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The Birational Geometry of K-Moduli Spaces

Author

Keller, Jacob James

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The Birational Geometry of K-Moduli Spaces

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Jacob Keller

Committee in charge:

Professor James McKernan, Chair
Professor Kenneth Intriligator
Professor Elham Izadi
Professor Kiran Kedlaya
Professor Dragos Oprea

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The dissertation of Jacob Keller is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

University of California San Diego

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DEDICATION

To my family.

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VITA

2019	Bachelor of Science in Mathematics, The Pennsylvania State University
2021	Candidate in Philosophy in Mathematics, University of California San Diego
2024	Doctor of Philosophy in Mathematics, University of California San Diego

ABSTRACT OF THE DISSERTATION

The Birational Geometry of K-Moduli Spaces

by

Jacob Keller

Doctor of Philosophy in Mathematics

University of California San Diego, 2024

Professor James McKernan, Chair

For C a smooth curve and ξ a line bundle on C, the moduli space $U_C(2,\xi)$ of semistable vector bundles of rank two and determinant ξ is a Fano variety. We show that $U_C(2,\xi)$ is K-stable for a general curve $C \in \overline{\mathcal{M}}_g$. As a consequence, there are irreducible components of the moduli space of K-stable Fano varieties that are birational to $\overline{M}_g$. In particular these components are of general type for $g \geq 22$.

Chapter 1

Introduction

The construction of moduli spaces parametrizing algebraic varieties is a central topic in algebraic geometry. One cannot expect to have reasonable moduli spaces for all varieties. Rather, attention should be restricted to varieties which are stable in some sense. Recently, the notion of K-stability has emerged as a powerful tool to construct moduli spaces of higher-dimensional algebraic varieties. All smooth canonically polarized varieties, and more generally, KSBA stable pairs are known to be K-stable, and there exist projective moduli spaces for these varieties (see [Kol23] for an extensive account of this theory). On the other hand, not every smooth Fano variety is K-stable, and a lot of research on K-stability centers on finding examples of K-stable Fano varieties.

Historically, it has been challenging to construct moduli spaces of Fano varieties since automorphism groups of Fano varieties are often positive-dimensional and not reductive, in contrast to the general type case. However, recent advances in the theory of

K-stability have allowed for the construction of projective moduli spaces that parameterize K-polystable Fano varieties ([LXZ22] finished the construction, but the proof is spread across many papers including but not limited to [XZ20], [BLX22], [ABHLC19], [LWX21], [BHLLX21]). The motivation for this paper is to study the birational geometry of these moduli spaces.

There have been a series of results establishing strong restrictions on rational curves in the moduli spaces of canonically polarized varieties (e.g. see [Kov03]). These give interesting examples of moduli spaces that inherit properties from the varieties they parameterize. We investigate the analogous question for Fano varieties. In particular, this project started by asking if moduli spaces of K-stable Fanos were uniruled. The answer to this question is no, as we produce irreducible components of the moduli space of K-stable Fano varieties that are of general type.

These components are the ones parametrizing moduli spaces $U_C(2,\xi)$ of rank-two vector bundles on a smooth projective curve C with fixed determinant bundle ξ of odd degree. These are Fano varieties with a long history, and they play a role in various fields such as enumerative geometry, conformal field theory, and representation theory. Our main theorem is

Theorem 1.1. *The moduli space $U_C(2,\xi)$ is K-stable for a general curve $C \in \mathcal{M}_g$ and any determinant line bundle ξ.*

We are interested in the consequences this has for the geometry of the moduli of K-stable Fano varieties. In particular, from the results of [NR75] and [MN68] we deduce the following:

Corollary 1.2. *Let M^{Kps} denote the projective moduli space of K-polystable Fano varieties. The irreducible component of M^{Kps} whose general points parameterize spaces $U_C(2,\xi)$ is birational to $\overline{M}_g$ when ξ has odd degree. In particular, by [EH87] and [FJP20], it is of general type for $g \geq 22$.*

We prove this corollary in Chapter 8 by constructing a rational map

$$\overline{M}_g \dashrightarrow M^{Kps}$$

and using [MN68] to say that it is injective on its domain and [NR75] to say that it induces an isomorphism on tangent spaces when the degree of ξ is odd, which implies that it is dominant. This result is not true for even degree determinants, because the results of [NR75] do not hold in this case. By [NR84], for C a smooth non-hyperelliptic curve of genus 3, the moduli space $U_C(2,\mathcal{O}_C)$ is isomorphic to a Coble quartic in $\mathbb{P}^7$. Because $U_C(2,\mathcal{O}_C)$ is not smooth, but the general quartic is, the component of the K-moduli space parameterizing the varieties $U_C(2,\mathcal{O}_C)$ is larger than $\overline{M}_3$.

We briefly outline the paper.

In Chapter 2, we recall background from the paper [Man18] which constructs toric degenerations of $U_C(2,\xi)$ using the theory of conformal blocks. These toric varieties are the central objects of study for this paper. The goal of Chapters 3 and 4 will be to show they are K-polystable for suitable curves C.

In Chapter 3 we use Hecke transformations to construct a finite group of automorphisms of our toric varieties, which normalize the torus and thus act on the associated fan of the toric varieties. In Chapter 4, this action on the fan will almost immediately let us apply a well-known criterion for toric varieties to be K-polystable (see [Ber16]). The toric varieties are constructed using the Cox rings of moduli stacks parametrizing parabolic bundles. Therefore to construct a group action on the toric varieties, we construct an action on the stacks, and construct a lift of that action to the Cox ring.

In Chapter 4 we use the results of the previous chapter to give conditions for the toric varieties to be K-polystable. We also give examples in this chapter, and show that for each genus there is at least one toric variety which is K-polystable.

In Chapter 5 we start building towards the proof of K-stability for $U_C(2, \xi)$. For this, we use the Luna slice theorem for stacks from [AHR20]. For X_Γ a K-polystable toric degeneration of $U_C(2, \xi)$, the Luna slice theorem relates the K-stability of $U_C(2, \xi)$ to the GIT stability of a point of a scheme $\mathcal{A}_\Gamma$ with respect to an action of $\mathrm{Aut}(X_\Gamma)$. First, we give lemmas that we will need to precisely compute stabilizers of points in $\mathcal{A}_\Gamma$. Then we give lemmas to guarantee GIT stability of certain points in $\mathcal{A}_\Gamma$. The goal of the rest of the paper will be to verify that these lemmas hold in our case.

In Chapter 6 we verify an important hypothesis for our GIT lemmas. In particular, we need to show that the automorphism group of the toric variety X_Γ is not too big. In fact, we show it is generated by the torus and a finite group containing the Hecke transformations from Chapter 3. This is checked combinatorially using the theory of

Demazure roots.

In Chapter 7 we study specific points in $\mathcal{A}_\Gamma$ along with their stabilizers. The space $\mathcal{A}_\Gamma$ is (an equivariant Artin approximation of) the versal deformation space of X_Γ. Therefore its points parametrize nearby deformations of X_Γ. If a point $a \in \mathcal{A}_\Gamma$ parameterizes a variety X_a, then the stabilizer of a under the $\mathrm{Aut}(X_\Gamma)$-action is isomorphic to the automorphism group of X_a. Thus, what we do in this chapter is construct interesting deformations of X_Γ and study their automorphism groups.

In Chapter 8 we combine everything from the previous chapters to prove Theorem 1.1. In particular we show that the deformations constructed in Chapter 7 have the properties needed to give the existence of GIT-stable points in $\mathcal{A}_\Gamma$. More precisely, we show there are GIT-stable points in $\mathcal{A}_\Gamma$ that parameterize the spaces $U_C(2, \xi)$, which, by the Luna slice theorem, proves the K-stability of these varieties.

Note that the strategy of using a toric degeneration and the Luna étale slice theorem for stacks for proving the K-stability of a different family of varieties has appeared in [KP21]. A significant difference between our strategy and theirs is that we do not need an explicit computation of the versal deformation space of the toric variety, which would be quite difficult. We simply have to produce enough deformations and understand their automorphism groups. Hopefully this means our strategy can be used to understand the K-stability of more families of varieties that admit toric degenerations.

Everything in the paper will be over $\mathbb{C}$ and by a torus we always mean an algebraic torus $T \cong (\mathbb{C}^\times)^n$. This comes equipped with two dual lattices: the character lattice

$$M(T) \coloneqq \mathrm{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^n$$

and the lattice of one-parameter subgroups

$$N(T) \coloneqq \mathrm{Hom}(\mathbb{C}^\times, T).$$

1.1 Open Questions

There are many questions that are suggested by the results of this paper. First we mention

Question 1.3. *Is the moduli space $U_C(2, \xi)$ K-stable for every smooth curve C?*

This type of problem can be quite challenging. Analogous results in [LX19] [SS17] rely on the varieties being complete intersections, as they show that K-polystable degenerations are complete intersections of the same type. The determinant of cohomology line bundle on the moduli spaces $U_C(2, \xi)$ gives embeddings into projective spaces whose dimensions are very large. Therefore adapting their strategy to this case could be very difficult.

Another strategy could be to appeal more directly to the definition of K-stability, e.g. to show that the δ-invariant is larger than one. Because the δ-invariant is a measure of the singularities of divisors on $U_C(2, \xi)$, such a bound on the δ-invariant could be interpreted as a question of higher-rank Brill-Noether theory.

We now know that the moduli space M^{Kps} has projective irreducible components. Therefore we can study the closure of the image of M_g in M^{Kps}, which we will call KM_g. If Question 1.3 is true, then KM_g would contain a dense open set isomorphic to M_g.

Question 1.4. *Is KM_g isomorphic to $\overline{M}_g$? If not, is it isomorphic to any other compactification of M_g known in the literature?*

Question 1.5. *Does every point in KM_g correspond to a moduli space of sheaves (perhaps with extra structure) on a projective curve?*

For Question 1.4, one could start by modifying our techniques to study the K-polystability of the varieties $X_{(C,\vec{\lambda}/L)}$ as defined in equation 2.2. When C is a so-called graph curve, these varieties are the toric varieties discussed above, but when C is an arbitrary nodal curve, they still have torus actions and it is reasonable to expect that they could be K-polystable under suitable conditions. The most accessible case would be the varieties X_{C_t} studied in Chapter 7 because these are complexity 1 T-varieties. Also, we only show that the toric degenerations are K-polystable when the dual graph of the underlying curve has no bridges, and it would be very interesting to know what happens when this restriction is lifted.

With regard to Question 1.5, our toric degenerations as well as the more general varieties $X_{(C,\vec{\lambda}/L)}$, can be interpreted as moduli spaces of framed sheaves on the underlying curves as in [HJ00].

Chapter 2

Conformal Blocks and Degenerations of Moduli Spaces of Vector Bundles

In order to study the K-stability of $U_C(2,\xi)$ for a general curve, we degenerate the curve to a nodal curve. For this paragraph, let $\xi = \mathcal{O}_C$. Then as C varies over $\mathcal{M}_g$ the spaces $U_C(2,\mathcal{O}_C)$ are fibers of a flat morphism $\mathcal{V}ec_{2,\mathcal{O}_C,g} \to \mathcal{M}_g$. In order to use degeneration arguments, we wish to extend this to a family over $\overline{\mathcal{M}}_g$. The extension of this family that we will consider in this paper is the one given by the theory of conformal blocks. This theory constructs a sheaf of algebras over $\overline{\mathcal{M}}_g$ whose fibers are called algebras of conformal blocks. The algebra of conformal blocks over a smooth curve C is the homogeneous coordinate ring of $U_C(2,\mathcal{O}_C)$, but over a nodal curve it is not obviously related to any moduli space of bundles. With more work, the theory of conformal blocks can be used to construct an analogous family over $\overline{\mathcal{M}}_{g,1}$ whose fibers over smooth curves are moduli spaces of vector bundles with fixed determinant of odd degree. We explain this in Chapter 4.

We start by reviewing the relevant aspects of the theory of conformal blocks. To be clear, none of the results in this chapter are due to the author, we are simply giving an exposition of the constructions that we will use for the rest of the paper. In general, the theory of conformal blocks starts with the choice of a simple complex Lie group, but we will state all the results for the simple case of $\mathrm{SL}_2 = \mathrm{SL}_2(\mathbb{C})$. The starting point is a construction going back to [TUY89]. Namely, the theory of conformal blocks associates a vector space, $\mathcal{V}^{\dagger}_{(C,\vec{p})}\left(\vec{\lambda}, L\right)$ to the following data:

1. A pointed stable curve $(C, \vec{p}) \in \overline{\mathcal{M}}_{g,n}$.
2. A positive integer λ_i, called a weight, for each marked point p_i.
3. An integer L called the level.

We are interested in the global situation as the curve varies.

Proposition 2.1. *[Man18] Section 2, [TUY89]: The vector spaces $\mathcal{V}^{\dagger}_{(C,\vec{p})}\left(\vec{\lambda}, L\right)$ are fibers of a vector bundle $\mathcal{V}^{\dagger}\left(\vec{\lambda}, L\right)$ over $\overline{\mathcal{M}}_{g,n}$. Further, the direct sum*

$$\mathcal{V}^{\dagger} = \bigoplus_{\vec{\lambda}, L} \mathcal{V}^{\dagger}\left(\vec{\lambda}, L\right)$$

admits the structure of a sheaf of multigraded algebras. This sheaf of algebras is relatively finitely generated, in particular each fiber is finitely generated.

To connect this theory with moduli spaces of vector bundles on curves, first we need to recall the notion of parabolic bundles.

Definition 2.2. A rank-two quasi-parabolic bundle $(E, E_\bullet)$ on C is a rank-two vector bundle

$$\pi : E \to C$$

equipped with one-dimensional subspace $E_i \subseteq E|_{p_i}$ for each marked point. The bundle $\mathcal{E}$ is also equipped with a trivialization of its determinant

$$\det(\mathcal{E}) \tilde{\to} \mathcal{O}_C.$$

A rank-two parabolic bundle is a rank-two quasi-parabolic bundle along with a choice of integer weights λ_i (one per marked point) and level L.

Remark 2.3. Usually a parabolic bundle is assigned weights which are rational numbers between 0 and 1. These numbers correspond to λ_i/L in our notation, as we will always have $\lambda_i \leq L$.

Remark 2.4. We will not discuss principal G-bundles in this paper, but we note that the reason we include a trivialization of the determinant in the definition is to make $\mathcal{E}$ the associated bundle of a principal SL_2-bundle.

The following theorem combines results from many authors ([Fal94], [Kum97], [BL94], [BLS98], [Pau96]) and connects conformal blocks with moduli spaces of bundles.

Theorem 2.5. *Let $(C,\vec{p}) \in \overline{\mathcal{M}}_{g,n}$ be a smooth stable curve, and $\mathcal{M}_{C,\vec{p}}$ be the moduli stack of quasi-parabolic bundles of rank 2. We have:*

1. *As groups,*

$$\mathrm{Pic}(\mathcal{M}_{C,\vec{p}}) \cong \mathbb{Z}^{n+1}.$$

   Elements of $\mathbb{Z}^{n+1}$ correspond to choices $(\vec{\lambda}, L)$ of weights and level. Given $(\vec{\lambda}, L)$, we denote by $\mathcal{L}(\vec{\lambda}, L)$ the corresponding line bundle.

2. *The sections of these line bundles are naturally identified with conformal blocks:*

$$\mathcal{V}^{\dagger}_{(C,\vec{p})}\left(\vec{\lambda}, L\right) \cong \mathrm{H}^0\left(\mathcal{M}_{C,\vec{p}}, \mathcal{L}\left(\vec{\lambda}, L\right)\right).$$

3. *The last item implies*

$$\mathrm{Proj}\left(\bigoplus_k \mathcal{V}^{\dagger}_{(C,\vec{p})}\left(k\vec{\lambda}, kL\right)\right) = M_{(C,\vec{p},\vec{\lambda}/L)}$$

   Where $M_{(C,\vec{p},\vec{\lambda}/L)}$ denotes the moduli space of parabolic bundles that are semi-stable with respect to the weights $\vec{\lambda}$ and level L.

Note that we use $\vec{\lambda}/L$ in the notation because this space only depends on the ratio, and because this ratio is the system of weights that usually appear in the definition of parabolic bundles. See [MS80] for the original construction of these moduli spaces of parabolic bundles, or [Bho89] for a more algebraic construction.

Remark 2.6. If there are no marked points then we have

$$\operatorname{Proj}\left(\bigoplus_k \mathcal{V}_C^\dagger(kL)\right) = U_C(2, \mathcal{O}_C).$$

In other words, the algebra of conformal blocks on a smooth curve C is the homogeneous coordinate ring of the moduli space of rank-two vector bundles on C with trivial determinant.

If there is one marked point $p \in C$ and the corresponding weight and level are both 1 then

$$\operatorname{Proj}\left(\bigoplus_k \mathcal{V}_C^\dagger(k, k)\right) = M_{(C,p,1)}$$

In fact, the moduli space of rank-two bundles with parabolic structure of weight 1 at p, $M_{(C,p,1)}$, is isomorphic to the moduli space $U_C(2, \mathcal{O}(-p))$ of semistable bundles. This is Theorem 9.4 of [BL94].

In summary, the propositions say that

$$V_{g,n}(\vec{\lambda}, L) := \operatorname{Proj}_{\overline{\mathcal{M}}_{g,n}}\left(\bigoplus_{k\in\mathbb{N}} \mathcal{V}^\dagger\left(k\vec{\lambda}, kL\right)\right)$$

is a flat family over $\overline{\mathcal{M}}_{g,n}$ whose fiber over a smooth curve is the moduli space of parabolic bundles with weights $\vec{\lambda}$ and level L. We will henceforth denote the fiber of this family over a stable curve $(C, \vec{p})$ by

$$V_{(C,\vec{p})}(\vec{\lambda}, L)$$

and refer to this as a conformal blocks space.

Remark 2.7. Sometimes we do not assign a weight λ to every marked point. In that case we define

$$V_{(C,\vec{p})}(\vec{\lambda}, L) := \mathrm{Proj}\left(\bigoplus_{k\in\mathbb{N}}\bigoplus_{\vec{\lambda}'}\mathcal{V}^{\dagger}\left(\vec{\lambda}', k\vec{\lambda}, kL\right)\right)$$

where λ' runs over all weightings of marked points not assigned a weight.

For fixed $k \in \mathbb{N}$, all but finitely many choices of λ' will give $\mathcal{V}^{\dagger}\left(\vec{\lambda}', k\vec{\lambda}, kL\right) = 0$. In particular, if any component of $\vec{\lambda}$ is negative or larger than L then it will be zero.

Now that we have extended the family of moduli spaces of bundles over the boundary of $\overline{\mathcal{M}}_g$, we wish to study the conformal blocks spaces over nodal curves. For this, we recall the main results of the paper [Man18]. This paper relates conformal blocks spaces on a nodal curve to conformal blocks spaces over the normalization of the curve. This relationship allows Manon to construct toric degenerations of $V_{(C,\vec{p})}(\vec{\lambda}, L)$ for special curves C, and these toric varieties will be the main objects of study for the rest of the paper.

Specifically, the goal of the rest of this chapter is to explain the following.

