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New Examples of Collapse With Lower Curvature Bound

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To my Tiny Miss.

ABSTRACT OF THE DISSERTATION

New Examples of Collapse With Lower Curvature Bound

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In this work, we describe a method to construct new examples of collapse with a lower curvature bound inspired by Cheeger and Gromov. Unlike with collapse with an upper and lower curvature bound, which is now completely understood, the structure of collapse with a lower curvature bound is still a mystery.

In [4], Grove and Petersen showed that if a sequence of Riemannian manifolds (M_i, g_i) has uniform lower curvature bound, k, and $M_i \to X$, then X is an Alexandrov space with lower curvature bound k. Petersen, Wilhelm, and Zhu then showed that the converse is false [7]. Perelman showed that given a sequence of n-dimensional Alexandrov spaces with a uniform lower curvature bound, with limit space X such that $\dim X = n$, all but finitely many of the prelimits are homeomorphic to X.

In 1985, Cheeger and Gromov introduced the concept of an F-structure, which can be thought as a generalized torus action on a manifold [1]. They showed that a manifold collapses with bounded curvature if and only if it admits an F-structure [2].

An F-structure is an example of a more general construct known as a $\mathfrak{g}$-structure, which is a sheaf of Lie groups actions, and is one of the main tools used in our approach.

The second main tool we use was also defined by Cheeger and is known as a *Cheeger deformation*. Cheeger generalized the method used by Berger in his classic example now known as the *Berger spheres*, which helped create the study of collapse in Riemannian geometry. Berger showed that scaling the metric of S^3 along the Hopf circles collapses S^3 to the 2-sphere of radius $\frac{1}{2}$.

In this work, using $\mathfrak{g}$-structures and Cheeger deformations, we construct new examples of collapse with a lower curvature bound.

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Chapter 0

Introduction

0.1 Motivation

The structure of collapse with a lower curvature bound is to date unknown. However, there have been many advances made in the field of collapse while preserving lower curvature bound In [4], Grove and Petersen showed that given a class of Riemannian manifolds with lower curvature bound k, the limit space is a length space (Alexandrov) with lower curvature bound k . Wilhelm, Petersen, and Zhu then showed that the converse is false [7]. By generalizing Berger's method, Cheeger came up with a method to construct collapse with a lower curvature bound on G-manifolds. Cheeger's method has since become known as a *Cheeger deformation*, which is a type scaling of the metric along the G-orbits. A related and very significant result comes from Perelman's stability theorem, which is as follows:

Theorem 1. *Let $\{X_i\}_i$ be a sequence of n-dimensional Alexandrov spaces with uniform lower curvature bound k. If X_i converge to X such that $\dim X = n$, then all but finitely many of the prelimits are homeomorphic to X.*

Perelman's stability theorem also implies the finiteness theorem given by Grove-Petersen-Wu [5].

The concept of collapsing a manifold is pretty intuitive. Viewed from a large scale, the collapsed manifold appears to be of lower dimension. To put it in a more geometric context, the volume of the collapsed manifold is zero.

The study of collapse began with the classic Berger spheres example. We take a more modern approach to this classic example by considering the *Hopf map*, as defined by Wilhelm [11]. Consider S^3 as a subset of $\mathbb{C} \oplus \mathbb{C}$ and $S^2 \subset \mathbb{R} \oplus \mathbb{C}$:

$$
\begin{aligned}
h &: S^3 \to S^2\left(\frac{1}{2}\right) \\
(z, w) &\mapsto \left(z\overline{w}, \frac{1}{2}(|z|^2 - |w|^2)\right).
\end{aligned}
$$

Let us first make a few observations on how this example and map can help us illustrate collapse. First, note that h is a Riemannian submersion. Second, we have a natural S^1-action on S^3, known as the *Hopf action*, via

$$\omega \cdot (z, w) = (\omega z, \omega w),$$

where $\omega \in S^1$ and $(z, w) \in S^3$. Note that the orbits of the S^1-action coincide with the fibers of the Hopf map.

Next, consider the following frame on S^3:

$$\partial\theta_1 + \partial\theta_2, \partial t, \frac{\cos^2 t\partial\theta_1 - \sin^2 t\partial\theta_2}{\cos t \sin t}.$$

We can now split the metric on (S^3, g_{can}) by vectors tangent and perpendicular to the S^1-orbits, where g_{can} is the standard unit metric on S^3. Let $\mathscr{V}$ = the distribution of vectors tangent to the S^1-orbits and $\mathscr{H}$ = the distribution of vectors perpendicular to the S^1-orbits, which are respectively given by

$$\mathscr{V} = \text{span}\{\partial\theta_1 + \partial\theta_2\},$$
$$\mathscr{H} = \text{span}\left\{\partial t, \frac{\cos^2 t \cdot \partial\theta_1 - \sin^2 t \cdot \partial\theta_2}{\cos t \sin t}\right\}.$$

Accordingly, we consider the metric as follows,

$$g_{\text{can}} = g^t + g^p,$$

where

$$g^t = g_{\text{can}}|_{\mathscr{V}}$$
$$g^p = g_{\text{can}}|_{\mathscr{H}}.$$

We now scale the metric along the vertical distribution and consider the following metric on S^3:

$$g_\varepsilon = \varepsilon^2 g^t + g^p.$$

Using the fact that the metric converges to zero along the Hopf fibers, it can be shown that

$$(S^3, g_\varepsilon) \to \left(S^2 \left(\frac{1}{2} \right), dr^2 + \frac{\sin^2(2r)}{4} d\theta^2 \right)$$

in the Gromov-Hausdorff topology. The former being the metric on the 2-sphere of radius $\frac{1}{2}$.

Though we have yet to define collapse explicitly, this example illustrates it nicely. When we let ε limit to zero, we see how the metric changes from three to two components and how this new metric is actually equivalent to the 2-sphere of radius $\frac{1}{2}$. We will generalize this process of scaling and limiting the scaling factor to zero when we introduce $\mathfrak{g}$-structures and a local type of Cheeger deformation. We now give an overview of the content covered in this work.

0.2 Overview

In Chapter 1, we list all the necessary definitions and background required by our approach. Many of the definitions we list that are needed to define a $\mathfrak{g}$-structure come from [1], but we list them here for convenience and ease of reading. We also list the curvature tensor equations from Gromoll-Walschap [3] , again for convenience, which are used heavily to show our main results. We also define the Cheeger deformation and give an example of how this works. To properly discuss collapse, we define the Gromov Hausdorff distance, as this makes the class of manifolds with a lower curvature bound a metric space, and therefore, allows us to view collapse as taking a limit in a metric space.

In Chapter 2, we state two of our main results. The first is the ideal case and is reminiscent of the classic Berger sphere example. To get collapse with a lower curvature bound in this case, we can simply scale the metric as we would with a G-manifold with some additional assumptions on our $\mathfrak{g}$-manifolds. The next main result is an analog of the first, though it does differ in obvious ways. For instance, the scaling factor along the orbits is a function on the charts of the atlas of the $\mathfrak{g}$-manifold. The atlas not only covers the manifold in the usual way, but each chart has its respective group action, $\mathfrak{g}_\alpha$. In addition, we must blow up the metric on the manifold in order to control the orbits. Blowing up the metric becomes necessary, as the contents of the next chapter will illustrate.

In Chapter 3, we go over an important example, which we commonly refer throughout this paper as the *Graph Manifold* example. This example has many purposes, but it mainly serves to illustrate the need to blow up the metric as we did for the second main result in Chapter 2 and for the third and strongest result in Chapter 4.

In Chapter 4, we state our strongest result yet as alluded above. It is named the *Hormetic result*, as the theorem exemplifies the concept of hormesis (Greek): effectively it is the process of strengthening by moderately weakening. Though hormesis refers to a biological process, its purpose in this paper is purely philosophical.

Lastly, Chapter 5 is where we state our conclusions and summarize the work done. We also list questions that were left unanswered and could also lead to future work in the search for an answer.

Chapter 1

Background and Formulas

We begin this chapter by defining a $\mathfrak{g}$-structure. As stated before, we have listed the definitions here for convenience, but they are defined originally by Cheeger and Gromov [1]. We then proceed with specifying what we mean by "collapsing a manifold" by defining the Gromov-Hausdorff metric along with a Cheeger deformation. Lastly, we list the Gromoll-Walschap curvature tensor equations [3], as well as define some useful definitions we will use throughout the thesis.

1.1 $\mathfrak{g}$-structure

In this section, we introduce the concepts needed to define a $\mathfrak{g}$-structure. The $\mathfrak{g}$ refers to "general Lie group", which extends the idea of a group acting on a manifold. A local action, so to speak. We list all the required definitions to define a $\mathfrak{g}$-structure and begin with the definition of a partial action:

Definition 2. *A **partial action**, A, of a topological group G on a Hausdorff space X is given by:*

1. *The domain of the action: a neighborhood $D \subset G \times X$ of $\{e\} \times X$.*

2. *A continuous map $A : D \to X$, also written $(g, x) \to g \cdot x$, such that $(g_1 * g_2) \cdot x = g_1 \cdot (g_2, x)$ whenever $(g_1 * g_2, x)$ and $(g_1, g_2 \cdot x)$ lie in D, and such that $e \cdot x = x$ for all $x \in X$.*

We denote a partial action, as well as emphasize its corresponding domain, by (A, D). As such, we can define an equivalence class, known as a local action as follows,

Definition 3. *Two partial actions (A_1, D_1), (A_2, D_2) are called (locally) equivalent if there is a domain $D \subset D_1 \cap D_2$, containing $\{e\} \times X$, such that $A_1|_D = A_2|_D$. A **local action**, $[A]$, is defined as the equivalence class of a partial action A of G on X. Every global action A defines an obvious local action A_{loc}. A local action which can be obtained in this way is called **complete**.*

Definition 4. *A subset $X_0 \subset X$ is called (locally) $[\mathbf{A}]$**-invariant** if for some representative $(A, D) \in [A]$, one has $gx_0 \in X_0$ for all $(g, x_0) \in D$. Since the intersection of $[A]$-invariant sets is $[A]$-invariant, it follows that each point $x \in X$ is contained in a unique minimal $[A]$-invariant subset called the **orbit**, denoted by $\mathscr{O} = \mathscr{O}_x \subset X$, and that the orbits partition the space X.*

In order to generalize a group action, we require a way to define group actions relative to some condition on a subset of our manifold. For this, it is only natural to discuss sheaves:

Definition 5. *A* ***sheaf*** $\mathfrak{F}$*, on a topological space* X *is an association between open sets* $U \subset X$ *and groups that satisfy the following three axioms:*

1. $\mathfrak{F}(U)$ *is a group whenever* U *is an open subset of* X *and* $\mathfrak{F}(\emptyset) = \{0\}$.
2. *If* $V \subset U$ *is an inclusion of open sets, there is a homomorphism (****the restriction homomorphism****)* $\rho_{UV} : \mathfrak{F}(U) \to \mathfrak{F}(V)$ *subject to the restrictions that:*
   (i) $\rho_{UU} = id$.
   (ii) $W \subset V \subset U$ *implies* $\rho_{UW} = \rho_{UV} \circ \rho_{VW}$.
3. *If* $\{U_\alpha\}$ *is an open covering of* U *and* $s_\alpha \in \mathfrak{F}(U_\alpha)$ *satisfies* $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, *then there exists a unique element* $s \in \mathfrak{F}(U)$ *so that* $s|_{U_\alpha} = s_\alpha$.

Definition 6. *Let* $p \in M$ *be fixed. Consider two neighborhoods* U *and* V *of* p*. Then* $g \sim h$ *if and only if* $\exists W \subset U \cap V$ *such that* $\rho_{UW}(g) = \rho_{VW}(h)$*, where* $g \in \mathfrak{F}(U)$ *and* $h \in \mathfrak{F}(V)$ *. The equivalence classes form the* ***stalk****,* $\mathfrak{F}_p$.

Definition 7. *An* ***action of a sheaf*** $\mathfrak{g}$ *on* X *is given by a local action of the group* $\mathfrak{g}(U)$ *on* U *for every connected open set* $U \subset M$*, such that the restriction homomorphism* $\mathfrak{g}(U) \to \mathfrak{g}(U')$*, where* $U' \subset U$*, agree with the restrictions of the local actions from* U *to* U'*, i.e. when* $x \in U' \subset U$ *and* $g \in \mathfrak{g}(U)$ *we have* $g \cdot x = \rho_{UU'}(g) \cdot x$*, wherever* $g \cdot x$ *and* $\rho_{UU'}(g) \cdot x$ *are defined.*

Definition 8. *A set S is called invariant if for all open sets U, the intersection $S \cap U$ is invariant for $\mathfrak{g}(U)$. Again, these minimal sets that partition M, are called orbits. A set which is the disjoint union of orbits is called* ***saturated****.*

Definition 9. *An action of a sheaf $\mathfrak{F}$ on connected groups on X is called a* ***complete local action*** *if whenever $x \in M$ there exists a neighborhood $V(x)$ of x and a local homeomorphism $\pi : \widetilde{V}(x) \to V(x)$ so that $\widetilde{V}(x)$ is Hausdorff and:*

1. *If $\widetilde{x} \in \pi^{-1}(x)$, then for any open neighborhood $W \subset \widetilde{V}(x)$ of $\widetilde{x}$, the restriction homomorphism $(\pi^*\mathfrak{F})(W) \to \widetilde{\mathfrak{g}}_{(\widetilde{x})}$ is an isomorphism, where $\pi^*\mathfrak{F}$ denotes the pullback sheaf,*

$$\left(\pi^*\mathfrak{F}\right)(W) \equiv \mathfrak{F}\left(\pi(W)\right). \tag{1.1}$$

2. *The local action of $(\pi^*\mathfrak{F})(\widetilde{V}(x))$ on $\widetilde{V}(x)$ is complete.*

Definition 10. *A $\mathfrak{g}$-structure $\mathscr{G} = (\mathfrak{g}, \cdot)$ on M is a sheaf, $\mathfrak{g}$, of connected topological groups and a complete local action of $\mathfrak{g}$ on X such that the sets $V(x)$ and $\widetilde{V}(x)$ can be chosen so that:*

1. *$\pi : \widetilde{V}(x) \to V(x)$ is a normal covering map.*

2. *For all x, $V(x)$ is saturated (w.r.t $\mathscr{G}$).*

3. *For all orbits $\mathscr{O}$, if $x, y \in \overline{\mathscr{O}}$, then $V(x) = V(y)$.*

Consider a $\mathfrak{g}$-structure on a manifold M. A subsheaf $\mathfrak{g}' \subset \mathfrak{g}$ along with the induced action on a given manifold M naturally define a *substructure* denoted by $\mathscr{G}'$.

Definition 11. *If $\mathscr{G}$ is locally constant on $V(x)$ for all x, then $\mathscr{G}$ is called pure, i.e. $\mathfrak{g}|_{V(x)}$ is locally isomorphic to the sheaf $\mathfrak{f}$, where $\mathfrak{f}$ is the sheaf of locally constant maps of $V(x)$ to the group $\mathfrak{g}_x$.*

Definition 12. *The rank of a $\mathfrak{g}$-structure, $\mathscr{G}$, at x is denoted by $\dim \mathscr{O}_x$. We say $\mathscr{G}$ has positive rank if $\dim \mathscr{O}_x > 0$ for all x.*

Definition 13. *If $\mathscr{G}$ is an effective $\mathfrak{g}$-structure, a collection $\{(U_\alpha, \mathscr{G}_\alpha)\}$ is called an **atlas** for $\mathscr{G}$ if*

1. *The U_α are connected, saturated (w.r.t. $\mathscr{G}$), and open, and form a locally finite covering of X.*
2. *Each $\mathscr{G}_\alpha \subset \mathscr{G}|_{U_\alpha}$.*
3. *Given any x, there is an α with $\mathscr{G}_{\alpha,x} = \mathscr{G}_x$.*

Definition 14. *A substructure $\mathscr{P} \subset \mathscr{G}$ is called a polarization if it has an atlas, $\mathscr{A} = \{(U_\alpha, \mathscr{P}_\alpha)\}$, such that for all α, the rank of $\mathscr{P}_\alpha$ is the same positive number for all $x \in \mathscr{P}_\alpha$ (although the rank of $\mathscr{P}_\alpha$ might depend on α).*

The concept of a polarized $\mathfrak{g}$-structure and a pure $\mathfrak{g}$-structure are closely related. A pure $\mathfrak{g}$-structure is such that the dimension of all $\mathfrak{g}$-orbits is constant. A polarized $\mathfrak{g}$-structure is such that the dimension of all the $\mathfrak{g}$-orbits is positive, though the dimensions may vary. Let us look at an example of a pure polarized $\mathfrak{g}$-structure:

Example 15. *Consider* $A \in SO(n)$ *such that* $A \neq \pm I$*. Define* $F_A : SO(n) \to SO(n)$ *such that* $F_A(B) = A \cdot B$*. Note that* F_A *is an automorphism on* $SO(n)$*. Now consider the mapping torus of* F_A*,* $M = {}^{SO(n)\times I}/_{\sim}$*, where* $(X, 0) \sim (F_A(X), 1)$*. Then* M *admits a polarization of rank* $\frac{n(n-1)}{2}$*. Note that since* F_A *does not commute with the* $SO(n)$*-action (except when* $A = \pm I$*), the action on* M *is not global.*

Remark 16. *Note that the above example also holds on any subgroup of* $SO(n)$ *and so we can get pure polarizations of a different rank. We can also construct an analogous example of a polarization using a Riemannian manifold along with a diffeomorphism on itself as follows:*

Example 17. *Similarly, consider* $A \in SO(n+1)$ *such that* $A \neq \pm I$*. Let* $F_A : S^n \to S^n$ *be defined as* $F_A(x) = A \cdot x$*. Note that* F_A *is a diffeomorphism. Consider the mapping torus for* F_A*, denoted by* M*. Then* M *admits a polarization, where the sheaf action is given by* $SO(n+1)$*. As with the previous example, the action on* M *is not global.*

