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On extremizers for adjoint Fourier restriction inequalities and a result in incidence geometry

by

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Abstract

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Whenever we have a bounded linear operator $T : X \to Y$ between two Banach spaces X, Y we can ask what nonzero elements $x^* \in X$ satisfy $\|Tx^*\| = \|T\|\|x^*\|$. Such elements of X are called extremizers for the inequality $\|Tx\| \leqslant \|T\|\|x\|$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X satisfying $\|x_n\| \leqslant 1$ and $\|Tx_n\| \to \|T\|$, as $n \to \infty$, is called an extremizing sequence. For extremizing sequences we can ask whether they are precompact after the application of symmetries of the operator T. We can also ask for the value of the operator norm of T, $\|T\|$.

The adjoint Fourier restriction operator associated to a hypersurface S with measure σ in $\mathbb{R}^d$, $f \mapsto \widehat{f\sigma}$, is known to be bounded from L^2 to L^p in the case of the cone, the hyperboloid and the paraboloid in $\mathbb{R}^d$, for a certain range of exponents $p \in [1, \infty]$. Existence and nonexistence of extremizers, precompactness of extremizing sequences, Euler-Lagrange equations for extremizers and best constants is what we study in the first three parts of this dissertation.

In the first part we study the adjoint restriction inequality on the cone, $\Gamma^2 \subset \mathbb{R}^3$. We prove that extremizing sequences for the inequality from $L^2(\Gamma^2)$ to $L^6(\mathbb{R}^3)$ are precompact up to the natural symmetries of the cone, dilations and Lorentz transformations.

In the second part we study extremizers on the hyperboloid in dimensions 3 and 4. We prove that in both cases extremizers do not exist and compute the best constant in the adjoint Fourier restriction inequality.

In the third part, in a joint work with Michael Christ, we consider the case of the paraboloid, or equivalently, Strichartz inequalities for the Shrödinger equation. It is shown there that a natural class of functions, the Gaussians, known to extremize the $L^2 \to L^p$ adjoint Fourier restriction inequalities in dimensions 2 and 3 are no longer critical points, and thus are not extremizers, of the nonlinear functional associated to the $L^q \to L^p$ inequalities for $q \neq 2$. The case of mixed norms is also studied.

In the last chapter we look at an incidence geometry problem, the problem of counting noncoplanar intersections of lines in $\mathbb{R}^d$. The problem can be seen as a discrete version of

the Kakeya problem, an open problem in real analysis. There we prove a sharp upper bound for the number of transverse intersections of a collection of lines.

To my families,

the one in Galvarino, the one in Santiago and the one in Berkeley.

Contents

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Chapter 1

Introduction to extremals for Fourier restriction inequalities

We give here an introduction to the topic of extremals for Fourier restriction inequalities. Without trying to be exhaustive we will give an overview of recent results in this subject as well as discuss the general ideas in some of the results which are relevant for this dissertation.

Let $d \geqslant 2$ and $S \subset \mathbb{R}^d$ be a an n-dimensional submanifold of the Euclidean space and σ a positive Borel measure on S. The adjoint restriction operator, or extension operator, associated to (S, σ) is

$$Tf(x) = \int_S e^{-ix\cdot y} f(y) d\sigma(y), \tag{1.1}$$

defined for $x \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$. The operator T is the formal adjoint of the operator R, defined by $Rg = \hat{g}|_S$, for $g \in \mathcal{S}(\mathbb{R}^d)$, i.e. the restriction of the Fourier transform of g to S.

With the Fourier transform in $\mathbb{R}^d$ defined to be $\hat{g}(x) = \int_{\mathbb{R}^d} e^{-ix\cdot y} g(y) dy$ we have $Tf(x) = \widehat{f\sigma}(x)$.

We are interested in the case where S is a hypersurface in $\mathbb{R}^d$ and the measure $\sigma = \psi \cdot d\mu$, where μ is the surface measure on S and $\psi \in C^\infty(\mathbb{R}^d)$. Examples include the paraboloid, sphere, hyperboloid and cone endowed with their natural measures.

Under conditions on S that include smoothness and nonvanishing Gaussian curvature an estimate of the kind[1]

$$\|Tf\|_{L^p(\mathbb{R}^d)} \leqslant C\|f\|_{L^2(S)} \tag{1.2}$$

holds for a certain range of $p \in [1, \infty]$. Suppose p is such that (1.2) holds and let $\mathcal{R} = \sup_{0 \neq f \in L^2(S)} \|Tf\|_{L^p(\mathbb{R}^d)} \|f\|^{-1}_{L^2(S)}$ be the best constant, then we can talk about extremizers and extremizing sequences:

Definition 1.1. An extremizing sequence for the inequality (1.2) is a sequence $\{f_n\}$ of functions in $L^2(S, \sigma)$ satisfying $\|f_n\|_{L^2(S)} \leqslant 1$, such that $\|\widehat{f_n\sigma}\|_{L^p(\mathbb{R}^d)} \to \mathcal{R}$ as $n \to \infty$.

[1] For different kinds of conditions an range of exponent p we refer to Chapter 8 in [45].

An extremizer for (1.2) is a function $f \neq 0$ which satisfies $\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^d)} = \mathcal{R}\|f\|_{L^2(S)}$.

Our main purpose here is to mention different results that prove the existence of extremizers, the precompactness of extremizing sequences and/or that compute the best constant.

We start with the work of Kunze [31] where he considers the adjoint Fourier restriction on the parabola in $\mathbb{R}^2$, $\mathbb{P}^1 = \{(y, \frac{1}{2}y^2) : y \in \mathbb{R}\}$ with measure $\sigma(y, y') = \delta(y' - \frac{1}{2}y^2)dydy'$, defined as

$$\int f d\sigma = \int_{\mathbb{R}} f(y, \tfrac{1}{2}y^2)dy, \tag{1.3}$$

for $f \in S(\mathbb{R}^2)$.

One has the following estimate [47]

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leqslant C\|f\|_{L^2(\mathbb{P}^1)}. \tag{1.4}$$

Using the technique of concentration-compactness of Lions [32], Kunze proves

Theorem 1.2 ([31])**.** *There exists an extremizer for inequality* (1.4)*. Moreover, nonnegative extremizing sequences are precompact, after the application of symmetries.*

1.1 The method of Foschi

The work of Foschi [20] is of special importance for us. He relies on the equivalent formulation of the restriction inequality in terms of a convolution inequality, obtained via the Fourier transform. In [20], he considers the case of the restriction on the paraboloid (or equivalently, Strichartz estimates for the Schrödinger equation) in dimensions 2 and 3, and the case of Strichartz inequalities for the wave equation in dimensions 3 and 4. This last ones are related to the restriction on the cone, case considered by Carneiro [7], who used the method developed by Foschi. We mention here that Hundertmark and Zharnitsky obtained the same result as Foschi for the restriction on the paraboloid, using a different method. To state their theorems we need to introduce the measures on the respective manifolds.

For $d \geqslant 1$, let $\mathbb{P}^d = \{(y, \frac{1}{2}|y|^2) : y \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$ denote the paraboloid in $\mathbb{R}^{d+1}$. In $\mathbb{P}^d$ we consider the scale invariant measure $\sigma(y, y') = \delta(y' - \frac{1}{2}|y|^2)dydy'$, i.e.

$$\int_{\mathbb{P}^d} f(y, y')d\sigma(y, y') = \int_{\mathbb{R}^d} f(y, \tfrac{1}{2}|y|^2)dy,$$

for all $f \in C_0(\mathbb{R}^{d+1})$. The adjoint Fourier restriction operator is then

$$Tf(x, t) = \widehat{f\sigma}(x, t) = \int_{\mathbb{R}^2} e^{-ix\cdot y} e^{-\frac{1}{2}it|y|^2} f(y)dy,$$

We denote by $\Gamma^d = \{(y, |y|) : y \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$ the cone in $\mathbb{R}^{d+1}$ with measure $\sigma(y, y') = \delta(y' - |y|)|y|^{-1}dydy'$.

In the case of the paraboloid, if $(d,p)=(2,4)$ or $(d,p)=(1,6)$ then (see [47])

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^{d+1})} \leqslant C\|f\|_{L^2(\mathbb{P}^d)}, \tag{1.5}$$

and in the case of the cone, if $(d,p)=(2,6)$ or $(d,p)=(3,4)$ then

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^{d+1})} \leqslant C\|f\|_{L^2(\Gamma^d)}. \tag{1.6}$$

Theorem 1.3 ([20],[28]). *The following inequalities are sharp*

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leqslant \frac{(2\pi)^{\frac{1}{2}}}{3^{\frac{1}{12}}}\|f\|_{L^2(\mathbb{P}^1)},$$
$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant 2^{3/4}\pi\|f\|_{L^2(\mathbb{P}^2)},$$

and there is equality if and only if $f(y)=e^{-a|y|^2+b\cdot y+c}$, *for* $a,c\in\mathbb{C}$, $\mathfrak{Re}\,a>0$ *and* $b\in\mathbb{C}^2$ *or* $\mathbb{C}^3$ *depending on the case.*

Theorem 1.4 ([7]). *The following inequalities are sharp*

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^3)} \leqslant (2\pi)^{5/6}\|f\|_{L^2(\Gamma^2)},$$
$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^4)} \leqslant (2\pi)^{5/4}\|f\|_{L^2(\Gamma^3)},$$

and there is equality if and only if $f(y)=e^{-a|y|+b\cdot y+c}$, *for* $a,c\in\mathbb{C}$, $\mathfrak{Re}\,a>0$ *and* $b\in\mathbb{C}^2$ *or* $\mathbb{C}^3$ *depending on the case.*

By considering the inequality in convolution form, one can reduce the problem to the computation of the convolutions $\sigma*\sigma$ for the inequality from L^2 to L^4, and $\sigma*\sigma*\sigma$ for the inequality from L^2 to L^6.

We will give an idea of Foschi's method with two examples. In the first we will sketch the proof of Theorem 1.3 in the case of the paraboloid in $\mathbb{R}^3$ and in the second, that extremizers do not exist for a perturbation of the paraboloid.

1.1.1 An existence result

Let us consider Theorem 1.3 in three dimensions

Theorem 1.5 ([20], [28]). *For any* $f\in L^2(\mathbb{P}^2)$,

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant 2^{3/4}\pi\|f\|_{L^2(\mathbb{P}^2)}.$$

The inequality is sharp and there is equality if and only if $f(y)=e^{-a|y|^2+b\cdot y+c}$, *for* $a,c\in\mathbb{C}$, $\mathfrak{Re}\,a>0$ *and* $b\in\mathbb{C}^2$.

Foshi's argument relies on the following lemma

Lemma 1.6. *Let $f \in L^2(\mathbb{P}^2)$, then for all $(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R}$ we have*

$$|f\sigma * f\sigma(\xi,\tau)|^2 \leqslant |f|^2\sigma * |f|^2\sigma(\xi,\tau) \cdot \sigma * \sigma(\xi,\tau). \tag{1.7}$$

Before we prove the lemma we give the proof of the theorem

Proof of Theorem 1.5. Using Plancherel's theorem,

$$\|\widehat{f\sigma}\|_{L^4}^2 = \|(\widehat{f\sigma})^2\|_{L^2} = \|(f\sigma * f\sigma)\check{}\|_{L^2} = (2\pi)^{3/2}\|f\sigma * f\sigma\|_{L^2},$$

so we need to show that $\|f\sigma * f\sigma\|_{L^2} \leqslant \pi^{1/2}\|f\|_{L^2}^2$ for all $f \in L^2(\mathbb{R}^2)$ with equality only if f is a Gaussian. Applying the Lemma, and denoting $\mathcal{P}$ the support of $\sigma * \sigma$, gives

$$\begin{aligned}\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)}^2 &\leqslant \int_{\mathcal{P}} |f|^2\sigma * |f|^2\sigma(\xi,\tau) \cdot \sigma * \sigma(\xi,\tau) d\xi d\tau &(1.8)\\ &\leqslant \sup_{(\xi,\tau)\in\mathcal{P}} \sigma * \sigma \cdot \int_{\mathcal{P}} |f|^2\sigma * |f|^2\sigma(\xi,\tau) d\xi d\tau &(1.9)\\ &= \|\sigma * \sigma\|_{L^\infty(\mathbb{R}^3)}\|f\|_{L^2(\mathbb{P}^2)}^4. &(1.10)\end{aligned}$$

It is not hard to compute $\sigma * \sigma$ explicitly. The symmetries of the paraboloid can be used to simplify the calculations. One gets

$$\sigma * \sigma(\xi,\tau) = \pi\chi_{\{\tau \geqslant \frac{1}{4}|\xi|^2\}},$$

and therefore

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leqslant \pi^{1/2}\|f\|_{L^2(\mathbb{P}^2)}^2 \tag{1.11}$$

as desired.

We now show that the inequality is sharp. To have equality in (1.11) we must have equality in (1.8) and (1.9). Since $\sigma * \sigma$ is constant in its support, (1.9) is an equality. For equality in (1.8), we need equality in (1.7) for a.e. (ξ,τ) in the support of $f\sigma * f\sigma$. On the one hand it is easy to show that for $f(y) = e^{-a|y|^2 + b\cdot y + c}$, with a, b, c as in the statement of the theorem, (1.7) becomes equality for all (ξ,τ). Foschi shows that the converse holds by studying a certain functional equation obtained from (1.7) by imposing the Cauchy-Schwarz inequality (1.12) below to be equality. □

Proof of Lemma 1.6. Note that

$$\begin{aligned}f\sigma * f\sigma(\xi,\tau) &= \int_{\mathbb{R}^2} f(y)f(\xi - y)\delta(\tau - \tfrac{1}{2}|y|^2 - \tfrac{1}{2}|\xi - y|^2)dy\\ &= \int_{\mathbb{R}^2} f(y)f(z)\delta(\tau - \tfrac{1}{2}|y|^2 - \tfrac{1}{2}|z|^2)\delta(\xi - y - z)dzdy\\ &= \int_{\mathbb{R}^2} f(y)f(z)\,\boldsymbol{\delta}\begin{pmatrix}\tau - \frac{1}{2}|y|^2 - \frac{1}{2}|z|^2\\ \xi - y - z\end{pmatrix}dzdy.\end{aligned}$$

For each (ξ,τ), the measure

$$\mu_{(\xi,\tau)} = \boldsymbol{\delta}\begin{pmatrix}\tau-\frac12|y|^2-\frac12|z|^2\\ \xi-y-z\end{pmatrix}dzdy$$

is defined as the pullback of the Dirac delta on $\mathbb{R}\times\mathbb{R}^2$ by the function $\Phi_{(\xi,\tau)}:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}\times\mathbb{R}^2$ given by

$$\Phi_{(\xi,\tau)}(y,z) = (\tau-\tfrac12|y|^2-\tfrac12|z|^2,\xi-y-z).$$

Denote by $(f\otimes g)(y,z)=f(y)g(z)$, so that

$$f\sigma * f\sigma(\xi,\tau) = \langle f\otimes f, 1\otimes 1\rangle_{(\xi,\tau)}.$$

Using the Cauchy-Schwarz inequality we obtain

$$|f\sigma * f\sigma(\xi,\tau)|^2 \leqslant \|f\otimes f\|^2_{(\xi,\tau)}\|1\otimes 1\|^2_{(\xi,\tau)}. \tag{1.12}$$

Now,

$$\begin{aligned}\|f\otimes g\|^2_{(\xi,\tau)} &= \int |f|^2(y)|g|^2(z)\,\boldsymbol{\delta}\begin{pmatrix}\tau-\frac12|y|^2-\frac12|z|^2\\ \xi-y-z\end{pmatrix}dzdy\\ &= |f|^2\sigma * |g|^2\sigma(\xi,\tau)\end{aligned}$$

for all $f,g\in\mathcal{S}(\mathbb{R}^3)$, and the result follows. □

1.1.2 A nonexistence result

Here we consider a perturbed paraboloid. For $a>0$ let $S_a=\{(y,\frac12|y|^2+a|y|^4):y\in\mathbb{R}^2\}\subset\mathbb{R}^3$, endowed with the measure $\sigma_a(y,y')=\delta(y'-\frac12|y|^2-a|y|^4)dydy'$. There exists $C=C_a<\infty$ such that for all $f\in L^2(S_a)$ the following inequality holds

$$\|\widehat{f\sigma_a}\|_{L^4(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(S_a)}. \tag{1.13}$$

We have the following result,

Theorem 1.7. *Let $a>0$. For all $f\in L^2(S_a)$*

$$\|\widehat{f\sigma_a}\|_{L^4(\mathbb{R}^3)} \leqslant 2^{3/4}\pi\|f\|_{L^2(S_a)}.$$

The inequality is sharp and there are no extremizers.

Remark 1.8. Note that the measure σ_a is not the surface measure on S_a. On the one hand, if surface measure is used, then it is not hard to see that an inequality as (1.13) does not hold. On the other hand, if surface measure, $\sigma_{a,\rho}$, is used on the truncation of S_a, $S_{a,\rho}=\{(y,\frac12|y|^2+a|y|^4):y\in\mathbb{R}^2,\ |y|\leqslant\rho\}$ for $\rho>0$, then an inequality as (1.13) holds.

The existence of extremizers depends on the relationship between a and ρ. The first step in proving precompactness of extremizing sequences is to obtain a lower bound for the best constant, of the form

$$\mathbf{C}_{a,\rho} := \sup_{f\in L^2(S_{a,\rho},\sigma_{a,\rho})} \|\widehat{f\sigma_{a,\rho}}\|_{L^4(\mathbb{R}^3)}\|f\|^{-1}_{L^2(S_{a,\rho})} > 2^{3/4}\pi. \tag{1.14}$$

Inequality (1.14) is a necessary condition for the argument in [9] to work. A stronger condition is needed to prove that every extremizing sequence is precompact, as one needs to rule out concentration at every point on $S_{a,\rho}$, not just at the vertex $(0,0)\in S_{a,\rho}$ (see Proposition 1.17). A sufficient condition to dismiss concentration seems to be

$$\mathbf{C}_{a,\rho} > (2\pi)^{3/4}\sup_{y\in\mathcal{P}} |\sigma_{a,\rho}*\sigma_{a,\rho}(y)|^{1/4},$$

where $\mathcal{P}$ is the part of the boundary of the support of $\sigma_{a,\rho}*\sigma_{a,\rho}$, contained in the surface $\{(y,\frac14|y|^2+\frac{a}{8}|y|^8) : y\in\mathbb{R}^2\}$. Observe that (1.14) is equivalent to the weaker condition $\mathbf{C}_{a,\rho} > (2\pi)^{3/4}|\sigma_{a,\rho}*\sigma_{a,\rho}(0)|^{1/4}$.

If ρ is sufficiently large, independent of a, then (1.14) holds for all $a>0$. For small ρ, it is possible to show that if a is small, then (1.14) holds, and if a is large, then (1.14) does not hold. More precisely, there exists $a_0\in[\frac18,\frac14]$ with the property that for all $a>a_0$, there exists $\rho_0>0$ such that $\mathbf{C}_{a,\rho}=2^{3/4}\pi$ if $\rho<\rho_0$ and $\mathbf{C}_{a,\rho}>2^{3/4}\pi$ if $\rho>\rho_0$, and if $a<a_0$ then $\mathbf{C}_{a,\rho}>2^{3/4}\pi$ for all $\rho>0$. We do not prove this here.

As for the case of the paraboloid, the proof of Theorem 1.7 reduces to the calculation of $\sigma_a*\sigma_a$.

Lemma 1.9. *For any $a>0$ we have*

$$\sigma_a*\sigma_a(\xi,\tau) \leqslant \frac{\pi\chi(\tau\geqslant\frac14|\xi|^2+\frac{a}{8}|\xi|^4)}{\big(8a(\tau-\frac14|\xi|^2-\frac{a}{8}|\xi|^4)+(1+a|\xi|^2)^2\big)^{1/2}}. \tag{1.15}$$

Sketch of proof. In the case $a>0$ there are no exact symmetries, so we write

$$\sigma*\sigma(\xi,\tau)=\int_{\mathbb{R}^2}\delta(\tau-\tfrac12|\xi-y|^2-\tfrac12|y|^2-a|\xi-y|^4-a|y|^4)dy.$$

The use of the change of variables $\eta=\frac12\xi-y$, polar coordinates and a few more changes of variables gives the formula

$$\sigma*\sigma(\xi,\tau)=\chi(\tau\geqslant\tfrac14|\xi|^2+\tfrac{a}{8}|\xi|^4)\int_0^{\pi/2}\frac{d\theta}{(2a(\tau-\frac{|\xi|^2}{4}-\frac{a|\xi|^4}{8}+2ah^2(a,\xi,\theta)))^{1/2}}, \tag{1.16}$$

where $h(a,\xi,\theta)=\frac{1}{4a}(1+a|\xi|^2(1+2\cos^2\theta))$.

For $0 \leqslant \theta \leqslant \frac{\pi}{2}$,

$$\begin{aligned} 2a(\tau - \tfrac{1}{4}|\xi|^2 - \tfrac{a}{8}|\xi|^4) + 4a^2h^2(a,\xi,\theta) &= 2a(\tau - \tfrac{1}{4}|\xi|^2 - \tfrac{a}{8}|\xi|^4) + \tfrac{1}{4}(1 + a|\xi|^2(1 + 2\cos^2\theta))^2 \\ &\geqslant 2a(\tau - \tfrac{1}{4}|\xi|^2 - \tfrac{a}{8}|\xi|^4) + \tfrac{1}{4}(1 + a|\xi|^2)^2. \end{aligned}$$

Then

$$\sigma_a * \sigma_a(\xi,\tau) \leqslant \frac{\pi\chi(\tau \geqslant \frac{1}{4}|\xi|^2 + \frac{a}{8}|\xi|^4)}{\left(8a(\tau - \frac{1}{4}|\xi|^2 - \frac{a}{8}|\xi|^4) + (1 + a|\xi|^2)^2\right)^{1/2}}.$$

□

Proof of Theorem 1.7. The same argument as for the paraboloid gives the inequality

$$\|f\sigma_a * f\sigma_a\|_{L^2(\mathbb{R}^3)} \leqslant \|\sigma_a * \sigma_a\|_{L^\infty(\mathbb{R}^3)}^{1/2} \|f\|_{L^2(S_a)}^2. \tag{1.17}$$

Lemma 1.9 and (1.16) imply that $\|\sigma_a * \sigma_a\|_{L^\infty(\mathbb{R}^3)} = \pi$ for all $a > 0$, just as the case of the paraboloid, that corresponds to $a = 0$. Hence

$$\|f\sigma_a * f\sigma_a\|_{L^2(\mathbb{R}^3)} \leqslant \pi^{1/2} \|f\|_{L^2(S_a)}^2.$$

To show that the inequality is sharp we consider the extremizing sequence $\{f_n/\|f_n\|_2\}_{n\in\mathbb{N}}$, where $f_n(y) = e^{-n(\frac{1}{2}|y|^2 + a|y|^4)}$. Since f_n is the restriction of the linear function in $\mathbb{R}^3$, e^{-nx_3}, to S_a, one sees that $f_n\sigma_a * f_n\sigma_a(\xi,\tau) = e^{-n\tau}\sigma_a * \sigma_a(\xi,\tau)$. A calculation shows

$$\lim_{n\to\infty} \|f_n\sigma_a * f_n\sigma_a\|_{L^2} \|f_n\|_{L^2}^{-2} = \pi^{1/2}.$$

To prove that extremizers do not exist we note that (1.15) implies that

$$\sigma_a * \sigma_a(\xi,\tau) < \pi \text{ for all } (\xi,\tau) \neq (0,0),$$

and therefore (1.17) is a strict inequality whenever $f \neq 0$ as can be seen from the equivalent of (1.9) for S_a. □

This very same method allows us to prove that for the hyperboloid $\mathbb{H}^d = \{(y, \sqrt{1+|y|^2}) : y \in \mathbb{R}^d\}$ with Lorentz invariant measure $\sigma(y,y') = \delta(y' - \sqrt{1+|y|^2})\frac{dydy'}{\sqrt{1+|y|^2}}$, extremizers do not exist for the inequalities

$$\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^{d+1})} \leqslant C\|f\|_{L^2(\mathbb{H}^d)} \tag{1.18}$$

for (d,p) equal to $(2,4), (2,6)$ and $(3,4)$. We also compute the value of the best constant for those three cases of (d,p).

1.2 A refinement of the adjoint Fourier restriction inequality

Moyua, Vargas and Vega gave in [34] a refinement to the Fourier extension inequality. This refinement will be important for the proof of Christ and Shao discussed in Section 1.3 and we will obtain an analogous result for the cone in $\mathbb{R}^3$. To state their result we let $\Phi : B(0,2) \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function satisfying

$$\det(\partial^2_{x_i x_j}\Phi) > 1,$$

for all $x \in B(0,2) = \{x \in \mathbb{R}^2 : |x| \leqslant 2\}$. This implies that the surface $S = \{(y, \Phi(y)) : |y| \leqslant 1\}$ has nonvanishing Gaussian curvature and thus Tf defined by

$$Tf(x,t) = \widehat{f\sigma}(x,t) = \int_{|y|\leqslant 1} e^{-ix\cdot y} e^{-it\Phi(y)} f(y)dy \tag{1.19}$$

satisfies

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(|y|\leqslant 1)}, \tag{1.20}$$

for all $f \in L^2(B(0,1))$, for $C < \infty$ independent of f.

For $1 < p < 2$ the cap space X_p is defined as

$$X_p = \Big\{ f : \|f\|_{X_p} := \Big(\sum_{\delta,k} \delta^4 \Big(\frac{1}{|\tau^k_\delta|} \int_{\tau^k_\delta} |f|^p dx \Big)^{4/p} \Big)^{1/4} < \infty \Big\}, \tag{1.21}$$

where, for $\delta = 2^{-j}$, $j = 1, 2, \ldots$, $\{\tau^k_\delta\}_{k\in\mathbb{N}}$ denotes a grid of squares with disjoint interior and dimensions $\delta \times \delta$.

Moyua, Vargas and Vega prove the following,

Theorem 1.10. *For every $p \geqslant 12/7$ there exists $C_p < \infty$ with the property that for all $f \in X_p$,*

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant C_p\|f\|_{X_p}. \tag{1.22}$$

They also prove that X_p is a bigger space than L^2, provided that $p < 2$,

Proposition 1.11. *Given $1 < p < 2$ there exists $C < \infty$ such that for every $f \in X_p$, $\|f\|_{X_p} \leqslant C\|f\|_{L^2}$.*

Rogers and Vargas [40] obtain the same type of refinement for the adjoint Fourier restriction inequality on the saddle $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_1 x_2\}$ where the measure $\delta(x_3 - x_1 x_2)dx_1dx_2dx_3$ is used. Moreover they refine the inequality in Proposition 1.11 in an analogous way to that obtained by Christ and Shao for the case of the sphere. In [9], that refinement is used to obtain a "cap decomposition" of a given function $f \in L^2(S^2)$. We will discuss this in the next section.

1.3 The concentration-compactness argument of Christ and Shao

In this section we give a general idea of the proof given by Christ and Shao for the existence of extremizers for the Fourier extension inequality on the sphere. The work of Christ and Shao [9] is the first that considers the case of a compact manifold, in this case the sphere $S^2 \subset \mathbb{R}^3$. They develop a general concentration compactness argument suitable to be used for other manifolds.

Let $S^2 = \{y \in \mathbb{R}^3 : |y| = 1\}$ denote the sphere in $\mathbb{R}^3$ and σ the surface measure on S^2. The adjoint Fourier restriction operator on S^2 is defined by

$$Tf(x) = \widehat{f\sigma}(x) = \int_{S^2} e^{-ix\cdot y} f(y) d\sigma(y). \tag{1.23}$$

for $x \in \mathbb{R}^3$ and $f \in L^2(S^2)$. With the Fourier transform defined to be $\hat{g}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} g(x)dx$ we see that $Tf(x) = \widehat{f\sigma}(x)$. The Thomas-Stein inequality for the adjoint Fourier restriction operator states that there exists $C < \infty$ such that

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(S^2)}, \tag{1.24}$$

for all $f \in L^2(S^2)$.

Denote by $\mathcal{R}$ the best constant in (1.24),

$$\mathcal{R} = \sup_{0\neq f\in L^2(S^2)} \frac{\|Tf\|_{L^4(\mathbb{R}^3)}}{\|f\|_{L^2(S^2)}}. \tag{1.25}$$

The use of Plancherel's Theorem allows us to rewrite (1.24) as a convolution inequality,

$$\|\widehat{f\sigma}\|_{L^4}^2 = \|(\widehat{f\sigma})^2\|_{L^2} = \|(f\sigma * f\sigma)\widehat{}\,\|_{L^2} = (2\pi)^{3/2}\|f\sigma * f\sigma\|_{L^2}.$$

Then (1.24) is equivalent to

$$\|f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(S^2)}^2. \tag{1.26}$$

The convolution form of (1.24) implies that $\|\widehat{|f|\sigma}\|_4 \geqslant \|\widehat{f\sigma}\|_4$ for all $f \in L^2(S^2)$ and so in the analysis of extremizers and extremizing sequences we can restrict to the case where the functions are nonnegative.

The main theorem in [9] is

Theorem 1.12. *There exists an extremizer in $L^2(S^2)$ for the inequality* (1.24). *Moreover, any extremizing sequence of nonnegative functions in $L^2(S^2)$ for the inequality* (1.24) *is precompact.*

A symmetry of the functional T allows to reduce to the case of even functions, functions satisfying $f(x) = f(-x)$ for all $x \in S^2$. So in what follows we will assume that the extremizing sequences are of even functions.

The argument in [9] relies on a refinement of (1.24) coming from [34] as stated in Theorem 1.10. We will restate the result for the case of the sphere.

A cap $\mathcal{C} = \mathcal{C}(z, r)$ with center $z \in S^2$ and radius $r \in (0, \sqrt{2}]$ is the set of all points $y \in S^2$ such that $|y - z| < r$ where $|\cdot|$ denotes the euclidean distance in $\mathbb{R}^3$. For each $k \geqslant 0$ choose a maximal subset $\{z_k^j\} \subset S^2$ satisfying $|z_k^j - z_k^i| \geqslant 2^{-k}$ for all $i \neq j$. Then the caps $\mathcal{C}_k^j = \mathcal{C}(z_k^j, 2^{-k})$ cover S^2 for each k and they have finite overlap, that is, there exists a constant C, independent of k, such that a point in S^2 belongs to no more than C caps $\mathcal{C}_k^j$. This is the analog of the grid $\{\tau_\delta^k\}$ of Section 1.2. For $p \geqslant 1$ the X_p norm is defined by

$$\|f\|_{X_p}^4 = \sum_{k=0}^{\infty} \sum_j 2^{-4k} \Big(\frac{1}{|\mathcal{C}_k^j|} \int_{\mathcal{C}_k^j} |f|^p d\sigma \Big)^{4/p}.$$

Theorem 1.10 implies that for $p \geqslant 12/7$ there exists $C < \infty$ such that for any $f \in L^2(S^2)$,

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant C \|f\|_{X_p}. \tag{1.27}$$

What is important about p is that one can take it in the interval $(1, 2)$. From Proposition 1.11 the X_p norm is bounded by the L^2 norm, but a further refinement is needed for the argument to work. The following is proved in [9]

Lemma 1.13. *For any $p \in [1, 2)$ there exists $C < \infty$ and $\gamma > 0$ such that for any $f \in L^2(S^2)$,*

$$\|f\|_{X_p} \leqslant C \|f\|_{L^2}^{1-\gamma} \Big(\sup_{k,j} |\mathcal{C}_k^j|^{-1/2} \int_{\mathcal{C}_k^j} |f| \Big)^{\gamma}. \tag{1.28}$$

Putting together (1.28) and (1.27) gives

$$\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant C \|f\|_{L^2}^{1-\gamma} \Big(\sup_{k,j} |\mathcal{C}_k^j|^{-1/2} \int_{\mathcal{C}_k^j} |f| \Big)^{\gamma}.$$

This tells us that if $\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)}$ is large then there exists a cap $\mathcal{C}$ such that f puts comparatively large mass in $\mathcal{C}$. This can be made rigorous and the result is a "cap decomposition" of a function $f \in L^2(S^2)$, $f = \sum_{\nu=0}^{\infty} f_\nu$, where, for each ν, f_ν is supported in a pair of antipodal caps and the f_ν's have disjoint supports. The previous sum is L^2 convergent. More useful properties of the decomposition can be obtained if f is a δ-nearly extremal, i.e.

$$\|\widehat{f\sigma}\|_{L^4} \geqslant (1 - \delta)\mathcal{R} \|f\|_{L^2}.$$

The final purpose of the decomposition is to prove a concentration-compactness result in the spirit of that of Lions [32]. This we record in Proposition 1.17 below, but before we introduce some definitions and general results in measure theory, taken from [26].

Let $(X, \mathcal{B}, \mu)$ be a measure space,

Definition 1.14. Let $p \in [1, \infty)$. A subset $\mathcal{H}$ of $L^p(X)$ is called equiintegrable of order p if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable subset A of X of μ-measure at most δ,

$$\int_A |f|^p d\mu \leqslant \varepsilon, \text{ for all } f \in \mathcal{H}.$$

It is an easy exercise to show

Proposition 1.15. *Let $\mathcal{H}$ be a bounded subset of $L^p(X)$. Then, $\mathcal{H}$ is equiintegrable of order p if and only if*

$$\lim_{R\to\infty} \int_{\{|f|>R\}} |f|^p d\mu = 0, \tag{1.29}$$

uniformly with respect to $f \in \mathcal{H}$.

In the case of finite measure the following holds,

Proposition 1.16. *Suppose $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence in $L^p(X)$ and let $f \in L^p(X)$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f in L^p if and only if the following two conditions are satisfied:*

1. *$\{f_n\}_{n\in\mathbb{N}}$ converges in measure to f,*
2. *The family $\{f_n : n \in \mathbb{N}\}$ is equiintegrable of order p.*

Now returning to the extremizers for the sphere, for $z \in S^2$ we say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $L^2(S^2)$, satisfying $\|f_n\|_2 \to 1$ as $n \to \infty$, is concentrating at the pair $\{z, -z\}$ if for every $\varepsilon, r > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$

$$\int_{\min(|x-z|,|x+z|)\geqslant r} |f_n(x)|^2 dx < \varepsilon.$$

The cap decomposition argument applied to δ-nearly extremals imply the following concentration-compactness result.

Proposition 1.17. *Let $\{f_n\}_{n\in\mathbb{N}}$ be an extremizing sequence for* (1.24) *of even nonnegative functions in $L^2(S^2)$. Then there exists a subsequence, again denoted $\{f_n\}_{n\in\mathbb{N}}$, and a decomposition $f_n = F_n + G_n$ where F_n, G_n are even and nonnegative with disjoint supports, $\lim_{n\to\infty} \|G_n\|_2 = 0$, and $\{F_n\}_{n\in\mathbb{N}}$ satisfies one of the two possibilities:*

(i) $\{F_n : n \in \mathbb{N}\}$ is equiintegrable of order 2.

(ii) $\{F_n\}_{n\in\mathbb{N}}$ concentrates at a pair of antipodal points.

Of course, since $\|G_n\|_2 \to 0$ as $n \to \infty$, then $\{f_n\}_{n\in\mathbb{N}}$ also satisfies one of the two possibilities above. We write the proposition in that way so one sees it follows from Proposition 2.7 in [9]. The dichotomy comes from Proposition 2.7 in [9] where the decomposition $f_n = F_n + G_n$ is accompanied by the existence of a cap $\mathcal{C}_n = \mathcal{C}(z_n, r_n)$. If $\limsup_{n\to\infty} r_n > 0$, then (i) holds in Proposition 1.17, while if $\limsup_{r\to\infty} r_n = 0$, (ii) holds.

The desirable case is the equiintegrability of $\{F_n\}_{n\in\mathbb{N}}$ and the concentration must be ruled out. To prove the precompactness we will use the following result of Fanelli, Vargas and Visciglia from [18], for the case of the sphere.

Proposition 1.18 ([18]). *Let $T : L^2(S^2) \to L^4(\mathbb{R}^3)$ be the Fourier extension operator defined in* (1.23). *Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(S^2)$ such that:*

(i) $\|f_n\|_2 = 1$;

(ii) $\lim_{n\to\infty} \|Tf_n\|_{L^4(\mathbb{R}^3)} = \|T\|_{\mathcal{L}(L^2(S^2),L^4(\mathbb{R}^3))}$;

(iii) $f_n \rightharpoonup f \neq 0$;

(iv) $Tf_n \to Tf$ *a.e. in* $\mathbb{R}^3$.

Then $f_n \to f$ in $L^2(S^2)$, in particular $\|f\|_2 = 1$ and $\|Tf\|_{L^4(\mathbb{R}^3)} = \|T\|_{\mathcal{L}(L^2(S^2),L^4(\mathbb{R}^3))}$.

This result says that the only obstruction for the precompactness of an extremizing sequence is that every weak limit is zero. We show that under the equiintegrability condition, nonnegative extremizing sequences have nonzero weak limits.

Proposition 1.19. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of L^2-normalized nonnegative functions. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is equiintegrable of order 2, then every L^2-weak limit is nonzero.*

Proof. Since $\|f_n\|_2 = 1$ for all n, the set of L^2-weak limits is nonempty. After passing to a subsequence we can assume $f_n \rightharpoonup f$, as $n \to \infty$, for some $f \in L^2(S^2)$. Suppose $f = 0$ a.e. in S^2. Then, by the weak convergence it follows that

$$\int_{S^2} f_n(y)d\sigma(y) \to \int_{S^2} f(y)d\sigma(y) = 0, \text{ as } n \to \infty.$$

Since $f_n \geqslant 0$ this tells us that $\{f_n\}_{n\in\mathbb{N}}$ converges to 0 in $L^1(S^2)$ and thus $f_n \to 0$ in measure. In view of Proposition 1.16, $f_n \to 0$ in $L^2(S^2)$, and so $1 = \|f_n\|_2 \to 0$ as $n \to \infty$, which is a contradiction. Thus, $f \neq 0$ as was to be shown. □

This easily implies

Corollary 1.20. *Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$ be as in Proposition 1.17. Suppose that $\{F_n\}_{n\in\mathbb{N}}$ satisfies condition (i). Then $\{f_n\}_{n\in\mathbb{N}}$ is precompact.*

Proof. Since $f_n = F_n + G_n$ and F_n, G_n have disjoint supports, we have $\|F_n\|_2 \to 1$ as $n \to \infty$. Also,

$$\|TF_n\|_4 \geqslant \|Tf_n\|_4 - \|TG_n\|_4 \geqslant \|Tf_n\|_4 - \mathcal{R}\|G_n\|_2,$$

thus $\|TF_n\|_4 \to \mathcal{R}$ as $n \to \infty$. Therefore $\{F_n\}_{n\in\mathbb{N}}$ is an extremizing sequence of nonnegative functions. By assumption $\{F_n\}_{n\in\mathbb{N}}$ is equiintegrable of order 2, so every L^2 weak limit is nonzero. For a subsequence, called the same, $F_n \rightharpoonup F$, for some $0 \neq F \in L^2(S^2)$. Every condition in Proposition 1.18 is satisfied by $\{F_n\}_{n\in\mathbb{N}}$ and so $F_n \to F$ in $L^2(S^2)$ and F is an extremizer to (1.24). It follows that $f_n \to F$ as $n \to \infty$ in $L^2(S^2)$, and so $\{f_n\}_{n\in\mathbb{N}}$ is precompact. □

To show that the second possibility in Proposition 1.17 is not possible we will use Foschi's argument. Let $\mathcal{C} = \mathcal{C}(z, r)$ be a cap in S^2 and consider the adjoint Fourier restriction operator on $\mathcal{C}$, $T_{z,r}$,

$$T_{z,r}f(x) = \int_{\mathcal{C}} e^{-ix\cdot y} f(y)d\sigma(y).$$

We are interested in $\|T_{z,r}\| = \sup_{0\neq f\in L^2(\mathcal{C})} \|T_{z,r}f\|_{L^4}\|f\|_{L^2}^{-1}$.

By rotation invariance, it is enough to analyze $T_{z,r}$ when z is the north pole of S^2, and so we will drop the subscript z and write T_r for $T_{z,r}$. We prove the following proposition.

Proposition 1.21. *For any $r > 0$, $\|T_r\| \geqslant 2^{3/4}\pi$ and $\lim_{r\to 0^+} \|T_r\| = 2^{3/4}\pi$.*

It will be important for the argument to consider the norm of the adjoint restriction operator associated to two antipodal caps, $\mathcal{C}(z, r) \cup \mathcal{C}(-z, r)$. This can be written as $\widetilde{T}_{z,r} := T_{z,r} + T_{-z,r}$, when $r < \sqrt{2}$, so the two caps are disjoint. When considering the norm we can drop the subscript z. From Proposition 1.21 we obtain

Corollary 1.22. *Let $r \in (0, 1/2)$. Then $\|\widetilde{T}_r\| = (3/2)^{1/4}\|T_r\|$. Thus $\lim_{r\to 0^+} \|\widetilde{T}_r\| = 2^{1/2}3^{1/4}\pi$.*

Before proving Proposition 1.21 and Corollary 1.22 we will prove that condition (ii) in Proposition 1.17 can not happen.

Lemma 1.23. *Let $\{f_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ and $\{F_n\}_{n\in\mathbb{N}}$ be as in Proposition 1.17. Then $\{F_n\}_{n\in\mathbb{N}}$ satisfies condition (i).*

Proof. By using the function 1 one obtains a lower bound on $\|T\| = \mathcal{R} \geqslant \|T1\|_{L^4}\|1\|_{L^2(S^2)}^{-1} = 2\pi$. Since $2\pi > 2^{1/2}3^{1/4}\pi$ we get $\|T\| > \lim_{r\to 0^+} \|\widetilde{T}_{z,r}\|$, for any $z \in S^2$.

By contradiction, suppose $\{F_n\}_{n\in\mathbb{N}}$ satisfies condition (ii), that is, it is concentrating at the pair $\{z, -z\}$, for some $z \in S^2$. Let $r_0 > 0$ be such that

$$\|\widetilde{T}_{z,r}\| < \|T\|, \text{ for all } r < r_0.$$

Consider $h_n := (\chi_{\mathcal{C}(z,r_0)} + \chi_{\mathcal{C}(-z,r_0)})F_n$. Then

$$\|Th_n\|_{L^4} \geqslant \|TF_n\|_{L^4} - \|T(F_n - h_n)\|_{L^4} \geqslant \|TF_n\|_{L^4} - \mathcal{R}\|F_n - h_n\|_2. \tag{1.30}$$

By the concentration assumption

$$\lim_{n\to\infty} \|F_n - h_n\|_2 = 0. \tag{1.31}$$

As noted in the proof of Corollary 1.20, $\{F_n\}_{n\in\mathbb{N}}$ is an extremizing sequence. This together with (1.30) and (1.31) imply that $\{h_n\}_{n\in\mathbb{N}}$ is an extremizing sequence for (1.24). On the other hand, $Th_n = \widetilde{T}_{z,r}h_n$, therefore $\mathcal{R} = \lim_{n\to\infty} \|\widetilde{T}_{z,r_0}h_n\| < \mathcal{R}$, which is a contradiction. □

Proof of Corollary 1.22. The condition $r < 1/2$ ensures that if f is supported on $\mathcal{C}(z,r)$ and g in $\mathcal{C}(-z,r)$, then $f\sigma * f\sigma, g\sigma * g\sigma$ and $f\sigma * g\sigma$ have disjoint supports. The rest follows as in [20, pg. 754-755] or [37]. We give the argument again here.

For a function $f \in L^2(\mathcal{C}(z,r) \cup \mathcal{C}(-z,r))$ we can write $f = f_+ + f_-$, where f_+ is supported on $\mathcal{C}(z,r)$, and f_- on $\mathcal{C}(-z,r)$. One then has $\|f\|^2_{L^2(S^2)} = \|f_+\|^2_{L^2(S^2)} + \|f_-\|^2_{L^2(S^2)}$.