Proposition 2.8. *([Man18] Theorem 1.3) Let $(C, \vec{p})$ be a graph curve with dual graph Γ (cf. Definition 2.11), L a positive integer, $\vec{\lambda}$ an assigment of integer weights to some subset of the marked points $\vec{p}$, and denote by $U(\Gamma)$ the half edges of Γ not assigned a weight. The conformal block space $V_{(C,\vec{p})}(\vec{\lambda}, L)$ degenerates to the toric variety $X_{(\Gamma,\vec{\lambda})}$ defined as follows. A choice of level L determines an ample line bundle whose associated moment* *polytope $P_{\Gamma,\vec{\lambda}}$ sits inside $M_{\mathbb{R}} = \mathbb{R}^{E(\Gamma)\cup U(\Gamma)}$, the space of real-valued weightings of the edges of Γ as well as the half edges that have not been assigned a weight λ. For $w \in M_{\mathbb{R}}$ and vertex $v \in V(\Gamma)$, let $w_1(v)$, $w_2(v)$, and $w_3(v)$ be the weights of the three edges or half edges adjacent to v, replacing the w values by λ when appropriate. Then the polytope $P_{\Gamma,\vec{\lambda}}$ is described by the inequalities*

$$\begin{aligned}
-w_1(v) - w_2(v) - w_3(v) &\geq -L/2 \\
-w_1(v) + w_2(v) + w_3(v) &\geq -L/2 \\
w_1(v) - w_2(v) + w_3(v) &\geq -L/2 \\
w_1(v) + w_2(v) - w_3(v) &\geq -L/2
\end{aligned}$$

where this set of inequalities appears for every vertex $v \in V(\Gamma)$. The lattice $M \subset M_{\mathbb{R}}$ is

$$M = \left\{(w_e)_{e\in E(\Gamma)} \in M_{\mathbb{R}} | w_e \in \mathbb{Z}, w_1(v) + w_2(v) + w_3(v) \in 2\mathbb{Z} \textit{ for each } v \in V(\Gamma)\right\}.$$

We start explaining the above proposition by setting our conventions for dual graphs of nodal curves.

Definition 2.9. The dual graph, Γ, of a pointed nodal projective curve $(C, \vec{p})$ has one vertex per irreducible component of C, and two vertices are connected by k edges if the corresponding components meet in k nodes. For each marked point p_i on an irreducible component of C corresponding to a vertex v, the graph has a half edge which connects only to the vertex v. We denote by $E(\Gamma)$ the set of full edges. Similarly we denote the vertex set by $V(\Gamma)$. Sometimes we assign integer weights to the half-edges of Γ, and in that case we denote by $U(\Gamma)$ the set of half-edges are not assigned a weight.

For a given edge $e \in E(\Gamma)$, one may normalize the corresponding node in the curve to obtain a curve $(\tilde{C}, \vec{p}, q_1, q_2)$ with two extra marked points corresponding q_1 and q_2 to the two preimages of the node. In terms of the graph, this corresponds to splitting e into two half-edges. Now if one assigns a weight $\alpha \in \mathbb{Z}$ to these two half edges, there is a map of vector spaces

$$\hat{\rho}_\alpha : \mathcal{V}^\dagger_{(\tilde{C}, \vec{p}, q_1, q_2)} \left(\vec{\lambda}, \alpha, \alpha, L\right) \to \mathcal{V}^\dagger_{(C, \vec{p})} \left(\vec{\lambda}, L\right) \tag{2.1}$$

which was proven to be injective in [TUY89]. We refer to the image of this map as the space of conformal blocks with weight α along e. Now we take $(\tilde{C}, \vec{p}, \vec{q})$ to be the partial normalization of C along a subset $I \in E(\Gamma)$ and choose weights $\vec{\alpha}$ to attach to the extra marked points $\vec{q}$. The images of the above maps are linear subspaces

$$\mathcal{V}^\dagger_{(C, \vec{p})} \left(\vec{\lambda}, L\right)_{\vec{\alpha}} \subseteq \mathcal{V}^\dagger_{(C, \vec{p})} \left(\vec{\lambda}, L\right)$$

and as a vector space, $\mathcal{V}^\dagger_{(C, \vec{p})} \left(\vec{\lambda}, L\right)$ is the direct sum of these subspaces as $\vec{\alpha}$ varies.

Manon in [Man18] Definition 3.2 now uses this to produce valuations on the (function field of the) ring

$$\bigoplus_{\vec{\lambda},L} \mathcal{V}^{\dagger}_{(C,\vec{p})}\left(\vec{\lambda},L\right) = \mathcal{V}^{\dagger}_{(C,\vec{p})}.$$

To do this one fixes a coweighting on the graph. For us this simply means an assignment of a number $\theta_e \in [0,1]$ to each edge $e \in I$. The valuation $\vec{\theta}$ is defined by giving an element of $\mathcal{V}^{\dagger}_{(C,\vec{p})}\left(\vec{\lambda},L\right)_{\vec{\alpha}}$ the value

$$\vec{\theta}\cdot\vec{\alpha} = \sum_{e\in I}\theta_e\alpha_e,$$

and to a sum of such elements the maximum valuation of the summands. Manon in [Man18] proves that this actually defines a valuation.

We note that the corresponding filtration $\mathcal{F}_{\vec{\theta}}$ on the algebra $\mathcal{V}_{(C,\vec{p})}$ has filtered pieces

$$\mathcal{F}^{\le a}_{\vec{\theta}} = \bigoplus_{\vec{\lambda},L}\bigoplus_{\substack{\vec{\alpha}\in\mathbb{Z}^I\\ \vec{\theta}\cdot\vec{\alpha}\le a}} \mathcal{V}^{\dagger}_{(C,\vec{p})}\left(\vec{\lambda},L\right)_{\vec{\alpha}}$$

for $a \in \mathbb{N}$.

Proposition 2.10. *([Man18] Proposition 1.7) Let $(\tilde{C},\vec{p},\vec{q})$ be the partial normalization of $(C,\vec{p})$ along a set of nodes $I \subseteq E(\Gamma)$, and let $(C_v,\vec{p}_v,\vec{q}_v)$, be the connected components of $\tilde{C}$. Also let θ be a coweighting of the edges in I. If all θ_e are strictly positive then the associated graded algebra*

$$gr_{\vec{\theta}}\left(\bigoplus_{\vec{\lambda},L} \mathcal{V}^{\dagger}_{(C,\vec{p})}\left(\vec{\lambda},L\right)\right)$$

is isomorphic to the subalgebra of

$$\mathcal{V}^{\dagger}_{(\tilde{C},\vec{p},\vec{q})} \cong \bigotimes \mathcal{V}^{\dagger}_{(C_v,\vec{p}_v,\vec{q}_v)}$$

which is generated by the subspaces whose weights agree on marked points that map to the same node in C and whose levels on all C_v agree.

If we fix an orientation on the graph, then we can realize this subalgebra as the invariant subring of the action of a torus. The torus is $(\mathbb{C}^{\times 2})^{|I|}$, and for a given edge $e \in I$ the corresponding $\mathbb{C}^{\times 2}$ factor acts on the subspace $\mathcal{V}_{(C,\vec{p},\vec{q})}(\vec{\lambda}, \alpha(\vec{q}))$ with the character

$$(t_1, t_2) \mapsto t_1^{(L_i - L_j)} t_2^{(\alpha(q_i) - \alpha(q_j))}$$

whenever q_i and q_j are two half-edges of the dual graph of $\tilde{C}$ corresponding to the edge e of Γ, and the level on the connected component of q_i is L_i. The character is determined by the orientation by letting q_i be the half-edge containing the endpoint of e and q_j be the one containing the starting point.

By the Rees algebra construction which is standard in K-stability, see e.g. Section 2 of [BX19], these valuations correspond to $\mathbb{C}^{\times}$-degenerations of the spaces $V_{(C,\vec{p})}(\vec{\lambda}, L)$. By a $\mathbb{C}^{\times}$-degeneration of a variety X we mean a flat morphism

$$\pi : \mathcal{X} \to \mathbb{C}$$

where $\mathcal{X}$ is a scheme with $\mathbb{C}^\times$-action such that π is equivariant with respect to the standard $\mathbb{C}^\times$ action on $\mathbb{C}$ and $\pi^{-1}(1) \cong X$. If we define $X_0 := \pi^{-1}(0)$ we denote this situation by

$$X \Rightarrow X_0$$

and think of this as a degeneration of X to X_0.

The most important case for us will be when $\tilde{C}$ is the normalization of $(C, \vec{p})$. In this case we have that a valuation $\vec{\theta}$, when $\vec{\theta}$ has strictly positive entries, induces a $\mathbb{C}^\times$-degeneration

$$V_{(C,\vec{p})}(\vec{\lambda}, L) \Rightarrow X_{(C,\vec{\lambda}/L)} := \left(\prod_{v \in V(\Gamma)} V_{(C_v, \vec{p}_v, \vec{q}_v)}(\vec{\lambda}, L) \right) /\!\!/ (\mathbb{C}^{\times 2})^{|E(\Gamma)|}. \tag{2.2}$$

Here it is important that the only prescribed weights in $V_{(C_v, \vec{p}_v, \vec{q}_v)}(\vec{\lambda}, L)$ are the $\vec{\lambda}$ attached to the original marked points $\vec{p}$. Also note that this variety only depends on the ratio of the weights λ and the level, so we only use that ratio in the notation.

The simplest degenerations arising from this construction are in the case when C is a graph curve.

Definition 2.11. A graph curve is a nodal projective curve with marked smooth points $(C, \vec{p})$ such that each irreducible component C_v is a rational curve with exactly three special points counted with multiplicity. A special point can be

1. An intersection point between C_v and another component $C_{v'}$: such a point has

multiplicity 1

2. A marked point which counts with multiplicity 1

3. A node of C_v which counts with multiplicity 2.

The dual graph of a graph curve is a trivalent graph, and the graph determines the curve up to isomorphism.

In the case of a graph curve, the components of the normalization are smooth rational curves and have exactly three marked points. Thus we are reduced to studying conformal blocks on the curve $(\mathbb{P}^1, p, q, r)$. In this case we have

Proposition 2.12. *The space $\mathcal{V}^{\dagger}_{(\mathbb{P}^1,p,q,r)}(a,b,c,L)$ has dimension either 0 or 1. If $a+b+c \leq 2L$ then we have an isomorphism*

$$\mathcal{V}^{\dagger}_{(\mathbb{P}^1,p,q,r)}(a,b,c,L) \cong (S_a \otimes S_b \otimes S_c)^{\mathrm{SL}_2}$$

where S_a is the irreducible representation of SL_2 of highest weight a, i.e. homogeneous polynomials of degree a in two variables. The SL_2 invariants are taken with respect to the diagonal action.

As described in Lemma 4.2 of [Bea96], the space $\mathcal{V}^{\dagger}_{(\mathbb{P}^1,p,q,r)}(a,b,c,L)$ has dimension one exactly when

$$|a-b| \leq c \leq a+b \tag{2.3}$$

$$a+b+c \leq 2L \tag{2.4}$$

$$a+b+c \in 2\mathbb{Z} \tag{2.5}$$

and is 0 otherwise.

This means that the ring

$$\mathcal{V}^{\dagger}_{(\mathbb{P}^1,p,q,r)} = \bigoplus_{a,b,c,L\in\mathbb{Z}} \mathcal{V}^{\dagger}_{(\mathbb{P}^1,p,q,r)}(a,b,c,L)$$

is the semigroup algebra associated to the semigroup of lattice points in the cone inside $\mathbb{R}^4$ defined by the inequalities (2.3) and (2.4) with respect to the lattice of integer vectors satisfying (2.5).

This semigroup algebra is graded by L, and if we fix a value for L the corresponding hyperplane section of the cone is a polytope defining the projective variety $V_{(\mathbb{P}^1,p,q,r)}$ as a toric variety. Once a level L is fixed, we will wish to translate the polytope so that it contains the origin in its interior. The necessary translation is

$$(a,b,c) \mapsto (w_1 := a - L/2, w_2 := b - L/2, w_3 := c - L/2)$$

after which the polytope is defined by the inequalities

$$-w_1 - w_2 - w_3 \geq -L/2$$

$$-w_1 + w_2 + w_3 \geq -L/2$$

$$w_1 - w_2 + w_3 \geq -L/2$$

$$w_1 + w_2 - w_3 \geq -L/2$$

Now for a graph curve C we will now study the variety

$$X_{(\Gamma,\vec{\lambda}/L)} = \left(\prod_{v \in V(\Gamma)} V_{(C_v,\vec{p}_v,\vec{q}_v)}(\vec{\lambda}, L) \right) /\!\!/ (\mathbb{C}^{\times 2})^{|E(\Gamma)|}.$$

in more depth. Note that for a graph curve C we will often replace the C in our notation with Γ because C and Γ are equivalent data.

Before taking the quotient by the torus, the variety

$$\prod_{v \in V(\Gamma)} V_{(C_v,\vec{p}_v,\vec{q}_v)}(\vec{\lambda}, L)$$

is the projective toric variety associated to the product of the above polytopes which sits inside $(\mathbb{R}^3)^{|V(\Gamma)|}$. In this setting, for a component of $\tilde{C}$ with corresponding vertex $v \in V(\Gamma)$, we label the coordinates of $\mathbb{R}^3$ as $(w_1(v), w_2(v), w_3(v))$. If one of these weights, say $w_1(v)$, corresponds to a half-edge with weight λ_i then we require $w_1(v) = \lambda_i - L/2$. Thus to get a full-dimensional polytope we restrict to the $6g-6-n$-dimensional affine subspace of $(\mathbb{R}^3)^{|V(\Gamma)|}$ whose corresponding components $w_1(v)$ are fixed as above.

Now the quotient of this space by the torus is the subalgebra from Proposition 2.10. In terms of polytopes, this means that we restrict to the subspace of $(\mathbb{R}^3)^{|V(\Gamma)|}$ where the weights and levels on half edges coming from the same edge of Γ are required to agree. Thus we have finished explaining Proposition 2.8.

Remark 2.13. Throughout the rest of the paper, we will mainly use $L = 4$, because in the case with no marked points (and in the other cases given certain caveats) this defines a polytope associated to the anticanonical divisor of the toric variety X_Γ. This is because for $U_C(2, \mathcal{O})$, the anticanonical divisor is four times the determinant of cohomology line bundle ($\mathcal{L}(1)$ in the notation of Theorem 2.5) by [MN89].

Chapter 3

Hecke Transformations

According to the description of the Picard group from Theorem 2.5, the algebras of conformal blocks, $\mathcal{V}^{\dagger}_{(C,\vec{p})}$, are the Cox rings of the moduli stacks $\mathcal{M}_{C,\vec{p}}$. The objective of this chapter is to recall the notion of Hecke transformations and to show that they act as automorphisms of these Cox rings. In particular we will use them to construct automorphisms of the varieties $X_{(C,\vec{\lambda})}$ from Chapter 2 when the components of C are rational curves. We work with the stacks of parabolic bundles instead of their moduli spaces because the Cox rings of the moduli spaces don't always recover the full Cox rings of the stacks. This is especially problematic when the curve is $\mathbb{P}^1$ with four marked points, which is our most important case.

We will first apply these automorphisms to the case where C is a graph curve in Chapter 4. In that case, the existence of these automorphisms will immediately imply the K-polystability of the toric variety X_C when the dual graph of C has no bridges. Because the variety is toric, these automorphisms can be seen simply as linear transformations of the M-lattice which preserve the polytope, and we could describe these without the full machinery developed in this chapter.

However, to prove the K-stability of $U_C(2,\xi)$ we will need to study deformations of the toric varieties X_C and their automorphisms. Specifically, in Chapter 7, given a graph curve C, we will smooth one node of C to get a family of stable curves $(C_t)_{t\in\mathbb{P}^1}$. Then the family of varieties X_{C_t} will be a one-parameter deformation of the toric variety X_C. In order to use the Luna slice theorem to study the K-stability of $U_C(2,\xi)$, we will need that the Hecke transformations act on the varieties X_{C_t}. To show this, we need to dive into the details of moduli stacks of parabolic bundles on rational curves and their Cox rings.

Now we will introduce the Hecke transformations. They go back at least to [NR75] and are described in more modern languange in [AG21].

Definition 3.1. Let $(E,E_\bullet)$ be a rank-two quasi-parabolic bundle on a smooth marked curve $(C,\vec{p})$. We define a new quasi-parabolic bundle $(H_{p_i}(E,E_\bullet),H_\bullet)$ called the Hecke transformation of $(E,E_\bullet)$ at the point p_i. The underlying vector bundle of the Hecke transformation is the kernel in the exact sequence

$$0\to H_p(E,E_\bullet)\to E\to E|_{p_i}/E_i\to 0.$$

To define the quasi-parabolic structure we restrict this exact sequence to p_i and extend to the left using Tor to get the exact sequence

$$0 \to \operatorname{Tor}_1^{\mathcal{O}_C}(\mathcal{O}|_{p_i}, E|_{p_i}/E_i) \to H_p(E, E_\bullet)|_{p_i} \to E|_{p_i} \to E|_{p_i}/E_i \to 0.$$

We define the parabolic subspace of $H_p(E, E_\bullet)|_{p_i}$ as the image of $\operatorname{Tor}_1^{\mathcal{O}_C}(\mathcal{O}|_{p_i}, E|_{p_i}/E_i)$. To define the parabolic structure at a marked point $p_j \neq p_i$, note that we have an equality of subsheaves $H_{p_i}(E, E_\bullet)|_{C\setminus\{p_i\}} = E|_{C\setminus\{p_i\}}$, so we take the same parabolic subspace as on E.

Note that on the level of determinant bundles we have a natural isomorphism

$$\det(H_p(E, E_\bullet)) = \det(E)(-p_i).$$

We prove the following lemma to show that the Hecke transformation is invertible and to calculate its inverse.

Lemma 3.2. *Let $(E, E_\bullet)$ be a quasi-parabolic bundle on a curve C. The twice-iterated Hecke transformation of $(E, E_\bullet)$ at p_i is naturally isomorphic to*

$$(E \otimes \mathcal{O}_C(-p_i), E_\bullet \otimes \mathcal{O}(-p_i)).$$

In particular, to invert the Hecke transformation, you perform the same Hecke transformation and then tensor by $\mathcal{O}_C(p_i)$.

Proof. If we simply denote the second Hecke transformation of $(E, E_\bullet)$ by $(H^2, H^2_\bullet)$ then we note that H^2 is a subsheaf of E and the inclusion fits into the exact sequence

$$0 \to H^2 \to E \to E|_{p_i} \to 0$$

where the map on the right is the natural restriction map. This exact sequence is also obtained by taking

$$0 \to \mathcal{O}_C(-p_i) \to \mathcal{O}_C \to \mathcal{O}_{p_i} \to 0$$

and tensoring by E. This gives a natural isomorphism between the underlying bundles $H^2 \cong E \otimes \mathcal{O}_C(-p_i)$. For the parabolic structure, we note that the parabolic structure on H^2 is naturally given by the short exact sequence

$$0 \to \mathrm{Tor}_1^{\mathcal{O}_C}(\mathcal{O}_p, E_i) \to \mathrm{Tor}_1^{\mathcal{O}_C}(\mathcal{O}_p, E|_{p_i}) \to \mathrm{Tor}_1^{\mathcal{O}_C}(\mathcal{O}_p, E|_{p_i}/E_i) \to 0.$$

Now, this sequence is naturally isomorphic to

$$0 \to E_i \otimes \mathcal{O}_C(-p_i) \to E|_{p_i} \otimes \mathcal{O}_C(-p_i) \to E|_{p_i}/E_i \otimes \mathcal{O}_C(-p_i) \to 0.$$

To see this, we take the exact sequence

$$0 \to \mathcal{O}_C(-p_i) \to \mathcal{O}_C \to \mathcal{O}_{p_i} \to 0$$

and tensor by E_i, $E|_{p_i}$, and $E|_{p_i}/E_i$ respectively. Tensoring like this gives the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Tor}_1^{\mathcal{O}_C}(\mathcal{O}_p, E_i) & \longrightarrow & \mathrm{Tor}_1^{\mathcal{O}_C}(\mathcal{O}_p, E|_{p_i}) & \longrightarrow & \mathrm{Tor}_1^{\mathcal{O}_C}(\mathcal{O}_p, E|_{p_i}/E_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_i \otimes \mathcal{O}_C(-p_i) & \longrightarrow & E|_{p_i} \otimes \mathcal{O}_C(-p_i) & \longrightarrow & E|_{p_i}/E_i \otimes \mathcal{O}_C(-p_i) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_i & \longrightarrow & E|_{p_i} & \longrightarrow & E|_{p_i}/E_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_i \otimes \mathcal{O}_{p_i} & \longrightarrow & E|_{p_i} \otimes \mathcal{O}_{p_i} & \longrightarrow & E|_{p_i}/E_i \otimes \mathcal{O}_{p_i} & \longrightarrow & 0
\end{array}$$

and the maps between the second and third rows are all 0 because it is given by multiplication by a defining equation of p_i and all sheaves in the diagram are supported on p_i. □

Remark 3.3. Note that at the end of the previous proof we want a natural isomorphism to make sure things work well in families of bundles over a fixed marked curve, and we do not identify $E_i \otimes \mathcal{O}_C(-p_i)$ with E_i because we would need to choose a trivialization of $\mathcal{O}_C(-p_i)$ around p_i. We will actually need to choose such trivializations later, but this needs care because different choices of trivialization can introduce scalars which may get in the way of certain diagrams commuting. This is only really important for Lemma 3.5.

