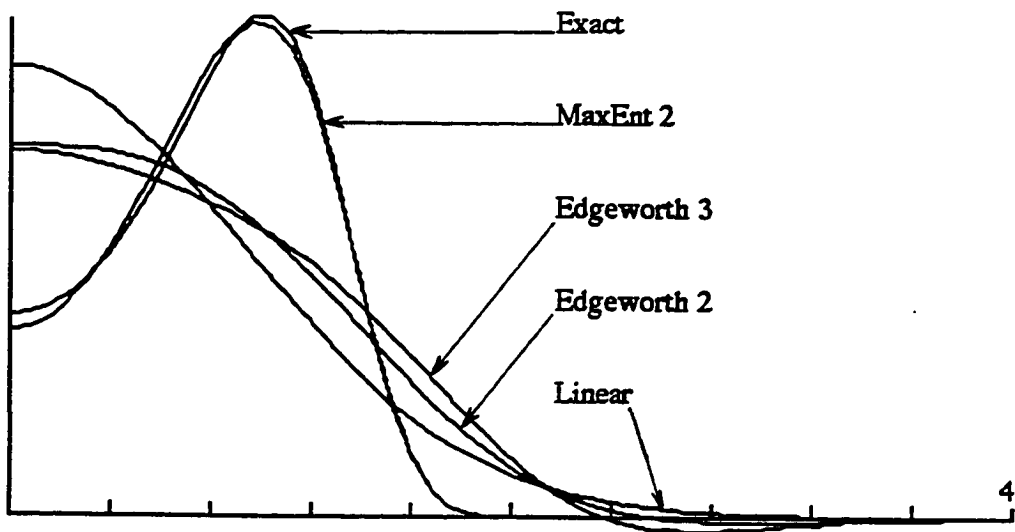


Figure 4.2. Response Probability Density Function, Static Case.  
 $\sigma_{y \text{ linear}}^2 = 5$

## Applications: Marine Vehicle Rigid Body Behavior

Two applications to the rigid body behavior of marine vehicles will serve to exemplify and to assess the accuracy of the different approximations discussed. The approximations appear at two levels. First in the prediction of the response statistics (Chapter 3), and also in the construction of the response distributions from knowledge of these statistics (Chapter 4).

The purpose of these applications is not to pretend that such mathematical models will invariably apply to the behavior of marine vehicles, but rather to show that our methods can be useful, at least in some situations involving those structures.

### Surge Motion of a Tension Leg Platform

The case of the surge motion of a tension leg platform is considered first. Ochi and Malakar (1984) assumed that such a motion could be modeled by the single degree of freedom equation of motion:

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2y + ry^3 = x \quad (4.15)$$

This assumption may be satisfied, provided that the other modes are decoupled from surge. The nonlinearity in  $y^3$  appearing in the equation of motion models the nonlinear behavior of the mooring system.

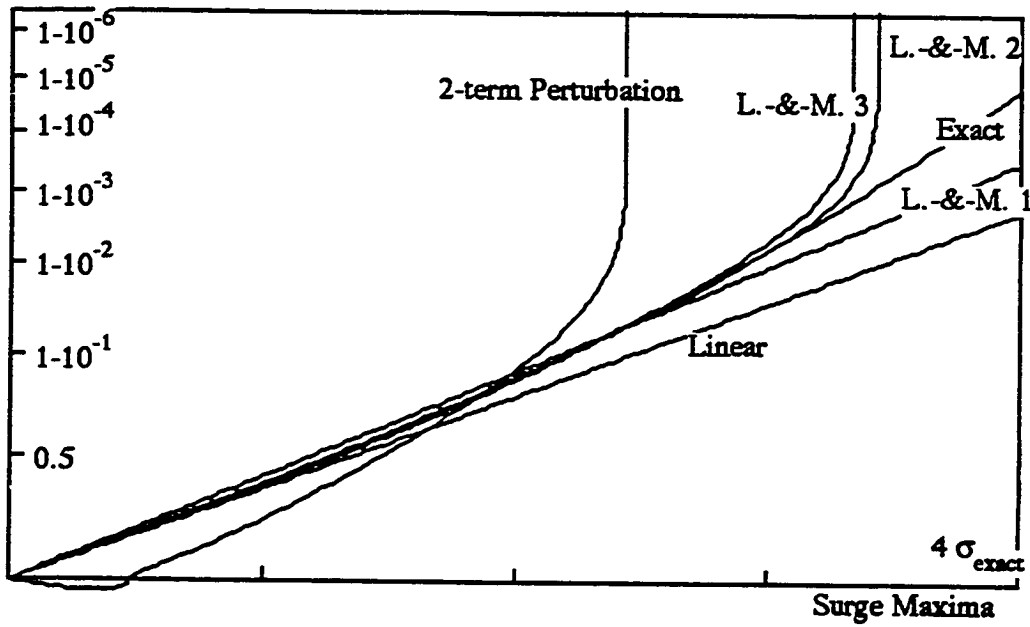


Figure 4.3. Surge Probability Distribution of Maxima, Edgeworth-Type Distributions,  $H_s = 9.15$  m.

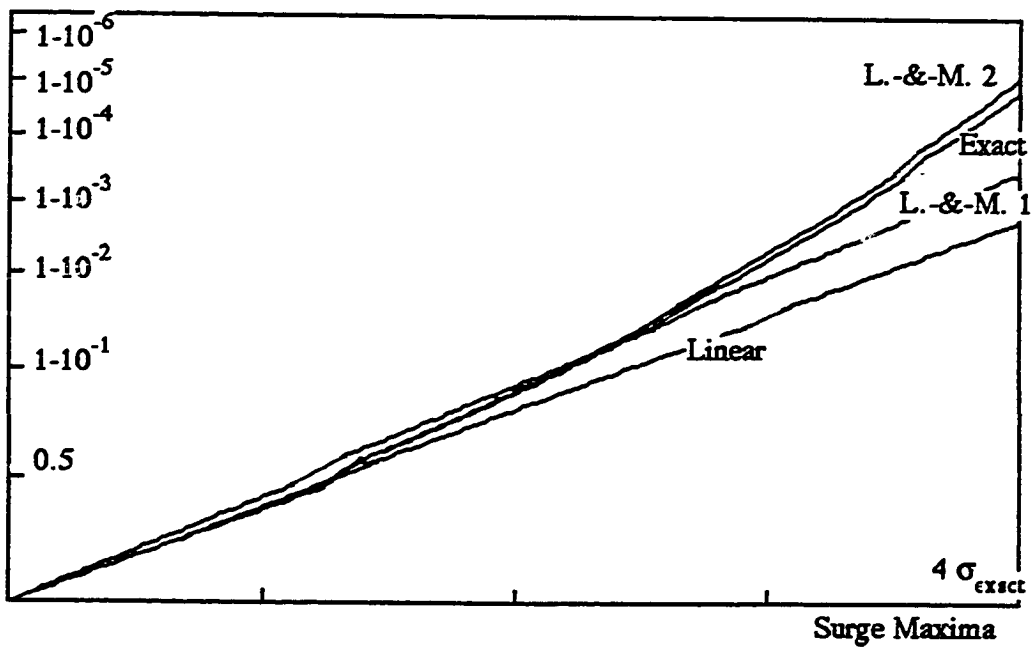


Figure 4.4. Surge Probability Distribution of Maxima, Maximum Entropy Distributions,  $H_s = 9.15$  m.