Observe that

$$\begin{aligned}\|\widetilde{T}_{z,r}f\|^4_{L^4} &= \|Tf_+ + Tf_-\|^4_{L^4} = \|(Tf_+ + Tf_-)^2\|^2_{L^2}\\ &= \|(Tf_+)^2 + (Tf_-)^2 + 2(Tf_+)(Tf_-)\|^2_{L^2}.\end{aligned}$$

Using that product transforms into convolution under the Fourier transform we see that the Fourier transforms of $(Tf_+)^2$, $(Tf_-)^2$ and $(Tf_+)(Tf_-)$ are supported on disjoint sets, therefore

$$\begin{aligned}\|\widetilde{T}_{z,r}f\|^4_{L^4} &= \|Tf_+\|^4_{L^4} + \|Tf_-\|^4_{L^4} + 4\|(Tf_+)(Tf_-)\|^2_{L^2}\\ &\leqslant \|Tf_+\|^4_{L^4} + \|Tf_-\|^4_{L^4} + 4\|(Tf_+)\|^2_{L^2}\|(Tf_-)\|^2_{L^2} \qquad (1.32)\\ &\leqslant \|T_r\|^4(\|f_+\|^4_{L^2} + \|f_-\|^4_{L^2} + 4\|f_+\|^2_{L^2}\|f_-\|^2_{L^2}) \qquad (1.33)\\ &\leqslant \frac{3}{2}\|T_r\|^4(\|f_+\|^2_{L^2} + \|f_-\|^2_{L^2})^2 \qquad (1.34)\\ &= \frac{3}{2}\|T_r\|^4\|f\|^4_{L^2},\end{aligned}$$

where we have used the sharp inequality (as in [20])

$$X^2 + Y^2 + 4XY \leqslant \frac{3}{2}(X+Y)^2, \quad X, Y \geqslant 0$$

where equality holds if and only if $X = Y$. Thus, for all $f \in L^2(\mathcal{C}(z,r) \cup \mathcal{C}(-z,r))$

$$\|\widetilde{T}_{z,r}f\|^4_{L^4}\|f\|^{-4}_{L^2(S^2)} \leqslant \frac{3}{2}\|T_r\|^4.$$

To obtain the lower bound we observe that if $\{f_n\}_{n\in\mathbb{N}}$ is an extremizing sequence for $T_{z,r}$, then $\{\frac{1}{\sqrt{2}}(f_n + g_n)\}_{n\in\mathbb{N}}$ is an extremizing sequence for $\widetilde{T}_{z,r}$, where $g_n(x) = f_n(-x)$. This is so because with this choice of g_n, (1.32) and (1.34) become an equality and (1.33) becomes an equality in the limit as $n \to \infty$. □

Proof of Proposition 1.21. For σ, the surface measure on S^2, one calculates that

$$\sigma * \sigma(x) = \frac{2\pi}{|x|}\chi_{|x|\leqslant 2}.$$

Let σ_r be the restriction of σ to $\mathcal{C}(z,r)$. Clearly $\sigma_r * \sigma_r(x) \leqslant \sigma * \sigma(x)$ for all $x \in \mathbb{R}^3$ and $\sigma_r * \sigma_r(x) = \sigma * \sigma(x)$ for all x in a neighborhood of z. From the formula of the double convolution of σ we obtain

$$\|\sigma_r * \sigma_r\|_{L^\infty(\mathbb{R}^3)} \leqslant \pi + o_r(1), \text{ where } o_r(1) \to 0 \text{ as } r \to 0^+. \tag{1.35}$$

Foschi's argument implies that $\|T_r\| \leqslant (2\pi)^{3/4}\|\sigma_r * \sigma_r\|_{L^\infty(\mathbb{R}^3)}^{1/4}$ and by using a family of linear functions in $\mathbb{R}^3$ restricted to $\mathcal{C}(z,r)$ which concentrate at z we obtain $\|T_r\| \geqslant (2\pi)^{3/4}\pi^{1/4}$. Thus $\|T_r\| \to 2^{3/4}\pi$ as $r \to 0^+$. □

1.4 Existence of extremals in the nonendpoint case

In this section we discuss the results of Fanelli, Vega and Visciglia contained in [18]. In [18] they consider the problem of existence of extremizers for Fourier restriction inequalities in the nonendpoint case, for compactly supported measures.

In this section, $d \geqslant 1$ and μ will denote a finite, positive and compactly supported measure on $\mathbb{R}^d$. The Fourier extension or adjoint Fourier restriction operator T_μ is defined by

$$T_\mu f(x) = \int_{\mathbb{R}^d} e^{-ix\cdot y}h(y)d\mu(y), \tag{1.36}$$

for all $x \in \mathbb{R}^d$ and $f \in S(\mathbb{R}^d)$. We will say that μ satisfies the restriction condition with respect to $p \in [1,\infty]$, denoted as $(RC)_p$, if T_μ is a bounded operator from $L^2(\mu)$ to $L^p(\mathbb{R}^d)$.

Since μ is finite, we have $\|T_\mu f\|_\infty \leqslant \|f\|_2\|\mu\|^{1/2}$, where $\|\mu\| = \mu(\mathbb{R}^d)$, thus μ has the $(RC)_\infty$. An interpolation argument shows that if μ has the $(RC)_p$ then it has the $(RC)_q$ for all $p \leqslant q \leqslant \infty$.

Examples of measures are surface measure on a compact hypersurface in $\mathbb{R}^d$ of nonvanishing Gaussian curvature, $d \geqslant 2$. Such a measure μ satisfies $(RC)_p$ for all $p \geqslant 2(d+1)/(d-1)$, [45, Chapter 8].

If μ satisfies the $(RC)_p$ for some $1 \leqslant p \leqslant \infty$, there exists $C_p < \infty$ such that

$$\|Tf\|_{L^p(\mathbb{R}^d)} \leqslant C_p\|f\|_{L^2(\mu)}. \tag{1.37}$$

and so we can study the problem of existence of extremizers and precompactness of extremizing sequences as in Definition 1.1 with $\mathcal{R}$ the best constant in (1.37). The following theorem is proved

Theorem 1.24. *Let μ be a finite, positive and compactly supported measure on $\mathbb{R}^d$ and let*

$$p_0(\mu) = \inf\{p \in [1,\infty] : (RC)_p \text{ holds for } \mu\}.$$

Then for every p satisfying $\max(2, p_0(\mu)) < p \leqslant \infty$ there exists an extremizer for (1.37) with respect to p. Moreover, for every extremizing sequence $\{f_n\}_{n\in\mathbb{N}}$ for T_μ w.r.t. p, there exists $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that $\{e^{ix_n\cdot y}h_n(y)\}_{n\in\mathbb{N}}$ is precompact in $L^2(\mu)$.

Note that this theorem does not apply to the $L^2(S^2) \to L^4(\mathbb{R}^3)$ case of the sphere (S^2, σ) studied in [9] since $p = 4$ equals $p_0(\sigma)$. The theorem is sharp in the sense that for the endpoint $p_0(\mu)$ the conclusion does not hold in general. Examples of this are a truncated cone in $\mathbb{R}^3$ or $\mathbb{R}^4$ with $p = 6$ and 4 resp., a truncated paraboloid in $\mathbb{R}^2$ or $\mathbb{R}^3$ with $p = 6$ and 4 resp., or a truncated hyperboloid in $\mathbb{R}^3$ with $p = 4$.

They prove a very interesting proposition of which Proposition 1.18 is a special case,

Proposition 1.25 ([18])**.** *Let $\mathcal{H}$ be a Hilbert space, $p \in (2,\infty)$ and $T : \mathcal{H} \to L^p(\mathbb{R}^d)$ be a bounded linear operator. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that:*

(i) $\|f_n\|_2 = 1$;

(ii) $\lim_{n\to\infty} \|Tf_n\|_{L^4(\mathbb{R}^3)} = \|T\|_{\mathcal{L}(\mathcal{H},L^p(\mathbb{R}^d))}$;

(iii) $f_n \rightharpoonup f \neq 0$;

(iv) $Tf_n \to Tf$ *a.e. in* $\mathbb{R}^d$.

Then $f_n \to f$ in $\mathcal{H}$, in particular $\|f\|_{\mathcal{H}} = 1$ and $\|Tf\|_{L^p(\mathbb{R}^d)} = \|T\|_{\mathcal{L}(\mathcal{H},L^p(\mathbb{R}^d))}$.

Note that in the case of $\mathcal{H} = L^2(\mu)$, and $T = T_\mu$, the adjoint Fourier restriction operator, conditions (i) and (ii) are satisfied, by definition, by and extremizing sequence $\{f_n\}_{n\in\mathbb{N}}$. Condition (iii) follows after passing to a subsequence, by the Banach-Alaoglu Theorem. If $f_n \rightharpoonup f$ in $L^2(\mu)$, and if μ is compactly supported, then condition (iv) follows. Thus, Proposition 1.25 states that for compactly supported measures, the only obstruction to the existence of extremizers is that every L^2-weak limit of every extremizing sequence is zero.

We used this proposition to give an alternative approach to the existence of extremizers for S^2. We will use it in the next chapter for the case of the cone to show that extremizing sequences are precompact up to symmetries of the cone.

1.5 Extremizers for Strichartz inequalities

In [17], Fanelli, Vargas and Visciglia consider the problem of extremizers for Strichartz inequalities. Their result includes a large family of such inequalities.

Consider the Cauchy Problem

$$\begin{cases} i\partial_t u + h(D)u = 0, \\ u(0,x) = f(x), \end{cases} \tag{1.38}$$

where $u(t,x) = (u_1(t,x), \ldots, u_n(t,x)) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^n$, $f(x) = (f_1(x), \ldots, f_n(x)) : \mathbb{R}^d \to \mathbb{C}^n$. In (1.38), $h(D)u$ is the the multiplier

$$\widehat{h(D)u}(t,\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} h(x) u(t,x) dx,$$

where the symbol $h(\xi) = (h_{i,j}(\xi))_{i,j=1,\ldots,n}$ is a matrix valued function.

We make the following assumptions:

(H1) there exists $0 < s < \frac{d}{2}$ such that (1.38) is globally well-posed in $\dot{H}^s$, and the unique solution given via the propagator, $u(t,x) = e^{ith(D)} f(x)$,

(H2) the flow $e^{ith(D)}$ is unitary onto $\dot{H}^s$, that is $\|e^{ith(D)} f\|_{\dot{H}^s} = \|f\|_{\dot{H}^s}$, for all $t \in \mathbb{R}$, where s is the same as in (H1).

For the function h we will assume:

(H3) h is homogeneous of degree k, for some $k > 0$, i.e. $h(\lambda\xi) = \lambda^k h(\xi)$, for all $\lambda > 0$ and $\xi \in \mathbb{R}^n$.

Suppose that for s as in (H1) a Strichartz estimate

$$\|e^{ith(D)} f\|_{L^p_t L^q_x} \leqslant C \|f\|_{\dot{H}^s} \tag{1.39}$$

holds. Then, using the homogeneity condition (H3) p and q have to satisfy the relation

$$\frac{k}{p} + \frac{d}{q} = \frac{d}{2} - s.$$

Condition (H2) implies

$$\|e^{ith(D)} f\|_{L^\infty_t \dot{H}^s_x} = \|f\|_{\dot{H}^s}$$

that together with the Sobolev embedding $\dot{H}^s \subset L^{\frac{2d}{d-2s}}$, for $0 < s < d/2$, gives

$$\|e^{ith(D)} f\|_{L^\infty_t L_x^{\frac{2d}{d-2s}}} \leqslant C \|f\|_{\dot{H}^s}.$$

Thus, if an estimate like (1.39) holds with $p < q$, then interpolating with the last inequality gives (see [1] for interpolation of mixed norms)

$$\|e^{ith(D)} f\|_{L^r_{t,x}} \leqslant C\|f\|_{\dot{H}^s},$$

for $r = 2(d + k)/(d - 2s)$.

It is extremizers for this last kind of estimate that is considered in [17]. Their theorem states

Theorem 1.26. *Let assumptions (H1), (H2) be satisfied for some* $0 < s < d/2$ *and let (H3) be satisfied for some* $k > 0$*. Moreover, assume that for some* $2 \leqslant p < q \leqslant \infty$

$$\|e^{ith(D)} f\|_{L^p_t L^q_x} \leqslant C\|f\|_{\dot{H}^s},$$

so that, for $r = \frac{2(d+k)}{d-2s}$*, we also have*

$$\|e^{ith(D)} f\|_{L^r_{t,x}} \leqslant \mathcal{R}\|f\|_{\dot{H}^s}, \tag{1.40}$$

with

$$\mathcal{R} = \sup_{\|f\|_{\dot{H}^s}=1} \|e^{ith(D)} f\|_{L^r_{t,x}}.$$

Then, there exists $f_0 \in \dot{H}^s$ *such that*

$$\|f_0\|_{\dot{H}^s} = 1 \quad \text{and} \quad \|e^{ith(D)} f_0\|_{L^r_{t,x}} = \mathcal{R},$$

that is, there exists an extremizer for (1.40)*. Moreover, extremizing sequences are precompact, after the application of symmetries.*

The proof is short and simple and uses Proposition 1.25 together with a result of Gérard [21] about the Sobolev embedding $\dot{H}^s \subset L^{\frac{2d}{d-2s}}$.

Of interest to us is the case $h(\xi) = |\xi|$, that gives estimates for the adjoint Fourier restriction operator on the cone. The Strichartz estimates for $d \geqslant 2$ are

$$\|e^{it|D|} f\|_{L^p_t L^q_x} \leqslant C\|f\|_{\dot{H}^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}}, \tag{1.41}$$

under the admissibility condition

$$\frac{1}{p} + \frac{d-1}{q} = \frac{d-1}{2}, \quad p \geqslant 2, \quad (p, q) \neq (2, \infty).$$

Taking $p = q$ gives

$$\left\|e^{it|D|} f\right\|_{L_{t,x}^{\frac{2(d+1)}{d-1}}} \leqslant C\|f\|_{\dot{H}^{\frac{1}{2}}}, \quad d \geqslant 2.$$

Using the Sobolev embedding as before gives

$$\left\| e^{it|D|} f \right\|_{L^{\frac{2(d+1)}{d-1-2\sigma}}_{t,x}} \leqslant C \|f\|_{\dot{H}^{\frac{1}{2}+\sigma}}, \quad 0 \leqslant \sigma < \frac{d-1}{2}, \quad d \geqslant 2. \tag{1.42}$$

In the cone $\Gamma^d = \{(y, |y|) : y \in \mathbb{R}^d\} \subset \mathbb{R}^{d+1}$ with measure $\sigma(y, y') = \delta(y' - |y|)\frac{dydy'}{|y|}$ considered in Section 1.1 the adjoint Fourier restriction operator is

$$Tf(x,t) = \widehat{f\sigma}(x,t) = \int_{\mathbb{R}^d \times \mathbb{R}} e^{-ix\cdot y} e^{-it|y|} f(y) \frac{dy}{|y|}.$$

Then $Tf(x,t) = e^{-it|D|}g$, where $\hat{g}(y) = f(y)|y|^{-1}$. For the norms we have the equality $\|g\|_{\dot{H}^s} = \|f \cdot |y|^{s-\frac{1}{2}}\|_{L^2(\Gamma^d)}$, so (1.42) gives the weighted estimate

$$\|\widehat{f\sigma}\|_{L^{\frac{2(d+1)}{d-1-2\sigma}}(\mathbb{R}^{d+1})} \leqslant C \|f \cdot |y|^\sigma\|_{L^2(\Gamma^2)}, \quad 0 \leqslant \sigma < \frac{d-1}{2}, \quad d \geqslant 2.$$

We will study the case $d = 2$ and $\sigma = 0$,

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^3)} \leqslant C \|f\|_{L^2(\Gamma^2)} \tag{1.43}$$

which is not covered by Theorem 1.26. The existence of extremals and the value of the best constant are known, [7]. The part of this dissertation dedicated to the cone proves that extremizing sequences are precompact, after the application of symmetries of the cone. The argument can be used for other manifolds, for example, for the $L^2 \to L^6$ estimates for Fourier restriction on curves in $\mathbb{R}^2$.

Our argument for the cone will use the method of Christ and Shao in [9] and also some of the techniques developed by Fanelli, Vargas and Visciglia in [18] and [17], that allow a simplification of the argument in [9].

1.6 A few more references

There have been several results on existence of extremizers and/or computation of best constants for Fourier extension operators and Strichartz inequalities. We mention here some of those not already discussed: [4],[6],[12],[27],[42].

1.7 The results proved in this dissertation related to Fourier restriction inequalities

We prove three results concerning Fourier restriction inequalities (and one result on incidence geometry that we introduce later). We have already mentioned two, the cone and hyperboloid. In brief the results are

Theorem 1.27. *Any extremizing sequence of nonnegative functions in $L^2(\Gamma^2)$ for the inequality* (1.43) *is precompact up to symmetries, that is, every subsequence of an extremizing sequence has sub-subsequence that converges in $L^2(\Gamma^2)$ after the application of symmetries of the cone.*

Here, the symmetries we refer to are dilations and Lorentz transformations. Concerning inequality (1.18) for the restriction on the hyperboloid $\mathbb{H}^d$, we define

$$\mathbf{H}_{d,p} = \sup_{0\neq f\in L^2(\mathbb{H}^d)} \frac{\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{H}^d)}},$$

as well as

$$\bar{\mathbf{H}}_{d,p} = \sup_{0\neq f\in L^2(\mathbb{H}^d)} \frac{\|\widehat{f\sigma}\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\bar{\mathbb{H}}^d)}},$$

where $\bar{\mathbb{H}}^d$ is the two sheeted hyperboloid $\bar{\mathbb{H}}^d = \mathbb{H}^d \cup -\mathbb{H}^d = \{(y,y')\in\mathbb{R}^{d+1} : y'^2 = 1+|y|^2\}$. We prove

Theorem 1.28. *The values of the best constants are, $\mathbf{H}_{2,4} = 2^{3/4}\pi$, $\mathbf{H}_{2,6} = (2\pi)^{5/6}$ and $\mathbf{H}_{3,4} = (2\pi)^{5/4}$. In each of the three cases of pairs (d,p) extremizers do not exist.*

For the two sheeted hyperboloid the best constants are, $\bar{\mathbf{H}}_{2,4} = (3/2)^{1/4}\mathbf{H}_{2,4}$, $\bar{\mathbf{H}}_{2,6} = (5/2)^{1/3}\mathbf{H}_{2,6}$ and $\bar{\mathbf{H}}_{3,4} = (3/2)^{1/4}\mathbf{H}_{3,4}$. Here extremizers do not exist either.

In a joint work with Michael Christ, we consider the adjoint restriction inequality on the paraboloid, or equivalently, Strichartz inequalities for the Schrödinger equation. As stated in Theorem 1.3, Gaussians extremize the adjoint Fourier restriction inequality in dimensions $d=2$, $\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leqslant C\|f\|_{L^2(\mathbb{P}^1)}$, and $d=3$, $\|\widehat{f\sigma}\|_{L^4(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(\mathbb{P}^2)}$.

There are $L^p \to L^q$ estimates for $\widehat{f\sigma}$,

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^{d+1})} \leqslant C\|f\|_{L^p(\mathbb{P}^d)}, \tag{1.44}$$

where $q = q(p,d)$ is specified by

$$q^{-1} = \frac{d}{d+2}(1-p^{-1}),$$

for $1 \leqslant p \leqslant p(d)$, for a certain $2 < p(d) < 2(d+1)/d$.

There are mixed norm estimates,

$$\|\widehat{f\sigma}\|_{L^r_t L^q_x(\mathbb{R}^{1+d})} \leqslant C\|f\|_{L^2(\mathbb{P}^d)}$$

where $q, r \geqslant 2$ satisfy

$$\frac{2}{r} + \frac{d}{q} = \frac{d}{2},$$

with endpoint $q = \infty$ excluded for $d = 2$.

The main result is

Theorem 1.29. *Let $d \geq 1$, let $1 < p < 2(d+1)/d$, and set $q = q(p,d)$. Radial Gaussians are critical points for the $L^p \to L^q$ adjoint Fourier restriction inequalities if and only if $p = 2$. Radial Gaussians are critical points for the $L^2 \to L^r_t L^q_x$ Strichartz inequalities for all admissible pairs $(r,q) \in (1,\infty)^2$.*

This tells us that Gaussians are not extremizers for (1.44) if $p \neq 2$. It is a conjecture of Foschi that Gaussians are extremizers if $p = 2$ and $d \geqslant 1$. As for the mixed norms, we can mention that Carneiro [7] proved that Gaussians are extremizers for the $L^2 \to L^8_t L^4_x$ estimate in dimension $d = 1$.

Chapter 2

On extremizing sequences for the adjoint restriction inequality on the cone

It is known that extremizers for the L^2 to L^6 adjoint Fourier restriction inequality on the cone in $\mathbb{R}^3$ exist. Here we show that nonnegative extremizing sequences are precompact, after the application of symmetries of the cone. If we use the knowledge of the exact form of the extremizers, as found by Carneiro, then we can show that nonnegative extremizing sequences converge, after the application of symmetries.

2.1 Introduction

We study the properties of extremizing sequences for the Fourier restriction inequality on the cone in dimension 3 for which the adjoint restriction inequality can be rewritten equivalently as a convolution inequality. Carneiro [7], using the method developed by Foschi [20], found the exact form of the extremizers for the adjoint Fourier restriction inequalities in dimensions 3 and 4 but there seems to be no mention in the literature as to whether extremizing sequences are precompact after appropriate rescaling[1]. That is the question we try to answer in this paper using the methods developed by Christ and Shao [9] to analyze the corresponding inequality for the sphere in three dimensions.

We denote $\Gamma^2 = \{(y, y') \in \mathbb{R}^2 \times \mathbb{R} : y' = |y|\}$, the cone in $\mathbb{R}^3$. A function f on Γ^2 can be identified, and we will do so, with a function from $\mathbb{R}^2$ to $\mathbb{R}$. On Γ^2 we consider the measure

[1] [17] answers this question in the nonendpoint case and appeared while this manuscript was being prepared. We comment on that later in the introduction.

$\sigma(y, y') = \delta(y' - |y|)\frac{dydy'}{|y|}$, that is, for a function f on the cone

$$\int_{\Gamma^2} f d\sigma = \int_{\mathbb{R}^2} f(y)\frac{dy}{|y|}.$$

We will denote the $L^p(\Gamma^2, \sigma)$ norm of a function f as $\|f\|_{L^p(\Gamma^2)}$, $\|f\|_{L^p(\sigma)}$ or $\|f\|_p$.

The extension or adjoint Fourier restriction operator for the cone is given by

$$Tf(x,t) = \int_{\mathbb{R}^2} e^{ix\cdot y}e^{it|y|} f(y)|y|^{-1}dy \tag{2.1}$$

where $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^2)$. With the Fourier transform $\hat{g}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi}g(x)dx$ we see that $Tf(x,t) = \widehat{f\sigma}(-x, -t)$.

A well known bound, [47], for Tf is given in the following theorem

Theorem 2.1. *There exists $C < \infty$ such that for all $f \in L^2(\Gamma^2)$ the following inequality holds*

$$\|Tf\|_{L^6(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(\Gamma^2)}. \tag{2.2}$$

Denote by $\mathbf{C}$ the best constant in (2.2), that is

$$\mathbf{C} = \sup_{0\neq f\in L^2(\Gamma^2)} \frac{\|Tf\|_{L^6(\mathbb{R}^3)}}{\|f\|_{L^2(\Gamma^2)}}. \tag{2.3}$$

The use of the Fourier transform allows us to write (2.2) in "convolution form", namely

$$\begin{aligned}\|Tf\|^3_{L^6(\mathbb{R}^3)} &= \|(Tf)^3\|_{L^2(\mathbb{R}^3)} = \|(\widehat{f\sigma})^3\|_{L^2(\mathbb{R}^3)} = \|(f\sigma * f\sigma * f\sigma)\check{}\|_{L^2(\mathbb{R}^3)}\\ &= (2\pi)^{3/2}\|f\sigma * f\sigma * f\sigma\|_{L^2(\mathbb{R}^3)},\end{aligned} \tag{2.4}$$

thus $\|T(f)\|_{L^6} \leqslant \|T(|f|)\|_{L^6}$. This implies that if $\{f_n\}_{n\in\mathbb{N}}$ is an extremizing sequence then so is $\{|f_n|\}_{n\in\mathbb{N}}$.

In what follows we will restrict attention to nonnegative functions $f \in L^2(\Gamma^2)$.

Definition 2.2. An extremizing sequence for the inequality (2.2) is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $L^2(\Gamma^2)$ satisfying $\|f_n\|_{L^2(\Gamma^2)} \leqslant 1$, such that $\|Tf_n\|_{L^6(\mathbb{R}^3)} \to \mathbf{C}$ as $n \to \infty$.

An extremizer for (2.2) is a function $f \neq 0$ which satisfies $\|Tf\|_{L^6(\mathbb{R}^3)} = \mathbf{C}\|f\|_{L^2}$.

The main theorem of this chapter is

Theorem 2.3. *Any extremizing sequence of nonnegative functions in $L^2(\Gamma^2)$ for the inequality (2.2) is precompact up to symmetries, that is, every subsequence of an extremizing sequence has a sub-subsequence that converges in $L^2(\Gamma^2)$ after the application of symmetries of the cone.*

The symmetries of the cone we refer to are dilations and Lorentz transformations that will be studied in Section 2.7, and Theorem 2.3 will be stated in a more precise form as Theorem 2.29 below.

With the knowledge of the exact form of the extremizers to (2.2) given by Carneiro in [7] one can improve Theorem 2.3 to obtain

Theorem 2.4. *Any extremizing sequence of nonnegative functions in $L^2(\Gamma^2)$ for the inequality* (2.2) *converges in $L^2(\Gamma^2)$, after the application of symmetries of the cone.*

Define the function g by its Fourier transform as $\hat{g}(y) = f(y)|y|^{-1}$. Then

$$e^{it\sqrt{-\Delta}}g(x) := \frac{1}{(2\pi)^2}\int e^{ix\cdot y}e^{it|y|}\hat{g}(y)dy = \frac{1}{(2\pi)^2}Tf(x,t), \tag{2.5}$$

and

$$\|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} = \|f\|_{L^2(\Gamma^2)},$$

where we used the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)$ norm

$$\|g\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)} = \int_{\mathbb{R}^2}|\hat{g}(y)|^2|y|dy.$$

We see that

$$(2\pi)^{-2}\|Tf\|_{L^6(\mathbb{R}^2)}\|f\|^{-1}_{L^2(\Gamma^2)} = \|e^{it\sqrt{-\Delta}}g\|_{L^6(\mathbb{R}^3)}\|g\|^{-1}_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}, \tag{2.6}$$

and (2.2) is equivalent to

$$\|e^{it\sqrt{-\Delta}}g\|_{L^6_{x,t}(\mathbb{R}^3)} \leqslant \frac{C}{(2\pi)^2}\|g\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}. \tag{2.7}$$

From (2.6), $\{f_n\}_{n\in\mathbb{N}}$ is an extremizing sequence for (2.2) if and only if $\{g_n\}_{n\in\mathbb{N}}$, with $\hat{g}_n(y) = f_n(y)|y|^{-1}$, is an extremizing sequence for (2.7).

The problem of computing the best constant in (2.2) and the exact form of the extremizers was solved by Carneiro in [7]. With the normalization of the Fourier transform discussed earlier, Carneiro proves

Theorem 2.5 ([7]). *For all $f \in L^2(\Gamma^2)$,*

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^3)} \leqslant (2\pi)^{5/6}\|f\|_{L^2(\Gamma^2)}. \tag{2.8}$$

and equality occurs in (2.8) *if and only if $f(y,|y|) = e^{-a|y|+b\cdot y+c}$, where $a, c \in \mathbb{C}$, $b \in \mathbb{C}^2$, and $|\mathfrak{Re}\, b| < \mathfrak{Re}\, a$.*

We will use this result to prove Theorem 2.4.

Fanelli, Vega and Visciglia proved in [17] a general existence theorem for extremizers of Strichartz inequalities. We state here the case of the cone, in its equivalent form via (2.5). For $d \geqslant 2$ and $0 \leqslant \sigma < \frac{d-1}{2}$ the following Strichartz estimates hold (see [17, Example 1.1])

$$\|e^{it\sqrt{-\Delta}}g\|_{L_{t,x}^{\frac{2(d+1)}{d-1-2\sigma}}(\mathbb{R}^{d+1})} \leqslant C\|g\|_{\dot{H}^{\frac{1}{2}+\sigma}(\mathbb{R}^d)}. \tag{2.9}$$

In [17], using "remodulation" (equation after [17, equation 2.12]) "rescaling" and "translation" ([17, equation 2.15]) the following theorem is proved,

Theorem 2.6 ([17])**.** *Let $d \geqslant 2$ and $0 < \sigma < \frac{d-1}{2}$. Then there exists an extremizer for* (2.9)*. Moreover, extremizing sequences are precompact, after the application of symmetries: "remodulation", "rescaling" and "translation".*

We point out here that their method does not apply to the endpoint case studied in this paper, $\sigma = 0$ and $d = 2$, because of the existence of further symmetries, Lorentz invariance, as discussed in Section 2.7. The symmetries referred to in Theorem 2.6, when expressed in the dual formulation for $f \in L^2(\Gamma^2)$ are, in respective order:

- $f(y) \rightsquigarrow e^{is|y|}f(y)$, $s \in \mathbb{R}$,
- $f(y) \rightsquigarrow \lambda^{1/2}f(\lambda y)$, $\lambda > 0$ and
- $f(y) \rightsquigarrow e^{iy\cdot y_0}f(y)$, $y_0 \in \mathbb{R}^2$.

From the Lorentz invariance of inequality (2.2), and the fact that the Lorentz group is not generated modulo a compact subgroup by the elements listed above, it follows that the final conclusion of Theorem 2.6 cannot be true in the endpoint case $d = 2$, $\sigma = 0$. This indicates that the proof in [17] likewise cannot apply to this endpoint case.

On the one hand, for $d \geqslant 2$, under admissibility conditions in (p, q) one has the Strichartz estimates

$$\|e^{it\sqrt{-\Delta}}g\|_{L_t^pL_x^q(\mathbb{R}^{d+1})} \leqslant C\|g\|_{\dot{H}^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}(\mathbb{R}^d)},$$

so that for the case of the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)$ one needs $p = q$ which then makes Theorem 1.1 in [17] not applicable.

On the other hand, at the level of the proof of [17, Theorem 1.1], one sees that [17, equation 2.12] does not hold for $\sigma = 0$ (or $s = 1/2$ as appears there) and $d = 2$. For this we show that there are extremizing sequences $\{g_n\}_{n\in\mathbb{N}}$ such that $\|e^{it\sqrt{-\Delta}}g_n\|_{L_t^\infty L_x^4} \to 0$ as $n \to \infty$.

This is the same as having extremizing sequence $\{f_n\}_{n\in\mathbb{N}}$ such that $\|Tf_n\|_{L_t^\infty L_x^4} \to 0$ as $n \to \infty$. For this we use the Lorentz invariance and the characterization of extremizers for the cone given in Theorem 2.5.

From Section 2.7, $\|T(f\circ L)\|_{L^6(\mathbb{R}^3)} = \|Tf\|_{L^6(\mathbb{R}^3)}$, for every Lorentz transformation L preserving Γ^2. Let f be an L^2-normalized extremizer, say $f(x_1,x_2,x_3)=c_0e^{-x_3}$. We take a sequence of Lorentz transformations L^s and $f\circ L^s$ is also an L^2-normalized extremizer. We now compute $\|(Tf)\circ L^s\|_{L^\infty_t L^4_x}$. We have

$$Tf(x,t)=\frac{2\pi c_0}{\sqrt{(1-it)^2+|x|^2}},$$

and

$$|Tf(x,t)|^4=\frac{(2\pi)^4c_0^4}{(1-t^2+|x|^2)^2+4t^2}.$$

Now we use $L^s(x,t)=(\frac{x_1+st}{(1-s^2)^{1/2}},x_2,\frac{t+sx_1}{(1-s^2)^{1/2}})$ and note that by making the change of variables $u=(x_1+st)(1-s^2)^{-1/2}$, $v=x_2$ we obtain

$$\int|(Tf)\circ L^s(x,t)|^4dx=(1-s^2)^{1/2}\int|Tf(x_1,x_2,sx_1+t(1-s^2)^{1/2})|^4dx$$

Then, if $s\neq 0$

$$\begin{aligned}\sup_{t\in\mathbb{R}}\int_{\mathbb{R}^2}|(Tf)\circ L^s(x,t)|^4dx&=(1-s^2)^{1/2}\sup_{t\in\mathbb{R}}\int_{\mathbb{R}^2}|Tf(x_1,x_2,s(x_1+t))|^4dx\\ &=(2\pi)^4c_0^4(1-s^2)^{1/2}\sup_{t\in\mathbb{R}}\int_{\mathbb{R}^2}\frac{dx_1dx_2}{(1-s^2(x_1+t)^2+x_1^2+x_2^2)^2+4s^2(x_1+t)^2}\\ &=(2\pi)^4c_0^4(1-s^2)^{1/2}\sup_{t\in\mathbb{R}}\int_{\mathbb{R}^2}\frac{dx_1dx_2}{(1-s^2x_1^2+(x_1+t)^2+x_2^2)^2+4s^2x_1^2}.\end{aligned}$$

It is not hard to show that for $(s,t)\in[1/2,1]\times\mathbb{R}$

$$\int_{\mathbb{R}^2}\frac{dx_1dx_2}{(1-s^2x_1^2+(x_1+t)^2+x_2^2)^2+4s^2x_1^2}\leqslant C,$$

with C independent of s and t. Therefore

$$\sup_{t\in\mathbb{R}}\int_{\mathbb{R}^2}|(Tf)\circ L^s(x,t)|^4dx\leqslant C(1-s^2)^{1/2}.$$

Hence $\lim_{s\to 1^-}\|T(f\circ L^s)\|_{L^\infty_t L^4_x}=0$.

Notation: We will write $X\lesssim Y$ or $Y\gtrsim X$ to denote an estimate of the form $X\leqslant CY$, and $X\asymp Y$ to denote an estimate of the form $cY\leqslant X\leqslant CY$, where $0<c,C<\infty$ are constants depending on fixed parameters of the problem, but independent of X and Y.

When writing integrals, we will sometimes drop the domain of integration or the measure when it is clear from context.

2.2 The structure of the paper and the idea of the proof

The proof of Theorem 2.3 follows the lines of the proof of precompactness of extremizing sequences for the adjoint Fourier operator on the sphere $S^2 \subset \mathbb{R}^3$ given in [9].

In Section 2.3 we give a (known [47], [44, Chapter 2]) proof of Theorem 2.1, with a view towards a refinement in terms of a cap space, as used in [9] and proved in [18] for compact surfaces in $\mathbb{R}^3$ of nonvanishing Gaussian curvature. In Section 2.4 we obtain bounds that we will use in Section 2.5 to obtain the following cap estimate,

$$\|Tf\|_{L^6(\mathbb{R}^3)} \lesssim \|f\|_{L^2(\Gamma^2)}^{1-\gamma/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2} d\sigma\Big)^{\gamma/3}, \tag{2.10}$$

where the supremum ranges over all "caps" $\mathcal{C} \subset \Gamma^2$ and $\gamma > 0$ is a small universal constant. This is the analog of Lemma 6.1 in [9].

For a function satisfying $\|Tf\|_{L^6(\mathbb{R}^3)} \geqslant \delta \mathbf{C}\|f\|_{L^2}$, the estimate in (2.10) allows the extraction of a cap $\mathcal{C}$ with good properties:

$$|f(x)| \leqslant C_\delta \|f\|_2 |\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(x), \text{ and } \|f\chi_{\mathcal{C}}\|_2 \geqslant \eta_\delta \|f\|_2.$$

This is the content of Section 2.6. In Section 2.7 we discuss symmetries of the cone. This includes dilations and Lorentz transformations and they allow us to take a cap $\mathcal{C}$ and transform it into a cap $\mathcal{C}'$ with better properties: $\mathcal{C}'$ is contained in a bounded region, independent of the extremizing sequence, and has big measure.

The existence of symmetries of (Γ^2, σ) simplifies the argument, compared to [9]. Two ways are possible, use the arguments of Fanelli, Vega and Visciglia contained in [18] and [17] carried out in Section 2.8; or use the decomposition algorithm as done by Christ and Shao and carried out in Section 2.9.

For the argument based on [18] and [17], a single extraction of a cap and the use of symmetries is enough to prove precompactness. In the case of the argument based on [9], a cap decomposition is needed. For an extremizing sequence, the cap decomposition is used to show that after dilations and Lorentz transformations, the extremizing sequence has a uniform L^2-decay at infinity. The uniform decay plus a result inspired from [18] allows us to complete the proof of precompactness.

In the last section, we prove that extremizing sequences converge, after the application of symmetries of the cone. This is an easy task, that follows from the fact that the extremizers for (2.2) are known and that the group of symmetries of the cone acts transitively in the set of extremizers.

2.3 The adjoint Fourier restriction inequality

Abusing notation we will write $f(r,\theta) = f(x)$, where $x = (r\cos\theta, r\sin\theta)$, that is the polar representation of x. Note that in polar coordinates the measure $|y|^{-1}dy$ becomes $dr\,d\theta$.

In the proof of Theorem 2.1 we will need the following standard lemma.

Lemma 2.7 (Fractional integration). *Let $1 < p,q < \infty$. Then for any $g \in L^p(\mathbb{R})$, $h \in L^q(\mathbb{R})$ the following holds*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |g(s)h(t)||t-s|^{-\alpha} ds dt \leqslant C_{p,q}\|g\|_{L^p}\|h\|_{L^q},$$

where $\alpha = 2 - \frac{1}{p} - \frac{1}{q}$ and $\frac{1}{p} + \frac{1}{q} > 1$.

From Lemma 2.7 we have

Lemma 2.8. *Let $1 < p,q < \infty$ with $\frac{1}{p} + \frac{1}{q} > 1$ and let $\alpha = 2 - \frac{1}{p} - \frac{1}{q}$. Then for any $g \in L^p([0,2\pi])$, $h \in L^q([0,2\pi])$ the following holds*

$$\int_0^{2\pi}\int_0^{2\pi} |g(s)h(t)||\sin(t-s)|^{-\alpha} ds dt \leqslant C_{p,q}\|g\|_{L^p}\|h\|_{L^q} \tag{2.11}$$

Proof. We split the integral in sixteen pieces according to $[0,2\pi] = [0,\pi/2] \cup [\pi/2,\pi] \cup [\pi,3\pi/2] \cup [3\pi/2,2\pi]$, and then it will be enough to show that

$$\int_{m\pi/2}^{(m+1)\pi/2}\int_{n\pi/2}^{(n+1)\pi/2} |g(s)h(t)||\sin(t-s)|^{-\alpha} ds dt \leqslant C_{p,q}\|g\|_{L^p}\|h\|_{L^q},$$

for all $m,n \in \{0,1,2,3\}$. For this we use a simple change of variable that allows us to use Lemma 2.7.

If $t,s \in [j\pi/2,(j+1)\pi/2]$, for some $j \in \{0,1,2,3\}$, then $|t-s| \leqslant \pi/2$ and we use that $\frac{2}{\pi}|t-s| \leqslant |\sin(t-s)| \leqslant |t-s|$.

If $s \in [0,\pi/2]$ and $t \in [\pi,3\pi/2]$ we can use the change of variables $t' = t-\pi$ so that $t' \in [0,\pi/2]$. We note that $|\sin(t-s)| = |\sin(t'-s)|$.

If $s \in [0,\pi/2]$ and $t \in [\pi/2,\pi]$ we further split the intervals as $[0,\pi/2] = [0,\pi/4] \cup [\pi/4,\pi/2]$ and $[\pi/2,\pi] = [\pi/2,3\pi/4] \cup [3\pi/4,\pi]$. If $s \in [0,\pi/4]$ and $t \in [\pi/2,3\pi/4]$ or if $s \in [\pi/4,\pi/2]$ and $t \in [3\pi/4,\pi]$, then $|\sin(t-s)| \geqslant 1/\sqrt{2}$ and the desired inequality follows from an application of Hölder's inequality. If $s \in [\pi/4,\pi/2]$ and $t \in [\pi/2,3\pi/4]$, then $|t-s| \leqslant \pi/2$ and we can use the inequality $\frac{2}{\pi}|t-s| \leqslant |\sin(t-s)| \leqslant |t-s|$ as in the first case discussed. Finally, if $s \in [0,\pi/4]$ and $t \in [3\pi/4,\pi]$ we use the substitution $t' = t-\pi$ so that $t' \in [-\pi/4,0]$. Since $|\sin(t-s)| = |\sin(t'-s)|$ and $|t'-s| \leqslant \pi/2$ we can conclude as before.

The other cases follow in the same way. □

Proof of Theorem 2.1. We split $f(y) = \sum_{k\in\mathbb{Z}} f_k(y)$ where $f_k(y) = f(y)\chi_{2^{k-1}\leqslant|y|<2^k}$. Then

$$(Tf)^2(x,t) = \sum_{k,k'\in\mathbb{Z}} Tf_k \cdot Tf_{k'}.$$

Taking L^3 norm in both sides, using the triangle inequality and Lemma 2.9 below we get

$$\|Tf\|_{L^6}^2 \leqslant C\sum_{k,k'} 2^{-|k-k'|/6}\|f_k\|_{L^2(\sigma)}\|f_{k'}\|_{L^2(\sigma)}.$$

To conclude we use the Cauchy-Schwarz inequality

$$\|Tf\|_{L^6}^2 \leqslant C(\sum_{k,k'} 2^{-|k-k'|/6}\|f_k\|_{L^2(\sigma)}^2)^{1/2}(\sum_{k,k'} 2^{-|k-k'|/6}\|f_{k'}\|_{L^2(\sigma)}^2)^{1/2} \leqslant C\|f\|_{L^2(\sigma)}^2. \qquad \square$$

Lemma 2.9. *There exists a constant $C < \infty$ with the following property. Let $k, k' \in \mathbb{Z}$ and $f, g \in L^2(\Gamma^2)$ with f and g supported in the regions $2^{k-1} \leqslant |y| < 2^k$ and $2^{k'-1} \leqslant |y| < 2^{k'}$ respectively, then*

$$\|Tf \cdot Tg\|_{L^3} \leqslant C2^{-|k-k'|/6}\|f\|_{L^2}\|g\|_{L^2}. \tag{2.12}$$

Proof. We can split

$$f(r,\theta)g(r',\theta') = f(r,\theta)g(r',\theta')(\chi_{r>r'} + \chi_{r<r'})(\chi_{\theta>\theta'} + \chi_{\theta<\theta'}) \text{ for a.e } (r,r',\theta,\theta').$$

Thus by the triangle inequality we can assume, without loss of generality, that $\theta > \theta'$ and $r < r'$ in the support of $f(r,\theta)g(r',\theta')$.