Now that we have established basic facts about the Hecke transform at one point, we will assume our curve is $\mathbb{P}^1$ with $n \geq 3$ marked points and introduce a larger group of Hecke transformations which will act on $\mathcal{M}_{\mathbb{P}^1,\vec{p}}$.

Definition 3.4. Let $\vec{e} = (e_1, \dots, e_n) \in \{0,1\}^n$ be such that $\sum_{i=1}^n e_i = 0 \pmod 2$, where n is the number of marked points on $\mathbb{P}^1$. Denote the corresponding vector in $(\mathbb{Z}/2)^n$ by $\overline{e} = (\overline{e_1}, \dots, \overline{e_n})$, and denote the sum of all e_i by $|\vec{e}|$. Given a quasi-parabolic bundle $(\mathcal{E}, \mathcal{E}_\bullet)$ on $\mathbb{P}^1$, we define the Hecke transform $H_{\overline{e}}(\mathcal{E}, \mathcal{E}_\bullet)$ as the kernel in the exact sequence

$$0 \to H_{\overline{e}}(\mathcal{E}, \mathcal{E}_\bullet) \to \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\frac{|\vec{e}|}{2}\right) \to \frac{\mathcal{E}|_{\sum e_i p_i}}{\bigoplus_{e_i=1} \mathcal{E}_i} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\frac{|\vec{e}|}{2}\right) \to 0.$$

Its quasi-parabolic structure is determined at each point exactly as in Definition 3.1.

Lastly, choose an affine chart $\mathbb{A}^1 \subseteq \mathbb{P}^1$ with coordinate t which contains all p_i. We use this to define isomorphisms

$$\mathcal{O}_{\mathbb{P}^1}(1) \tilde{\to} \mathcal{O}_{\mathbb{P}^1}(p_i)$$

which send a local section s to $\frac{s}{t-p_i}$. Note that this defines isomorphisms

$$\mathcal{O}_{\mathbb{P}^1}(p_i) \tilde{\to} \mathcal{O}_{\mathbb{P}^1}(p_j)$$

for all i and j from 1 to n, and these satisfy the cocycle condition. Therefore we can safely identify any line bundle of the form $\mathcal{O}_{\mathbb{P}^1}(\sum a_i p_i)$ for $a_i \in \mathbb{Z}$ with $\mathcal{O}_{\mathbb{P}^1}(\sum a_i)$.

Lemma 3.5. *The Hecke transformations define an action of the group*

$$H_0 := \{\overline{e} \in (\mathbb{Z}/2)^n \mid \sum e_i = 0 \ \textit{(mod 2)}\}$$

on the stack $\mathcal{M}_{\mathbb{P}^1,\vec{p}}$ in the sense of [Rom05].

Proof. Let three elements $\overline{e}, \overline{f}, \overline{g} \in H_0$ be given and let $H_{\overline{e}}$, $H_{\overline{f}}$, and $H_{\overline{g}}$ denote the corresponding functors. We need to construct natural isomorphisms

$$\alpha_{\overline{e},\overline{f}} : H_{\overline{e}} \circ H_{\overline{f}} \tilde{\to} H_{\overline{e}+\overline{f}}$$

satisfying the associativity condition

$$\begin{array}{ccc} H_{\overline{e}} \circ H_{\overline{f}} \circ H_{\overline{g}} & \longrightarrow & H_{\overline{e}} \circ H_{\overline{f}+\overline{g}} \\ \downarrow & & \downarrow \\ H_{\overline{e}+\overline{f}} \circ H_{\overline{g}} & \longrightarrow & H_{\overline{e}+\overline{f}+\overline{g}} \end{array} .$$

We define $\alpha_{\overline{e},\overline{f}}$ by identifying $H_e(H_f((\mathcal{E},\mathcal{E}_\bullet)))$ as the kernel of the restriction sequence

$$0 \to H_{\overline{e}}(H_{\overline{f}}((\mathcal{E},\mathcal{E}_\bullet))) \to \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\frac{|\vec{e}| + |\vec{f}|}{2}\right) \to \mathcal{E}|_{\sum(e_i+f_i)p_i} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\frac{|\vec{e}| + |\vec{f}|}{2}\right) \to 0$$

with the quasi-parabolic structure at p_i determined by tensoring with $\mathcal{O}_{\mathbb{P}^1}|_{p_i}$ and extending to the left using Tor. Then we note that whenever $e_i = f_i = 1$, the sequence, locally around the point p_i, simply looks like a twist of the restriction sequence

$$0 \to \mathcal{E} \otimes \mathcal{O}^1_{\mathbb{P}}(-p_i) \to \mathcal{E} \to \mathcal{E}|_{p_i} \to 0.$$

Thus we canonically identify $H_{\overline{e}} \circ H_{\overline{f}}$ with

$$H_{\overline{e}+\overline{f}} \otimes \mathcal{O}^1_{\mathbb{P}}\left(\frac{|\vec{e}| + |\vec{f}|}{2} - \sum(e_i f_i p_i) - \sum(\overline{e}_i + \overline{f}_i)p_i\right) \cong H_{\overline{e}+\overline{f}}$$

where the last isomorphism is given by the isomorphisms between line bundles $\mathcal{O}^1_{\mathbb{P}}(p_i)$ that were chosen earlier.

Now the associativity diagram commutes because the isomorphisms $\alpha_{\overline{e},\overline{f}}$ are defined by including all the quasi-parabolic bundles involved into a twist of the original bundle $(\mathcal{E}, \mathcal{E}_\bullet)$ and identifying them as subsheaves. The quasi-parabolic structure is determined by the inclusion into the twist of $(\mathcal{E}, \mathcal{E}_\bullet)$ using Tor. The isomorphisms between the twisting line bundles do not affect commutativity because they satisfy the cocycle condition. □

Now we wish to see how this group of Hecke transforms acts on the Picard group of $\mathcal{M}_{\mathbb{P}^1,\vec{p}}$. For this we need to use the standard description of the line bundles $\mathcal{L}(L, \vec{\lambda})$ as in [LS97]. We choose a universal quasi-parabolic bundle $(\mathcal{E}, \mathcal{E}_\bullet)$ on $\mathcal{M} \times \mathbb{P}^1$ where $\mathcal{M} := \mathcal{M}_{\mathbb{P}^1,\vec{p}}$. We denote the quotients $\mathcal{E}_{\mathcal{M}\times\{p_i\}}/\mathcal{E}_i$ by $\mathcal{Q}_i$. We think of $\mathcal{Q}_i$ and $\mathcal{E}_i$ both as sheaves on $\mathcal{M}\times\mathbb{P}^1$ supported on $\mathcal{M}\times\{p_i\}$ and as line bundles on $\mathcal{M}$. Using the projection $\pi_{\mathcal{M}} : \mathcal{M} \times \mathbb{P}^1$, our line bundles are defined as

$$\mathcal{L}(\vec{\lambda}, L) := \det(R\pi_{\mathcal{M}*}\mathcal{E})^{-L} \otimes \bigotimes_{i=1}^{n} (\mathcal{Q}_i)^{\lambda_i}.$$

Lemma 3.6. *There are isomorphisms*

$$H_{\vec{e}}^{*}\mathcal{L}(\vec{\lambda}, L) \cong \mathcal{L}(H_{\vec{e}}^{-1}(\vec{\lambda}, L), L)$$

where

$$H_{\vec{e}}^{-1}(\vec{\lambda}, L)_k = H_{\vec{e}}(\vec{\lambda}, L)_k := \begin{cases} L - \lambda_k, & \text{if } e_k = 1 \\ \lambda_k, & \text{if } e_k = 0. \end{cases}$$

Proof. First, we should establish a few points about the determinant of cohomology functor $\det(R\pi_{\mathcal{M}*}\cdot)$.

- ([KM76] Page 46) Suppose we are given a short exact sequence of sheaves on $\mathcal{M} \times \mathbb{P}^1$,

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

  where all sheaves involved are perfect. Then there is an isomorphism $\det(R\pi_{\mathcal{M}*}\mathcal{F}) \cong \det(R\pi_{\mathcal{M}*}\mathcal{F}') \otimes \det(R\pi_{\mathcal{M}*}\mathcal{F}'')$.

- If $\mathcal{F}$ is a sheaf on $\mathcal{M} \times \mathbb{P}^1$ which is supported on $\mathcal{M} \times \{p\}$ for some $p \in \mathbb{P}^1$, then

$$\det(R\pi_{\mathcal{M}*}\mathcal{F}) \cong \det(\mathcal{F})$$

  where we think of $\mathcal{F}$ as a sheaf on $\mathcal{M}$. This is because

$$\pi_{\mathcal{M}} : \mathcal{M} \times \{p\} \to \mathcal{M}$$

is essentially the identity.

- For a sheaf $\mathcal{F}$ on $\mathcal{M} \times \mathbb{P}^1$ with no torsion along $\mathcal{M} \times \{p\}$ and an integer n there is an isomorphism

$$\det(R\pi_{\mathcal{M}*}(\mathcal{F} \otimes \pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(n))) \cong \det(R\pi_{\mathcal{M}*}\mathcal{F}) \otimes \det(\mathcal{F})^n|_{\mathcal{M}\times\{p\}}.$$

This is established inductively using the short exact sequence

$$0 \to \mathcal{F} \otimes \pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(n-1) \to \mathcal{F} \otimes \pi_{\mathbb{P}^1}^* \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{F}|_{\mathcal{M}\times\{p\}} \to 0$$

and applying the last two bullet points.

Note that at a point p_i such that $e_i = 1$ the parabolic structure of $H_{\overline{e}}((\mathcal{E}, \mathcal{E}_\bullet))$ is given by the quotient map

$$H_{\overline{e}}\mathcal{E}|_{\mathcal{M}\times\{p_i\}} \to \mathcal{E}_i \otimes \pi_{\mathbb{P}^1}^* \left(\mathcal{O}_{\mathbb{P}^1}\left(\frac{|\vec{e}|}{2}\right)\right) \cong \mathcal{E}_i.$$

Therefore we have that

$$H_{\overline{e}}^*(\mathcal{L}(\vec{\lambda}, L)) = \det(R\pi_{\mathcal{M}*}H_{\overline{e}}\mathcal{E})^{-L} \otimes \bigotimes_{e_i=1}(\mathcal{E}_i)^{\lambda_i} \otimes \bigotimes_{e_i=0}(\mathcal{Q}_i)^{\lambda_i}.$$

Note that by definition $(\mathcal{E}, \mathcal{E}_\bullet)$ is equipped with an isomorphism

$$\det(\mathcal{E}) \tilde{\to} \mathcal{O}_{\mathcal{M}}.$$

Thus we have that $\mathcal{E}_i \cong \mathcal{Q}_i^*$. Therefore we have

$$H_{\overline{e}}^*(\mathcal{L}(\vec{\lambda}, L)) = \det(R\pi_{\mathcal{M}*} H_{\overline{e}}\mathcal{E})^{-L} \otimes \bigotimes_{e_i=1} (\mathcal{Q}_i)^{-\lambda_i} \otimes \bigotimes_{e_i=0} (\mathcal{Q}_i)^{\lambda_i}.$$

To deal with the determinant of cohomology portion, we apply the above bullet points to the defining exact sequence

$$0 \to H_{\overline{e}}(\mathcal{E}, \mathcal{E}_\bullet) \to \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\frac{|\vec{e}|}{2}\right) \to \frac{\mathcal{E}|_{\sum e_i p_i}}{\bigoplus_{e_i=1} \mathcal{E}_i} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\frac{|\vec{e}|}{2}\right) \to 0.$$

This says that

$$\det(R\pi_{\mathcal{M}*} H_{\overline{e}}\mathcal{E})^{-L} \cong det\left(R\pi_{\mathcal{M}*}\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^1}\left(\frac{|\vec{e}|}{2}\right)\right)^{-L} \otimes \bigotimes_{e_i=1} \mathcal{Q}_i^L \cong det\left(R\pi_{\mathcal{M}*}\mathcal{E}\right)^{-L} \otimes \bigotimes_{e_i=1} \mathcal{Q}_i^L.$$

This last isomorphism follows from the third bullet point because of the isomorphism

$$\det(\mathcal{E}) \tilde{\to} \mathcal{O}_{\mathcal{M}}.$$

Altogether this gives

$$H_{\overline{e}}^*(\mathcal{L}(\vec{\lambda}, L)) = \det(R\pi_{\mathcal{M}*}\mathcal{E})^{-L} \otimes \bigotimes_{e_i=1} (\mathcal{Q}_i)^{L-\lambda_i} \otimes \bigotimes_{e_i=0} (\mathcal{Q}_i)^{\lambda_i}.$$

□

Ideally we would like to say that the action of the Hecke transforms on $\mathcal{M}$ lifts to an action on the Cox ring of $\mathcal{M}_{\mathbb{P}^1,\vec{p}}$. We will show that this is true as long as we extend the group of Hecke transforms by a torus. Before stating the theorem we should review some concepts from [AMF+22].

Definition 3.7. Let G be an abelian group. A G-family of line bundles on an algebraic stack $\mathcal{X}$ is a collection of line bundles $\vec{\mathcal{L}} = (\mathcal{L}_v)_{v\in G}$ equipped with isomorphisms $\mathcal{L}_v \otimes \mathcal{L}_{v'} \tilde{\to} \mathcal{L}_{v+v'}$ satisfying some coherence conditions which can be found in [AMF+22] Section 1.

This definition is motivated by the fact that a G-family of line bundles has a Cox ring defined as

$$\mathrm{Cox}(\vec{\mathcal{L}}) := \bigoplus_{v\in G} \mathrm{H}^0(\mathcal{X}, \mathcal{L}_v)$$

with the ring structure defined by the maps

$$\mathrm{H}^0(\mathcal{X}, \mathcal{L}_v) \otimes \mathrm{H}^0(\mathcal{X}, \mathcal{L}_{v'}) \to \mathrm{H}^0(\mathcal{X}, \mathcal{L}_v \otimes \mathcal{L}_{v'}) \tilde{\to} \mathrm{H}^0(\mathcal{X}, \mathcal{L}_{v+v'}).$$

It also defines a Cox sheaf of $\mathcal{O}_{\mathcal{X}}$-algebras

$$\mathcal{C}ox(\vec{\mathcal{L}}) := \bigoplus_{v\in G} \mathcal{L}_v$$

whose relative spectrum is written

$$\mathcal{C}(\vec{\mathcal{L}}) := \mathrm{Spec}_{\mathcal{X}}(\mathcal{C}ox(\vec{\mathcal{L}})).$$

Furthermore, if $G \subseteq \operatorname{Pic}(\mathcal{X})$ is free, then by [AMF+22] Theorem 3.3, there is a unique G-family of line bundles $\vec{\mathcal{L}}_G$ that is compatible with the inclusion map $G \hookrightarrow \operatorname{Pic}(\mathcal{X})$. This result is crucial for the main results of this section.

Definition 3.8. Let $\alpha : G \to H$ be a morphism of abelian groups, $\vec{\mathcal{L}}$ a G-family of line bundles on $\mathcal{X}$ and $\vec{\mathcal{K}}$ an H-family. A morphism of families of line bundles with respect to α is a collection of morphisms

$$\mathcal{L}_v \to \mathcal{K}_{\alpha(v)}$$

for all $v \in G$ which are compatible with the tensor isomorphisms.

Fix a finitely generated free subgroup $G \subseteq \operatorname{Pic}(\mathcal{X})$ with corresponding G-family of line bundles $\vec{\mathcal{L}}_G$, and let $T := \operatorname{Hom}(G, \mathbb{C}^*)$ be the Néron-Severi torus for the subgroup G.

Definition 3.9. Let $G \subseteq \operatorname{Pic}(\mathcal{X})$ be a subgroup. We denote by $\operatorname{Aut}_G(\mathcal{X})$ the subgroup of automorphisms whose induced automorphism of $\operatorname{Pic}(\mathcal{X})$ sends G to G.

We want to lift objects of $\operatorname{Aut}_G(\mathcal{X})$ to automorphisms of the space $\mathcal{C}(\vec{\mathcal{L}}_G)$. An important subtlety that we will need to monitor is the fact that the collection of maps $\mathcal{X} \to \mathcal{X}$ is itself a groupoid, not just a set. Accordingly we will need to study $\operatorname{Aut}_G(\mathcal{X})$ and $\operatorname{Aut}(\mathcal{C}(\vec{\mathcal{L}}_G))$ as 2-groups (really a stack valued in 2-groups, but we are really only interested in the $\mathbb{C}$-points), so we will review some fundamental concepts of 2-group theory before moving on.

Definition 3.10. A 2-group, $\mathfrak{G}$, is a group object in the 2-category of small groupoids. Its set of objects up to isomorphism is naturally a group, called $\pi_1(\mathfrak{G})$ and each object itself has an automorphism group. The automorphism group of the identity is denoted $\pi_2(\mathfrak{G})$. This is an abelian group by the Eckmann-Hilton argument, and is isomorphic to the automorphism group of any object. Any group can be considered as a 2-group with trivial π_2.

Definition 3.11. A short exact sequence of 2-groups is given by functors

$$1 \to \mathfrak{G}' \to \mathfrak{G} \to \mathfrak{G}'' \to 1$$

inducing a short exact sequence on the groups π_1 and π_2.

Definition 3.12. Let $\mathfrak{H} \hookrightarrow \mathfrak{G}$ be a sub-2-group. We define the normalizer of $\mathfrak{H}$ in $\mathfrak{G}$ to be the full subcategory of objects whose isomorphism classes lie in the normalizer of $\pi_1(\mathfrak{H})$ inside $\pi_1(\mathfrak{G})$.

Now we start towards the proof of the main theorem of this chapter with the following lemma, which relates morphisms of families of line bundles to automorphisms of the Cox sheaves of algebras.

Lemma 3.13. *The normalizer of the torus T inside the 2-group $\mathrm{Aut}(\mathcal{C}(\vec{\mathcal{L}}_G))$, which we denote by $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G))$, is equivalent to the groupoid of pairs of isomorphisms $(\varphi : \mathcal{X} \tilde{\to} \mathcal{X}, \psi : \varphi^* \vec{\mathcal{L}} \tilde{\to} \vec{\mathcal{L}})$ where $\varphi \in \mathrm{Aut}_G(\mathcal{X})$ and ψ is an isomorphism of families of line bundles with respect to the automorphism $\varphi^* : G \to G$.*

Proof. First, we see how given a pair $(\varphi : \mathcal{X} \tilde{\to} \mathcal{X}, \psi : \varphi^* \vec{\mathcal{L}} \tilde{\to} \vec{\mathcal{L}})$, we can construct an automorphism $\tilde{\varphi} \in \mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G))$. Because the construction of $\mathcal{C}(\vec{\mathcal{L}}_G)$ is functorial with respect to morphisms of G-families of line bundles, we get the following isomorphisms:

$$\mathcal{C}(\vec{\mathcal{L}}_G) \to \mathcal{C}(\varphi^*\vec{\mathcal{L}}) \to \mathcal{C}(\vec{\mathcal{L}}_G).$$

Here the first map is induced by ψ and the second is the natural projection from the pullback. This automorphism normalizes the torus because it is given on the graded pieces of $\mathcal{C}ox(\vec{\mathcal{L}}_G)$ by isomorphisms $(\varphi^*\vec{\mathcal{L}})_v \to \mathcal{L}_{\varphi^*(v)}$ and then extended by linearity. This construction is clearly functorial with respect to isomorphisms of pairs (φ, ψ).