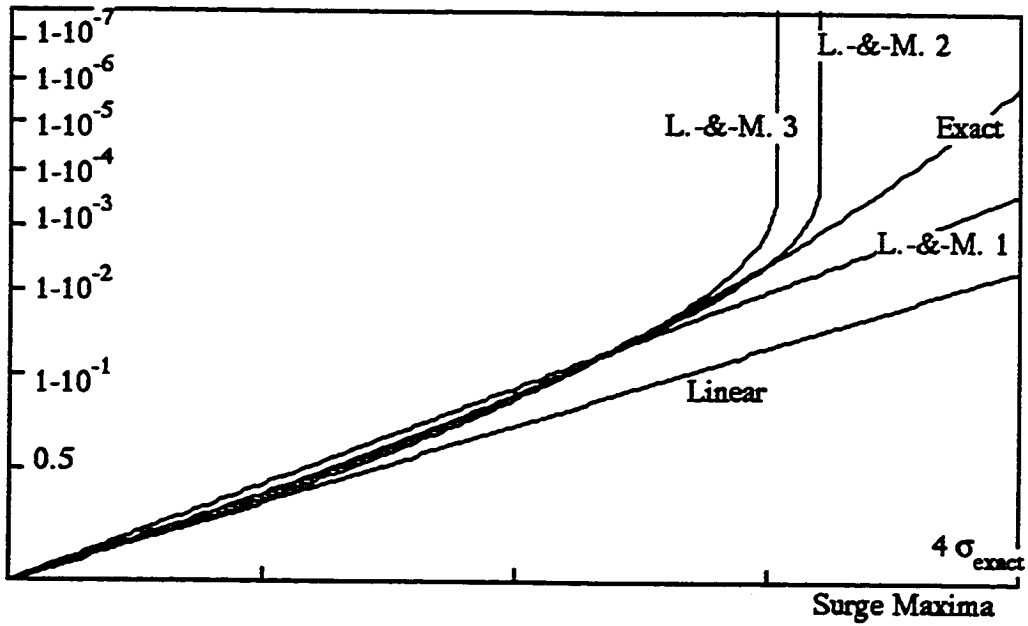


Figure 4.5. Surge Probability Distribution of Maxima, Edgeworth-Type Distributions,  $H_s = 14$  m.

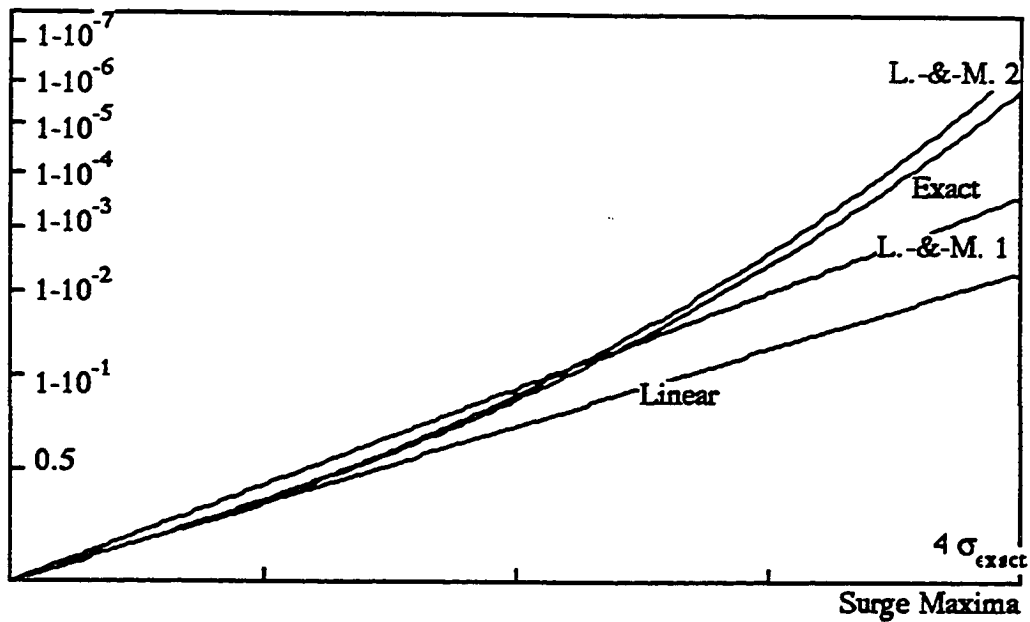


Figure 4.6. Surge Probability Distribution of Maxima, Maximum Entropy Distributions,  $H_s = 14$  m.

The American Bureau of Shipping provided them with the force resulting from a sea state described by the Pierson-Mosckowitz wave spectrum. The significant wave height is 9.15 m, with modal period 10 s. We also consider a significant wave height of 14 m, by simply extrapolating linearly Ochi's data.

Ochi and Malakar (1984) made the further assumption that the wave excitation could be approximated by a Gaussian white noise process. They introduced a method to evaluate the magnitude of an equivalent white noise spectrum based on equating the energy with the one associated with the corresponding linear system.

The following data were also used:  $\omega_n=0.1 \text{ rad.s}^{-1}$ ;  $\xi=20\%$ ;  $r=0.0057 \text{ m}^{-2}.\text{s}^{-2}$ . Such configurations respectively correspond to adimensionalized linear output variance  $\sigma_{y\text{linear}}^2=0.12$  and 0.28 for the incoming significant wave height of 9.15 m and 14.0 m.

The surge motion distribution of maxima can be obtained, assuming that the surge power spectral density is sufficiently narrow-banded (Duthoit and Armand, 1987). Figures 4.3, 4.4, 4.5 and 4.6 show the surge distribution of maxima on Rayleigh probability paper<sup>6</sup>, in order to emphasize deviation from the theoretical distribution of the maxima of a narrow-banded Gaussian process.

The case  $H_s=9.15 \text{ m}$  is considered on Figures 4.3 and 4.4, whereas the more severe case  $H_s=14 \text{ m}$  is shown on Figures 4.5 and 4.6. Figures 4.3 and 4.5 show the Edgeworth-type distributions evaluated from the response cumulants predicted by the linear, the perturbation as well as the linearize-and-match methods. Figures 4.4 and 4.6 show the corresponding maximum entropy distributions.

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<sup>6</sup>I.e. Rayleigh distributions will appear as straight lines.