Using polar coordinates and Fubini's Theorem we have

$$\begin{aligned} Tf \cdot Tg(x,t) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{ix\cdot(y+y')}e^{it(|y|+|y'|)}f(y)g(y')|y|^{-1}|y'|^{-1}\,dydy' \\ &= \int e^{ix\cdot(r\cos\theta+r'\cos\theta',r\sin\theta+r'\sin\theta')}e^{it(r+r')}f(r,\theta)g(r',\theta')\,d\theta d\theta' dr dr'. \end{aligned}$$

We make the following change of variables

$$(r,r',\theta,\theta') \mapsto (u,s,\varrho) = (r\cos\theta + r'\cos\theta', r\sin\theta + r'\sin\theta', r+r', r),$$

which is injective in the region where $\theta > \theta'$, $r < r'$. The Jacobian of the transformation is

$$J^{-1} = \frac{\partial(u,s,\varrho)}{\partial(r,r',\theta,\theta')} = rr'\sin(\theta-\theta').$$

Using the change of variables

$$Tf\cdot Tg(x,t)=\int_{\mathbb{R}}\Big(\int_{\mathbb{R}^3}e^{ix\cdot u}e^{its}f(y)g(y')Jduds\Big)d\varrho,$$

and by Minkowski's inequality and Hausdorff-Young inequality,

$$\begin{aligned}\|Tf\cdot Tg\|_{L^3}&\leqslant\int_{\mathbb{R}}\Big\|\int_{\mathbb{R}^3}e^{ix\cdot u}e^{its}f(y)g(y')Jduds\Big\|_{L^3}d\varrho\\&\leqslant C\int_{\mathbb{R}}\Big(\int|f(y)g(y')J|^{3/2}duds\Big)^{2/3}d\varrho\\&=C\int_{\mathbb{R}}\Big(\int|f(y)g(y')|^{3/2}(rr')^{-\frac12}|\sin(\theta-\theta')|^{-\frac12}Jduds\Big)^{2/3}d\varrho.\end{aligned}$$

We now use that $r\asymp 2^k$, $r'\asymp 2^{k'}$ and Hölder's inequality to obtain

$$\begin{aligned}\|Tf\cdot Tg\|_{L^3}&\leqslant C(2^k2^{k'})^{-1/3}(2^k)^{1/3}\Big(\int|f(y)g(y')|^{3/2}|\sin(\theta-\theta')|^{-\frac12}Jdudsd\varrho\Big)^{2/3}\\&=C(2^k2^{k'})^{-1/3}(2^k)^{1/3}\Big(\int|f(y)g(y')|^{3/2}|\sin(\theta-\theta')|^{-\frac12}d\theta d\theta'drdr'\Big)^{2/3}.\end{aligned}\tag{2.13}$$

On the other hand, by Lemma 2.8

$$\begin{aligned}&\int|f(y)g(y')|^{3/2}|\sin(\theta-\theta')|^{-\frac12}d\theta d\theta'drdr'\\&\qquad\leqslant C\int\Big(\int|f(r,\theta)|^2d\theta\Big)^{3/4}dr\cdot\int\Big(\int|g(r,\theta')|^2d\theta'\Big)^{3/4}dr'\\&\qquad\leqslant C(2^k2^{k'})^{1/4}\Big(\int|f(r,\theta)|^2drd\theta\Big)^{3/4}\Big(\int|g(r',\theta')|^2dr'd\theta'\Big)^{3/4}.\end{aligned}$$

Then, as $2^k\leqslant 2^{k'}$

$$\begin{aligned}\|Tf\cdot Tg\|_{L^3}&\leqslant C(2^k2^{k'})^{-1/3}\min((2^k)^{1/3},(2^{k'})^{1/3})(2^k2^{k'})^{1/6}\|f\|_{L^2_{r,\theta}}\|g\|_{L^2_{r,\theta}}\\&=C2^{-(k+k')/6}\min((2^k)^{1/3},(2^{k'})^{1/3})\|f\|_{L^2(\sigma)}\|g\|_{L^2(\sigma)}.\end{aligned}$$

We note that $2^{-(k+k')/6}\min((2^k)^{1/3},(2^{k'})^{1/3})=2^{-|k-k'|/6}$, so

$$\|Tf\cdot Tg\|_{L^3}\leqslant C2^{-|k-k'|/6}\|f\|_{L^2(\sigma)}\|g\|_{L^2(\sigma)}.$$

□

Proposition 2.10. *There exists a constant $C<\infty$ with the following property. Let $f\in L^2(\Gamma^2)$ and for $k\in\mathbb{Z}$ let $f_k(y)=f(y)\chi_{\{2^{k-1}\leqslant|y|<2^k\}}$. Then*

$$\|Tf\|_{L^6(\mathbb{R}^3)}\leqslant C\Big(\sum_{k\in\mathbb{Z}}\|f_k\|_{L^2}^3\Big)^{1/3}.$$

Proof. By rewriting $\|Tf\|^3_{L^6(\mathbb{R}^3)}$ as the L^2 norm of a trilinear form and using the triangle inequality we have

$$\|Tf\|^3_{L^6} = \|Tf \cdot Tf \cdot Tf\|_{L^2} = \Big\|\sum_{i,j,k} Tf_i \cdot Tf_j \cdot Tf_k\Big\|_{L^2} \leqslant \sum_{i,j,k} \|Tf_i \cdot Tf_j \cdot Tf_k\|_{L^2}.$$

Now for each i, j, k, without loss of generality we can assume that $|j - k| = \max(|i' - j'| : i', j' \in \{i, j, k\})$. Using Hölder's inequality, Theorem 2.1 and Lemma 2.9 we get

$$\|Tf_i \cdot Tf_j \cdot Tf_k\|_{L^2} \leqslant \|Tf_i\|_{L^6}\|Tf_j \cdot Tf_k\|_{L^3} \leqslant C2^{-|j-k|/6}\|f_i\|_{L^2}\|f_j\|_{L^2}\|f_k\|_{L^2}. \tag{2.14}$$

Now, using the maximality of $|j - k|$ we see that $|j - k| \geqslant \frac{1}{3}|i - j| + \frac{1}{3}|j - k| + \frac{1}{3}|k - i|$, and hence from (2.14),

$$\|Tf_i \cdot Tf_j \cdot Tf_k\|_{L^2} \leqslant 2^{-|i-j|/18}2^{-|j-k|/18}2^{-|k-i|/18}\|f_i\|_{L^2}\|f_j\|_{L^2}\|f_k\|_{L^2}.$$

Then

$$\|Tf\|^3_{L^6} \leqslant C\sum_{i,j,k} 2^{-|i-j|/18}2^{-|j-k|/18}2^{-|k-i|/18}\|f_i\|_{L^2}\|f_j\|_{L^2}\|f_k\|_{L^2},$$

and a final application of Hölder's inequality gives the desired conclusion

$$\|Tf\|^3_{L^6} \leqslant C\sum_{i,j,k} 2^{-|i-j|/18}2^{-|j-k|/18}2^{-|k-i|/18}\|f_k\|^3_{L^2} \leqslant C\sum_{k\in\mathbb{Z}} \|f_k\|^3_{L^2}.$$ □

2.4 Preliminaries for the cap bound for the adjoint Fourier operator

Recall that in the computation of $\|(Tf)^2\|_{L^3}$, in equation (2.13) with $g = f$, we came across the expression

$$\int |f(r,\theta)f(r',\theta')|^{3/2}|\sin(\theta - \theta')|^{-1/2}d\theta d\theta' dr dr'.$$

By assuming the angular support of f is contained in the region $0 \leqslant \theta \leqslant \frac{\pi}{2}$, that is $f(r,\theta) = 0$ if $\theta \notin [0, \frac{\pi}{2}]$, we can study instead the comparable expression

$$\int |f(r,\theta)f(r',\theta')|^{3/2}|\theta - \theta'|^{-1/2}d\theta d\theta' dr dr'.$$

Instead of using fractional integration in θ, θ' and Hölder's inequality in r, r' we want to obtain a "cap type" inequality for T of the form in Theorem 4.2 in [34].

Definition 2.11. By a cap $\mathcal{C}$ we mean a set $\mathcal{C} \subset \Gamma$ whose projection to the plane $\mathbb{R}^2 \times \{0\}$ is of the form $[2^{k-1}, 2^k] \times J$, when written in polar coordinates (r, θ), where $k \in \mathbb{Z}$ and $J \subset [0, 2\pi]$ is an interval. We will identify the cap $\mathcal{C}$ with its projection to the xy-plane and write $\mathcal{C} = [2^{k-1}, 2^k] \times J$.

For a cap $\mathcal{C} = [2^{k-1}, 2^k] \times J$, $|\mathcal{C}| := \sigma(\mathcal{C}) = 2^{k-1}|J|$, and for any $\lambda \geqslant 0$, $\lambda\mathcal{C} = [\lambda 2^{k-1}, \lambda 2^k] \times J$, so $\sigma(\lambda\mathcal{C}) = \lambda\sigma(\mathcal{C})$.

Definition 2.12. Let $0 < \alpha < 1$ and $p = 2/(2-\alpha)$. Define, for $f, g \in L^p(\mathbb{R})$, the bilinear operator

$$B(f,g) = \int_{\mathbb{R}^2} f(x)g(x')|x-x'|^{-\alpha}dxdx'. \tag{2.15}$$

Note that the kernel $x \in \mathbb{R} \mapsto |x|^{-\alpha}$ has a strictly positive Fourier transform and thus B is nondegenerate and satisfies the Cauchy-Schwarz inequality $|B(f,g)|^2 \leqslant B(f,f)B(g,g)$.

Lemma 2.7 implies that $|B(f,f)| \leqslant C_p\|f\|^2_{L^p(\mathbb{R})}$. We can say more if we work with the Lorentz spaces $L^{p,q}(\mathbb{R})$ (see [46] for an introduction to Lorentz spaces). We have the following bound for B [36]

$$|B(f,f)| \lesssim \|f\|^2_{L^{p,2}(\mathbb{R})}.$$

This bound will allow us to prove the following

Proposition 2.13. *Let $0 < \alpha < 1$ and $p = 2/(2-\alpha)$. There exist constants $C < \infty$ and $\delta \in (0,2)$ such that for all $f \in L^p(\mathbb{R})$ the following inequality holds,*

$$B(f,f) \leqslant C\|f\|^{2-\delta}_{L^p} \sup_{k,I} \|f_k\|^\delta_{L^p}\Big(\frac{|E_k \cap I|}{|E_k| + |I|}\Big)^\delta,$$

where I ranges over all compact intervals of $\mathbb{R}$, $E_k = \{x \in \mathbb{R} : 2^k \leqslant |f(x)| < 2^{k+1}\}$ and $f_k = f\chi_{E_k}$, $k \in \mathbb{Z}$.

Proof. We will use the following characterization of the $L^{p,2}$ norm. If we decompose f as in the statement of the proposition, $f = \sum_{k\in\mathbb{Z}} f_k$ where f_k have disjoint supports, E_k, and $2^k\chi_{E_k} \leqslant |f_k| < 2^{k+1}\chi_{E_k}$, then

$$\|f\|^2_{L^{p,2}} \asymp \sum_k \|f_k\|^2_{L^p}. \tag{2.16}$$

It follows from (2.16) that $\|f\|^2_{L^{p,2}} \lesssim \|f\|^p_{L^p} \sup_k \|f_k\|^{2-p}_{L^p}$, from where the following bound is obtained

$$|B(f,f)| \lesssim \|f\|^p_{L^p} \sup_k \|f_k\|^{2-p}_{L^p}.$$

We can improve the previous estimate. For this, let $\eta > 0$, $S = \{k : \|f_k\|_p \geqslant \eta\|f\|_p\}$, and $g = \sum_{k\in S} f_k$. Then $|B(f-g, f-g)| \lesssim \eta^{2-p}\|f\|^2_{L^p}$. Since $\|f\|^p_{L^p} = \sum_k \|f_k\|^p_{L^p}$ we obtain that $|S| \leqslant \eta^{-p}$. Therefore, by Cauchy-Schwarz

$$|B(g,g)|^{1/2} \leqslant \sum_{k\in S} |B(f_k,f_k)|^{1/2} \leqslant |S| \max_{k\in S} |B(f_k,f_k)|^{1/2} \leqslant \eta^{-p} \max_{k\in S} |B(f_k,f_k)|^{1/2}.$$

We deduce that

$$|B(f,f)|^{1/2} \leqslant |B(f-g,f-g)|^{1/2} + |B(g,g)|^{1/2} \leqslant \eta^{(2-p)/2}\|f\|_{L^p} + \eta^{-p}\max_{k\in S}|B(f_k,f_k)|^{1/2},$$

and squaring we obtain that for all $\eta > 0$

$$|B(f,f)| \lesssim \eta^{2-p}\|f\|_{L^p}^2 + \eta^{-2p}\max_{k\in S}|B(f_k,f_k)|.$$

Optimizing in η gives

$$|B(f,f)| \lesssim \max_k |B(f_k,f_k)|^{\delta/2}\|f\|_{L^p}^{2-\delta}, \tag{2.17}$$

for some $\delta \in (0,1)$ (the optimization gives $\delta = 2(2-p)/(2+p)$). Thus it is then enough to obtain a bound on $B(f,f)$ where $f = \chi_E$.

Lemma 2.14. *There exist $C < \infty$ and $\gamma \in (0,1)$ with the following property. For every E subset of $\mathbb{R}$ of finite measure*

$$B(\chi_E,\chi_E) \leqslant C\|\chi_E\|_{L^p}^2\Big(\sup_I \frac{|E\cap I|}{|E|+|I|}\Big)^{\gamma}, \tag{2.18}$$

where the supremum ranges over all compact intervals I of $\mathbb{R}$.

Proof. Let $\{I_j^k\}_{j\in\mathbb{Z}}$ be a partition of the real line into intervals of equal length 2^k. Then

$$\begin{aligned} B(\chi_E,\chi_E) &= \iint \frac{\chi_E(x)\chi_E(y)}{|x-y|^{\alpha}}\,dxdy = \sum_k \iint_{\{2^{k-1}\leqslant|x-y|<2^k\}} \frac{\chi_E(x)\chi_E(y)}{|x-y|^{\alpha}}\,dxdy \\ &\asymp \sum_k\sum_j 2^{-k\alpha}|E\cap I_j^k||E\cap \tilde{I}_j^k| \\ &\lesssim \sum_k\sum_j 2^{-k\alpha}|E\cap \tilde{I}_j^k|^2 \end{aligned}$$

where $\tilde{I}_j^k$ has the same center as I_j^k and double length. From now on we will rename $\tilde{I}_j^k$ by I_j^k.

Now we fix k and estimate $\sum_j 2^{-k\alpha}|E\cap I_j^k|^2$. Let n be such that $2^n \leqslant |E| < 2^{n+1}$. We will divide the analysis into the cases where $k \leqslant n$ and $k > n$. Recall that $p = 2/(2-\alpha)$, and let $\gamma \in (0,1)$ be a number to be determined later. We first consider the case $k \leqslant n$. We have

$$
\begin{aligned}
\sum_j 2^{-k\alpha}|E\cap I_j^k|^2 &\leqslant \sum_j |E\cap I_j^k| 2^{-k\alpha}\sup_i |E\cap I_i^k|\\
&\lesssim |E|2^{-k\alpha}\Big(\sup_i \frac{|E\cap I_i^k|}{|E|+|I_i^k|}\Big)^{\gamma} 2^{k(1-\gamma)}|E|^{\gamma}\\
&\asymp |E|^{1+\gamma}2^{k(1-\alpha-\gamma)}\Big(\sup_i \frac{|E\cap I_i^k|}{|E|+|I_i^k|}\Big)^{\gamma}\\
&= |E|^{2-\alpha}|E|^{-1+\alpha+\gamma}2^{k(1-\alpha-\gamma)}\Big(\sup_i \frac{|E\cap I_i^k|}{|E|+|I_i^k|}\Big)^{\gamma}\\
&\lesssim \|\chi_E\|_{L^p}^2 2^{-(n-k)(1-\alpha-\gamma)}\Big(\sup_i \frac{|E\cap I_i^k|}{|E|+|I_i^k|}\Big)^{\gamma}.
\end{aligned}
$$

Now if $k>n$ we will have

$$
\begin{aligned}
\sum_j 2^{-k\alpha}|E\cap I_j^k|^2 &\leqslant \sum_j |E\cap I_j^k| 2^{-k\alpha}\sup_i |E\cap I_i^k|\\
&\lesssim |E|2^{-k\alpha}\Big(\sup_i \frac{|E\cap I_i^k|}{|E|+|I_i^k|}\Big)^{\gamma} 2^{k\gamma}|E|^{1-\gamma}\\
&\asymp |E|^{2-\gamma}2^{-k(\alpha-\gamma)}\Big(\sup_i \frac{|E\cap I_i^k|}{|E|+|I_i^k|}\Big)^{\gamma}\\
&= |E|^{2-\alpha}|E|^{\alpha-\gamma}2^{-k(\alpha-\gamma)}\Big(\sup_i \frac{|E\cap I_i^k|}{|E|+|I_i^k|}\Big)^{\gamma}\\
&\lesssim \|\chi_E\|_{L^p}^2 2^{-(k-n)(\alpha-\gamma)}\Big(\sup_i \frac{|E\cap I_i^k|}{|E|+|I_i^k|}\Big)^{\gamma}.
\end{aligned}
$$

Thus if we choose $\gamma>0$ smaller than $\min(1-\alpha,\alpha)$ we obtain the desired conclusion after adding over k

$$
B(\chi_E,\chi_E)\lesssim \|\chi_E\|_{L^p}^2\Big(\sup_I \frac{|E\cap I|}{|E|+|I|}\Big)^{\gamma}. \qquad \square
$$

By combining Lemma 2.14 and (2.17) we obtain that for $f\in L^p$

$$
B(f,f)\lesssim \|f\|_{L^p}^{2-\delta}\sup_{k,I}\|f_k\|_{L^p}^{\delta}\Big(\frac{|E_k\cap I|}{|E_k|+|I|}\Big)^{\delta\gamma/2},
$$

that implies (after we rename $\delta\gamma/2$ by δ)

$$
B(f,f)\lesssim \|f\|_{L^p}^{2-\delta}\sup_{k,I}\|f_k\|_{L^p}^{\delta}\Big(\frac{|E_k\cap I|}{|E_k|+|I|}\Big)^{\delta},
$$

since $\|f_k\|_p/\|f\|_p\leqslant 1$ and so $(\|f_k\|_p/\|f\|_p)^{\delta}\leqslant(\|f_k\|_p/\|f\|_p)^{\delta\gamma/2}$. $\square$

We note that $\sup_{k,I} \|f_k\|_{L^p}^{\delta} \Big(\frac{|E_k \cap I|}{|E_k|+|I|}\Big)^{\delta}$ is bounded by $\big(\sup_I |I|^{-1+1/p} \int_I |f|\big)^{\delta}$. Indeed, we have

$$\sup_{k,I} \|f_k\|_{L^p}^{\delta} \Big(\frac{|E_k \cap I|}{|E_k| + |I|}\Big)^{\delta} \lesssim \Big(\sup_{k,I} |I|^{-1+1/p} \int_I |f_k|\Big)^{\delta}.$$

To see this, we rewrite $\|f_k\|_{L^p}^{\delta} \asymp 2^{k\delta}|E_k|^{\delta/p}$ and $\int_I |f_k| \asymp 2^k |E_k \cap I|$. It suffices to show that for all k, I

$$|E_k|^{\delta/p} \Big(\frac{|E_k \cap I|}{|E_k| + |I|}\Big)^{\delta} \leqslant |I|^{(-1+1/p)\delta} |E_k \cap I|^{\delta},$$

which is equivalent to $|E_k|^{\delta/p}|I|^{\delta} \leqslant |I|^{\delta/p}(|E_k| + |I|)^{\delta}$. This holds trivially in the case $|E_k| \leqslant |I|$, while in the case $|E_k| > |I|$ we rewrite the inequality as

$$1 \leqslant \Big(1 + \frac{|I|}{|E_k|}\Big)^{\delta} \Big(\frac{|I|}{|E_k|}\Big)^{\delta(-1+1/p)},$$

which holds because $-1 + 1/p < 0$.

We have proved the following proposition

Proposition 2.15. *Let $0 < \alpha < 1$ and $p = 2/(2 - \alpha)$. There exist $C < \infty$ and $\delta \in (0, 2)$ with the following property. For all $f \in L^p(\mathbb{R})$*

$$B(f, f) \leqslant C\|f\|_{L^p(\mathbb{R})}^{2-\delta} \Big(\sup_I |I|^{-1+1/p} \int_I |f| dx\Big)^{\delta}. \tag{2.19}$$

where I ranges over all compact intervals of $\mathbb{R}$.

Using the Cauchy-Schwarz inequality for B and a decomposition as in Lemma 2.8 we obtain the corollary,

Corollary 2.16. *Let $0 < \alpha < 1$ and $p = 2/(2 - \alpha)$. There exist $C < \infty$ and $\delta \in (0, 2)$ such that for all $f \in L^p([0, 2\pi])$,*

$$\int_{[0,2\pi]^2} f(x)f(y)|\sin(x - y)|^{-\alpha} dxdy \leqslant C\|f\|_{L^p([0,2\pi])}^{2-\delta} \Big(\sup_I |I|^{-1+1/p} \int_I |f| dx\Big)^{\delta},$$

where I ranges over all intervals of $[0, 2\pi]$.

We now consider the operator we will use to control the adjoint Fourier operator T.

Definition 2.17. Let $0 < \alpha < 1$ and $p = 2/(2 - \alpha)$. We define the bilinear operator $Q : L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2) \to \mathbb{R}$ by

$$Q(f, g) = \int_{(\mathbb{R}^2)^2} f(r, x)g(r', x')|x - x'|^{-\alpha} dxdx'drdr', \tag{2.20}$$

Note that we can write $Q(f,f) = B(\int f(r,x)dr, \int f(r',x')dr')$.

For $f \in L^p(r,x)$ with $\|\int f(r,x)dr\|_{L^p_x} < \infty$ we use (2.19) to obtain

$$Q(f,f) \lesssim \Big\| \int |f(r,x)|dr \Big\|_{L^p_x}^{2-\delta} \Big(\sup_I |I|^{-1+1/p} \int_I \int |f(r,x)|drdx \Big)^{\delta}. \tag{2.21}$$

Suppose that $f(r,x)$ is supported where $2^{k-1} \leqslant r < 2^k$, then $\int_I \int f(r,x)drdx = \int_{\mathcal{C}} f(r,x)$, where $\mathcal{C} = [2^{k-1}, 2^k] \times I$, and $\|\int f(r,x)dr\|_{L^p_x} \leqslant 2^{(k-1)(1-1/p)} \|f\|_{L^p(r,x)}$. Thus, it follows from (2.21) that

$$Q(f,f) \lesssim 2^{2k(1-1/p)} \|f\|_{L^p(r,x)}^{2-\delta} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1+1/p} \int_{\mathcal{C}} |f(r,x)|drdx \Big)^{\delta}, \tag{2.22}$$

where we used $2^{k-1}|I| = |\mathcal{C}|$.

In the case we are interested in we will need to estimate $Q(f_k^{3/2}, f_k^{3/2})$ with the support of f_k as before and $f_k \in L^2$, with $\alpha = 1/2$ and $p = 4/3$.

Corollary 2.18. *There exist $C < \infty$ and $\delta \in (0,2)$ with the following property. Let $k, k' \in \mathbb{Z}$ and $f, g \in L^{4/3}(\mathbb{R}^2)$ and suppose that $f(r,x), g(r,x)$ are supported in the regions $[2^{k-1}, 2^k] \times \mathbb{R}$ and $[2^{k'-1}, 2^{k'}] \times \mathbb{R}$ respectively. Then*

$$|Q(f,g)|^2 \leqslant C\, 2^{2(k+k')/4} \|f\|_{L^{4/3}(r,x)}^{2-\delta} \|g\|_{L^{4/3}(r,x)}^{2-\delta} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f| \Big)^{\delta} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |g| \Big)^{\delta}. \tag{2.23}$$

Proof. This follows from (2.22) and the Cauchy-Schwarz inequality for Q,

$$Q(f,g)^2 \leqslant Q(f,f)Q(g,g).$$

□

For $f_k, f_{k'} \in L^2(\mathbb{R}^2_{(r,x)})$ supported where $2^{k-1} \leqslant r < 2^k$ and $2^{k'-1} \leqslant r < 2^{k'}$ respectively we obtain

$$Q(|f_k|^{3/2}, |f_{k'}|^{3/2})^2 \lesssim 2^{2(k+k')/4} \|f_k\|_{L^2(r,x)}^{3(2-\delta)/2} \|f_{k'}\|_{L^2(r,x)}^{3(2-\delta)/2} \cdot \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} \Big)^{\delta} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_{k'}|^{3/2} \Big)^{\delta}. \tag{2.24}$$

The use of the Cauchy-Schwarz inequality for Q, and a decomposition as in Lemma 2.8 implies that for $f_k, f_{k'} \in L^2(\mathbb{R}_r \times [0,2\pi]_x)$ supported where $2^{k-1} \leqslant r < 2^k$ and $2^{k'-1} \leqslant r < 2^{k'}$ the following estimate holds

$$\Big(\int |f_k(r,x) f_{k'}(r'x')|^{3/2} |\sin(x-x')|^{-1/2} dxdx'drdr' \Big)^2 \lesssim 2^{2(k+k')/4} \|f_k\|_{L^2(r,x)}^{3(2-\delta)/2} \|f_{k'}\|_{L^2(r,x)}^{3(2-\delta)/2} \cdot \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} \Big)^{\delta} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_{k'}|^{3/2} \Big)^{\delta}. \tag{2.25}$$

2.5 The cap bound for the adjoint Fourier restriction operator

Proposition 2.19. *There exist $C < \infty$ and $\delta \in (0,2)$ with the following property. Let $k, k' \in \mathbb{Z}$ and $f, g \in L^2(\Gamma^2)$, with f and g supported in the regions $2^{k-1} \leqslant |y| < 2^k$ and $2^{k'-1} \leqslant |y| < 2^{k'}$ respectively, then*

$$\|Tf \cdot Tg\|_{L^3} \leqslant C\, 2^{-\frac{1}{6}|k-k'|} \|f\|_{L^2(\Gamma^2)}^{\frac{1}{2}(2-\delta)} \|g\|_{L^2(\Gamma^2)}^{\frac{1}{2}(2-\delta)} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{\frac{3}{2}}\Big)^{\frac{\delta}{3}} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |g|^{\frac{3}{2}}\Big)^{\frac{\delta}{3}}. \tag{2.26}$$

Proof. Recall from Section 2.3, equation (2.13), that we have the inequality

$$\|Tf \cdot Tg\|_{L^3} \leqslant C(2^k 2^{k'})^{-1/3} \min(2^k, 2^{k'})^{1/3} \cdot \Big(\int |f(y)g(y')|^{3/2} |\sin(\theta - \theta')|^{-1/2} d\theta d\theta' dr dr'\Big)^{2/3}.$$

From (2.25) we obtain

$$\|Tf \cdot Tg\|_{L^3} \lesssim (2^k 2^{k'})^{-1/3} \min(2^k, 2^{k'})^{1/3} 2^{(k+k')/6} \|f\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \|g\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \cdot \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2}\Big)^{\delta/3} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |g|^{3/2}\Big)^{\delta/3},$$

which as in the proof of Lemma 2.9 can be rewritten as

$$\|Tf \cdot Tg\|_{L^3} \lesssim 2^{-|k-k'|/6} \|f\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \|g\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2}\Big)^{\delta/3} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |g|^{3/2}\Big)^{\delta/3}. \qquad \square$$

Corollary 2.20. *There exist $C < \infty$ and $\delta > 0$ with the following property. If $f \in L^2(\Gamma^2)$ and $f_k = f\chi_{2^{k-1} \leqslant |y| < 2^k}$, $k \in \mathbb{Z}$, then*

$$\|Tf\|_{L^6(\mathbb{R}^3)}^2 \leqslant C \sum_{k \in \mathbb{Z}} \|f_k\|_{L^2(\Gamma^2)}^{2-\delta} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} d\sigma\Big)^{2\delta/3}. \tag{2.27}$$

Proof. We start by writing $\|Tf\|_{L^6}^2 = \|Tf \cdot Tf\|_{L^3}$ and $Tf = \sum_{k \in \mathbb{Z}} Tf_k$, so the triangle inequality gives

$$\|Tf \cdot Tf\|_{L^3} \leqslant \sum_{k,k'} \|Tf_k \cdot Tf_{k'}\|_{L^3}$$

that together with Proposition 2.19 gives

$$\|Tf\|_{L^6}^2 \lesssim \sum_{k,k'} 2^{-|k-k'|/6} \|f_k\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \|f_{k'}\|_{L^2(\Gamma^2)}^{(2-\delta)/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2}\Big)^{\delta/3} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_{k'}|^{3/2}\Big)^{\delta/3}.$$

The desired conclusion follows by the Cauchy-Schwarz inequality. □

By using Proposition 2.19 instead of Lemma 2.9 we can obtain an analog of Proposition 2.10, that is

Proposition 2.21. *There exist $C < \infty$ and $\delta \in (0, 2)$ with the following property. Let $f \in L^2(\Gamma^2)$ and for $k \in \mathbb{Z}$ let $f_k(y) = f(y)\chi_{\{2^{k-1} \leqslant |y| < 2^k\}}$. Then*

$$\|Tf\|_{L^6(\mathbb{R}^3)} \leqslant C\Big(\sum_{k\in\mathbb{Z}} \|f_k\|_{L^2(\Gamma^2)}^{3-3\delta/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} d\sigma\Big)^{\delta}\Big)^{1/3}. \tag{2.28}$$

Proposition 2.22 (Cap estimate)**.** *There exist $C < \infty$ and $\delta \in (0, 2)$ such that for all $f \in L^2(\Gamma^2)$ the following estimate holds*

$$\|Tf\|_{L^6(\mathbb{R}^3)} \leqslant C\|f\|_{L^2(\Gamma^2)}^{1-\delta/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2} d\sigma\Big)^{\delta/3}, \tag{2.29}$$

Proof. From Proposition 2.21 we have

$$\|Tf\|_{L^6} \lesssim \Big(\sum_{k} \|f_k\|_{L^2}^{3-3\delta/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} d\sigma\Big)^{\delta}\Big)^{1/3}.$$

For each k, using that $\delta \leqslant 2/3$ (δ can be taken as small as desired by changing the corresponding implicit constants C in the inequalities) we have

$$\begin{aligned} \|f_k\|_{L^2}^{3-3\delta/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} d\sigma\Big)^{\delta} &= \|f_k\|_{L^2}^2 \|f_k\|_{L^2}^{1-3\delta/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f_k|^{3/2} d\sigma\Big)^{\delta} \\ &\leqslant \|f_k\|_{L^2}^2 \|f\|_{L^2}^{1-3\delta/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2} d\sigma\Big)^{\delta}. \end{aligned}$$

Then, adding over k,

$$\|Tf\|_{L^6} \lesssim \|f\|_{L^2}^{1-\delta/2} \Big(\sup_{\mathcal{C}} |\mathcal{C}|^{-1/4} \int_{\mathcal{C}} |f|^{3/2} d\sigma\Big)^{\delta/3}.$$

□

2.6 Using the cap bound

We will prove the analog of [9, Lemma 2.6].

Lemma 2.23. *For any $\delta > 0$ there exist $C_\delta < \infty$ and $\eta_\delta > 0$ with the following property. If $f \in L^2(\Gamma^2)$ satisfies $\|Tf\|_6 \geqslant \delta \mathbf{C} \|f\|_2$ then there exists a decomposition $f = g + h$ and a cap $\mathcal{C}$ satisfying*

$$0 \leqslant |g|, |h| \leqslant |f|, \tag{2.30}$$

$$g, h \text{ have disjoint supports}, \tag{2.31}$$

$$|g(x)| \leqslant C_\delta \|f\|_2 |\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(x), \text{ for all } x, \tag{2.32}$$

$$\|g\|_2 \geqslant \eta_\delta \|f\|_2. \tag{2.33}$$

Proof. For convenience, normalize so that $\|f\|_{L^2(\Gamma^2)} = 1$. By Proposition 2.22 there exists a cap $\mathcal{C}$ such that

$$\int_{\mathcal{C}} |f|^{3/2} dr d\theta \geqslant \tfrac{1}{2} c(\delta) |\mathcal{C}|^{1/4}.$$

Let $R \geqslant 1$ and define $E = \{x \in \mathcal{C} : |f(x)| \leqslant R\}$. Set $g = f\chi_E$ and $h = f - f\chi_E$. Then g, h have disjoint supports, $g + h = f$, g is supported on $\mathcal{C}$, and $\|g\|_\infty \leqslant R$. Since $|h(x)| \geqslant R$ for almost every $x \in \mathcal{C}$ for which $h(x) \neq 0$ we have

$$\int_{\mathcal{C}} |h|^{3/2} \leqslant R^{-1/2} \int_{\mathcal{C}} h^2 \leqslant R^{-1/2} \|f\|_2^2 = R^{-1/2}.$$

If we choose R by setting $R^{-1/2} = \frac{1}{4} c(\delta) |\mathcal{C}|^{1/4}$, then

$$\int_{\mathcal{C}} |g|^{3/2} = \int_{\mathcal{C}} |f|^{3/2} - \int_{\mathcal{C}} |h|^{3/2} \geqslant \tfrac{1}{4} c(\delta) |\mathcal{C}|^{1/4}.$$

By Hölder's inequality, since g is supported on $\mathcal{C}$,

$$\|g\|_2 \geqslant |\mathcal{C}|^{-1/6} \Big(\int |g|^{3/2} \Big)^{2/3} \geqslant c'(\delta) = c'(\delta) \|f\|_2 > 0.$$ □

We note that the conditions $|g(x)| \leqslant C_\delta \|f\|_2 |\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(x)$ and $\|g\|_2 \geqslant \eta_\delta \|f\|_2$ easily imply a lower bound on the L^1 norm of g.

Lemma 2.24. *Let $g \in L^2(\Gamma^2)$ satisfying $|g(x)| \leqslant a |\mathcal{C}|^{-1/2} \chi_{\mathcal{C}}(x)$ and $\|g\|_2 \geqslant b$, for some $a, b > 0$ and $\mathcal{C} \subset \Gamma^2$. Then there is a constant $C = C(a, b) > 0$ such that*

$$\|g\|_{L^1(\Gamma^2)} \geqslant C |\mathcal{C}|^{1/2}.$$

Proof. The hypotheses on g imply that $|a^{-1} |\mathcal{C}|^{1/2} g(x)| \leqslant \chi_{\mathcal{C}}(x) \leqslant 1$ and thus $\|a^{-1} |\mathcal{C}|^{1/2} g\|_2^2 \leqslant \|a^{-1} |\mathcal{C}|^{1/2} g\|_1$. Therefore

$$\|g\|_1 \geqslant a^{-1} |\mathcal{C}|^{1/2} \|g\|_2^2 \geqslant a^{-1} b^2 |\mathcal{C}|^{1/2}.$$ □

2.7 Using the group of symmetries

A Lorentz transformation, L, in $\mathbb{R}^3$ is an invertible linear map that preserves the bilinear form

$$A(x,y) = x_1y_1 + x_2y_2 - x_3y_3,$$

$x = (x_1,x_2,x_3), y = (y_1,y_2,y_3) \in \mathbb{R}^3$, i.e.

$$A(x,y) = A(Lx,Ly), \text{ for all } x,y \in \mathbb{R}^3.$$

Examples of Lorentz transformations are L^t, M^t and R_θ given next. For $t \in (-1,1)$ we define the linear map $L^t : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$L^t(x_1,x_2,x_3) = \Big(\frac{x_1+tx_3}{\sqrt{1-t^2}}, x_2, \frac{x_3+tx_1}{\sqrt{1-t^2}}\Big).$$

$\{L^t\}_{t\in(-1,1)}$ is a one parameter subgroup of Lorentz transformations. Similarly,

$$M^t(x_1,x_2,x_3) = \Big(x_1, \frac{x_2+tx_3}{\sqrt{1-t^2}}, \frac{x_3+tx_2}{\sqrt{1-t^2}}\Big)$$

is a Lorentz transformation.

One computes that L^t and M^t preserve the cone for all $t \in (-1,1)$, that is, $L^t(\Gamma^2) = M^t(\Gamma^2) = \Gamma^2$. For $\lambda > 0$ we define the dilation $D_\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ by $D_\lambda(x) = \lambda x$ that clearly satisfies $D_\lambda(\Gamma^2) = \Gamma^2$ for every $\lambda > 0$. For $\theta \in [0,2\pi]$ we denote by R_θ the rotation in $\mathbb{R}^3$ by angle θ about the x_3-axis

$$R_\theta(x_1,x_2,x_3) = (x_1\cos\theta - x_2\sin\theta, x_1\sin\theta + x_2\cos\theta, x_3).$$

R_θ preserves the cone for all $0 \leqslant \theta \leqslant 2\pi$.

Associated to L^t, M^t, D_λ, R_θ are the operators L^{t*}, M^{t*}, D^*_λ and R^*_θ acting on a function $f \in L^2(\Gamma^2)$ by

$$L^{t*}f = f\circ L^t,\ M^{t*}f = f\circ M^t,\ D^*_\lambda f = \lambda^{1/2} f\circ D_\lambda,\ R^*_\theta f = f\circ R_\theta, \tag{2.34}$$

where "$\circ$" denotes composition. We also define $L_t = \sqrt{1-t^2}L^t = D_{\sqrt{1-t^2}}L^t$ and L^*_t by

$$L^*_t f(x_1,x_2,x_3) = (1-t^2)^{1/4} f\circ L_t(x_1,x_2,x_3) = (1-t^2)^{1/4} f(x_1+tx_3, \sqrt{1-t^2}\,x_2, x_3+tx_x).$$

The measure σ is invariant under the action of Lorentz transformations that preserve the cone, and in fact is the only one with that property, up to multiplication by constant; for this we refer to [39] where the case of the cone in $\mathbb{R}^4$ is considered. In this paper we only need to know that for every $t \in (-1,1)$, L^t and M^t preserve the measure σ and this can be done directly using the change of variables formula and seeing that the Jacobians work out. We write it in the next proposition and include the proof for completeness.

Proposition 2.25. *For any $t \in (-1,1)$ the linear maps L^t, M^t are invertible, preserve Γ^2, that is $L^t(\Gamma^2) = M^t(\Gamma^2) = \Gamma^2$, and preserve σ, that is, for any $f \in L^1(\Gamma^2)$*

$$\int_{\Gamma^2} f \circ L^t d\sigma = \int_{\Gamma^2} f \circ M^t d\sigma = \int_{\Gamma^2} f d\sigma.$$

Proof. Letting $P(x_1, x_2, x_3) = (x_2, x_1, x_3)$ we see that $M^t = P \circ L^t \circ P$ and so it is enough to prove the statements for L^t. The inverse of L^t is L^{-t}. That $L^t(\Gamma^2) \subseteq \Gamma^2$ follows from the equality

$$\Big(\frac{x_3 + tx_1}{\sqrt{1-t^2}}\Big)^2 = \Big(\frac{x_1 + tx_3}{\sqrt{1-t^2}}\Big)^2 + x_2^2$$

and the inequality

$$\frac{x_3 + tx_1}{\sqrt{1-t^2}} \geqslant 0$$

whenever $x_3^2 = x_1^2 + x_2^2$ and $x_3 \geqslant 0$. Since the same is true for L^{-t}, it follows that $L^t(\Gamma^2) = \Gamma^2$. For the invariance of the measure, let $f \in L^1(\Gamma^2)$. We have

$$\int f \circ L^t(x_1, x_2, x_3) d\sigma(x_1, x_2, x_3) = \int_{\mathbb{R}^2} f\Big(\frac{y_1 + ty_3}{\sqrt{1-t^2}}, y_2, \frac{y_3 + ty_1}{\sqrt{1-t^2}}\Big) \frac{dy_1 dy_2}{\sqrt{y_1^2 + y_2^2}}$$

where $y_3 = \sqrt{y_1^2 + y_2^2}$. We use the change of variables $u = \frac{y_1 + ty_3}{\sqrt{1-t^2}} = \frac{y_1 + t\sqrt{y_1^2 + y_2^2}}{\sqrt{1-t^2}}$, $v = y_2$. We note that the Jacobian is

$$\frac{\partial(u,v)}{\partial(y_1,y_2)} = \frac{1}{\sqrt{1-t^2}}\Big(1 + \frac{ty_1}{\sqrt{y_1^2 + y_2^2}}\Big),$$

or equivalently

$$\frac{\partial(y_1,y_2)}{\partial(u,v)} = \sqrt{y_1^2 + y_2^2}\frac{\sqrt{1-t^2}}{\sqrt{y_1^2 + y_2^2} + ty_1}.$$

Now, since $L^t(y_1, y_2, y_3)$ lies in Γ^2 we also have

$$\sqrt{u^2 + v^2} = \frac{y_3 + ty_1}{\sqrt{1-t^2}}.$$

It follows that the Jacobian factor can be rewritten as

$$\frac{\partial(y_1,y_2)}{\partial(u,v)} = \frac{\sqrt{y_1^2 + y_2^2}}{\sqrt{u^2 + v^2}}.$$

Therefore, letting $w = \sqrt{u^2 + v^2}$,

$$\int_{\mathbb{R}^2} f\Big(\frac{y_1 + ty_3}{\sqrt{1-t^2}}, y_2, \frac{y_3 + ty_1}{\sqrt{1-t^2}}\Big) \frac{dy_1 dy_2}{\sqrt{y_1^2 + y_2^2}} = \int_{\mathbb{R}^2} f(u, v, w) \frac{du dv}{\sqrt{u^2 + v^2}}$$

or equivalently,

$$\int f \circ L^t d\sigma = \int f d\sigma.$$

□

The Lorentz invariance of the measure implies invariance of the L^2 norm, for $f \in L^2(\Gamma^2)$

$$\|L^{t*}f\|_{L^2(\sigma)} = \|M^{t*}f\|_{L^2(\sigma)} = \|D^*_\lambda f\|_{L^2(\sigma)} = \|R^*_\theta f\|_{L^2(\sigma)} = \|L^*_t f\|_{L^2(\sigma)} = \|f\|_{L^2(\sigma)}. \quad (2.35)$$

Using the Lorentz invariance of σ it is direct to check that for all $p \in [1,\infty]$ the L^p norm of Tf does not change under Lorentz transformations in the sense that

$$\|T(f\circ L)\|_{L^p(\mathbb{R}^3)} = \|Tf\|_{L^p(\mathbb{R}^3)}. \quad (2.36)$$

Indeed, writing

$$Tf(x,t) = \int e^{ix\cdot y}e^{ity'}f(y,y')d\sigma(y,y') = \int e^{iA((x,-t),(y,y'))}f(y,y')d\sigma(y,y'),$$

thus

$$\begin{aligned}T(f\circ L)(x,t) &= \int e^{iA((x,-t),(y,y'))}f\circ L(y,y')d\sigma(y,y')\\ &= \int e^{iA(L(x,-t),L(y,y'))}f\circ L(y,y')d\sigma(y,y')\\ &= \int e^{iA(L(x,-t),(y,y'))}f(y,y')d\sigma(y,y').\end{aligned}$$

Since for a Lorentz transformation L, $|\det L| = 1$, (2.36) follows by change of variables in the case $p \in [1,\infty)$. When $p = \infty$, (2.36) follows since L is invertible.

We can use the group of symmetries to widen caps, that is, we have

Lemma 2.26. *Let $\mathcal{C} \subset [1/2,1]\times[0,2\pi]$ be a cap in Γ^2. Then there exist $t \in [0,1)$ and $\theta \in [0,2\pi]$ such that $L_t^{-1}R_\theta^{-1}(\mathcal{C})$ satisfies*

$$\sigma(L_t^{-1}R_\theta^{-1}(\mathcal{C})) \geqslant \frac{1}{2},\ \textit{and}\ L_t^{-1}R_\theta^{-1}(\mathcal{C}) \subseteq [1/4,1]\times[0,2\pi]. \quad (2.37)$$

Proof. Let $\theta \in [0,2\pi]$ be such that $R_\theta^{-1}\mathcal{C} = [1/2,1]\times[-\varepsilon,\varepsilon]$, for some $\varepsilon \in [0,\pi]$. The measure of $\mathcal{C}$ is $\sigma(\mathcal{C}) = |\mathcal{C}| = \varepsilon$, and so we can assume $\varepsilon < 1/2$, otherwise we are done by taking $t = 0$.

The inverse of L_t is $L_t^{-1} = (1-t^2)^{-1/2}L^{-t}$ and the measure of $L_t^{-1}R_\theta^{-1}(\mathcal{C})$ is

$$\begin{aligned}\sigma(L_t^{-1}R_\theta^{-1}(\mathcal{C})) = \sigma((1-t^2)^{-1/2}L^{-t}R_\theta^{-1}(\mathcal{C})) &= \sigma((L^t)^{-1}R_\theta^{-1}((1-t^2)^{-1/2}\mathcal{C}))\\ &= \sigma((1-t^2)^{-1/2}\mathcal{C}) = (1-t^2)^{-1/2}\sigma(\mathcal{C}),\end{aligned}$$

where we used the invariance of σ under Lorentz transformations and that $\sigma(\lambda\mathcal{C}) = |\lambda|\sigma(\mathcal{C})$ for any $\lambda \in \mathbb{R}$.

Let t be such that $\sigma(L_t^{-1}R_\theta^{-1}(\mathcal{C})) = 1$, that is $t = (1-|\mathcal{C}|^2)^{1/2} = (1-\varepsilon^2)^{1/2}$. Now we write $R_\theta^{-1}\mathcal{C} = \{(r\cos\varphi, r\sin\varphi, r) : 1/2 \leqslant r \leqslant 1, -\varepsilon \leqslant \varphi \leqslant \varepsilon\}$, so that

$$L_t^{-1}R_\theta^{-1}(\mathcal{C}) = \Big\{r(1-t^2)^{-1/2}\Big(\frac{\cos\varphi - t}{(1-t^2)^{1/2}}, \sin\varphi, \frac{1-t\cos\varphi}{(1-t^2)^{1/2}}\Big) : 1/2 \leqslant r \leqslant 1, -\varepsilon \leqslant \varphi \leqslant \varepsilon\Big\}.$$

Note that for $1/2 \leqslant r \leqslant 1$ and $-\varepsilon \leqslant \varphi \leqslant \varepsilon$ we have

$$r(1-t^2)^{-1/2}\frac{|\cos\varphi - t|}{(1-t^2)^{1/2}} \leqslant \frac{1-t}{1-t^2} = \frac{1}{1+t} \leqslant 1$$

because $\cos\varphi \geqslant \cos\varepsilon \geqslant t$. Similarly

$$r(1-t^2)^{-1/2}|\sin\varphi| \leqslant \frac{\sin\varepsilon}{\varepsilon} \leqslant 1$$

and

$$\frac{1}{4} \leqslant \frac{1}{2(1+t)} = \frac{1-t}{2(1-t^2)} \leqslant r\frac{1-t\cos\varphi}{1-t^2} \leqslant 1$$

Then $t = (1-\varepsilon^2)^{1/2}$ gives the desired conclusion. □

Corollary 2.27. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative functions in $L^2(\Gamma^2)$ with $\|f_n\|_{L^2(\Gamma^2)} = 1$ and such that there exists a cap $\mathcal{C}_n \subset [1/2, 1] \times [0, 2\pi]$ with the property*

$$\int_{\mathcal{C}_n} f_n d\sigma \geqslant c|\mathcal{C}_n|^{1/2}, \tag{2.38}$$

where $c > 0$ is independent of n. Then there exist sequences $\{t_n\}_{n\in\mathbb{N}} \subset [0, 1)$ and $\{\theta_n\}_{n\in\mathbb{N}} \subset [0, 2\pi]$ such that $\{L^*_{t_n} R^*_{\theta_n} f_n\}_{n\in\mathbb{N}}$ satisfies that every weak limit in $L^2(\Gamma^2)$ is nonzero.