Now we construct an essential inverse functor from $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G))$ to the groupoid of pairs (φ, ψ). Firstly, we need to start with an automorphism $\tilde{\varphi} \in \mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G))$ and construct the corresponding automorphism $\varphi \in \mathrm{Aut}_G(\mathcal{X})$. Note that $\mathcal{C}(\vec{\mathcal{L}}_G)$ is a T-torsor over $\mathcal{X}$, and therefore we have that $[\mathcal{C}(\vec{\mathcal{L}}_G)/T] \cong \mathcal{X}$. Thus, we get a functor $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G)) \to \mathrm{Aut}(\mathcal{X})$ by simply composing an automorphism with the projection $\mathcal{C}(\vec{\mathcal{L}}_G) \to \mathcal{X}$ and noting that because the automorphism normalizes the T-action, the resulting map is invariant for the T action and thus descends to $\mathcal{X}$.

Now we justify why the image of this functor lands in the subcategory $\mathrm{Aut}_G(\mathcal{X})$. Denote the automorphism $\tilde{\varphi}$ induces on $\mathcal{X}$ by φ, so our automorphisms fit into a diagram

$$\begin{array}{ccc} \mathcal{C}(\vec{\mathcal{L}}_G) & \xrightarrow{\tilde{\varphi}} & \mathcal{C}(\vec{\mathcal{L}}_G) \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{\varphi} & \mathcal{X}. \end{array}$$

Therefore $\tilde{\varphi}$ induces an isomorphism $\varphi^*\mathcal{C}ox(\vec{\mathcal{L}}_G) \to \mathcal{C}ox(\vec{\mathcal{L}}_G)$ of sheaves of rings over

$\mathcal{X}$. That the automorphism $\tilde{\varphi}$ normalizes T means that this isomorphism is T equivariant as long as you twist the pullback T-action on $\varphi^*\mathcal{C}(\vec{\mathcal{L}}_G)$ by the automorphism $\tilde{\varphi}$ induces on T by conjugation in $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G))$. Therefore this isomorphism sends the graded pieces of $\varphi^*\mathcal{C}ox(\vec{\mathcal{L}}_G)$ to the graded pieces of $\mathcal{C}ox(\vec{\mathcal{L}}_G)$. In other words, the pullback of line bundles in G under φ are still in G, and therefore φ is an object of $\mathrm{Aut}_G(\mathcal{X})$ as desired.

In fact, that the isomorphism of sheaves of algebras $\varphi^*\mathcal{C}ox(\vec{\mathcal{L}}_G) \to \mathcal{C}ox(\vec{\mathcal{L}}_G)$ respects graded pieces means that it is determined by a system of isomorphisms $\varphi^*\mathcal{L}_v \tilde{\to} \mathcal{L}_{\varphi^* v}$ which respect the ring structures on the Cox rings, i.e. a morphism ψ of G-families of line bundles $\varphi^*(\vec{\mathcal{L}}) \tilde{\to} \vec{\mathcal{L}}$ with respect to the automorphism $\varphi^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})$. Thus we have constructed our pair (φ, ψ). This construction is functorial and gives the desired essential inverse. □

Now we get to the main theorem of this chapter.

Theorem 3.14. *Let $\mathcal{X}$ be an algebraic stack and $G \subseteq \mathrm{Pic}(\mathcal{X})$ be a subgroup. Assume that G is free and finitely generated. Let $T := \mathrm{Hom}(G, \mathbb{C}^*)$ be the subtorus of the Néron-Severi torus corresponding to the subgroup G (sometimes we abusively call this the Néron-Severi torus of G). Let $\mathcal{C}ox(\vec{\mathcal{L}}_G)$ be the Cox sheaf of algebras as in [AMF+22] and denote its relative spectrum $\mathcal{C}(\vec{\mathcal{L}}_G) := \mathrm{Spec}_{\mathcal{X}}(\mathcal{C}ox(\vec{\mathcal{L}}_G))$. If we define $Aut_T(\mathcal{C}(\vec{\mathcal{L}}_G))$ as the normalizer of T, then we have a short exact sequence of 2-groups:*

$$1 \to T \to \mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G)) \to \mathrm{Aut}_G(\mathcal{X}) \to 1.$$

Here a sequence of 2-groups is exact if it induces exact sequences on the group of isomorphism classes of objects and on the automorphism groups of each object.

Proof. By Lemma 3.13 we can replace $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G))$ with the groupoid of pairs

$$(\varphi \in \mathrm{Aut}_G(\mathcal{X}), \psi : \varphi^* \vec{\mathcal{L}}_G \to \vec{\mathcal{L}}_G).$$

We start by checking that the map $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G)) \to \mathrm{Aut}_G(\mathcal{X})$ is surjective on objects. Take an automorphism $\varphi : \mathcal{X} \to \mathcal{X}$ inside $\mathrm{Aut}_G(\mathcal{X})$. Now, because G is free, by Theorem 3.3 of [AMF+22] there is only one structure of G-family of line bundles on $\vec{\mathcal{L}_G}$ that is compatible with the inclusion $G \subseteq \mathrm{Pic}(\mathcal{X})$. Thus we can choose an isomorphism $\psi : \varphi^*(\vec{\mathcal{L}_G}) \tilde{\to} \vec{\mathcal{L}_G}$ with respect to the automorphism $\varphi^* : G \to G$.

This shows that the morphism of 2-groups $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G)) \to \mathrm{Aut}_G(\mathcal{X})$ is surjective on objects.

Now we see that $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G)) \to \mathrm{Aut}_G(\mathcal{X})$ induces isomorphisms between spaces of morphisms. Suppose we have an isomorphism $\varphi \to \varphi'$. Then there is a unique way to fill in the commutative diagram

$$\begin{array}{ccc} \varphi^*(\vec{\mathcal{L}}) & \xrightarrow{\psi} & \vec{\mathcal{L}} \\ \downarrow & & \downarrow{\scriptstyle id} \\ \varphi'^*(\vec{\mathcal{L}}) & & (\vec{\mathcal{L}}) \end{array}$$

where the left arrow is induced by the isomorphism $\varphi \to \varphi'$. Therefore isomorphisms between pairs $(\varphi, \psi : \varphi^* \vec{\mathcal{L}} \tilde{\to} \vec{\mathcal{L}})$ are uniquely determined by what they do to φ.

Lastly we should check that the kernel of the morphism $\mathrm{Aut}_T(\mathcal{C}(\vec{\mathcal{L}}_G)) \to \mathrm{Aut}_G(\mathcal{X})$ is T considered as a 2-group with no automorphisms. To see this, note that the kernel is the collection of pairs $(id, \psi : \vec{\mathcal{L}} \tilde{\to} \vec{\mathcal{L}})$. Such a thing is given by a collection of isomorphisms $\mathcal{L}_v \tilde{\to} \mathcal{L}_v$ for all $v \in G$ which is compatible with tensor products. In other words, it is a group homomorphism from $G \to \mathbb{C}^*$. □

Corollary 3.15. *The ring* $\mathrm{Cox}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$ *admits an action by a group* H_T *which is an extension of the Hecke transformation group* H_0 *by a quotient* T/μ_2 *of the Neron-Severi torus* T *of* $\mathcal{M}_{\mathbb{P}^1,\vec{p}}$*. The ring* $\mathrm{Cox}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$ *is graded by a sublattice of* $\mathrm{Pic}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$ *and* H_T *permutes the graded components in the same way* H_0 *permutes* $\mathrm{Pic}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$ *as described in Lemma 3.6.*

Proof. Because of Lemma 3.5, we have a map $H_0 \to \mathrm{Aut}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$, so we can define the group H_T as the fiber product in this diagram of fundamental groups

$$\begin{array}{ccc} H_T & \longrightarrow & H_0 \\ \downarrow & & \downarrow \\ \pi_1(\mathrm{Aut}_T(\mathcal{C}(\mathcal{M}_{\mathbb{P}^1,\vec{p}}))) & \longrightarrow & \pi_1(\mathrm{Aut}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})). \end{array}$$

Recall that the fundamental group of a 2-group is its set of isomorphism classes. More concretely, one can think of points of H_T as pairs $([(\varphi, \psi : \varphi^* \vec{\mathcal{L}} \tilde{\to} \vec{\mathcal{L}})], h)$ such that the automorphism φ of $\mathcal{M}_{\mathbb{P}^1,\vec{p}}$ is isomorphic to the Hecke transform $h \in H_0$. Here the square brackets denote the isomorphism class.

We will now explain that the top row fits into a short exact sequence

$$1 \to T/\mu_2 \to H_T \to H_0 \to 1.$$

The projection $H_T \to H_0$ is surjective because the map

$$\pi_1(\mathrm{Aut}_T(\mathcal{C}(\mathcal{M}_{\mathbb{P}^1,\vec{p}}))) \to \pi_1(\mathrm{Aut}(\mathcal{M}_{\mathbb{P}^1,\vec{p}}))$$

is. This in turn is true because $\mathrm{Aut}_T(\mathcal{C}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})) \to \mathrm{Aut}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$ is surjective on the group of objects, and therefore it is surjective on the group of isomorphism classes.

Now we calculate the kernel of the map $H_T \to H_0$. A priori, the kernel is the group of isomorphism classes of the form $[(id, \psi : \vec{\mathcal{L}} \tilde{\to} \vec{\mathcal{L}})]$. Now, the group of pairs $(id, \psi : \vec{\mathcal{L}} \tilde{\to} \vec{\mathcal{L}})$ is T, as explained in the proof of Theorem 3.14. Also in that proof we saw that isomorphisms between those pairs are the same as isomorphisms between the automorphisms $\varphi : \mathcal{M}_{\mathbb{P}^1,\vec{p}} \tilde{\to} \mathcal{M}_{\mathbb{P}^1,\vec{p}}$. This means that the kernel of $H_T \to H_0$ is the quotient of T by the action of the automorphism group of

$$id : \mathcal{M}_{\mathbb{P}^1,\vec{p}} \to \mathcal{M}_{\mathbb{P}^1,\vec{p}}$$

i.e. the automorphism group of the universal parabolic bundle $(\mathcal{E}, \mathcal{E}_\bullet)$ on $\mathcal{M}_{\mathbb{P}^1,\vec{p}} \times \mathbb{P}^1$. We claim that as long as there are at least two marked points $\vec{p}$, then this automorphism group is $\mathbb{Z}/2$, the center of SL_2. This is because the general point of $\mathcal{M}_{\mathbb{P}^1,\vec{p}}$ parameterizes a bundle of the form $\mathcal{O} \oplus \mathcal{O}$ where no global section passes through more than one of the parabolic subspaces. Such a parabolic bundle is not a direct sum of parabolic sub-bundles, and therefore its only automorphisms are scalars. However, points of $\mathcal{M}_{\mathbb{P}^1,\vec{p}}$ also come with trivializations of their determinants, so the only scalars compatible with these trivializations are ± 1.

Now we claim that the automorphism -1 of $id : \mathcal{M}_{\mathbb{P}^1,\vec{p}} \to \mathcal{M}_{\mathbb{P}^1,\vec{p}}$ acts on the line bundle $\mathcal{L}(m_1, \ldots, m_n, L)$ by $(-1)^{\sum_{i=1}^n m_i}$. For the line bundles given by the universal parabolic quotients $\mathcal{Q}_i$, this is clear because the scalar -1 acting on $\mathcal{E}$ acts by -1 on its fiber $\mathcal{E}|_{\mathcal{M}_{\mathbb{P}^1,\vec{p}} \times \{p_i\}}$ and so by -1 on its quotient $\mathcal{Q}_i$. Thus we only have to check that -1 acts trivially on the determinant of cohomology line bundle.

To see this note that the complex $R\pi_{\mathcal{M}_{\mathbb{P}^1,\vec{p}}*}\mathcal{E}$ is perfect of amplitude $[0,1]$, and that the scalar -1 acts by -1 on each term of the complex. If you represent this complex locally by a complex of vector bundles $\mathcal{F}_1 \to \mathcal{F}_2$, then the difference in ranks between $\mathcal{F}_1$ and $\mathcal{F}_2$ is the Riemann Roch number $\chi(\mathcal{E}|_{\mathbb{P}^1 \times \{(E, E_\bullet)\}}) = 2$ for any $(E, E_\bullet) \in \mathcal{M}_{\mathbb{P}^1,\vec{p}}$. This means that -1 acts by $(-1)^2 = 1$ on the determinant of this complex.

Therefore we have shown that if we write $\text{Pic}(\mathcal{M}_{\mathbb{P}^1,\vec{p}}) \cong \mathbb{Z}^n \times \mathbb{Z}$ then the action of the scalar -1 on the torus $T \cong (\mathbb{C}^\times)^n \times \mathbb{C}^\times$ is trivial on the component that corresponds to the level and scales each of the components that correspond to the marked points by -1. Thus the kernel of $H_T \to H_0$ is the quotient of T by this μ_2 action. The torus T/μ_2 is the torus whose M-lattice is the sublattice of $M(T)$ consisting of vectors $(\vec{m}, l)$ such that the sum $m_1 + \ldots + m_n$ is even. This explains the strange lattice used to define the toric varieties in Proposition 2.8.

By Theorem 3.14 the first statement of the corollary is true with

$$\mathcal{C}(\mathcal{M}_{\mathbb{P}^1,\vec{p}}) = \mathrm{Spec}_{\mathcal{M}}(\mathcal{C}ox(\mathcal{M}_{\mathbb{P}^1,\vec{p}}))$$

replacing $\mathrm{Cox}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$. Because $\mathrm{Cox}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$ is the ring of global sections of the structure sheaf of $\mathcal{C}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$, we get a map

$$\mathrm{Aut}_T(\mathcal{C}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})) \to \mathrm{Aut}_T(\mathrm{Cox}(\mathcal{M}_{\mathbb{P}^1,\vec{p}}))$$

and therefore an action of H_T on $\mathrm{Cox}(\mathcal{M}_{\mathbb{P}^1,\vec{p}})$. The last statement of the Corollary follows immediately from the constructions. □

Now we apply Corollary 3.15 to study the degenerations $X_{(C,\vec{\lambda})}$ in the case where $(C,\vec{p})$ is a stable curve each of whose components is a rational curve. As before, we denote by $(C_v,\vec{p}_v,\vec{q}_v)$, $v \in V(\Gamma)$, the connected components of the normalization of C. Here $\vec{p}_v$ lists the preimages of the marked points $\vec{p}$ and $\vec{q}_v$ lists the preimages of the nodes. Suppose we choose two marked points on C_v, $q_{v,i}$, and $q_{v,j}$. Then we will be working with Hecke transformations on C_v of the form $H_{\vec{e}}$, where only e_i and e_j equal 1, and we refer to these as Hecke transformations from $q_{v,i}$ to $q_{v,j}$.

Proposition 3.16. *Let $(C,\vec{p})$ be a stable curve whose irreducible components are rational curves, and Γ be its dual graph. The variety $X_{C,\vec{\lambda}}$ defined by a choice of weighting $\vec{\lambda}$ of* *marked points $\vec{p}$ admits the action of a group $H_T(C,\vec{p})$ which is an extension*

$$0 \to (\mathbb{C}^\times)^{E(\Gamma)}/(\mu_2)^{V(\Gamma)} \to H_T(C,\vec{p}) \to H_1(\Gamma,\mathbb{Z}/2) \to 0.$$

The induced action of $H_1(\Gamma,\mathbb{Z}/2)$ on the character lattice $\mathbb{Z}^{E(\Gamma)}$ of $(\mathbb{C}^\times)^{E(\Gamma)}$ is given by the map

$$H_1(\Gamma,\mathbb{Z}/2) \times \mathbb{Z}^{E(\Gamma)} \to \mathbb{Z}^{E(\Gamma)}$$

$$(\gamma,(m_e)_{e\in E(\Gamma)}) \mapsto H_\gamma((m_e)_{e\in E(\Gamma)})$$

where

$$(H_\gamma((m_e)_{e\in E(\Gamma)}))_e = \begin{cases} -m_e & \text{if } \gamma \text{ has coefficient 1 along } e \\ m_e & \text{if } \gamma \text{ has has coefficient 0 along } e. \end{cases}$$

Proof. Using the concept of a G-family of line bundles, we can rephrase the definition of $X_{C,\vec{\lambda}}$. In particular, consider the abelian group $\mathbb{Z}^{|E(\Gamma)|+1}$ consisting of vectors $(\vec{m},k)$ where $\vec{m}$ is an integer weighting of the full edges of Γ, and k will describe the level. Such a weighting determines a line bundle $\mathcal{L}(k\vec{\lambda}_v,\vec{m}_v,kL)$ on $\mathcal{M}_{C_v,\vec{p}_v,\vec{q}_v}$ giving the marked points $\vec{p}_v$ the weights from $\vec{\lambda}$ scaled by k, and giving the preimages of the nodes the $\vec{m}$-weights of the corresponding edges. Recall that when we write $X_{C,\vec{\lambda}}$ we imagine all the weights $\vec{\lambda}$ to be at a fixed level L.

Therefore we get a $\mathbb{Z}^{|E(\Gamma)|+1}$-family of line bundles, $\vec{\mathcal{L}}_C$ on $\prod_{v\in V(\Gamma)}\mathcal{M}_{C_v,\vec{p}_v,\vec{q}_v}$ by assigning to $(\vec{m}, k)$ the exterior tensor product

$$\mathcal{L}_C(\vec{m}, k) := \bigotimes_{v \in V(\Gamma)} \pi_v^* \mathcal{L}(k\vec{\lambda}_v, \vec{m}_v, kL).$$

In this language, $X_{C,\vec{\lambda}}$ is defined as the Proj of $\mathrm{Cox}(\vec{\mathcal{L}}_C)$ with respect to the grading by k.

Choose a cycle $\gamma = (e_1, \dots, e_l)$ in Γ, label the vertex at the end of e_i as v_i, and denote by $q_{i,1}$ and $q_{i,2}$ the marked points on C_{v_i} which correspond to the edges e_i and e_{i+1}. We define the "Hecke transform along the cycle γ" as the automorphism of $\prod_{v \in V(\Gamma)} \mathcal{M}_{C_v, \vec{p}_v, \vec{q}_v}$ which performs the Hecke transformation from $q_{i,1}$ to $q_{i,2}$ on the i component. If the cycle visits a vertex multiple times then we take the product of the Hecke transformations. This defines an action of the group $H_1(\Gamma, \mathbb{Z}/2)$ on $\prod_{v \in V(\Gamma)} \mathcal{M}_{C_v, \vec{p}_v, \vec{q}_v}$.

Further, this action preserves the collection of line bundles

$$\bigotimes_{v \in V(\Gamma)} \pi_v^* \mathcal{L}(k\vec{\lambda}_v, \vec{m}_v, kL)$$

because it does not change the level or the weights $\vec{\lambda}$, and performs the same Hecke transformation at both preimages of the same node. Because of this, we get an action of $H_1(\Gamma, \mathbb{Z}/2)$ on $\mathbb{Z}^{|E(\Gamma)|+1}$. Explicitly, by Lemma 3.5, a cycle γ sends $(\vec{m}, k)$ to the vector with e-component $kL - m_e$ whenever e is in γ, m_e when e is not in γ, and level component k. This means that $\vec{\mathcal{L}}_C$ is isomorphic, with respect to the action of γ on $\mathbb{Z}^{|E(\Gamma)|+1}$, to its pullback by the Hecke transformation around γ.