1.2 Collapse Definitions

1.2.1 Gromov-Hausdorff Distance

Note that the definitions we list here come from P. Petersen's book, *Riemannian Geometry* [6]. As stated earlier, we take the metric space approach to collapse. Let

$\mathscr{M}$ be a class of closed Riemannian manifolds. In the context of metric spaces, collapsing a manifold equates to taking a limit of a "sequence", or more properly, a class of manifolds. The limit of this class of manifolds is considered as the collapsed manifold. The metric that allows for this is called the *Gromov-Hausdorff* metric. Before defining this metric, we need a couple of definitions. Let (X, d_X) be a metric space. Consider two subsets $A, B \subset X$. Then their distance is defined as:

$$d(A, B) \equiv \inf\{d_X(a, b) \ : \ a \in A, \ b \in B\}. \tag{1.2}$$

We can also define a ball with a subset as the center, as follows:

$$B(A, \varepsilon) \equiv \{x \in X \ : \ d_X(x, A) < \varepsilon\}.$$

Now, we define the *Hausdorff* metric:

Definition 18. *Let (X, d_X) be a metric space. The Hausdorff distance between any two subsets $A, B \subset$ is defined as:*

$$d_H(A, B) \equiv \inf\{\varepsilon \mid A \subset B(B, \varepsilon), \ B \subset B(A, \varepsilon)\}. \tag{1.3}$$

Note the difference between the two metrics on subsets of X defined above. The metric (1.2) is small if some points of A and B are close, while (1.3) is small iff all points of A are close to a point in B, and vice versa. Gromov took this definition and expanded the concept of distances between abstract metric spaces. The idea is to consider spaces which admit an *admissible* metric, i.e. given two metric spaces (X, d_X) and (Y, d_Y), an admissible metric on $(X \sqcup Y, d)$ is such that

$$d|_X = d_X, \qquad \text{and}$$

$$d|_Y = d_Y.$$

Finally, we can define the Gromov-Hausdorff distance:

Definition 19. *Given two metric spaces (X, d_X) and (Y, d_Y), which admit an admissible metric, the Gromov-Hausdorff distance is defined as:*

$$d_{GH}(X, Y) \equiv \inf\{d_H(X, Y) \mid \textit{admissible metrics on } X \sqcup Y\}.$$

An equivalent and more intuitive definition relies on the following notion. Let $f : X \hookrightarrow Z$ and $g : Y \hookrightarrow Z$ be isometric embeddings. Then, the Gromov-Hausdorff distance is defined as:

$$d_{GH}(X, Y) \equiv \inf\{\ d_H(f(X), g(Y)) \mid f, g \text{ are isometric embeddings }\}. \tag{1.4}$$

Definition 20. *Let (Z, d) be a metric space. Consider $X, Y \subset Z$. Let $x \in X,\ y \in Y$ such that $d(x, y) < r$, for some small r. Then the pointed Gromov-Hausdorrf distance* *is defined as*

$$d_{GH}\left((X,x),(Y,y)\right) = \inf\{d_H(X,Y) + d(x,y)\}.$$

The Gromov-Hausdorff distance is very useful in that it allows us to view collapsing a manifold as taking a limit, where a given class of manifolds are analogous to sequences of points or functions. However, two spaces can be GH-close, but still be very different topologically, as the following two examples illustrate:

Example 21. *Consider the GH-distance between* $\mathbb{R}$ *and* $\mathbb{Q}$*, namely*

$$d_{GH}(\mathbb{R}, \mathbb{Q}) = 0.$$

Example 22. *Consider the n-dimensional sphere,* S^n*. Let* Σ *be* S^n *with* $\varepsilon > 0$ *length handles attached. Then,*

$$d_{GH}(S^n, \Sigma) = \varepsilon.$$

1.2.2 Cheeger Deformation

We now define a method to perturb a metric, g_M, on a given Riemannian manifold, M, with non-negative curvature. Let G be a compact Lie group acting on M with corresponding bi-invariant metric, g_{bi}. Then consider the one parameter family of metrics $\ell^2 g_{\text{bi}} + g_M$ on $G \times M$. Then G acts on $(G \times M, \ell^2 g_{\text{bi}} + g_M)$ via

$$(h;(p,m)) \mapsto (ph^{-1}, h\cdot m), \tag{1.5}$$

which is known as the *Cheeger action*. Now, modding out by (1.5), we get the diffeomorphism $M \cong {}^{(G\times M)}/_{G}$. The induced metric on M is denoted by g_ℓ.

To get a better sense of this concept, consider the following example:

Example 23. *Consider* $(S^1\times\mathbb{R}^2, \ell^2 d\varphi^2 + dr^2 + r^2 d\theta^2) = (S^1\times\mathbb{R}^2, \widetilde{g})$*, for some* $\ell > 0$*. We get a natural* S^1*-action on* $S^1\times\mathbb{R}^2$ *as follows:*

$$S^1 \times (S^1\times\mathbb{R}^2) \longrightarrow S^1\times\mathbb{R}^2,$$
$$\left(e^{i\varphi};(e^{i\theta_1}, re^{i\theta_2})\right) \mapsto \left(e^{i(\varphi+\theta_1)}, re^{i(\varphi+\theta_2)}\right).$$

Modding out by the S^1*-action, we get the following diffeomorphism:*

$$\left(S^1\times\mathbb{R}^2\right)/S^1 \cong \left(\mathbb{R}^2, g_\ell\right).$$

where g_ℓ *is the notation used for the induced metric on the quotient. We now calculate the sectional curvature of* $(\mathbb{R}^2, g_\ell)$ *using the result from [cite O'Neill formula], namely*

$$\sec_{g_B}(X^*,Y^*) = \sec_{g_M}(X,Y) + 3|A_XY|_{g_M}. \tag{1.6}$$

First, we define the vertical space for the quotient map

$$q : \left(S^1 \times \mathbb{R}^2, \widetilde{g}\right) \to \left(\mathbb{R}^2, g_\ell\right),$$

namely

$$\mathscr{V} = \text{span}\{(-\partial_\varphi, 0, \partial_\theta)\} = \text{span}\{V\}. \tag{1.7}$$

The horizontal space is then given by:

$$\mathscr{H} = \text{span}\left\{(0, \partial_r, 0), \left(\frac{r^2}{\ell^2}\partial_\varphi, 0, \partial_\theta\right)\right\} = \text{span}\{X, Y\}. \tag{1.8}$$

To see that Y is indeed horizontal,

$$\begin{aligned}\widetilde{g}(Y, V) &= (\ell^2 d\varphi^2 + dr^2 + r^2 d\theta^2)\left(\left(\frac{r^2}{\ell^2}\partial_\varphi, 0, \partial_\theta\right), (-\partial_\varphi, 0, \partial_\theta)\right)\\ &= \ell^2\left(\frac{-r^2}{\ell^2}\right) + r^2 = 0.\end{aligned}$$

Now, we need to calculate the A-tensor (w.r.t. $\widetilde{g}$), but for this we will consider horizontal unit vectors $|X|_{g_M} = 1$ and $\hat{Y} = \frac{Y}{|Y|_{\widetilde{g}}}$ and the vertical vector $\hat{V} = \frac{V}{|V|_{\widetilde{g}}}$, to simplify the calculation. Then,

$$|Y|^2_{\widetilde{g}} = \left(\ell^2 d\varphi^2 + dr^2 + r^2 d\theta^2\right)\left((\frac{r^2}{\ell^2}\partial_\varphi, 0, \partial_\theta), (\frac{r^2}{\ell^2}\partial_\varphi, 0, \partial_\theta)\right)$$

$$= \ell^2 \left(\frac{r^4}{\ell^4}\right) + r^2$$

$$= r^2 \left(\frac{r^2+\ell^2}{\ell^2}\right),$$

which then gives us that,

$$\hat{Y} = \frac{Y}{|Y|_{\tilde{g}}}$$

$$= \frac{\ell}{r(r^2+\ell^2)^{\frac{1}{2}}} \left(\frac{r^2}{\ell^2}\partial_\varphi, 0, \partial_\theta\right).$$

Now, we need to calculate the norm of V*:*

$$|V|^2_{\tilde{g}} = \left(\ell^2 d\varphi^2 + dr^2 + r^2 d\theta^2\right)\left((-\partial_\varphi, 0, \partial_\theta)\right), \left(-(\partial_\varphi, 0, \partial_\theta)\right)$$

$$= \ell^2 + r^2,$$

which gives us that,

$$\hat{V} = \frac{1}{(\ell^2+r^2)^{\frac{1}{2}}}\left(-\partial_\varphi, 0, \partial_\theta\right).$$

Now, in order to calculate the A-tensor, we need to have the vertical component of $\nabla_X Y$*, which using Koszul's formula appropriately gives us,*

$$\nabla^v_X Y = \left(\frac{2r}{\ell^2}\partial_\varphi, 0, \frac{1}{r}\partial_\theta\right).$$

Then the A-tensor is given by:

$$
\begin{aligned}
|A_X\hat{Y}|^2_{\widetilde{g}} &= \widetilde{g}\left(\nabla_X\hat{Y}, \hat{V}\right)^2 \\
&= \frac{1}{|Y|_{\widetilde{g}}|V|_{\widetilde{g}}}\widetilde{g}\left(\nabla_X Y, V\right)^2 \\
&= \frac{\ell^2}{r^2(\ell^2+r^2)}\left(\ell^2 d\varphi^2 + dr^2 + d\theta^2\right)\left(\left(\frac{2r}{\ell^2}\partial_\varphi, 0, \frac{1}{r}\partial_\theta\right), (-\partial_\varphi, 0, \partial_\theta)\right)^2 \\
&= \frac{\ell^2}{r^2(\ell^2+r^2)}\frac{r^2}{(\ell^2+r^2)} \\
&= \frac{\ell^2}{(\ell^2+r^2)^2}.
\end{aligned}
$$

Then, the sectional curvature of $(\mathbb{R}^2, g_\ell)$ *is given by*

$$
\begin{aligned}
\sec_{g_\ell}(\partial_r, \partial_\theta) &= \sec_{\widetilde{g}}(X, \hat{Y}) + 3|A_X\hat{Y}|^2_{\widetilde{g}} \\
&= \frac{3\ell^2}{(\ell^2+r^2)^2}.
\end{aligned} \tag{1.9}
$$

1.3 Fundamental Equations

Note that here we list many equations from [3] for ease of reading. Let (M, g_M) be a smooth Riemannian manifold. Let $\pi : M \to B$ be a Riemannian submersion. Let $v, w \in T_pM$ be such that $v = u_1 + x_1, w = u_2 + x_2$, where u_1, u_2 are vertical vectors and x_1, x_2 are horizontal vectors. Given a smooth map, $\phi : M \to \mathbb{R}$, we can define a new metric on M, g_ϕ, via,

$$g_\phi(v,w) = e^{2\phi} g^t(u_1,u_2) + g^p(x_1,x_2). \tag{1.10}$$

where g^t is the metric g_M restricted to vertical vectors and g^p is g_M restricted to horizontal vectors.

Unless stated otherwise, from here on, u, v, w will denote vertical vectors and x, y, z will denote horizontal vectors. We now list the covariant derivatives w.r.t the metric g_ϕ.

$$\widetilde{\nabla}_u v = \nabla_u v + (e^{2\phi} - 1)\sigma(u,v) - e^{2\phi}\langle u, v\rangle \operatorname{grad}_{g_M}(\phi) \tag{1.11}$$

$$\widetilde{\nabla}_u x = \nabla_u x + (1 - e^{2\phi})A^*_x u + \left\langle \operatorname{grad}_{g_M}(\phi), x\right\rangle u \tag{1.12}$$

$$\widetilde{\nabla}_x u = \nabla_x u + (1 - e^{2\phi})A^*_u x + \left\langle \operatorname{grad}_{g_M}(\phi), u\right\rangle x \tag{1.13}$$

$$\widetilde{\nabla}_x y = \nabla_x y \tag{1.14}$$

1.3.1 The Curvature Tensor

Below we calculate the curvature tensor. Before we proceed, let us define some terms used in the formulas:

1. The second fundamental tensor on the fibers is defined as

$$\sigma(v,w) \equiv \nabla^h_V W, \qquad \text{for } v, w \in T_pM, \tag{1.15}$$

where V, W are the local vertical extensions of v, w.

2. The second fundamental form of a leaf, $S : \mathscr{H} \times \mathscr{V} \to \mathscr{V}$ is such that

$$S_X V \equiv -\nabla^v_V X \tag{1.16}$$

3. Let $X, Y \in \mathscr{H}$. Then we can write the A tensor as follows:

$$A_X Y = \frac{1}{2}[X, Y]^v \tag{1.17}$$

Let A^* denote the adjoint of the A-tensor, which is defined as:

$$g_M(A^*_X U, Y) \equiv g_M(A_X Y, U) \tag{1.18}$$

$\underline{\widetilde{R}(x,y)z}$: Vertical component.

$$\begin{aligned} \widetilde{R}^v(x,y)z = R^v(x,y)z + \left\langle \operatorname{grad}_{g_M}(\phi), x \right\rangle A_y z - \left\langle \operatorname{grad}_{g_M}(\phi), y \right\rangle A_x z \\ - 2 \left\langle \operatorname{grad}_{g_M}(\phi), z \right\rangle A_x y \end{aligned} \tag{1.19}$$

$\underline{\widetilde{R}(x,y)z}$: Horizontal component.

$$\widetilde{R}^h(x,y)z = R^h(x,y)z + (1 - e^{2\phi})(A^*_x A_y z - A^*_y A_x z - 2A^*_z A_x y) \tag{1.20}$$

$\underline{\widetilde{R}(u,v)w}$: Horizontal component.

$$\widetilde{R}^h(u,v)w = e^{2\phi}\left[(2-e^{2\phi})R^h(u,v)w \right. \\ \left. +e^{2\phi}A^*_{\mathrm{grad}_{g_M}(\phi)}\left(\langle v,w\rangle u - \langle u,w\rangle v\right)\right] \tag{1.21}$$

$\underline{\widetilde{R}(u,v)w}$: Vertical component.

$$\begin{aligned}\widetilde{R}^v(u,v)w = R^v(u,v)w &+ (1-e^{2\phi})\{S_{\sigma(v,w)}u - S_{\sigma(u,w)}v\} \\ &+ e^{2\phi}\left(S_{\mathrm{grad}_{g_M}(\phi)} - \left|\mathrm{grad}_{g_M}(\phi)\right|^2_{g_M} I\right)\left(\langle v,w\rangle u - \langle u,w\rangle v\right) \\ &+ e^{2\phi}\left(\left\langle \mathrm{grad}_{g_M}(\phi), \sigma(v,w)\right\rangle u - \left\langle \mathrm{grad}_{g_M}(\phi), \sigma(u,w)\right\rangle v\right)\end{aligned} \tag{1.22}$$

$\underline{\widetilde{R}(x,u)y}$: Vertical component.

$$\begin{aligned}\widetilde{R}^v(x,u)y = R^v(x,u)y &+ (1-e^{2\phi})A_x A^*_y u \\ &+ \left\{\mathrm{Hess}_{g_M}\phi(x,y) + \left\langle \mathrm{grad}_{g_M}(\phi), x\right\rangle \left\langle \mathrm{grad}_{g_M}(\phi), y\right\rangle\right\} u \\ &- \left(\left\langle \mathrm{grad}_{g_M}(\phi), x\right\rangle S_y u + \left\langle \mathrm{grad}_{g_M}(\phi), y\right\rangle S_x u\right)\end{aligned} \tag{1.23}$$

$\underline{\widetilde{R}^h(x,u)y}$: Horizontal component.

$$\begin{aligned}e^{-2\phi}\widetilde{R}^h(x,u)y = R^h(x,u)y &- \left\langle \mathrm{grad}_{g_M}(\phi), y\right\rangle A^*_x u \\ &- 2\left\langle \mathrm{grad}_{g_M}(\phi), x\right\rangle A^*_y u + \langle A_x y, u\rangle \,\mathrm{grad}_{g_M}(\phi)\end{aligned} \tag{1.24}$$

$\underline{\widetilde{R}(u,x)v}$: Horizontal component.

$$
\begin{aligned}
e^{-2\phi}\widetilde{R}^h(u,x)v = {} & R^h(u,x)v - (1-e^{2\phi})A^*_{A^*_x v}u \\
& + \langle u,v\rangle \left\langle \operatorname{grad}_{g_M}(\phi), x\right\rangle \operatorname{grad}_{g_M}(\phi) \\
& - \langle x, \sigma(u,v)\rangle \operatorname{grad}_{g_M}(\phi) - \left\langle \operatorname{grad}_{g_M}(\phi), x\right\rangle \sigma(u,v) \\
& + \langle u,v\rangle \left(\nabla_x \operatorname{grad}_{g_M}(\phi)\right)^h
\end{aligned} \tag{1.25}
$$

$\underline{\widetilde{R}(u,v)x}$: Horizontal component.

$$
e^{-2\phi}\widetilde{R}^h(u,v)x = R^h(u,v)x + (1-e^{2\phi})\left(A^*_{A^*_x u}v - A^*_{A^*_x v}u\right) \tag{1.26}
$$

$\underline{\widetilde{R}(x,y)u}$: Vertical component.

$$
\widetilde{R}^v(x,y)u = R^v(x,y)u + (1-e^{2\phi})\left(A_x A^*_y - A_y A^*_x\right)u \tag{1.27}
$$

$\underline{\widetilde{R}(x,y)u}$: Horizontal component.

$$
\begin{aligned}
e^{-2\phi}\widetilde{R}^h(x,y)u = {} & R^h(x,y)u - \left\langle \operatorname{grad}_{g_M}(\phi), x\right\rangle A^*_y u + \left\langle \operatorname{grad}_{g_M}(\phi), y\right\rangle A^*_x u \\
& + 2\langle A_x y, u\rangle \operatorname{grad}_{g_M}(\phi)
\end{aligned} \tag{1.28}
$$