Proof. L^*_t and R^*_θ preserve the $L^2(\Gamma^2)$ norm thus $\|L^*_t R^*_\theta f_n\|_{L^2(\Gamma^2)} = 1$ for any $t \in [0, 1)$ and $\theta \in [0, 2\pi]$. It follows that for any sequences $\{t_n\}_{n\in\mathbb{N}} \subset [0, 1)$ and $\{\theta_n\}_{n\in\mathbb{N}} \subset [0, 2\pi]$, the set of L^2-weak limits of $\{L^*_{t_n} R^*_{\theta_n} f_n\}$ is nonempty.

Under the action of $L^*_t R^*_\theta$ the integral of a function f changes according to

$$\int L^*_t R^*_\theta f d\sigma = (1-t^2)^{-1/4} \int R^*_\theta f d\sigma = (1-t^2)^{-1/4} \int f d\sigma. \tag{2.39}$$

By Lemma 2.26, for each n there exist $t_n \in [0, 1)$ and $\theta_n \in [0, 2\pi]$ such that

$$\sigma(L^{-1}_{t_n} R^{-1}_{\theta_n}(\mathcal{C}_n)) \geqslant \frac{1}{2} \text{ and } L^{-1}_{t_n} R^{-1}_{\theta_n}(\mathcal{C}_n) \subseteq [1/4, 1] \times [0, 2\pi].$$

Suppose that for a subsequence (that we call the same) $L^*_{t_n} R^*_{\theta_n} f_n \rightharpoonup f$, as $n \to \infty$, for some $f \in L^2(\Gamma^2)$. Using (2.38) and (2.39) we have

$$\begin{aligned} \int_{[1/4,1]\times[0,2\pi]} L^*_{t_n} R^*_{\theta_n} f_n d\sigma &\geqslant (1-t_n^2)^{-1/4} \int_{\mathcal{C}_n} f_n d\sigma \\ &\geqslant c(1-t_n^2)^{-1/4}|\mathcal{C}_n|^{1/2} = c(\sigma(L^{-1}_{t_n} R^{-1}_{\theta_n}(\mathcal{C}_n)))^{1/2} \geqslant \frac{c}{\sqrt{2}}. \end{aligned} \tag{2.40}$$

From (2.40) and the weak convergence it follows that

$$\int_{[1/4,1]\times[0,2\pi]} f d\sigma \geqslant \frac{c}{\sqrt{2}} > 0$$

and so $f \neq 0$. □

2.8 The proof of the precompactness

In this section we prove that up to symmetries of the cone, an extremizing sequence is precompact.

We will give two proofs. In this section the proof will be based on [18] and [17]; the other will be based on [9] with a modification coming from [18] and is given in Section 2.9.

Recall that $\mathbf{C}$, given in (2.3), denotes the best constant in the inequality (2.2), in other words, $\mathbf{C} = \|T\|$, the norm of the operator T as a map from $L^2(\Gamma^2)$ to $L^6(\mathbb{R}^3)$.

We start by stating Proposition 1.1 of [18] for the cone.

Proposition 2.28 ([18]). *Let $T : L^2(\Gamma^2, \sigma) \to L^6(\mathbb{R}^3)$ be the Fourier extension operator defined in* (2.1). *Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\Gamma^2)$ such that:*

(i) $\lim\limits_{n\to\infty} \|f_n\|_2 = 1$;

(ii) $\lim\limits_{n\to\infty} \|Tf_n\|_{L^6(\mathbb{R}^3)} = \mathbf{C}$;

(iii) $f_n \rightharpoonup f \neq 0$;

(iv) $Tf_n \to Tf$ *a.e. in* $\mathbb{R}^3$.

Then $f_n \to f$ in $L^2(\Gamma^2)$, in particular $\|f\|_2 = 1$ and $\|Tf\|_{L^6(\mathbb{R}^3)} = \mathbf{C}$.

We have changed slightly condition (i) in Proposition 1.1 of [18] from $\|f_n\|_2 = 1$ to $\lim_{n\to\infty} \|f_n\|_2 = 1$, but the proposition as stated here is easily shown to be equivalent to the one in [18] by considering $f_n/\|f_n\|_2$.

We now restate the precompactness theorem, Theorem 2.3, in a more precise way,

Theorem 2.29. *Let $\{f_n\}_{n\in\mathbb{N}}$ be an extremizing sequence for* (2.2) *of nonnegative functions in $L^2(\Gamma^2)$. Then there exist sequences $\{t_n\}_{n\in\mathbb{N}} \subset (-1,1)$, $\{\theta_n\}_{n\in\mathbb{N}} \subset [0,2\pi]$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ such that $\{L^*_{t_n} R^*_{\theta_n} D^*_{\lambda_n} f_n\}_{n\in\mathbb{N}}$ is precompact, that is, any subsequence has a convergent sub-subsequence in $L^2(\Gamma^2)$.*

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is an extremizing sequence, for all n large enough $\|Tf_n\|_6 \geqslant \frac{\mathbf{C}}{2}\|f_n\|_2$. By Lemma 2.23 with $\delta = 1/2$ there exists $C < \infty$ and $\eta > 0$, a decomposition $f_n = g_n + h_n$ and a cap $\mathcal{C}_n$ satisfying (2.30), (2.31), (2.32) and (2.33). Using that $\|f_n\|_{L^2} \to 1$, as $n \to \infty$, and Lemma 2.24 for g_n gives

$$\|g_n\|_{L^1(\Gamma^2)} \geqslant C|\mathcal{C}_n|^{1/2},$$

where C is independent of n.

Now there exists $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ such that $\lambda_n^{-1}\mathcal{C}_n \subset [1/2,1]\times[0,2\pi]$ and $\lambda_n^{-1}\mathcal{C}_n$ is a cap as in Definition 2.11. By dilation invariance $\{D^*_{\lambda_n} f_n\}_{n\in\mathbb{N}}$ is also an extremizing sequence, with $\|D^*_{\lambda_n} f_n\|_2 = \|f_n\|_2$. The decomposition for f_n gives a decomposition for $D^*_{\lambda_n} f_n$, $D^*_{\lambda_n} f_n = D^*_{\lambda_n} g_n + D^*_{\lambda_n} h_n$, and

$$\int_{\lambda_n^{-1}\mathcal{C}_n} D^*_{\lambda_n} f_n d\sigma \geqslant \|D^*_{\lambda_n} g_n\|_{L^1(\Gamma^2)} \geqslant C|\lambda_n^{-1}\mathcal{C}_n|^{1/2}.$$

We now apply Corollary 2.27 to $\{D^*_{\lambda_n} f_n\}_n$ to obtain sequences $\{t_n\}_{n\in\mathbb{N}} \subset [0,1)$ and $\{\theta_n\}_{n\in\mathbb{N}} \subset [0,2\pi]$ such that every weak limit of $\{L^*_{t_n} R^*_{\theta_n} D^*_{\lambda_n} f_n\}_n$ in $L^2(\Gamma^2)$ is nonzero.

In view of Proposition 2.28 the theorem is proved if we show that, after passing to a subsequence, if $L^*_{t_n} R^*_{\theta_n} D^*_{\lambda_n} f_n \rightharpoonup f$, as $n \to \infty$, then $TL^*_{t_n} R^*_{\theta_n} D^*_{\lambda_n} f_n \to Tf$ a.e. in $\mathbb{R}^3$. We will do this by using the following proposition. □

Proposition 2.30. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a uniformly bounded sequence in $L^2(\Gamma^2)$, i.e., $\sup_n \|u_n\|_2 =: c < \infty$. Suppose there exists $u \in L^2(\Gamma^2)$ such that $u_n \rightharpoonup u$ as $n \to \infty$. Then, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $Tu_{n_k} \to Tu$ a.e. in $\mathbb{R}^3$.*

Proof. The proof of this is contained in the proof of Theorem 1.1 in [17]. We repeat it here for the convenience of the reader (and the author). We start by defining $v_n(y)$ by its Fourier transform

$$\hat{v}_n(y) = u_n(y)|y|^{-1},$$

and $\hat{v}(y) = u(y)|y|^{-1}$.

Since $\|u_n\|^2_{L^2(\Gamma^2)} = \int_{\mathbb{R}^2} |u_n(y)|^2 \frac{dy}{|y|} \leqslant c^2$ we see that $\|v_n\|^2_{\dot{H}^{1/2}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\hat{v}_n(y)|^2 |y| dy \leqslant c^2$. The operator T applied to u_n equals $e^{it\sqrt{-\Delta}} v_n$. Fix $t \in \mathbb{R}$, by the continuity of $e^{it\sqrt{-\Delta}}$ in $\dot{H}^{1/2}(\mathbb{R}^2)$, we have

$$e^{it\sqrt{-\Delta}} v_n \rightharpoonup e^{it\sqrt{-\Delta}} v$$

weakly in $\dot{H}^{1/2}(\mathbb{R}^2)$, as $n \to \infty$. Then, by the Rellich Theorem ([11, Theorem 1.5]), for any $R > 0$

$$e^{it\sqrt{-\Delta}} v_n \to e^{it\sqrt{-\Delta}} v$$

strongly in $L^2(B(0,R))$, as $n \to \infty$. Denote by

$$F_n(t) := \int_{|x|<R} \left| e^{it\sqrt{-\Delta}} (v_n - v) \right|^2 dx = \|e^{it\sqrt{-\Delta}} (v_n - v)\|^2_{L^2(B(0,R))}.$$

By Hölder's inequality and Sobolev embedding, $\dot{H}^{1/2}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$, we obtain

$$F_n(t) \leqslant CR \|e^{it\sqrt{-\Delta}} (v_n - v)\|^2_{\dot{H}^{1/2}(\mathbb{R}^2)} \leqslant 2CR,$$

consequently, by the Fubini and dominated convergence Theorems we have that

$$\int_{-R}^{R} F_n(t) dt = \int_{-R}^{R} \int_{|x|<R} \left| e^{it\sqrt{-\Delta}} (v_n - v) \right|^2 dx dt \to 0$$

as $n \to \infty$. This implies that, up to a subsequence,

$$e^{it\sqrt{-\Delta}}(v_n - v) \to 0 \quad \text{a.e. in } B(0,R) \times (-R,R).$$

Repeating the argument on a discrete sequence of radii R_n such that $R_n \to \infty$, as $n \to \infty$ we conclude, by a diagonal argument, that there exists a subsequence v_{n_k} of v_n such that

$$e^{it\sqrt{-\Delta}}(v_{n_k} - v)(x) \to 0 \quad \text{a.e. for } (x,t) \in \mathbb{R}^2 \times \mathbb{R},$$

or equivalently, in terms of the sequence $\{u_n\}_{n\in\mathbb{N}}$,

$$Tu_{n_k} - Tu \to 0 \quad \text{a.e. in } \mathbb{R}^3.$$

□

This concludes the proof of Theorem 2.29. For the proof of Theorem 2.29 using the Christ-Shao argument we will need the next proposition, an analog of Proposition 2.28, which of course follows from Propositions 2.28 and 2.30, but the idea is to give an alternative approach.

We denote $B(0,R)^c := \{x \in \mathbb{R}^2 : |x| \geqslant R\}$, the complement of the ball $B(0,R)$.

Proposition 2.31. *Let $T : L^2(\Gamma^2, \sigma) \to L^6(\mathbb{R}^3)$ be the Fourier extension operator defined in* (2.1)*. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\Gamma^2)$ such that:*

(i) $\lim_{n\to\infty} \|f_n\|_2 = 1$;

(ii) $\lim_{n\to\infty} \|Tf_n\|_{L^6(\mathbb{R}^3)} = \mathbf{C}$;

(iii) $f_n \rightharpoonup f \neq 0$;

(iv) $\sup_{n\in\mathbb{N}} \|f_n\|_{L^2(B(0,R)^c)} \leqslant \Theta(R)$, *where* $\Theta(R) \to 0$ *as* $R \to \infty$.

Then $f_n \to f$ in $L^2(\Gamma^2)$, in particular $\|f\|_2 = 1$ and $\|Tf\|_{L^6(\mathbb{R}^3)} = \mathbf{C}$.

Proof. Our proof follows that of Proposition 1.1 in [18]. We will denote by $o_n(1)$ a quantity depending on n only that satisfies $\lim_{n\to\infty} o_n(1) = 0$. We will allow $o_n(1)$ to change from line to line without changing its name.

Let $R > 0$. Note that because of the weak convergence we also have $\|f\|_{L^2(B(0,R)^c)} \leqslant \Theta(R)$. Denote $f^R = f\chi_{B(0,R)}$ and $f_n^R = f_n\chi_{B(0,R)}$. Because of the weak convergence and the compact support of f^R and f_n^R, we have

$$T(f_n^R) \to T(f^R), \text{ a.e. in } \mathbb{R}^3,$$

and because of the continuity of T,

$$\|T(f_n - f_n^R)\|_6, \|T(f - f^R)\|_6 \leqslant C\Theta(R). \tag{2.41}$$

Thus by triangular inequality, using that $\|Tf_n^R - Tf^R\|_6 \leqslant C$ for all n and the binomial expansion

$$\begin{aligned}\|Tf_n - Tf\|_6^6 &\leqslant (\|Tf_n - Tf_n^R\|_6 + \|T(f_n^R - f^R)\|_6 + \|T(f^R - f)\|_6)^6 \\ &\leqslant \|Tf_n^R - Tf^R\|_6^6 + C\Theta(R), \end{aligned} \tag{2.42}$$

and similarly

$$\|Tf_n^R - Tf^R\|_6^6 \leqslant \|Tf_n - Tf\|_6^6 + C\Theta(R). \tag{2.43}$$

Using the Brézis and Lieb Lemma as in [18] we get

$$\|Tf_n^R - Tf^R\|_6^6 = \|Tf_n^R\|_6^6 - \|Tf^R\|_6^6 + o_{n,R}(1), \tag{2.44}$$

where $o_{n,R}(1) \to 0$ as $n \to \infty$, when we keep R fixed. Using (2.41), (2.42) and (2.44) we obtain

$$|\|Tf_n - Tf\|_6^6 - (\|Tf_n\|_6^6 - \|Tf\|_6^6)| \leqslant o_{n,R}(1) + C\Theta(R), \tag{2.45}$$

By the weak convergence

$$\|f_n - f\|_2^2 = \|f_n\|_2^2 - \|f\|_2^2 + o_n(1) \tag{2.46}$$

or equivalently

$$\lim_{n\to\infty} \|f_n - f\|_2^2 = 1 - \|f\|_2^2. \tag{2.47}$$

Using that $\{f_n\}_{n\in\mathbb{N}}$ is a maximizing sequence for T we get

$$\|T\|^2 = \frac{\|Tf_n\|_6^2}{\|f_n\|_2^2} + o_n(1) \leqslant \frac{(\|Tf_n - Tf\|_6^6 + \|Tf\|_6^6 + o_{n,R}(1) + C\Theta(R))^{1/3}}{\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)} + o_n(1) \tag{2.48}$$

$$\leqslant \frac{\|Tf_n - Tf\|_6^2 + \|Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}}{\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)} + o_n(1), \tag{2.49}$$

where we used the inequality

$$(a+b+c)^t \leqslant a^t + b^t + c^t, \text{ for all } a,b,c \geqslant 0 \text{ and } 0 \leqslant t \leqslant 1.$$

The continuity of T and (2.49) imply

$$\|T\|^2 \leqslant \frac{\|T\|^2\|f_n - f\|_2^2 + \|Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}}{\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)} + o_n(1),$$

and hence

$$\|T\|^2(\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)) \leqslant \|T\|^2\|f_n - f\|_2^2 + \|Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}},$$

that after canceling terms implies

$$\|T\|^2(\|f\|_2^2 + o_n(1)) \leqslant \|Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}.$$

Since we also have the inequality $\|Tf\|_6^2 \leqslant \|T\|^2\|f\|_2^2$ we can take the limit as $n \to \infty$ in the previous inequality followed by a limit as $R \to \infty$ to obtain

$$\|Tf\|_6 = \|T\|\|f\|_2. \tag{2.50}$$

Note that this implies that f is an extremizer for T since $f \neq 0$ by hypothesis. It remains to prove that $f_n \to f$ in $L^2(\Gamma^2)$. Using (2.49) and (2.50) we get

$$\|T\|^2 \leqslant \frac{\|T\|^2\|f\|_2^2 + \|Tf_n - Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}}{\|f_n - f\|_2^2 + \|f\|_2^2 + o_n(1)} + o_n(1),$$

which as before implies

$$\|T\|^2(\|f_n - f\|_2^2 + o_n(1)) \leqslant \|Tf_n - Tf\|_6^2 + o_{n,R}(1) + C\Theta(R)^{\frac{1}{3}}. \tag{2.51}$$

and by continuity of T

$$\|Tf_n - Tf\|_6^2 \leqslant \|T\|^2\|f_n - f\|_2^2. \tag{2.52}$$

Using (2.51), (2.52) and (2.47) we can take the limit as $n \to \infty$ and then the limit as $R \to \infty$ to obtain

$$\lim_{n\to\infty} \|Tf_n - Tf\|_6^2 = \|T\|^2(1 - \|f\|_2^2).$$

Now, using this last equality together with (2.45) and the fact that $\{f_n\}_{n\in\mathbb{N}}$ is an L^2-normalized extremizing sequence gives

$$\|T\|^6 = \lim_{n\to\infty} \|Tf_n\|_6^6 = \|T\|^6(1 - \|f\|_2^2)^3 + \|T\|^6\|f\|_2^6,$$

therefore

$$(1 - \|f\|_2^2)^3 + \|f\|_2^6 = 1 \text{ and } \|f\|_2 \leqslant 1.$$

This easily implies that either $\|f\|_2 = 1$ or $\|f\|_2 = 0$. The latter case does not hold since $f \neq 0$ by assumption. Thus $\|f\|_2 = 1$ and $f_n \to f$ in $L^2(\Gamma^2)$. □

2.9 The Christ-Shao concentration compactness argument

We are now ready to use the Christ-Shao concentration compactness argument to gain control over extremizing sequences. We follow the same lines as in [9] so many of the arguments are the same. We will indicate when changes are needed, but will not go over the entire argument. We note that there will be one part we will do in a different way, namely we will not study "cross terms", section 14 in [9], but instead we will use an argument in [18] that to the author seems much easier. In this way we avoid the need of the use of Fourier integral operators (sections 7.3 and 7.4 in [9]).

We now state the results from [9] of interest for us. We indicate the changes needed in our case and if we just state the result without proof is because it is exactly as in [9] with the possible exception of changing norms from L^4 in their case to L^6 in our case.

Definition 2.32. A nonzero function $f \in L^2(\Gamma^2)$ is said to be a δ-nearly extremal for (2.2) if

$$\|Tf\|_{L^6(\mathbb{R}^3)} \geqslant (1-\delta)\mathbf{C}\|f\|_{L^2(\Gamma^2)}.$$

Lemma 2.33. *Let $f = g + h \in L^2(\Gamma^2)$. Suppose that $g \perp h$, $g \neq 0$, and that f is a δ-nearly extremal for some $\delta \in (0, \frac{1}{4}]$. Then*

$$\frac{\|h\|_2}{\|f\|_2} \leqslant C \max\Big(\frac{\|Th\|_6}{\|h\|_2}, \delta^{1/2}\Big). \tag{2.53}$$

Here $C < \infty$ is a constant independent of g, h.

Let $\mathcal{M}$ be the set of all caps modulo the relation $\mathcal{C} \sim \mathcal{C}'$ if there exists $k \in \mathbb{Z}$ such that $\mathcal{C}, \mathcal{C}' \subseteq [2^{k-1}, 2^k] \times [0, 2\pi]$. We define the following metric on $\mathcal{M}$.

Definition 2.34. For any two caps $\mathcal{C}, \mathcal{C}' \subseteq \Gamma^2$,

$$\varrho([\mathcal{C}], [\mathcal{C}']) = |k - k'| \tag{2.54}$$

where $\mathcal{C} = [2^{k-1}, 2^k] \times J$ and $\mathcal{C}' = [2^{k'-1}, 2^{k'}] \times J'$ and $[\mathcal{C}]$ denotes the equivalent class $[\mathcal{C}] = \{[2^{k-1}, 2^k] \times I : I \subseteq [0, 2\pi]$ and I is an interval $\}$.

We will also write $\varrho(\mathcal{C}, \mathcal{C}') = \varrho([\mathcal{C}], [\mathcal{C}'])$.

The equivalent of [9, Lemma 7.5] is the bilinear estimate in Lemma 2.9. We restate it in the language of caps.

Lemma 2.35. *Let $f, g \in L^2(\Gamma^2)$ supported in the caps $\mathcal{C}, \mathcal{C}'$ respectively, then*

$$\|Tf \cdot Tg\|_{L^3} \leqslant C 2^{-\varrho(\mathcal{C},\mathcal{C}')/6}\|f\|_2\|g\|_2,$$

in particular

$$\|T\chi_{\mathcal{C}} \cdot T\chi_{\mathcal{C}'}\|_{L^3} \leqslant C 2^{-\varrho(\mathcal{C},\mathcal{C}')/6}|\mathcal{C}|^{1/2}|\mathcal{C}'|^{1/2}.$$

Here $C < \infty$ is a universal constant.

We now move to the decomposition algorithm, [9, Section 8]. Note that the decomposition algorithm does not depend on the specific manifold we are dealing with, it just requires Lemma 2.23.

Given a nonnegative function $f \in L^2(\Gamma^2)$, the decomposition algorithm gives, in brief, a sequence of disjoint caps $\{\mathcal{C}_\nu\}_{\nu\in\mathbb{N}}$, constants $\{C_\nu\}_{\nu\in\mathbb{N}}$, nonnegative functions f_ν supported on $\mathcal{C}_\nu$ and nonnegative functions G_ν, whose support is disjoint from $f_0 + \cdots + f_{\nu-1}$, such that $f_\nu \leqslant C_\nu|\mathcal{C}_\nu|^{-1/2}\chi_{\mathcal{C}_\nu}$, $f = \sum_{\nu=0}^{N-1} f_\nu + G_N$, for all $N \geqslant 0$, and $f = \sum_{\nu=0}^{\infty} f_\nu$, where the sum is $L^2(\Gamma^2)$-convergent, [9, Lemma 8.1].

Other useful properties can be obtained if f is nearly extremal for (2.2). Lemmas 8.2, 8.3 and 8.4 in [9] have exact analogs for the cone. We mention here the ones we will use.

The analog of Lemma 8.3 in [9] for the cone implies

Lemma 2.36. *There exists a sequence of positive constants $\gamma_\nu \to 0$ and a function $N : (0, \frac{1}{2}] \to \mathbb{Z}^+$ satisfying $N(\delta) \to \infty$ as $\delta \to 0$ such that for any nonnegative $f \in L^2(\Gamma^2)$ that is δ-nearly extremal, the functions $\{f_\nu, G_\nu\}_{\nu\in\mathbb{N}}$ obtained from the decomposition algorithm satisfy*

$$\|G_\nu\|_2, \|f_\nu\|_2 \leqslant \gamma_\nu \|f\|_2 \text{ for all } \nu \leqslant N(\delta).$$

This lemma will be used in the following way: given $\varepsilon > 0$ we can find $\nu(\varepsilon)$ such that $\gamma_\nu < \varepsilon^3$ for all $\nu \geqslant \nu(\varepsilon)$. If we let $\delta(\varepsilon)$ be such that $N(\delta) \geqslant \nu_0$ for all $\delta \leqslant \delta(\varepsilon)$ it follows that an inequality $\|G_N\|_2 \geqslant \varepsilon^3$ applied to a $\frac{1}{2}\delta(\varepsilon)$-nearly extremal f whose decomposition is $\{f_\nu, G_\nu\}_{\nu\in\mathbb{N}}$, implies $N \leqslant N(\delta(\varepsilon))$ or $N > N(\frac{1}{2}\delta(\varepsilon))$. The fact that $\{\|G_\nu\|_2\}_{\nu\in\mathbb{N}}$ is a nonincreasing sequence discards the second possibility, hence $N \leqslant N(\delta(\varepsilon)) =: M_\varepsilon$.

From Lemma 2.23, the analog of Lemma 8.4 in [9] follows

Lemma 2.37. *For any $\varepsilon > 0$ there exist $\delta_\varepsilon > 0$ and $C_\varepsilon < \infty$ such that for every δ_ε-nearly extremal nonnegative function $f \in L^2(\Gamma^2)$, the functions f_ν, G_ν and the caps $\mathcal{C}_\nu$ associated to f via the decomposition algorithm satisfy $f_\nu \leqslant C_\varepsilon\|f\|_2|\mathcal{C}_\nu|^{-1/2}\chi_{\mathcal{C}_\nu}$ and $\|f_\nu\|_2 \geqslant \delta_\varepsilon\|f\|_2$ whenever $\|G_\nu\|_2 \geqslant \varepsilon\|f\|_2$.*

We now move to the analog of [9, Lemma 9.2]. The only difference in the proof compared to that in the Christ-Shao paper is that we need to replace the L^4 norm by the L^6 norm.

Lemma 2.38. *For any $\varepsilon > 0$ there exists $\delta > 0$ and $\lambda < \infty$ such that for any $0 \leqslant f \in L^2(\Gamma^2)$ which is δ-nearly extremal, the summands f_ν produced by the decomposition algorithm and the associated caps $\mathcal{C}_\nu$ satisfy*

$$\varrho(\mathcal{C}_j, \mathcal{C}_k) \leqslant \lambda \text{ whenever } \|f_j\|_2 \geqslant \varepsilon\|f\|_2 \text{ and } \|f_k\|_2 \geqslant \varepsilon\|f\|_2. \tag{2.55}$$

Proof. It suffices to prove this for all sufficiently small ε. Let f be a nonnegative L^2 function which satisfies $\|f\|_2 = 1$ and is δ-nearly extremal for a sufficiently small $\delta = \delta(\varepsilon)$, and let $\{G_\nu, f_\nu\}_{\nu\in\mathbb{N}}$ be associated to f via the decomposition algorithm. Set $F = \sum_{\nu=0}^{N} f_\nu$.

Suppose that $\|f_{j_0}\|_2 \geqslant \varepsilon$ and $\|f_{k_0}\|_2 \geqslant \varepsilon$. Let N be the smallest integer such that $\|G_{N+1}\|_2 < \varepsilon^3$. Since $\|G_\nu\|_2$ is a nonincreasing function of ν, and since $\|f_\nu\|_2 \leqslant \|G_\nu\|_2$, necessarily $j_0, k_0 \leqslant N$. Moreover, from the comment after Lemma 2.36, there exists $M_\varepsilon < \infty$ depending only on ε such that $N \leqslant M_\varepsilon$. By Lemma 2.37, if δ is chosen to be a sufficiently small function of ε then since $\|G_\nu\|_2 \geqslant \varepsilon^3$ for all $\nu \leqslant N$, $f_\nu \leqslant \theta(\varepsilon)|\mathcal{C}|^{-1/2}\chi_{\mathcal{C}}$ for all such ν, where θ is a continuous, strictly positive function on $(0,1]$.

Now let $\lambda < \infty$ be a large quantity to be specified. It suffices to show that if $\delta(\varepsilon)$ is sufficiently small, an assumption that $\varrho(\mathcal{C}_j, \mathcal{C}_k) > \lambda$ implies an upper bound, which depends only on ε, for λ.

There exists a decomposition $F = F_1 + F_2 = \sum_{\nu\in S_1} f_\nu + \sum_{\nu \in S_2} f_\nu$ where $[0,N] = S_1 \cup S_2$ is a partition of $[0,N]$, $j_0 \in S_1$, $k_0 \in S_2$, and $\varrho(\mathcal{C}_j, \mathcal{C}_k) \geqslant \lambda/2N \geqslant \lambda/2M_\varepsilon$ for all $j \in S_1$ and $k \in S_2$. Certainly $\|F_1\|_2 \geqslant \|f_{j_0}\|_2 \geqslant \varepsilon$ and similarly $\|F_2\| \geqslant \varepsilon$. The cross term satisfies

$$\|TF_1 \cdot TF_2\|_3 \leqslant \sum_{j\in S_1}\sum_{k\in S_2} \|Tf_j \cdot Tf_k\|_3 \leqslant M_\varepsilon^2 \gamma(\lambda/2M_\varepsilon)\theta(\varepsilon)^2,$$

where $\gamma(\lambda) \to 0$ as $\lambda \to \infty$ by Lemma 2.9. Expand

$$\begin{aligned}\|TF \cdot TF\|_3^3 \leqslant \|TF_1\|_6^6 + \|TF_2\|_6^6 + 15\|(TF_1)^2 \cdot TF_2\|_2^2 + 15\|TF_1 \cdot (TF_2)^2\|_2^2 \\ + 20\|TF_1 \cdot TF_2\|_3^3 + 6\|(TF_1)^5 \cdot TF_2\|_1 + 6\|TF_1 \cdot (TF_2)^5\|_1.\end{aligned}$$

By using Hölder's inequality

$$\begin{aligned}\|TF \cdot TF\|_3^3 \leqslant \|TF_1\|_6^6 + \|TF_2\|_6^6 + 20\|TF_1 \cdot TF_2\|_3^3 \\ + 6(\|TF_1\|_6^4 + \|TF_2\|_6^4)\|TF_1 \cdot TF_2\|_3 \\ + 15(\|TF_1\|_6^2 + \|TF_2\|_6^2)\|TF_1 \cdot TF_2\|_3^2,\end{aligned}$$

and using that T is continuous and denoting $\mathbf{C} = \|T\|$ we get

$$\begin{aligned}\|TF \cdot TF\|_3^3 \leqslant \mathbf{C}^6(\|F_1\|_2^6 + \|F_2\|_2^6) + 20\|TF_1 \cdot TF_2\|_3^3 \\ + 6\mathbf{C}^4(\|F_1\|_2^4 + \|F_2\|_2^4)\|TF_1 \cdot TF_2\|_3 \\ + 15\mathbf{C}^2(\|F_1\|_2^2 + \|F_2\|_2^2)\|TF_1 \cdot TF_2\|_3^2,\end{aligned}$$

Since F_1 and F_2 have disjoint supports, $\|F_1\|_2^2 + \|F_2\|_2^2 \leqslant \|f\|_2 = 1$ and consequently

$$\begin{aligned}\|F_1\|_2^4 + \|F_2\|_2^4 \leqslant \max(\|F_1\|_2^2, \|F_2\|_2^2) \cdot (\|F_1\|_2^2 + \|F_2\|_2^2) \leqslant (1-\varepsilon^2)\cdot 1 = 1-\varepsilon^2, \\ \|F_1\|_2^6 + \|F_2\|_2^6 \leqslant \max(\|F_1\|_2^2, \|F_2\|_2^2)^2 \cdot (\|F_1\|_2^2 + \|F_2\|_2^2) \leqslant (1-\varepsilon^2)^2\cdot 1 = (1-\varepsilon^2)^2.\end{aligned}$$

Thus

$$\begin{aligned}\|TF \cdot TF\|_3^3 \leqslant \mathbf{C}^6(1-\varepsilon^2)^2 + 20(M_\varepsilon^2\gamma(\lambda/2M_\varepsilon)\theta(\varepsilon)^2)^3 \\ + 6\mathbf{C}^4(1-\varepsilon^2)M_\varepsilon^2\gamma(\lambda/2M_\varepsilon)\theta(\varepsilon)^2 \\ + 15\mathbf{C}^2(M_\varepsilon^2\gamma(\lambda/2M_\varepsilon)\theta(\varepsilon)^2)^2.\end{aligned}$$

On the other hand repeating the previous calculations with $F_1 = F$ and $F_2 = f - F$ and using that $\|f\|_2 = 1$, $\|f - F\|_2 \leqslant \varepsilon^3 < 1$ we get

$$(1-\delta)^6\mathbf{C}^6 \leqslant \|Tf \cdot Tf\|_3^3 \leqslant \|TF \cdot TF\|_3^3 + C\|f\|_2\|f-F\|_2 \\ \leqslant \|TF \cdot TF\|_3^3 + C\varepsilon^3.$$

Hence

$$(1-\delta)^6\mathbf{C}^6 \leqslant C\varepsilon^3 + \mathbf{C}^6(1-\varepsilon^2)^2 + 20(M_\varepsilon^2\gamma(\lambda/2M_\varepsilon)\theta(\varepsilon)^2)^3 \\ + 6\mathbf{C}^4(1-\varepsilon^2)M_\varepsilon^2\gamma(\lambda/2M_\varepsilon)\theta(\varepsilon)^2 + 15\mathbf{C}^2(M_\varepsilon^2\gamma(\lambda/2M_\varepsilon)\theta(\varepsilon)^2)^2.$$

Since $\gamma(t) \to 0$ as $t \to \infty$, for all sufficiently small $\varepsilon > 0$ this implies an upper bound, which depends on ε, for λ, as was to be proved. □

Proposition 2.39. *There exists a function $\Theta : [1,\infty) \to (0,\infty)$ satisfying $\Theta(R) \to 0$ as $R \to \infty$ with the following property. For any $\varepsilon > 0$ there exists $\delta > 0$ such that any nonnegative function $f \in L^2(\Gamma^2)$ satisfying $\|f\|_2 = 1$ which is δ-nearly extremal may be decomposed as $f = F + G$ where F, G are nonnegative with disjoint supports, $\|G\|_2 < \varepsilon$, and there exists $k \in \mathbb{Z}$ such that*

$$\int_{|x|>2^kR} |F(x)|^2 d\sigma(x) \leqslant \Theta(R)\ \forall R \geqslant 1,$$
$$\int_{|x|<2^kR^{-1}} |F(x)|^2 d\sigma(x) \leqslant \Theta(R)\ \forall R \geqslant 1.$$

Remark 2.40. It will be clear from the proof of Lemma 2.41 that if there exists a cap $\mathcal{C} \subset [2^{k_0}, 2^{k_0+1}] \times [0, 2\pi]$ such that $g := f\chi_{\mathcal{C}}$ satisfies

$$|g(x)| \leqslant C\|f\|_2|\mathcal{C}|^{-1/2}\chi_{\mathcal{C}},\ \forall x \text{ and}$$
$$\|g\|_2 \geqslant c\|f\|_2$$

for C, c universal constants, then we can take $k = k_0$, and $F \geqslant g$.

Lemma 2.41. *There exists a function $\Theta : [1,\infty) \to (0,\infty)$ satisfying $\Theta(R) \to 0$ as $R \to \infty$ with the following property. For any $\varepsilon > 0$ and $\bar{R} \in [1,\infty)$ there exists $\delta > 0$ such that any nonnegative function $f \in L^2(\Gamma^2)$ satisfying $\|f\|_2 = 1$ which is δ-nearly extremal may be decomposed as $f = F + G$ where F, G are nonnegative with disjoint supports, $\|G\|_2 < \varepsilon$, and there exists $k \in \mathbb{Z}$ such that for any $R \in [1, \bar{R}]$*

$$\int_{|x|>2^kR} |F(x)|^2 d\sigma(x) \leqslant \Theta(R)\ , \text{ and} \tag{2.56}$$

$$\int_{|x|<2^kR^{-1}} |F(x)|^2 d\sigma(x) \leqslant \Theta(R). \tag{2.57}$$

Proof that Lemma 2.41 implies Proposition 2.39. Let Θ be the function promised by the lemma. Let ε, f be given, and assume without loss of generality that ε is small. Assuming as we may that Θ is a continuous, strictly decreasing function, define $\bar{R} = \bar{R}(\varepsilon)$ by the equation $\Theta(\bar{R}) = \varepsilon^2/2$. Let $k, \delta = \delta(\varepsilon, \bar{R}(\varepsilon))$ along with F, G satisfy the conclusions of the lemma relative to $\varepsilon, \bar{R}(\varepsilon)$. Define χ to be the characteristic function of the set of all $x \in \mathbb{R}^2$ which satisfy $|x| > 2^k\bar{R}$ or $|x| < 2^k\bar{R}^{-1}$. Redecompose $f = \tilde{F} + \tilde{G}$, where $\tilde{F} = (1-\chi)F$ and $\tilde{G} = G + \chi F$. Then $\|\tilde{G}\|_2 < 2\varepsilon$, while $\tilde{F}$ satisfies the required inequalities. For $R > \bar{R}$ we have $\tilde{F} = 0$, and if $R \leqslant \bar{R}$, then,

$$\int_{|x|>2^kR} |\tilde{F}(x)|^2 d\sigma(x) \leqslant \int_{|x|>2^kR} |F(x)|^2 d\sigma(x) \leqslant \Theta(R),$$

and similarly for the other integral. □

We now prove Lemma 2.41, the analog of Lemma 10.1 of [9]

Proof. Let $\eta : [1, \infty) \to (0, \infty)$ be a function to be chosen below, satisfying $\eta(t) \to 0$ as $t \to \infty$. This function will not depend on the quantity $\bar{R}$.

Let $\bar{R} \geqslant 1$, $R \in [1, \bar{R}]$, and $\varepsilon > 0$ be given. Let $\delta = \delta(\varepsilon, \bar{R}) > 0$ be a small quantity to be chosen below. Let $0 \leqslant f \in L^2(\Gamma^2)$ be a δ-nearly extremal, with $\|f\|_2 = 1$.

Let $\{f_\nu\}_{\nu\in\mathbb{N}}$ be the sequence of functions obtained by applying the decomposition algorithm to f. Choose $\delta = \delta(\varepsilon) > 0$ sufficiently small and $M = M(\varepsilon)$ sufficiently large to guarantee $\|G_{M+1}\|_2 < \varepsilon/2$ and that f_ν, G_ν satisfy the conclusions of the analog of Lemma 8.4 and Lemma 8.3 in [9] for $\nu \leqslant M$. Set $F = \sum_{\nu=0}^{M} f_\nu$. Then $\|f - F\|_2 = \|G_{M+1}\|_2 < \varepsilon/2$.

Let $N \in \{0, 1, 2, \dots\}$ be the minimum of M, and the smallest number such that $\|f_{N+1}\|_2 < \eta$. N is bounded above by a quantity which depends only on η. Set $\mathcal{F} = \mathcal{F}_N = \sum_{k=0}^{N} f_\nu$. It follows from Lemma 8.4 in [9], that

$$\|F - \mathcal{F}\|_2 < \gamma(N) \text{ where } \gamma(\eta) \to 0 \text{ as } \eta \to 0. \tag{2.58}$$

This function γ is independent of $\varepsilon, \bar{R}$.

To prove the lemma, we must produce an integer k and must establish the existence of Θ. To do the former is simple: To f_0 is associated a cap $\mathcal{C}_0 \subset [2^{k_0-1}, 2^{k_0}] \times [0, 2\pi]$ such that $f_0 \leqslant C|\mathcal{C}_0|^{-1/2}\chi_{\mathcal{C}_0}$, for some universal constant C. $k = k_0$ is the required integer. Note that by Lemma 2.23, $\|f_0\|_2 \geqslant c$ for some positive universal constant c. This implies, by Lemma 2.24, that $\|f_0\|_1 \geqslant c'|\mathcal{C}_0|^{1/2}$, for some universal constant c'. This last remark will be of use after rescaling.

Suppose that functions $R \mapsto \eta(R)$ and $R \mapsto \Theta(R)$ are chosen so that

$$\eta(R) \to 0 \text{ as } R \to \infty,$$
$$\gamma(\eta(R)) \leqslant \Theta(R) \text{ for all } R.$$

Then by (2.58), $F - \mathcal{F}$ already satisfies the desired inequalities in $L^2(\Gamma^2)$, so it suffices to show that $\mathcal{F}(x) \equiv 0$ whenever $|x| \geqslant 2^{k_0}R$ or $|x| \leqslant 2^{k_0}R^{-1}$.

Each summand satisfies $f_k \leqslant C(\eta)|\mathcal{C}_k|^{-1/2}\chi_{\mathcal{C}_k}$, where $C(\eta)<\infty$ depends only on η, and in particular, f_k is supported in $\mathcal{C}_k$. $\|f_k\|_2 \geqslant \eta$ for all $k \leqslant N$, by definition of N. Therefore by Lemma 2.38, there exists a function $\eta \mapsto \lambda(\eta)<\infty$ such that if δ is sufficiently small as a function of η then $\varrho(\mathcal{C}_k,\mathcal{C}_0) \leqslant \lambda(\eta)$ for all $k \leqslant N$. This is needed for $\eta=\eta(R)$ for all R in the compact set $[1,\bar{R}]$, so such a δ may be chosen as a function of $\bar{R}$ alone; conditions already imposed on δ above make it a function of both $\varepsilon,\bar{R}$. The independence from ε is needed since for $\delta<\delta'$, if f is a δ'-nearly extremal then is is also a δ-nearly extremal.

Let $\tau_k \in \mathbb{Z}$ be such that $\mathcal{C}_k \subset [2^{\tau_k}, 2^{\tau_k+1}]\times[0,2\pi]$. Then $|\tau_k-k_0| \leqslant \lambda(\eta)$, so $2^{\tau_k} \leqslant 2^{k_0}2^{\lambda(\eta)}$ and $2^{\tau_k} \geqslant 2^{k_0}2^{-\lambda(\eta)}$. Choosing $R \mapsto \eta(R)$ so that $2^{\lambda(\eta(R))} \leqslant R$ gives $\mathcal{F}(x)\equiv 0$ when $|x| \geqslant 2^{k_0}R$ and when $|x| \leqslant 2^{k_0}R^{-1}$. □

We are now ready to give a proof of Theorem 2.29 based in the Christ-Shao concentration compactness argument.

Alternative proof of Theorem 2.29. Let $\{f_n\}_{n\in\mathbb{N}}$ be an extremizing sequence. We start as in the proof of Theorem 2.29 by using Lemma 2.23 with $\delta=1/2$ to decompose $f_n=g_n+h_n$ and to obtain a cap $\mathcal{C}_n$ satisfying the conclusions of Lemma 2.23. We then find $\{\lambda_n\}_{n\in\mathbb{N}}$, $\{t_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ such that the support of $L^*_{t_n}R^*_{\theta_n}D^*_{\lambda_n}g_n$ is contained in a bounded region independent of n and has measure comparable to 1.

Define $\tilde{f}_n=L^*_{t_n}R^*_{\theta_n}D^*_{\lambda_n}f_n$, $\tilde{g}_n=L^*_{t_n}R^*_{\theta_n}D^*_{\lambda_n}g_n$, $\tilde{h}_n=L^*_{t_n}R^*_{\theta_n}D^*_{\lambda_n}h_n$ and $\tilde{\mathcal{C}}_n=L^{-1}_{t_n}R^{-1}_{\theta_n}D^{-1}_{\lambda_n}\mathcal{C}_n$. Then $\tilde{g}_n$ and $\tilde{h}_n$ have disjoint supports, $\tilde{g}_n$ is supported on $\tilde{\mathcal{C}}_n \subset [1/4,1]\times[0,2\pi]$, $\sigma(\tilde{\mathcal{C}}_n)\geqslant \frac{1}{2}$ and there exist $0<c,C<\infty$ independent of n such that

$$|\tilde{g}_n(x)| \leqslant C\|\tilde{f}_n\|_2|\tilde{\mathcal{C}}_n|^{-1/2}\chi_{\tilde{\mathcal{C}}_n}(x), \quad \text{and} \quad \|\tilde{g}_n\|_2 \geqslant c\|\tilde{f}_n\|_2. \tag{2.59}$$

We now apply Proposition 2.39 to $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ with $\varepsilon_n=1/n$, $n\geqslant 1$, to obtain a subsequence of $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ (that we call the same), that satisfies the following. Each $\tilde{f}_n$ can be decomposed as $\tilde{f}_n=F_n+G_n$, with F_n,G_n nonnegative with disjoint supports, $\|G_n\|_2<\frac{1}{n}$ and F_n satisfies both (2.56) and (2.57) for certain $k=k_n\in\mathbb{Z}$.

Denote by $\Omega_n := \operatorname{supp} F_n$ the support of F_n. For all n large enough so that $1/n<c/8$ and $\|\tilde{f}_n\|_2>1/2$ we have $\Omega_n\cap\tilde{\mathcal{C}}_n\neq\emptyset$ and $\|\tilde{g}_n\chi_{\Omega_n}\|_2>\frac{c}{4}$. Then $\|F_n\chi_{\tilde{\mathcal{C}}_n}\|_2=\|\tilde{g}_n\chi_{\Omega_n}\|_2>\frac{c}{4}$ and $|F_n\chi_{\tilde{\mathcal{C}}_n}(x)|=|\tilde{g}_n\chi_{\Omega_n}(x)|\leqslant C|\tilde{\mathcal{C}}_n|^{-1/2}\chi_{\tilde{\mathcal{C}}_n\cap\Omega_n}(x)\leqslant 2C\chi_{\tilde{\mathcal{C}}_n\cap\Omega_n}(x)$. The lower bound in the L^2 norm of $F_n\chi_{\tilde{\mathcal{C}}_n}$ imply $|\tilde{\mathcal{C}}_n\cap\Omega_n|>c'>0$ for c' independent of n. As in Lemma 2.24 this implies

$$\int_{\tilde{\mathcal{C}}_n}|F_n|d\sigma>c''>0, \tag{2.60}$$

with c'' independent of n.