Now we apply Theorem 3.14 to the Cox ring of our $\mathbb{Z}^{|E(\Gamma)|+1}$-family of line bundles $\vec{\mathcal{L}}_C$. This gives the group G which is an extension of $H_1(\Gamma, \mathbb{Z}/2)$ by the torus $\mathrm{Hom}(\mathbb{Z}^{|E(\Gamma)|+1}, \mathbb{C}^\times)/(\mu_2)^{V(\Gamma)}$. This group $(\mu_2)^{V(\Gamma)}$ is the automorphism group of the identity map of $\prod_{v \in V(\Gamma)} \mathcal{M}_{C_v, \vec{p}_v, \vec{q}_v}$. An element in $(\epsilon_v)_{v \in V(\Gamma)} \in (\mu_2)^{V(\Gamma)}$ acts on an element of $\mathrm{Hom}(\mathbb{Z}^{|E(\Gamma)|+1}, \mathbb{C}^\times)$ by scaling the e component by $\epsilon_{v_1}\epsilon_{v_2}$ when e is the edge between two vertices v_1 and v_2. The M-lattice of this quotient torus is the sublattice of $\mathbb{Z}^{E(\Gamma)+1}$ consisting of vectors where the sum of the weights of all edges adjacent to any given vertex must be even. The previous paragraph tells us how $H_1(\Gamma, \mathbb{Z}/2)$ acts on this M-lattice. Note that the $\mathbb{C}^\times$-factor of the torus of G that corresponds to the level acts trivially on $X_{C,\vec{\lambda}}$ because $X_{C,\vec{\lambda}}$ is the Proj of the Cox ring of $\vec{\mathcal{L}}_C$ with respect to that one-parameter subgroup. This means that the G action factors through the quotient group $H_T(C, \vec{p})$ whose torus is $\mathrm{Hom}(\mathbb{Z}^{|E(\Gamma)|}, \mathbb{C}^\times)/(\mu_2)^{V(\Gamma)}$. We identify the M-lattice of this quotient torus as the sublattice of $\mathbb{Z}^{E(\Gamma)+1}$ where the level component k is 0 and the weightings of the edges satisfy the above parity condition. The Hecke action on this sublattice is the one described in the statement of the corollary, so this finishes the proof. $\square$

Chapter 4

K-semistability of moduli spaces of vector bundles

In this chapter we will prove the K-semistability of the moduli spaces $U_C(2,\xi)$ for a general curve C. We do this using the openness of K-semistability as proved in [BLX22] and proving the K-polystability of the toric degenerations X_Γ of $U_C(2,\xi)$ described in Chapter 2.

First we will show how graphs with half-edges furnish toric degenerations of the moduli spaces $U_C(2,\xi)$ when the line bundle ξ is non-trivial. More specifically, given a curve with many marked points $p_1,\dots,p_n$, a level L, and weightings $\vec{\lambda}$ of the marked points, if all the weights satisfy $\lambda_i = 0$ or $\lambda_i = L$ then $X_{(\Gamma,\vec{\lambda}/L)}$ is a degeneration of $U_C(2,\mathcal{O}(-\sum_{i=1}^n \frac{\lambda_i}{L}p_i))$. For simplicity, we will only explain this degeneration in the case of curves with one marked point, as that is all we need.

To understand this degeneration, consider a specific level $L \in \mathbb{N}$, and the sheaf of algebras $\mathcal{V}^{\dagger}(L, L)$ over $\overline{\mathcal{M}}_{g,1}$ as in Proposition 2.1. Over a fixed smooth pointed curve (C, p), the fiber of this sheaf of algebras is the section ring of the line bundle $\mathcal{L}(L, L)$ on the stack $\mathcal{M}_{(C,\vec{p})}$. By [BL94] Section 8, this is the homogeneous coordinate ring of the moduli space $U_C(2, \mathcal{O}(-p))$. Therefore the relative proj of this sheaf of algebras is a flat family over $\overline{\mathcal{M}}_{g,1}$ with fiber $U_C(2, \mathcal{O}_C(-p))$ over smooth (C, p) and $V_{(C,p)}(L, L)$ (the conformal blocks space) over nodal (C, p). Then as before, when (C, p) is a graph curve, $V_{(C,p)}(L, L)$ admits a $\mathbb{C}^{\times}$ equivariant degeneration to the toric variety $X_{C,p,1}$ (i.e. $X_{\Gamma,1}$ when we identify (C, p) with its dual graph Γ).

Remark 4.1. This looks strange, because Theorem 2.5 says that this algebra of conformal blocks of weight L on (C, p) is actually the homogeneous coordinate ring of the moduli space $M_{(C,p,1)}$ of parabolic bundles with weight equal to the level. However, this space is in fact isomorphic to the space $U_C(2, \mathcal{O}(-p))$. We claim that the explicit isomorphism is given by taking a parabolic bundle with trivial determinant and taking its Hecke transform at p to obtain a bundle with determinant $\mathcal{O}_C(-p)$ and simply forgetting its parabolic structure. We could prove this is an isomorphism if we extended the results of Chapter 3 to cover moduli spaces of quasiparabolic bundles of fixed non-trivial determinant. In particular we would consider the isomorphism from the stack of quasiparabolic bundles with trivial determinant to the stack of quasiparabolic bundles with determinant $\mathcal{O}_C(-p)$ given by the Hecke transform at p. The we would show that the pullback of the line bundle $\mathcal{L}(0, L)$ (defined appropriately) under this transformation is $\mathcal{L}(L, L)$, and taking the corresponding isomorphism of section rings would yield the desired isomorphism. While this is an interesting result, we do not give the full proof as it is not important for what follows.

We want to have a better understanding of the toric variety $X_{\Gamma,\vec{\lambda}}$ and its anti-canonical polytope in case the graph Γ has half edges with weights equal to L or 0. We start by analyzing what happens when there is a vertex v with a half-edge of weight $\lambda_i = 0$ i.e. $w_i = -L/2$ in the shifted coordinates w. In this case, for $m \in P_{(\Gamma,\vec{\lambda})}$, i.e. an integer weighting of the edges satisfying the conditions of Proposition 2.8, the components of m on the two edges touching v must be equal. Thus, if we remove v from the graph and replace those two edges by one edge we get a new graph defining the same polytope. This means we can ignore half-edges with weights $w_i = -L/2$.

Now we analyze the case where a vertex v connects to a half edge with weight $w = L/2$. Then the inequalities 2.8 imply that a lattice point $m \in P_{(\Gamma,\vec{\lambda})}$ must be of the form on the left of Figure 4.1. In this case, as before, we remove v and replace the two

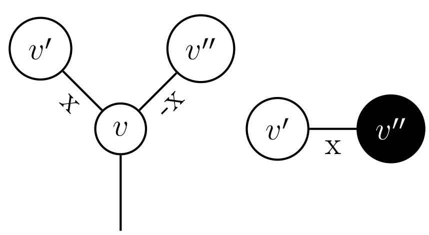

Figure 4.1: Weight $w_i = L/2$ and the corresponding colored graph

edges touching v with one edge. Furthermore, we arbitrarily choose one of the vertices touching v and color it black. To one such colored graph, we can associate a polytope such that the inequalities supported on the colored vertex have $-w_i$ in place of w_i as in Figure 4.2. Here we are identifying an inequality of the form $\langle n, m\rangle \geq -1$ with the vector $n \in N$ and are also choosing level 4. This polytope will be the same as the polytope of

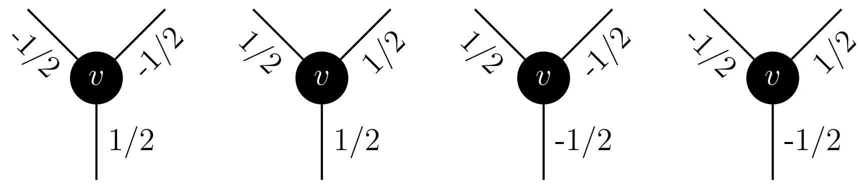

Figure 4.2: Inequalities at a colored vertex.

the original graph.

In conclusion, when studying degenerations of $U_C(2, \xi)$, especially when ξ has odd degree, we will work with a colored graph with no half-edges. This colored graph construction is from [BGM20].

Remark 4.2. In chapter 7 we will need to distinguish between edges that came from the original graph, and the new edges that were introduced in this construction to replace the two edges attached to the vertex with a half edge. We will call this set of edges $C(\Gamma) \subseteq E(\Gamma)$.

Now consider a connected graph curve of genus g and its associated dual graph Γ, which may have colored vertices. The main result of this chapter is the following.

Theorem 4.3. *X_Γ is K-polystable if Γ does not contain a bridge, i.e. an edge whose removal would disconnect Γ.*

Proof. We use the theorem from [Ber16] that a projective toric variety is K-polystable if and only if the barycenter of the polytope associated to the anti-canonical divisor is 0.

For X_Γ, the barycenter being 0 is a simple consequence of the action of

$$H_\Gamma := H_1(\Gamma, \mathbb{Z}/2)$$

on the M-lattice as described in Proposition 3.16. In particular this action comes from the action of $H_T(C_\Gamma)$ on X_Γ, so it preserves the anticanonical polytope of X_Γ. This implies the barycenter is fixed by the H_Γ-action. To finish the proof we will show that $0 \in M$ is the only fixed point for the H_Γ-action.

It is a basic fact in graph theory that Γ having no bridges is equivalent to every edge in Γ being contained in a cycle. See for instance [Cha77]. Therefore for $(w_e)_{e \in E(\Gamma)} \in M$ and any edge e we can take a cycle containing e and the corresponding automorphism changes w_e to $-w_e$. Consequently, if this vector were fixed by H_Γ, all w_e would have to be 0. □

An example of a family of bridge-less graphs are the ladder graphs as in Figure 4.3. Because there is a ladder graph for each genus, Theorem 4.3 shows that for each genus there is at least one toric variety which is K-polystable.

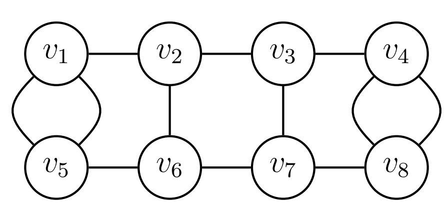

Figure 4.3: A ladder graph of genus 5

Example 4.4. *We give an example of a toric variety X_Γ that is not K-polystable. The graph is the dumbbell graph of Figure 4.4. If we denote by (w_1, w_2, w_3) the weights on the leftmost edge, central edge, and rightmost edge respectively, then the inequalities from Proposition 2.8 for $L = 4$ give*

$$-2w_1 - w_2 \geq -2$$

$$w_2 \geq -2$$

$$2w_1 - w_2 \geq -2$$

at the left vertex. This is because a weight of w_1 at the loop gives the weight w_1 to both of the marked points in the normalization of the curve that map to the corresponding node. The inequalities for the right vertex are the same but with w_3 instead of w_1.

The polytope described by these inequalities is the square pyramid with vertices $(0,2,0), (\pm 2, -2, \pm 2)$ whose barycenter is $(0,-1,0)$. In particular its barycenter is not 0 or even a lattice point.

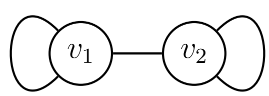

Figure 4.4: The dumbbell graph.

Example 4.5. *Now we give an example of a colored graph Γ which has a bridge but X_Γ is nevertheless K-polystable. For this, we take the dumbbell graph with a colored vertex of*

Figure 4.5. Its polytope for $L = 4$ has the same inequalities as Example 4.4 for the left vertex but the inequalities

$$-2w_3 + w_2 \geq -2$$

$$-w_3 \geq -2$$

$$2w_3 + w_2 \geq -2$$

for the right one.

This describes a tetrahedron with vertices $(\pm 2, -2, 0)$ and $(0, 2, \pm 2)$ whose barycenter is 0.

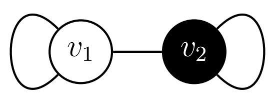

Figure 4.5: The dumbbell graph with extra half-edge.

Corollary 4.6. *The moduli space $U_C(2, \xi)$ of rank-two semistable vector bundles with fixed determinant ξ is K-semistable for general C.*

Proof. As previously noted, these moduli spaces are isomorphic to either $U_C(2, \mathcal{O}_C)$ or $U_C(2, \mathcal{O}_C(-p))$ depending on if $deg(\xi)$ is even or odd respectively. As C degenerates to a graph curve C', we have a flat family with generic fibers $U_C(2, \xi)$ and special fiber the conformal block space $V_{C'}$ (in the odd-degree case, $V_{(C',p)}(1,1)$). Now since the conformal block space admits a $\mathbb{C}^\times$-degeneration to X_Γ which is a K-polystable Fano, it is

K-semistable because K-semistability is open in families [BLX22]. Then again for the same reason, $U_C(2, \xi)$ is K-semistable for a general curve C. □

Remark 4.7. In the preceding argument, to apply the openness of K-semistability we technically needed to make sure the relevant families are $\mathbb{Q}$-Gorenstein. We will usually not check this hypothesis explicitly because it will always be satisfied for us. One way to see this is to note that for a trivalent graph Γ, the toric variety X_Γ is Gorenstein by [FM19]. Thus by the results of [WITO69], any flat family over a scheme S with X_Γ as its central fiber is a Gorenstein family in some neighborhood of $0 \in S$ as long as S is Gorenstein at 0.

Chapter 5

The Luna Slice Theorem and GIT Stability

By the Luna Slice Theorem for Algebraic Stacks [AHR20], we know that for each K-polystable Fano variety X, there is an affine scheme of finite type over $\mathbb{C}$, $\operatorname{Spec}(R)$, equipped with an action of $\operatorname{Aut}(X)$ which fixes a distinguished point $x \in \operatorname{Spec}(R)$, and an étale map

$$f_{\operatorname{Aut}(X)}: [\operatorname{Spec}(R)/\operatorname{Aut}(X)] \to \mathcal{M}^{Kss}$$

sending x to the class of X. Furthermore, as explained in [ABHLC19] and [AHLH18] Proposition 4.3, we can assume that the stabilizer of a point $p \in \operatorname{Spec}(R)$ is the automorphism group of the Fano variety $f(p)$ and that the preimage of the K-semistable, K-polystable, and K-stable loci in $\mathcal{M}^{Kss}$ are the GIT-semistable, GIT-polystable, and GIT-stable loci respectively in $\operatorname{Spec}(R)$. Thus, by studying GIT stability on $\operatorname{Spec}(R)$, we can study which of the irreducible components of $\mathcal{M}^{Kss}$ contain K-stable points.

For Γ a colored graph with no bridges and $X = X_\Gamma$ we will fix a choice of such a scheme $\mathrm{Spec}(R)$ for the rest of the paper and call it $\mathcal{A}_\Gamma = \mathrm{Spec}(R_\Gamma)$. The scheme $\mathcal{A}_\Gamma$ is an equivariant Artin approximation of the versal deformation space of X_Γ so we will refer to it simply as the Artin approximation space.

In order to work with $\mathcal{A}_\Gamma$, we now prove some lemmas concerning the GIT stability of affine varieties under actions of certain reductive groups. The main result of the chapter is Lemma 5.6, and the rest of the paper will be devoted to applying this Lemma to be able to find GIT-stable points in $\mathcal{A}_\Gamma$ that parameterize moduli spaces of vector bundles on smooth curves.

First we recall the formulation of the Hilbert-Mumford criterion in terms of weight polytopes.

Definition 5.1. Let G be a reductive group with linear representation V. For every vector $v \in V$ and maximal torus $T \subseteq G$, we will define a weight polytope $W_{v,T} \subseteq M(T)_\mathbb{R}$. Let

$$V \cong \bigoplus_{\alpha \in M(T)} V_\alpha$$

be a decomposition of V into irreducible representations with respect to T. With respect to this decomposition we have

$$v = \sum v_\alpha$$

and $W_{v,T} \subseteq M(T)_\mathbb{R}$ is defined as the convex hull of all α so that $v_\alpha \neq 0$.

Proposition 5.2. *([Mum65] Theorem 2.1) Let G be a reductive group and V a linear representation. A vector $v \in V$ is*

1. *Stable if $W_{v,T}$ contains the origin in its interior for every maximal torus T. Here the interior is the union of open subsets of $M(T)_{\mathbb{R}}$ contained in $W_{v,T}$.*

2. *Polystable if $W_{v,T}$ contains the origin in its relative interior for every T. Here the relative interior means that we take the interior in the induced topology on $W_{v,T}$.*

3. *Semistable if $W_{v,T}$ contains the origin.*

Remark 5.3. This formulation seems to be well-known, but for lack of a suitable reference, we indicate how to translate the formulation of the Hilbert-Mumford criterion in [Mum65] to the one we have here. First of all, the criterion is originally stated for projective varieties with an ample linearization. Our criterion for stability can then be considered as projective stability for $\mathbb{P}(V)$ with the natural linearization on $\mathcal{O}(1)$. This makes sense because our criterion does not change if we scale the vector v.

The other significant difference is that in [Mum65], one-parameter subgroups are considered individually. For us, a one-parameter subgroup in G can be considered as a vector $\lambda \in N(T)$ for some maximal torus $T \subseteq G$. Then the weights of V for the $\mathbb{C}^\times$-action induced by λ are given by the natural pairing $\langle \lambda, \alpha \rangle$ where α runs over the T-weights of V. For a non-zero vector $v \in V$, we consider the family of lines $\lambda(t)\mathbb{C}v$ for $t \in \mathbb{C}^*$. As $t \to 0$, this line becomes a weight space for λ of weight $r = \max_{\alpha \in W_{v,T}} \langle \lambda, \alpha \rangle$. The original Hilbert-Mumford criterion for semistability is then that $r \geq 0$ for all one-parameter subgroups λ

and $\mathbb{C}v \in \mathbb{P}(V)$. Considering all $\lambda \in N(T)$, this is the same as saying that $W_{v,T}$ contains the origin. The other criteria follow similarly.

This criterion becomes difficult to apply in the case where G has more than one maximal torus. Therefore for the rest of the chapter we will work under the following assumption:

Assumption 5.4. *G is a group of the form $T \rtimes H$, where T is a torus and H is a finite group such that the only one-parameter subgroup (equivalently, character) of T which is fixed by the conjugation action of H is the constant one, i.e. $0 \in N(T)$.*

In Proposition 6.2 we will show that the automorphism group of the toric variety X_Γ satisfies 5.4 under some mild conditions.

Lemma 5.5. *Let G satisfy assumption 5.4, and let V be a finite-dimensional representation of G. Then the general point of V is GIT-polystable. Further, if no 1-parameter subgroup of G acts trivially on V, then the general point of V is GIT-stable.*

Proof. Choose a basis of V diagonalizing the action of T. Consider the union of all coordinate hyperplanes with respect to this basis. If a point $v \in V$ is not contained in this union, then v has a non-zero component for each weight that appears in the T action on V. This means that the weight polytope W_v (we supress T from the notation as it is the only maximal torus) is the convex hull of all weights appearing in V. We call this generic weight polytope W.

Now let $V_\alpha \subseteq V$ be the weight space for a character $\alpha \in \mathrm{Hom}(T, \mathbb{C}^\times) = M$. We have that for $h \in H$, $h(V_\alpha) = V_{h*\alpha}$ where $h * \alpha(t) = \alpha(h^{-1}th)$ is the H-action on M. In particular, the H action on M preserves W and therefore fixes its barycenter. This implies the barycenter of W is the origin, and the barycenter is always in the relative interior. Thus we know that the generic $v \in V$ is polystable.