1.3.2 The Sectional Curvature

Let v, w denote vertical fields and x, y horizontal fields, neither of which being necessarily orthonormal. Then,

$$k(v,w) \equiv \langle R(v,w)w, v\rangle \tag{1.29}$$

$$\widetilde{k}(v,w) \equiv \langle R(v,w)w, v\rangle_{\phi} \tag{1.30}$$

$$k_B(x,y) \equiv \langle R_B(\pi_* x, \pi_* y)\pi_* y, \pi_* x\rangle_B \tag{1.31}$$

The three main components of the curvature tensor w.r.t. (1.10) are given by,

$$\widetilde{k}(x,y) = (1-e^{2\phi})k_B(x,y) + e^{2\phi}k(x,y) \tag{1.32}$$

$$\begin{aligned}\widetilde{k}(x,u) = k(x,u) &- (1-e^{2\phi})|A_x^* u|^2 \left\langle \operatorname{grad}_{g_M}(\phi), x\right\rangle \langle x, \sigma(u,u)\rangle \\ &- \left(\operatorname{Hess}_{g_M}\phi(x,x) + \left\langle \operatorname{grad}_{g_M}(\phi), x\right\rangle^2\right)|u|^2\end{aligned} \tag{1.33}$$

$$\begin{aligned}e^{-4\phi}\widetilde{k}(u,v) = e^{-2\phi}(1-e^{2\phi})k_F(u,v) &+ k(u,v) - \left\langle \operatorname{grad}_{g_M}(\phi), \sigma(u,u)\right\rangle |v|^2 \\ &- \left\langle \operatorname{grad}_{g_M}(\phi), \sigma(v,v)\right\rangle |u|^2 + \left\langle \operatorname{grad}_{g_M}(\phi), \sigma(u,v)\right\rangle \langle u, v\rangle \\ &- \left|\operatorname{grad}_{g_M}(\phi)\right|^2 \left(|u|^2|v|^2 - \langle u,v\rangle^2\right)\end{aligned} \tag{1.34}$$

Chapter 2

Main Result

2.1 Pure Polarized $\mathfrak{g}$-structure

Pure polarized $\mathfrak{g}$-structures are almost global group actions except possibly on a given region, (15). This is the ideal case in the context of $\mathfrak{g}$-structures. We begin this section with a nice result for these structures, which will set the theme for subsequent results and the thesis:

Theorem 24. *Let $\mathscr{P}$ be a pure polarized $\mathfrak{g}$-structure on (M, g_M), such that M is compact and g_M is $\mathscr{P}$-invariant. Suppose that the intrinsic metric on the fibers of the orbits of the $\mathscr{P}$ action is nonnegative. Further, suppose that the horizontal component of the $(1,3)$-tensor, $R(U,V)W$ vanishes, for vectors U, V, W tangent to the orbits of $\mathscr{P}$. Then M admits a family of metrics, g_δ for $\delta > 0$, such that:*

1. $\operatorname{diam}_{g_\delta}(M) < \operatorname{diam}_{g_M}(M) < \infty$,

2. $\mathrm{vol}_{g_\delta}(M) < C \cdot \delta^\ell \cdot \mathrm{vol}_{g_M}(M)$*, where* $\ell = \dim \mathscr{O}$ *and for some* $C \in \mathbb{Z}$*,*

3. $\sec_{g_\delta}(M) \geq k$*, for some* $k \in \mathbb{R}$*.*

Remark 25. *This result is particularly nice in that the limit space is compact. As stated before, this is the best case scenario.*

Before we begin the proof, we state a useful lemma, which will not only aid in proving Theorem 24, but the next major result in the following section, as well.

Lemma 26. *Let* (M, g_M) *be closed and let* G *be a Lie group acting on* M*. Let* $g_M = g^t + g^p$ *represent the splitting of the metric on* M*, where* $g^t = g_M|_{\mathscr{V}}$ *and* $g^p = g_M|_{\mathscr{H}}$*. For* $\varepsilon > 0$*, define* $g_\varepsilon = \varepsilon^2 g^t + g^p$*. Then,*

$$II^\varepsilon(u_\varepsilon, v_\varepsilon) = II(u, v),$$

for vectors u, v *tangent to the orbits of* G*.*

Proof. Let X be perpendicular to the orbits of G and U, V tangent to the orbits of G. First, we show that the shape operator is independent of ε, that is:

$$S_X^\varepsilon U = S_X^0 U, \tag{2.1}$$

where

$$S_X^\varepsilon U = -\left(\nabla_U^{g_\varepsilon} X\right)^v = -\left(\nabla_U^\varepsilon X\right)^v$$

and

$$S^0_X U = -\left(\nabla^{g_M}_U X\right)^v = -\left(\nabla_U X\right)^v.$$

Then applying Koszul's formula, we have

$$\begin{aligned}
2\ g_\varepsilon(S^\varepsilon_X U, V) &= 2\ g_\varepsilon(-\nabla^\varepsilon_U X, V) \\
&= -[D_X\ g_\varepsilon(U,V) + D_U\ g_\varepsilon(V,X) - D_V\ g_\varepsilon(X,U) \\
&\quad - g_\varepsilon([X,U],V) - g_\varepsilon([U,V],X) + g_\varepsilon([V,X],U)],
\end{aligned}$$

since $[U,V]$ is tangent to the fibers of G and with X being orthogonal to the fibers, we have

$$\begin{aligned}
2g_\varepsilon(S^\varepsilon_X U, V) &= -[\varepsilon^2\ D_X g_M(U,V) - \varepsilon^2 g_M([V,X],U) + \varepsilon^2 g_M([X,U],V)] \\
&= 2\varepsilon^2\ g_M(-\nabla_U X, V) \\
&= 2g_\varepsilon(S^0_X U, V).
\end{aligned}$$

Repeating the above calculation and normalizing U, V w.r.t. g_ε, we get

$$g_\varepsilon\left(S^\varepsilon_X U_\varepsilon, V_\varepsilon\right) = g_M\left(S^0_X U, V\right), \tag{2.2}$$

where $U_\varepsilon = \frac{U}{\varepsilon}$. What this tells us is that the Gaussian curvature is independent of ε. To see this, let $\{E_i\}_i$ be a basis for the normal space to the fibers, i.e. $\mathscr{H}$, and consider the second fundamental form on u, v, such that $|u|_{g_M} = |v|_{g_M} = 1$, where U and V are extensions of u, v respectively:

$$\begin{aligned} II^\varepsilon(u_\varepsilon, v_\varepsilon) &= \left(\nabla^\varepsilon_{U_\varepsilon} V_\varepsilon\right)^\perp \\ &= \sum_i g_\varepsilon(\nabla^\varepsilon_{U_\varepsilon} V_\varepsilon, E_i) E_i \\ &= -\sum_i g_\varepsilon(V_\varepsilon, \nabla^\varepsilon_U E_i) E_i \\ &= -\sum_i g_M(V, \nabla_U E_i) E_i, \qquad \text{by (2.2)} \\ &= \sum_i g_M(\nabla_U V, E_i) E_i \\ &= (\nabla_U V)^\perp \\ &= II(u, v). \end{aligned} \tag{2.3}$$

□

And now, we prove Theorem 24.

Proof. Let $\mathscr{V}$ denote the distribution of vectors tangent to the orbits of $\mathscr{P}$ and $\mathscr{H}$ the distribution of vectors orthogonal to the orbits of $\mathscr{P}$. Then define

$$g^t = g_M|_{\mathscr{V}}, \qquad \text{and}$$

$$g^p = g_M|_{\mathscr{H}}$$

and write g_M such that,

$$g_M = g^t + g^p.$$

Let $\delta \in \mathbb{R}$. Then define the metric g_δ as follows:

$$g_\delta = \delta^2 g^t + g^p.$$

Next, to prove (1), notice that g_δ decreases as $\delta \to 0$, and so

$$\operatorname{diam}_{g_\delta}(M) < \operatorname{diam}_{g_M}(M) < \infty.$$

Note that (2) is a consequence of scaling the metric and we also have that $\exists C$ such that:

$$\operatorname{vol}_{g_\delta}(M) < C \cdot \delta^\ell \cdot \operatorname{vol}_{g_M}(M) < \infty,$$

where $\ell = \dim \mathscr{O}$.

Next, let $\phi = \log \delta$, (1.10). Then $\operatorname{grad}_{g_\delta}(\phi) = 0$. Likewise, $\operatorname{Hess}_{g_\delta}\phi(v, w) = 0$, for any $v, w \in T_pM$. To see that $\sec_{g_\delta}(M)$ is bounded below, consider $v, w \in T_pM$. We can write these vectors as a sum of the tangential and orthogonal components with respects to $\mathscr{P}$, $v = u_1 + x_1$, $w = u_2 + x_2$. Extending these vectors to vector fields, we get $V = U_1 + X_1$, $W = U_2 + X_2$. Then,

$$\begin{aligned}
\left(\widetilde{R} - R\right)(V, W, W, V) &= \left(\widetilde{R} - R\right)(U_1 + X_1, U_2 + X_2, U_2 + X_2, U_1 + X_1) \\
&= \left(\widetilde{R} - R\right)(U_1, U_2, U_2, U_1) + \left(\widetilde{R} - R\right)(U_1, U_2, U_2, X_1) \\
&+ \left(\widetilde{R} - R\right)(U_1, U_2, X_2, U_1) + \left(\widetilde{R} - R\right)(U_1, U_2, X_2, X_1) \\
&+ \left(\widetilde{R} - R\right)(U_1, X_2, U_2, U_1) + \left(\widetilde{R} - R\right)(U_1, X_2, U_2, X_1) \\
&+ \left(\widetilde{R} - R\right)(U_1, X_2, X_2, U_1) + \left(\widetilde{R} - R\right)(U_1, X_2, X_2, X_1) \\
&+ \left(\widetilde{R} - R\right)(X_1, U_2, U_2, U_1) + \left(\widetilde{R} - R\right)(X_1, U_2, U_2, X_1) \\
&+ \left(\widetilde{R} - R\right)(X_1, U_2, X_2, U_1) + \left(\widetilde{R} - R\right)(X_1, U_2, X_2, X_1) \\
&+ \left(\widetilde{R} - R\right)(X_1, X_2, U_2, U_1) + \left(\widetilde{R} - R\right)(X_1, X_2, U_2, X_1) \\
&+ \left(\widetilde{R} - R\right)(X_1, X_2, X_2, U_1) + \left(\widetilde{R} - R\right)(X_1, X_2, X_2, X_1)
\end{aligned} \tag{2.4}$$

Since the gradient and hessian of ϕ vanish, the tensor equations (1.19) - (1.28) simplify nicely. Now we proceed with the curvature calculation. The following are the tensor equations which are bounded below and/or converge to 0 as $\delta \to 0$:

1. (see (1.19)):

$$\widetilde{R}(X_1, X_2, X_2, U_\delta) = \delta R(X_1, X_2, X_2, U)$$

2. (see (1.20)):

$$(\widetilde{R} - R)(X_1, X_2, X_2, X_1) = -3(1-\delta^2)\ g_\delta(A^*_{X_2} A_{X_1} X_2, X_1)$$
$$= -3(1-\delta^2) g_M(A^*_{X_2} A_{X_1} X_2, X_1) \qquad (2.5)$$

3. (see (1.23)):

$$\widetilde{R}(X, U_\delta, X, U_\delta) = R(X, U_\delta, X, U_\delta) + (1-\delta^2)\ g_\delta(A_X A^*_X U_\delta, U_\delta)$$
$$= R(X, U, X, U) + (1-\delta^2)\ g_M(A_X A^*_X U, U) \qquad (2.6)$$

4. (see (1.26)):

$$\widetilde{R}(U_\delta, V_\delta, X, Y) = \delta^2 R(U_\delta, V_\delta, X, Y)$$
$$+ \delta^2(1-\delta^2)\ g_\delta\left(A^*_{A^*_X U_\delta}(V_\delta) - A^*_{A^*_X V_\delta} U_\delta, Y\right)$$
$$= R(U, V, X, Y) + (1-\delta^2)\ g_M(A^*_{A^*_X U} V - A^*_{A^*_X V} U, Y) \qquad (2.7)$$

We calculate the remaining tensor equations, which converge to $+\infty$ as $\delta \to 0$

(1.21):

$$\begin{aligned}\widetilde{R}(U_\delta, V_\delta, V_\delta, X) &= \delta^2(2-\delta^2)R(U_\delta, V_\delta, V_\delta, X)\\ &= \left(\frac{2-\delta^2}{\delta}\right)R(U,V,V,X) = 0,\end{aligned}$$

since $R(U,V,V,X) = 0$ by assumption. Lastly, we have:

$$\begin{aligned}\widetilde{R}(U_\delta, V_\delta, V_\delta, U_\delta) = \widetilde{R}^{\text{int}}&(U_\delta, V_\delta, V_\delta, U_\delta) - II^\delta(U_\delta, U_\delta)\ II^\delta(V_\delta, V_\delta)\\ &+ II^\delta(U_\delta, V_\delta)\ II^\delta(V_\delta, U_\delta)\\ = \frac{1}{\delta^2}&R^{\text{int}}(U,V,V,U) - II(V,V)\ II(U,U)\\ &+ II(U,V)\ II(V,U), \qquad \text{by Lemma 26},\end{aligned}$$

which goes to $+\infty$ as $\delta \to 0$. As we can see, we have the desired lower curvature bound. □

2.2 Polarized $\mathfrak{g}$-structure

Let $\mathscr{A} = \{(U_\alpha, \mathscr{G}_\alpha)\}_\alpha$ be an atlas for $\mathscr{G}$. We now consider the case where our $\mathfrak{g}$-structure is not pure, or in other words, the dimension of the orbits of $\mathscr{G}_\alpha$ possibly varies with α. As before, we split the metric into its vertical and horizontal components. However, now our scaling factor is no longer a constant function on

U_α, hence, scaling the metric brings up the issue of the metric blowing up as we transition from one chart to another. By choosing an atlas such that the charts admit a transnormal function, which will be constant on the fibers of a given sheaf action, we no longer have an issue. Consequently, the following result is a stronger analog of Theorem 24.

Theorem 27. *Let $\mathscr{G}$ be a polarized $\mathfrak{g}$-structure on a compact manifold (M, g_M), where g_M is $\mathscr{G}$-invariant. Suppose further that the intrinsic metric on the fibers of the orbits of the $\mathscr{G}$ action on M are normal homogeneous. Let U, V, W be vectors tangent to the orbits of $\mathscr{G}$ and suppose that the horizontal component of $R(U,V)W$ vanishes. Then M admits a sequence of metrics g_δ such that:*

1. $\operatorname{diam}_{g_\delta}(M) < \operatorname{diam}_{g_M}(M) |\log \delta|$,
2. $\operatorname{vol}_{g_\delta}(M) < \operatorname{vol}_{g_M}(M) \cdot |\log \delta|^n \cdot \delta^l$*, where* $l = \min_{i=1,\dots,m} \dim \mathscr{O}_i$*, and* $n = \dim M$*,*
3. $\sec_{g_\delta}(M) \geq k$*, for* $k \in \mathbb{R}$*.*

Remark 28. *Note that $\operatorname{vol}_{g_\delta}(M) \to 0$ as $\delta \to 0$, by applying L'hôpital's rule appropriately. Although for a fixed δ we have that $\operatorname{diam}_{g_\delta}(M) < \infty$, we do not get the desired compact limit as $\delta \to 0$ as with (24). We will comment on this later.*

To prove this result, we will prove a local version first. In order to prove the lemma, or the local the version of the main result, we will state and prove additional results. Then let

$$g_{\delta_0} = \log^2 \delta \cdot g_M, \tag{2.8}$$

for some $\delta \in (0,1)$.

Lemma 29. *Let G be a Lie group acting on M. Let $W \subset M$ be G-invariant. Suppose further that the intrinsic metric on the fibers of the orbits of the G action on W are normal homogenous. Let $V \subset W$ be such that $\overline{V} \subset W$ and suppose that V is G-invariant. Let $U = B(V,b)$, where $b > 0$ is sufficiently small. Suppose that the horizontal component of the $(1,3)$-tensor $R(U_1,U_2)U_3$ vanishes, for vectors U_i tangent to the orbits of G. Then, M admits a sequence of metrics g_δ on U such that*

1. $\operatorname{diam}_{g_\delta}(M) < \operatorname{diam}_{g_M}(M) \cdot |\log \delta|$,
2. $\operatorname{vol}_{g_\delta}(V) < \operatorname{vol}_{g_M}(V) \cdot |\log \delta|^n \cdot \delta^\ell$, *where* $\ell = \dim \mathscr{O}$,
3. $\sec_{g_\delta}(M) \geq k$, *for* $k \in \mathbb{R}$,
4. $g_\delta = \log^2 \delta \; g_M$ *on* $M \setminus U$.

Lemma 30. *Let F be a real-valued map on M. Consider the metric (2.8) on M. Then we have the following:*

1. $\operatorname{grad}_{g_{\delta_0}}(F) = \frac{1}{\log^2 \delta} \operatorname{grad}_g(F)$
2. $\operatorname{Hess}_{g_{\delta_0}} F(v,w) = \operatorname{Hess}_g F(v,w)$ *for all* $v, w \in T_pM$.