The conditions $\|F_n\chi_{\tilde{\mathcal{C}}_n}\|_2>\frac{c}{4}$, $\tilde{\mathcal{C}}_n\subset[1/4,1]\times[0,2\pi]$ together with the L^2-decay estimates (2.56) and (2.57) imply that $|k_n|\leqslant C'<\infty$ for a constant C' independent of n.

As $\|G_n\|_2\to 0$ as $n\to\infty$, $\{F_n\}_{n\in\mathbb{N}}$ is an extremizing sequence of nonnegative functions. After passing to a subsequence $F_n \rightharpoonup F$ for some $F\in L^2(\Gamma^2)$ and $F\neq 0$ since the F_n's satisfy (2.60). Therefore $\{F_n\}_{n\in\mathbb{N}}$ satisfies all the conditions in Proposition 2.31 and thus $F_n \to F$ in $L^2(\Gamma^2)$ and also $\tilde{f}_n \to F$ in $L^2(\Gamma^2)$.

This shows that $\{f_n\}_{n\in\mathbb{N}}$ is precompact up to symmetries of the cone as needed. □

2.10 On convergence of extremizing sequences

In this section we prove Theorem 2.4. We start with a general discussion.

Let $(X, \mathcal{B}, \mu)$ be a measure space and let G be a group acting on $L^2(X)$, with an action that preserves the L^2 norm, that is $\|g^* f\|_{L^2(X)} = \|f\|_{L^2(X)}$ for all $g \in G$ and $f \in L^2(X)$. For an element $f \in L^2(X)$ we consider its orbit under G, $G(f) := \{g^* f : g \in G\}$.

Proposition 2.42. *Let $f \in L^2(X)$ and $\{f_n\}_{n\in\mathbb{N}}$ a sequence in $L^2(X)$ with the property that every subsequence has an L^2-convergent subsequence whose limit lies on $G(f)$. Then there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset G$ such that $g_n^* f_n \to f$ in $L^2(X)$, as $n \to \infty$.*

Proof. For each n let $g_n \in G$ be such that

$$\|g_n^* f_n - f\|_{L^2(X)} \leqslant \inf_{g\in G} \|g^* f_n - f\|_{L^2(X)} + \frac{1}{n}.$$

We show that $\{g_n^* f_n\}_{n\in\mathbb{N}}$ converges to f by showing that every subsequence has a further subsequence that converges to f. Take a subsequence (that we call the same), $\{g_n^* f_n\}_{n\in\mathbb{N}}$. By hypothesis, f_n has a convergent subsequence (that we call the same) to an element in $G(f)$. That is $f_n \to g^* f$, as $n \to \infty$, for some $g \in G$. By the definition of g_n and the invariance of the norm under the action of G we get

$$\|g_n^* f_n - f\|_{L^2(X)} \leqslant \|(g^{-1})^* f_n - f\|_{L^2(X)} + \frac{1}{n} = \|f_n - g^* f\|_{L^2(X)} + \frac{1}{n} \to 0$$

as $n \to \infty$. □

From Theorem 2.5 the extremizers for (2.2) are all of the form

$$g(x_1, x_2, x_3) = e^{-ax_3 - bx_2 - cx_1 + d}, \tag{2.61}$$

where $a, b, c, d \in \mathbb{C}$ and $|(\mathfrak{Re}\, b, \mathfrak{Re}\, c)| < \mathfrak{Re}\, a$, and here $x_3 = \sqrt{x_1^2 + x_2^2}$. As indicated in [20], any extremizer can be obtained from $g_0(x_1, x_2, x_3) = e^{-x_3}$ by applying Lorentz transformations and dilations.

We define G as the group generated by L^t, M^s and D_r, $s, t \in (-1, 1)$, $r > 0$ under composition. The action of G is given by the action of the generators as in (2.34) : $L^{t*} f = f \circ L^t$, $M^{s*} f = f \circ M^s$ and $D_r^* f = r^{1/2} f \circ D^r$. That G preserves the $L^2(\Gamma^2)$ norm follows from the Lorentz invariance of σ.

Lemma 2.43. *The set of real, L^2-normalized extremizers for inequality (2.2) equals the orbit of $g_0(y) = \pi^{-1/2} e^{-|y|}$, $y \in \mathbb{R}^2$, under the group G.*

Proof. A computation shows

$$L^t \circ M^s(x_1,x_2,x_3) = \Big(\frac{x_1 + t\frac{x_3+sx_2}{(1-s^2)^{1/2}}}{(1-t^2)^{1/2}}, \frac{x_2+sx_3}{(1-s^2)^{1/2}}, \frac{\frac{x_3+sx_2}{(1-s^2)^{1/2}} + tx_1}{(1-t^2)^{1/2}}\Big).$$

Then

$$g_0 \circ L^t \circ M^s \circ D_r = r^{\frac{1}{2}}\pi^{-\frac{1}{2}} \exp\Big(-\frac{rx_3}{(1-s^2)^{1/2}(1-t^2)^{1/2}} - \frac{srx_2}{(1-s^2)^{1/2}(1-t^2)^{1/2}} - \frac{trx_1}{(1-t^2)^{1/2}}\Big).$$

For given $a>0$ and $b,c\in\mathbb{R}$ with $|(b,c)|<a$ we want to solve the equations

$$\begin{aligned}\frac{r}{(1-s^2)^{1/2}(1-t^2)^{1/2}} &= a,\\ \frac{sr}{(1-s^2)^{1/2}(1-t^2)^{1/2}} &= b,\\ \frac{tr}{(1-t^2)^{1/2}} &= c.\end{aligned}$$

Since $a\neq 0$ and $|b|<a$ we have $b/a = s\in(-1,1)$. Also $c/a = t(1-s^2)^{1/2}$, so $t = \frac{c}{a(1-s^2)^{1/2}} = \frac{c}{(a^2-b^2)^{1/2}}$ and we see that $|t|<1$. Finally $r = a(1-s^2)^{1/2}(1-t^2)^{1/2} = (a^2-b^2-c^2)^{1/2}$. The L^2-norm is preserved by the action of G thus a normalized, real extremizer $g(y) = e^{-a|y|-by_2-cy_1+d}$ can be obtained from g_0 by composing with $L^t\circ M^s\circ D_r$ for the computed values of t, s and r. □

Proof of Theorem 2.4. From the previous discussion we have that the group G gives all real extremizers as the orbit of g_0. Proposition 2.42, Theorem 2.3 and Theorem 2.5 give a proof of Theorem 2.4. □

Chapter 3

Nonexistence of extremals for the adjoint restriction inequality on the hyperboloid

We study the problem of existence of extremizers for the L^2 to L^p adjoint Fourier restriction inequalities on the hyperboloid in dimensions 3 and 4, in which cases p is an even integer. We will use the method developed by Foschi in [20] to show that extremizers do not exist.

3.1 Introduction

For $d \geqslant 1$ let $\mathbb{H}^d$ denote the hyperboloid in $\mathbb{R}^{d+1}$, $\mathbb{H}^d = \{(y, \sqrt{1+|y|^2}) : y \in \mathbb{R}^d\}$, equipped with the measure $\sigma(y, y') = \delta(y' - \sqrt{1+|y|^2})\frac{dydy'}{\sqrt{1+|y|^2}}$ defined by duality as

$$\int_{\mathbb{H}^d} g(y, y')d\sigma(y, y') = \int_{\mathbb{R}^d} g(y, \sqrt{1+|y|^2})\frac{dy}{\sqrt{1+|y|^2}}.$$

for all $g(y, y') \in C_0(\mathbb{H}^d)$.

A function $f : \mathbb{H}^d \to \mathbb{R}$ can be identified with a function from $\mathbb{R}^d$ to $\mathbb{R}$ and in what follows we will do so. We will denote the $L^p(\mathbb{H}^d, \sigma)$ norm of a function f as $\|f\|_{L^p(\mathbb{H}^d)}$, $\|f\|_{L^p(\sigma)}$ or $\|f\|_p$.

The extension or adjoint Fourier restriction operator for $\mathbb{H}^d$ is given by

$$Tf(x, t) = \int_{\mathbb{R}^d} e^{ix\cdot y}e^{it\sqrt{1+|y|^2}} f(y)(1+|y|^2)^{-\frac{1}{2}}dy \tag{3.1}$$

where $(x, t) \in \mathbb{R}^d \times \mathbb{R}$ and $f \in \mathcal{S}(\mathbb{R}^d)$. With the Fourier transform in $\mathbb{R}^{d+1}$ defined to be $\hat{g}(\xi) = \int_{\mathbb{R}^{d+1}} e^{-ix\cdot\xi}g(x)dx$, we see that $Tf(x, t) = \widehat{f\sigma}(-x, -t)$.

It is known, [47], that there exists $C_{d,p} < \infty$ such that for all $f \in L^2(\mathbb{H}^d)$ the following estimate for Tf holds

$$\|Tf\|_{L^p(\mathbb{R}^{d+1})} \leqslant C_{d,p}\|f\|_{L^2(\mathbb{H}^d)} \tag{3.2}$$

provided that

$$\begin{gathered} 2(d+2)/d \leqslant p \leqslant 2(d+1)/(d-1), \text{ if } d>1 \\ 6 \leqslant p < \infty, \text{ if } d=1. \end{gathered} \tag{3.3}$$

For p satisfying (3.3) we will denote by $\mathbf{H}_{d,p}$ the best constant in (3.2),

$$\mathbf{H}_{d,p} = \sup_{0\neq f\in L^2(\mathbb{H}^d)} \frac{\|Tf\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{H}^d)}}.$$

We will also look at the two sheeted hyperboloid $\bar{\mathbb{H}}^d = \{(y,y') \in \mathbb{R}^d \times \mathbb{R} : y'^2 = 1 + |y|^2\}$. We endow it with the measure $\bar{\sigma} = \sigma^+ + \sigma^-$ where

$$\begin{aligned} \sigma^+(y,y') = \sigma(y,y') &= \delta(y' - \sqrt{1+|y|^2})\frac{dydy'}{\sqrt{1+|y|^2}}, \\ \sigma^-(y,y') &= \delta(y' + \sqrt{1+|y|^2})\frac{dydy'}{\sqrt{1+|y|^2}}. \end{aligned}$$

We denote by $\bar{T}$ the corresponding adjoint Fourier restriction operator, $\bar{T}f = \widehat{f\sigma^+} + \widehat{f\sigma^-}$. If (d,p) satisfies (3.3), then the following constant is finite,

$$\bar{\mathbf{H}}_{d,p} = \sup_{f\in L^2(\bar{\mathbb{H}}^d)} \frac{\|\bar{T}f\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\bar{\mathbb{H}}^d)}}. \tag{3.4}$$

Definition 3.1. An extremizing sequence for the inequality (3.2) is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $L^2(\mathbb{H}^d)$ satisfying $\|f_n\|_{L^2(\sigma)} \leqslant 1$, such that $\|Tf_n\|_{L^p(\mathbb{R}^{d+1})} \to \mathbf{H}_{d,p}$ as $n \to \infty$.

An extremizer for (3.2) is a function $f \neq 0$ which satisfies $\|Tf\|_{L^p(\mathbb{R}^{d+1})} = \mathbf{H}_{d,p}\|f\|_{L^2(\sigma)}$.

An analogous definition of extremizing sequence and extremizer will be used for (3.4).

We will be interested in the following pairs of (d,p): $(2,4)$, $(2,6)$ and $(3,4)$, which are the only cases for $d>1$ where p is an even integer. The main result of this chapter is:

Theorem 3.2. *The values of the best constants are,* $\mathbf{H}_{2,4} = 2^{3/4}\pi$, $\mathbf{H}_{2,6} = (2\pi)^{5/6}$ *and* $\mathbf{H}_{3,4} = (2\pi)^{5/4}$. *In each of the three cases of pairs* (d,p) *extremizers do not exist.*

For the two sheeted hyperboloid the best constants are, $\bar{\mathbf{H}}_{2,4} = (3/2)^{1/4}\mathbf{H}_{2,4}$, $\bar{\mathbf{H}}_{2,6} = (5/2)^{1/3}\mathbf{H}_{2,6}$ *and* $\bar{\mathbf{H}}_{3,4} = (3/2)^{1/4}\mathbf{H}_{3,4}$. *Here extremizers do not exist either.*

We normalize the Fourier transform in $\mathbb{R}^d$ as

$$\hat{g}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} g(x)dx.$$

With this, the convolution and $L^2(\mathbb{R}^d)$ norm satisfy

$$\widehat{f*g} = \hat{f}\,\hat{g}, \text{ and } \|\hat{f}\|_{L^2(\mathbb{R}^d)} = (2\pi)^{d/2}\|f\|_{L^2(\mathbb{R}^d)}.$$

When p is an even integer we can write (3.2) in "convolution form". If $p = 2k$ then

$$\begin{aligned}\|Tf\|^k_{L^{2k}(\mathbb{R}^{d+1})} &= \|(Tf)^k\|_{L^2(\mathbb{R}^{d+1})} = \|(\widehat{f\sigma})^k\|_{L^2(\mathbb{R}^{d+1})} = \|(f\sigma * \cdots * f\sigma)\hat{}\,\|_{L^2(\mathbb{R}^{d+1})}\\ &= (2\pi)^{(d+1)/2}\|f\sigma * \cdots * f\sigma\|_{L^2(\mathbb{R}^{d+1})},\end{aligned} \tag{3.5}$$

where $f\sigma * \cdots * f\sigma$ is the k^{th}-fold convolution of $f\sigma$ with itself. Therefore, for p even integer, (3.2) is equivalent to

$$\|f\sigma * \cdots * f\sigma\|^{1/k}_{L^2(\mathbb{R}^{d+1})} \leqslant (2\pi)^{-(d+1)/(2k)} C_{d,2k}\|f\|_{L^2(\mathbb{H}^d)}, \text{ for all } f \in \mathcal{S}(\mathbb{R}^d).$$

For reference, we write here the best constants in convolution form,

$$\begin{aligned}\sup_{f\in L^2(\mathbb{H}^2)} \|f\sigma * f\sigma\|^{1/2}_{L^2(\mathbb{R}^3)}\|f\|^{-1}_{L^2(\mathbb{H}^2)} &= \pi^{1/4},\\ \sup_{f\in L^2(\mathbb{H}^2)} \|f\sigma * f\sigma * f\sigma\|^{1/3}_{L^2(\mathbb{R}^3)}\|f\|^{-1}_{L^2(\mathbb{H}^2)} &= (2\pi)^{1/3},\\ \sup_{f\in L^2(\mathbb{H}^3)} \|f\sigma * f\sigma\|^{1/2}_{L^2(\mathbb{R}^4)}\|f\|^{-1}_{L^2(\mathbb{H}^3)} &= (2\pi)^{1/4}.\end{aligned}$$

It would be interesting to analyze the case $d = 1$ for even integers greater or equal to 6. Our argument relies on the explicit computation of the n^{th}-fold convolution of the measure σ with itself and this seems to be computationally involving if $n \geqslant 3$.

Interpolation shows that for $d = 2$ and $p \in [4, 6]$ we have $\mathbf{H}_{2,p} \leqslant \mathbf{H}^{\theta}_{2,4}\mathbf{H}^{1-\theta}_{2,6}$, where $\frac{1}{p} = \frac{\theta}{4} + \frac{1-\theta}{6}$. We do not know whether extremizers exist for $p \in (4, 6)$ as our method needs p to be an even integer.

One could consider, for $s > 0$, the hyperboloid $\mathbb{H}^d_s = \{(y, \sqrt{s^2 + |y|^2}) : y \in \mathbb{R}^d\}$ equipped with the measure

$$\sigma_s(y, y') = \delta(y' - \sqrt{s^2 + |y|^2})\frac{dydy'}{\sqrt{s^2 + |y|^2}}. \tag{3.6}$$

As we mention in Section 3.2 this measure is natural since it is the only Lorentz invariant measure on $\mathbb{H}^d_s$. Let $T_s f(x, t) = \widehat{f\sigma_s}(x, t)$. For (d, p) satisfying (3.3) the estimate

$$\|T_s f\|_{L^p(\mathbb{R}^{d+1})} \leqslant \mathbf{H}_{d,p,s}\|f\|_{L^2(\mathbb{H}^d_s)} \tag{3.7}$$

holds, where

$$\mathbf{H}_{d,p,s} := \sup_{f\in L^2(\mathbb{H}^d_s)} \frac{\|T_s f\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mathbb{H}^d_s)}}. \tag{3.8}$$

is a finite constant.

Simple scaling, as shown in Appendix 1, relates $\mathbf{H}_{d,p,s}$ and $\mathbf{H}_{d,p}$:

$$\mathbf{H}_{d,p,s} = s^{(d-1)/2-(d+1)/p}\mathbf{H}_{d,p}. \tag{3.9}$$

Moreover $\{f_n\}_{n\in\mathbb{N}}$ is a extremizing sequence for (3.2) if and only if $\{s^{-(d-1)/2}f_n(s^{-1}\cdot)\}_{n\in\mathbb{N}}$ is an extremizing sequence for the inequality for T_s, $s>0$. Thus for the problem of extremizers and properties of extremizing sequences it is enough to study $s=1$.

For any $\rho\in(0,\infty)$ we can consider the truncated hyperboloid $\mathbb{H}^d_{s,\rho} = \{(y,\sqrt{s^2+|y|^2}) : y\in\mathbb{R}^d,\ |y|\leqslant\rho\}$ endowed with the measure which is the restriction of σ_s to $\mathbb{H}^d_{s,\rho}$. We denote by T_ρ the corresponding adjoint Fourier restriction operator, $T_\rho f = Tf$ for $f\in L^2(\mathbb{H}^d_{s,\rho})$. Since one has the estimate $\|T_\rho f\|_{L^\infty(\mathbb{R}^d)} \leqslant C\|f\|_{L^2(\mathbb{H}^d_{s,\rho})}$, it follows that for $d\geqslant 1$ and $p\geqslant 2(d+2)/d$ the estimate

$$\|T_\rho f\|_{L^p(\mathbb{R}^{d+1})} \leqslant C\|f\|_{L^2(\mathbb{H}^d_{s,\rho})} \tag{3.10}$$

holds for some constant $C=C(d,p,s,\rho)<\infty$.

A theorem of Fanelli, Vega and Visciglia in [18] implies that if $d\geqslant 1$ and $p>2(d+2)/d$, then complex valued extremizers for (3.10) exist. There are nonnegative extremizers if p is an even integer as can be seen from the equivalent "convolution form" of (3.10). This shows that for $(d,p)=(2,6)$ and $(d,p)=(3,4)$ there are extremizers for (3.10). The case $(d,p)=(2,4)$ does not follow from the result in [18] since it is the endpoint. Our argument here shows that in this case extremizers do not exist.

3.2 The Lorentz invariance

The Lorentz group is defined as the group of invertible linear transformations in $\mathbb{R}^{d+1}$ preserving the bilinear form $(x,y)\in\mathbb{R}^{d+1}\times\mathbb{R}^{d+1}\mapsto x\cdot Jy$, where J is the $(d+1)\times(d+1)$ matrix $J_{i,j}=0$ if $i\neq j$, $J_{i,i}=-1$ if $1\leqslant i\leqslant d$ and $J_{d+1,d+1}=1$.

Let us denote by $\mathcal{L}^+$ the subgroup of Lorentz transformations in $\mathbb{R}^{d+1}$ that preserve $\mathbb{H}^d_s$. It is known that σ_s is invariant under the action of $\mathcal{L}^+$ and moreover is the unique measure on $\mathbb{H}^d_s$ invariant under such Lorentz transformations, up to multiplication by scalar; for this we refer to [39] where the case $d=3$ is considered, but the same argument can be adapted to $d\geqslant 2$.

For $t\in(-1,1)$ we define the linear map $L^t:\mathbb{R}^{d+1}\to\mathbb{R}^{d+1}$ by

$$L^t(\xi_1,\dots,\xi_d,\tau) = \Big(\frac{\xi_1+t\tau}{\sqrt{1-t^2}},\xi_2\dots,\xi_d,\frac{\tau+t\xi_1}{\sqrt{1-t^2}}\Big).$$

$\{L^t\}_{t\in(-1,1)}$ is a one parameter subgroup of Lorentz transformations, contained in $\mathcal{L}^+$.

For $i,j \in \{1,\dots,d\}$ we let $P_{i,j}$ be the linear transformation that swaps the i^{th} and j^{th} components of a vector in $\mathbb{R}^{d+1}$.

For any orthogonal matrix $A \in O(d,\mathbb{R})$ the transformation $(\xi,\tau) \mapsto R_A(\xi,\tau) = (A\xi,\tau)$ belongs to $\mathcal{L}^+$.

By composing the transformations $P_{i,j}$ and L^t for suitable i,j's and t's it is not hard to see that if $(\xi,\tau) \in \mathbb{R}^{d+1}$ satisfies $\tau > |\xi|$, then there exists $L \in \mathcal{L}^+$ such that $L(\xi,\tau) = (0,\sqrt{\tau^2-\xi^2})$. Alternatively, this can be achieved by using the transformations R_A and L^t: we first find $A \in O(d,\mathbb{R})$ such that $A\xi = (|\xi|,0,\dots,0)$. We take $t = -|\xi|\tau^{-1}$ and note that $L^t(R_A(\xi,\tau)) = L^t(|\xi|,0,\dots,0,\tau) = (0,\sqrt{\tau^2-|\xi|^2})$.

For $p \in [1,\infty]$, $L \in \mathcal{L}^+$ and $f \in L^p(\mathbb{H}^d_s)$ we define

$$L^* f = f \circ L,$$

where "$\circ$" denotes composition. The invariance of the measure σ_s under the action of $\mathcal{L}^+$ implies that for all $p \in [1,\infty)$ we have $\|f\|_{L^p(\mathbb{H}^d_s)} = \|L^* f\|_{L^p(\mathbb{H}^d_s)}$, and the equality for $p=\infty$ holds since Lorentz transformations are invertible. It is also direct to check that for $p \in [1,\infty]$ we have $\|T_s(L^* f)\|_{L^p(\mathbb{R}^{d+1})} = \|T_s f\|_{L^p(\mathbb{R}^{d+1})}$. Therefore, if $\{f_n\}_{n\in\mathbb{N}}$ is an extremizing sequence for (3.7) and $\{L_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^+$, then $\{L_n^* f_n\}_{n\in\mathbb{N}}$ is also an extremizing sequence for (3.7).

The Lorentz transformations we will use in this paper are the $P_{i,j}$, R_A and L^t. The invariance of σ_s with respect to these transformations can be seen directly by using the change of variables formula and seeing that the Jacobians work out.

3.3 On Foschi's argument

For ease of writing we will define $\psi_s : \mathbb{R} \to \mathbb{R}$ by

$$\psi_s(x) = \sqrt{s^2 + x^2}.$$

We let $\psi := \psi_1$. We will abuse notation and write $\psi_s(y)$ to mean $\psi_s(|y|)$, for $y \in \mathbb{R}^d$.

For measures μ,ν in $\mathbb{R}^d$, their convolution is defined by duality as

$$\int g\, d(\mu * \nu) = \int g(x+y)\, d\mu(x)\, d\nu(y),$$

for all $g \in C_0(\mathbb{R}^d)$.

For a measure μ in $\mathbb{R}^d$ and $n \geqslant 1$ we will denote $\mu^{(*n)} = \mu * \cdots * \mu$, the n^{th}-fold convolution of μ with itself.

The measure σ_s on $\mathbb{H}^d_s$ satisfies that the n^{th}-fold convolution $\sigma_s^{(*n)}$ is supported in the closure of the region $\mathcal{P}_{d,n} = \{(\xi,\tau) : \tau > \sqrt{(ns)^2 + |\xi|^2}\}$. For any fixed $(\xi,\tau) \in \mathcal{P}_{d,n}$ we define the measure on $(\mathbb{R}^d)^n$ by

$$\mu_{(\xi,\tau)} = \boldsymbol{\delta}\begin{pmatrix} \tau - \psi_s(x_1) - \cdots - \psi_s(x_n) \\ \xi - x_1 - \cdots - x_n \end{pmatrix} dx_1 \dots dx_n.$$

With the Dirac delta, δ_0, on $\mathbb{R}^d \times \mathbb{R}$ defined as

$$\langle \delta_0, f\rangle = f(0), \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}),$$

$\mu_{(\xi,\tau)}$ is the pullback of δ_0 on $\mathbb{R}^d \times \mathbb{R}$ by the function $\Phi_{(\xi,\tau)} : (\mathbb{R}^d)^n \to \mathbb{R}^d \times \mathbb{R}$ given by

$$\Phi_{(\xi,\tau)}(x_1, \dots, x_n) = (\xi - x_1 - \dots - x_n, \tau - \psi_s(x_1) - \dots - \psi_s(x_n)).$$

As discussed in [20], the pullback is well defined as long as the differential of $\Phi_{(\xi,\tau)}$ is surjective at the points where $\Phi_{(\xi,\tau)}$ vanishes. The differential of $\Phi_{(\xi,\tau)}$ is surjective at a point $(x_1, \dots, x_n)$ if and only if $x_1, \dots, x_n$ are not all equal. Now $\Phi_{(\xi,\tau)}(x, \dots, x) = 0$ if and only if $\tau^2 = (ns)^2 + |\xi|^2$, that is, at the boundary of $\mathcal{P}_{d,n}$. Thus, the pullback is well defined on $\mathcal{P}_{d,n}$.

For each $(\xi, \tau) \in \mathcal{P}_{d,n}$ we define the inner product $\langle \cdot, \cdot\rangle_{(\xi,\tau)}$ and norm $\| \cdot \|_{(\xi,\tau)}$ associated to $\mu_{(\xi,\tau)}$ as

$$\begin{aligned}\langle F, G\rangle_{(\xi,\tau)} &= \int_{(\mathbb{R}^d)^n} F(x_1, \dots, x_n)\overline{G(x_1, \dots, x_n)}\, d\mu_{(\xi,\tau)}(x_1, \dots, x_n),\\ \|F\|^2_{(\xi,\tau)} &= \int_{(\mathbb{R}^d)^n} |F(x_1, \dots, x_n)|^2\, d\mu_{(\xi,\tau)}(x_1, \dots, x_n),\end{aligned}$$

What connects this inner product with inequality (3.2) is the following identity. For $f_1, \dots, f_n \in L^2(\mathbb{H}^d_s)$

$$\begin{aligned}f_1\sigma_s * \dots * f_n\sigma_s &= \int_{(\mathbb{R}^d)^n} \frac{f_1(x_1)\dots f_n(x_n)}{\psi_s(x_1)\dots\psi_s(x_n)}\delta(\xi - x_1 - \dots - x_n)\cdot \\ &\qquad\qquad \delta(\tau - \psi_s(x_1) - \dots - \psi_s(x_n))\, dx_1 \dots dx_n\\ &= \int_{(\mathbb{R}^d)^n} \frac{f_1(x_1)\dots f_n(x_n)}{\psi_s(x_1)\dots\psi_s(x_n)} d\mu_{(\xi,\tau)}(x_1 \dots x_n)\\ &= \langle F, G\rangle_{(\xi,\tau)},\end{aligned}$$

where $F(x_1, \dots, x_n) = \frac{f_1(x_1)\dots f_n(x_n)}{\psi_s(x_1)^{1/2}\dots\psi_s(x_n)^{1/2}}$ and $G(x_1, \dots, x_n) = \frac{1}{\psi_s(x_1)^{1/2}\dots\psi_s(x_n)^{1/2}}$.

Lemma 3.3. *Let $f \in \mathcal{S}(\mathbb{R}^d)$, then the n^{th}-fold convolution of $f\sigma_s$ with itself satisfies*

$$\|(f\sigma_s)^{(*n)}\|_{L^2(\mathbb{R}^d)} \leqslant \|\sigma_s^{(*n)}\|^{1/2}_{L^\infty(\mathbb{R}^d)}\|f\|^n_{L^2(\mathbb{H}^d_s)}. \tag{3.11}$$

Moreover, for $f \neq 0$, for equality to hold in (3.11) it is necessary that $\sigma_s^{(*n)}(\xi, \tau) = \|\sigma_s^{(*n)}\|_{L^\infty(\mathbb{R}^d)}$ for a.e. (ξ, τ) in the support of $(f\sigma_s)^{(*n)}$.

Proof. Let $g \in \mathcal{S}(\mathbb{R}^{d+1})$, then by definition of the convolution we have

$$
\begin{aligned}
&\langle g, f\sigma_s^{(*n)}\rangle = \int g(x_1+\cdots+x_n, \psi_s(x_1)+\cdots+\psi_s(x_n))\frac{f(x_1)\dots f(x_n)}{\psi_s(x_1)\dots\psi_s(x_n)}dx_1\dots dx_n\\
&= \int \frac{g(x_1+\cdots+x_n, \psi_s(x_1)+\cdots+\psi_s(x_n))}{\psi_s(x_1)^{1/2}\dots\psi_s(x_n)^{1/2}}\frac{f(x_1)\dots f(x_n)}{\psi_s(x_1)^{1/2}\dots\psi_s(x_n)^{1/2}}dx_1\dots dx_n\\
&\leqslant \left|\int \frac{g^2(x_1+\cdots+x_n, \psi_s(x_1)+\cdots+\psi_s(x_n))}{\psi_s(x_1)\dots\psi_s(x_n)}dx_1\dots dx_n\right|^{\frac{1}{2}}\left|\int\frac{f(x_1)^2\dots f(x_n)^2}{\psi_s(x_1)\dots\psi_s(x_n)}dx_1\dots dx_n\right|^{\frac{1}{2}}\\
&= \langle g^2, \sigma_s^{(*n)}\rangle^{1/2}\|f\|^n_{L^2(\mathbb{H}^d_s)}\\
&\leqslant \|g\|_{L^2(\mathbb{R}^d)}\|\sigma_s^{(*n)}\|^{1/2}_{L^\infty(\mathbb{R}^d)}\|f\|^n_{L^2(\mathbb{H}^d_s)},
\end{aligned}
\tag{3.12}
$$

which proves (3.11) by taking the supremum over $g \in L^2(\mathbb{R}^{d+1})$.

Now if

$$\|f\sigma_s^{(*n)}\|_{L^2(\mathbb{R}^d)} = \|\sigma_s^{(*n)}\|^{1/2}_{L^\infty(\mathbb{R}^d)}\|f\|^n_{L^2(\mathbb{H}^d_s)},$$

then, taking $g = f\sigma_s^{(*n)}$, we must have equality in (3.12)

$$\langle (f\sigma_s^{(*n)})^2, \sigma_s^{(*n)}\rangle = \|f\sigma_s^{(*n)}\|^2_{L^2(\mathbb{H}^d_s)}\|\sigma_s^{(*n)}\|_{L^\infty(\mathbb{R}^d)},$$

which occurs if and only if

$$\sigma_s^{(*n)}(\xi,\tau) = \|\sigma_s^{(*n)}\|_{L^\infty(\mathbb{R}^d)}$$

for a.e. (ξ,τ) in the support of $f\sigma_s^{(*n)}$. □

From Lemma 3.3 and (3.5) we obtain

Corollary 3.4. *Let* (d,p) *satisfy* (3.3) *and suppose* $p = 2k$ *is an even integer. Then*

$$\|T_s f\|_{L^p(\mathbb{R}^{d+1})} \leqslant (2\pi)^{(d+1)/p}\|\sigma_s^{(*k)}\|^{1/p}_{L^\infty(\mathbb{R}^{d+1})}\|f\|_{L^2(\mathbb{H}^d_s)}, \tag{3.13}$$

and thus

$$\mathbf{H}_{d,p,s} \leqslant (2\pi)^{(d+1)/p}\|\sigma_s^{(*k)}\|^{1/p}_{L^\infty(\mathbb{R}^{d+1})}. \tag{3.14}$$

In the three cases of pairs (d,p) that interest us in this paper, (3.14) gives

$$
\begin{aligned}
&\mathbf{H}_{2,4,s} \leqslant (2\pi)^{3/4}\|\sigma_s * \sigma_s\|^{1/4}_{L^\infty(\mathbb{R}^3)},\\
&\mathbf{H}_{2,6,s} \leqslant (2\pi)^{1/2}\|\sigma_s * \sigma_s * \sigma_s\|^{1/6}_{L^\infty(\mathbb{R}^3)}, \text{ and}\\
&\mathbf{H}_{3,4,s} \leqslant 2\pi\|\sigma_s * \sigma_s\|^{1/4}_{L^\infty(\mathbb{R}^4)}.
\end{aligned}
$$

For the nonexistence of extremizers we will be using the following result,

Corollary 3.5. *Let* (d,p) *satisfy* (3.3) *and suppose* $p=2k$ *is an even integer. Suppose that* $\mathbf{H}_{d,p}=(2\pi)^{(d+1)/p}\|\sigma^{(*k)}\|^{1/p}_{L^\infty(\mathbb{R}^{d+1})}$ *and that* $\sigma^{(*k)}(\tau,\xi)<\|\sigma^{(*k)}\|_{L^\infty(\mathbb{R}^{d+1})}$ *for a.e.* (ξ,τ) *in the support of* $\sigma^{(*k)}$. *Then extremizers for* (3.2) *do not exist for the pair* (d,p).

Proof. This is direct from the last assertion in Lemma 3.3. □

Lemma 3.6. *Let* $f\in\mathcal{S}(\mathbb{R}^d)$, *then the* n^{th}*-fold convolution of* $f\sigma_s$ *with itself satisfies*

$$\|f\sigma_s^{(*n)}\|_2^2\leqslant\int_{(\mathbb{R}^d)^n}\frac{f^2(x_1)\dots f^2(x_n)}{\psi_s(x_1)\dots\psi_s(x_n)}\sigma_s^{(*n)}(x_1+\cdots+x_n,\psi_s(x_1)+\cdots+\psi_s(x_n))\,dx_1\dots dx_n. \tag{3.15}$$

Proof. We will prove the case $n=2$ as the general case is analogous requiring only more notation. Following Foschi's argument we write

$$\begin{aligned} f\sigma_s*f\sigma_s(\xi,\tau)&=\int_{(\mathbb{R}^d)^2}\frac{f(x)f(y)}{\psi_s(x)\psi_s(y)}\delta(\xi-x-y)\delta(\tau-\psi_s(x)-\psi_s(y))\,dx\,dy\\ &=\int_{(\mathbb{R}^d)^2}\frac{f(x)f(y)}{\psi_s(x)\psi_s(y)}d\mu_{(\tau,\xi)}(x,y). \end{aligned}\tag{3.16}$$

From Cauchy-Schwarz, for $(\xi,\tau)\in\mathcal{P}_{d,2}$,

$$|f\sigma_s*f\sigma_s(\tau,\xi)|\leqslant\Big\|\frac{f(x)f(y)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}}\Big\|_{(\tau,\xi)}\Big\|\frac{1}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}}\Big\|_{(\tau,\xi)}. \tag{3.17}$$

Now

$$\Big\|\frac{1}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}}\Big\|^2_{(\tau,\xi)}=\sigma_s*\sigma_s(\xi,\tau) \tag{3.18}$$

as can be seen from (3.16) by taking $f\equiv 1$. Then,

$$\begin{aligned} \|f\sigma_s*f\sigma_s\|_2^2&\leqslant\int_{\mathcal{P}_{d,2}}\Big\|\frac{f(x)f(y)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}}\Big\|^2_{(\tau,\xi)}\sigma_s*\sigma_s(\xi,\tau)d\tau\,d\xi\\ &=\int_{\mathcal{P}_{d,2}}\int_{(\mathbb{R}^d)^2}\frac{f^2(x)f^2(y)}{\psi_s(x)\psi_s(y)}\boldsymbol{\delta}\begin{pmatrix}\tau-\psi_s(x)-\psi_s(y)\\ \xi-x-y\end{pmatrix}\sigma_s*\sigma_s(\xi,\tau)\,dx\,dy\,d\tau\,d\xi\\ &=\int_{(\mathbb{R}^d)^2}\frac{f^2(x)f^2(y)}{\psi_s(x)\psi_s(y)}\int_{\mathcal{P}_{d,2}}\boldsymbol{\delta}\begin{pmatrix}\tau-\psi_s(x)-\psi_s(y)\\ \xi-x-y\end{pmatrix}\sigma_s*\sigma_s(\xi,\tau)\,d\tau\,d\xi\,dx\,dy\\ &=\int_{(\mathbb{R}^d)^2}\frac{f^2(x)f^2(y)}{\psi_s(x)\psi_s(y)}\sigma_s*\sigma_s(x+y,\psi_s(x)+\psi_s(y))\,dx\,dy. \end{aligned}$$

□

3.4 Nonexistence of extremizers

In this section we prove Theorem 3.2. We start with the computation of the double and triple convolution of σ_s with itself.

Lemma 3.7. *Let $d = 2$, $s > 0$ and σ_s the measure on $\mathbb{H}_s^2$ given in (3.6). Then, for $(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R}$*

$$\sigma_s * \sigma_s(\xi,\tau) = \frac{2\pi}{\sqrt{\tau^2 - |\xi|^2}} \chi_{\{\tau \geqslant \sqrt{(2s)^2 + |\xi|^2}\}}, \tag{3.19}$$

$$\sigma_s * \sigma_s * \sigma_s(\xi,\tau) = (2\pi)^2 \Big(1 - \frac{3s}{\sqrt{\tau^2 - |\xi|^2}}\Big) \chi_{\{\tau \geqslant \sqrt{(3s)^2 + |\xi|^2}\}}. \tag{3.20}$$

In particular, $\|\sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^3)} = \frac{\pi}{s}$ and for all (ξ,τ) in the interior of the support of $\sigma_s * \sigma_s$ we have $\sigma_s * \sigma_s(\xi,\tau) < \|\sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^3)}$.

Also, $\|\sigma_s * \sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^3)} = (2\pi)^2$ and for all $(\xi,\tau) \in \mathbb{R}^{d+1}$, $\sigma_s * \sigma_s * \sigma_s(\xi,\tau) < \|\sigma_s * \sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^3)}$.

Proof. It is easy to compute the convolution,

$$\sigma_s * \sigma_s(0,\tau) = \int_{\mathbb{R}^2} \delta(\tau - 2\sqrt{s^2 + |y|^2}) \frac{dy}{s^2 + |y|^2} = 2\pi \int_0^\infty \delta(\tau - 2\sqrt{s^2 + r^2}) \frac{r dr}{s^2 + r^2}.$$

Let $u = 2\sqrt{s^2 + r^2}$, then

$$\sigma_s * \sigma_s(0,\tau) = 2\pi \int_{2s}^\infty \delta(\tau - u) \frac{du}{u} = 2\pi \chi(\tau > 2s) \frac{1}{\tau}.$$

By Lorentz invariance we obtain

$$\sigma_s * \sigma_s(\xi,\tau) = \frac{2\pi}{\sqrt{\tau^2 - |\xi|^2}} \chi_{\{\tau \geqslant \sqrt{(2s)^2 + |\xi|^2}\}}.$$

For the triple convolution we use the expression we just obtained for the double convolution,

$$\begin{aligned}
\sigma_s * \sigma_s * \sigma_s(0,\tau) &= \int_{\mathbb{R}^2} \sigma * \sigma(\tau - \sqrt{s^2 + |y|^2}, -y) \frac{dy}{\sqrt{s^2 + |y|^2}} \\
&= (2\pi)^2 \int_0^\infty \frac{\chi(\tau - \sqrt{s^2 + r^2} \geqslant \sqrt{(2s)^2 + r^2})}{((\tau - \sqrt{s^2 + r^2})^2 - r^2)^{1/2}} \frac{r dr}{\sqrt{s^2 + r^2}}.
\end{aligned}$$

Let $u = \sqrt{s^2 + r^2}$, then

$$\begin{aligned}
\sigma_s * \sigma_s * \sigma_s(0,\tau) &= (2\pi)^2 \chi_{\{\tau \geqslant 3s\}} \int_s^{\frac{\tau^2 - 3s^2}{2\tau}} \frac{du}{\sqrt{(\tau - u)^2 - (u^2 - s^2)}} \\
&= (2\pi)^2 \chi_{\{\tau \geqslant 3s\}} \int_s^{\frac{\tau^2 - 3s^2}{2\tau}} \frac{du}{\sqrt{\tau^2 - 2\tau u + s^2}} \\
&= (2\pi)^2 \Big(1 - \frac{3s}{\tau}\Big) \chi_{\{\tau \geqslant 3s\}}.
\end{aligned}$$

By Lorentz invariance,

$$\sigma_s * \sigma_s * \sigma_s(\xi,\tau) = (2\pi)^2\Big(1 - \frac{3s}{\sqrt{\tau^2 - |\xi|^2}}\Big)\chi_{\{\tau \geqslant \sqrt{(3s)^2+|\xi|^2}\}}. \qquad \square$$

A different proof of Lemma 3.7 is given in Appendix 2.

Lemma 3.8. *Let $d = 3$ and $s > 0$. Then for $(\xi,\tau) \in \mathbb{R}^3 \times \mathbb{R}$*

$$\sigma_s * \sigma_s(\xi,\tau) = 2\pi\Big(1 - \frac{4s^2}{\tau^2 - |\xi|^2}\Big)^{1/2}\chi_{\{\tau \geqslant \sqrt{(2s)^2+|\xi|^2}\}}. \tag{3.21}$$

In particular, $\|\sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^4)} = 2\pi$ and for all $(\xi,\tau) \in \mathbb{R}^4$, $\sigma_s * \sigma_s(\xi,\tau) < \|\sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^4)}$.

Proof.

$$\sigma_s * \sigma_s(0,\tau) = \int_{\mathbb{R}^3} \delta(\tau - 2\sqrt{s^2+|y|^2})\frac{dy}{s^2+|y|^2} = 4\pi\int_0^\infty \delta(\tau - 2\sqrt{s^2+r^2})\frac{r^2dr}{s^2+r^2}.$$

Let $u = 2\sqrt{s^2+r^2}$, then

$$\begin{aligned}\sigma_s * \sigma_s(0,\tau) &= 2\pi\int_{2s}^\infty \delta(\tau - u)\frac{\sqrt{u^2-4s^2}}{u}du = 2\pi\frac{\sqrt{\tau^2-4s^2}}{\tau}\chi_{\{\tau\geqslant 2s\}} \\ &= 2\pi\Big(1 - \frac{4s^2}{\tau^2}\Big)^{1/2}\chi_{\{\tau\geqslant 2s\}}.\end{aligned}$$

Therefore, by the Lorentz invariance,

$$\sigma_s * \sigma_s(\xi,\tau) = 2\pi\Big(1 - \frac{4s^2}{\tau^2 - |\xi|^2}\Big)^{1/2}\chi_{\{\tau \geqslant \sqrt{(2s)^2+|\xi|^2}\}}. \qquad \square$$

A different proof of Lemma 3.7 is given in Appendix 3. From Corollary 3.4, Lemma 3.7 and Lemma 3.8 we obtain

Corollary 3.9. *We have the following upper bounds for the best constants,*

$$\mathbf{H}_{2,4} \leqslant 2^{3/4}\pi, \quad \mathbf{H}_{2,6} \leqslant (2\pi)^{5/6}, \quad \textit{and} \quad \mathbf{H}_{3,4} \leqslant (2\pi)^{5/4}.$$

For the lower bound for the best constants we will exhibit explicit extremizing sequences.

Lemma 3.10. *Let $d = 2$ and $s > 0$. For $a > 0$ we let $f_a(y) = e^{-a\sqrt{s^2+|y|^2}}$, $y \in \mathbb{R}^2$. Then*

$$\lim_{a\to\infty} \|T_sf_a\|_{L^4(\mathbb{R}^3)}\|f_a\|^{-1}_{L^2(\mathbb{H}^2_s)} = \frac{2^{3/4}\pi}{s^{1/4}}, \tag{3.22}$$

$$\lim_{a\to 0+} \|T_sf_a\|_{L^6(\mathbb{R}^3)}\|f_a\|^{-1}_{L^2(\mathbb{H}^2_s)} = (2\pi)^{5/6}. \tag{3.23}$$

A proof of this can be found in Appendix 2. For the case $d = 3$ we have an analogous result.

Lemma 3.11. *Let $d = 3$ and $s > 0$. For $a > 0$ let $f_a(y) = e^{-a\sqrt{s^2+|y|^2}}$, $y \in \mathbb{R}^3$. Then*

$$\lim_{a\to 0^+} \|T_s f_a\|_{L^4(\mathbb{R}^4)} \|f_a\|^{-1}_{L^2(\sigma_s)} = (2\pi)^{5/4}.$$

For a proof of Lemma 3.11 see Appendix 3.

Note that Corollary 3.9, Lemma 3.10 and Lemma 3.11 imply that for $(d,p) = (2,4)$ the sequence $\{f_a/\|f_a\|_{L^2(\sigma_s)}\}_{a>0}$ is an extremizing sequence as $a \to \infty$, for $(d,p) = (2,3)$ $\{f_a/\|f_a\|_{L^2(\sigma_s)}\}_{a>0}$ is an extremizing sequence as $a \to 0^+$ and for $(d,p) = (3,6)$, $\{f_a/\|f_a\|_{L^2(\sigma_s)}\}_{a>0}$ is an extremizing sequence as $a \to 0^+$.