Now assume that no 1-parameter subgroup acts trivially. If the generic $v \in V$ is not stable, that means that W has empty interior, and because W contains the origin, it is therefore contained in a linear hyperplane. Furthermore, since W is a rational polytope, this hyperplane can be taken as the annihilator of an element $\lambda \in N_{\mathbb{Q}}$ and by scaling we assume $\lambda \in N$. But this means that λ acts trivially on V, because for $\alpha \in M \cap W$, the weight of λ acting on V_α is $\langle \lambda, \alpha \rangle = 0$. Thus if no 1-parameter subgroup acts trivially, W must be full-dimensional, and therefore the generic point in V is stable. □

Lemma 5.6. *Let G satisfy assumption 5.4, V a G-representation on which no 1-parameter subgroup acts trivially, and $Y \subseteq V$ an irreducible affine variety not contained in any hyperplane. Then Y contains GIT-stable points for the G-action.*

Proof. By the proof of the above lemma, V contains GIT-stable points, and the points that aren't stable are contained in the union of coordinate hyperplanes for a basis diagonalizing T. Because Y is irreducible, if Y is contained in the union of these hyperplanes, then it is completely contained in one of them, which is impossible by hypothesis. Thus, Y must contain stable points for the action on V. □

Chapter 6

The Automorphism Group of the Toric Degenerations

Our strategy to prove the K-stability of $U_C(2, \xi)$ is to use the Luna slice theorem. This reduces the problem to determining the GIT stability of points in $\mathcal{A}_\Gamma$ with respect to its action by $\mathrm{Aut}(X_\Gamma)$. In particular, we want to apply Lemma 5.6. The goal of this chapter is to show that $\mathrm{Aut}(X_\Gamma) \cong T \rtimes H$ where T is the torus acting on X_Γ and H is finite, as required by the lemma. To do this, we will apply the theory of Demazure roots. These are defined using the rays of the fan of a toric variety, so we start with a lemma describing the rays of the fan Σ_Γ corresponding to one of our toric degenerations X_Γ.

We start by recalling the lattices M and N associated to X_Γ. The lattice M is defined as the set of integer weightings of the edges such that for any vertex the sum of the three weights adjacent to it is even. Thus, the lattice N is described as the set of weightings of the edges such that the three weights around a vertex are either all integers or all half-integers.

Now let $\Sigma_\Gamma(1)$ denote the set of rays of Σ_Γ, and for $\tau \in \Sigma_\Gamma(1)$ denote by $v_\tau \in N$ the primitive lattice vector generating the ray. We will refer to the vectors v_τ as primitive generators.

Lemma 6.1. *Let Γ be a colored, connected, trivalent graph that has no half-edges, no bridges, and is not the theta graph of Figure 6.2. The primitive generators of Σ_Γ are split into groups of four, one group for each vertex of Γ. If the vertex is colored white then these are the vectors of Figure 6.1, and if the vertex is colored black they are the ones of Figure 4.2. These vectors are implied to have weight 0 along all edges that are not pictured.*

Proof. Firstly, note that we are assuming our graphs have no loops, because a trivalent graph with a loop automatically has a bridge.

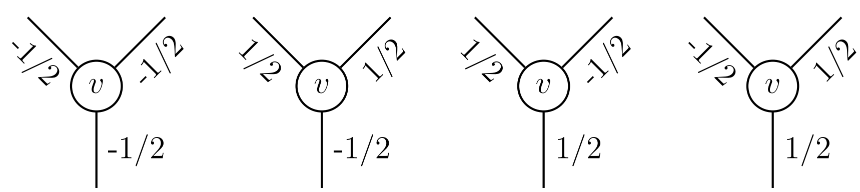

Figure 6.1: Possible primitive generators.

By construction, the polytope P_Γ described by Proposition 2.8 corresponds to an ample line bundle on X_Γ. This means that the fan over the faces of the dual polytope $\check{P}_\Gamma \subseteq N_\mathbb{R}$ is the fan Σ_Γ defining X_Γ. For $L = 4$, the polytope $\check{P}_\Gamma$ is the convex hull of the vectors in Figure 6.1 (or 4.2 at colored vertices). This is because for L=4, the inequalities defining P_Γ are of the form $\langle v, m\rangle \geq -1$ for $m \in M$ and v a vector from Figure 6.1 or 4.2. This means the primitive generators are the vectors from Figures 6.1 or 4.2 which are vertices of this dual polytope. We see they are all vertices because they all have norm $\sqrt{3}/2$ in the standard norm on $N_\mathbb{R}$. □

Now we use this description of Σ_Γ to prove the main proposition of this chapter.

Proposition 6.2. *Let Γ be a colored, connected, trivalent graph that has no half-edges, no bridges, and is not the theta graph of Figure 6.2. Then the toric variety X_Γ satisfies* $\mathrm{Aut}(X_\Gamma) = T \rtimes H$ *where H is the group of lattice automorphisms of N leaving the fan invariant.*

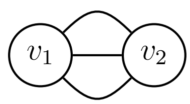

Figure 6.2: The Theta Graph.

Proof. It is known ([BG99] Theorem 5.4) that there is a short exact sequence

$$1 \to \mathrm{Aut}^0(X_\Gamma) \to \mathrm{Aut}(X_\Gamma) \to \mathrm{Aut}(\Sigma_\Gamma)/\left(\mathrm{Aut}^0(X_\Gamma) \cap \mathrm{Aut}(\Sigma_\Gamma)\right) \to 1$$

where $\mathrm{Aut}^0(X_\Gamma)$ denotes the connected component of the identity. This short exact sequence follows from [BG99] Theorem 5.4 by the second isomorphism theorem, because that theorem states that there is a subgroup of $\mathrm{Aut}(X_\Gamma)$ isomorphic to $\mathrm{Aut}(\Sigma_\Gamma)$ such that $\mathrm{Aut}(\Sigma_\Gamma)$ and $\mathrm{Aut}^0(X_\Gamma)$ together generate the entire automorphism group.

We note that if $\mathrm{Aut}^0(X_\Gamma) = T$ then this short exact sequence becomes the desired semidirect product decomposition

$$\mathrm{Aut}(X_\Gamma) \cong T \rtimes \mathrm{Aut}(\Sigma_\Gamma).$$

This is because T acts trivially on the fan, so the intersection $\big(\mathrm{Aut}^0(X_\Gamma) \cap \mathrm{Aut}(\Sigma_\Gamma)\big)$ is trivial, and we already have an embedding of $\mathrm{Aut}(\Sigma_\Gamma)$ as a subgroup of $\mathrm{Aut}(X_\Gamma)$ so the sequence splits.

As described in [Nil06] and [Cox92], we can show $\mathrm{Aut}^0(X_\Gamma) = T$ by checking that the set of Demazure roots is empty. This is the set

$$\mathcal{R} = \{m \in M | \text{For one } \tau \in \Sigma_\Gamma(1), \langle v_\tau, m\rangle = -1, \text{and for } \sigma \in \Sigma_\Gamma(1) \setminus \{\tau\}, \langle v_\sigma, m\rangle \geq 0\}.$$

The semisimple roots are defined as

$$\mathcal{R}^s := \mathcal{R} \cap -\mathcal{R}$$

and when the toric variety has reductive automorphism group all the Demazure roots are semisimple, as explained in [Nil06]. By [ABHLC19] this is true whenever X_Γ is K-polystable, for instance when Γ has no bridges. Note that if a root m is semisimple then there are primitive generators v_τ and $v_{\tau'}$ such that $\langle v_\tau, m\rangle = -1$, $\langle v_{\tau'}, -m\rangle = -1$ and $\langle v_{\tau''}, m\rangle = 0$ for any other $\tau'' \in \Sigma_\Gamma(1)$. This is a very restrictive condition that we will explicitly prove cannot happen for any graphs satisfying our hypotheses.

Now we prove that there are no Demazure roots. Observe that if an element of M has a non-zero component on an edge e then it has a non-zero pairing with at least two primitive generators based at a vertex v incident to e. This is because the four primitive generators in either Figure 6.1 or 4.2 span a 3-d subspace of $N_{\mathbb{R}}$ such that any three form a basis. Thus if an element of m pairs to 0 with three of them it must have components equal to 0 on all edges touching v.

Now suppose for contradiction that there exists a semisimple root m. By the assumption that Γ is not the theta graph, there are no pairs of vertices that are incident to the exact same set of edges. Thus, a primitive generator based at one vertex cannot equal a primitive generator based at another vertex. This means by the above paragraph, if m has a non-zero component on an edge e then it will pair non-trivially with at least four primitive generators, two for each vertex touching e. This proves there can be no Demazure roots.

$\square$

Chapter 7

Deformations of the Toric Varieties and Their Automorphisms

To use the Luna slice theorem we need to better understand the Artin approximation $\mathcal{A}_\Gamma$ along with the action of $\mathrm{Aut}(X_\Gamma)$. In particular we need to understand certain deformations of X_Γ, and how automorphisms of X_Γ extend to automorphisms of the deformations. Specifically, in Chapter 8 we will equivariantly embed $\mathcal{A}_\Gamma$ into a linear representation, V, of $\mathrm{Aut}(X_\Gamma)$, but we will not be able to apply Lemma 5.6 directly to this representation because there isn't an appropriate irreducible subvariety to use. Instead, we will use the deformations studied in this chapter to span a subspace, V', of this linear representation. The results of this chapter will then be used to show that this subspace is stabilized by the group $H_T(C_\Gamma)$ for the graph curve C_Γ, and that no subtorus of this group acts trivially on this subrepresentation. We will then be able to apply Lemma 5.6 to the group $H_T(C_\Gamma)$, representation V', and an irreducible subvariety that will be constructed in Chapter 8.

To construct our deformations we first note that for each non-loop edge of Γ, smoothing the corresponding node gives a 1-parameter family of curves $\{C_t\}_{t\in\mathbb{P}^1\smallsetminus\{0,1,\infty\}}$ of genus g with rational irreducible components. The dual graphs of these curves are denoted by Γ_e and look like Figure 7.1 near e.

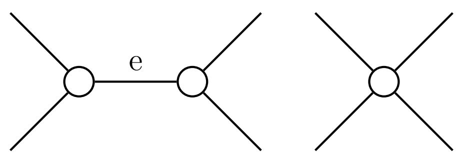

Figure 7.1: The graph Γ near e (left) and the smoothing, Γ_e (right).

For the curves C_t we can then consider the varieties X_{C_t} constructed in (2.2). When considering odd degree vector bundles then we will consider a family of pointed curves (C_t, p_t) and the corresponding varieties $X_{(C_t,1)}$ where the weight on the marked point p_t is 1. For now we will drop p_t and its weight from the notation except where it makes a difference. Since all irreducible components of C_t are rational, by Proposition 3.16 the group $H_T(C_t)$ acts on X_{C_t} and the torus subgroup of $H_T(C_t)$ is naturally identified with the subtorus T_e of $T \subseteq \mathrm{Aut}(X_\Gamma)$ induced by the inclusion

$$N(T_e)_\mathbb{Q} = \mathbb{Q}^{E(\Gamma_e)} \subseteq \mathbb{Q}^{E(\Gamma)} = N(T)_\mathbb{Q}.$$

Thus as the curve varies over the 1-parameter family of curves C_t with dual graph Γ_e, the varieties X_{C_t} form a family of complexity-1 T-varieties. Further, because we have not smoothed a loop of Γ, we have

$$H_\Gamma = H_1(\Gamma, \mathbb{Z}/2) \cong H_1(\Gamma_e, \mathbb{Z}/2).$$

In other words, the group $H_T(C_t)$ of Proposition 3.16 can be naturally embedded as the subgroup

$$T_e \rtimes H_\Gamma \subseteq T \rtimes H_\Gamma.$$

The main goal of this chapter is to show that the Luna slice $\mathcal{A}_\Gamma$ has points corresponding to X_{C_t} and that these points are stabilized by $T_e \rtimes H_\Gamma$ but not all of $T \rtimes H_\Gamma$.

An important detail when studying the family X_{C_t} is that the theory of conformal blocks does not naturally give the toric variety X_Γ as the limit as t tends to zero. Rather, it is more natural to place a certain K-semistable variety $X_{\Gamma,e}$ as the central fiber. Then X_Γ will be the K-polystable degeneration of $X_{\Gamma,e}$. We will need to study these varieties $X_{\Gamma,e}$ in order to make sure that the varieties X_{C_t} appear in the Artin approximation space and are not stabilized by all of T.

When constructing the toric variety X_Γ we needed to choose a coweighting $\vec{\theta}$ on Γ. This is a choice of rational number in $[0,1]$ for each edge of Γ. Such a choice then gave a valuation, also denoted $\vec{\theta}$, on the conformal blocks space V_Γ. Whenever all components of $\vec{\theta}$ were positive, the associated graded ring of the algebra of conformal blocks was the homogeneous coordinate ring of X_Γ. Now in contrast, we will be interested in coweightings whose components might vanish. In particular, we make the following definition.

Definition 7.1. Let Γ be a trivalent graph. For $e \in E(\Gamma)$, we consider the coweighting $\vec{\theta}_e$ of $E(\Gamma)$ which is 0 on e and 1 along all other edges. The variety we get as the Proj of the associated graded ring will be called $X_{\Gamma,e}$.

Remark 7.2. The coweighting θ_e on $E(\Gamma)$ gives the same valuation on $\mathcal{V}_C^{\dagger}$ as the coweighting on $E(\Gamma) \setminus \{e\}$ assigning coweight 1 to all edges.

We will now show that the varieties X_{C_t} naturally degenerate to $X_{\Gamma,e}$ as t tends to 0, and then we will compute the stabilizers of points in $\mathcal{A}_\Gamma$ that correspond to $X_{\Gamma,e}$. Understanding these stabilizers will then give us enough control over the GIT-stable locus in $\mathcal{A}_\Gamma$ to finish the proof of the main theorem.

Proposition 7.3. *The varieties X_{C_t} fit into a flat family over $\mathbb{A}^1$ whose fiber over 0 is $X_{\Gamma,e}$.*

Proof. The smoothing of the node of the graph curve C corresponding to the edge e gives a flat family

$$\mathcal{C}_e \to \mathbb{P}^1,$$

a so-called F-curve in the moduli space of stable curves, where

1. The fiber over 0 is the graph curve C.

2. The fibers over 1 and ∞ are the two other graph curves whose dual graph is Γ_e after smoothing an appropriate edge.

3. The fibers over any other point $t \in \mathbb{P}^1$ is C_t.

The family $\mathcal{C}_e \to \mathbb{P}^1$ has $|E(\Gamma)| - 1$ sections corresponding to all the nodes that aren't smoothed. Taking the normalization along these sections, we get a flat family of curves

$$\tilde{\mathcal{C}}_e \to \mathbb{P}^1$$

whose fiber over any point t other than $0, 1$, or ∞ will be the normalization of C_t and the fiber over 0 will be the partial normalization $C_e \to C$ which resolves all nodes except the one labeled by $e \in E(\Gamma)$.

By the theory of conformal blocks, the conformal blocks spaces associated to the fibers of the family $\mathcal{C}_e$ themselves fit into a flat family

$$V_{\mathcal{C}_e} \to \mathbb{P}^1.$$

Now consider a level L, a weighting $(a_f)_{f \in E(\Gamma_e)}$ of the edges of Γ_e, the corresponding map $\hat{\rho}_{\vec{a},L}$ (2.1) for the normalized family $\tilde{\mathcal{C}}_e$, and the image subsheaf $(\mathcal{V}^{\dagger}_{\mathcal{C}_e})_{\vec{a},L} \subseteq \mathcal{V}^{\dagger}_{\mathcal{C}_e}$. By Proposition 2.1 this is a vector bundle inside the flat sheaf of algebras $\mathcal{V}^{\dagger}_{\mathcal{C}_e}$ over $\mathbb{P}^1$ whose fiber over 0 is $\mathcal{V}^{\dagger}_{C}$ and whose fiber over $t \in \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$ is $\mathcal{V}^{\dagger}_{C_t}$. The vector bundle $(\mathcal{V}^{\dagger}_{\mathcal{C}_e})_{\vec{a},L}$ over $\mathbb{P}^1$ has fiber over 0 equal to $(\mathcal{V}^{\dagger}_{C})_{\vec{a},L}$ where the weighting $\vec{a}$ on $E(\Gamma) \smallsetminus \{e\}$ assigns the weight a_f to all edges $f \in E(\Gamma) \smallsetminus \{e\}$.

Now we choose the coweighting $\vec{\theta}$ on Γ_e with all entries equal to 1. The value of $\vec{\theta}$

on $(\mathcal{V}^\dagger_{C_t})_{\vec{a},L}$ is the same as the value of $\vec{\theta}_e$ on $(\mathcal{V}^\dagger_C)_{\vec{a},L}$. Therefore, the sheaf

$$\bigoplus_{L=0}^{\infty}\bigoplus_{j=0}^{\infty}\bigoplus_{\vec{\theta}(\vec{a})=j}(\mathcal{V}^\dagger_{\mathcal{C}_e})_{\vec{a},L}$$

is a flat sheaf of $\mathbb{Z}^2$-graded algebras whose fiber over $t \neq 0, 1, \infty$ is $gr_{\vec{\theta}}(\mathcal{V}^\dagger_{C_t})$ and whose fiber over 0 is $gr_{\vec{\theta}_e}(\mathcal{V}^\dagger_C)$. Taking the Proj relative to $\mathbb{P}^1$ with respect to the grading by L, we prove the proposition.

□

Now we move on to study $X_{\Gamma,e}$. We wish to work in a way that includes all edges at once. For this, we enumerate the edges of Γ by numbers between 1 and $|E(\Gamma)|$, and consider the $\mathbb{Z}^{|E(\Gamma)|+1}$-graded $\mathbb{C}[t_1, \dots, t_{|E(\Gamma)|}]$-algebra

$$\bigoplus_{L=0}^{\infty}\bigoplus_{\vec{a}\in\mathbb{Z}^{|E(\Gamma)|}}\left(\bigoplus_{\vec{b}\leq\vec{a}}(\mathcal{V}^\dagger_C)_{\vec{b},L}\right)t_1^{a_1}\dots t_{|E(\Gamma)|}^{a_{|E(\Gamma)|}} \tag{7.1}$$

where the partial order $\leq$ on $\mathbb{Z}^{E(\Gamma)}$ is given by

$$(a_e)_{e\in E(\Gamma)} \leq (b_e)_{e\in E(\Gamma)} \Leftrightarrow a_e \leq b_e \text{ for all } e \in E(\Gamma)$$

Proposition 7.4. *The Proj of the $\mathbb{C}[t_1, \dots, t_{|E(\Gamma)|}]$-algebra 7.1 with respect to the grading by L, is a $(\mathbb{C}^\times)^{|E(\Gamma)|}$-equivariant flat family $\mathcal{X}_\Gamma \to \mathbb{A}^{|E(\Gamma)|}$ with the following properties.*

1. *The fiber over any point which is not in a coordinate hyperplane is V_Γ.*

2. *For any edge $e \in E(\Gamma)$, the fiber over a non-zero point on the e-axis of $\mathbb{A}^{|E(\Gamma)|}$ is X_{Γ_e}.*

3. *The fiber over the origin is the toric variety X_Γ.*

4. *For any $\vec{\theta} \in ([0,1] \cap \mathbb{Q})^{|E(\Gamma)|}$ and $m \in \mathbb{N}$ such that $m\vec{\theta} \in \mathbb{Z}^{|E(\Gamma)|}$, the base change under the inclusion*

$$\mathbb{A}^1 \to \mathbb{A}^{|E(\Gamma)|}$$
$$t \mapsto (t^{m\theta_1}, \dots, t^{m\theta_{|E(\Gamma)|}})$$

is the $\mathbb{C}^\times$-degeneration of V_Γ associated to the valuation $\vec{\theta}$ on $\mathcal{V}_\Gamma^\dagger$.

5. *The induced $(\mathbb{C}^\times)^{|E(\Gamma)|}$-action on X_Γ is the action induced by the inclusion $M \hookrightarrow \mathbb{Z}^{|E(\Gamma)|}$ where M is the character lattice of the torus $T \subseteq X_\Gamma$.*

Proof. Flatness follows from Theorem 9.9 of chapter 3 of Hartshorne [Har13]. This is because for each $L \in \mathbb{N}$, the L-th graded piece

$$\bigoplus_{\vec{a} \in \mathbb{Z}^{|E(\Gamma)|}} \left(\bigoplus_{\vec{b} \leq \vec{a}} (\mathcal{V}_C^\dagger)_{\vec{b},L} \right) t_1^{a_1} \dots t_{|E(\Gamma)|}^{a_{|E(\Gamma)|}}$$

is a free $\mathbb{C}[t_1, \dots, t_{|E(\Gamma)|}]$-module with basis

$$\{ s_{\vec{a}} t^{\vec{a}} | \vec{a} \in \mathbb{Z}^{E(\Gamma)}, s_{\vec{a}} \in \mathcal{V}_{\vec{a},L}, \mathcal{V}_{\vec{a},L} \neq \{0\} \}.$$

The first two points follow from the fourth point by considering the valuation $\vec{\theta}_e$.