$y \geq$	$\sigma_{\text{exact}}$	$2 \sigma_{\text{exact}}$	$3 \sigma_{\text{exact}}$	$4 \sigma_{\text{exact}}$
Linear	0.6722	0.2041	0.0280	$1.7 \cdot 10^{-3}$
Eq. Linearization	0.6010	0.1305	0.0102	$2.9 \cdot 10^{-4}$
2-term MaxEnt	0.6594	0.1457	0.0049	$0.7 \cdot 10^{-5}$
Exact Solution	0.6595	0.1507	0.0060	$1.3 \cdot 10^{-5}$

Table 4.1. Probabilities of Exceedence  
9.15 m Significant Wave Height.

$y \geq$	$\sigma_{\text{exact}}$	$2 \sigma_{\text{exact}}$	$3 \sigma_{\text{exact}}$	$4 \sigma_{\text{exact}}$
Linear	0.7148	0.2610	0.0487	$4.6 \cdot 10^{-3}$
Eq. Linearization	0.5949	0.1253	0.0093	$2.5 \cdot 10^{-4}$
2-term MaxEnt	0.6887	0.1439	0.0024	$0.3 \cdot 10^{-6}$
Exact Solution	0.6924	0.1570	0.0037	$1.3 \cdot 10^{-6}$

Table 4.2. Probabilities of Exceedence  
14 m Significant Wave Height.

As already mentioned in Chapter 3, the perturbation method yields rather poor estimates. The Edgeworth-type distributions together the linearize-and-match cumulants provide accurate results over a reasonable range of maxima ( $0-2.5\sigma_{\text{exact}}$ ). For larger values of the surge maxima, these distributions may reach values greater than 1 and thus are of little interest.

The figures clearly emphasize that the distributions of maximum entropy together with the linearize-and-match first two moment constraints are the most accurate when compared with the exact distributions<sup>7</sup> obtained from Fokker-Planck equation (equation (3.37)). Such agreement is equally good for the probability of exceedence (Tables 4.1 and 4.2).

The improvement over the linear and the equivalent linear models is clear and is the most significant for extreme values, i.e.  $\Pr\{y \geq 4\sigma_{\text{exact}}\}$ , which corresponds to rare events of the order  $10^{-5}$  or  $10^{-6}$ .

### Roll Motion of a Ship

The form of the equation governing ship rolling has been discussed through the literature for some time. There exists at least one important configuration where ship rolling can be modeled as a single degree of freedom equation, this is the case of beam waves.

In such a case, it is generally agreed that the differential equation that prevails is of the form:

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<sup>7</sup>Exact within the narrow-band hypothesis.

$$\ddot{y} + a\dot{y} + y + b\dot{y}|\dot{y}| - y^3 = x \quad (4.16)$$

where the roll angle  $y$  as well as the incoming wave slope have been both adimensionalized relatively to the half range of positive static stability of the ship (Dalzell, 1971 & 1973).

The technique presented above is not directly applicable to this type of differential equation. In order to transform this differential equation into an analytic system, the technique of equivalent nonlinearization is used (Barrett, 1963 and Caughey, 1984). One nonlinear analytic equation "equivalent" to (4.16) is:

$$\ddot{y} + 2\xi\dot{y} + \Xi y^3 + y - y^3 = x \quad (4.17)$$

where:

$$2\xi = a + b \sqrt{\frac{2E[\dot{y}^2]}{\pi}} \quad \text{and} \quad \Xi = b \sqrt{\frac{2}{9\pi E[\dot{y}^2]}} \quad (4.18)$$

The extent to which such an approximation be reasonable clearly depends on the kind of statistics sought. Keeping in mind that equivalence in the mean square sense is invoqued in order to substitute (4.17) to (4.16), equation (4.17) should yield second order statistics within reasonable accuracy. On the other hand, one can predict that less accurate higher order statistics will result.

In order to assess the validity of both the output moment prediction technique and the distribution of maximum entropy approximation, comparisons with Dalzell (1971 & 1973, Roberts, 1982a) simulation of nonlinear ship rolling have been performed.

In his work, three different wave input processes (labelled 1, 2 and 3) were considered, wave process 1 corresponds to a narrow-band wave spectrum, while wave process 3 has the largest bandwidth. Unfortunately, the spectral density of wave process 1 do not have an analytic form, thus only the two others are considered here. Only the configurations of highest severity are shown and discussed here. The average error of our method on the roll angle standard deviation was found to be less than 3%.

In Dalzell's work, those configurations where the deviation of the distribution of roll maxima from those of a Gaussian process are the most sensible will now be considered. In order to evaluate the distribution of maxima, the spectral density in roll is assumed to be narrow-banded. Such an assumption turns out to be justified by the bandwidth parameters calculated by Dalzell, which for practical purpose, can be set equal to zero.

Figures 4.7, 4.8, 4.9 and 4.10 show Dalzell's simulation results together with the linear and the maximum entropy<sup>8</sup> distributions on Rayleigh probability paper. Figures 4.7, 4.9 and 4.10 correspond to wave process 3 with modal frequency  $\omega_0 = 0.90$ , and standard deviation 0.0351, while  $a = 0.1$ ; 0.01 and 0 respectively and  $b = 1$ . Figure 4.8 corresponds to wave process 2 with modal frequency  $\omega_0 = 0.95$ , and standard deviation 0.0261, while  $a = 0.1$  and  $b = 1$ .

Clearly, the linear approximation does not yield any useful information on the distribution of roll maxima in these cases, while the distribution of maximum entropy

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<sup>8</sup>Labelled L.-&M. N, where N denotes the number of terms in the distribution of maximum entropy, except for the normalizing constant.

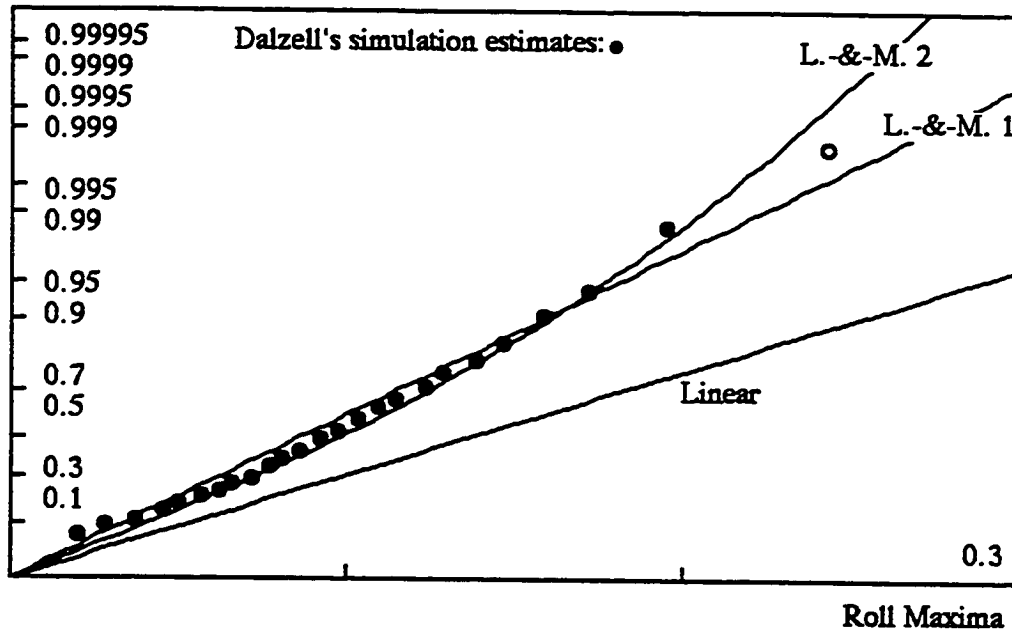


Figure 4.7. Roll Probability Distribution of Maxima, Maximum Entropy Distributions,  $a=0.1$ ,  $b=1$ , Wave Process 3.