Proof. 1.: We have,

$$\begin{aligned} g_{\delta_0}(\text{grad}_{g_{\delta_0}}(F), v) &= D_v(F) \\ &= g(\text{grad}_g(F), v) \\ &= \frac{1}{\log^2 \delta} \cdot g_{\delta_0}(\text{grad}_g(F), v) \end{aligned}$$

i.e.,

$$\text{grad}_{g_{\delta_0}}(F) = \frac{\text{grad}_g(F)}{\log^2 \delta} \tag{2.9}$$

2.: Now we will show that $\text{Hess}_{g_{\delta_0}} F = \text{Hess}_g F$. For $v, w \in T_pM$, we have

$$\begin{aligned} \text{Hess}_{g_{\delta_0}} F(v, w) &= g_{\delta_0}(\nabla_v \text{grad}_{g_{\delta_0}}(F), w) \\ &= g_{\delta_0}(\nabla_v \left(\frac{\text{grad}_g(F)}{\log^2 \delta} \right), w), \qquad \text{by (2.9)} \\ &= \log^2 \delta \cdot g(\nabla_v \left(\frac{\text{grad}_g(F)}{\log^2 \delta} \right), w) \\ &= g(\nabla_v \text{grad}_g(F), w) \\ &= \text{Hess}_g F(v, w) \end{aligned}$$

□

Lemma 31. *Let (M, g) be a Riemannian manifold. Consider the map $\phi = \log F$, for* *some* $F : M \to \mathbb{R}^+$. *Then,*

$$\mathrm{Hess}_g\phi(v,w) + g(\mathrm{grad}_g(\phi), v)\ g(\mathrm{grad}_g(\phi), w) = \frac{\mathrm{Hess}_g F(v,w)}{F} \tag{2.10}$$

for all $v, w \in T_pM$.

Proof. Before we prove the claim, we will calculate the gradient of ϕ. For any $v \in T_pM$, we have

$$\begin{aligned} g(\mathrm{grad}_g(\phi), v) &= D_v(\phi) \\ &= \frac{D_v(F)}{F} \\ &= g\left(\frac{\mathrm{grad}_g(F)}{F}, v\right). \end{aligned}$$

Then,

$$\mathrm{grad}_g(\phi) = \frac{\mathrm{grad}_g(F)}{F}$$

Now, we proceed with proving the claim:

$$\begin{aligned} \mathrm{Hess}_g\phi(v,w) &= g\left(\nabla_v \mathrm{grad}_g(\phi), w\right) \\ &= g\left(\nabla_v\left(\frac{\mathrm{grad}_g(F)}{F}\right), w\right) \\ &= g\left(\frac{\nabla_v \mathrm{grad}_g(F)}{F} - \frac{D_v(F)\cdot \mathrm{grad}_g(F)}{F^2}, w\right) \end{aligned}$$

$$= \frac{\mathrm{Hess}_g F(v,w)}{F} - g\left(\frac{\mathrm{grad}_g(F)}{F}, v\right) g\left(\frac{\mathrm{grad}_g(F)}{F}, w\right)$$
$$= \frac{\mathrm{Hess}_g F(v,w)}{F} - g(\mathrm{grad}_g(\phi), v)\ g(\mathrm{grad}_g(\phi), w).$$

Thus,

$$\mathrm{Hess}_g \phi(v,w) + g(\mathrm{grad}_g(\phi), v)\ g(\mathrm{grad}_g(\phi), w) = \frac{\mathrm{Hess}_g F(v,w)}{F}.$$

□

Proposition 32. *Let G be a Lie group acting on M. Consider a G-invariant open set $W \subset M$. Let $V \subset W$ be a precompact G-invariant set. Define $U = B(V, b)$, for sufficiently small $b > 0$. Then there exists a transnormal function $f : U \to [1, e]$ such that*

1. $f|_V \equiv e$,
2. $f|_{\partial U} \equiv 1$,
3. *f is constant along each of the fibers of G,*
4. $\frac{|\mathrm{grad}_{g_M}(f)|_{g_M}}{f} < \infty$,
5. $\frac{\mathrm{Hess}_{g_M} f}{f} < \infty$.

Proof. By [9], there exists a transnormal function $r : U \to [0,1]$ such that:

$$r(x) = \frac{\text{dist}(V, x)}{\text{dist}(V, x) + \text{dist}(M \setminus U, x)}. \tag{2.11}$$

Note that $r|_V \equiv 0$ and $r \to 1$ as $x \to \partial U$. Next define $\psi : [0,1] \to [0,1]$ such that:

$$\psi(t) = \begin{cases} 0 & \text{if} \quad t \in [0, \frac{1}{100}] \\ 1 & \text{if} \quad t \in [\frac{99}{100}, 1] \end{cases}, \tag{2.12}$$

where $\psi'(t) > 0$ for all $t \in (\frac{1}{100}, \frac{99}{100})$. We choose b small enough so that $\text{dist}(V, \cdot)$ and $\text{dist}(M \setminus U, \cdot)$ are smooth on a neighborhood of $U \setminus V$. Consequently, the condition on b is such that $\psi \circ r$ is smooth. On V, we have that the gradient and hessian of r are bounded. Near ∂U, we also get that the gradient and hessian of r are bounded. Note that $\psi \in C^\infty(\mathbb{R})$, and so its gradient and hessian are bounded on its domain.

Next, consider $\overline{f} : [0,1] \to [1, e]$ such that:

$$\overline{f}(t) = (1-e)t + e.$$

Then, our transnormal function, $f : U \to [1, e]$ is defined as:

$$f(\cdot) \equiv (\overline{f} \circ \psi \circ r)(\cdot) = (1-e)(\psi \circ r)(\cdot) + e. \tag{2.13}$$

Note that f is smooth and bounded.

It is easy to see that $f|_V \equiv e$, since

$$f|_V = \overline{f}\left(\psi(r|_V)\right) = \overline{f}\left(\psi(0)\right) = \overline{f}(0) = e. \tag{2.14}$$

Next, we have that as $x \to \partial U$,

$$f(x) = \overline{f}\left(\psi(r(x))\right) \to \overline{f}(\psi(1)) = \overline{f}(1) = 1, \tag{2.15}$$

which shows (2).

Lastly, observe that (3) easily follows by observing that f is constant along the orbits of G, since V and U are G-invariant.

To see 4), consider the gradient of f:

$$\begin{aligned}\operatorname{grad}_{g_M}(f) &= \overline{f}'\left(\psi(r(x))\right) \cdot \operatorname{grad}_{g_M}(\psi)\\ &= (1-e) \cdot \operatorname{grad}_{g_M}(\psi).\end{aligned}$$

Then,

$$\begin{aligned}\frac{|\operatorname{grad}_{g_M}(f)|_{g_M}}{f} &= \frac{(1-e)|\operatorname{grad}_{g_M}(\psi)|_{g_M}}{(1-e)\psi(r(x))+e}\\ &< \frac{(1-e)C_1}{(1-e)\psi(r(x))+e}, \qquad \text{since } \operatorname{grad}_{g_M}(\psi) \text{ is bounded,}\\ &\overset{x\to\partial U}{\longrightarrow} (1-e)C_1 < \infty.\end{aligned}$$

Let $v, w \in T_pM$. Before showing 5), let us calculate the hessian of f:

$$\begin{aligned}
\mathrm{Hess}_{g_M} f(v,w) &= g_M(\nabla_v \mathrm{grad}_{g_M}(f), w) \\
&= g_M(\nabla_v (1-e)\mathrm{grad}_{g_M}(\psi), w) \\
&= (1-e)\mathrm{Hess}_{g_M}\psi(v,w).
\end{aligned}$$

Then, to see 5) we have:

$$\begin{aligned}
\frac{\mathrm{Hess}_{g_M} f(v,w)}{f} &= \frac{(1-e)\mathrm{Hess}_{g_M}\psi(v,w)}{(1-e)\psi(r(x))+e} \\
&< \frac{(1-e)C_2}{(1-e)\psi(r(x))+e}, \qquad \text{since } \mathrm{Hess}_{g_M}\psi \text{ is bounded,} \\
&\overset{x\to\partial U}{\longrightarrow} (1-e)C_2 < \infty.
\end{aligned}$$

□

Now we proceed with the proof of Lemma 29.

Proof. Let $f : U \to [1, e]$ be the transnormal function from Proposition 32. Let $\delta \in (0,1)$. Consider the metric defined in (2.8) on M. Define

$$\rho = \delta^{\log f}. \tag{2.16}$$

Let $\mathscr{V}$ = distribution of vectors tangent to the orbits of G; $\mathscr{H}$ = distribution of vectors perpendicular to the orbits of G. Let

$$g^t = g_{\delta_0}|_{\mathscr{V}}, \quad \text{and}$$

$$g^p = g_{\delta_0}|_{\mathscr{H}}.$$

Then we define

$$g_\delta = \begin{cases} \rho^2 \ g^t + g^p & \text{if} \quad U \\ g_{\delta_0} & \text{if} \quad M \setminus U. \end{cases}$$

For $v, w \in T_pU$, we have that:

$$g_{\delta_0}(v, w) \geq g_\delta(v, w), \tag{2.17}$$

since g_δ decreases as $\delta \to 0$. Then,

$$\text{diam}_{g_{\delta_0}}(M) = |\log \delta| \cdot \text{diam}_{g_M}(M) > \text{diam}_{g_\delta}(M).$$

Again, by (2.17), we have

$$\text{vol}_{g_\delta}(V) < \text{vol}_{g_{\delta_0}}(V) = |\log^n \delta| \cdot \delta^\ell \cdot \text{vol}_{g_M}(V).$$

Now, let

$$\phi = \log \rho. \tag{2.18}$$

The goal now is to show that the Gromoll-Walschap equations are bounded below. For convenience in our calculations, we begin by listing some notation and calculating certain terms that appear in the tensor equation calculations. Let us begin by calculating the gradient of (2.18). For $v \in T_pM$:

$$\begin{aligned} D_v(\rho) &= D_v\left(e^{\log\delta\cdot\log(f)}\right) \\ &= \log\delta\frac{D_v(f)}{f}e^{\log\delta\cdot\log(f)} \\ &= \log\delta\cdot\rho\cdot\frac{D_v(f)}{f}. \end{aligned}$$

Then, for any metric, g, on M, we have

$$\begin{aligned} g(\mathrm{grad}(\phi), v) &= D_v(\phi) \\ &= \frac{D_v(\rho)}{\rho} \\ &= \log\delta\frac{D_v(f)}{f} \\ &= g(\log\delta\frac{\mathrm{grad}(f)}{f}, v), \end{aligned}$$

i.e.

$$\mathrm{grad}(\phi) = \log\delta\frac{\mathrm{grad}(f)}{f}. \tag{2.19}$$

Next, we calculate the hessian of ϕ. For $v, w \in T_pM$, we have by Lemma 31,

$$\mathrm{Hess}_g\phi(v,w) = \frac{\mathrm{Hess}_g\rho(v,w)}{\rho} - g(\mathrm{grad}_{g_M}(\phi), v)\ g(\mathrm{grad}_{g_M}(\phi), w). \tag{2.20}$$

Let us make a short side note and calculate the hessian of ρ:

$$\begin{aligned}
\mathrm{Hess}_g\rho(v,w) &= g(\nabla_v \mathrm{grad}(\rho), w) \\
&= g\left(\nabla_v\left(\log\delta\cdot\rho\cdot\frac{\mathrm{grad}(f)}{f}\right), w\right) \\
&= g\left(\log^2\delta\cdot\rho\cdot\frac{D_v(f)}{f}\mathrm{grad}(f) - \log\delta\cdot\rho\cdot\frac{D_v(f)}{f^2}\mathrm{grad}(f)\right. \\
&\quad \left. + \log\delta\cdot\rho\cdot\frac{\nabla_v\mathrm{grad}(f)}{f}, w\right) \\
&= \rho\left[\log^2\delta g\left(\frac{\mathrm{grad}(f)}{f}, v\right) g\left(\frac{\mathrm{grad}(f)}{f}, w\right)\right. \\
&\quad - \log\delta g\left(\frac{\mathrm{grad}(f)}{f}, v\right)\ g\left(\frac{\mathrm{grad}(f)}{f}, w\right) \\
&\quad \left. + \log\delta\frac{\mathrm{Hess}_g f(v,w)}{f}\right].
\end{aligned}$$

Dividing by ρ, we get

$$\begin{aligned}
\frac{\mathrm{Hess}_g\rho(v,w)}{\rho} &= \log^2\delta g\left(\frac{\mathrm{grad}(f)}{f}, v\right) g\left(\frac{\mathrm{grad}(f)}{f}, w\right) \\
&\quad - \log\delta g\left(\frac{\mathrm{grad}(f)}{f}, v\right)\ g\left(\frac{\mathrm{grad}(f)}{f}, w\right) \\
&\quad + \log\delta\frac{\mathrm{Hess}_g f(v,w)}{f}.
\end{aligned} \tag{2.21}$$

Now, (2.20) simplifies to:

$$\mathrm{Hess}_g\phi(v,w) = \log\delta\left[\frac{\mathrm{Hess}_g f(v,w)}{f} - g\left(\frac{\mathrm{grad}(f)}{f}, v\right)\ g\left(\frac{\mathrm{grad}(f)}{f}, w\right)\right]. \tag{2.22}$$

Again for convenience, we calculate a term that we will refer to shortly when we calculate the tensor equations to show l.c.b. Then,

$$\begin{aligned}
\left|\mathrm{grad}_{g_{\delta_0}}(\phi)\right|^2_{g_{\delta_0}} &= g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), \mathrm{grad}_{g_{\delta_0}}(\phi)) \\
&= \log^2\delta\ g_M\left(\mathrm{grad}_{g_{\delta_0}}(\phi), \mathrm{grad}_{g_{\delta_0}}(\phi)\right) \\
&= \log^2\delta\ g_M\left(\frac{\mathrm{grad}_{g_M}(\phi)}{\log^2\delta}, \frac{\mathrm{grad}_{g_M}(\phi)}{\log^2\delta}\right), \qquad \text{by Lemma 30,} \\
&= \frac{1}{\log^2\delta} g_M\left(\log\delta\frac{\mathrm{grad}_{g_M}(f)}{f}, \log\delta\frac{\mathrm{grad}_{g_M}(f)}{f}\right) \\
&= \frac{\left|\mathrm{grad}_{g_M}(f)\right|^2_{g_M}}{f^2}.
\end{aligned}$$

Then, taking the square root of both sides gives us

$$\left|\mathrm{grad}_{g_{\delta_0}}(\phi)\right|_{g_{\delta_0}} = \frac{\left|\mathrm{grad}_{g_M}(f).\right|_{g_M}}{f}. \tag{2.23}$$

In order to distinguish the curvature tensors that will appear in our tensor calculations and to correctly apply the Gromoll-Walschap tensor equations, consider the following note:

1. Let $\widetilde{R}^{\rho}$ correspond to $g_{\delta} = \rho^2\ g^t + g^p$ (on U),

2. R^{δ} correspond to $g_{\delta_0} = g^t + g^p$ (on M), and

3. R correspond to g_M.

Let

$$X = \frac{\text{grad}_{g_M}(\phi)}{|\text{grad}_{g_M}(\phi)|_{g_M}} = \frac{\log\delta}{|\log\delta|}\frac{\text{grad}_{g_M}(f)}{|\text{grad}_{g_M}(f)|_{g_M}} = \frac{\log\delta}{|\log\delta|}\overline{X},$$

where $\overline{X} = \frac{\text{grad}_{g_M}(f)}{|\text{grad}_{g_M}(f)|_{g_M}}$ and X a unit vector in g_M. The corresponding unit vector in g_{δ_0} is

$$X_{\delta} = \frac{X}{\log\delta}.$$

Given that X is a horizontal vector, we do not need to write the equivalent unit vector in g_{δ}, but as we do need to have notation for these vectors, consider the unit vertical vector in g_{δ}:

$$U_{\rho} = \frac{U_{\delta}}{\rho} = \frac{U}{\rho\cdot\log\delta},$$

where U is unit in g_M, and U_{δ} is unit in g_{δ_0}. Let Y be a horizontal vector orthogonal to $\text{grad}_{g_M}(\phi)$. Then, we have:

1. (see (1.19)):

$$
\begin{aligned}
\widetilde{R}^{\rho}(X_\delta, Y_\delta, X_\delta, U_\rho) = R^{\delta}(X_\delta, Y_\delta, X_\delta, U_\rho) \; + g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), X_\delta) g_\delta(A_{Y_\delta} X_\delta, U_\rho) \\
- g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), Y_\delta)\; g_\delta(A_{X_\delta} X_\delta, U_\rho) - 2 g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), X_\delta)\; g_\delta(A_{X_\delta} Y_\delta, U_\rho) \\
= \frac{\rho}{\log^4 \delta} R^{\delta}(X, Y, X, U) - \frac{3\rho}{\log^3 \delta} \left| \mathrm{grad}_{g_{\delta_0}}(\phi) \right|_{g_{\delta_0}} g_{\delta_0}(A_X Y, U) \\
= \frac{\rho}{\log^2 \delta} \frac{\log^2 \delta}{|\log \delta|^2} R(\overline{X}, Y, \overline{X}, U) - \frac{3\rho}{\log \delta} \frac{\log \delta}{|\log \delta|} \frac{|\mathrm{grad}_{g_M}(f)|_{g_M}}{f} g_M(A_{\overline{X}} Y, U),
\end{aligned}
$$

by (2.23),

$$
= \frac{\rho}{\log^2 \delta} R(\overline{X}, Y, \overline{X}, U) - \frac{3\rho}{|\log \delta|} \frac{|\mathrm{grad}_{g_M}(f)|_{g_M}}{f} g_M(A_{\overline{X}} Y, U), \tag{2.24}
$$

and by applying Proposition 32, we see that tensor goes to zero as $\delta \to 0$.

2. (see (1.20))

$$
\begin{aligned}
\widetilde{R}^{\rho}(X_\delta, Y_\delta, X_\delta, Y_\delta) = R^{\delta}(X_\delta, Y_\delta, X_\delta, Y_\delta) + (1-\rho^2)\left[g_\delta(A^*_{X_\delta} A_{Y_\delta} X_\delta \right. \\
\left. - A^*_{Y_\delta} A_{X_\delta} X_\delta - 2 A^*_{X_\delta} A_{X_\delta} Y_\delta, Y_\delta \right] \\
= \frac{1}{\log^4 \delta} R^{\delta}(X, Y, X, Y) - \frac{3(1-\rho^2)}{\log^4 \delta} g_{\delta_0}(A^*_X A_X Y, Y) \\
= \frac{1}{\log^2 \delta} R(X, Y, X, Y) - \frac{3(1-\rho^2)}{\log^2 \delta} g_M(A^*_X A_X Y, Y),
\end{aligned} \tag{2.25}
$$

which goes to zero as $\delta \to 0$.