The third point follows from the fourth by considering the valuation $\vec{\theta} = (1, \ldots, 1)$.

The fourth point is true because the map

$$\mathbb{A}^1 \to \mathbb{A}^{|E(\Gamma)|}$$

on the level of rings sends the monomial $t^{\vec{a}}$ to $t^{\vec{\theta}\cdot\vec{a}}$. Therefore the homogeneous coordinate ring of the base change is

$$\bigoplus_{L=0}^{\infty}\bigoplus_{n\in\mathbb{Z}}\sum_{\vec{\theta}\cdot\vec{a}=n}\left(\bigoplus_{\vec{b}\leq\vec{a}}(\mathcal{V}_C^{\dagger})_{\vec{b},L}\right)t^n = \bigoplus_{L=0}^{\infty}\bigoplus_{n\in\mathbb{Z}}\left(\bigoplus_{\vec{\theta}\cdot\vec{b}\leq\vec{\theta}\cdot n}(\mathcal{V}_C^{\dagger})_{\vec{b},L}\right)t^n$$

which is the Rees algebra of the valuation $\vec{\theta}_e$ on V_Γ.

The fifth point follows from standard toric geometry. □

Proposition 7.5. *The family $\mathcal{X}_\Gamma$ induces a morphism*

$$\alpha_1 : \mathbb{A}^{|E(\Gamma)|} \to \mathcal{A}_\Gamma$$

which is equivariant with respect to the morphism $(\mathbb{C}^\times)^{|E(\Gamma)|} \to T$ and fits into a commutative diagram

$$\begin{array}{ccc} & & \mathcal{A}_\Gamma \\ & \overset{\alpha_1}{\nearrow} & \downarrow f \\ \mathbb{A}^{|E(\Gamma)|} & \xrightarrow{\alpha_0} & \mathcal{M}^{Kss} \end{array}$$

where α_0 is the classifying morphism of the family $\mathcal{X}_\Gamma$.

Proof. Because $\mathcal{X}_\Gamma$ is a $(\mathbb{C}^\times)^{|E(\Gamma)|}$-equivariant family of K-semistable Fano varieties, we get a classifying morphism

$$\overline{\alpha_0} : [\mathbb{A}^{|E(\Gamma)|}/(\mathbb{C}^\times)^{|E(\Gamma)|}] \to \mathcal{M}^{Kss}.$$

We claim that the theory of coherent completeness developed in [AHR20] implies there is a lifting

$$\overline{\alpha_1} : [\mathbb{A}^{|E(\Gamma)|}/(\mathbb{C}^\times)^{|E(\Gamma)|}] \to [\mathcal{A}_\Gamma/\operatorname{Aut}(X_\Gamma)].$$

The proof is that by Proposition 5.18 of [AHR20], $[\mathbb{A}^{|E(\Gamma)|}/T]$ is coherently complete along the image of $0 \in \mathbb{A}^{|E(\Gamma)|}$. We also know from [AHR20] that $[\mathcal{A}_\Gamma/\operatorname{Aut}(X_\Gamma)] \to \mathcal{M}^{Kss}$, induces an isomorphism of the coherent completions at X_Γ. Putting this together, we see that the morphism $\overline{\alpha_0}$ factors through the coherent completion of $\mathcal{M}^{Kss}$ at X_Γ, and therefore lifts to the morphism $\overline{\alpha_1}$.

Now, Theorem 1.4.8 of the paper [HL14] shows that $\overline{\alpha_1}$ lifts to the equivariant morphism α_1 that we wished to construct. In particular, our morphism $\overline{\alpha_1}$ is a $\mathbb{C}$-point of the stack $\operatorname{Filt}^n([\mathcal{A}_\Gamma/\operatorname{Aut}(X_\Gamma)])$ and the desired lift α_1 is any point of the space $\mathcal{A}_\Gamma^+ \subseteq \operatorname{Map}(\mathbb{A}^n, \mathcal{A}_\Gamma)$ which maps to $\overline{\alpha_1}$. The space $\mathcal{A}_\Gamma^+$ is defined in Proposition 1.4.1 of [HL14] as the space of equivariant maps $\mathbb{A}^n \to \mathcal{A}_\Gamma$, and Theorem 1.4.8 writes $\operatorname{Filt}^n([\mathcal{A}_\Gamma/\operatorname{Aut}(X_\Gamma)])$ as an explicit group quotient of that space.

□

Proposition 7.6. *Let* Γ *be a (possibly colored) trivalent graph which is not the theta* *graph and has no bridges. If Γ has colored vertices consider the set of edges inside Let $L \subseteq \mathbb{A}^{|E(\Gamma)|}$ denote the union of the axes. Then α_1 is injective on L, except it may contract axes to a point if they correspond to edges in the special set of edges $C(\Gamma)$ (Remark 4.2) that were introduced in the construction of the colored graph.*

Proof. Consider the axis $\mathbb{A}^1_e \subseteq \mathbb{A}^{|E(\Gamma)|}$ corresponding to the edge $e \in E(\Gamma)$. This axis corresponds to a one-parameter subgroup λ_e of T, given by the vector in $N(T)$ which is 1 on e and 0 elsewhere. By the Rees construction, this further corresponds to a valuation, also called λ_e, on the homogeneous coordinate ring of $X_{\Gamma,e}$, as well as a filtration, $\mathcal{F}_e$. We claim that this axis is contracted to a point if and only if $\mathcal{F}_e$ is induced by a grading on the coordinate ring of $X_{\Gamma,e}$.

To prove this, note that α_1 is constant on $\mathbb{A}^1_e$ if and only if the induced map

$$\psi_1 : [\mathbb{A}^1_e/\lambda_e] \to [\mathcal{A}_\Gamma/\operatorname{Aut}(X_\Gamma)]$$

being pulled back from the map

$$\psi_2 : [pt/\lambda_e] \to [\mathcal{A}_\Gamma/\operatorname{Aut}(X_\Gamma)]$$

(coming from the λ_e action on X_Γ) along the constant map

$$[\mathbb{A}^1_e/\lambda_e] \to [pt/\lambda_e].$$

By the Rees construction, the data of the family

$$[\mathcal{X}_\Gamma|_{\mathbb{A}^1_e}/\lambda_e] \to [\mathbb{A}^1_e/\lambda_e]$$

associated to the map ψ_1 is equivalent to the data of the filtration $\mathcal{F}_e$ on the homogeneous coordinate ring of X_{Γ_e}.

Similarly the map

$$[X_\Gamma/\lambda_e] \to [pt/\lambda_e]$$

associated to ψ_2 corresponds to a grading on the homogeneous coordinate ring of X_Γ. If ψ_1 is pulled back from ψ_2 then that means $X_{\Gamma_e} \cong X_\Gamma$ and the filtration $\mathcal{F}_e$ comes from the grading induced by the λ_e action on X_Γ.

Lemma 7.16 will show that in fact $\mathcal{F}_e$ does not come from a grading unless e belongs to the special set $C(\Gamma)$.

□

Before finishing the proof of Proposition 7.6 by studying the filtration $\mathcal{F}_e$ we prove the main results that we will need about the varieties X_{C_t}.

Lemma 7.7. *Suppose Γ is the dual graph of a graph curve with no bridges which is not the theta graph, and let $e \in \Gamma$ be an edge not in the special set of edges $C(\Gamma)$. Suppose $Y_e \subseteq \mathcal{A}_\Gamma$ is an irreducible component of the locus parameterizing varieties of the form X_{C_t}, and the closure of Y_e contains the image of the e-axis under the map α_1 of Proposition 7.5. The subgroup $T_e \subseteq \mathrm{Aut}(X_\Gamma)$ stabilizes all points of Y_e but the entire torus T does not* *stabilize the general point.*

Proof. By Proposition 7.6 we know that T_e stabilizes the points in the image of α_1 parameterizing $X_{\Gamma,e}$, but no element $\alpha \in T \smallsetminus T_e$ stabilizes them. We also know that a torus of dimension $|E(\Gamma)| - 1$ stabilizes any point parameterizing X_{C_t} for any $t \in \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. Because the dimension of the stabilizer is upper semicontinuous, this implies that only a strict subtorus of T stabilizes the general point of Y_e. Because there are only countably many subtori of T, they cannot vary in an algebraic family, and therefore T_e is exactly the maximal torus of the stabilizer of the general point of Y_e. Note that the last statement is only true because of the irreducibility of Y_e. □

Remark 7.8. There actually is a non-empty variety Y_e as in the statement. This is because the toric variety X_Γ is K-polystable, K-semistability is open in families and the map $\mathcal{A}_\Gamma \to \mathcal{M}^{Kss}$ is open (it is smooth). More specifically, the openness of K-semistability tells us that any neighborhood of X_Γ in $\mathcal{M}^{Kss}$ contains a point parameterizing $X_{\Gamma,e}$ and any neighborhood of that point contains points parameterizing varieties of the form $X_{(C_t,p_t),1}$.

Proposition 7.9. *Consider* Γ *and* $Y_e \subseteq \mathcal{A}_\Gamma$ *as in the statement of Lemma 7.7, except* e *can be in the set* $C(\Gamma)$. *Each point of* Y_e *is stabilized by some conjugate of the Hecke-transformation subgroup* $H_\Gamma \subseteq \mathrm{Aut}(X_\Gamma)$ *under* T.

Proof. The subgroup H_Γ is determined up to conjugacy by T by its action on the fan Σ_Γ. Because of Proposition 3.16, we know the group $T_e \rtimes H_\Gamma$ acts on X_{C_t} and that H_Γ acts on $N(T_E) \subseteq N(T)$ compatibly with its action on $N(T)$. Therefore we will show that any automorphism γ of Σ_Γ that on $N(T_e)$ acts as a Hecke transform around a loop in Γ_e is equal to the Hecke transform around the corresponding loop in Γ. By multiplying by the inverse of the Hecke transform around the corresponding loop in Γ we can assume that γ is trivial on $N(T_e)$ and we will show that this implies it is trivial on $N(T)$.

Because γ is trivial on $N(T_e)$ we know its action on all rays of Σ except for the eight distinct rays attached to the two vertices of e. We denote the set of primitive generators of these eight rays by $\Sigma(1)_e$. For $v_1 \in \Sigma(1)_e$, consider the set

$$S_{v_1} := \{v_1 + v_2 \mid v_2 \in \Sigma(1)_e, v_1 + v_2 \in N(T_e)\} \subseteq N(T_e).$$

Note by inspecting Figure 6.1 that the map $\Sigma(1)_e \to \mathcal{P}(N(T_e))$ sending v to S_v is equivariant for the action of the stabilizer of $N(T_e)$ inside $\mathrm{Aut}(\Sigma_\Gamma)$ and injective when the graph satisfies our hypotheses. This means that γ acts trivially on $\Sigma(1)_e$ as desired. □

7.1 The Filtration $\mathcal{F}_e$ and the Algebra of Conformal Blocks

The rest of this section is dedicated to finishing the proof of Proposition 7.6 by giving a more explicit description of the ring $gr_{\vec{\theta}_e}\mathcal{V}_\Gamma^\dagger$ and filtration $\mathcal{F}_e$.

We start by applying Proposition 2.10 to the coweighting on $E(\Gamma) \setminus \{e\}$ assigning 1 to all edges. We can interpret this as saying that $gr_{\vec{\theta}_e}\mathcal{V}_\Gamma^\dagger$ is a subalgebra of a tensor product of two rings as follows.

1. The first ring is the homogeneous coordinate ring of the toric variety $X_{\tilde{\Gamma}_e}$ associated to the graph with half-edges, $\tilde{\Gamma}_e$, which is Γ without with the edge e and the two vertices that it connects to. We keep the edges that used to connect to those vertices as half edges. We will write this ring as $\mathbb{C}[C_{\tilde{\Gamma}_e}]$ because it is the semigroup algebra corresponding to the cone $C_{\tilde{\Gamma}_e}$ over the polytope $P_{\tilde{\Gamma}_e}$. In other words $C_{\tilde{\Gamma}_e}$ is the cone inside $\mathcal{M}_{\mathbb{R}} \times \mathbb{R}$ defined by the inequalities 2.8 where L is thought of as a variable, not a fixed value.

2. The second ring is the algebra of conformal blocks associated to the subgraph, Γ', of Γ pictured in the left side of Figure 7.1. Here we interpret that subgraph as having only two vertices and the four exterior edges are half edges. If any pair of those exterior edges connected to each other inside Γ we split that edge into half edges, so that Γ' always has four half edges.

3. Consider a lattice point $m \in M(\tilde{\Gamma}_e) \times \mathbb{Z}$ contained inside the cone over the polytope $P_{\tilde{\Gamma}_e}$. In other words inside the cone defined by the inequalities 2.8 where L is thought of as a variable, not a fixed value. Then the homogenous coordinate ring of $X_{\tilde{\Gamma}_e}$ is a direct sum of one-dimensional spaces $\mathbb{C}m$ as m varies across that cone. Futhermore, the ring $gr_{\vec{\theta}_e}\mathcal{V}_{\Gamma}^{\dagger}$ is the direct sum of spaces $\mathbb{C}m \otimes \mathcal{V}_{\Gamma'}^{\dagger}(\vec{a}(m), L(m))$ where $\vec{a}(m)$ is the weighting of the four half edges of Γ' by the weights that m assigns to the corresponding half edges of $\tilde{\Gamma}_e$ and $L(m)$ is the level of m.

Now, by considering the base change of the Rees family $\mathcal{X}_\Gamma$ to the e-axis, we see that for $n \in \mathbb{N}$ the filtration $\mathcal{F}_e$ has n-th filtered piece

$$\mathcal{F}_e^{\leq n}\left(gr_{\vec{\theta}_e}\mathcal{V}_\Gamma^\dagger\right) = \bigoplus_m \bigoplus_{\substack{a_e \in \mathbb{N} \\ a \leq n}} \left(\mathbb{C}m \otimes \mathcal{V}_{\Gamma'}^\dagger\left(\vec{a}(m), L(m)\right)_{a_e}\right)$$

where a_e denotes the weight of the edge e. Therefore we will refer to $\mathcal{F}_e$ as the filtration by the weight along e. We will also denote by $\mathcal{F}_e$ the filtration by the weight along e in the algebra $\mathcal{V}_{\Gamma'}^\dagger$. Note that this is the filtration associated to the coweighting of Γ' that assigns the coweight 1 to e.

Therefore to finish proving that $\mathcal{F}_e$ does not come from a grading, we must give a more explicit description of the ring $\mathcal{V}_{\Gamma'}^\dagger$. We do this by recalling the presentation of this ring given in [Man15].

Definition 7.10. The ribbon graph R_Γ associated to a graph Γ is a 2-d C^∞ manifold with boundary which deformation retracts to a subspace homeomorphic to (the CW complex associated to) Γ. To the ribbon graph we associate the following data.

1. For each half edge h of Γ, a segment $B_h \subseteq \partial R_\Gamma$. This is the portion of the boundary that retracts to h, which is considered as a subset of R_Γ.

2. For each vertex $v \in V(\Gamma)$, a region $R_v \subseteq R_\Gamma$. This is the region inside R_Γ consisting of points which are closer to $v \in \Gamma \subseteq R_\Gamma$ than they are to any other vertex (with respect to some fixed metric on R_Γ).

3. For any edge $e \in E(\Gamma)$ connecting vertices $v_1, v_2 \in V(\Gamma)$ a curve segment denoted L_e. This is the boundary between the two regions R_{v_1} and R_{v_2}.

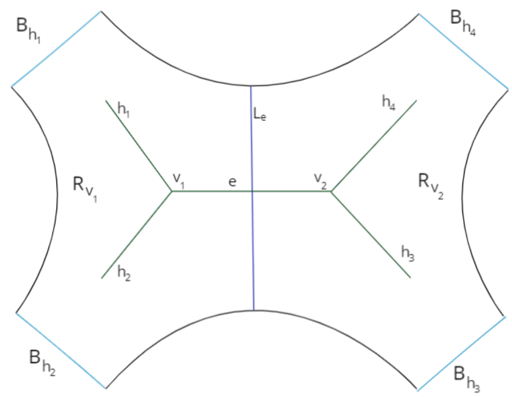

Figure 7.2: The ribbon graph of Γ' with its embedded copy of Γ'.

Definition 7.11. Consider a possibly disconnected, oriented path $l \subseteq R_\Gamma$ with the following properties.

1. The interior points of l are contained in the interior of R_Γ.
2. The boundary points of l are each contained in some boundary segment B_h of R_Γ.
3. The path l is transverse to each curve segment L_e.

Denote the half edges of Γ by $h_1, \dots, h_n$ and let $a_1, \dots, a_n \in \mathbb{N}$ denote the number of points of l contained in the respective segments B_{h_i}. In the discussion after Definition 8.10 of [Man15], for an integer L, Manon associates to the pair (L, l) an element of $\mathcal{V}_\Gamma^\dagger(L, a_1, \dots, a_n)$. These elements are called Γ-tensors. They will be 0 when L is smaller than a certain threshold level.

Remark 7.12. Technically Manon constructs Γ-tensors as elements of a larger space

$$\mathcal{V}_\Gamma^\dagger(L, a_1, \ldots, a_n) \otimes S_{a_1} \otimes \ldots \otimes S_{a_n}$$

where for $a \in \mathbb{N}$, S_a is the irreducible representation of SL_2 of dimension $a+1$ as before. The direct sum of these spaces as the weights a_i vary is a ring denoted W_Γ. This ring has a linear action of ${\mathrm{SL}_2}^n$. The conformal blocks algebra is reconstructed as the invariant subring under the subgroup U_+^n of n-tuples of strictly upper-triangular matrices. Manon's construction of Γ-tensors also includes the data of an orientation, which is either UP or DOWN, for each endpoint of the path l. We denote a basis of torus-invariant vectors in S_a by

$$\{x^i y^j \mid i, j \in \mathbb{N}, i + j = a\}.$$

Then if a path l has i_h endpoints oriented UP and j_h endpoints oriented DOWN for a half edge h, then the Γ tensor is a pure tensor with S_{a_h}-component equal to $x^{i_h} y^{j_h}$. Because the only U_+-invariant vector in S_a, up to scale, is x^a, we can get elements of $\mathcal{V}_\Gamma^\dagger(L, a_1, \ldots, a_n)$ by considering Γ-tensors with only UP orientiations, and therefore we simply forget the orientations altogether. We also will be ignoring the orientation of the entire curve l because a change of orientation only changes the sign of the corresponding conformal block.

In [Man15] Proposition 8.17 Manon constructs a $\mathbb{C}$-basis of the entire conformal blocks algebra $\mathcal{V}_{\Gamma,h_1,\ldots,h_n}^\dagger$ whose elements are pairs (L, a) where $L \in \mathbb{N}$ is the level and a is what is called a planar Γ-tensor. Planar Γ-tensors are exactly the Γ-tensors coming from smooth paths.

Proposition 7.13. *([Man15] Corollary 1.3) The product of two planar* Γ*-tensors given by* (L_1, a_1)*,* (L_2, a_2) *is a sum of planar* Γ*-tensors of level* $L_1 + L_2$ *described as follows. First assume that the paths* a_1 *and* a_2 *meet transversely. Then take their union and resolve each crossing using the Skein relations (these are exactly given by Figure 7.3). The signs in the Skein relations don't seem well defined, but that is fixed if one is careful with the orientations of the curves. This gives a linear combination of paths with smooth paths, and which therefore represents a linear combination of planar* Γ*-tensors.*

Proof. The important parts of this proposition are explained in [Man15] Section 9. However, we'd like to clarify why it is possible to take transverse paths a_1 and a_2. Firstly the equivalence relation on paths paths in R_Γ given by regarding two paths equivalent if they define the same Γ-tensor should be isotopy relative to the boundary, but we do not prove that here. Instead we just note that the Γ-tensor coming from a path (that intersects itself transversely) is determined by the sequence (as we traverse along a parameterization of the path) of intersections of the path with the boundary, the segments L_e, and itself.