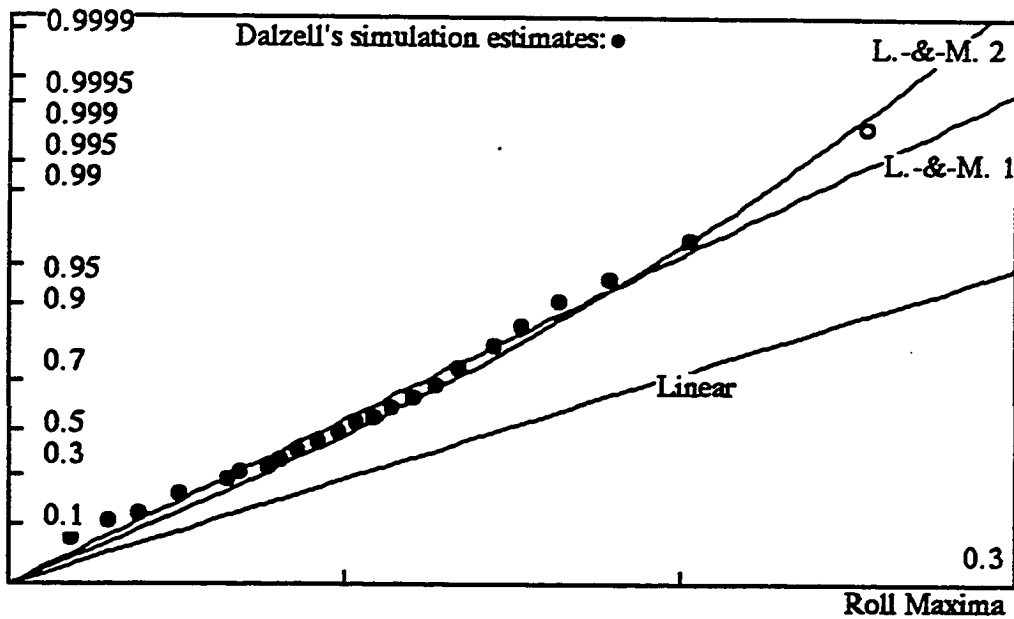


Figure 4.8. Roll Probability Distribution of Maxima, Maximum Entropy Distributions,  $a=0.1$ ,  $b=1$ , Wave Process 2.

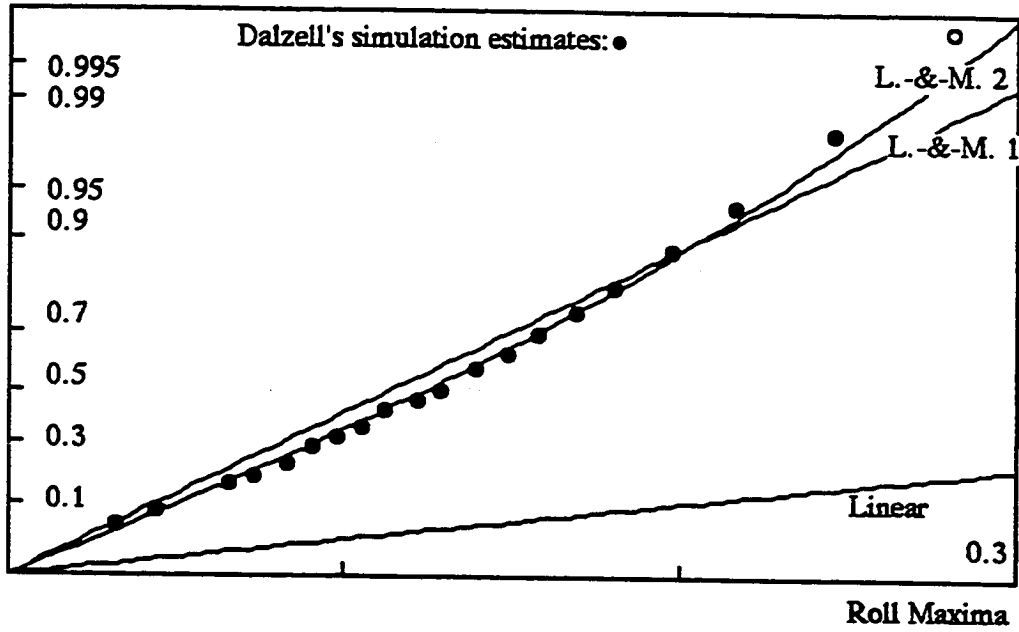


Figure 4.9. Roll Probability Distribution of Maxima, Maximum Entropy Distributions,  $a=0.01$ ,  $b=1$ , Wave Process 3.

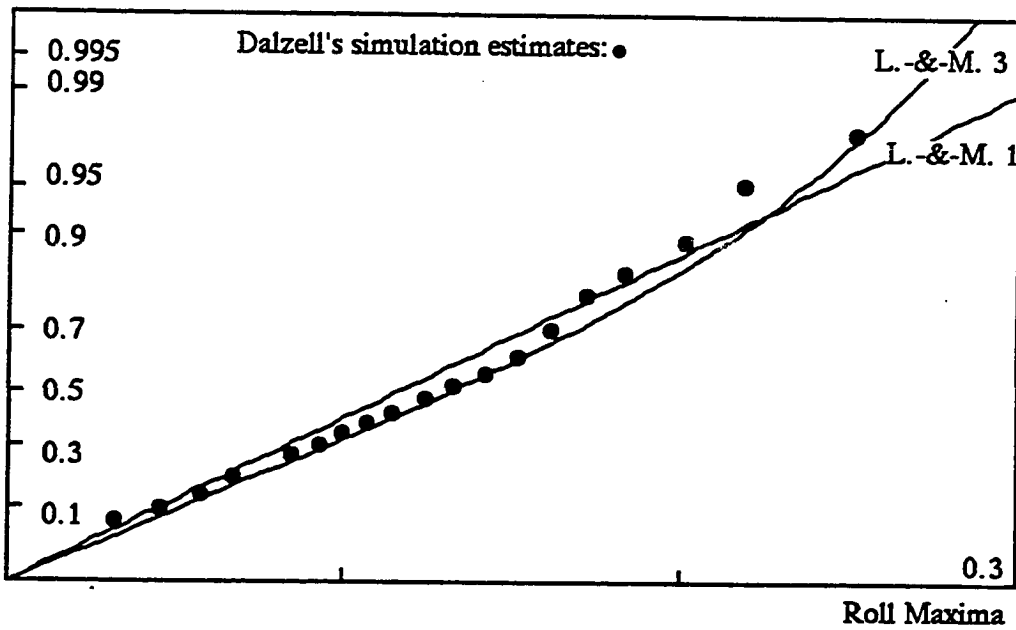


Figure 4.10. Roll Probability Distribution of Maxima, Maximum Entropy Distributions,  $a=0$ ,  $b=1$ , Wave Process 3.

with two moment constraints derived from the theory exposed in Chapter 3 yields good agreement with simulation estimates.