3. (see (1.21))

$$
\begin{aligned}
\widetilde{R}^{\rho}(U_\rho, V_\rho, W_\rho, Z_\delta) &= \rho^2 \Big[(2-\rho^2) R^\delta(U_\rho, V_\rho, W_\rho, Z_\delta) \\
&+ \rho^2 g_{\delta_0}(V_\rho, W_\rho) g_\delta(A^*_{\mathrm{grad}_{g_{\delta_0}}(\phi)} U_\rho, Z_\delta) \\
&- \rho^2 g_{\delta_0}(U_\rho, W_\rho) g_\delta(A^*_{\mathrm{grad}_{g_{\delta_0}}(\phi)} V_\rho, Z_\delta)\Big] \\
&= \frac{\rho^2(2-\rho^2)}{\rho^3 \cdot \log^2\delta} R^\delta(U,V,W,Z) + \frac{\rho^4}{\rho^3 \cdot \log^3 \delta}\Bigg[g_{\delta_0}(V,W) g_{\delta_0}(A^*_{\frac{\mathrm{grad}_{g_{\delta_0}}(f)}{f}} U, Y) \\
&\quad - g_{\delta_0}(U,W) g_{\delta_0}(A^*_{\frac{\mathrm{grad}_{g_{\delta_0}}(f)}{f}} V, Y)\Bigg] \\
&= \frac{(2-\rho^2)}{\rho \cdot \log\delta} R(U,V,W,Z) + \frac{\rho}{f \cdot \log\delta}\Big[g_M(V,W) g_M(A^*_{\mathrm{grad}_{g_M}(f)} U, Z) \\
&\quad - g_M(U,W) g_M(A^*_{\mathrm{grad}_{g_M}(f)} V, Z)\Big], \quad \text{by assumption, } R(U,V,W,Z) = 0, \\
&= \frac{\rho}{f \cdot \log\delta}\Big[g_M(V,W) g_M(A^*_{\mathrm{grad}_{g_M}(f)} U, Z) - g_M(U,W) g_M(A^*_{\mathrm{grad}_{g_M}(f)} V, Z)\Big],
\end{aligned} \tag{2.26}
$$

which goes to zero as $\delta \to 0$.

4. (see (1.23))

   (i) $Y \perp \mathrm{grad}_{g_M}(\phi)(\propto X)$:

$$
\begin{aligned}
\widetilde{R}^{\rho}(X_\delta, U_\rho, Y_\delta, V_\rho) &= R^\delta(X_\delta, U_\rho, Y_\delta, V_\rho) + (1-\rho^2)\ g_\delta(A_{X_\delta} A^*_{Y_\delta} U_\rho, V_\rho) \\
&+ \Big[\mathrm{Hess}_{g_{\delta_0}} \phi(X_\delta, Y_\delta) + g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), X_\delta) g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), Y_\delta)\Big]\ g_\delta(U_\rho, V_\rho) \\
&- \Big[g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), X_\delta) g_\delta(S^\delta_{Y_\delta} U_\rho, V_\rho) + g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), Y_\delta) g_\delta(S^\delta_{X_\delta} U_\rho, V_\rho)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\log^2\delta} R(X,U,Y,V) + \frac{(1-\rho^2)}{\log^2\delta} g_M(A_X A_Y^* U,V) \\
&\quad + \log\delta \frac{\mathrm{Hess}_{g_{\delta_0}} f(X_\delta,Y_\delta)}{f} \cdot g_M(U,V) \\
&\quad - \frac{1}{\log^2\delta} \frac{|\mathrm{grad}_{g_M}(f)|_{g_M}}{f} g_M(S_Y^0 U,V), \qquad \text{by (2.22) and (2.23)} \\
&= \frac{1}{\log^2\delta} \frac{\log\delta}{|\log\delta|} R(\overline{X},U,Y,V) + \frac{(1-\rho^2)}{\log^2\delta} \frac{\log\delta}{|\log\delta|} g_M(A_{\overline{X}} A_Y^* U,V) \\
&\quad + \frac{1}{\log\delta} \frac{\log\delta}{|\log\delta|} \frac{\mathrm{Hess}_{g_M} f(\overline{X},Y)}{f} \cdot g_M(U,V) \\
&\quad - \frac{1}{\log^2\delta} \frac{|\mathrm{grad}_{g_M}(f)|_{g_M}}{f} g_M(S_Y^0 U,V) \\
&= \frac{1}{\log\delta|\log\delta|} R(\overline{X},U,Y,V) + \frac{(1-\rho^2)}{\log\delta|\log\delta|} g_M(A_{\overline{X}} A_Y^* U,V) \\
&\quad + \frac{1}{|\log\delta|} \frac{\mathrm{Hess}_{g_M} f(\overline{X},Y)}{f} \cdot g_M(U,V) \\
&\quad - \frac{1}{\log^2\delta} \frac{|\mathrm{grad}_{g_M}(f)|_{g_M}}{f} g_M(S_Y^0 U,V),
\end{aligned}
\tag{2.27}
$$

by using Proposition 32, the tensor goes to zero as $\delta \to 0$.

(ii) Horizontal vectors are both X :

$$
\begin{aligned}
\widetilde{R}^\rho(X_\delta,U_\rho,X_\delta,V_\rho) &= R^\delta(X_\delta,U_\rho,X_\delta,V_\rho) + (1-\rho^2) g_\delta(A_{X_\delta} A_{X_\delta}^* U_\rho,V_\rho) \\
&\quad + \left[\mathrm{Hess}_{g_{\delta_0}}\phi(X_\delta,X_\delta) + g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi),X_\delta)^2\right] g_\delta(U_\rho,V_\rho) \\
&\quad - \left[2 g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi),X_\delta) g_\delta(S_{X_\delta} U_\rho,V_\rho)\right] \\
&= \frac{1}{\log^2\delta}\left(\frac{\log\delta}{|\log\delta|}\right)^2 R(\overline{X},U,\overline{X},V) + \frac{(1-\rho^2)}{\log^2\delta}\left(\frac{\log\delta}{|\log\delta|}\right)^2 g_M(A_{\overline{X}} A_{\overline{X}}^* U,V) \\
&\quad + \left[\frac{1}{\log\delta}\left(\frac{\log\delta}{|\log\delta|}\right)^2 \frac{\mathrm{Hess}_{g_{\delta_0}} f(\overline{X},\overline{X})}{f} - \frac{1}{\log^2\delta}\frac{|\mathrm{grad}_{g_M}(f)|_{g_M}^2}{f^2}\right] g_M(U,V)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{\log^2\delta}\frac{\log\delta}{|\log\delta|}\frac{|\mathrm{grad}_{g_M}(f)|_{g_M}}{f} g_M(S^0_X U, V) \\
&= \frac{1}{\log^2\delta} R(\overline{X}, U, \overline{X}, V) + \frac{(1-\rho^2)}{\log^2\delta} g_M(A_{\overline{X}} A^*_{\overline{X}} U, V) \\
&+ \left[\frac{1}{\log\delta}\frac{\mathrm{Hess}_{g_M} f(\overline{X}, \overline{X})}{f} - \frac{1}{\log^2\delta}\frac{|\mathrm{grad}_{g_M}(f)|^2_{g_M}}{f^2}\right] g_M(U, V) \\
&- \frac{2}{\log\delta|\log\delta|}\frac{|\mathrm{grad}_{g_M}(f)|_{g_M}}{f} g_M(S^0_{\overline{X}} U, V),
\end{aligned}
\tag{2.28}
$$

and using Proposition 32, the tensor goes to zero as $\delta \to 0$.

(iii) Horizontal vectors are both Y :

$$
\begin{aligned}
\widetilde{R}^\rho(Y_\delta, U_\rho, Y_\delta, V_\rho) &= R^\delta(Y_\delta, U_\rho, Y_\delta, V_\rho) + (1-\rho^2)\ g_\delta(A_{Y_\delta} A^*_{Y_\delta} U_\rho, V_\rho) \\
&+ \left[\mathrm{Hess}_{g_{\delta_0}} \phi(Y_\delta, Y_\delta) + g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), Y_\delta)^2\right] g_\delta(U_\rho, V_\rho) \\
&- 2 g_{\delta_0}(\mathrm{grad}_{g_{\delta_0}}(\phi), Y_\delta) g_\delta(S^\delta_{Y_\delta} U_\rho, V_\rho) \\
&= \frac{1}{\log^2\delta} R(Y, U, Y, V) + \frac{(1-\rho^2)}{\log^2\delta} g_M(A_Y A^*_Y U, V) \\
&+ \log\delta \frac{\mathrm{Hess}_{g_{\delta_0}} f(Y_\delta, Y_\delta)}{f} \cdot g_M(U, V) \\
&= \frac{1}{\log^2\delta} R(Y, U, Y, V) + \frac{(1-\rho^2)}{\log^2\delta} g_M(A_Y A^*_Y U, V) \\
&+ \frac{1}{\log\delta}\frac{\mathrm{Hess}_{g_M} f(Y, Y)}{f} \cdot g_M(U, V),
\end{aligned}
\tag{2.29}
$$

yet again, using Proposition 32, the tensor goes to zero as $\delta \to 0$.

5. (see (1.27))

$$\begin{aligned}\widetilde{R}^\rho(X_\delta,Y_\delta,U_\rho,V_\rho) &= R^\delta(X_\delta,Y_\delta,U_\rho,V_\rho) + (1-\rho^2)g_\delta(A_{X_\delta}A^*_{Y_\delta}U_\rho - A_{Y_\delta}A^*_{X_\delta}U_\rho, V_\rho)\\ &= \frac{1}{\log^2\delta}R(X,Y,U,V) + \frac{(1-\rho^2)}{\log^2\delta}g_M(A_XA^*_YU - A_YA^*_XU,V), \qquad (2.30)\end{aligned}$$

which goes to zero as $\delta \to 0$.

Lastly, we calculate the all vertical vector tensor equation using Lemma 26.

$$\begin{aligned}\widetilde{R}^\rho(U_\rho,V_\rho,V_\rho,U_\rho) &= (\widetilde{R}^\rho)^{\text{int}}(U_\rho,V_\rho,V_\rho,U_\rho) - II^\rho(U_\rho,U_\rho)II^\rho(V_\rho,V_\rho)\\ &\quad + II^\rho(U_\rho,V_\rho)II^\rho(V_\rho,U_\rho)\\ &= \frac{1}{\rho^2\cdot\log^2\delta}R^{\text{int}}(U,V,V,U) - II(U,U)II(V,V) + II(U,V)II(V,U),\end{aligned} \qquad (2.31)$$

which goes to $+\infty$ as $\delta \to 0$.

With this last calculation, we have shown that the sectional curvature w.r.t. g_δ is indeed bounded below. □

Lemma 33. *Let (M, g_M) be such that it admits a $\mathfrak{g}$-structure. As before, let $g_{\delta_0} = \log^2\delta g_M$. Consider an atlas on $\mathfrak{g}$ as follows: : $\mathscr{A} = \{(W_i, \mathscr{G}_i)\}_{i=1}^2$ with $\dim\mathscr{G}_1 \leq \dim\mathscr{G}_2$. Let $\widetilde{g}$ be the metric we get by applying Lemma 29 on W_1. Let $r : W_2 \to I$ be a distance function with respect to g_{δ_0}, whose gradient is orthogonal to the orbits of $\mathscr{G}_2$. Then,*

1. $\mathrm{grad}_{\widetilde{g}} r = \mathrm{grad}_{g_{\delta_0}} r$, *and set* $X = \mathrm{grad}_{\widetilde{g}} r$,

2. *For* $U \in \mathscr{V}_1$,

$$\widetilde{\nabla}_{U_\rho} X = (\nabla_U X)^{\mathscr{V}_1} + \rho_1 \cdot (\nabla_U X)^{\mathscr{V}_1^\perp},$$

   where $U_\rho = \frac{U}{\rho_1}$, $\rho_1 = \delta^{\log r}$ *and* $\delta \in (0,1)$.

3. *For* $V \in \left(\mathscr{V}_1 \oplus \{\mathrm{grad}_{\widetilde{g}} r\}\right)^\perp$,

$$\widetilde{\nabla}_V X = (\nabla_V X)^{\mathscr{V}_1} + (\nabla_V X)^{\mathscr{V}_1^\perp}.$$

Proof. We obtained $\widetilde{g}$ by changing g_{δ_0} along a distribution orthogonal to X, therefore

$$\mathrm{grad}_{\widetilde{g}} r = \mathrm{grad}_{g_{\delta_0}} r.$$

Next, we show that the hessian of r in terms of g_{δ_1} remains bounded. We consider the following cases:

1. $U, V \in \mathscr{V}_1$: To simplify the Koszul formula calculation, we can suppose that $[X, U] = [X, V] = 0$:

$$\begin{aligned} 2\mathrm{Hess}_{\widetilde{g}} r(U_\rho, V_\rho) &= 2\widetilde{g}(\widetilde{\nabla}_{U_\rho} X, V_\rho) \\ &= \frac{1}{\rho_1^2} D_X \widetilde{g}(U, V) \end{aligned}$$

$$= D_X g_{\delta_0}(U,V)$$

$$= 2g_{\delta_0}(\nabla_U X, V)$$

$$= 2\text{Hess}_{g_{\delta_0}} r(U,V).$$

2. U, V are orthogonal to $\mathscr{V}_1$ and X:

$$\begin{aligned}
2\text{Hess}_{\widetilde{g}} r(U,V) &= 2\widetilde{g}(\widetilde{\nabla}_U X, V) \\
&= D_X \widetilde{g}(U,V) - \widetilde{g}([X,U],V) + \widetilde{g}([V,X],U) \\
&= D_X g_{\delta_0}(U,V) - g_{\delta_0}([X,U],V) + g_{\delta_0}([V,X],U)] \\
&= 2g_{\delta_0}(\nabla_U X, V) \\
&= 2\text{Hess}_{g_{\delta_0}} r(U,V).
\end{aligned}$$

3. $U \in \mathscr{V}_1$ and $V \in (\mathscr{V}_1 \oplus \{X\})^{\perp}$ and consider U such that $[X, U] = 0$:

$$\begin{aligned}
2\text{Hess}_{\widetilde{g}} r(U_\rho, V) &= 2\widetilde{g}(\widetilde{\nabla}_{U_\rho} X, V) \\
&= -\frac{1}{\rho_1}\widetilde{g}([X,V],U) \\
&= -\rho_1 g_{\delta_0}([X,V],U) \\
&= 2\rho_1 \cdot g_{\delta_0}(\nabla_U X, V) \\
&= 2\rho_1 \cdot \text{Hess}_{g_{\delta_0}} r(U,V).
\end{aligned}$$

$\square$

Corollary 34. *Assume all the hypotheses of Lemma 33 and set $f = \lambda \circ r$, where $\lambda : I \to I$. Then,*

1. $\mathrm{grad}_{\widetilde{g}} f = \mathrm{grad}_{g_{\delta_0}} f$,

2. *For $U \in \mathscr{V}_1$,*

$$\widetilde{\nabla}_{U_\rho} \mathrm{grad}_{\widetilde{g}} f = \left(\nabla_U \mathrm{grad}_{g_{\delta_0}} f\right)^{\mathscr{V}_1} + \rho_1 \cdot \left(\nabla_U \mathrm{grad}_{g_{\delta_0}} f\right)^{\mathscr{V}_1^\perp},$$

   where $U_\rho = \frac{U}{\rho_1}$, $\rho_1 = \delta^{\log f}$ and $\delta \in (0,1)$.

3. *For $V \in \left(\mathscr{V}_1 \oplus \{\mathrm{grad}_{\widetilde{g}} f\}\right)^\perp$,*

$$\widetilde{\nabla}_V \mathrm{grad}_{\widetilde{g}} f = \left(\nabla_V \mathrm{grad}_{g_{\delta_0}} f\right)^{\mathscr{V}_1} + \left(\nabla_V \mathrm{grad}_{g_{\delta_0}} f\right)^{\mathscr{V}_1^\perp}.$$

4. $\nabla_{\mathrm{grad}_{\widetilde{g}} f} \mathrm{grad}_{\widetilde{g}} f = \lambda'' X$, *where $X = \mathrm{grad}_{\widetilde{g}} r$.*

The last claim of the above corollary is a consequence of the following lemma:

Lemma 35. *Let (M, g_M) be a closed manifold. Let $f = \lambda \circ r$ be a transnormal function on $U \subset M$, $\lambda : I \to I$, and $r : U \to I$ is a distance function. Then the gradient $\mathrm{grad}_{g_M} f$ is an eigenvector for $\mathrm{Hess}_{g_M} f$, with eigenvalue λ''.*

Proof. Let $W \in T_pU$. We have $\text{grad}_{g_M} f = \lambda' \cdot X$, where $X = \text{grad}_{g_M} r$. Then we have,

$$\begin{aligned}\text{Hess}_{g_M} f(X, W) &= g_M\left(\nabla_X(\lambda' \cdot X), W\right) \\ &= g_M\left(\lambda'' X + \lambda' \nabla_X X, W\right) \\ &= \lambda'' g_M\left(X, W\right),\end{aligned}$$

since $\nabla_X X = 0$ and since the hessian is linear, we multiply both sides of the equation by λ' to get the desired the result. □

Corollary 36. *Assume the hypotheses of Lemma 33. Then, applying the deformation of Lemma 29 first to W_1 and second to W_2, the resulting metric has lower curvature bound that converges to zero.*

Now that we have all the necessary tools, let us prove Theorem 27.