Proof of the first part of Theorem 3.2. Combining Corollary 3.9, Lemma 3.10 and Lemma 3.11 we obtain the first part of Theorem 3.2, namely the value of the best constants,

$$\mathbf{H}_{2,4} = 2^{3/4}\pi, \quad \mathbf{H}_{2,6} = (2\pi)^{5/6}, \quad \text{and} \quad \mathbf{H}_{3,6} = (2\pi)^{5/4}.$$

That extremizers do not exist is a consequence of Corollary 3.5 and the last assertions about the infinity norm of the double and triple convolution of σ with itself, contained in Lemma 3.7 and Lemma 3.8. □

We now prove the assertion given in the introduction about extremizers for the truncated operator T_ρ for $d = 2$ and $p = 4$.

Proposition 3.12. *Let $(d,p) = (2,4)$ and $s > 0$. For any $\rho > 0$, the best constant in (3.10) equals $2^{3/4}\frac{\pi}{s^{1/4}}$ and there are no extremizers for (3.10).*

Proof. The nonexistence of extremizers follows from the nonexistence for (3.2) if we prove that the best constant for the truncated hyperboloid equals the best constant for the entire hyperboloid, $\mathbf{H}_{2,4,s}$. For this we need a lower bound.

Since the extremizing sequence $\{f_a/\|f_a\|_2\}_{a>0}$ given in Lemma 3.10 concentrates at $y = 0$ as $a \to \infty$, one easily sees that for the sequence $\{f_a\chi_{|y|\leqslant\rho}/\|f_a\chi_{|y|\leqslant\rho}\|_2\}_{a>0}$,

$$T_\rho(f_a\chi_{|y|\leqslant\rho}/\|f_a\chi_{|y|\leqslant\rho}\|_2) \to 2^{3/4}\pi/s^{1/4} \text{ , as } a \to \infty,$$

giving the desired lower bound. □

3.5 On extremizing sequences

We are interested here in properties of extremizing sequences for (3.2). The Lorentz invariance of σ_s implies that given an extremizing sequence $\{f_n\}_{n\in\mathbb{N}}$ for (3.2), and a sequence of Lorentz transformations $\{L_n\}_{n\in\mathbb{N}}$ preserving $\mathbb{H}^d_s$, then $\{f_n \circ L_n\}_{n\in\mathbb{N}}$ is also an extremizing sequence. Even though there is this symmetry group we can obtain some general properties concerning concentration of extremizing sequences.

Consider first the case $d = 2$ and $p = 6$. From Lemma 3.10 it follows that the sequence of functions $\{f_a/\|f_a\|_2\}_{a>0}$ is an extremizing sequence as $a \to 0^+$. This particular extremizing sequence concentrates at spatial infinity, that is, for any $\varepsilon, R > 0$ there exists $a_0 > 0$ such that for all $0 < a < a_0$, $\|f_a/\|f_a\|_2\|_{L^2(B(0,R))} < \varepsilon$, where $B(0,R) = \{y \in \mathbb{R}^2 : |y| < R\}$. Next we show that this is the case for any extremizing sequence.

Proposition 3.13. *Let $\{f_n\}_{n\in\mathbb{N}}$ be an extremizing sequence for (3.2) in the case $(d,p) = (2,6)$, then for any $\varepsilon, R > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$*

$$\|f_n\|_{L^2(B(0,R))} < \varepsilon,$$

that is, the sequence concentrates at spatial infinity.

Proof. Let $\varepsilon, R > 0$ be given. From the proof of Lemma 3.6 and from Lemma 3.7, for the inequality in convolution form, we have

$$\begin{aligned}\|f_n\sigma_s * f_n\sigma_s * f_n\sigma_s\|^2_{L^2(\mathbb{R}^3)} &\leqslant \int_{\mathcal{P}_{2,3}} \Big\|\frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}}\Big\|^2_{(\tau,\xi)} \sigma_s*\sigma_s*\sigma_s(\tau,\xi)d\tau d\xi \\ &= (2\pi)^2 \int_{\mathcal{P}_{2,3}} \Big\|\frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}}\Big\|^2_{(\tau,\xi)} \Big(1 - \frac{3s}{\sqrt{\tau^2-|\xi|^2}}\Big)d\tau d\xi \\ &= (2\pi)^2\|f_n\|^6_{L^2(\sigma_s)} - (2\pi)^2 \int_{\mathcal{P}_{2,3}} \Big\|\frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}}\Big\|^2_{(\tau,\xi)} \frac{3s\ , d\tau d\xi}{\sqrt{\tau^2-|\xi|^2}}.\end{aligned}$$

Since $\|f_n\sigma_s * f_n\sigma_s * f_n\sigma_s\|^2_{L^2(\mathbb{R}^3)} \to (2\pi)^2$ as $n \to \infty$ we obtain

$$\int_{\mathcal{P}_{2,3}} \Big\|\frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}}\Big\|^2_{(\tau,\xi)} \frac{d\tau d\xi}{\sqrt{\tau^2-|\xi|^2}} \to 0 \ \text{ as } n \to \infty, \tag{3.24}$$

and thus there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$

$$\int_{\mathcal{P}_{2,3}} \Big\|\frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}}\Big\|^2_{(\tau,\xi)} \frac{d\tau d\xi}{\sqrt{\tau^2-|\xi|^2}} < \frac{\varepsilon}{3\sqrt{s^2+R^2}}.$$

From Lemma 3.6 the expression in the left hand side can be written as

$$\begin{aligned}&\int_{\mathcal{P}_{2,3}} \Big\|\frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}}\Big\|^2_{(\tau,\xi)} \frac{d\tau d\xi}{\sqrt{\tau^2-|\xi|^2}} \\ &= \int \frac{f_n^2(x)f_n^2(y)f_n^2(z)}{\psi_s(x)\psi_s(y)\psi_s(z)} \int_{\mathcal{P}_{2,3}} \boldsymbol{\delta}\begin{pmatrix}\tau-\psi_s(x)-\psi_s(y)-\psi_s(z)\\ \xi-x-y-z\end{pmatrix} \frac{d\tau d\xi}{\sqrt{\tau^2-|\xi|^2}}\, dx\, dy\, dz \\ &\geqslant \int \frac{f_n^2(x)f_n^2(y)f_n^2(z)}{\psi_s(x)\psi_s(y)\psi_s(z)} \int_{\mathcal{P}_{2,3}} \boldsymbol{\delta}\begin{pmatrix}\tau-\psi_s(x)-\psi_s(y)-\psi_s(z)\\ \xi-x-y-z\end{pmatrix} \frac{1}{\tau} d\tau d\xi\, dx\, dy\, dz \\ &\geqslant \int_{(B(0,R))^3} \frac{f_n^2(x)f_n^2(y)f_n^2(z)}{\psi_s(x)\psi_s(y)\psi_s(z)} \frac{dx\, dy\, dz}{\psi_s(x)+\psi_s(y)+\psi_s(z)}.\end{aligned}$$

If $x, y, z \in B(0,R)$, then $3s < \psi_s(x)+\psi_s(y)+\psi_s(z) \leqslant 3\psi_s(R) = 3\sqrt{s^2+R^2}$. Therefore, for all $n \geqslant N$

$$\frac{\varepsilon}{3\sqrt{s^2+R^2}} > \int_{\mathcal{P}_{2,3}} \Big\| \frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}} \Big\|^2_{(\tau,\xi)} \frac{d\tau d\xi}{\sqrt{\tau^2-|\xi|^2}} \geqslant \frac{1}{3\sqrt{s^2+R^2}} \|f_n\|^6_{L^2(B(0,R))},$$

and so, $\sup_{n\geqslant N} \|f_n\|_{L^2(B(0,R))} < \varepsilon$ as desired. □

We now turn to the case $d=3$ and $p=4$. Here we can also prove that extremizing sequences must concentrate at spatial infinity, the analog of Proposition 3.13.

Proposition 3.14. *Let $\{f_n\}_{n\in\mathbb{N}}$ be an extremizing sequence for (3.2) in the case $(d,p) = (3,4)$, then for any $\varepsilon, R > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$*

$$\|f_n\|_{L^2(B(0,R))} < \varepsilon,$$

that is, the sequence concentrates at spatial infinity.

Proof. The proof follows the same lines as the one for Proposition 3.13. Using the convolution form of the inequality we obtain the analog of equation (3.24),

$$\int_{\mathcal{P}_{3,2}} \Big\| \frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}} \Big\|^2_{(\tau,\xi)} \Big(1-\Big(1-\frac{4s^2}{\tau^2-|\xi|^2}\Big)^{1/2}\Big) d\tau d\xi \to 0 \text{ as } n\to\infty.$$

If we use the bound $1-\big(1-\frac{4s^2}{\tau^2-|\xi|^2}\big)^{1/2} \geqslant 1-\big(1-\frac{4s^2}{\tau^2}\big)^{1/2}$ and $0 < \psi_s(x)+\psi_s(y) \leqslant 2\psi_s(R)$ whenever $|x|,|y| \leqslant R$ we obtain

$$\int_{\mathcal{P}_{3,2}} \Big\| \frac{f_n(x)f_n(y)f_n(z)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}\psi_s(z)^{\frac12}} \Big\|^2_{(\tau,\xi)} \Big(1-\Big(1-\frac{4s^2}{\tau^2-|\xi|^2}\Big)^{1/2}\Big) d\tau d\xi$$
$$\geqslant \Big(1-\Big(\frac{R^2}{R^2+s^2}\Big)^{1/2}\Big) \|f_n\|^2_{L^2(B(0,R))}.$$

The conclusion follows as in the proof of Proposition 3.13. □

We now analyze the last case $(d,p) = (2,4)$. Since $\sigma_s * \sigma_s(\xi,\tau) = \|\sigma_s * \sigma_s\|_{L^\infty(\mathbb{R}^3)}$ whenever $\tau = \sqrt{s^2+|\xi|^2}$, that it, at the boundary of the support, it is not hard to see that there are extremizing sequences that concentrate at any given point in $\mathbb{H}^2_s$. For the example of extremizing sequence given in Lemma 3.10, the concentration occurs at the vertex of the hyperboloid, $(\xi,\tau) = (0,s) =: P$. We want to show that any extremizing sequences must concentrate.

Since one can have extremizing sequences concentrating at any point in the boundary it is possible to construct an extremizing sequence that concentrates on a dense set of $\mathbb{H}^2_s$ in the sense that given a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{H}^2_s$ there exists $\{f_n\}_{n\in\mathbb{N}} \subset L^2(\mathbb{H}^2_s)$, extremizing sequence, with the property that for any $\varepsilon > 0$ and $r > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$

$$\int_{|y-y_n|>r} |f_n(y)|^2 \, d\sigma_s(y) \leqslant \varepsilon. \tag{3.25}$$

Equivalently, by taking a Lorentz transformation $L_n \in \mathcal{L}^+$ with $L_n^{-1}(y_n) = (0, s) = P$ and using the Lorentz invariance of the measure, (3.25) can be written as

$$\int_{|L_n(y-P)|>r} |L_n^* f_n(y)|^2 \, d\sigma_s(y) \leqslant \varepsilon,$$

where $L_n^* f_n(y) = f_n(L_n y)$. We show that this is the only possibility for an extremizing sequence.

Proposition 3.15. *Let $\{f_n\}_{n\in\mathbb{N}}$ be an extremizing sequence for (3.2). Then there exists a sequence $\{L_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^+$ satisfying that for any $\varepsilon, r > 0$ there exist $N \in \mathbb{N}$ such that for all $n \geqslant N$*

$$\int_{|y-P|>r} |L_n^* f_n(y)|^2 \, d\sigma_s(y) \leqslant \varepsilon, \tag{3.26}$$

where $P = (0, s)$ is the vertex of the hyperboloid $\mathbb{H}_s^2$.

For the proof of the proposition we will need to introduce the function $d_s : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by the formula

$$d_s(x, y) = \frac{1}{2s}((\psi_s(x) + \psi_s(y))^2 - |x + y|^2)^{1/2} - 1.$$

Elementary properties of d_s are contained in the next lemma whose proof is left to the reader.

Lemma 3.16.

(i) For all $x, y \in \mathbb{R}^2$, $d_s(x, y) = d_s(y, x) \geqslant 0$ and $d_s(x, y) = 0$ if and only if $x = y$.

(ii) For all $x \in \mathbb{R}^2$, $\lim_{|y|\to\infty} d_s(x, y) = \infty$.

(iii) For every $R > 0$ there exists $0 < C_1(R)$, $C_2(R) < \infty$ such that

$$C_1(R)|x - y|^2 \leqslant d_s(x, y) \leqslant C_2(R)|x - y|.$$

for all x, y with $|x|, |y| \leqslant R$.

Property (ii) implies that for given $y \in \mathbb{R}^2$, the d_s-ball of radius $R > 0$ and center y, $B_{d_s}(y, R) := \{x \in \mathbb{R}^2 : d_s(x, y) \leqslant R\}$, is a bounded set. Property (iii) relates the d_s-ball with the Euclidean ball, for y with $|y| \leqslant R$ and $r > 0$

$$B(y, cr) \subset B_{d_s}(y, r) \subset B(y, c'\sqrt{r}), \tag{3.27}$$

for some constants c, c' depending on R and r only.

Proof of Proposition 3.15. The first task is to find a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{H}^2_s$ such that an analog of (3.25) is satisfied. It will be convenient, for notation only, to identify functions from $\mathbb{H}^2_s$ to $\mathbb{R}$ with functions $\mathbb{R}^2$ to $\mathbb{R}$ and points in $\mathbb{H}^2_s$ with points in $\mathbb{R}^2$. This is done via the projection of $\mathbb{H}^2_s$ onto $\mathbb{R}^2 \times \{0\}$.

From Lemma 3.6 and from Lemma 3.7, for the inequality in convolution form, we have

$$\begin{aligned}\|f_n\sigma_s * f_n\sigma_s\|^2_{L^2(\mathbb{R}^3)} &\leqslant \int_{\mathcal{P}_{2,2}} \Big\| \frac{f_n(x)f_n(y)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}} \Big\|^2_{(\tau,\xi)} \sigma_s * \sigma_s(\tau,\xi)\,d\tau d\xi \\ &= \frac{\pi}{s}\int_{\mathcal{P}_{2,2}} \Big\| \frac{f_n(x)f_n(y)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}} \Big\|^2_{(\tau,\xi)} \frac{2s}{\sqrt{\tau^2-|\xi|^2}}\,d\tau d\xi \\ &\leqslant \frac{\pi}{s}\|f_n\|^4_{L^2}.\end{aligned}$$

Since $\|f_n\sigma_s * f_n\sigma_s\|^2_{L^2(\mathbb{R}^3)} \to \frac{\pi}{s}$ as $n\to\infty$ we obtain

$$\int_{\mathcal{P}_{2,2}} \Big\| \frac{f_n(x)f_n(y)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}} \Big\|^2_{(\tau,\xi)} \frac{2s}{\sqrt{\tau^2-|\xi|^2}}\,d\tau d\xi \to 1 \text{ as } n\to\infty, \tag{3.28}$$

Similarly as in the proof of Lemma 3.6 the expression in the left hand side can be written as

$$\begin{aligned}&\int_{\mathcal{P}_{2,2}} \Big\| \frac{f_n(x)f_n(y)}{\psi_s(x)^{\frac12}\psi_s(y)^{\frac12}} \Big\|^2_{(\tau,\xi)} \frac{2s}{\sqrt{\tau^2-|\xi|^2}}\,d\tau d\xi \\ &= \int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)f_n^2(y)}{\psi_s(x)\psi_s(y)} \int_{\mathcal{P}_2} \boldsymbol{\delta}\begin{pmatrix}\tau-\psi_s(x)-\psi_s(y)\\ \xi-x-y\end{pmatrix} \frac{2s}{\sqrt{\tau^2-|\xi|^2}}\,d\tau d\xi\,dx\,dy \\ &= \int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)f_n^2(y)}{\psi_s(x)\psi_s(y)} \frac{2s}{((\psi_s(x)+\psi_s(y))^2-|x+y|^2)^{1/2}}\,dx\,dy \\ &= \int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)f_n^2(y)}{\psi_s(x)\psi_s(y)} K_s(x,y)\,dx\,dy.\end{aligned}$$

Observe that

$$\int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)f_n^2(y)}{\psi_s(x)\psi_s(y)}\,dx\,dy = \|f_n\|^2_{L^2(\mathbb{H}^2_s)} = 1$$

and

$$K_s(x,y) := \frac{2s}{((\psi_s(x)+\psi_s(y))^2-|x+y|^2)^{1/2}} = \frac{1}{d_s(x,y)+1} \leqslant 1$$

for all $x,y\in\mathbb{R}^2$. Equation (3.28) implies that

$$\int_{(\mathbb{R}^2)^2} \frac{f_n^2(x)f_n^2(y)}{\psi_s(x)\psi_s(y)} K_s(x,y)\,dx\,dy \to 1 \text{ as } n\to\infty.$$

Denote $h_n(y) = f_n(y)^2/\psi_s(y)$, so that $\int_{\mathbb{R}^2} h_n(y)dy = 1$. For $\varepsilon > 0$ we can write

$$\begin{aligned}\int_{(\mathbb{R}^2)^2} h_n(x)h_n(y)K_s(x,y)dx\,dy &= \int_{d_s(x,y)\leqslant\varepsilon} h_n(x)h_n(y)K_s(x,y)dx\,dy \\ &\quad + \int_{d_s(x,y)>\varepsilon} h_n(x)h_n(y)K_s(x,y)dx\,dy \\ &\leqslant \|h_n\|_{L^1(\mathbb{R}^2)}^2 - \Big(1-\frac{1}{\varepsilon+1}\Big)\int_{d_s(x,y)>\varepsilon} h_n(x)h_n(y)dx\,dy.\end{aligned}$$

Then, as the left hand side tends to 1 as $n\to\infty$ we conclude that

$$\lim_{n\to\infty}\int_{d_s(x,y)\leqslant\varepsilon} h_n(x)h_n(y)dx\,dy = 1.$$

Using the Fubini Theorem we can write

$$\begin{aligned}\int_{d_s(x,y)\leqslant\varepsilon} h_n(x)h_n(y)dx\,dy &= \int_{\mathbb{R}^2} h_n(y)\int_{d_s(x,y)\leqslant\varepsilon} h_n(x)dxdy \\ &\leqslant \|h\|_1 \sup_{y\in\mathbb{R}^2}\int_{d_s(x,y)\leqslant\varepsilon} h_n(x)dx.\end{aligned}$$

Then

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_{d_s}(y,\varepsilon)} h_n(x)dx = 1. \tag{3.29}$$

Fix a continuous function $\gamma : (0,\infty)\to(0,1)$ satisfying $\gamma(t)\to 0$ as $t\to 0^+$. Then (3.29) implies that there exists $N(\varepsilon)\in\mathbb{N}$ such that for all $n\geqslant N(\varepsilon)$

$$\sup_{y\in\mathbb{R}^2}\int_{B_{d_s}(y,\varepsilon)} h_n(y)dy \geqslant 1-\gamma(\varepsilon),$$

and so there exists $\{y_n^\varepsilon\}_{n\geqslant N(\varepsilon)}\subset\mathbb{R}^2$ such that

$$\int_{B_{d_s}(y_n^\varepsilon,\varepsilon)} h_n(y)dy \geqslant 1-2\gamma(\varepsilon).$$

In this way, each $\varepsilon > 0$ we have a number $N(\varepsilon)$, and a sequence $\{y_n^\varepsilon\}_{n\geqslant N(\varepsilon)}$.

The construction of the sequence $\{y_n\}_{n\in\mathbb{N}}$ will be obtained by a diagonal process. We take a strictly decreasing sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ with $\varepsilon_k\to 0$ as $k\to\infty$. This gives a sequence $\{N(k)\}_{k\in\mathbb{N}}$ and $\{y_n^k\}_{n\geqslant N(k),k\geqslant 0}$. We can assume that the sequence $\{N(k)\}_{k\in\mathbb{N}}$ is strictly increasing. For each $n\geqslant N(1)$ we let $l(n) = \inf\{k : N(k)\leqslant n\}$

Define $\{y_n\}_{n\in\mathbb{N}}$ by

$$y_n = \begin{cases} y_n^{l(n)} & , \text{ if } n\geqslant N(1) \\ y_0 & , \text{ if } n < N(1)\end{cases}$$

where $y_0 \in \mathbb{R}^2$ is arbitrary, but fixed.

Now let $\varepsilon > 0$ be given. Take k such that $\varepsilon_k < \varepsilon$. For $n \geqslant N(k)$ we have $l(n) \geqslant k$, so $\varepsilon_{l(n)} \leqslant \varepsilon_k < \varepsilon$ and

$$\int_{B_{d_s}(y_n^{l(n)},\varepsilon_{l(n)})} h_n(y)dy \geqslant 1 - 2\gamma(\varepsilon_{l(n)}),$$

and hence

$$\int_{B_{d_s}(y_n,\varepsilon)} h_n(y)dy \geqslant \int_{B_{d_s}(y_n^{l(n)},\varepsilon_{l(n)})} h_n(y)dy \geqslant 1 - 2\gamma(\varepsilon_{l(n)}) \geqslant 1 - 2\gamma(\varepsilon).$$

Since $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ we have just proved that for any $\varepsilon, r > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$

$$\int_{B_{d_s}(y_n,\varepsilon)} f_n(y)\frac{dy}{\sqrt{s^2+|y|^2}} \geqslant 1 - \varepsilon. \tag{3.30}$$

To finish we need to use the Lorentz invariance. This is better done without the identification of $\mathbb{H}_s^2$ with $\mathbb{R}^2$ that we have been using, so we now lift everything to $\mathbb{H}_s^2$. Let $D_s : \{(\xi,\tau) \in \mathbb{R}^2 \times \mathbb{R} : \tau > |\xi|\} \to \mathbb{R}$ be defined by

$$D_s((\xi_1,\tau_1),(\xi_2,\tau_2)) = (2s)^{-1}((\tau_1+\tau_2)^2 - |\xi_1+\xi_2|^2)^{1/2} - 1,$$

and observe that for every $L \in \mathcal{L}^+$, $D_s(L(\xi_1,\tau_1),L(\xi_2,\tau_2)) = D_s((\xi_1,\tau_1),(\xi_2,\tau_2))$.

Let $z_n = (y_n, \psi_s(y_n)) \in \mathbb{H}_s^2$. We can write (3.30) equivalently as

$$\int_{D_s(z,z_n)>r} |f(z)|^2 d\sigma(z) < \varepsilon.$$

By the Lorentz invariance of D_s and σ, for $L_n \in \mathcal{L}^+$ such that $L_n^{-1}(z_n) = (0,s) = P$ we have that for every $\varepsilon, r > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$

$$\int_{D_s(z,P)\leqslant r} |L_n^* f(z)|^2 d\sigma(z) \geqslant 1 - \varepsilon$$

which implies

$$\int_{|z-P|\leqslant\sqrt{r}} |L_n^* f(z)|^2 d\sigma(z) \geqslant 1 - \varepsilon$$

for all $r > 0$ sufficiently small, independent of ε and n, proving the proposition. □

3.6 The two sheeted hyperboloid

In this section we consider the two sheeted hyperboloid

$$\bar{\mathbb{H}}_s^d = \{(y,y') \in \mathbb{R}^d \times \mathbb{R} : y'^2 = s^2 + |y|^2\}$$

with measure

$$\bar{\sigma}_s(y,y') = \delta(y' - \sqrt{s^2+|y|^2})(s^2+|y|^2)^{-\frac{1}{2}}dydy' + \delta(y' + \sqrt{s^2+|y|^2})(s^2+|y|^2)^{-\frac{1}{2}}dydy'$$

and adjoint Fourier restriction operator defined by $\bar{T}_s f = \widehat{f\bar{\sigma}_s}$, for $f \in \mathcal{S}(\mathbb{R}^{d+1})$.

$\bar{\mathbb{H}}_s^d$ is the union of the two sheets

$$\mathbb{H}_s^{d,\pm} = \{(y,y') \in \mathbb{R}^d \times \mathbb{R} : y' = \pm(s^2+|y|^2)^{1/2}\}.$$

What in this section we are calling $\mathbb{H}_s^{d,+}$ is what before we denoted by $\mathbb{H}_s^d$ (that change of notation is convenient here). In the previous section we proved that for $\mathbb{H}_s^{d,+}$ (and thus also for $\mathbb{H}_s^{d,-}$) extremizers do not exist for the cases $(d,p) = (2,4), (2,6)$ and $(3,4)$. Here we show that extremizers for $\bar{\mathbb{H}}_s^d$ do not exist either and compute the best constants.

The adjoint Fourier restriction operator on $\mathbb{H}_s^{d,+}$ is denoted by T_s and the one on $\mathbb{H}_s^{d,-}$ will be denoted T_s^-. For $s = 1$ we will drop the subscript s.

For sets $A, B \subset \mathbb{R}^d$ we denote $A + B = \{a + b : a \in A,\, b \in B\}$, the algebraic sum of A and B, and $-A = \{-a : a \in A\}$. We start with the following lemma.

Lemma 3.17. *For $d \geqslant 1$ we have*

$$\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} \subset \{(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R} : \tau \geqslant \sqrt{(2s)^2+|\xi|^2}\}, \tag{3.31}$$

$$\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,-} \subset \{(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R} : |\tau| \leqslant \sqrt{(2s)^2+|\xi|^2}\}, \tag{3.32}$$

$$\mathbb{H}_s^{d,-} + \mathbb{H}_s^{d,-} \subset \{(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R} : \tau \leqslant -\sqrt{(2s)^2+|\xi|^2}\}. \tag{3.33}$$

Proof. The first assertion was already proved when we computed the double convolution $\sigma_s * \sigma_s$. We do it again. If $\xi = x + y$ and $\tau = \psi_s(x) + \psi_s(y)$, then $\tau \geqslant 2s > 0$ and squaring

$$\tau^2 = (\psi_s(x) + \psi_s(y))^2 = 2s^2 + |x|^2 + |y|^2 + 2(s^2+|x|^2)^{1/2}(s^2+|y|^2)^{1/2}.$$

On the other hand,

$$|\xi|^2 = |x+y|^2 = |x|^2 + |y|^2 + 2x\cdot y.$$

Using $x \cdot y = |x||y|\cos\theta$, with θ the angle between x and y we see that (3.31) is equivalent to the inequality for real numbers $a, b, s \geqslant 0$

$$(s^2+a^2)^{1/2}(s^2+b^2)^{1/2} \geqslant s^2 + ab \tag{3.34}$$

which is easily shown to hold, by squaring both sides.

We proceed in a similar way for the second part. Let $\xi = x + y$ and $\tau = \psi_s(x) - \psi_s(y)$. Then

$$\tau^2 = 2s^2 + |x|^2 + |y|^2 - 2(s^2+|x|^2)^{1/2}(s^2+|y|^2)^{1/2}.$$

As before we see that (3.32) is equivalent to

$$-(s^2+|x|^2)^{1/2}(s^2+|y|^2)^{1/2} \leqslant s^2 + x\cdot y,$$

which in turn is equivalent to the easy to verify inequality for real numbers $a, b, s \geqslant 0$

$$(s^2 + a^2)^{1/2}(s^2 + b^2)^{1/2} \geqslant ab - s^2.$$

As for (3.33), it follows from (3.31) by observing that $\mathbb{H}_s^{d,-} = -\mathbb{H}_s^{d,+}$. □

Lemma 3.18. *Let $d \geqslant 1$, then*

$$\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} \subseteq \{(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R} : \tau \geqslant \sqrt{(3s)^2 + |\xi|^2}\}, \tag{3.35}$$
$$\mathbb{H}_s^{d,-} + \mathbb{H}_s^{d,-} + \mathbb{H}_s^{d,-} \subseteq \{(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R} : \tau \leqslant -\sqrt{(3s)^2 + |\xi|^2}\}, \tag{3.36}$$
$$\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,-} \subseteq \{(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R} : \tau \geqslant -\sqrt{(3s)^2 + |\xi|^2}\}, \tag{3.37}$$
$$\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,-} + \mathbb{H}_s^{d,-} \subseteq \{(\xi,\tau) \in \mathbb{R}^d \times \mathbb{R} : \tau \leqslant \sqrt{(3s)^2 + |\xi|^2}\}. \tag{3.38}$$

Proof. We know from Lemma 3.17 that

$$\mathbb{H}_s^{d,+} + \mathbb{H}_s^{d,+} \subset \{(\xi,\tau) : \tau \geqslant \sqrt{(2s)^2 + |\xi|^2}\}.$$

We start with (3.35). Let $\xi = x + y$ and $\tau \geqslant \psi_{2s}(x) + \psi_s(y) > 0$. By squaring,

$$\tau^2 \geqslant 5s^2 + |x|^2 + |y|^2 + 2(4s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2}.$$

Then (3.35) would follow from

$$(4s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2} \geqslant 2s^2 + x \cdot y,$$

which is equivalent to the easy to verify inequality for real numbers $a, b, s \geqslant 0$

$$(4s^2 + a^2)^{1/2}(s^2 + b^2)^{1/2} \geqslant 2s^2 + ab.$$

We now establish (3.37). Let $\xi = x+y$, $\tau \geqslant \psi_{2s}(x) - \psi_s(y)$. If $\tau \geqslant 0$ we are done, so suppose that $0 \geqslant \tau \geqslant \psi_{2s}(x) - \psi_s(y)$, then

$$\tau^2 \leqslant 5s^2 + |x|^2 + |y|^2 - 2(4s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2},$$

and (3.37) would follow from

$$-(4s^2 + |x|^2)^{1/2}(s^2 + |y|^2)^{1/2} \leqslant 2s^2 + x \cdot y,$$

which is equivalent to show

$$(4s^2 + a^2)^{1/2}(s^2 + b^2)^{1/2} \geqslant ab - 2s^2$$

for all $a, b, s \geqslant 0$ and this last inequality holds.

Both (3.36) and (3.38) can be proved in a similar way. □

For a function $f \in L^2(\bar{\mathbb{H}}^d_s)$ we can write $f = f_+ + f_-$, where f_+ is supported on $\mathbb{H}^{d,+}_s$, and f_- on $\mathbb{H}^{d,-}_s$. One then has $\|f\|^2_{L^2(\bar{\mathbb{H}}^d_s)} = \|f_+\|^2_{L^2(\mathbb{H}^{d,+}_s)} + \|f_-\|^2_{L^2(\mathbb{H}^{d,-}_s)}$.

Proposition 3.19. *Let $d \in \{2,3\}$ and $s > 0$. Then*

$$\bar{\mathbf{H}}_{d,4,s} = (3/2)^{1/4}\mathbf{H}_{d,4,s},$$

and extremizers for the $L^2(\bar{\mathbb{H}}^d_s)$ to $L^4(\mathbb{R}^d)$ adjoint restriction inequality on $\bar{\mathbb{H}}^d_s$ do not exist. Moreover, if $\{f_n\}_{n\in\mathbb{N}}$ is an extremizing sequence for $\bar{T}$ then $\{f_{n,+}/\|f_{n,+}\|_2\}_{n\in\mathbb{N}}$ and $\{f_{n,-}/\|f_{n,-}\|_2\}_{n\in\mathbb{N}}$ are extremizing sequences for T_s and T^-_s in $\mathbb{H}^{d,+}_s$ and $\mathbb{H}^{d,-}_s$ respectively.

Proof. We want to show

$$\sup_{0\neq f\in L^2(\bar{\mathbb{H}}^d_s)} \|\bar{T}_s f\|^4_{L^4}\|f\|^{-4}_{L^2(\bar{\mathbb{H}}^d_s)} = \frac{3}{2}\mathbf{H}^4_{d,4,s}. \tag{3.39}$$

For the inequality $\|\bar{T}_s f\|^4_{L^4}\|f\|^{-4}_{L^2(\bar{\mathbb{H}}^d_s)} \leqslant \frac{3}{2}\mathbf{H}^4_{d,4,s}$ we use the argument in [20, pg. 754-755]. We will restrict attention to the case $s = 1$, but the other cases follow in the same way, or by the use of scaling. Observe that

$$\begin{aligned}\|\bar{T}f\|^4_{L^4} &= \|Tf_+ + T^-f_-\|^4_{L^4} = \|(Tf_+ + T^-f_-)^2\|^2_{L^2}\\ &= \|(Tf_+)^2 + (T^-f_-)^2 + 2(Tf_+)(T^-f_-)\|^2_{L^2}.\end{aligned}$$

Using that product transforms into convolution under the Fourier transform we see that the Fourier transforms of $(Tf_+)^2$, $(T^-f_-)^2$ and $(Tf_+)(T^-f_-)$ are supported on $\mathbb{H}^{d,+} + \mathbb{H}^{d,+}$, $\mathbb{H}^{d,-}+\mathbb{H}^{d,-}$ and $\mathbb{H}^{d,+}+\mathbb{H}^{d,-}$ respectively. Those three sets have intersection of measure zero by Lemma 3.17, therefore

$$\|\bar{T}f\|^4_{L^4} = \|Tf_+\|^4_{L^4} + \|T^-f_-\|^4_{L^4} + 4\|(Tf_+)(T^-f_-)\|^2_{L^2} \tag{3.40}$$

$$\leqslant \mathbf{H}^4_{d,4}(\|f_+\|^4_{L^2} + \|f_-\|^4_{L^2} + 4\|f_+\|^2_{L^2}\|f_-\|^2_{L^2}) \tag{3.41}$$

$$\leqslant \frac{3}{2}\mathbf{H}^4_{d,4}(\|f_+\|^2_{L^2} + \|f_-\|^2_{L^2})^2 \tag{3.42}$$

$$= \frac{3}{2}\mathbf{H}^4_{d,4}\|f\|^4_{L^2}, \tag{3.43}$$

where we have used the sharp inequality (as in [20])

$$X^2 + Y^2 + 4XY \leqslant \frac{3}{2}(X+Y)^2, \quad X, Y \geqslant 0 \tag{3.44}$$

where equality holds if and only if $X = Y$. Thus,

$$\|\bar{T}f\|^4_{L^4}\|f\|^{-4}_{L^2(\bar{\mathbb{H}}^d)} \leqslant \frac{3}{2}\mathbf{H}^4_{d,4}. \tag{3.45}$$

To establish (3.39) in the case $s = 1$, let $\{f_{n,+}\}_{n\in\mathbb{N}}$ be an extremizing sequence for T. By identifying a function on $\mathbb{H}^{d,\pm}$ with a function from $\mathbb{R}^d$ to $\mathbb{R}$ we let $f_{n,-}(y) = \overline{f_{n,+}}(y)$, $y \in \mathbb{R}^d$, that is, the complex conjugate of $f_{n,+}$. Then $f_n = \frac{1}{\sqrt{2}}(f_{n,+} + f_{n,-})$ is an extremizing sequence for $\bar{T}$ on $\bar{\mathbb{H}}^d$ since (3.42) becomes equality and (3.41) becomes equality in the limit $n \to \infty$.

To prove the nonexistence of extremizers we note that for $f \neq 0$ there is equality in (3.45) if and only if there is equality in (3.41) and (3.42). There is equality in (3.42) if and only if $\|f_+\|_2 = \|f_-\|_2$. There is equality in (3.41) if and only if

$$\|Tf_+\|_{L^4} = \mathbf{H}_{d,4}\|f_+\|_{L^2(\mathbb{H}^{d,+})}\,,\ \|T^- f_-\|_{L^4} = \mathbf{H}_{d,4}\|f_-\|_{L^2(\mathbb{H}^{d,-})}, \tag{3.46}$$

and $|Tf_+| = |T^- f_-|$ a.e. in $\mathbb{R}^d$.

By Theorem 3.2 we know that (3.46) can not hold for nonzero f_+ and f_- proving the nonexistence of extremizers for the $L^2(\bar{\mathbb{H}}^d)$ to $L^4(\mathbb{R}^d)$ adjoint restriction inequality on $\bar{\mathbb{H}}^d$.

Now let $\{f_n\}_{n\in\mathbb{N}}$ be an extremizing sequence for $\bar{T}$, i.e. $\|f_n\|_2 \leqslant 1$ and $\lim\limits_{n\to\infty} \|\bar{T}f_n\|_{L^4(\mathbb{R}^d)} = (3/2)^{1/4}\mathbf{H}_{d,4}$. For the decomposition $f_n = f_{n,+} + f_{n,-}$, we see that

$$\lim_{n\to\infty}(\|f_{n,+}\|_{L^2}^4 + \|f_{n,-}\|_{L^2}^4 + 4\|f_{n,+}\|_{L^2}^2\|f_{n,-}\|_{L^2}^2) = \frac{3}{2}.$$

This implies that if $\lim_{n\to\infty} \|f_{n,+}\|_{L^2}$ and $\lim_{n\to\infty} \|f_{n,-}\|_{L^2}$ exist then they must be equal, and so equal to $1/\sqrt{2}$. Therefore any subsequence has a convergent subsequence with limit $1/\sqrt{2}$. This implies the existence of both limits and $\lim_{n\to\infty} \|f_{n,+}\|_{L^2} = \lim_{n\to\infty} \|f_{n,-}\|_{L^2} = 1/\sqrt{2}$.

If we write $\|Tf_{n,+}\|_4 = a_n\mathbf{H}_{d,4}\|f_+\|_{L^2(\mathbb{H}^{d,+})}$ and $\|T^- f_{n,-}\|_4 = b_n\mathbf{H}_{d,4}\|f_-\|_{L^2(\mathbb{H}^{d,-})}$. Then, as before, $\lim_{n\to\infty} a_n\|f_{n,+}\|_2 = \frac{1}{\sqrt{2}}$, and so $\lim_{n\to\infty} a_n = 1$, and similarly $\lim_{n\to\infty} b_n = 1$.

Hence, $\{f_{n,+}/\|f_{n,+}\|_2\}_{n\in\mathbb{N}}$ and $\{f_{n,-}/\|f_{n,-}\|_2\}_{n\in\mathbb{N}}$ are extremizing sequences for T_s and T_s^- in $\mathbb{H}_s^{d,+}$ and $\mathbb{H}_s^{d,-}$ respectively. □

Proposition 3.20. *Let $d \in \{1, 2\}$ and $s > 0$. Then*

$$\bar{\mathbf{H}}_{2,6,s} = (5/2)^{1/3}\mathbf{H}_{2,6,s}, \tag{3.47}$$

and extremizers for the $L^2(\bar{\mathbb{H}}_s^d)$ to $L^6(\mathbb{R}^d)$ adjoint restriction inequality on $\bar{\mathbb{H}}_s^d$ do not exist. Moreover, if $\{f_n\}_{n\in\mathbb{N}}$ is an extremizing sequence for $\bar{T}$ then $\{f_{n,+}/\|f_{n,+}\|_2\}_{n\in\mathbb{N}}$ and $\{f_{n,-}/\|f_{n,-}\|_2\}_{n\in\mathbb{N}}$ are extremizing sequences for T_s and T_s^- in $\mathbb{H}_s^{d,+}$ and $\mathbb{H}_s^{d,-}$ respectively.

Proof. A proof of this is contained in [20, pg. 758-760]. It follows the same lines as Proposition 3.19. One first writes $\|\bar{T}_s f\|_{L^6(\mathbb{R}^3)}^6 = \|(\bar{T}_s f)^3\|_{L^2(\mathbb{R}^3)}^2$ and $f = f_+ + f_-$. Expanding $(\bar{T}_s f_+ + \bar{T}_s f_-)^3$ and using Plancherel's Theorem together with Lemma (3.18) plus Hölder's inequality one obtains

$$\|\bar{T}_s f\|_{L^6}^6 \|f\|_{L^2(\bar{\mathbb{H}}_s^d)}^{-6} \leqslant \frac{25}{4}\mathbf{H}_{d,6,s}^6,$$

proving $\bar{\mathbf{H}}_{2,6,s} \leqslant (5/2)^{1/3}\mathbf{H}_{2,6,s}$. The reverse inequality in (3.47), the nonexistence of extremizers and the property of extremizing sequences stated in the proposition are handled as in the proof of Proposition 3.19. We skip the details. □

Proposition 3.19 and Proposition 3.20 give the proof of the second part of Theorem 3.2.

3.7 Appendix 1: Scaling

Here we record the scaling for the family of operators $\{T_s\}_{s>0}$. Recall from the introduction that for $s>0$, $\mathbb{H}^d_s := \{(y,\sqrt{s^2+|y|^2}) : y\in\mathbb{R}^d\}$ equipped with the measure $\sigma_s(y,y') = \delta(y'-\sqrt{s^2+|y|^2})\frac{dydy'}{\sqrt{s^2+|y|^2}}$.

The operator T_s defined on $\mathcal{S}(\mathbb{R}^2)$ by

$$T_sf(x,t)=\widehat{f\sigma_s}(-x,-t)=\int_{\mathbb{R}^d} e^{ix\cdot y}e^{it\sqrt{s^2+|y|^2}}f(y)\frac{dy}{\sqrt{s^2+|y|^2}}.$$

We want to show that $\mathbf{H}_{d,p,s}$ defined in (3.8) satisfies (3.9). If we make the change of variables $v=sy$ in the expression defining $Tf(x,t)$, then

$$\begin{aligned}Tf(x,t)&=\int_{\mathbb{R}^d} e^{ix\cdot y}e^{it\sqrt{1+|y|^2}}f(y)\frac{dy}{\sqrt{1+|y|^2}}\\&=\int_{\mathbb{R}^d} e^{is^{-1}x\cdot y}e^{it\sqrt{1+s^{-2}|y|^2}}f(s^{-1}y)\frac{s^{-d}dy}{\sqrt{1+s^{-2}|y|^2}}\\&=s^{-d+3/2}\int_{\mathbb{R}^d} e^{is^{-1}x\cdot y}e^{is^{-1}t\sqrt{s^2+|y|^2}}s^{-1/2}f(s^{-1}y)\frac{dy}{\sqrt{s^2+|y|^2}}\end{aligned}$$

from where $s^{d-3/2}Tf(sx,st)=T_s(s^{-1/2}f(s^{-1}\cdot))(x,t)$ and it follows that

$$s^{d-3/2-(d+1)/p}\|Tf\|_{L^p(\mathbb{R}^{d+1})}=\|T_ss^{-1/2}f(s^{-1}\cdot)\|_{L^p(\mathbb{R}^{d+1})}.$$

On the other hand

$$\begin{aligned}\int_{\mathbb{R}^d}|f(y)|^2\frac{dy}{\sqrt{1+|y|^2}}&=\int_{\mathbb{R}^d}|f(s^{-1}y)|^2\frac{s^{-d}dy}{\sqrt{1+s^{-2}|y|^2}}\\&=s^{-d+2}\int_{\mathbb{R}^d}|s^{-1/2}f(s^{-1}y)|^2\frac{dy}{\sqrt{s^2+|y|^2}}\end{aligned}$$

that is $\|f\|_{L^2(\sigma)}=s^{-(d-2)/2}\|s^{-1/2}f(s^{-1}\cdot)\|_{L^2(\sigma_s)}$, thus

$$s^{(d-1)/2-(d+1)/p}\|Tf\|_{L^p(\mathbb{R}^{d+1})}\|f\|^{-1}_{L^2(\sigma)}=\|T_ss^{-1/2}f(s^{-1}\cdot)\|_{L^p(\mathbb{R}^{d+1})}\|s^{-1/2}f(s^{-1}\cdot)\|^{-1}_{L^2(\sigma_s)},$$

and it follows that for all $s>0$

$$\mathbf{H}_{d,p,s}=s^{(d-1)/2-(d+1)/p}\mathbf{H}_{d,p}. \tag{3.48}$$

3.8 Appendix 2: some explicit calculations for the case $d = 2$

The exponential integral function $\mathrm{Ei}(x)$, for $x \neq 0$, is defined by

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt = \int_{-\infty}^{x} \frac{e^{t}}{t} dt \tag{3.49}$$

where the principal value is taken for $x > 0$.

Lemma 3.21. *Let $a > 0$ and $f_s(y) = e^{-a\sqrt{s^2+|y|^2}}$, $y \in \mathbb{R}^2$. Then*

$$\|T_s f_a\|_{L^6(\mathbb{R}^3)}^6 \|f_a\|_{L^2(\sigma_s)}^{-6} = (2\pi)^5(1 - 6as - 36a^2s^2e^{6as}\,\mathrm{Ei}(-6as)), \text{ and} \tag{3.50}$$

$$\|T_s f_a\|_{L^4(\mathbb{R}^3)}^4 \|f_a\|_{L^2(\sigma_s)}^{-4} = 2^3\frac{\pi^4}{s}(-4ase^{4as}\,\mathrm{Ei}(-4as)). \tag{3.51}$$

Proof of Lemma 3.10. Using the expressions in Lemma 3.21

$$\lim_{a\to 0^+} \|T_s f_a\|_{L^6}^6 \|f_a\|_{L^2(\sigma_s)}^{-6} = \lim_{a\to 0^+} (2\pi)^5(1 - 6as - 36a^2s^2e^{6as}\,\mathrm{Ei}(-6as)) = (2\pi)^5$$

and

$$\lim_{a\to \infty} \|T_s f_a\|_{L^4}^4 \|f_a\|_{L^2(\sigma_s)}^{-4} = \lim_{a\to \infty} 2^3\frac{\pi^4}{s}(-4ase^{4as}\,\mathrm{Ei}(-4as)) = 2^3\frac{\pi^4}{s}. \qquad \square$$

Remark 3.22.