When more moment constraints are taken into account in the maximum entropy distribution, the accuracy is improved. Such a better agreement with Dalzell's simulation estimates is sensible only in the last case (Figure 4.10).

Recalling that the roll angle is scaled relatively to the half range of positive static stability of the ship, Dalzell's simulation estimates tend to prove that our method yields rather accurate predictions at least up to 25 or 35 degree amplitudes.

As already mentioned, the 1-term distribution of maximum entropy coincides with the one obtained from the method of equivalent linearization. Thus, our method provides further insights into the probabilistic structure of non-Gaussian processes which are not, in any way, predictable within the framework of a linear model.

Such deviations from the distribution of the maxima of a Gaussian process are clearly depicted on the figures and accurately predicted by the proposed method.

It is presumably possible to further refine our model in adding higher order terms in the equivalent nonlinear system, equation (4.17). Further improvements could be gained, at some computational expense, in removing the narrow-band assumption, thereby predicting the distribution of maxima of a general Gaussian stochastic process.

## Conclusion

A new heuristic technique for the determination of the nonlinear response to a stochastic process has been proposed. Essentially, the linearize-and-match method consists in constructing an infinite series of linear systems. Each one of these systems is defined so as to predict the response statistics of a given order by matching the Volterra functional model response statistics of the same order. The linear model for predicting statistics of order two coincides with the one defined by the method of equivalent linearization. Some new results related to the Volterra transfer functions and response cumulants have been derived along the way.

This linearize-and-match method overcomes many of the drawbacks of perturbation and other functional representation techniques related to the questions of convergence and accuracy of the resulting series. Furthermore, this method provides added versatility over methods based on the theory of Markov processes such as the Fokker-Planck equation and Itô stochastic calculus. Finally, the linearize-and-match method is characterized by an extreme simplicity of implementation, when compared with the more excessive digital as well as analog simulation techniques.



The response statistics, resulting from the linearize-and-match method, can in turn be combined to derive response probability distributions. Several methods to address this underdetermined moment problem have been described and compared. The distribution of maximum entropy appears to consistently yield the best accuracy, based on incomplete information. The global method applies to a quite broad variety of nonlinear dynamic systems driven by general stochastic excitation.

Various mathematical idealization of the functional relationship governing the rigid body behavior of marine vehicles served both to exemplify and to assess the accuracy of this technique. Comparisons with exact solutions or time domain digital simulations (whenever exact methods do not apply) of the assumed mathematical models show the accuracy of the proposed method in predicting response statistics of various order as well as probability distributions.

Furthermore, we have shown how this method can be generalized in several directions to address asymmetric as well as coupled multi degree of freedom systems. Finally, the method is quite simple in principle as well as in practice.

The applications discussed were all implemented on Apple's Macintosh™ microcomputer.

On the other hand, the main drawback of the linearize-and-match method is that, although its accuracy appears quite reasonable regarding the configurations considered, the resulting error on the response statistics is nevertheless irremovable. Therefore it may not be acceptable in some other situations.

Although the mathematical models considered represent idealized and simple models of the behavior of ocean structures, it is believed they can yield further insight

into the complicated mechanisms leading to large amplitude response and eventually capsizing.

Further assesement of the accuracy of the proposed method cannot be achieved unless comprehensive model as well as full-scale experiments and measurements (both digital and analog) are performed. The double objective of identifying the parameters of the dynamic systems in hand as well as characterizing the approximations which hold the most promises should govern such experiments.

Finally, further research should be conducted toward the integration of such techniques within a global dynamic system reliability and safety model not yet available. Such a model would be an essential step towards the rationally-based design of ocean structures.



## APPENDIX A

### Volterra Transfer Functions

Let us consider a nonlinear system described by the functional input-output relationship (1.1). Expanding the nonlinear functional  $\mathcal{F}$  in Volterra series, excitation process  $x$  can be written in terms of response process  $y$  in the frequency domain:

$$X(\omega) = \sum_{j=1}^{+\infty} \iint \dots \int G_j(\omega_1, \omega_2, \dots, \omega_j) \delta(\omega - \omega_1 - \omega_2 - \dots - \omega_j) \prod_{k=1}^j Y(\omega_k) d\omega_k \quad (\text{A.1})$$

where,  $G_j$  are known as the transfer functions,  $X$  and  $Y$  respectively denote the Fourier transforms of  $x$  and  $y$ , and  $\delta$  stands for the Dirac distribution.

Clearly, equation (A.1) is the frequency domain equivalent to the time domain series (1.10) and (1.11). Since this is an inverse formulation of our input-output relationship, it is generally required to determine the explicit one:

$$Y(\omega) = \sum_{i=1}^{+\infty} \iint \dots \int H_i(\omega_1, \omega_2, \dots, \omega_i) \delta(\omega - \omega_1 - \omega_2 - \dots - \omega_i) \prod_{k=1}^i X(\omega_k) d\omega_k \quad (\text{A.2})$$

In other words, transfer functions  $H_i$  are to be determined from transfer functions  $G_j$ . This is an inversion problem which can be solved in a variety of ways (some of the methods available have been discussed in chapter 1). For instance, the direct expansion method (Bedrosian and Rice, 1971) consists in expanding functional  $\mathcal{F}$  and its unknown inverse  $\mathcal{F}^{-1}$  in the time domain, substituting and identifying the terms of same order, and finally Fourier transforming the resulting kernels.