Proof. Consider an atlas $\mathscr{A} = \{(U_i, \mathscr{G}_i)\}_{i=1}^n$ for $\mathscr{G}$. Note that each chart is such that $\dim \mathscr{G}_i =$ constant, though the constant varies for each i. WLOG, we can order the charts of $\mathscr{A}$ by dimension, i.e. $\dim \mathscr{G}_j \leq \dim \mathscr{G}_{j+1}$ for $j = 1, \cdots, m-1$.

Let $f_i : U_i \to [1, e]$ be the transnormal function in Proposition 32, for each i. Let g_{δ_0} be as in (2.8). We now define the metric on M in an iterative fashion. First, notation. Let $\mathscr{V}_i =$ the distribution of vectors tangent to the orbits of $\mathscr{G}_i$. Likewise $\mathscr{H}_i =$ the distribution of vectors perpendicular to the orbits of $\mathscr{G}_i$. Then, consider the following metric restrictions:

$$g_i^t \equiv g_{\delta_0}|_{\mathscr{V}_i}, \qquad \text{and}$$

$$g_i^p \equiv g_{\delta_0}|_{\mathscr{H}_i}.$$

We begin the iterative process by first considering the metric, g_{δ_0}, on V_1 such that

$$g_{\delta_0} = g_1^t + g_1^p.$$

Next, we define the scaling factor for each i,

$$\rho_i = \delta^{\log f_i}.$$

Then, we scale the metric as follows:

$$g_{\delta_1} = \begin{cases} \rho_1^2 g_1^t + g_1^p & \text{on} \quad U_1 \\ g_{\delta_0} & \text{on} \quad M \setminus U_1 \end{cases}$$

Now we proceed inductively and write $g_{\delta_{i-1}} = g_i^t + g_i^p$ and

$$g_{\delta_i} = \begin{cases} \rho_i^2 g_i^t + g_i^p & \text{on} \quad U_i \\ g_{\delta_{i-1}} & \text{on} \quad M \setminus U_i. \end{cases} \tag{2.32}$$

Letting $g_\delta = g_{\delta_m}$, we claim that g_δ is the metric that satisfies the statement of the theorem. To see this, let us first show 1). By definition of g_δ,

$$\operatorname{diam}_{g_\delta}(M) < \operatorname{diam}_{\mathrm{g}_{\delta_0}}(M) = |\log\delta| \cdot \operatorname{diam}_{g_M}(M).$$

The volume condition also follows by the definition of the metric:

$$\operatorname{vol}_{g_\delta}(M) < \operatorname{vol}_{g_{\delta_0}}(M) < C \cdot |\log^n \delta| \cdot \delta^d \cdot \operatorname{vol}_{g_M}(M),$$

where $d = \min_{i=1,\ldots,m} \dim \mathscr{O}_i$, and where $C \in \mathbb{R}^+$. To verify the lower curvature bound, we apply Corollary 36 successively on each U_i. □

Chapter 3

The Graph Manifold

The graph manifold is central to our approach because it helps to illustrate the need to blow up the metric on (M, g_M).

3.1 Graph Manifold Example

Example 37 (Graph Manifold)**.** *We define the graph manifold in parts. The end result being a manifold that could be thought of as a "dumb bell" of sorts, composed of three pieces: the left (l), the center(c), and the right (r).*

Let Σ^n be an n-manifold, such that $\partial\Sigma^n = S^{n-1}$. Let the "left piece" of the manifold be defined as:

$$M_l = \Sigma^n \times S^{q-1}, \tag{3.1}$$

where $q, n \in \mathbb{Z}^+$. *Next, we can define the "center piece" as follows:*

$$M_c = S^{n-1} \times S^{q-1} \times [-5, 5]. \tag{3.2}$$

Lastly, we define the "right piece":

$$M_r = S^{n-1} \times \Sigma^q, \tag{3.3}$$

where Σ^q *is such that* $\partial\Sigma^q = S^{q-1}$.

Note that we get an $SO(n)$ *action on* M_l, *an* $SO(q)$ *action on* M_r, *and an* $SO(n) \times SO(q)$ *action on* M_c. *The graph manifold then, is defined as follows:*

$$M = {}^{M_l \sqcup M_c \sqcup M_r}/_{\sim}, \tag{3.4}$$

which is obtained by identifying the boundaries of (3.1) *and* (3.3) *with* (3.2) *via the identity map on* $S^{n-1} \times S^{q-1}$.

Next, we will define a sheaf on M. *In order to do this, we will add some height to the left and right pieces as follows. Let*

$$\widetilde{M_l} \equiv M_l \cup \left(S^{n-1} \times S^{q-1} \times [-5, -4]\right), \qquad \text{and} \tag{3.5}$$

$$\widetilde{M_r} \equiv M_r \cup \left(S^{n-1} \times S^{q-1} \times [4, 5]\right) \tag{3.6}$$

Define a sheaf on M as follows: For an open set $U \subset M$,

$$\mathfrak{g}(U) = \begin{cases} \{0\} & \text{if} \quad M_l \cap U \neq \emptyset \text{ and } M_r \cap U \neq \emptyset \text{ or } U = \emptyset, \\ SO(q) & \text{if} \quad U \subset \widetilde{M_l}, \\ SO(n) & \text{if} \quad U \subset \widetilde{M_r}, \\ SO(n) \times SO(q) & \text{otherwise} \end{cases} \tag{3.7}$$

In fact, this is a $\mathfrak{g}$-structure over M. The corresponding covering map over each piece is the identity map.

The sheaf action on M is a complete local action. We have that the pullback sheaf is simply $\mathfrak{g}$, since the corresponding map is the identity. When $U = \emptyset$, this is clear. For $U \subset \widetilde{M_l}$ (or $U \subset \widetilde{M_r}$), the corresponding group is $SO(n)$ (or $SO(q)$), which acts globally on S^{n-1} (or S^{q-1}), respectively, and so, likewise on any subset $V(x)$. Lastly, for $U \subset M_c$, the corresponding group action is $SO(n) \times SO(q)$. A similar note applies here, in that $SO(n) \times SO(q)$ acts globally on $S^{n-1} \times S^{q-1}$.

To see that $\mathfrak{g}$ actually defines a $\mathfrak{g}$-structure, notice that the corresponding covering maps are all trivially normal. Let $x \in M$ and consider any neighborhood, $V(x)$. Then $V(x)$ can be written as a disjoint union of the trivial group, $SO(n)$, $SO(q)$, or $SO(n) \times SO(q)$. Hence, $V(x)$ is saturated.

<u>Remarks</u>:

1. First note that the above example generalizes nicely. Namely, we can choose Σ^n such that $\partial\Sigma^n$ is a homogenous manifold.

2. In this example, we want to think of pointed Gromov-Hausdorff convergence and not Gromov-Hausdorff convergence.

For the above example, we consider three cases for $p_i \in M$. We consider $p_i = p$, but what follows holds for non-constant sequences with the appropriate technical modifications. If $p_i \in \text{int}(M_l)$, then (M, p_i, g_i) collapses to $T_p\Sigma^n$. If $p_i \in \text{int}(M_c)$, then (M, p_i, g_i) collapses to the real line. If $p_i \in \text{int}(M_r)$, then M collapses to $T_p\Sigma^q$.

3.2 Reason For Blowing Up Metric

In general, a $\mathfrak{g}$-structure cannot be collapsed with uniform lower curvature bound and uniform diameter bound. We saw this in the statement and proof of Theorem 27, where we had to scale the metric up by a factor of $\log^2\delta$, for $\delta > 0$. This arises again in the following chapter, in which we state our strongest result. The graph manifold shows us what occurs when we collapse along the orbits without blowing up the metric first:

Example 38. *Let M be the graph manifold defined in the example above. Consider* *the quotient of the graph manifold,* $\widetilde{M} = {}^{M}/_{\sim}$*, or more specifically:*

$$\widetilde{M} = (\Sigma^n \times \{m_1\}) \sqcup (\{m_1\} \times \{m_2\} \times [-5,5]) \sqcup (\{m_2\} \times \Sigma^q)\,,$$

Notice that $\widetilde{M}$ *is not an Alexadrov space as we get bifurcating geodesics at the one point unions.*

Chapter 4

Hormesis

4.1 Stronger Result

We reach the final chapter and the final layers of the onion we began peeling with Theorem 24. In this section, we state a result that allows us to drop the polarized $\mathfrak{g}$-structure hypothesis. In order to accomplish this, we again employ transnormal functions, which play the crucial role of controlling the metric blow up. In addition, we also use a powerful tool mentioned previously, namely Cheeger deformations. This has the immediate advantage of simplifying the Gromoll-Walshchap tensor equations, since the A and T tensors vanish on a product manifold.

We now weaken the condition on Lemma 29, and therefore, on our main result Theorem 27. Specifically, we can drop the condition that the intrinsic metric on the fibers be normal homogeneous, as the transnormal functions we have mentioned have the additional bonus of controlling the intrinsic metric blow up on the fibers. Note that the transnormal functions we will construct differ from earlier in that they are no longer bounded.

Now, we can state our strongest result yet:

Theorem 39. *Let $\mathscr{G}$ be a $\mathfrak{g}$-structure on a compact manifold (M, g_M), where g_M is $\mathscr{G}$-invariant. Then M admits a sequence of metrics g_δ, for $\delta > 0$, such that:*

1. $\operatorname{diam}_{g_\delta}(M) < \operatorname{diam}_{g_M}(M)|\log\delta|^m$, *where m corresponds to the number of charts that cover M,*

2. $\operatorname{vol}_{g_\delta}(M) < C \cdot \operatorname{vol}_{g_M}(M) \cdot |\log\delta|^{mn} \cdot \delta^d$, *where* $d = \min_{i=1,\ldots,m} \dim \mathscr{O}_i$, $n = \dim M$, *and where* $C \in \mathbb{R}^+$,

3. $\sec_{g_\delta}(M) \geq k$, *for some* $k \in \mathbb{R}$.

As before, proving the local version of this theorem will prove useful, no pun intended.

Lemma 40. *Let G be a Lie group acting on (M, g_M). Let $W \subset M$ be G-invariant. Consider a G-invariant subset $V \subset W$ such that $\overline{V} \subset W$. Then M admits a sequence of metrics g_i on W such that*

1. $\operatorname{diam}_{g_i}(M) < \operatorname{diam}_{g_M}(M) \cdot |\log\delta|$

2. $\operatorname{vol}_{g_i}(V) < \operatorname{vol}_{g_M}(V) \cdot |\log\delta|^n \cdot \delta^d$, *where* $d = \dim \mathscr{O}$.

3. $\sec_{g_i}(M) \geq k$, *for some* $k \in \mathbb{Z}$,

4. $g_i = \log^2 \delta \cdot g_M$ *on* $M \setminus W$.

The manifolds we are interested in admit $\mathfrak{g}$-structures. As a result of scaling on each chart for the corresponding atlas of our $\mathfrak{g}$-structure, a blow up occurs as we move from one chart to the next. The following proposition proves the existence of a transnormal function, which gives us allows us to control that blow up. Ultimately, this proposition is the reason why we can drop the normal homogeneous hypothesis along with the polarized $\mathfrak{g}$-structure hypotheses from Theorem 27.

Proposition 41. *Let* G *be a Lie group acting on* M*, where* (M, g_M) *is a closed manifold. Consider a* G*-invariant open set* $W \subset M$*. Let* $V \subset W$ *be a precompact* G*-invariant set. Define* $U = B(V, b)$*, for sufficiently small* $b > 0$*. Then there exists a transnormal function* $f : U \to [1, \infty)$ *satisfying the following:*

1. $f|_V = 1$

2. $f \to \infty$ *near* ∂U

3. f *is constant along each of the orbits of* G.

4. $\frac{|\mathrm{grad}_{g_M}(f)|^2_{g_M}}{f^6} < \infty$.

5. $\frac{\mathrm{Hess}_{g_M} f}{f^3} < \infty$.

Proof. As in [9], there exists a transnormal function $r : U \to [0, 1]$ such that

$$r(x) = \frac{\mathrm{dist}(V, x)}{\mathrm{dist}(V, x) + \mathrm{dist}(M \setminus U, x)} .$$

Note that $r|_V \equiv 0$ and as $x \to \partial U$, we get that $r \to 1$. Define $\psi : [0,1] \to [0,1]$ such that:

$$\psi(t) = \begin{cases} 0 & \text{if} \quad t \in [0, \frac{1}{100}] \\ 1 & \text{if} \quad t \in [\frac{99}{100}, 1] \end{cases},$$

where $\psi'(t) > 0$ for all $t \in \left(\frac{1}{100}, \frac{99}{100}\right)$. We choose b small enough so that $\text{dist}(V, \cdot)$ and $\text{dist}(M \setminus U, \cdot)$ are smooth on a neighborhood of $U \setminus V$. Consequently, the condition on b is such that $\psi \circ r$ is smooth. On V, we have that the gradient and hessian of r are bounded. Near ∂U, we also get that the gradient and hessian of r are bounded. Note that $\psi \in C^\infty(\mathbb{R})$ and thus its gradient and hessian are bounded on its domain.

Next, consider $\bar{f} : [0,1] \to [1, \infty)$ be such that

$$\bar{f}(t) = \left(\frac{1}{1-t}\right)^p,$$

for some $p \geq \frac{3}{2}$. Now we define our transnormal function, $f : U \to [1, \infty)$ such that

$$f(\cdot) \equiv \left(\bar{f} \circ \psi \circ r\right)(\cdot) = \left(\frac{1}{1 - \psi(r(\cdot))}\right)^p \tag{4.1}$$

To see 1), note that it is clear to see that $f|_V = 1$, since

$$f|_V = \overline{f}\left(\psi\left(r|_V\right)\right) = \overline{f}\left(\psi(0)\right) = \overline{f}(0) = 1$$

We see that $f \to \infty$ as $x \to \partial U$, since

$$f(x) = \overline{f}\left(\psi\left(r(x)\right)\right) \to \infty,$$

which shows 2).

3) is easily shown by observing that f is constant along the orbits of G since, V and U are G-invariant.

To see 4), we first calculate the gradient of f:

$$\begin{aligned} \operatorname{grad}_{g_M}(f) &= \overline{f}'\left(\psi\left(r(x)\right)\right) \cdot \operatorname{grad}_{g_M}(\psi) \\ &= \frac{p}{(1-\psi\left(r(x)\right))^{p+1}} \cdot \operatorname{grad}_{g_M}(\psi) \end{aligned}$$

On a compact set away from ∂U, r and ψ are smooth. On a neighborhood of $U \setminus V$, we have,

$$\begin{aligned} \frac{\left|\operatorname{grad}_{g_M}(f)\right|_{g_M}^2}{f^6} &= \frac{\left[\overline{f}'\left(\psi\left(r(x)\right)\right)\right]^2 \cdot \left|\operatorname{grad}_{g_M}(\psi)\right|_{g_M}^2}{\left[\overline{f}\left(\psi\left(r(x)\right)\right)\right]^6} \\ &= \frac{p^2 \cdot |\operatorname{grad}_{g_M}(\psi)|_{g_M}^2}{\left(1-\psi\left(r(x)\right)\right)^{2p+2}} \cdot \left(1-\psi\left(r(x)\right)\right)^{6p} \\ &< C \cdot p^2 \cdot \left(1-\psi\left(r(x)\right)\right)^{4p-2} \longrightarrow 0, \end{aligned}$$

as $x \to \partial U$.

Lastly, to show 5), we calculate the hessian of f. Let $v, w \in T_pU$, then

$$\begin{aligned}
\mathrm{Hess}_{g_M}(f)(v,w) &= g_M\left(\nabla_v \mathrm{grad}_{g_M}(f), w\right) \\
&= g_M\left(\nabla_v\left(\overline{f}'\left(\psi\left(r(x)\right)\right)\cdot \mathrm{grad}_{g_M}(\psi)\right), w\right) \\
&= g_M\Big(\overline{f}''\left(\psi\left(r(x)\right)\right)\cdot D_v(\psi)\cdot \mathrm{grad}_{g_M}(\psi) \\
&\qquad + \overline{f}'\left(\psi(r(x))\right)\nabla_v \mathrm{grad}_{g_M}(\psi), w\Big) \\
&= \frac{p(p+1)}{(1-\psi(r(x)))^{p+2}}\, g_M(\mathrm{grad}_{g_M}(\psi), v)\ g_M(\mathrm{grad}_{g_M}(\psi), w) \\
&\quad + \frac{p}{(1-\psi(r(x)))^{p+1}}\,\mathrm{Hess}_{g_M}\psi(v,w)
\end{aligned}$$

Then we have,

$$\begin{aligned}
\frac{\mathrm{Hess}_{g_M} f(v,w)}{f^3} &= \frac{1}{\left[\overline{f}\left(\psi(r(x))\right)\right]^3}\left[\frac{p(p+1)}{(1-\psi(r(x)))^{p+2}}\, g_M(\mathrm{grad}_{g_M}(\psi), v)\ g_M(\mathrm{grad}_{g_M}(\psi), w)\right. \\
&\qquad\left. + \frac{p}{(1-\psi(r(x)))^{p+1}}\,\mathrm{Hess}_{g_M}\psi(v,w)\right] \\
&= (1-\psi(r(x)))^{3p}\left[\frac{p(p+1)}{(1-\psi(r(x)))^{p+2}}\, g_M(\mathrm{grad}_{g_M}(\psi), v)\ g_M(\mathrm{grad}_{g_M}(\psi), w)\right. \\
&\qquad\left. + \frac{p}{(1-\psi(r(x)))^{p+1}}\,\mathrm{Hess}_{g_M}\psi(v,w)\right] \\
&= p(p+1) g_M(\mathrm{grad}_{g_M}(\psi), v)\ g_M(\mathrm{grad}_{g_M}(\psi), w)\cdot(1-\psi(r(x)))^{2p-2} \\
&\quad + p\ \mathrm{Hess}_{g_M}\psi(v,w)\cdot(1-\psi(r(x)))^{2p-1}. \qquad (4.2)
\end{aligned}$$

Letting $x \to \partial U$, we see that (4.2) goes to 0. □

Recall that when we Cheeger deform the product $(G \times U, g_G + g_M)$, we scale the group component of the metric with f. Also note that $f \in C^\infty$, but as the value of p increases, the smoother the Cheeger deformed metric $(G \times U, f^2\ g_G + g_M)$ becomes. The following proposition establishes this claim.