1. It is not hard to see that the function $a \mapsto 1 - a + a^2e^a\,\mathrm{Ei}(-a)$ is a strictly decreasing function for $a \in [0,\infty)$ which tends to 0 as $a \to \infty$ and to 1 as $a \to 0^+$. Then $\|T_s f_a\|_{L^6}^6 \|f_a\|_{L^2(\sigma_s)}^{-6}$ is a strictly decreasing function of a, for each fixed s.

2. The function $a \mapsto -ae^a\,\mathrm{Ei}(-a)$ is strictly increasing for $a \in [0,\infty)$, tends to 0 as $a \to 0^+$, and to 1 as $a \to \infty$. Then $\|T_s f_a\|_{L^4}^4 \|f_a\|_{L^2(\sigma_s)}^{-4}$ is a strictly increasing function of a, for each fixed s.

Proof of Lemma 3.21. We first compute the $L^2(\sigma_s)$-norm of f_a,

$$\begin{aligned}\|f_a\|_{L^2(\sigma_s)}^2 &= \int_{\mathbb{R}^2} e^{-2a\sqrt{s^2+|y|^2}} \frac{dy}{\sqrt{s^2+|y|^2}} = 2\pi\int_0^\infty e^{-2a\sqrt{s^2+r^2}}\frac{r}{\sqrt{s^2+r^2}}dr \\ &= 2\pi\int_s^\infty e^{-2ar}dr = \frac{\pi}{a}e^{-2as}.\end{aligned}$$

The formulas in (3.50) and (3.51) are easier to compute in their equivalent convolution form. Let $g_a(\xi,\tau) = e^{-a\tau}$ and observe that $f_a\sigma_s * f_a\sigma_s = g_a\sigma_s * g_a\sigma_s$ and $f_a\sigma_s * f_a\sigma_s * f_a\sigma_s =$

$g_a\sigma_s * g_a\sigma_s * g_a\sigma_s$. Then, because g_a is the exponential of a linear function, $g_a\sigma_s * g_a\sigma_s(\xi,\tau) = g_a(\xi,\tau)\sigma_s * \sigma_s(\xi,\tau)$ and $g_a\sigma_s * g_a\sigma_s * g_a\sigma_s(\xi,\tau) = g_a(\xi,\tau)\sigma_s * \sigma_s * \sigma_s(\xi,\tau)$, therefore

$$\begin{aligned}
\|f_a\sigma_s * f_a\sigma_s * f_a\sigma_s\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}\times\mathbb{R}^2} e^{-2a\tau}(2\pi)^4\Big(1-\frac{3s}{\sqrt{\tau^2-|\xi|^2}}\Big)^2\chi_{\{\tau\geqslant\sqrt{(3s)^2+|\xi|^2}\}}d\tau d\xi \\
&= (2\pi)^5\int_{3s}^\infty\int_0^{\sqrt{\tau^2-(3s)^2}} e^{-2a\tau}\Big(1-\frac{3s}{\sqrt{\tau^2-r^2}}\Big)^2 r dr d\tau \\
&= (2\pi)^5\int_{3s}^\infty\int_0^{\sqrt{\tau^2-(3s)^2}} e^{-2a\tau}(r+(3s)^2\frac{r}{\tau^2-r^2}-6s\frac{r}{\sqrt{t^2-r^2}})drd\tau \\
&= (2\pi)^5\int_{3s}^\infty e^{-2a\tau}(\tfrac{1}{2}(\tau^2-(3s)^2)+(3s)^2(\log\tau-\log(3s))-6s(\tau-3s))d\tau \\
&= (2\pi)^5\Big(\frac{1}{2}\int_{3s}^\infty e^{-2a\tau}\tau^2 d\tau+(3s)^2\int_{3s}^\infty e^{-2a\tau}\log\tau d\tau \\
&\qquad -6se^{-6as}\int_0^\infty e^{-2a\tau}\tau d\tau-(\frac{9}{2}s^2+(3s)^2\log(3s))\int_{3s}^\infty e^{-2a\tau}d\tau\Big) \\
&= (2\pi)^5\Big(\frac{e^{-6as}(1+6as(1+3as))}{8a^3}+(3s)^2\frac{e^{-6as}\log(3s)-\mathrm{Ei}(-6as)}{2a}-\frac{6se^{-6as}}{4a^2} \\
&\qquad -(\frac{9}{2}s^2+(3s)^2\log(3s))\frac{e^{-6as}}{2a}\Big).
\end{aligned}$$

Rearranging the terms we have

$$\|f_a\sigma_s * f_a\sigma_s * f_a\sigma_s\|^2_{L^2(\mathbb{R}^3)} = (2\pi)^5 e^{-6as}\Big(\frac{1}{8a^3}-(3s)^2\frac{\mathrm{Ei}(-6as)e^{6as}}{2a}-\frac{6s}{8a^2}\Big).$$

Then

$$\begin{aligned}
\|f_a\sigma_s * f_a\sigma_s * f_a\sigma_s\|^2_{L^2(\mathbb{R}^3)}\|f_a\|^{-6}_{L^2(\sigma_s)} &= (2\pi)^5\pi^{-3}a^3\Big(\frac{1}{8a^3}-(3s)^2\frac{\mathrm{Ei}(-6as)e^{6as}}{2a}-\frac{6s}{8a^2}\Big) \\
&= (2\pi)^2(1-6as-36a^2s^2e^{6as}\,\mathrm{Ei}(-6as)).
\end{aligned}$$

For the case of L^4,

$$\begin{aligned}
\|f_a\sigma_s * f_a\sigma_s\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}\times\mathbb{R}^2} e^{-2a\tau}\frac{(2\pi)^2}{\tau^2-|\xi|^2}\chi_{\{\tau\geqslant\sqrt{(2s)^2+|\xi|^2}\}}d\tau d\xi \\
&= (2\pi)^3\int_{2s}^\infty\int_0^{\sqrt{\tau^2-(2s)^2}} e^{-2a\tau}\frac{r}{\tau^2-r^2}drd\tau \\
&= (2\pi)^3\Big(\frac{e^{-4as}\log(2s)-\mathrm{Ei}(-4as)}{2a}-\log(2s)\frac{e^{-4as}}{2a}\Big) \\
&= -(2\pi)^3\frac{\mathrm{Ei}(-4as)}{2a}.
\end{aligned}$$

Then

$$\|f_a\sigma_s * f_a\sigma_s\|^2_{L^2(\mathbb{R}^3)}\|f_a\|^{-4}_{L^2(\sigma_s)} = -(2\pi)^3 a e^{4as}\frac{\mathrm{Ei}(-4as)}{2\pi^2}$$
$$= \frac{\pi}{s}(-4ase^{4as}\,\mathrm{Ei}(-4as)).$$ □

Our aim now is to give an alternative proof of Lemma 3.7.

Lemma 3.23. *Let $d=2$ and $s>0$. For $a>0$ let $f_a(y) := e^{-a\sqrt{s^2+|y|^2}}$, $y\in\mathbb{R}^2$. Then*

$$T_s f_a(x,t) = 2\pi\frac{e^{-s\sqrt{(a-it)^2+|x|^2}}}{\sqrt{(a-it)^2+|x|^2}}. \tag{3.52}$$

Proof. To compute the function $T_s f_a(x,t)$ we use polar coordinates where the polar axis is parallel to x, so that $x\cdot y = |x||y|\cos\theta$, with θ the polar angle.

$$T_s f_a(x,t) = \int e^{ix\cdot y}e^{it\sqrt{s^2+|y|^2}}e^{-a\sqrt{s^2+|y|^2}}\frac{dy}{\sqrt{s^2+|y|^2}}$$
$$= \int_0^\infty\int_0^{2\pi} e^{i|x|r\cos\theta}e^{it\sqrt{s^2+r^2}}e^{-a\sqrt{s^2+r^2}}\frac{r}{\sqrt{s^2+r^2}}d\theta dr.$$

Now from [33, pg. 26] we have that for $b\geqslant 0$

$$\int_0^{2\pi} e^{ib\cos\theta}d\theta = 2\pi J_0(b),$$

where J_0 is the Bessel function of the first kind of order zero. Thus

$$T_s f_a(x,t) = 2\pi\int_0^\infty e^{it\sqrt{s^2+r^2}}e^{-a\sqrt{s^2+r^2}}J_0(|x|r)\frac{r}{\sqrt{s^2+r^2}}\,dr$$
$$= 2\pi\int_0^\infty e^{-(a-it)\sqrt{s^2+r^2}}J_0(|x|r)\frac{r}{\sqrt{s^2+r^2}}\,dr$$
$$= 2\pi\int_s^\infty e^{-(a-it)u}J_0(|x|\sqrt{u^2-s^2})\,du$$

which is the Laplace transform of the function $J_0(|x|\sqrt{u^2-s^2})$. It is known ,[35, pg. 129], that the Laplace transform of $J_0(a\sqrt{u^2-b^2})$, for $a,b>0$, is given by

$$\int_b^\infty e^{-\lambda u}J_0(a\sqrt{u^2-b^2})du = \frac{e^{-b\sqrt{\lambda^2+a^2}}}{\sqrt{\lambda^2+a^2}}. \tag{3.53}$$

for λ with $\mathfrak{Re}(\lambda)>0$, and the branch of the square root is the one that is real in the positive real line. For a derivation of this formula we refer the reader to [49, pg. 416].

Using (3.53) we conclude that

$$T_s f_a(x,t) = 2\pi \frac{e^{-s\sqrt{(a-it)^2+|x|^2}}}{\sqrt{(a-it)^2+|x|^2}}.$$

and the lemma is proved. □

Alternative proof of Lemma 3.7. Let $a > 0$ be fixed and $f_s(y) = e^{-a\sqrt{s^2+|y|^2}}$. From Lemma 3.23 we have

$$T_s f_s(x,t) = 2\pi \frac{e^{-s\sqrt{(a-it)^2+|x|^2}}}{\sqrt{(a-it)^2+|x|^2}}.$$

We note that if we let $g(\xi,\tau) = e^{-a\tau}$, then $g\sigma_s * g\sigma_s * g\sigma_s(\xi,\tau) = g(\xi,\tau)\cdot\sigma_s*\sigma_s*\sigma_s(\xi,\tau) = f_s\sigma_s * f_s\sigma_s * f_s\sigma_s(\xi,\tau)$, and thus

$$(T_s f_s(x,t))^3 = (g\sigma_s * g\sigma_s * g\sigma_s)\widehat{\ }(x,t).$$

Applying the inverse Fourier transform gives

$$g(\xi,\tau)\cdot\sigma_s*\sigma_s*\sigma_s(\xi,\tau) = ((T_s f_s)^3)\check{\ }(\xi,\tau)$$

from where

$$\sigma_s*\sigma_s*\sigma_s(\xi,\tau) = e^{a\tau}((T_s f_s)^3)\check{\ }(\xi,\tau).$$

From the explicit expression for $T_s f_s$,

$$(T_s f_s(x,t))^3 = (2\pi)^3 \frac{e^{-3s\sqrt{(a-it)^2+|x|^2}}}{(\sqrt{(a-it)^2+|x|^2})^3}$$

and on the other hand, if $\mathfrak{Re}\, b > 0$

$$\int_{3s}^{\infty}\int_{\lambda}^{\infty} e^{-b\lambda'}d\lambda' d\lambda = \frac{e^{-3sb}}{b^2}$$

so then

$$(2\pi)^3\int_{3s}^{\infty}\int_{\lambda}^{\infty} \frac{e^{-\lambda'\sqrt{(a-it)^2+|x|^2}}}{\sqrt{(a-it)^2+|x|^2}} d\lambda' d\lambda = (2\pi)^3 \frac{e^{-3s\sqrt{(a-it)^2+|x|^2}}}{(\sqrt{(a-it)^2+|x|^2})^3} = (T_s f_s(x,t))^3$$

and thus

$$(2\pi)^2\int_{3s}^{\infty}\int_{\lambda}^{\infty} T_{\lambda'} f_{\lambda'}(x,t) d\lambda' d\lambda = (T_s f_s(x,t))^3. \tag{3.54}$$

Using the representation of $T_{\lambda'} f_{\lambda'}$ in terms of the Fourier transform gives

$$(2\pi)^2\int_{3s}^{\infty}\int_{\lambda}^{\infty} T_{\lambda'} f_{\lambda'}(x,t) d\lambda' d\lambda = (2\pi)^2\int_{3s}^{\infty}\int_{\lambda}^{\infty} \widehat{f_{\lambda'}\sigma_{\lambda'}}(x,t) d\lambda' d\lambda \tag{3.55}$$

Combining (3.54) and (3.55) and using Fubini's theorem gives

$$((T_s f_s)^3)\check{}(\xi,\tau) = (2\pi)^2 \int_{3s}^{\infty}\int_{\lambda}^{\infty} \delta(\tau - \sqrt{(\lambda')^2+|\xi|^2})e^{-a\sqrt{(\lambda')^2+|\xi|^2}} \frac{d\lambda' d\lambda}{\sqrt{(\lambda')^2+|\xi|^2}}$$
$$= (2\pi)^2 \int_{3s}^{\infty} \delta(\tau - \sqrt{(\lambda')^2+|\xi|^2})(\lambda' - 3s)e^{-a\sqrt{(\lambda')^2+|\xi|^2}} \frac{d\lambda'}{\sqrt{(\lambda')^2+|\xi|^2}}.$$

We now make the change of variables $v = \sqrt{(\lambda')^2+|\xi|^2}$, so $\frac{dv}{\sqrt{v^2-|\xi|^2}} = \frac{d\lambda'}{\sqrt{\lambda'^2+|\xi|^2}}$ to get

$$((T_s f_s)^3)\check{}(\xi,\tau) = (2\pi)^2 \int_{\sqrt{(3s)^2+|\xi|^2}}^{\infty} \delta(\tau - v)(\sqrt{v^2-|\xi|^2} - 3s)\frac{e^{-av}}{\sqrt{v^2-|\xi|^2}}dv$$
$$= (2\pi)^2\Big(1 - \frac{3s}{\sqrt{\tau^2-|\xi|^2}}\Big)e^{-a\tau}\chi_{\{\tau\geqslant\sqrt{(3s)^2+|\xi|^2}\}}.$$

It follows that

$$\sigma_s * \sigma_s * \sigma_s(\xi,\tau) = (2\pi)^2\Big(1 - \frac{3s}{\sqrt{\tau^2-|\xi|^2}}\Big)\chi_{\{\tau\geqslant\sqrt{(3s)^2+|\xi|^2}\}}.$$

The case of the double convolution can be done in the same way using $\int_{2s}^{\infty} e^{-b\lambda}d\lambda = \frac{e^{-2sb}}{b}$, for all b with $\mathfrak{Re}\, b > 0$. □

Remark 3.24. For any $n \geqslant 1$ and b with $\mathfrak{Re}\, b > 0$ we have

$$\int_{\lambda_{n+1}}^{\infty} \cdots \int_{\lambda_2}^{\infty} e^{-b\lambda_1} d\lambda_1 \dots d\lambda_n = \frac{e^{-b\lambda_{n+1}}}{b^n}$$

and thus we can compute, in the same way as before, the n^{th}-fold convolution $\sigma_s^{(*n)}$ for any $n \geqslant 1$.

3.9 Appendix 3: some explicit calculations for the case $d = 3$

Proof of Lemma 3.11. For the L^2 norm we have

$$\|f\|_{L^2(\sigma_s)}^2 = \int_{\mathbb{R}^3} e^{-2a\sqrt{s^2+|y|^2}} \frac{dy}{\sqrt{s^2+|y|^2}} = 4\pi \int_0^{\infty} e^{-2a\sqrt{s^2+r^2}} \frac{r^2\, dr}{\sqrt{s^2+r^2}}$$
$$= 4\pi \int_s^{\infty} e^{-2au}\sqrt{u^2-s^2}du = \frac{4\pi}{a^2}\int_{as}^{\infty} e^{-2x}\sqrt{x^2-(as)^2}dx.$$

Then

$$\lim_{a\to 0^+}\frac{a^2}{\pi}\|f\|^2_{L^2(\sigma_s)}=1.$$

Using the convolution form of the inequality, our goal is to show

$$\lim_{a\to 0^+}a^4\|f\sigma_s * f\sigma_s(\xi,\tau)\|^2_{L^2(\mathbb{R}^4)}=2\pi^3.$$

As in the proof of Lemma 3.21

$$\|f_a\sigma_s * f_a\sigma_s(\xi,\tau)\|^2_{L^2(\mathbb{R}^4)}=\int_{\mathbb{R}\times\mathbb{R}^3}e^{-2a\tau}(2\pi)^2\Big(1-\frac{4s^2}{\tau^2-|\xi|^2}\Big)\chi_{\{\tau\geqslant\sqrt{|\xi|^2+(2s)^2}\}}d\tau d\xi$$
$$=(2\pi)^2 4\pi\int_{2s}^{\infty}\int_0^{\sqrt{\tau^2-(2s)^2}}e^{-2a\tau}\Big(1-\frac{4s^2}{\tau^2-r^2}\Big)r^2 dr d\tau$$
$$=16\pi^3\int_{2s}^{\infty}e^{-2a\tau}\Big(\frac{1}{3}(\tau^2-(2s)^2)^{\frac{3}{2}}+4s^2((\tau^2-(2s)^2)^{\frac{1}{2}}-\tau\log(\frac{\tau+\sqrt{\tau^2-(2s)^2}}{2s})\Big)d\tau$$
$$=\frac{16\pi^3}{a}\int_{2as}^{\infty}e^{-2\tau}\Big(\frac{1}{3a^3}(\tau^2-(2as)^2)^{\frac{3}{2}}+\frac{4s^2}{a}((\tau^2-(2as)^2)^{\frac{1}{2}}-\frac{\tau}{a}\log(\frac{\tau+\sqrt{\tau^2-(2as)^2}}{2as})\Big)d\tau.$$

Multiplying by a^4 and taking the limit as $a\to 0^+$ gives

$$\lim_{a\to 0^+}a^4\|f_a\sigma_s * f_a\sigma_s(\xi,\tau)\|^2_{L^2(\mathbb{R}^4)}=\frac{16\pi^3}{3}\int_0^{\infty}e^{-2\tau}\tau^3 d\tau=2\pi^3.$$ □

Alternative proof of Lemma 3.8. The two fold convolution of σ_s with itself can be computed directly by using changes of variables. We will use the method of an earlier version of Foschi's paper [20], available on the arXiv. Given $\xi\in\mathbb{R}^3\setminus\{0\}$ we can use spherical coordinates adapted to ξ, that is, we can write $\eta\in\mathbb{R}^3$ as

$$\eta=(\rho\sin\theta\cos\varphi,\rho\sin\theta\sin\varphi,\rho\cos\theta)$$

where $\rho=|\eta|\geqslant 0$, $\theta\in[0,\pi]$ is the angle between η and ξ and $\varphi\in[0,2\pi]$ is an angular variable. Then

$$d\eta=\rho^2\sin\theta\, d\rho d\theta d\varphi.$$

If we let $\varsigma=|\xi-\eta|$, then

$$\varsigma^2=|\xi-\eta|^2=|\xi|^2+\rho^2-2|\xi|\rho\cos\theta$$

and changing variables from θ to ς gives $2\varsigma d\varsigma=2|\xi|\rho\sin\theta d\theta$. The Jacobian of the change of variables $\eta\mapsto(\rho,\varsigma,\varphi)$ is

$$d\eta=\frac{\rho\varsigma}{|\xi|}d\rho d\varsigma d\varphi.$$

The variables ρ and ς are subject to the conditions $|\rho-\varsigma|\leqslant|\xi|\leqslant\rho+\varsigma$. With this we can write

$$\begin{aligned}\sigma_s*\sigma_s(\tau,\xi) &= \int_{\mathbb{R}^3}\frac{\delta(\tau-\sqrt{s^2+|\xi-\eta|^2}-\sqrt{s^2+|\eta|^2})}{\sqrt{s^2+|\xi-\eta|^2}\sqrt{s^2-|\eta|^2}}d\eta\\ &= \frac{2\pi}{|\xi|}\int_{\substack{|\rho-\varsigma|\leqslant|\xi|\\ \rho+\varsigma\geqslant|\xi|}}\frac{\delta(\tau-\sqrt{s^2+\varsigma^2}-\sqrt{s^2+\rho^2})}{\sqrt{s^2+\varsigma^2}\sqrt{s^2+\rho^2}}\rho\varsigma\,d\rho d\varsigma\\ &= \frac{2\pi}{|\xi|}\int_{R_s}\delta(\tau-u-v)\,du\,dv,\end{aligned}$$

where $u=\sqrt{s^2+\rho^2}$, $v=\sqrt{s^2+\varsigma^2}$ and R_s is the image of the region $\{(\rho,\varsigma)\in\mathbb{R}^2_+ : |\rho-\varsigma|\leqslant|\xi|\,,\rho+\varsigma\geqslant|\xi|\}$ under the transformation $(\rho,\varsigma)\mapsto(u,v)$. Using the change of variables $a=u-v$, $b=u+v$, so that $2du\,dv=da\,db$, we get

$$\sigma_s*\sigma_s(\tau,\xi)=\frac{\pi}{|\xi|}\int_{\tilde{R}_s}\delta(\tau-b)\,db\,da.$$

where $\tilde{R}_s$ is the image of R_s under the map $(u,v)\mapsto(a,b)$. Now it is not hard to see that $\tilde{R}_s$ is contained in the region $\{(a,b) : |a|\leqslant|\xi|,\, b\geqslant\sqrt{(2s)^2+|\xi|^2}\}$. Computing the region $\tilde{R}_s$ gives the explicit formula for $\sigma_s*\sigma_s$,

$$\sigma_s*\sigma_s(\xi,\tau)=\frac{2\pi}{|\xi|}|\tau-2u(\xi,\tau)|\chi_{\{\tau\geqslant\sqrt{|\xi|^2+(2s)^2}\}},$$

where $u(\xi,\tau)$ is implicitly defined by the equation $\tau=u(\xi,\tau)+((\sqrt{u(\xi,\tau)^2-s^2}-|\xi|)^2+s^2)^{1/2}$ and $u(\xi,\tau)\geqslant s$. Note that simple algebraic manipulation shows that

$$(\tau-2u(\xi,\tau))^2=|\xi|^2\Big(1-\frac{4s^2}{\tau^2-|\xi|^2}\Big).$$

This implies

$$\sigma_s*\sigma_s(\tau,\xi)=2\pi\Big(1-\frac{4s^2}{\tau^2-|\xi|^2}\Big)^{1/2}\chi_{\{\tau\geqslant\sqrt{|\xi|^2+(2s)^2}\}}.\qquad\square$$

Chapter 4

Gaussians rarely extremize adjoint Fourier restriction inequalities for paraboloids

This chapter is joint work with Michael Christ.

4.1 Introduction

Let $\mathbb{P}^{d-1}$ be the paraboloid in $\mathbb{R}^d$,

$$\mathbb{P}^{d-1} = \{(y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : y_d = \tfrac{1}{2}|y'|^2\}.$$

Equip $\mathbb{P}^{d-1}$ with the appropriately dilation-invariant measure σ on $\mathbb{R}^d$ defined by

$$\int_{\mathbb{R}^d} f(y', y_d)\, d\sigma(y', y_d) = \int_{\mathbb{R}^{d-1}} f(y', \tfrac{1}{2}|y'|^2)\, dy',$$

where dy' denotes Lebesgue measure on $\mathbb{R}^{d-1}$.

The adjoint Fourier restriction inequality states that for a certain range of exponents p,

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{P}^{d-1},\sigma)} \tag{4.1}$$

for some finite constant $C = C(p, d)$, where $q = q(p)$ is specified by

$$q^{-1} = \frac{d-1}{d+1}(1 - p^{-1}). \tag{4.2}$$

This inequality is known to be valid for $1 \leq p \leq p_0$ for a certain exponent $p_0 > 2$ depending on d, and is conjectured to be valid for all $p \in [1, \frac{2d}{d-1})$.

The case $p = 2$ is of special interest, since it gives a space-time upper bound for the solution of a linear Schrödinger equation with arbitrary initial data in the natural class $L^2(\mathbb{R}^{d-1})$. While the cases $p \neq 2$ also give such bounds, they are expressed in terms of less natural norms on initial data.

The more general Strichartz inequalities (4.3) are phrased in terms of mixed norms. For simplicity we restrict our discussion of mixed norm inequalities to the case $p = 2$. For $\mathbb{R}^d$, adopt coordinates $(x,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. For $r,q \in [1,\infty)$, for $u : \mathbb{R}^d_{x,t} \to \mathbb{C}$, define $\|u\|_{L^r_t L^q_x} = (\int(\int |u(x,t)|^q \, dx)^{r/q} \, dt)^{1/r}$. The Strichartz inequalities state [48] that

$$\|\widehat{f\sigma}\|_{L^r_t L^q_x} \leq C\|f\|_{L^2(\mathbb{P}^{d-1},\sigma)} \tag{4.3}$$

for all r,q,d satisfying $q,r \geq 2$ and

$$\frac{2}{r} + \frac{d-1}{q} = \frac{d-1}{2} \tag{4.4}$$

with the endpoint $q = \infty$ excluded for $d = 3$.

By a radial Gaussian we mean a function $f : \mathbb{P}^{d-1} \to \mathbb{C}$ of the form $f(y, |y|^2/2) = c\exp(-z|y-y_0|^2 + y \cdot v)$ for $y \in \mathbb{R}^{d-1}$, where $0 \neq c \in \mathbb{C}$, $y_0 \in \mathbb{R}^{d-1}$, and $v \in \mathbb{C}^{d-1}$ are arbitrary, and $z \in \mathbb{C}$ has positive real part. Radial Gaussians on $\mathbb{P}^{d-1}$ are simply restrictions to $\mathbb{P}^{d-1}$ of functions $F(x) = e^{x\cdot w + c}$ where $w = (w', w_d) \in \mathbb{C}^d$ satisfies $\mathfrak{Re}(w_d) < 0$.

Radial Gaussians extremize [20] inequality (4.1) for $p = 2$ in the two lowest-dimensional cases, $d = 2$ and $d = 3$. More than one proof of these facts is known. It is natural to ask whether these are isolated facts, or whether Gaussians appear as extremizers more generally. Additional motivation is provided by recent work of Christ and Shao [9],[10], who have shown the existence of extremizers for the corresponding inequalities for the spheres S^1 and S^2. Their analysis relies on specific information about extremizers for the paraboloid, which can be read off from explicit calculations for Gaussians, but which has not been shown to follow more directly from the inequality itself. If Gaussians were known to be extremizers for $\mathbb{P}^{d-1}$, it should then be possible to establish the existence of extremizers for S^{d-1}.

In this paper, we discuss a related question: Are radial Gaussians critical points for the nonlinear functionals associated to inequalities (4.1) and (4.3)? These functionals are defined as follows.

$$\Phi(f) = \Phi_{p,d}(f) = \frac{\|\widehat{f\sigma}\|_q^q}{\|f\|_p^q}, \tag{4.5}$$

where $q = q(p,d)$ is defined by (4.2) and

$$\Psi(f) = \Psi_{q,r,d}(f) = \frac{\|\widehat{f\sigma}\|^r_{L^r_t L^q_x}}{\|f\|_2^r}. \tag{4.6}$$

Φ is defined for all $0 \neq f \in L^p(\mathbb{P}^{d-1},\sigma)$, while Ψ is defined for all $0 \neq f \in L^2(\mathbb{P}^{d-1},\sigma)$. (4.1) and (4.3) guarantee that $\Phi_{p,d}, \Psi_{q,r,d}$ are bounded functionals, for the ranges of parameters indicated.

By a critical point of Φ is of course meant a function $0 \neq f \in L^p(\mathbb{P}^{d-1})$ such that for any $g \in L^p(\mathbb{P}^{d-1})$,

$$\Phi(f+\varepsilon g) = \Phi(f) + o(|\varepsilon|) \text{ as } \varepsilon \to 0. \tag{4.7}$$

Here $\varepsilon \in \mathbb{C}$. For $d \geq 3$, there is a range of exponents for which $\Phi_{p,d}$ is conjectured to be bounded but for which this is not known [2],[28]. But $\Phi_{p,d}(f)$ is well-defined and finite for any Schwartz function, so we may still ask whether (4.7) holds whenever f is a radial Gaussian and g is an arbitrary Schwartz function. This gives a definition of critical point which is equivalent whenever the functional is bounded.

It is a simple consequence of symmetries of these functionals that for fixed p, q, r, d, one radial Gaussian is a critical point if and only if all are critical points. Our main result is as follows.

Theorem 4.1. *Let $d \geq 2$, let $1 < p < 2d/(d-1)$, and set $q = q(p,d)$. Radial Gaussians are critical points for the $L^p \to L^q$ adjoint Fourier restriction inequalities if and only if $p = 2$. Radial Gaussians are critical points for the $L^2 \to L^r_t L^q_x$ Strichartz inequalities for all admissible pairs $(r,q) \in (1,\infty)^2$.*

For spheres S^{d-1}, the situation is different; constant functions are critical points for the analogues of both functionals.

4.2 Euler-Lagrange equations

We will show that extremizers must satisfy a certain Euler-Lagrange equation, then check by explicit calculation whether radial Gaussians satisfy this equation. In this section we formulate and justify the Euler-Lagrange equations.

Let $g^\vee$ denote the inverse Fourier transform of g.

Proposition 4.2. *Let $d \geq 2$, let $1 < p < 2d/(d-1)$, and set $q = q(p,d)$. A complex-valued function $f \in L^p(\mathbb{P}^{d-1})$ with nonzero $L^p(\mathbb{P}^{d-1})$ norm is a critical point of $\Phi_{p,d}$ if and only if there exists $\lambda > 0$ such that f satisfies the equation*

$$\left(|\widehat{f\sigma}|^{q-2}\widehat{f\sigma}\right)^\vee\Big|_{\mathbb{P}^{d-1}} = \lambda |f|^{p-2} f \text{ almost everywhere on } \mathbb{P}^{d-1}. \tag{4.8}$$

For S^{d-1}, the same equation likewise characterizes critical points, except of course that the restriction on the left-hand side is to S^{d-1}, and q can take on any value in $[q(p,d),\infty)$.

λ is determined by $\|f\|_p$ and $\Phi(f)$; multiply both sides of (4.8) by $\bar{f}$ and integrate with respect to σ.

Both exponents $q-1, p-1$ are strictly positive, and $q-2 > 0$. Moreover, since $f \in L^p$, $\widehat{f\sigma} \in L^q$, and therefore $|\widehat{f\sigma}|^{q-2}\widehat{f\sigma} \in L^{q/(q-1)}(\mathbb{R}^d)$. Therefore by the Fourier restriction inequality, the restriction to $\mathbb{P}^{d-1}$ of $\left(|\widehat{f\sigma}|^{q-2}\widehat{f\sigma}\right)^\vee$ is a well-defined element of $L^{p/(p-1)}(\mathbb{P}^{d-1},\sigma)$. Thus the left-hand side of (4.8) is well-defined for any $f \in L^p(\mathbb{P}^{d-1},\sigma)$.

Proposition 4.3. *Let (q, r, d) satisfy the necessary and sufficient conditions (4.4) for the Strichartz inequalities. A nonzero complex-valued function $f \in L^2(\mathbb{P}^{d-1})$ is a critical point of $\Psi_{p,d}$ if and only if there exists $\lambda > 0$ such that f satisfies the equation*

$$\Big(\widehat{f\sigma}(x,t)|\widehat{f\sigma}(x,t)|^{q-2}\|\widehat{f\sigma}(\cdot,t)\|_{L^q_x}^{r-q}\Big)^{\vee} = \lambda f \text{ a.e. on } \mathbb{P}^{d-1}. \tag{4.9}$$

The Euler-Lagrange equation for S^{d-1} takes the corresponding form. It follows at once that constant functions are critical points for S^{d-1}, because $\widehat{\sigma}|\widehat{\sigma}|^{q-2}$ is a radial function, the inverse Fourier transform of any radial function is radial, and the restriction of any radial function to S^{d-1} is constant.

Propositions 4.2 and 4.3 will follow from the following elementary fact.

Lemma 4.4. *For any exponents $q, r \in (1, \infty)$, there exists $\gamma > 1$ with the following property. Let $F, G \in L^r_t L^q_x$ of some measure space(s), and assume that $\|F\|_{L^r_t L^q_x} \neq 0$. Let $z \in \mathbb{C}$ be a small parameter. Then*

$$\|F + zG\|_{L^r_t L^q_x}^r = \|F\|_{L^r_t L^q_x}^r + r\iint \|F_t\|^{r-q}|F(x,t)|^q\,\mathfrak{Re}(zG(x,t)/F(x,t))\,dx\,dt + O\big(|z|^\gamma\big) \tag{4.10}$$

as $z \to 0$.

Here $F_t(x) = F(x,t)$ and $\|F_t\|^q = \int |F(x,t)|^q\,dx$. The constant implicit in the remainder term $O\big(|z|^\gamma\big)$ does depend on the norms of F, G. It is a consequence of Hölder's inequality that the double integral is absolutely convergent.

An immediate consequence is:

Proposition 4.5. *Let T be a bounded linear operator from L^p to $L^r_t L^q_x$ where $p, q, r \in (1, \infty)$. For $0 \neq f \in L^p$ define $\Phi(f) = \|Tf\|_{L^rL^q}^r / \|f\|_p^r$. Then any critical point f of Φ satisfies the equation*

$$T^*\Big(Tf(x,t)\ |Tf(x,t)|^{q-2}\ \|Tf(\cdot,t)\|_{L^q_x}^{r-q}\Big) = \lambda|f|^{p-2}f \tag{4.11}$$

for some $\lambda \in [0, \infty)$.

Again, it is a consequence of Hölder's inequality that the indicated function belongs to the domain $L^{r'}_t L^{q'}_x$ of the transposed operator T^*.

Proof of Lemma 4.4. Let $\varepsilon > 0$ be a small exponent, to be chosen below. Assume throughout the discussion that $|z| \leq 1$. Write $F_t(x) = F(x,t)$, $G_t(x) = G(x,t)$, $\|F_t\| = \|F(\cdot,t)\|_{L^q_x}$, and $\|G_t\| = \|G(\cdot,t)\|_{L^q_x}$. Define

$$\Omega_t = \{x \in \mathbb{R}^{d-1} : |zG(x,t)| \leq |z|^\varepsilon |F(x,t)| \text{ and } F(x,t) \neq 0\} \tag{4.12}$$

$$\omega = \{t : \|F_t\| \neq 0 \text{ and } |z|\,\|G_t\| \leq |z|^\varepsilon \|F_t\|\}. \tag{4.13}$$

Fix $\rho \in (0, q-1)$. For $x \in \Omega_t$, expand

$$|1 + zG(x,t)/F(x,t)|^q = 1 + q\,\mathfrak{Re}(zG(x,t)/F(x,t)) + O(|zG(x,t)/F(x,t)|^{1+\rho})$$

to obtain

$$\int_{\Omega_t} |(F+zG)(x,t)|^q\,dx = \int_{\Omega_t} |F(x,t)|^q\,dx + q\int_{\Omega_t} |F(x,t)|^q\,\mathfrak{Re}(zG(x,t)/F(x,t))\,dx \\ + O\big(|z|^{1+\rho}\|F_t\|^{q-1-\rho}\|G_t\|^{1+\rho}\big).$$

The contribution of $\mathbb{R}^{d-1} \setminus \Omega_t$ is negligible, because of the following three bounds:

$$\int_{\mathbb{R}^{d-1}\setminus\Omega_t} |F(x,t)|^q\,dx \le \int_{\mathbb{R}^{d-1}\setminus\Omega_t} |z|^{(1-\varepsilon)q}|G(x,t)|^q\,dx = |z|^{(1-\varepsilon)q}\|G_t\|^q; \tag{4.14}$$

similarly

$$\int_{\mathbb{R}^{d-1}\setminus\Omega_t} |F(x,t)|^q|\,\mathfrak{Re}(zG(x,t)/F(x,t))|\,dx \le C|z|^{q-C\varepsilon}\|G_t\|^q; \tag{4.15}$$

and

$$\begin{aligned}\int_{\mathbb{R}^{d-1}\setminus\Omega_t} |F(x,t)+zG(x,t)|^q\,dx &\le 2^q\int_{\mathbb{R}^{d-1}\setminus\Omega_t} \big(|F(x,t)|^q + |z|^q|G(x,t)|^q\big)\,dx \\ &\le C|z|^{q-C\varepsilon}\|G_t\|^q.\end{aligned} \tag{4.16}$$

Define

$$H(t) = \|F_t\| + \|G_t\|.$$

Then $\int_{\mathbb{R}} H(t)^r\,dt < \infty$. We have shown that if $\varepsilon > 0$ is chosen to be sufficiently small, depending on q, then

$$\begin{aligned}&\int_{\mathbb{R}^{d-1}} |F(x,t)+zG(x,t)|^q\,dx \\ &= \|F_t\|^q + q\int_{\mathbb{R}^{d-1}} |F(x,t)|^q\,\mathfrak{Re}(zG(x,t)/F(x,t))\,dx + O\big(|z|^{1+\sigma}\|G_t\|^{1+\sigma}H(t)^{q-1-\sigma}\big)\end{aligned} \tag{4.17}$$

for all sufficiently small $\sigma > 0$.

Suppose that $t \in \omega$. For any $|z| \ll 1$,

$$|z|^{1+\sigma}\|G_t\|^{1+\sigma}\|F_t\|^{-q}H(t)^{q-1-\sigma} \le |z|^{(1+\sigma)\varepsilon}\|F_t\|^{1+\sigma-q}H(t)^{q-1-\sigma} = O(|z|^{(1+\sigma)\varepsilon}) \ll 1.$$

Similarly, by Hölder's inequality,

$$\|F_t\|^{-q}\int_{\mathbb{R}^{d-1}} |F(x,t)|^q\,\mathfrak{Re}(zG(x,t)/F(x,t))\,dx = O\big(\min(|z|\|G_t\|\|F_t\|^{-1}, |z|^{\varepsilon})\big) \ll 1. \tag{4.18}$$

Therefore for all sufficiently small $z \in \mathbb{C}$,

$$\begin{aligned}\|F_t\|^{-r}(\int_{\mathbb{R}^{d-1}} |F + zG|^q \, dx)^{r/q} &= \Big(1 + q\|F_t\|^{-q} \int_{\mathbb{R}^{d-1}} |F(x,t)|^q \, \mathfrak{Re}(zG(x,t)/F(x,t)) \, dx \\ &\qquad + O(|z|^{1+\sigma}\|G_t\|^{1+\sigma}\|F_t\|^{-q}H(t)^{q-1-\sigma})\Big)^{r/q} \\ &= 1 + r\|F_t\|^{-q} \int_{\mathbb{R}^{d-1}} |F(x,t)|^q \, \mathfrak{Re}(zG(x,t)/F(x,t)) \, dx \\ &\quad + O(|z|^{1+\sigma}\|G_t\|^{1+\sigma}H(t)^{-1-\sigma}) \\ &\quad + O\big(\|F_t\|^{-q} \int_{\mathbb{R}^{d-1}} |F(x,t)|^q |zG(x,t)/F(x,t)| \, dx\big)^2 \\ &= 1 + r\|F_t\|^{-q} \int_{\mathbb{R}^{d-1}} |F(x,t)|^q \, \mathfrak{Re}(zG(x,t)/F(x,t)) \, dx \\ &\quad + O(|z|^{1+\sigma}\|G_t\|^{1+\sigma}\|F_t\|^{-1-\sigma}),\end{aligned}$$

provided that $\sigma < 1$, using (4.18) to deduce the final line. Provided that σ is chosen to satisfy $\sigma < \min(r-1, 1)$, an application of Hölder's inequality now yields

$$\begin{aligned}&\int_\omega (\int_{\mathbb{R}^{d-1}} |F + zG|^q \, dx)^{r/q} \\ &\quad = \int_\omega \|F_t\|^r + r\int_\omega \|F_t\|^{r-q} \int_{\mathbb{R}^{d-1}} |F(x,t)|^q \, \mathfrak{Re}(zG(x,t)/F(x,t)) \, dx + O\big(|z|^{1+\sigma}\big).\end{aligned} \tag{4.19}$$

It remains to verify that the contribution of $\mathbb{R} \setminus \omega$ is negligible. If $t \notin \omega$ then $\|F_t\| \le |z|^{1-\varepsilon}\|G_t\|$, so

$$(\int_{\mathbb{R}^{d-1}} |F(x,t) + zG(x,t)|^q \, dx)^{1/q} \le C|z|^{1-\varepsilon}\|G_t\| \tag{4.20}$$

and consequently

$$\int_{\mathbb{R}\setminus\omega} (\int_{\mathbb{R}^{d-1}} |F(x,t) + zG(x,t)|^q \, dx)^{q/r} \, dt \le C|z|^{(1-\varepsilon)r} \int_{\mathbb{R}\setminus\omega} \|G_t\|^r \, dt = O(|z|^{(1-\varepsilon)r}); \tag{4.21}$$

in the same way,

$$\int_{\mathbb{R}\setminus\omega} \|F_t\|^r \, dt \le \int_{\mathbb{R}\setminus\omega} |z|^{(1-\varepsilon)r}\|G_t\|^r \, dt = O(|z|^{(1-\varepsilon)r}). \tag{4.22}$$

Finally

$$\begin{aligned}\int_{\mathbb{R}\setminus\omega} \|F_t\|^{r-q} \int_{\mathbb{R}^{d-1}} |F(x,t)|^q |\, \mathfrak{Re}(zG(x,t)/F(x,t))| \, dx \, dt &\le |z| \int_{\mathbb{R}\setminus\omega} \|F_t\|^{r-1}\|G_t\| \, dt \\ &\le |z|^{r-C\varepsilon} \int_{\mathbb{R}\setminus\omega} \|G_t\|^r \, dt \\ &= O(|z|^{r-C\varepsilon}).\end{aligned} \tag{4.23}$$

In conjunction with (4.19), the three bounds (4.21),(4.22),(4.23) complete the proof once ε is chosen to be sufficiently small. □

4.3 The case $p = 2$ and $q = r$

Functions in $L^p(\mathbb{P}^{d-1}, \sigma)$ may be identified with functions in $L^p(\mathbb{R}^{d-1})$ via the correspondence $f(y, |y|^2/2) = g(y)$ for $y \in \mathbb{R}^{d-1}$. We will often make this identification without further comment. Thus a function $g \in L^p(\mathbb{R}^{d-1})$ is said to satisfy the equation (4.8), if the corresponding function $f(y, |y|^2/2) = g(y)$ does so. We will sometimes write $g\sigma$, for $g \in L^p(\mathbb{R}^{d-1})$, as shorthand for $f\sigma$, where f, g corresponding in this way.