However, a simpler approach can be undertaken in directly expanding functional  $\mathcal{F}$  and its inverse in the frequency domain (equations (A.1) and (A.2)). After substitution of equation (A.1) into equation (A.2), the following relation results:

$$Y(\omega) = \sum_{i=1}^{+\infty} \iint \dots \int H_i(\omega_1, \omega_2, \dots, \omega_i) \delta(\omega - \omega_1 - \omega_2 - \dots - \omega_i) \prod_{k=1}^i d\omega_k \sum_{j=1}^{+\infty} \iint \dots \int G_j(\omega_{i+1}, \omega_{i+2}, \dots, \omega_{i+j}) \delta(\omega_k - \omega_{i+1} - \omega_{i+2} - \dots - \omega_{i+j}) \prod_{l=1}^j Y(\omega_{i+l}) d\omega_{i+l} \quad (\text{A.3})$$

A series of  $i$  sums over the indices  $j_1, j_2, \dots, j_i$  can be substituted to the  $k$ -product of the  $j$ -sums above:

$$Y(\omega) = \sum_{i=1}^{+\infty} \sum_{j_1=1}^{+\infty} \sum_{j_2=1}^{+\infty} \dots \sum_{j_i=1}^{+\infty} \iint \dots \int H_i(\omega_1, \omega_2, \dots, \omega_i) \delta(\omega - \omega_1 - \omega_2 - \dots - \omega_i)$$

$$\left( \prod_{k=1}^i G_{j_k}(\omega_{i+j_1+\dots+j_{k-1}+1}, \dots, \omega_{i+j_1+\dots+j_k}) \delta(\omega_k - \omega_{i+j_1+\dots+j_{k-1}+1} - \dots - \omega_{i+j_1+\dots+j_k}) \right) \prod_{l=1}^{j_1+j_2+\dots+j_i} Y(\omega_{i+l}) \prod_{m=1}^{i+j_1+\dots+j_i} d\omega_m \tag{A.4}$$

We can further simplify the expression above in evaluating the  $i$  integrals corresponding to the first  $i$   $\omega_k$  variables, and simultaneously making the change of variables corresponding to the indicial transformation  $i+j \rightarrow j$ :

$$Y(\omega) = \sum_{n=1}^{+\infty} \sum_{i=1}^n \sum_{j_1+\dots+j_i=n} \iint \dots \int H_i(\omega_1+\dots+\omega_{j_1}, \omega_{j_1+1}+\dots+\omega_{j_1+j_2}, \dots, \omega_{j_1+\dots+j_{i-1}+1}+\dots+\omega_{j_1+\dots+j_i}) \delta(\omega - \omega_1 - \omega_2 - \dots - \omega_{j_1+j_2+\dots+j_i}) \prod_{k=1}^i G_{j_k}(\omega_{j_1+\dots+j_{k-1}+1}, \dots, \omega_{j_1+\dots+j_k}) \prod_{l=1}^{j_1+j_2+\dots+j_i} Y(\omega_l) d\omega_l \tag{A.5}$$

where, the rightmost summation is over all the unordered combinations such that  $j_1+j_2+\dots+j_i=n$ . For each different arrangement:

$$1 \leq j_1 \leq j_2 \leq \dots \leq j_i \tag{A.6}$$

There are:

$$\frac{i!}{p_1! p_2! \dots p_q!} \quad (\text{A.7})$$

distinct terms contributing to that sum, where  $p_r$  denotes the number of equal  $j$ 's in the  $r$ -th run of equalities in the set of inequalities (A.6). If no  $j$ 's are equal then the  $p_r$  all vanish (Bedrosian and Rice, 1971).

Finally, in order for equation (A.5) to hold, the following compatibility conditions must be satisfied:

$$\delta_{1n} = \sum_{i=1}^n \sum_{j_1+\dots+j_i=n} H_i(\omega_1+\dots+\omega_{j_1}, \dots, \omega_{j_1+\dots+j_{i-1}+1}, \dots, \omega_{j_1+\dots+j_i}) \prod_{k=1}^i G_{j_k}(\omega_{j_1+\dots+j_{k-1}+1}, \dots, \omega_{j_1+\dots+j_k}) \quad (\text{A.8})$$

where  $\delta_{ij}$  denotes the Kronecker delta.

These equalities lead to the desired transfer functions  $H_n(\omega_1, \omega_2, \dots, \omega_n)$ :

$$H_1(\omega_1) = \frac{1}{G_1(\omega_1)} \quad (\text{A.9})$$

$$H_n(\omega_1, \omega_2, \dots, \omega_n) = -H_1(\omega_1)H_1(\omega_2)\dots H_1(\omega_n) \sum_{i=1}^{n-1} \sum_{j_1+\dots+j_i=n} H_i(\omega_1+\dots+\omega_{j_1}, \dots,$$

$$\dots, \omega_{j_1+\dots+j_{i-1}+1+\dots+\omega_{j_1+\dots+j_i}}) \prod_{k=1}^i G_{j_k}(\omega_{j_1+\dots+j_{k-1}+1}, \dots, \omega_{j_1+\dots+j_k}) \quad (\text{A.10})$$

Such an expression however, does not generally result in the symmetric transfer functions required in the various derivations of response statistics throughout chapters 2 and 3. Whenever this is the case, the associated symmetric transfer functions can be derived from the simple transformation:

$$\text{Sym}\{H_n(\omega_1, \omega_2, \dots, \omega_n)\} = \frac{1}{n!} \sum_{\text{permutations of } \omega_x} H_n(\omega_1, \omega_2, \dots, \omega_n) \quad (\text{A.11})$$

Substituting in equation (A.10), we obtain:

$$H_n(\omega_1, \omega_2, \dots, \omega_n) = -H_1(\omega_1)H_1(\omega_2)\dots H_1(\omega_n) \sum_{i=1}^{n-1} \sum_{j_1+\dots+j_i=n}^* \frac{i! j_1! j_2! \dots j_i!}{n!} \sum_{\text{permutations of } \omega_j} H_i(\omega_1+\dots+\omega_{j_1}, \dots, \omega_{j_1+\dots+j_{i-1}+1+\dots+\omega_{j_1+\dots+j_i}}) \prod_{k=1}^i G_{j_k}(\omega_{j_1+\dots+j_{k-1}+1}, \dots, \omega_{j_1+\dots+j_k}) \quad (\text{A.12})$$

where the middle sum with a star superscript extends over all those partitions of  $n$  which have  $i$  parts. In other words, the sum is over all the different arrangements (A.6).

Whereas the rightmost sum extends over all the different terms which can be obtained in permuting the  $\omega_j$ . Identical is used in the sense that the  $H_i$  are symmetric in



their arguments and that for instance:  $\omega_1 + \omega_2 = \omega_2 + \omega_1$ . The number of terms in this sum is:

$$\frac{n!}{j_1! j_2! \dots j_i! p_1! p_2! \dots p_q!} \quad (\text{A.13})$$

Equation (A.12) as well as Bedrosian and Rice's formula are recurrence relationships since each transfer function  $H_n(\omega_1, \omega_2, \dots, \omega_n)$  depends upon lower order ones. However, our general inversion formula, equation (A.12) is not only simpler than Bedrosian and Rice's but also more general since it applies to general nonlinear Volterra systems.

Note that such expressions can easily be generalized to the case of a nonlinear system described by functional relationships of the type:

$$\mathcal{F}_1\{y(t)\} = \mathcal{F}_2\{x(t)\} \quad (\text{A.14})$$

where  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are both nonlinear and analytic functionals, i.e. expandable in Volterra series.