Proposition 42. *If $p \geq \frac{3}{2}$ and f is defined as in (4.1), then $g_f \in C^{2p-1}$.*

Proof. To see this, consider $v, w \in T_pM$. Then,

$$\begin{aligned}\frac{1}{f^2} g_f(f\ Ch_f(v), f\ Ch_f(w)) &= g_f\left(\frac{\kappa(v))}{f^2} + v, \frac{\kappa(w)}{f^2} + w\right) \\ &= g_f\left((1 - \psi(r(x)))^{2p}\ \kappa(v) + v, (1 - \psi(r(x)))^{2p}\ \kappa(w) + w\right).\end{aligned} \tag{4.3}$$

We also have

$$\begin{aligned}\frac{1}{f^2} g_f(f\ Ch_f(v), f\ Ch_f(w)) &= \left(f^2\ g_G + g_M\right)\left(\left(\frac{\kappa(v)}{f^2}, v\right), \left(\frac{\kappa(w)}{f^2}, w\right)\right) \\ &= \frac{1}{f^2}\ g_G(\kappa(v), \kappa(w)) + g_M(v, w) \\ &= [1 - \psi(r(x))]^{2p}\ \ g_G(\kappa(v), \kappa(w)) + g_M(v, w) \\ &\longrightarrow g_M(v, w) \text{ as } x \to \partial U.\end{aligned}$$

From this, we see that (4.3) converges to g_M as $f \to \infty$. □

Remark 43. *Although $g_f \in C^\infty$ is preferable, we note that any C^α function can be approximated to a C^∞ smooth function in the C^α-topology. In other words, $g_f \in C^{2p-1}$*

is good enough. Lastly, we call the metric g_f a ***relative Cheeger deformation****.*

Now, we can proceed with the proof of Lemma 40.

Proof. Let (M, g_M) be closed with G a Lie group acting on a G-invariant subset $W \subset M$. Consider the metric

$$g_\delta \equiv \log^2 \delta \; g_M,$$

on M, where $\delta \in (0,1)$. Let $U \equiv B(V, b)$, where $b > 0$ is sufficiently small. Let $f : U \to [1, \infty)$ be the transnormal function of Proposition 41. Let

$$\phi = \log f. \tag{4.4}$$

Then the directional derivative of ϕ is given by:

$$\begin{aligned} D_v(\phi) &= D_v(\log f) \\ &= \frac{D_v(f)}{f}. \end{aligned}$$

Similarly, we calculate the gradient of ϕ. Let g be any metric on M. Then,

$$\begin{aligned} g(\operatorname{grad}_g(\phi), V) &= D_V(\phi) \\ &= \frac{D_V(f)}{f} \end{aligned}$$

$$= \frac{g(\operatorname{grad}_g(f), V)}{f}, \tag{4.5}$$

which gives us

$$\operatorname{grad}_g(\phi) = \frac{\operatorname{grad}_g(f)}{f}. \tag{4.6}$$

For simplicity, we write the gradient of f as follows:

$$\operatorname{grad}_g(f) = \bar{f}' \cdot \overline{X}, \tag{4.7}$$

where $\overline{X} = \operatorname{grad}_g(r)$. Then,

$$\begin{aligned} \frac{\operatorname{grad}_g(\phi)}{\left|\operatorname{grad}_g(\phi)\right|_g} &= \frac{\frac{\operatorname{grad}_g(f)}{f}}{\frac{|\operatorname{grad}_g(f)|_g}{f}} \\ &= \frac{\operatorname{grad}_g(f)}{\left|\operatorname{grad}_g(f)\right|_g} = \frac{\overline{X}}{\left|\overline{X}\right|_g}. \end{aligned} \tag{4.8}$$

Our goal is to control the blow up on the fibers of the group action. We need to show that the sectional curvature of $(G \times U, f^2 \cdot g_G + g_\delta)$ is bounded from below. Let

$$\widetilde{g} = f^2 \cdot g_G + g_\delta \tag{4.9}$$

correspond to $\widetilde{R}$ and let

$$g' = g_G + g_\delta \tag{4.10}$$

correspond to R. Now we must show that the curvature on $(G \times U, \widetilde{g})$ has a lower bound. For this, we consider the tensor terms we get from the Gromoll-Walschap tensor equations (1.19)-(1.25). We verify that the non-vanishing tensor terms remain bounded.

First, since we have a product manifold, the A-tensor and the T-tensor vanish, or in the context of the Gromoll-Walschap equations, S and σ vanish. Second, note that $R \equiv 0$ except on a frame where vectors are all vertical or all horizontal. Lastly, using the symmetry of the curvature tensor, we can reduce the number of terms we have to calculate.

Let $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ be horizontal vectors and let $\widetilde{U}, \widetilde{V}, \widetilde{W}$ be vertical vectors. The following tensor terms are trivial:

1. (see (1.19)):

$$\widetilde{R}^v\left(\widetilde{X}, \widetilde{Y}\right)\widetilde{Z} = 0$$

2. (see (1.20)):

$$(\widetilde{R} - R)^h\left(\widetilde{X}, \widetilde{Y}\right)\widetilde{Z} = 0$$

3. (see (1.21)):

$$\widetilde{R}^h\left(\widetilde{U},\widetilde{V}\right)\widetilde{W}=0$$

4. (see (1.26)):

$$\widetilde{R}^h\left(\widetilde{U},\widetilde{V}\right)\widetilde{X}=0$$

5. (see (1.28)):

$$\widetilde{R}^h\left(\widetilde{X},\widetilde{Y}\right)\widetilde{U}=0$$

The following tensor terms are non-zero and non-trivial. As before, we can use the symmetry of the tensor so that we only need to check two terms.

1. (see (1.22)):

$$\begin{aligned}\widetilde{R}^v\left(\widetilde{U}_1,\widetilde{U}_2\right)\widetilde{U}_3 &= R^v(\widetilde{U}_1,\widetilde{U}_2)\widetilde{U}_3+f^2\cdot\left|\mathrm{grad}_{g_\delta}(\phi)\right|^2_{g_\delta}\Big[-\widetilde{U}_1\ g'\left(\widetilde{U}_2,\widetilde{U}_3\right)\\ &\qquad +\widetilde{U}_2\ g'\left(\widetilde{U}_1,\widetilde{U}_3\right)\Big]\\ &= R^v(U_1,U_2)U_3+f^2\ \frac{|\mathrm{grad}_{g_M}(\phi)|^2_{g_M}}{\log^2\delta}\Big[\widetilde{U}_2\ g_G(U_1,U_3)\\ &\qquad -\widetilde{U}_1\ g_G(U_2,U_3)\Big]\\ &= R^v(U_1,U_2)U_3+\frac{f^2}{\log^2\delta}\left|\frac{\mathrm{grad}_{g_M}(f)}{f}\right|^2_{g_M}\Big[\widetilde{U}_2\ g_G(U_1,U_3)\end{aligned}$$

$$
\begin{aligned}
&- \widetilde{U}_1 \; g_G(U_2, U_3)\Big] \\
&= R^v(U_1, U_2)U_3 + \frac{|\mathrm{grad}_{g_M}(f)|^2_{g_M}}{\log^2 \delta} \Big[\widetilde{U}_2 \; g_G(U_1, U_3) \\
&\qquad - \widetilde{U}_1 \; g_G(U_2, U_3)\Big] \qquad (4.11)
\end{aligned}
$$

2. (see (1.23)):

$$
\begin{aligned}
\widetilde{R}^v(\widetilde{X}, \widetilde{U})\widetilde{Y} &= \widetilde{U}\left[\widetilde{g}\left(\nabla_{\widetilde{X}}(0, \mathrm{grad}_{g_\delta}(\phi)), \widetilde{Y}\right)\right. \\
&\qquad \left. + g'\left((0, \mathrm{grad}_{g_\delta}(\phi)), \widetilde{X}\right) \; g'\left((0, \mathrm{grad}_{g_\delta}(\phi)), \widetilde{Y}\right)\right] \\
&= \widetilde{U}\left[g_\delta(\nabla_X \mathrm{grad}_{g_\delta}(\phi), Y) + g_\delta(\mathrm{grad}_{g_\delta}(\phi), X) \; g_\delta(\mathrm{grad}_{g_\delta}(\phi), Y)\right] \\
&= \widetilde{U}\left[\frac{\mathrm{Hess}_{g_\delta} f(X, Y)}{f}\right], \qquad \text{by Lemma 31 .} \qquad (4.12)
\end{aligned}
$$

To evaluate (4.12), we must consider two types of horizontal vectors: one proportional to the gradient of ϕ (or equivalently to f), and one that is orthogonal to the gradient of ϕ (and therefore to the gradient of f). We use (4.4)-(4.8) as follows.

Let $X = \frac{\mathrm{grad}_{g_M}(\phi)}{\left|\mathrm{grad}_{g_M}(\phi)\right|_{g_M}}$ be a unit vector with respect to g_M. The corresponding unit vector in g_δ is $X_\delta = \frac{\mathrm{grad}_{g_M}(\phi)}{\log \delta \cdot \left|\mathrm{grad}_{g_M}(\phi)\right|_{g_M}}$. Then,

$$
\begin{aligned}
\frac{\mathrm{Hess}_{g_\delta} f(X_\delta, X_\delta)}{f} &= \frac{g_\delta(\nabla_{X_\delta} \mathrm{grad}_{g_\delta}(f), X_\delta)}{f} \\
&= \frac{g_\delta\left(\nabla_{\frac{\mathrm{grad}_{g_M}(\phi)}{\log \delta |\mathrm{grad}_{g_M}(\phi)|_{g_M}}} \mathrm{grad}_{g_\delta}(f), \frac{\mathrm{grad}_{g_M}(\phi)}{\log \delta |\mathrm{grad}_{g_M}(\phi)|_{g_M}}\right)}{f}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\log^2\delta}\frac{1}{f} g_\delta\left(\nabla_{\frac{\mathrm{grad}_{g_M}(f)}{|\mathrm{grad}_{g_M}(f)|_{g_M}}} \mathrm{grad}_{g_\delta}(f), \frac{\mathrm{grad}_{g_M}(f)}{|\mathrm{grad}_{g_M}(f)|_{g_M}}\right), \quad \text{recall (4.8)} \\
&= \frac{1}{\log^2\delta} \frac{\mathrm{Hess}_{g_\delta} f\left(\frac{\mathrm{grad}_{g_M}(f)}{|\mathrm{grad}_{g_M}(f)|_{g_M}}, \frac{\mathrm{grad}_{g_M}(f)}{|\mathrm{grad}_{g_M}(f)|_{g_M}}\right)}{f}, \\
&= \frac{1}{\log^2\delta}\frac{\mathrm{Hess}_{g_M} f(X,X)}{f}, \qquad \text{by Lemma 30 .}
\end{aligned}
$$

Now, consider $Y \in T_pM$ such that Y is orthogonal to $\mathrm{grad}_{g_M}(\phi)$. Let $Y_\delta = \frac{Y}{\log\delta}$. Then,

$$
\begin{aligned}
\frac{\mathrm{Hess}_{g_\delta} f(Y_\delta, Y_\delta)}{f} &= \frac{1}{\log^2\delta}\cdot\frac{1}{f} g_\delta(\nabla_Y \mathrm{grad}_{g_\delta}(f), Y) \\
&= \frac{1}{\log^2\delta}\cdot\frac{1}{f} g_M(\nabla_Y \mathrm{grad}_{g_M}(f), Y) \\
&= \frac{1}{\log^2\delta}\frac{\mathrm{Hess}_{g_M} f(Y,Y)}{f}
\end{aligned}
$$

Lastly, we consider the cross terms, i.e. consider X_δ and Y_δ as defined above. We have:

$$
\begin{aligned}
\mathrm{Hess}_{g_\delta} f(X_\delta, Y_\delta) &= \left[\overline{f}'' + \overline{f}'\cdot\psi''\right] g_\delta\left(\frac{\mathrm{grad}_{g_\delta}(f)}{\log\delta|\mathrm{grad}_{g_\delta}(f)|_{g_\delta}}, \frac{Y}{\log\delta}\right) \\
&= \left[\frac{\overline{f}'' + \overline{f}'\cdot\psi''}{|\mathrm{grad}_{g_\delta}(f)|_{g_\delta}\cdot\log^2\delta}\right] g_\delta(\mathrm{grad}_{g_\delta}(f), Y), \qquad \text{by Lemma 35,} \\
&= 0,
\end{aligned}
$$

where the last equality holds since $\text{grad}_{g_\delta}(f) \perp Y$. Then, we get

$$\frac{\text{Hess}_{g_\delta} f(X_\delta, Y_\delta)}{f} = 0.$$

For convenience, we write (4.11) and (4.12) in terms of their equivalent $(0,4)$-tensor form.

1. (4.11):

$$\begin{aligned}\widetilde{R}(\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3, \widetilde{U}_4) &= f^2 R(\widetilde{U}_1, \widetilde{U}_2, \widetilde{U}_3, \widetilde{U}_4) + \frac{|\text{grad}_{g_M}(f)|^2_{g_M}}{\log^2 \delta} \Big[\widetilde{g}(\widetilde{U}_1, \widetilde{U}_3)\ g_G(U_2, U_4) \\ &\qquad - \widetilde{g}(\widetilde{U}_2, \widetilde{U}_3)\ g_G(U_1, U_4)\Big] \\ &= f^2 R(U_1, U_2, U_3, U_4) + f^2 \cdot \frac{|\text{grad}_{g_M}(f)|^2_{g_M}}{\log^2 \delta} [g_G(U_1, U_3)\ g_G(U_2, U_4) \\ &\qquad - g_G(U_2, U_3)\ g_G(U_1, U_4)] \end{aligned} \tag{4.13}$$

2. (4.12):

   (a) $\widetilde{X}_\delta, \widetilde{U}_1, \widetilde{X}_\delta, \widetilde{U}_2$:

$$\begin{aligned}\widetilde{R}(\widetilde{X}_\delta, \widetilde{U}_1, \widetilde{U}_2, \widetilde{X}_\delta) &= -\widetilde{R}(\widetilde{X}_\delta, \widetilde{U}_1, \widetilde{X}_\delta, \widetilde{U}_2) \\ &= -\frac{f^2}{\log^2 \delta} \frac{\text{Hess}_{g_M} f(X, X)}{f} \cdot g_G(U_1, U_2) \\ &= -\frac{f}{\log^2 \delta} \text{Hess}_{g_M} f(X, X) \cdot g_G(U_1, U_2). \end{aligned} \tag{4.14}$$

(b) $\widetilde{Y}_\delta, \widetilde{U}_1, \widetilde{Y}_\delta, \widetilde{U}_2$:

$$\begin{aligned}\widetilde{R}(\widetilde{Y}_\delta, \widetilde{U}_1, \widetilde{U}_2, \widetilde{Y}_\delta) &= -\widetilde{R}(\widetilde{Y}_\delta, \widetilde{U}_1, \widetilde{Y}_\delta, \widetilde{U}_2) \\ &= -\frac{f^2}{\log^2\delta}\frac{\mathrm{Hess}_{g_M} f(Y,Y)}{f} \cdot g_G(U_1, U_2) \\ &= -\frac{f}{\log^2\delta}\mathrm{Hess}_{g_M} f(Y,Y) \cdot g_G(U_1, U_2). \end{aligned} \tag{4.15}$$

(c) $\widetilde{X}_\delta, \widetilde{U}_1, \widetilde{Y}_\delta, \widetilde{U}_2$:

$$\widetilde{R}(\widetilde{X}_\delta, \widetilde{U}_1, \widetilde{Y}_\delta, \widetilde{U}_2) = (0,0). \tag{4.16}$$

Next, we consider Cheeger vertical/horizontal vectors . Consider the following types of vectors in $T_{(e,p)}(G \times U)$:

Type 1: Let X be perpendicular to the orbits G on U. Then, $\hat{X} = (0, X)$ is perpendicular to the orbits G on $G \times U$.

Type 2: Let V be tangent to the orbits of G on U such that $|V|_{g_M} = 1$, and where the corresponding unit vector in g_δ is $V_\delta = \frac{V}{\log\delta}$. Then, $\exists\ \kappa_{(e,p)} : T_pM \to T_eG$ such that $\hat{V}_1 = \left(\kappa_{(e,p)}\left(V_\delta\right), V_\delta\right)$ is perpendicular to the orbits G on $G \times U$. Then for any value of f, we have $\hat{V} = \hat{V}_f = \left(\frac{1}{\log\delta}\frac{\kappa_{(e,p)}(V)}{f^2}, V_\delta\right)$.