Lemma 4.6. *Fix p, d and let $q = q(p, d)$. Suppose that $f \in L^2(\mathbb{R}^{d-1})$ satisfies the Euler-Lagrange equation (4.8). Then so does the function $y' \mapsto \rho f(rAy' + v)e^{iy'\cdot w}$ for any $r > 0$, $\rho \in \mathbb{C} \setminus \{0\}$, $A \in O(d-1)$, and $v, w \in \mathbb{R}^{d-1}$.*

The proof is left to the reader. To prove our main result, it suffices to consider henceforth the radial Gaussian $f(y) = e^{-|y|^2/2}$, $y \in \mathbb{R}^{d-1}$, for which $\widehat{f\sigma}(x,t) = u(x,t)$ takes the form

$$\begin{aligned} u(x,t) = \widehat{f\sigma}(x,t) &= \int e^{-ix\cdot y} e^{-it|y|^2/2} e^{-|y|^2/2}\, dy \\ &= (2\pi)^{(d-1)/2}(1+it)^{-(d-1)/2} e^{-|x|^2/2(1+it)}. \end{aligned} \tag{4.24}$$

Throughout the discussion we will encounter real powers of $1 \pm it$ and of $q - 1 - it$. These are always interpreted as the corresponding powers of $\log(1 \pm it)$ and of $\log(q - 1 - it)$ respectively, where the branch of log is chosen so that $\log(1) = 0$ and $\log(1 + it)$ is analytic in the complement of the ray $\{is : s \in [1, \infty)\}$, while $\log(1 - it)$ and $\log(q - 1 - it)$ are both analytic in the complement of the ray $\{-is : s \in [1, \infty)\}$, with values 0 and $\log(q - 1)$ respectively when $t = 0$. Thus

$$\begin{aligned} |u|^{q-2}u &= (2\pi)^{(q-1)(d-1)/2}(1+t^2)^{-(d-1)(q-2)/4}(1+it)^{-(d-1)/2} e^{-|x|^2\left(\frac{1-it}{1+t^2} + \frac{q-2}{1+t^2}\right)/2} \\ &= (2\pi)^{(q-1)(d-1)/2}(1+t^2)^{-(d-1)(q-2)/4}(1+it)^{-(d-1)/2} e^{-|x|^2(q-1-it)/2(1+t^2)}. \end{aligned}$$

We now begin to analyze the inverse Fourier transform $\iint e^{ix\cdot y} e^{\frac{1}{2}it|y|^2} |u(x,t)|^{q-2} u(x,t)\, dx\, dt$ by calculating the integral with respect to $x \in \mathbb{R}^{d-1}$.

$$\int_{\mathbb{R}^{d-1}} e^{ix\cdot y} e^{-\frac{1}{2}|x|^2 \frac{q-1-it}{1+t^2}}\, dx = (2\pi)^{(d-1)/2}\left(\frac{q-1-it}{1+t^2}\right)^{-(d-1)/2} e^{-\frac{1}{2}|y|^2 \frac{1+t^2}{q-1-it}}.$$

Thus

$$\begin{aligned} \left(|u|^{q-2}u\right)^{\vee}(y, |y|^2/2) &= (2\pi)^{q(d-1)/2} \\ &\int_{\mathbb{R}} e^{it|y|^2/2}(1+t^2)^{-(d-1)(q-2)/4}(1+it)^{-(d-1)/2}\left(\frac{q-1-it}{1+t^2}\right)^{-(d-1)/2} e^{-\frac{1}{2}|y|^2 \frac{1+t^2}{q-1-it}}\, dt \end{aligned}$$

which simplifies to

$$(2\pi)^{q(d-1)/2}\int_{\mathbb{R}}(1+it)^{-(d-1)(q-2)/4}(1-it)^{-\frac{1}{4}(d-1)(q-2)+\frac{1}{2}(d-1)}(q-1-it)^{-(d-1)/2}e^{\frac{1}{2}|y|^2\left(it-\frac{1+t^2}{q-1-it}\right)}\,dt. \tag{4.25}$$

Consider first the case $p=2$. Then $q=2(d+1)/(d-1)=2+\frac{4}{d-1}$, so $(d-1)(q-2)/4=1$ and the integral with respect to $t\in\mathbb{R}$ becomes

$$(2\pi)^{q(d-1)/2}\int_{\mathbb{R}}(1+it)^{-1}(1-it)^{(d-3)/2}(q-1-it)^{-(d-1)/2}e^{a\left(it-\frac{1+t^2}{q-1-it}\right)}\,dt$$

where $a=|y|^2/2$. This may be evaluated by deformation of the contour of integration through the upper half-plane in $\mathbb{C}$. In the upper half-plane, the integrand is meromorphic with a single pole at $t=i$. Therefore the integral equals

$$\begin{aligned}(2\pi i)(2\pi)^{q(d-1)/2}i^{-1}2^{(d-3)/2}q^{-(d-1)/2}e^{a\left(i\cdot i-\frac{1+i^2}{q-1-i\cdot i}\right)} &= (2\pi)^{d+2}2^{(d-3)/2}q^{-(d-1)/2}e^{-a}\\ &= (2\pi)^{d+2}2^{(d-3)/2}q^{-(d-1)/2}e^{-|y|^2/2} = (2\pi)^{d+2}2^{(d-3)/2}q^{-(d-1)/2}f(y).\end{aligned}$$

Since $p=2$, $f\equiv|f|^{p-2}f$ for $p=2$ and thus the Euler-Lagrange equation (4.8) is indeed satisfied.

Now consider the general mixed-norm case. The Euler-Lagrange equation is modified via the factor $\|\widehat{f\sigma}(\cdot,t)\|_{L^q_x}^{r-q}$. By (4.24),

$$\|\widehat{f\sigma}(\cdot,t)\|_{L^q_x}^{r-q}=\frac{(2\pi)^{\frac{1}{2}(r-q)(d-1)(1+1/q)}}{q^{(d-1)(r-q)/2q}}(1+t^2)^{-\frac{1}{4q}(d-1)(r-q)(q-2)}.$$

Set

$$J(a)=\int_{\mathbb{R}}(1+it)^{-\frac{r}{4q}(d-1)(q-2)}(1-it)^{-\frac{r}{4q}(d-1)(q-2)+\frac{1}{2}(d-1)}\cdot(q-1-it)^{-\frac{1}{2}(d-1)}e^{a(it-\frac{1+t^2}{q-1-it})}dt. \tag{4.26}$$

Since $p=2$, the Euler-Lagrange equation (4.9) is satisfied if and only if $J(a)$ is a constant multiple of e^{-a}. Using the equation (4.4) which relates q to r, $J(a)$ simplifies to

$$J(a)=\int_{\mathbb{R}}(1+it)^{-1}(1-it)^{\frac{1}{2}(d-3)}(q-1-it)^{-\frac{1}{2}(d-1)}e^{a(it-\frac{1+t^2}{q-1-it})}dt,$$

which was shown above to be a constant multiple of e^{-a}.

4.4 The case $p \neq 2$

We will use the following simple lemma.

Lemma 4.7. *Let $H(t) : \mathbb{C} \to \mathbb{C}$ be holomorphic on the upper half plane $\{\mathfrak{Im}(t) > 0\}$ and continuous in its closure, and suppose that $|(1+it)^\gamma H(t)| = O(|t|^{-1-\delta})$ as $|t| \to \infty$, for some $\delta > 0$. Then for $\gamma > -1$,*

$$\int_{\mathbb{R}} (1+it)^\gamma H(t)dt = -2\sin(\gamma\pi)\int_0^\infty y^\gamma H(i+iy)dy,$$

and for $\gamma = -1$

$$\int_{\mathbb{R}} (1+it)^\gamma H(t)dt = 2\pi H(i).$$

This is obtained via contour integration in the region $\{\mathfrak{Im}(t) \geqslant 0\} \setminus \{iy : y \in [1,\infty)\}$. As a consequence of Lemma 4.7 we have the following: Suppose that H is real-valued, nonnegative when restricted to the imaginary axis, and satisfies $H(i) > 0$. If $\gamma \geqslant -1$, then $\int_{\mathbb{R}}(1+it)^\gamma H(t)dt = 0$ if and only if $\gamma \geqslant 2$ is an integer.

Define $I : [0,\infty) \to \mathbb{C}$ by

$$I(a) = \int_{\mathbb{R}} (1+it)^{-\frac{1}{4}(d-1)(q-2)}(1-it)^{-\frac{1}{4}(d-1)(q-2)+\frac{1}{2}(d-1)}(q-1-it)^{-\frac{1}{2}(d-1)}e^{a(it-\frac{1+t^2}{q-1-it})}dt \quad (4.27)$$

where $d \geqslant 2$, and $q > \frac{2d}{d-1}$ is defined by (4.2). The integrand is

$$O(t^{-\frac{(d-1)(q-2)}{2}}e^{-a(q-1)\frac{1+t^2}{(q-1)^2+t^2}})$$

and since $q > 2$ and $\frac{(d-1)(q-2)}{2} > 1$, it belongs to $L^1(\mathbb{R})$ for all $a \geqslant 0$. We note that $I(\frac{1}{2}|y|)$ equals the expression in (4.25) up to constant.

Our goal is to demonstrate:

Lemma 4.8. *As a function of $a \in [0,\infty)$, the function I is a constant multiple of $e^{-(p-1)a}$ only if $p = 2$.*

Proof. **Case 1** : $p < 2$. Consider

$$e^a I(a) = \int_{\mathbb{R}} (1+it)^{-\frac{1}{4}(d-1)(q-2)}(1-it)^{-\frac{1}{4}(d-1)(q-2)+\frac{1}{2}(d-1)}(q-1-it)^{-\frac{1}{2}(d-1)}e^{a\frac{(q-2)(1+it)}{q-1-it}}dt.$$

Expanding the exponential in power series and interchanging integral and sum gives

$$e^a I(a) = \sum_{k=0}^\infty \frac{a^k}{k!}(q-2)^k I_k,$$

where

$$I_k = \int_{\mathbb{R}} (1+it)^{k-\frac{1}{4}(d-1)(q-2)} H_k(t)dt,$$

with

$$H_k(t) = (1-it)^{-\frac{1}{4}(d-1)(q-2)+\frac{1}{2}(d-1)}(q-1-it)^{-k-\frac{1}{2}(d-1)}.$$

H_k satisfies the hypothesis of Lemma 4.7, and $H_k(iy) > 0$ for all $y \geqslant 0$.

Now $e^a I(a)$ is a constant multiple of $e^{-(p-2)a}$ if and only if there exists $c \in \mathbb{C}$ such that for all $k \geqslant 0$,

$$I_k = c\Big(\frac{2-p}{q-2}\Big)^k. \tag{4.28}$$

Let $k_0 = \lceil (d-1)(q-2)/4 \rceil$, the smallest integer $\geqslant (d-1)(q-2)/4$, and consider any $k \geqslant k_0$. By Lemma 4.7,

$$I_k = -2\sin(\alpha_k \pi) \int_0^\infty y^{k-\frac{1}{4}(d-1)(q-2)} H_k(i+iy)dy$$

where $\alpha_k = k-(d-1)(q-2)/4$.

Suppose first that p is such that $(d-1)(q-2)/4$ is not an integer, so $I_k \neq 0$. Now $\sin(\alpha_{k+1}\pi) = -\sin(\alpha_k\pi)$ and thus I_k is alternating while $c(2-p)^k(q-2)^{-k}$ is not. If p is such that $(d-1)(q-2)/4$ is an integer (necessarily $\geqslant 2$ as $p \neq 2$) and (4.28) holds we get that $c = 0$ since $J_k = 0$ for $k \geqslant k_0$. On the other hand, $k_0 - 1 \geqslant 1$ and $I_{k_0-1} \neq 0$, for

$$I_{k_0-1} = \pi 2^{-\frac{1}{4}(d-1)(q-2)+\frac{1}{2}(d-1)+1} q^{-k_0-\frac{1}{2}(d-1)+1},$$

by Lemma 4.7.

Case 2: $p > 2$. It is now convenient to work with

$$e^{(p-1)a}I(a) = \int_{\mathbb{R}} (1+it)^{-\frac{1}{4}(d-1)(q-2)}(1-it)^{-\frac{1}{4}(d-1)(q-2)+\frac{1}{2}(d-1)}(q-1-it)^{-\frac{1}{2}(d-1)} e^{a(p-1+it-\frac{1+t^2}{q-1-it})}dt;$$

we need to show that this expression is not constant, as a function of $a \in [0,\infty)$. For $2 < p \leqslant 2d/(d-1)$, $\frac{(d-1)(q-2)}{4} = \frac{d-1}{4(p-1)} - \frac{d-3}{4}$ lies in $[1/2, 1)$. Therefore the integrand has an integrable singularity at $t = i$, so we may expand the exponential factor in the integrand in power series to obtain an analogue of I_k:

$$e^{(p-1)a}I(a) = \sum_{k=0}^\infty \frac{a^k}{k!} I'_k$$

where

$$I'_k = \int_{\mathbb{R}} (1+it)^{-\frac{1}{4}(d-1)(q-2)} H'_k(t)dt,$$

with

$$H'_k(t) = (1-it)^{-\frac{1}{4}(d-1)(q-2)+\frac{1}{2}(d-1)}(q-1-it)^{-k-\frac{1}{2}(d-1)}(pq-p-q+(q-p)it)^k.$$

H'_k satisfies the hypothesis of Lemma 4.7, is real when restricted to the imaginary axis and nonnegative at least when k is an even integer.

Lemma 4.7 gives

$$I'_k = 2\sin(\tfrac{1}{4}(d-1)(q-2)\pi)\int_0^\infty y^{-\frac{1}{4}(d-1)(q-2)}H_k(i+iy)dy. \tag{4.29}$$

Since $(d-1)(q-2)/4 \in [\frac{1}{2},1)$, the factor $\sin(\frac{1}{4}(d-1)(q-2)\pi)$ is nonzero. If k is an even positive integer, then the integrand is nonnegative, so the integral in (4.29) is likewise nonzero. $\square$

Chapter 5

Introduction to the joints problem

The joints problem is a problem in incidence geometry. Let $d \geqslant 2$ and L a collection of lines in $\mathbb{R}^d$. A joint of L is a point that is the intersection of d lines in L, not all in a common hyperplane. We will call J the set of joints determined by L.

The joints problem asks the following: what is the maximum number of joints determined by a set of lines in $\mathbb{R}^d$ of a given cardinality?

The problem is interesting if $d \geqslant 3$ as for $d = 2$ one easily gets $|J| \leqslant |L|^2$ and this upper bound is sharp in the sense that there are sets of n lines such that $|J| \gtrsim n^2$. For instance, a set of $n/2$ vertical lines and $n/2$ horizontal lines gives $|J| = n^2/4$. Similarly, in $\mathbb{R}^d$ one has the trivial bound $|J| \leqslant |L|^d$. A much better upper bound exists.

For a lower bound, we can do as in the two dimensional case. Consider a set of $d \cdot n^{d-1}$ lines, divided in d sets $L_1, \ldots, L_d$ of the same cardinality n^{d-1}, where the lines in L_i are pairwise parallel, orthogonal to the coordinate plane $\{(x_1 \ldots, x_d) \in \mathbb{R}^d : x_i = 0\}$. We arrange them so that for each $(z_1, \ldots, z_d) \in \mathbb{Z}^d$ satisfying $1 \leqslant z_i \leqslant n$ for all $1 \leqslant i \leqslant d$, there exist $\ell_1 \in L_1, \ldots, \ell_d \in L_d$ such that $z = \cap_{i=1}^d \ell_i$. Then the number of joints equals $|J| = n^d = (n^{d-1})^{d/(d-1)} = d^{-d/(d-1)}|L|^{d/(d-1)}$. Therefore the maximum number of joints determined by a set of n lines in $\mathbb{R}^d$ is $\Omega(n^{d/(d-1)})$.

Here we discuss different developments that lead to the proof of the upper bound,

Theorem 5.1 ([29],[38])**.** *Let L be a collection of lines in $\mathbb{R}^d$. Then the cardinality of the set of joints of L, J satisfies $|J| = O(|L|^{d/(d-1)})$.*

This theorem was proved by the author in [38] and independently by Kaplan, Sharir and Shustin in [29] (the two papers appeared on the arXiv the same day).

5.1 A bit of a history of the problem

The problem seems to appear for the first time in [8], and it was considered in the three dimensional case only until [29] and [38] appeared and gave the optimal upper bound in all dimensions. For the three dimensional case the optimal bound is $|J| \lesssim |L|^{3/2}$. Partial results are:

1. In [8] it was proved that $|J| = O(|L|^{7/4})$.
2. This was improved by Sharir in [43] who obtained $|J| = O(|L|^{23/14} \log^{31/14} |L|)$, which is $O(|L|^{1.643})$.
3. Later, Feldman and Sharir [19] obtained $|J| = O(|L|^{112/69} \log^{6/23} |L|)$, which is $O(|L|^{1.6232})$.

The problem also appears in the book "Research Problems in Discrete Geometry" [5], in Chapter 7.1, problem 4.

It was recently that Guth and Katz [23] gave an affirmative answer for the three dimensional case,

Theorem 5.2. *The number of joints defined by a set of n lines in space is $O(n^{3/2})$.*

Their proof uses the ideas from [13] for the finite field Kakeya problem that we discuss in the next section where we also mention the result of Bennett, Carbery and Tao, that proves a weaker result by using a multilinear Kakeya estimate.

Our proof for $n \geqslant 3$ also uses the polynomial method of Dvir, but in a different way.

On one more development for the three dimensional case, Elekes, Kaplan and Sharir [14] simplified the proof of Guth and Katz and extended their techniques to obtain a bound on $I(J', L)$, the number of incidences between an arbitrary subset J' of J and L,

Theorem 5.3. *Let L be a set of n lines in $\mathbb{R}^3$ and J' be a set of m joints of L. Then*

$$I(J', L) = \min\{O(m^{1/3}n), O(m^{2/3}n^{2/3} + m + n)\}.$$

The bound is tight in the worst case.

5.2 A relation with the Kakeya problem

A Kakeya set $E \subset \mathbb{R}^d$ is a compact set containing a unit line segment in every direction, that is, for all $e \in S^{d-1}$ there exists $x \in \mathbb{R}^d$ such that $x + te \in E$ for all $t \in [0, 1]$. The Kakeya conjecture states,

Conjecture 5.4. A Kakeya set in $\mathbb{R}^d$ has Hausdorff dimension equal to d.

We refer to [50] for a survey on this problem. In [50] the following finite field analog of the Kakeya problem is proposed:

Let $\mathbb{F}_q$ be the field with q elements and let V be a d-dimensional vector space over $\mathbb{F}_q$. Let E be a subset of V which contains a line in every direction, that is

$$\forall\, 0 \neq e \in V,\ \exists x \in V : \ x + te \in E\,,\ \text{for all } t \in \mathbb{F}_q.$$

Does it follow that $|E| \geqslant C_d\, q^d$?

After several partial results, Dvir [13], gave a very simple proof of this finite field Kakeya problem. His method uses the following linear algebra result,

Lemma 5.5. *For $d, n \geqslant 1$ let $N = \binom{d+n}{n} - 1$, and $u_1, \dots, u_N \in \mathbb{F}_q^d$. Then there exists a nontrivial polynomial $P : \mathbb{F}^d \to \mathbb{F}$ of degree $\deg P \leqslant n$ that vanishes on all $u_1, \dots, u_N$,*

and his idea is to study the properties of a polynomial that vanishes on every point of a Kakeya set.

Dvir's method suggests the introduction of tools from algebraic geometry, properties of polynomials and their zero set, to treat incidence geometry problems and this has been developed by different authors, some examples being in [15], [14], [16], [23], [24].

A stronger statement than Conjecture 5.4 is the Kakeya maximal operator conjecture, a survey can be found in [30]. For $0 < \delta \ll 1$ we let a δ-tube denote a tube in $\mathbb{R}^d$ of length 1 and cross section of radius δ.

Conjecture 5.6. Let $\mathbf{T} = \{T_i : i \in I\}$ be any collection of δ-tubes in $\mathbb{R}^d$, whose orientations are δ-separated in S^{d-1}. Then

$$\Big\| \sum_{T \in \mathbf{T}} \chi_T \Big\|_{L^{d/(d-1)}} \leqslant C_\varepsilon \delta^{-\varepsilon} \Big(\sum_{T \in \mathbf{T}} |T| \Big)^{(d-1)/d}. \tag{5.1}$$

Inequality (5.1) can be written equivalently as

$$\Big\| \Big(\sum_{T \in \mathbf{T}} \chi_T \Big)^d \Big\|_{L^{1/(d-1)}} \leqslant C_\varepsilon \delta^{-\varepsilon} \big(\delta^{d-1} |\mathbf{T}| \big)^{(d-1)},$$

from where a multilinear version can be deduced.

It was Bennett, Carbery and Tao who proposed in [3] a multilinear version of the Kakeya problem. Suppose $\mathbf{T}_1, \dots, \mathbf{T}_d$ are families of δ-tubes in $\mathbb{R}^d$ and assume that for each $1 \leqslant i \leqslant d$, the tubes in $\mathbf{T}_i$ have the long sides pointing in directions belonging to some sufficiently small but fixed neighborhood of the i^{th} standard basis vector $e_i \in S^{d-1}$. We will refer to such family of tubes as transversal. In [3] it was proved

Theorem 5.7. *If $d/(d-1) < q \leqslant \infty$, then there exists a constant C, independent of δ and the transversal family of tubes $\mathbf{T}_1, \dots, \mathbf{T}_d$, such that*

$$\Big\| \prod_{j=1}^{d} \Big(\sum_{T_j \in \mathbf{T}_j} \chi_{T_j} \Big) \Big\|_{L^{q/d}(\mathbb{R}^d)} \leqslant C \prod_{j=1}^{d} \delta^{d/q} |\mathbf{T}_j|. \tag{5.2}$$

Furthermore, for each $\varepsilon > 0$ there is a similarly uniform constant C for which

$$\Big\|\prod_{j=1}^{d}\Big(\sum_{T_j\in\mathbf{T}_j}\chi_{T_j}\Big)\Big\|_{L^{1/(d-1)}(\mathbb{R}^n)} \leqslant C\delta^{-\varepsilon}\prod_{j=1}^{d}\delta^{d-1}|\mathbf{T}_j|. \tag{5.3}$$

The proof is based on the monotonicity of the inequalities under the heat flow. Using Theorem 5.7, they obtained a nearly optimal bound for joints in $\mathbb{R}^3$ under the assumption of transversality. More precisely, for $0 < \theta \leqslant 1$, we say that the lines ℓ_1, ℓ_2, ℓ_3 are θ-transverse if the parallelepiped generated by unit vectors parallel to the lines has volume at least θ. A θ-transverse joint of a collection of lines L is a joint that is the intersection of three θ-transverse lines in L.

Theorem 5.8. *For any $0 < \theta \leqslant 1$, the number of θ-transverse joints is*

$$O_\varepsilon(|L|^{3/2+\varepsilon}\theta^{-1/2-\varepsilon})$$

for any $\varepsilon > 0$, where the subscript of the O by ε means that the implicit constant can depend on ε.

The endpoint case of Theorem 5.7, $q = d/(d-1)$, without the ε loss, was settled by Guth in [22]. Note that by scaling (5.3) is equivalent to the inequality

$$\Big\|\prod_{j=1}^{d}\Big(\sum_{T_j\in\mathbf{T}_j}\chi_{T_j}\Big)\Big\|_{L^{1/(d-1)}(\mathbb{R}^n)} \leqslant C\delta^{-\varepsilon}\prod_{j=1}^{d}|\mathbf{T}_j|,$$

where the tubes T are "unit" cylinders, that is, have cross section of radius 1 and infinite length.

For $0 < \theta \leqslant 1$ we will say that the families of cylinders $\mathbf{T}_1, \ldots, \mathbf{T}_d$ are θ-transverse if any collection of lines $\ell_1, \ldots, \ell_d$ with ℓ_i parallel to some tube in $\mathbf{T}_i$, is θ-transverse, that is, the volume of the d-dimensional box generated by unit vectors parallel to the lines has volume at least θ.

Theorem 5.9 ([22])**.** *Let $0 < \theta \leqslant 1$ and let $\mathbf{T}_1, \ldots, \mathbf{T}_d$ be a collection of θ-transverse unit cylinders. Then*

$$\Big\|\prod_{j=1}^{d}\Big(\sum_{T_j\in\mathbf{T}_j}\chi_{T_j}\Big)\Big\|_{L^{1/(d-1)}(\mathbb{R}^d)} \leqslant C_n\theta^{-1/(d-1)}\prod_{j=1}^{d}|\mathbf{T}_j|.$$

Guth makes use of what can be seen as the continuum version of the polynomial method of Dvir. The correspondence can be seen reflected in the next proposition, the polynomial ham sandwich theorem,

Proposition 5.10. *Let $N = \binom{d+n}{n} - 1$ and $U_1, \ldots, U_N$ be finite volume open sets in $\mathbb{R}^d$. Then there exists a polynomial $P : \mathbb{R}^d \to \mathbb{R}$ of degree $\deg P \leqslant n$ such that the algebraic hypersurface $Z = \{x \in \mathbb{R}^d : P(x) = 0\}$ bisects each of the sets $U_1, \ldots, U_N$.*

Compared to Lemma 5.5, we change points by open sets of finite volume, and vanishing at the points by bisecting the open sets.

Using Proposition 5.10 and the pigeonhole principle only, Guth proved a weaker version of Theorem 5.9, namely

Proposition 5.11. *Let $\mathbf{T}_1, \ldots, \mathbf{T}_d$ be transverse families of unit cylinders in $\mathbb{R}^d$ of equal cardinality A. Let I be the set of points that belong to at least one cylinder in each direction, i.e.*

$$I = \bigcap_{j=1}^{d} \bigcup_{T_j \in \mathbf{T}_j} T_j,$$

then $Vol(I) \leqslant C_n A^{d/(d-1)}$.

A much stronger tool than Proposition 5.10 is needed for the general case and the techniques involve the use of algebraic topology. If Proposition 5.11 follows from Proposition 5.10 which is the analog of Lemma 5.5, then Theorem 5.9 can be said to be a consequence of the continuum analog of the following lemma

Lemma 5.12. *Let $d, N \geqslant 1$, $u_1, \ldots, u_N \in \mathbb{F}_q^d$ and $m_1, \ldots, m_N \in \mathbb{N}$. Then there exists a nontrivial polynomial $P : \mathbb{F}_q^d \to \mathbb{F}_q$ of degree $\deg P \lesssim (\sum_{j=1}^{N} m_j^d)^{1/d}$ that vanishes on all $u_1, \ldots, u_N$ at degree $m_1, \ldots, m_N$ respectively.*

The problem of counting joints can be seen as a discrete analog of Proposition 5.11. The families of tubes gets replaced by a single family of lines L, and the set I by the set of joints J. The condition on transversality is translated in the definition of J, we only count those intersection than come from "transverse" lines, lines not lying in the same hyperplane. The volume of I, $Vol(I)$, is replaced by the cardinality of J, $|J|$. The upper bound for both is exactly the same. Moreover, the proof of Theorem 5.1 given by the author in [38] was inspired by the proof of Proposition 5.11 and in its original form followed the same lines. A simplification to the proof was later pointed out by Fedor Nazarov.

Different problems can be suggested based on this analogy between joints and multilinear Kakeya. For a collection of lines L in $\mathbb{R}^d$, J the set of joints and for $x \in J$ we define $I(x)$ to be the number of lines in L passing through x and $B(x)$ to be the number of ways in which x can be written as the intersection $\cap_{i=1}^{d} \ell_i$ with $\ell_1, \ldots, \ell_d \in L$ not all lying in a common hyperplane, up to permutations. Note that $I(x) \leqslant d \cdot B(x) \leqslant \binom{I(x)}{d}$.

Problem 1. Are the following true?

$$\sum_{x \in J} I(x)^{d/(d-1)} \lesssim |L|^{d/(d-1)} \ ? \tag{5.4}$$

$$\sum_{x \in J} B(x)^{d/(d-1)} \lesssim |L|^{d/(d-1)} \ ? \tag{5.5}$$

Chapter 6

The joints problem in $\mathbb{R}^n$

We show that given a collection of A lines in $\mathbb{R}^n$, $n \geqslant 2$, the maximum number of their joints (points incident to at least n lines whose directions form a linearly independent set) is $O(A^{n/(n-1)})$. An analogous result for smooth algebraic curves is also proved.

6.1 Introduction

In a recent paper, Katz and Guth [23] proved that the number of joints determined by a given collection of A lines in $\mathbb{R}^3$ is $O(A^{3/2})$, where a joint (in $\mathbb{R}^3$) is a point which is incident to at least three noncoplanar lines of the given collection. Lately, Elekes, Kaplan, and Sharir [14] extended the results in [23] to obtain a bound on the number of incidences between a collection of lines and a given subset of their joints, in $\mathbb{R}^3$, which implies the result on the number of joints (they also consider a more general situation where joints are replaced by an arbitrary set of points satisfying that no plane contains more that $O(A)$ points and each point is incident to at least three lines). Both results make use of algebraic geometric properties of polynomials in three variables, which bound the number of critical lines (lines where the polynomial and its gradient both vanish) a polynomial can have in terms of its degree. For more references on this problem, consult [23] and [14].

Our proof does not require the algebraic geometric considerations in [23] and [14] about polynomials in n variables but just the fact that given m points in $\mathbb{R}^n$, there exists a nonzero polynomial $Q \in \mathbb{R}[x_1, \ldots, x_n]$ such that Q vanishes on all the given m points and whose degree is bounded by $d \lesssim m^{1/n}$. The method can be seen as, and was largely inspired by, an adaptation of the methods in [22] to the discrete case, more precisely the result in the section "warmup to multilinear Kakeya" of [22], together with the application of the polynomial method as in [13].

We point out that an independent proof of the bound on the number of joints, due to Kaplan, Sharir, and Shustin [29], appeared at the same time as the one presented in the first version of this work. Our proof has some similarities with the proof in [29] (for example, compare Lemma 6.2 and the "Differentiating" step in the proof of Theorem 1 in [29]).

6.2 The main result

For a given collection of lines L in $\mathbb{R}^n$ consider the set J of points of the form $\cap_{i=1}^n \ell_i$, where $\ell_i \in L$ for all $1 \leqslant i \leqslant n$ and the directions of the lines $\ell_1, \dots, \ell_n$ are linearly independent. We will refer to J as the set of transverse intersections, or joints, of L.

Notation. In this section the letters L and J will always be used with the same meaning, a set of lines in $\mathbb{R}^n$ and the set of joints determined by the set of lines, respectively. We will denote by $|S|$ the cardinality of the set S. We also use the notation $X \lesssim Y$, $Y \gtrsim X$, $Y = \Omega(X)$, or $X = O(Y)$ to denote any estimate of the form $X \leqslant CY$, where C is a constant that depends only on the dimension n. We use $X = \Theta(Z)$ to denote $X = O(Z)$ and $Z = O(X)$.

Our main theorem is the following.

Theorem 6.1. *Let L be a collection of lines in $\mathbb{R}^n$. Then the cardinality of the set of joints of L, J satisfies $|J| \lesssim |L|^{n/(n-1)}$.*

We start by proving the following lemma.

Lemma 6.2. *Let J' be a subset of J with the property that every line $\ell \in L$ with $\ell \cap J' \neq \emptyset$ contains at least m points of J', that is, $|\ell \cap J'| \geqslant m$ for some given constant m. Then $|J'| \geqslant C_n m^n$, where C_n is a constant depending on n only.*

Proof. By contradiction, assume there exists an arrangement of lines L and points J' as in the statement of Lemma 6.2, where $|J'| \leqslant \frac{m^n}{K}$, where K is a big constant depending on n only that we will choose later. Let $Q \in \mathbb{R}[x_1, \dots, x_n]$ be a nonzero polynomial that vanishes on every point of J'. We can choose Q of degree $\deg(Q) \leqslant c(n)|J'|^{1/n} \leqslant \frac{c(n)}{K^{1/n}} m$ (because the space of polynomials of degree $\leqslant d$ has dimension $\binom{d+n}{d} = \Theta(d^n)$). Choosing K sufficiently big depending on n only we can ensure that $\deg(Q) < m$. The restriction of Q to any line of L which intersects J' is a polynomial in one variable of degree $< m$ that vanishes on at least m points, hence it vanishes identically. From $Q|_\ell = 0$ we obtain $\nabla Q \cdot v|_\ell = 0$, where v is the direction of ℓ. Therefore at each point of J', ∇Q is orthogonal to a linearly independent set of n vectors, so it is zero. Now every component of ∇Q vanishes on J' and has degree $\deg(\nabla Q) < \deg(Q) < m$. We can apply the same argument to every component of ∇Q, so inductively we obtain $\frac{\partial^\alpha Q}{\partial x^\alpha} = 0$ on J' for every multi-index $\alpha \in \mathbb{N}^n$. From here it follows that Q is identically zero, which is a contradiction. $\square$

Following the initial publication of this work, Fedor Nazarov observed that the proof of Theorem 6.1 follows immediately from Lemma 6.2.

Proof of Theorem 6.1. Let $m = K|J|^{1/n}$, where K satisfies $K^n C_n > 1$ and C_n is the constant in the conclusion of Lemma 6.2 (hence K depends on n only). We start an iterative process to remove lines from L having a control in the number of joints removed at each step. Let $L^{(0)} = L$ and $J^{(0)} = J$. Suppose that $L^{(i)} \subseteq L$, $L^{(i)} \neq \emptyset$ has been defined, and let $J^{(i)} \subseteq J$ denote the set of joints determined by $L^{(i)}$. With the choice of m, there must be a line $\ell_i \in L^{(i)}$ that contains no more than m points of $J^{(i)}$; otherwise, by Lemma 6.2, we would have $|J| \geqslant |J^{(i)}| \geqslant C_n m^n = K^n C_n |J| > |J|$, which is a contradiction.

Define $L^{(i+1)} = L^{(i)} \backslash \{\ell_i\}$ and let $J^{(i+1)}$ be the set, possibly empty, of joints of $L^{(i+1)}$, which are necessarily contained in J. In this way we have $|J^{(i)}| \leqslant |J^{(i+1)}| + m$.

Since for $i \geqslant |L| - (n-1)$ we have $J^{(i)} = \emptyset$, we conclude that $|J| = |J^{(0)}| \leqslant m|L| = O(|J|^{1/n}|L|)$, whence we obtain $|J| \lesssim |L|^{n/(n-1)}$. □

6.3 The case of algebraic curves

A bound similar to the one in Theorem 6.1 can be proven if we replace lines by algebraic curves. By a smooth curve γ we mean a curve such that its tangent vector $\dot{\gamma}$ exists at every point of γ and is nonzero. Given a collection $\mathcal{C}$ of smooth curves we define the set of joints, J, determined by $\mathcal{C}$ as the set of incidences of at least n curves in $\mathcal{C}$ such that the tangent vectors of the curves at the intersection are linearly independent.

We start by considering a special case of algebraic curves. Let $\mathcal{C}$ be a set of smooth curves, each parametrized by polynomials; that is, if $\gamma \in \mathcal{C}$, we can parametrize it as $\gamma(t) = (P_1(t), \ldots, P_n(t))$, where each P_i is a polynomial in one variable of degree at most d for a given constant d. We let J denote the set of joints determined by $\mathcal{C}$.

A minor modification of Lemma 6.2 gives the following.

Lemma 6.3. *Let $\mathcal{C}$ and J be as in the previous paragraph, and let J' be a subset of J with the property that $|\gamma \cap J'| \geqslant m$ for every curve $\gamma \in \mathcal{C}$ with $\gamma \cap J' \neq \emptyset$ for some given constant m. Then $|J| = \Omega(m^n/d^n)$.*

The conclusion follows as in the case of lines, and the bound on the number of joints is $|J| \leqslant C_n |\mathcal{C}|^{n/(n-1)} d^{n/(n-1)}$, where C_n is a constant depending on n only.

More generally, if we consider an irreducible, smooth algebraic curve γ of degree d, and if $Q \in \mathbb{R}[x_1, \ldots, x_n]$ has degree $< m/d$, and its zero locus intersects γ on at least m different points, then the curve is contained in the zero set of Q, that is, $Q|_\gamma \equiv 0$, by an application of Bezout's theorem (see, for example, Chapter 1 in [25] or Chapter 3 in [41]). Hence the same conclusion as in Lemma 6.3 holds if we let $\mathcal{C}$ consist of irreducible, smooth algebraic curves of degree at most d. Therefore we have the following theorem.

Theorem 6.4. *Let $\mathcal{C}$ be a collection of irreducible, smooth algebraic curves of degree at most d in $\mathbb{R}^n$. Let J denote the set of joints determined by $\mathcal{C}$. Then the cardinality of J satisfies $|J| \leqslant C_n |\mathcal{C}|^{n/(n-1)} d^{n/(n-1)}$ for some constant C_n depending on n only.*

6.4 The original proof of Theorem 6.1

We include here the original proof we had of Theorem 6.1. We derive the following consequence from Lemma 6.2. Let $c : J \to L$ be a function satisfying $x \in c(x)$ for all $x \in J$, that is, for each x, c selects a line incident at x. Note that for each $x \in J$ we have at least n transverse lines intersecting at x. Thus at each x we have at least n different lines to choose from. We call such a function a coloring of J.

Proposition 6.5. *There exists a coloring c of J such that for every line $\ell \in L$, $|\{x \in \ell \cap J : c(x) = \ell\}| = O(|J|^{1/n})$.*

Short proof, sketch. We will use the same method as in the proof of Theorem 6.1. With the notation as in the proof of Theorem 6.1 we know that for $i \geqslant |L| - (n-1)$ we have $J^{(i)} = \emptyset$. Let $i_0 \leqslant |L| - (n-1)$ be the first time $J^{(i)}$ is empty. We let $\ell_i \in L$ be the line deleted at the $i-$th step, that is $\ell_i \in L^{(i)} \backslash L^{(i+1)}$. Every point in J is contained in some line ℓ_i, $1 \leqslant i \leqslant i_0$. For $x \in J$ let $i(x)$ be the first time a line containing x is deleted, ie, $x \in \ell_{i(x)}$ and $x \neq \ell_i$ for $i < i(x)$. Define $c(x) = \ell_{i(x)}$. Since ℓ_i is such that the number of joints of $L^{(i)}$ contained in ℓ_i is less than or equal to m, it follows that $|\{x \in J : c(x) = \ell_i\}| \leqslant m$, and the proposition is verified. ☐

The original proof. Let $m = |J|$ and note that for any coloring c of J, $W(\ell) := |\{x \in \ell \cap J : c(x) = \ell\}|$ satisfies $W(\ell) \leqslant |\ell \cap J|$. We use an inductive method to define the coloring c. Choose an ordering $J = \{x_1, \dots, x_m\}$. By a provisional coloring c_ν on $J_\nu := \{x_1, \dots, x_\nu\}$ we mean a function $c_\nu : J_\nu \to L$ with $x \in c_\nu(x)$ for all $x \in J_\nu$. Given a provisional coloring c_ν we define the provisional counting function, W_ν, on L by $W_\nu(\ell) = |\{x \in \ell \cap J_\nu : c_\nu(x) = \ell\}|$. We will say that the provisional coloring c_ν is acceptable if $W_\nu(\ell) \leqslant Km^{1/n}$ for all $\ell \in L$, for a given big constant K depending only on n that we will choose later. The Proposition is proven if we can find an acceptable coloring c_m.

Define the provisional coloring c_ν on $\{x_1, \dots, x_\nu\}$ inductively by setting $c_1(x_1) = \ell_1$, for an arbitrarily selected line $\ell_1 \in L$ intersecting x_1. It follows that $W_1(\ell_1) = 1$, $W_1(\ell) = 0$, for all $\ell \neq \ell_1$, which is acceptable if we choose $K > 1$.

We will show that if c_ν is an acceptable coloring on $\{x_1, \dots, x_\nu\}$ then, by possibly modifying c_ν, we can obtain an acceptable coloring $c_{\nu+1}$ on $\{x_1, \dots, x_{\nu+1}\}$.

Suppose c_ν is an acceptable coloring on $\{x_1, \dots, x_\nu\}$. The good case is the following: there is a line $\ell_{\nu+1}$ intersecting $x_{\nu+1}$ such that $W_\nu(\ell_{\nu+1}) + 1 \leqslant Km^{1/n}$. In this case we let $c_{\nu+1}$ on $\{x_1, \dots, x_\nu, x_{\nu+1}\}$ be defined by $c_{\nu+1}(x_i) = c_\nu(x_i)$ for all $1 \leqslant i \leqslant \nu$, and $c_{\nu+1}(x_{\nu+1}) = \ell_{\nu+1}$. It follows that $W_{\nu+1}(\ell) = W_\nu(\ell)$ for all $\ell \neq \ell_{\nu+1}$, and $W_{\nu+1}(\ell_{\nu+1}) = W_\nu(\ell_{\nu+1}) + 1$, so that $c_{\nu+1}$ is acceptable.

We now turn to the complementary case, the bad one. Here we have $W_\nu(\ell) \geqslant \frac{1}{2}Km^{1/n}$ (the $\frac{1}{2}$ is just because $Km^{1/n}$ may not be integer), for all $\ell \in L$ incident at $x_{\nu+1}$, and we note that there are at least n such lines with linearly independent directions. Now look at each point $x_i \in \ell \cap J_\nu$ with $c_\nu(x_i) = \ell$, where ℓ is a line incident at $x_{\nu+1}$. If we can change the value of $c_\nu(x_i)$ to say $c_\nu(x_i) = \ell'$, for $\ell' \neq \ell$, for some i, without violating the restriction on $W_\nu(\ell')$ (that is $W_\nu(\ell') + 1 \leqslant Km^{1/n}$), then we are done, as we define $c_{\nu+1}(x_j) = c_\nu(x_j)$ for $j \leqslant \nu$ and $x_j \neq x_i$, $c_{\nu+1}(x_i) = \ell'$, $c_{\nu+1}(x_{\nu+1}) = \ell$. If we can not find such x_i this means that for any ℓ incident at $x_{\nu+1}$, for any point $x_i \in \ell \cap J_\nu$ with $c_\nu(x_i) = \ell$, and any ℓ' incident at x_i we have $W_\nu(\ell') \geqslant \frac{1}{2}Km^{1/n}$.

We let

$$I^{(1)} = \{x \in J : x \in \ell \cap J_\nu, \text{ for some } \ell \text{ incident at } x_{\nu+1} \text{ and } c_\nu(x) = \ell\}$$

and if $I^{(\sigma)}$ is defined we let

$$I^{(\sigma+1)} = \{x \in J_\nu : \text{ there exists } x' \in I^{(\sigma)} \text{ and } \ell \text{ incident at } x' \text{ s.t. } x \in \ell \text{ and } c_\nu(x) = \ell\}.$$

We note that, similarly as we did for points in $I^{(1)}$, if $x \in I^{(\sigma)}$ and ℓ is such that $c_\nu(x) = \ell$ and there exists $\ell' \neq \ell$ incident at x such that $W_\nu(\ell') + 1 \leqslant Km^{1/n}$, then by modifying c_ν on the corresponding points on $I^{(1)} \cup \cdots \cup I^{(\sigma)}$, we can obtain an acceptable coloring $c_{\nu+1}$ on $\{x_1, \ldots, x_{\nu+1}\}$ as desired.

If this is not the case, that means that for any $x \in \bigcup_{\sigma=1}^{\infty} I^{(\sigma)} =: J'$ and any $\ell \in L$ of the at least n transverse lines incident at x we have $W_\nu(\ell) \geqslant \frac{1}{2}Km^{1/n}$. Note that $I^{(\sigma+1)} = I^{(\sigma)}$ for all sufficiently large σ, since these are nested subsets of the finite set J. We let L' denote the set of lines of L incident to some point of J'. Thus for all $\ell \in L'$ we have

$$\frac{1}{2}Km^{1/n} \leqslant W_\nu(\ell) \leqslant |\ell \cap J'|, \tag{6.1}$$

where the second inequality comes from the inclusion $\{x \in \ell \cap J_\nu : c_\nu(x) = \ell\} \subseteq \ell \cap J'$, that we show now. We first note that if $x \in I^{(\sigma)}$ is such that $c_\nu(x) = \ell$, then any $x' \in \ell \cap J_\nu$ with $c_\nu(x') = \ell$ is in $I^{(\sigma)}$. Now for $\ell \in L'$ we have $\ell \cap J' \neq \emptyset$, so let $x_{i_0} \in \ell \cap J'$. For x_{i_0} we have, $x_{i_0} \in J'$ hence $x_{i_0} \in I^{(\sigma)}$ for some $\sigma \geqslant 1$, then any $x \in \ell \cap J_\nu$ with $c_\nu(x) = \ell$ is in either in $I^{(\sigma)}$ or in $I^{(\sigma+1)}$ (depending whether $c_\nu(x_{i_0}) = \ell$ or not), thus $x \in J'$ and the inclusion follows.

Now use Lemma 6.2 together with (6.1) applied to L' and J', to obtain

$$|J'| \geqslant C(n)(\frac{1}{2}Km^{1/n})^n = \frac{1}{2^n}C(n)K^n m.$$

We now choose K big enough, depending on n only so that $\frac{1}{2^n}C(n)K^n > 1$. Hence we obtain $|J| \geqslant |J'| > m = |J|$ which is a contradiction. This means that in the bad case we can always modify c_ν to obtain an acceptable coloring $c_{\nu+1}$. Therefore the Proposition is proved, by induction. □

For those familiar with [22], a coloring as in Proposition 6.5 is the analog in "warmup to multilinear Kakeya" in [22] to finding directions $v_{j(k),a(k)}$ such that for the k-th cube Q_k, the directed volume $V_{Z\cap Q_k}(v_{j(k),a(k)})$ is large ($V_{Z\cap Q_k}(v_{j(k),a(k)}) \gtrsim 1$). The next proposition follows exactly as in the last paragraphs in the mentioned section of [22].

Proof of Theorem 6.1. By Proposition 6.5 there exists a coloring c satisfying $|\{x \in \ell \cap J : c(x) = \ell\}| = O(|J|^{1/n})$ for all $\ell \in L$. For each $x \in J$ we have a distinguished line, namely $c(x)$. We have just associated a line to any point $x \in J$. There are in total $|L|$ lines and $|J|$ points. By the pigeonhole principle, there is a line, ℓ^*, associated to $\gtrsim |J|/|L|$ different points, therefore $|\{x \in \ell^* \cap J : c(x) = \ell^*\}| \gtrsim |J|/|L|$.

On the other hand $|\{x \in \ell^* \cap J : c(x) = \ell^*\}| \lesssim |J|^{1/n}$. From here it follows that $|J| \lesssim |L|^{n/(n-1)}$. □

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