### Duffing Equation with Nonlinear Damping

In order to illustrate the general transfer function expressions, let us consider a variation of the Duffing equation, namely the Duffing equation with linear plus cubic

damping (equation (3.22)). Alternatively, this equation can be rewritten in terms of Volterra integrals:

$$\int_{-\infty}^{+\infty} g_1(t-\tau_1)y(\tau_1)d\tau_1 + \iiint_{-\infty}^{+\infty} [\Xi \delta'(t-\tau_1)\delta'(t-\tau_2) \delta'(t-\tau_3) + \delta(t-\tau_1)\delta(t-\tau_2)\delta(t-\tau_3)] y(\tau_1)y(\tau_2)y(\tau_3)d\tau_1d\tau_2d\tau_3 = x(t)$$

(A.15)

Equation (A.15) can be Fourier transformed, in order to determine the inverse transfer functions  $G_j$ .

$$\int_{-\infty}^{+\infty} G_1(\omega_1) \delta(\omega - \omega_1) Y(\omega_1) d\omega_1 + \iiint_{-\infty}^{+\infty} [\Xi \omega_1\omega_2\omega_3 + 1] \delta(\omega - \omega_1 - \omega_2 - \omega_3) Y(\omega_1) Y(\omega_2)Y(\omega_3) d\omega_1d\omega_2d\omega_3 = X(\omega)$$

(A.16)

where the linear and third order transfer functions are given by:

$$G_1(\omega) = -\omega^2 + 2i\xi\omega + 1$$

$$G_3(\omega_1, \omega_2, \omega_3) = \Xi\omega_1\omega_2\omega_3 + 1$$

(A.17)

Substituting the  $G_j$  in equations (A.12) leads to the symmetric version of the first few transfer functions  $H_j$ :

$$H_1(\omega) = \frac{1}{-\omega^2 + 2i\xi\omega + 1}$$
(A.18)

$$H_3(\omega_1, \omega_2, \omega_3) = -(1 + \Xi \omega_1 \omega_2 \omega_3) H_1(\omega_1) H_1(\omega_2) H_1(\omega_3) H_1(\omega_1 + \omega_2 + \omega_3)$$
(A.19)

$$\begin{aligned}
H_5(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) = & \frac{3}{10} \left( (1 + \Xi \omega_1 \omega_2 \omega_3) H_1(\omega_1 + \omega_2 + \omega_3) H_1(\omega_4) H_1(\omega_5) \right. \\
& + (1 + \Xi \omega_1 \omega_2 \omega_4) H_1(\omega_1 + \omega_2 + \omega_4) H_1(\omega_3) H_1(\omega_5) \\
& + (1 + \Xi \omega_1 \omega_2 \omega_5) H_1(\omega_1 + \omega_2 + \omega_5) H_1(\omega_3) H_1(\omega_4) \\
& + (1 + \Xi \omega_1 \omega_3 \omega_4) H_1(\omega_1 + \omega_3 + \omega_4) H_1(\omega_2) H_1(\omega_5) \\
& + (1 + \Xi \omega_1 \omega_3 \omega_5) H_1(\omega_1 + \omega_3 + \omega_5) H_1(\omega_2) H_1(\omega_4) \\
& + (1 + \Xi \omega_1 \omega_4 \omega_5) H_1(\omega_1 + \omega_4 + \omega_5) H_1(\omega_2) H_1(\omega_3) \\
& + (1 + \Xi \omega_2 \omega_3 \omega_4) H_1(\omega_2 + \omega_3 + \omega_4) H_1(\omega_1) H_1(\omega_5) \\
& + (1 + \Xi \omega_2 \omega_3 \omega_5) H_1(\omega_2 + \omega_3 + \omega_5) H_1(\omega_1) H_1(\omega_4) \\
& + (1 + \Xi \omega_2 \omega_4 \omega_5) H_1(\omega_2 + \omega_4 + \omega_5) H_1(\omega_1) H_1(\omega_3) \\
& \left. + (1 + \Xi \omega_3 \omega_4 \omega_5) H_1(\omega_3 + \omega_4 + \omega_5) H_1(\omega_1) H_1(\omega_2) \right) \\
& H_1(\omega_1) H_1(\omega_2) H_1(\omega_3) H_1(\omega_4) H_1(\omega_5) H_1(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5)
\end{aligned}$$
(A.20)

Although the derivations are straightforward, the number of terms in the transfer functions increases geometrically with its order. For instance, the seventh order transfer function is made up of 117 terms. This is partly due to the fact that symmetry is required. It is, however, important to be not misled, the simplification resulting from

not requiring symmetric transfer functions would have, at least partly, to be paid in a significant complication of the response statistics discussed throughout chapters 2 and 3.



**APPENDIX B**  
**I<sub>1</sub>, I<sub>2</sub>, J<sub>1</sub> and J<sub>2</sub> Integrals**

$$I_1 = \varepsilon^2 \int_{-\infty}^{+\infty} |H_1(\omega)|^2 \operatorname{Re}\{H_1(\omega)\} S_{xx}(\omega) d\omega \quad (\text{B.1})$$

$$I_2 = \varepsilon^2 \int_{-\infty}^{+\infty} \omega |H_1(\omega)|^2 \operatorname{Im}\{H_1(\omega)\} S_{xx}(\omega) d\omega \quad (\text{B.2})$$

$$J_1 = \varepsilon^6 \iiint_{-\infty}^{+\infty} \operatorname{Re}\{H_1(\omega_1 + \omega_2 + \omega_3)\} |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2 \\ S_{xx}(\omega_1) S_{xx}(\omega_2) S_{xx}(\omega_3) d\omega_1 d\omega_2 d\omega_3 \quad (\text{B.3})$$

$$J_2 = \varepsilon^6 \iiint_{-\infty}^{+\infty} \omega_1 \omega_2 \omega_3 \operatorname{Im}\{H_1(\omega_1 + \omega_2 + \omega_3)\} |H_1(\omega_1)|^2 |H_1(\omega_2)|^2 |H_1(\omega_3)|^2$$

$$S_{xx}(\omega_1)S_{xx}(\omega_2)S_{xx}(\omega_3)d\omega_1d\omega_2d\omega_3$$

(B.4)

While integrals  $I_1$  and  $I_2$  can be evaluated without too much effort,  $J_1$  and  $J_2$  are triple integrals and would involve significant computational effort if evaluated under this form. Instead, we seek the time-domain equivalents of integrals  $J_1$  and  $J_2$ . The following Fourier transform pairs are defined first:

$$\begin{aligned} R_{y_{lin}y_{lin}}(\tau) &= \int_{-\infty}^{+\infty} S_{y_{lin}y_{lin}}(\omega) e^{i\omega\tau} d\omega & S_{y_{lin}y_{lin}}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{y_{lin}y_{lin}}(\tau) e^{-i\omega\tau} d\tau \\ R_{y_{lin}y_{lin}}(\tau) &= \int_{-\infty}^{+\infty} S_{y_{lin}y_{lin}}(\omega) e^{i\omega\tau} d\omega & S_{y_{lin}y_{lin}}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{y_{lin}y_{lin}}(\tau) e^{-i\omega\tau} d\tau \\ f_1(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{Re}\{H_1(\omega)\} e^{i\omega\tau} d\omega & \mathcal{Re}\{H_1(\omega)\} &= \int_{-\infty}^{+\infty} f_1(\tau) e^{-i\omega\tau} d\tau \\ f_2(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathcal{Im}\{H_1(\omega)\} e^{i\omega\tau} d\omega & \mathcal{Im}\{H_1(\omega)\} &= \int_{-\infty}^{+\infty} f_2(\tau) e^{-i\omega\tau} d\tau \end{aligned}$$