Let V, W be unit with respect to g_M and tangent to the orbits of G on U. Then, the corresponding unit vectors in g_δ are V_δ and W_δ. Let

$$\hat{V}_\delta = \left(\frac{1}{\log \delta} \frac{\kappa_{(e,m)}(V)}{f^2}, V_\delta \right), \qquad \text{and}$$
$$\hat{W}_\delta = \left(\frac{1}{\log \delta} \frac{\kappa_{(e,m)}(W)}{f^2}, W_\delta \right)$$

be their Type 2 vectors. First, we calculate the $\widetilde{g}$-norm of $\hat{V}_\delta$ and $\hat{W}_\delta$, as these are terms that appear in (4.13). The $\widetilde{g}$-norm of $\hat{V}_\delta$ is given by,

$$\begin{aligned} |\hat{V}_\delta|^2_{\widetilde{g}} &= f^2 \cdot \left| \frac{1}{\log \delta} \frac{\kappa_{(e,p)}(V)}{f^2} \right|^2_{g_G} + |V_\delta|^2_{g_\delta} \\ &= \frac{1}{\log^2 \delta} \frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{f^2} + 1 \\ &= \frac{|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2 \delta \cdot f^2}{\log^2 \delta \cdot f^2} \end{aligned}$$

and the g'-norm of the group component is given by

$$\begin{aligned} |\hat{V}^G_\delta|^2_{g_G} &= \left| \frac{1}{\log \delta} \frac{\kappa_{(e,m)}(V)}{f^2} \right|^2_{g_G} \\ &= \frac{1}{\log^2 \delta} \frac{|\kappa_{(e,m)}(V)|^2_{g_G}}{f^4}. \end{aligned}$$

Then, we get that their ratio is given by:

$$\begin{aligned} \frac{|\hat{V}^G_\delta|^2_{g_G}}{|\hat{V}_\delta|^2_{\widetilde{g}}} &= \left(\frac{\log^2 \delta \cdot f^2}{|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2 \delta \cdot f^2} \right) \left(\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{\log^2 \delta \cdot f^4} \right) \\ &= \frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{f^2 \left(|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2 \delta \cdot f^2 \right)}. \end{aligned}$$

Letting $\widetilde{U}_1 = \widetilde{U}_4 = \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}$ and $\widetilde{U}_2 = \widetilde{U}_3 = \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}$, we can proceed with getting a lower bound for (4.13):

$$
\begin{aligned}
&\widetilde{R}\left(\frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}\right) = R^G\left(\frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}\right) \\
&\qquad + \frac{f^2}{\log^2\delta}\ |\mathrm{grad}_{g_M}(f)|^2_{g_M}\left[g_G\left(\frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}\right)^2 - \left|\frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}\right|^2_{g_G} \cdot \left|\frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}\right|^2_{g_G}\right] \\
&= \frac{f^2}{|\hat{V}_\delta|^2_{\widetilde{g}}|\hat{W}_\delta|^2_{\widetilde{g}}} R^G\left(\hat{V}_\delta, \hat{W}_\delta, \hat{W}_\delta, \hat{V}_\delta\right) \\
&+ \frac{f^2}{\log^2\delta}|\mathrm{grad}_{g_M}(f)|^2_{g_M}\left[\frac{\log^4\delta\cdot f^4\cdot g_G\left(\frac{1}{\log\delta}\frac{\kappa_{(e,p)}(V)}{f^2}, \frac{1}{\log\delta}\frac{\kappa_{(e,p)}(W)}{f^2}\right)^2}{\left(|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2\delta\cdot f^2\right)\left(|\kappa_{(e,p)}(W)|^2_{g_G} + \log^2\delta\cdot f^2\right)}\right. \\
&\qquad\left. - \left(\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{f^2\left(|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2\delta\cdot f^2\right)}\right)\left(\frac{|\kappa_{(e,p)}(W)|^2_{g_G}}{f^2\left(|\kappa_{(e,p)}(W)|^2_{g_G} + \log^2\delta\cdot f^2\right)}\right)\right] \\
&= \left(\frac{\log^2\delta\cdot f^2}{(|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2\delta\cdot f^2)}\frac{\log^2\delta\cdot f^2}{(|\kappa_{(e,p)}(W)|^2_{g_G} + \log^2\delta\cdot f^2)}\right) \\
&\qquad\cdot\frac{f^2}{f^8\cdot\log^4\delta}R^G(\kappa_{(e,p)}(V), \kappa_{(e,p)}(W), \kappa_{(e,p)}(W), \kappa_{(e,p)}(V)) \\
&+ \frac{|\mathrm{grad}_{g_M}(f)|^2_{g_M}}{f^2\ \log^2\delta}\frac{\left[g_G(\kappa_{(e,p)}(V), \kappa_{(e,p)}(W))^2 - |\kappa_{(e,p)}(V)|^2_{g_G}\cdot|\kappa_{(e,p)}(W)|^2_{g_G}\right]}{\left(|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2\delta\cdot f^2\right)\left(|\kappa_{(e,p)}(W)|^2_{g_G} + \log^2\delta\cdot f^2\right)} \\
&= \frac{1}{\log^4\delta\cdot f^6}\frac{R^G(\kappa_{(e,p)}(V), \kappa_{(e,p)}(W), \kappa_{(e,p)}(W), \kappa_{(e,p)}(V))}{\left(\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{\log^2\delta\cdot f^2} + 1\right)\left(\frac{|\kappa_{(e,p)}(W)|^2_{g_G}}{\log^2\delta\cdot f^2} + 1\right)} \\
&+ \frac{|\mathrm{grad}_{g_M}(f)|^2_{g_M}}{f^6\ \log^4\delta}\frac{\left[g_G(\kappa_{(e,p)}(V), \kappa_{(e,p)}(W))^2 - |\kappa_{(e,p)}(V)|^2_{g_G}\cdot|\kappa_{(e,p)}(W)|^2_{g_G}\right]}{\left(\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{\log^2\delta\cdot f^2} + 1\right)\left(\frac{|\kappa_{(e,p)}(W)|^2_{g_G}}{\log^2\delta\cdot f^2} + 1\right)}. \qquad (4.17)
\end{aligned}
$$

Next, we show (4.14) - (4.16) have a lower bound.

First, we consider (4.14) by letting $\widetilde{U}_1 = \widetilde{U}_2 = \widetilde{V}$, where U_1 and U_4 are defined as in the previous calculation. Then,

$$\begin{aligned}\widetilde{R}\left(\widetilde{X}_\delta, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \widetilde{X}_\delta\right) &= -\frac{f}{\log^2\delta}\ \mathrm{Hess}_{g_M} f(X,X)\frac{|\hat{V}_\delta^G|^2_{g_G}}{|\hat{V}_\delta|^2_{\widetilde{g}}} \\ &= -\frac{f}{\log^2\delta}\ \left(\mathrm{Hess}_{g_M} f(X,X)\right)\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{f^2\left(|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2\delta\cdot f^2\right)} \\ &= -\frac{1}{\log^4\delta}\frac{\mathrm{Hess}_{g_M} f(X,X)}{f^3}\cdot\left(\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{\log^2\delta\cdot f^2}+1}\right). \qquad (4.18)\end{aligned}$$

For (4.15), consider $\widetilde{Y}_\delta = \frac{\hat{Y}}{\log\delta}$, where $\hat{Y} = (0, Y)$ and with Y orthogonal to $\mathrm{grad}_{g_M}(\phi)$ (and therefore to $\mathrm{grad}_{g_M}(f)$). Let $\widetilde{U}_2 = \widetilde{U}_4 = \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}$, as before. Then

$$\begin{aligned}\widetilde{R}\left(\widetilde{Y}_\delta, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \widetilde{Y}_\delta\right) &= -\frac{f}{\log^2\delta}\ \mathrm{Hess}_{g_M} f(Y,Y)\frac{|\hat{V}_\delta^G|^2_{g_G}}{|\hat{V}_\delta|^2_{\widetilde{g}}} \\ &= -\frac{f}{\log^2\delta}\ \left(\mathrm{Hess}_{g_M} f(Y,Y)\right)\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{f^2\left(|\kappa_{(e,p)}(V)|^2_{g_G} + \log^2\delta\cdot f^2\right)} \\ &= -\frac{1}{\log^4\delta}\frac{\mathrm{Hess}_{g_M} f(Y,Y)}{f^3}\cdot\left(\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{\frac{|\kappa_{(e,p)}(V)|^2_{g_G}}{\log^2\delta\cdot f^2}+1}\right). \qquad (4.19)\end{aligned}$$

Given that (4.16) is identically zero, the corresponding tensor term for vectors $\widetilde{X}_\delta, \widetilde{Y}_\delta, \widetilde{U}_2 = \widetilde{U}_4 = \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}$ is trivially bounded below. By Proposition 41, the tensor equations (4.17) - (4.19) are bounded below, as $f \to \infty$.

Lastly, we consider the following metric on $G \times U$:

$$\widetilde{g}_\ell \equiv (\ell f)^2 g_G + g_\delta,$$

where ℓ is a nonnegative scalar. Fortunately, we do not need to repeat the above calculations for this metric. Simply adding ℓ's in the appropriate places saves us the trouble:

1.

$$\begin{aligned}
&\widetilde{R}\left(\frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}\right) \\
&= \frac{1}{\ell^6 \cdot f^6 \cdot \log^4 \delta} \frac{R^G(\kappa_{(e,p)}(V), \kappa_{(e,p)}(W), \kappa_{(e,p)}(W), \kappa_{(e,p)}(V))}{\left(\frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{\log^2 \delta \cdot \ell^2 \cdot f^2} + 1\right)\left(\frac{|\kappa_{(e,p)}(W)|_{g_G}^2}{\log^2 \delta \cdot \ell^2 \cdot f^2} + 1\right)} \\
&+ \frac{|\mathrm{grad}_{g_M}(f)|_{g_M}^2}{\ell^4 \cdot f^6 \ \log^4 \delta} \frac{\left[g_G(\kappa_{(e,p)}(V), \kappa_{(e,p)}(W))^2 - |\kappa_{(e,p)}(V)|_{g_G}^2 \cdot |\kappa_{(e,p)}(W)|_{g_G}^2\right]}{\left(\frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{\log^2 \delta \cdot \ell^2 \cdot f^2} + 1\right)\left(\frac{|\kappa_{(e,p)}(W)|_{g_G}^2}{\log^2 \delta \cdot \ell^2 \cdot f^2} + 1\right)}.
\end{aligned}$$

2.

$$\widetilde{R}\left(\widetilde{X}_\delta, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \widetilde{X}_\delta\right) = -\frac{1}{\log^4 \delta} \frac{\mathrm{Hess}_{g_M} f(X,X)}{\ell^2 f^3} \cdot \left(\frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{\frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{\log^2 \delta \cdot \ell^2 \cdot f^2} + 1}\right).$$

3.

$$\widetilde{R}\left(\widetilde{Y}_\delta \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \widetilde{Y}_\delta\right) = -\frac{1}{\log^4 \delta} \frac{\mathrm{Hess}_{g_M} f(Y,Y)}{\ell^2 f^3} \cdot \left(\frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{\frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{\log^2 \delta \cdot \ell^2 \cdot f^2} + 1}\right).$$

Now, we see the significance of blowing up the metric with $\log^2 \delta$ as this is the very thing that makes the above tensor equations converge to zero as $\ell \to 0$, by letting

$$\delta = e^{-\frac{1}{\ell^{1+}}}, \tag{4.20}$$

where 1+ is a number slightly greater than 1. Then our curvature equations become:

1.

$$\begin{aligned}&\widetilde{R}\left(\frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}, \frac{\hat{W}_\delta^G}{|\hat{W}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}\right)\\ &= \frac{1}{\ell^{2+}} \frac{1}{f^6} \frac{R^G(\kappa_{(e,p)}(V), \kappa_{(e,p)}(W), \kappa_{(e,p)}(W), \kappa_{(e,p)}(V))}{\left(\ell^{0+} \frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{f^2} + 1\right)\left(\ell^{0+} \frac{|\kappa_{(e,p)}(W)|_{g_G}^2}{f^2} + 1\right)}\\ &+ \ell^{0+} \frac{|\mathrm{grad}_{g_M}(f)|_{g_M}^2}{f^6} \frac{\left[g_G(\kappa_{(e,p)}(V), \kappa_{(e,p)}(W))^2 - |\kappa_{(e,p)}(V)|_{g_G}^2 \cdot |\kappa_{(e,p)}(W)|_{g_G}^2\right]}{\left(\ell^{0+} \frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{f^2} + 1\right)\left(\ell^{0+} \frac{|\kappa_{(e,p)}(W)|_{g_G}^2}{f^2} + 1\right)}.\end{aligned}$$

2.

$$\widetilde{R}\left(\widetilde{X}_\delta, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \widetilde{X}_\delta\right) = -\ell^{2+} \frac{\mathrm{Hess}_{g_M} f(X,X)}{f^3} \cdot \left(\frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{\ell^{0+} \frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{f^2} + 1}\right).$$

3.

$$\widetilde{R}\left(\widetilde{Y}_\delta \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \frac{\hat{V}_\delta^G}{|\hat{V}_\delta|_{\widetilde{g}}}, \widetilde{Y}_\delta\right) = -\ell^{2+} \frac{\mathrm{Hess}_{g_M} f(Y,Y)}{f^3} \cdot \left(\frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{\ell^{0+} \frac{|\kappa_{(e,p)}(V)|_{g_G}^2}{f^2} + 1}\right).$$

Then the metric which satisfies the claim of the lemma is :

$$g_i = \begin{cases} g_\ell & \text{on} \quad W \\ g_\delta & \text{on} \quad M \setminus W, \end{cases} \tag{4.21}$$

where g_ℓ is the induced metric on U obtained by modding out by the Cheeger action, i.e.

$$(G \times U)\,/G \cong (U, g_\ell)\,.$$

□

Remark 44. *Note that if we define (4.20) instead as,*

$$\delta = e^{-\frac{1}{\ell^{2+}}},$$

where $2+$ is a number slightly bigger than 2, we get that the sectional curvature from Lemma 40 goes to zero. With this, we now have that the sectional curvature of the class of manifolds satisfying the hypothesis of Theorem 39 have sectional curvature going to zero.

With Lemma 40 proven, we can now proceed with the proof of Theorem 39.

Proof. Recall that g_M is $\mathscr{G}$-invariant. As before, we consider the metric

$$g_{\delta_0} = \log^2 \delta \; g_M$$

on M. Let $\mathscr{A} = \{(U_i, \mathscr{G}_i)\}_{i=1}^m$ be an atlas for $\mathscr{G}$, where with $f_i : U_i \to [1, \infty)$, the transnormal function of Proposition 41, for each i.

We begin the iterative step to define the metric that satisfies the claim of the theorem. As before, for $\ell > 0$, consider

$$(G_1 \times U_1, \widetilde{g}_{\ell_1}) = (G_1 \times U_1, (\ell \cdot f_1)^2 \cdot g_{G_1} + g_{\delta_0}),$$

where $G_1 = \mathscr{G}_1$ and where g_{G_1} is a G_1 bi-invariant metric. Letting g_{ℓ_1} be the relative Cheeger deformation on U_1, we get a metric as in (4.21),

$$g_1 = \begin{cases} g_{\ell_1} & \text{on} \quad U_1 \\ g_{\delta_0} & \text{on} \quad M \setminus U_1. \end{cases}$$

Proceeding in this fashion, we get for each i,

$$g_i = \begin{cases} g_{\ell_i} & \text{on} \quad U_i \\ \log^2 \delta \cdot g_{i-1} & \text{on} \quad M \setminus U_i, \end{cases}$$

where again, g_{ℓ_i} is the relative Cheeger deformation on U_i. Let $g_\delta = g_m$. This metric satisfies the claim of the theorem by applying Lemma 40 on U_i, for each i.

We immediately get that the sectional curvature goes to zero. It still remains to show the diameter and volume conditions. However, these conditions are readily shown to be true:

$$\mathrm{diam}_{g_\delta}(M) < |\log\delta|^m \cdot \mathrm{diam}_{g_M}(M),$$

and the volume condition follows by,

$$\mathrm{vol}_{g_\delta}(M) \leq C|\log\delta|^{mn} \cdot \ell^d \cdot \mathrm{vol}_{g_M}(M),$$

where $d = \min_{i=1,\ldots,m} \dim \mathscr{O}_i$. $\square$

Chapter 5

Conclusion

5.1 Summarize

We shave shown how to collapse a manifold while preserving lower curvature bound via $\mathfrak{g}$-structures. With varying hypotheses, we showed three different results that give us a lower curvature bound. We also showed how we can use two current existing tools to accomplish our aims.

First, we showed that a manifold that admits a pure polarized $\mathfrak{g}$-structure, which is almost a global action, along with assuming additional structure on the intrinsic metric of the fibers and on a component of the curvature tensor, leads to a compact limit while preserving l.c.b. The pure polarized result was followed by an analogous result.

We dropped the pure assumption, meaning we were no longer guaranteed that the dimension of the $\mathfrak{g}$-orbits were the same. For this, we had to scale the metric up and choose support functions on the charts of the atlas for our $\mathfrak{g}$-manifold so that our metric remained smooth as we transitioned throughout the charts. However, we were not able to get a compact length space for our limit. The diameter of the limit space will not be finite, in general. This was a necessary trade off as our approach required blowing up the metric in order to preserve l.c.b.

The reason for blowing up the metric was made clear when we discussed the graph manifold. Its orbit space was not an Alexandrov space otherwise. We used the graph manifold to introduce our strongest result yet.

By utilizing transnormal functions to control the blow up from chart to chart and then use these functions to define a scaling factor on the vertical distribution, we were able to construct our final and strongest result. Considering transnormal support functions on the atlas of our $\mathfrak{g}$-manifold and by using relative Cheeger deformations, we were able to drop the polarized assumption, as well as any assumptions on the structure of the intrinsic metric of the fibers. Together, these tools proved powerful and show great potential in the field of collapse with lower curvature bound.

5.2 Further Research

As is to be expected, in our search for answers we came up with new questions. The answers to these questions can expand the present work or take a life of its own. We list these questions in the hopes that answers will one day be sought and found:

1. What conditions on $\mathfrak{g}$-structure allow collapse with diam $< \infty$ and lcb?
2. What conditions on Alexandrov space allow this to occur?
3. Can we get analogous results as in the lifting theorems [10], in the context of $\mathfrak{g}$-structures?
4. Can we get similar results if we replace the l.c.b. condition with Ric or scal bounded below?
5. Under what conditions can we drop the hypothesis that $R^h(U,V)W$ vanish in Theorem 24 and Theorem 27?
6. Can we get a tighter diameter and volume bound on Theorem 39 using the order of the cover as opposed to its cardinality?

The latter question is of particular interest, since Wilhelm and Pro showed that in general, Riemannian submersions need not preserve Ric curvature [8].

There is hope that with $\mathfrak{g}$-structures and possibly additional assumptions, these results can be proven. For now, these questions will remain unanswered.

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