Finally,  $J_1$  and  $J_2$  are now given in the time domain by single integrals involving Fourier transforms, for which very efficient algorithms are now well established (fast Fourier transform):

$$J_1 = \varepsilon^6 \int_{-\infty}^{+\infty} f_1(\tau) R_{y_{lin} y_{lin}}^3(\tau) d\tau$$

(B.5)

$$J_2 = \varepsilon^6 \int_{-\infty}^{+\infty} f_2(\tau) R_{y_{lin} y_{lin}}^3(\tau) d\tau$$

(B.6)



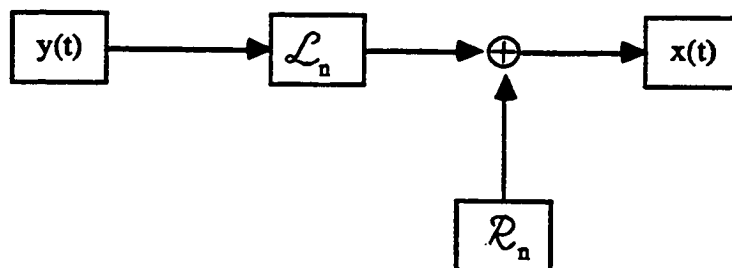


## APPENDIX C

### Generalization of the Linearize-and-Match Method

We will show how the linearize-and-match method can be generalized to the case of asymmetric nonlinear systems the excitation process of which does not necessarily possess a zero mean value. Moreover, we will not only consider nonlinear systems described by ordinary differential equations, but the more general case of nonlinear analytic systems with memory.

Let us consider again the case of a general nonlinear analytic system described by the functional input-output relationship (1.1). Such a functional relationship can be linearized such that an infinite series of systems ( $S_n$ ) characterized by the following diagram results:



Alternatively, the functional relationship describing these systems are:

$$\mathcal{L}_n\{y_n(t)\} + \mathcal{R}_n = x(t) \quad (\text{S}_n) \quad (\text{C.1})$$

where  $\mathcal{L}_n$  is a linear functional, whereas  $\mathcal{R}_n$  denotes an active filter independent of the instantaneous value of response process  $y$ .

Since (S) is an analytic system, the left hand side of equation (1.1) can be written as a Volterra series with kernels  $g_k(\tau_1, \tau_2, \dots, \tau_k)$ . Clearly, the general even-order and odd-order terms:

$$\begin{aligned} & \iint \dots \int g_{2k}^n(\tau_1, \tau_2, \dots, \tau_{2k}) E[y(t - \tau_1)y(t - \tau_2)\dots y(t - \tau_{2k})] d\tau_1 \dots d\tau_{2k} \\ & \text{and} \\ & \iint \dots \int g_{2k+1}^n(\tau_1, \tau_2, \dots, \tau_{2k+1}) E[y(t - \tau_1)y(t - \tau_2)\dots y(t - \tau_{2k})] y(t - \tau_{2k+1}) d\tau_1 \dots d\tau_{2k+1} \end{aligned} \quad (\text{C.2})$$

respectively appear in  $\mathcal{R}_n$  and  $\mathcal{L}_n$  and are substituted respectively to:

$$\begin{aligned} & \iint \dots \int g_{2k}(\tau_1, \tau_2, \dots, \tau_{2k}) y(t - \tau_1)y(t - \tau_2)\dots y(t - \tau_{2k}) d\tau_1 \dots d\tau_{2k} \\ & \text{and} \\ & \iint \dots \int g_{2k+1}(\tau_1, \tau_2, \dots, \tau_{2k+1}) y(t - \tau_1)y(t - \tau_2)\dots y(t - \tau_{2k+1}) d\tau_1 \dots d\tau_{2k+1} \end{aligned} \quad (\text{C.3})$$

in original system (S). The new kernels  $g_{2k}^n$  and  $g_{2k+1}^n$  are determined by matching of the n-th order response moment.

Averaging equation (C.1), we obtain:

$$\mathcal{L}_n\{E[y_n(t)]\} + \mathcal{R}_n = E[x(t)] \quad (C.4)$$

Defining the zero-mean variables  $x^*(t)=x(t)-E[x(t)]$  and  $y^*(t)=y(t)-E[y(t)]$ , and subtracting equation (C.4) from (C.1), the linear systems ( $S_n^*$ ) are obtained:

$$\mathcal{L}_n\{y_n^*(t)\} = x_n^*(t) \quad (S_n^*) \quad (C.5)$$

Equations (C.4) and (C.5), in turn, lead to the mean value and autocorrelation of systems ( $S_n$ ) and ( $S_n^*$ ) respectively:

$$E[y_n(t)] = \mathcal{L}_n^{-1}\{E[x(t)] - \mathcal{R}_n\} \quad (C.6)$$

$$E[y_n^*(t_1)y_n^*(t_2)] = E[\mathcal{L}_n^{-1}\{x^*(t_1)\} \mathcal{L}_n^{-1}\{x^*(t_2)\}] \quad (C.7)$$

Taking advantage of the Gaussian property of process  $y_n^*$ , equations (C.6) and (C.7) represent coupled integral equations in the desired response statistics  $E[y_n(t)]$  and  $E[y_n^*(t_1)y_n^*(t_2)]$ .

Raising  $y_n(t)$  to the  $n$ -th power and averaging, the  $n$ -th order power moment of system ( $S_n$ ) can be obtained:

$$E[y_n^n] = \sum_{k=0}^{E[\frac{n}{2}]} \frac{1}{\sqrt{\pi}} 2^k \Gamma(k+\frac{1}{2}) C_n^{2k} E^{k*2}[y_n] E^{n-2k}[y_n] \quad (C.8)$$

where  $E[n/2]$  denotes the largest integer smaller than  $n/2$ .

The response power moments of system (S) are approximately determined from:

$$E[\tilde{y}^n] = E[y_n^n] \quad (C.9)$$

Whereas, the central moments are:

$$\tilde{\mu}_n = E[(\tilde{y} - E[\tilde{y}])^n] \quad n \geq 2 \quad (C.10)$$

Finally, substituting equations (C.8) and (C.9) into (C.10), we get:

$$\tilde{\mu}_n = \sum_{m=0}^n C_n^m E^{n-m}[y_1] \sum_{k=0}^{E[\frac{m}{2}]} \frac{1}{\sqrt{\pi}} 2^k \Gamma(k+\frac{1}{2}) C_m^{2k} E^{k*2}[y_m] \quad (C.11)$$

These expressions are to be substituted to equations (3.29) in the case of an asymmetric nonlinear system. Whereas the matching procedure is similar to the one exposed throughout Chapter 3.



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