In particular, an isotopy of such a path that preserves the order of those crossings cannot change the resulting Γ-tensor. This means we can always assume any two planar Γ-tensors are represented by paths that meet each other and themselves transversely. □

Proposition 7.14. *For an edge* $e \in E(\Gamma)$*, consider the coweighting of* $E(\Gamma)$ *which is* 1 *on* e *and* 0 *on the rest. For an edge* $e \in E(\Gamma)$*, consider the coweighting of* $E(\Gamma)$ *which* *is 1 on e and 0 on the rest. The corresponding valuation λ_e (which corresponds to the filtration $\mathcal{F}_e$) is computed on planar Γ-tensors by counting the number of times the path crosses the segment L_e.*

Proof. This is Corollary 8.16 of [Man15]. That the valuation v_x from [Man15] Definition 8.1 agrees with the valuation $\vec{\theta}$ when $\vec{\theta} = x \in [0,1]^{E(\Gamma)} \cap \mathbb{Q}$ follows from the description in [Man15] section 5 of the factorization map

$$\hat{\rho}_\alpha : \mathcal{V}^\dagger_{(\tilde{C},\vec{p},q_1,q_2)}\left(\vec{\lambda},\alpha,\alpha,L\right) \to \mathcal{V}^\dagger_{(C,\vec{p})}\left(\vec{\lambda},L\right)$$

and correlation map

$$\mathcal{V}^\dagger_{(C,\vec{p})}\left(\vec{\lambda},L\right) \to \big(\bigotimes_{p_i \in C} S_{\lambda_i}\big)^{\mathrm{SL}_2{}^{V(\Gamma)}}.$$

Here the space S_{λ_i} is the SL_2 representation associated with the weight λ_i associated with the marked point p_i of the nodal curve $(C,\vec{P})$. The tensor product is taken over all the marked points of C. The correlation map is the global version of the map from Proposition 2.12 that we saw in the context of the curve $\mathbb{P}^1$ with three marked points. □

Lemma 7.15. *The basis of planar Γ-tensors is adapted to the filtration $\mathcal{F}_e$, i.e. they give bases of the filtered pieces of $\mathcal{F}_e$. Equivalently, it maps to a basis under the set-theoretic map*

$$gr : \mathcal{V}^\dagger_\Gamma \to gr_{\mathcal{F}_e}\left(\mathcal{V}^\dagger_\Gamma\right).$$

Proof. This follows from Proposition 8.17 of [Man15] which says that for each lattice point

$((m_e)_{e\in E(\Gamma)}, L)$ in the cone over the polytope P_Γ, there is exactly one planar Γ-tensor of level L whose λ_e-value is m_e for all $e \in E(\Gamma)$. Therefore for any level L and $m \in \mathbb{N}$, the number of planar Γ-tensors in the subspace $\mathcal{F}_e^{\leq m}(\mathcal{V}_\Gamma^\dagger)_L$ is exactly the dimension of that space. □

We finally have a sufficiently concrete description of the conformal blocks algebra associated to Γ' to prove our Lemma.

Lemma 7.16. *Let Γ be a trivalent graph with at most one colored vertex, no half edges, no bridges, and which is not the theta graph. Then for any edge $e \in E(\Gamma)$, the filtration $\mathcal{F}_e$ on $gr_{\vec{\theta}_e}(\mathcal{V}_C)$ is not induced from a grading.*

Proof. This is true for the corresponding filtration by the weight on e on the ring $\mathcal{V}_{\Gamma'}^\dagger$ by Figure 7.3. This is because of Lemma 7.15 because if $\mathcal{F}_e$ came from a grading, the Γ'-tensors (l_1, L) and (l_2, L) (with any positive level L) would be homogeneous elements because they live in the 1-dimensional filtered pieces $\mathcal{F}_e^{\leq 1}(\mathcal{V}_{\Gamma'}^\dagger((1,0,1,0), L))$ and $\mathcal{F}_e^{\leq 1}(\mathcal{V}_{\Gamma'}^\dagger((0,1,0,1), L))$ respectively. Here the four-tuple $(1,0,1,0)$ means weight 1 on the half edges h_1 and h_3 and 0 on the others. Then their product would also be a homogeneous element. The two planar Γ'-tensors on the right hand side of Figure 7.3 would have to be homogeneous as well because they are each products of two planar Γ'-tensors (corresponding to the the connected components of the paths) which would have to be homogeneous for the same reason that l_1 and l_2 were. Because the two planar Γ'-tensors in the figure have different λ_e-degrees, this means it would be impossible for $\mathcal{F}_e$ to come from a grading on $\mathcal{V}_{\Gamma'}^\dagger$.

Now to finish the lemma, we need to find lattice points $m_1, m_2 \in C_{\tilde{\Gamma}_e}$ such that $m_1 \otimes (l_1, L)$ and $m_2 \otimes (l_2, L)$ define elements of the ring $gr_{\vec{\theta}_e} \mathcal{V}_\Gamma^\dagger$ when it is considered as a subalgebra of $\mathbb{C}[C_{\tilde{\Gamma}_e}] \otimes \mathcal{V}_{\Gamma'}^\dagger$ as described in the start of this section. This is equivalent to finding Γ-tensors for the entire graph Γ that extend our Γ'-tensors in the smaller graph Γ'. Note that here we are considering the ribbon graph of Γ' as a subspace of the ribbon graph of Γ, although we may need to identify boundary components B_{h_i} and B_{h_j} if the half edges h_i and h_j connect in the original graph Γ.

Case 1: The half edge h_1 of Γ' is not connected to the half edge h_4 in the original graph Γ and the same for h_2 and h_3.

In this case, because e is not a bridge in Γ, $R_\Gamma \setminus R_{\Gamma'}$ is connected. If there are no colored vertices then we can connect the endpoints of the paths l_1 and l_2 inside $R_\Gamma \setminus R_{\Gamma'}$ to get the desired Γ-tensors.

If Γ has a colored vertex, then R_Γ has a boundary component B_h, and a path l needs to have L-many endpoints in B_h in order for the Γ-tensor (l, L) to be in the ring we want (the ring of conformal blocks where the weight on the half edge corresponding to the colored vertex is equal to the level). In this case, we don't connect the endpoints of l_1 and l_2 to each other, rather we connect them to the boundary segment B_h. Then each l_1 and l_2 will define Γ-tensors of level 2.

Case 2: Either the half edges h_2 and h_3 of Γ' connect inside the graph Γ or the half edges h_1 and h_4 do.

Without loss of generality we will assume that h_2 and h_3 connect, because if h_1

and h_4 do then we can simply flip our figures upside down to make it work. Also if both pairs of half edges are connected then Γ is the theta graph, where the statement is false.

For Case 2, we need to consider the different Γ' tensors l_1 and l_2 depicted in Figure 7.4. The same argument as before concerning homogeneity applies to these other Γ'-tensors, because each connected component of all depicted Γ'-tensors must be homogeneous.

These work when h_2 is connected to h_3 because the two connected components of the purple curve meet up when we identify the boundary segments B_{h_2} and B_{h_3} inside R_Γ, as do the components of the red curve. The same argument as before concerning homogeneity applies to these other Γ'-tensors, because each connected component of all depicted Γ'-tensors must be homogeneous. Then by the same reasoning as in Case 1, we can extend these connected curves to define the necessary Γ-tensors.

□

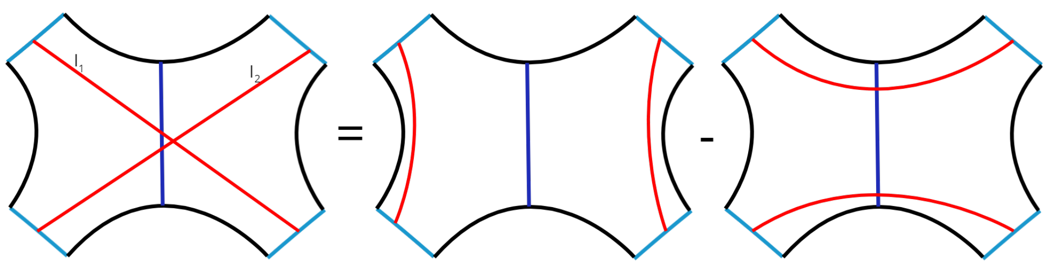

Figure 7.3: Product of Planar Γ'-tensors.

For the proof of the main theorem, we need to prove Lemma 7.7 when there is one marked point and the edge e is in the set $C(\Gamma)$.

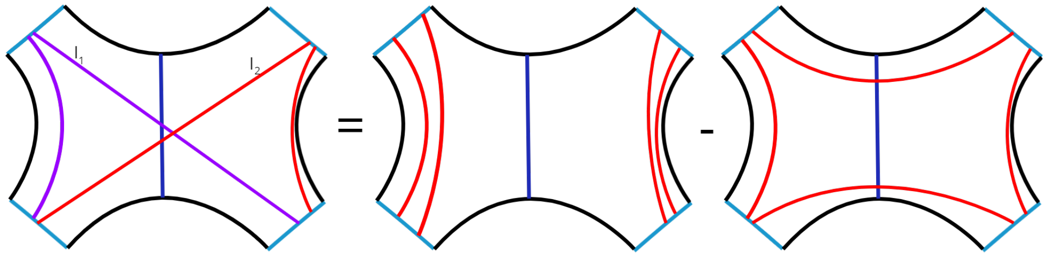

Figure 7.4: Another Product of Planar Γ'-tensors.

Lemma 7.17. *Suppose Γ is the dual graph of a graph curve with at most one colored vertex, no bridges, and which is not the theta graph. Let $e \in \Gamma$ be any edge. Suppose $Y_e \subseteq \mathcal{A}_\Gamma$ is an irreducible component of the locus parameterizing varieties of the form X_{C_t}, and the closure of Y_e contains the image of the e-axis under the map α_1 of Proposition 7.5. The subgroup $T_e \subseteq \mathrm{Aut}(X_\Gamma)$ stabilizes all points of Y_e but the entire torus T does not stabilize the general point.*

Proof. The only thing we have left to prove is that if there is a colored vertex, with special edge $e \in C(\Gamma)$, then the stabilizer of the general point of Y_e doesn't contain the entire torus T. We should start with a graph curve with one marked point (C, p). Then, instead of the family of curves

$$\mathcal{C}_e \to \mathbb{P}^1 \to \overline{\mathcal{M}}_g$$

we should consider the base change of this family under the map

$$\overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g.$$

Really, we only want the irreducible component of this family which contains (C, p), which we'll denote by

$$(\mathcal{C}_{e,1}, \mathcal{P}) \to T$$

where T is some rational surface. Then for a point $t \in T$ we can consider the variety $X_{(C_t,p_t),1}$. If we let (C_t, p_t) tend towards (C, p) then $X_{(C_t,p_t),1}$ will tend towards $X_{(C,p),e}$. However, our goal is only to show that the automorphism group of X_{C_t,p_t}, e contains T_e but not all of T. Therefore if we can't use Lemma 7.6 directly, we can degenerate to a different limit (C', p') where the same edge e is not in the special set of edges $C(\Gamma)$. The local picture of the dual graph of (C', p') around the edge e is shown in Figure 7.5. Now, the variety $X_{(C',p'),e,1}$ (the analogue of the variety $X_{\Gamma,e}$ for the graph curve C' with marked point p' with weight λ equal to the level, so that λ/L is 1) is not toric because of Proposition 7.6. Therefore by the upper semi-continuity of the dimension of the automorphism group, the variety $X_{(C_t,p_t)}$ is not toric for the general point $p_t \in C_0$.

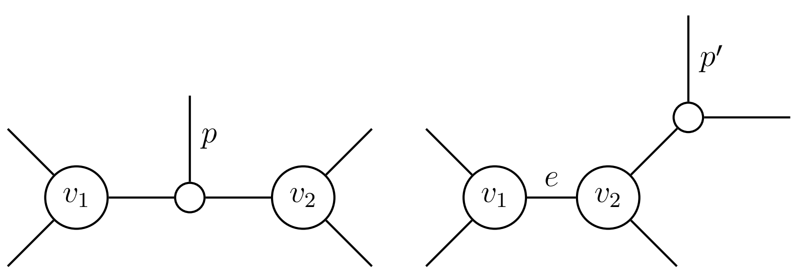

Figure 7.5: The original subgraph Γ' for (C, p) and the one for the other limit (C', p').

□

We finish the section with one last lemma we will need for the main proof.

Lemma 7.18. *Let Γ be a trivalent graph with at most one colored vertex which has no bridges and is not the theta graph. Consider the union $Y = \bigcup_{e \in E(\Gamma)} Y_e \subseteq \mathcal{A}_\Gamma$. No subtorus of T acts trivially on Y.*

Proof. We know that the stabilizer of the general point of Y_e-axis contains the torus generated by all $\lambda_{e'}$ where $e' \neq e$, but this stabilizer does not contain λ_e. Now, if a 1-parameter subgroup

$$\lambda = \sum a_e \lambda_e$$

acts on Y, with a_e not equal to 0 for some edge e, then λ acts like $a_e \lambda_e$ on Y_e and therefore cannot fix the general point in Y_e. Because no 1-parameter subgroup can act trivially on Y, no subtorus can act trivially on Y. □

Chapter 8

Proof of the Main Theorem

Using the Luna slice theorem, the relationship between GIT-stability and K-stability explained at the start of Chapter 5, and the openness of K-stability, the following theorem directly implies Theorem 1.1.

Theorem 8.1. *For a toric variety X_Γ with Γ satisfying the hypotheses of Proposition 6.2, the Artin approximation $\mathcal{A}_\Gamma$ contains GIT-stable points for the* $\mathrm{Aut}(X)$*-action. Further, for some smooth curve C, one of these stable points maps to the class of $U_C(2,\xi) \in \mathcal{M}^{Kss}$.*

Proof. We ignore the case of the theta graph because in that case $X_\Gamma = \mathbb{P}^3$ which is well understood. Assuming Γ is not the theta graph, we then know that $\mathrm{Aut}(X_\Gamma)$ is equal to $T \rtimes H$ by Proposition 6.2. We will also assume for convenience that there are no colored vertices. The proof goes through with insignificant changes if we consider colored graphs and moduli spaces $U_C(2, \mathcal{O}_C(-p))$ varying over $\mathcal{M}_{g,1}$.

Consider the flat family $V_g = \mathrm{Proj}(\mathcal{V}^\dagger) \to \overline{\mathcal{M}}_g$ from Chapter 2. For a trivalent graph

Γ with no bridges, the conformal block space V_Γ degenerates to X_Γ via $\mathbb{C}^\times$-degeneration, and therefore V_Γ is K-semistable. Thus, $\mathcal{X}$ defines a morphism

$$\mathcal{U} \to \mathcal{M}^{Kss}$$

where $\mathcal{U} \subseteq \overline{\mathcal{M}}_g$ is an open neighborhood of the curve corresponding to Γ.

Now consider the pullback square

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & \mathcal{A}_\Gamma \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathcal{M}^{Kss} \end{array} .$$

We know $\mathcal{A}_\Gamma$ is smooth over $\mathcal{M}^{Kss}$, and $\mathcal{U}$ is smooth, therefore $\mathcal{M}$ is smooth. However, it may have multiple connected components, so choose one and denote its image in $\mathcal{A}_\Gamma$ as $\mathcal{M}_0$. Each of these components map smoothly to $\mathcal{U}$ so we just have to make sure we choose one whose image contains a neighborhood of the point $[\Gamma]$.

Remark 8.2. If the image of $\mathcal{M}$ inside $\mathcal{A}_\Gamma$ were irreducible, then the proof would be much easier. The image of $\mathcal{M}$ would have an action of $T \rtimes H$, we could embed it equivariantly as an irreducible non-degenerate subvariety of a representation V of $T \rtimes H$, and we could then apply Lemma 5.6 directly. Unfortunately for us, H may non-trivially permute the components of $\mathcal{M}$. We will therefore have to work harder to find an appropriate set-up to apply the lemma.

Now, we claim for every edge $e \in E(\Gamma)$, the orbit closure $\overline{T\mathcal{M}_0}$ contains points parameterizing the varieties X_{C_t} of Chapter 7. First note that for a general $t \in \mathbb{P}^1$, X_{C_t} is K-semistable when t is general by the openness of K-stability. Specifically, Proposition 7.3 shows $X_{\Gamma,e}$ is K-semistable, and Proposition 7.5 shows that X_{C_t} is K-semistable for general t. Further, we know that X_{C_t} is a $\mathbb{C}^\times$-degeneration of V_{C_t} and $\mathcal{M}_0$ contains points parameterizing these varieties by definition. This $\mathbb{C}^\times$-degeneration means there is a map $[\mathbb{A}^1/\mathbb{C}^\times] \to \mathcal{M}^{Kss}$ such that the generic point parameterizes V_{C_t} and the special point parameterizes X_{C_t}. As in the proof of Proposition 7.5, we can lift this map to an equivariant map $\mathbb{A}^1 \to \mathcal{A}_\Gamma$ such that some non-zero point maps to $\mathcal{M}_0$. Therefore the origin maps into $\overline{T\mathcal{M}_0}$ which establishes the claim.

We write $\overline{Y_e} \subseteq \overline{T\mathcal{M}_0}$ for the closure of the locus parameterizing the varieties X_{C_t}. By Lemma 7.17 and Proposition 7.9, $\overline{Y_e}$ is stable under the action of T and the Hecke transformations H_Γ.

Now we claim that we can find a $\mathrm{Aut}(X_\Gamma) \cong T \rtimes H$-equivariant embedding

$$\mathcal{A}_\Gamma \hookrightarrow V$$

where V is a finite dimensional linear $T \rtimes H$ representation. Because $\mathcal{A}_\Gamma$ is a finite type affine scheme with $T \rtimes H$-action, we can find a finite list of generators $f_1, \ldots, f_k$ of the ring R_Γ and these generate a finite dimensional $T \rtimes H$-representation $V^* \subseteq R_\Gamma$. The embedding above is then given by the surjection of rings

$$\mathrm{Sym}(V^*) \to R_\Gamma.$$

Denote the linear span of $\overline{Y} = \bigcup_{e \in E(\Gamma)} Y_e$ by $V' \subseteq V$. This is in fact a $T \rtimes H_\Gamma$-subrepresentation because all $\overline{Y_e}$ are $T \rtimes H_\Gamma$-invariant. Because $T \rtimes H_\Gamma$ is reductive there is a $T \rtimes H_\Gamma$-equivariant projection

$$p : V \to V'.$$

Namely, V splits as a direct sum

$$V = V' \oplus V''$$

of $T \rtimes H_\Gamma$ representations where V'' is the kernel of p.

We now consider the closure of $p(\overline{T\mathcal{M}_0})$, which is an irreducible subvariety of V' which cannot be contained in a hyperplane because it contains a spanning set $\overline{Y}$ of V'. We wish to apply Lemma 5.6, but we must first show that V' satisfies the hypotheses of the lemma, and that stability in V' implies stability in V. This will then finish the proof, as it will show that $\mathcal{M}_0$ contains GIT-stable points, and a general point of $\mathcal{M}_0$ parameterizes a variety of the form $U_C(2, \mathcal{O}_C)$.

First we show that for $v \in V$, if $p(v) \in V'$ is GIT stable then v is GIT stable . We have that for $v \in V$, the weight polytope of $p(v)$ is contained in that of v. This is because we have simply removed any of the weights that appeared only in V''. This means that if the weight polytope of $p(v)$ contains the origin in its interior then so does the weight polytope of v.

As an aside, note that the same statement is not true for polystability: if the weight polytope of $p(v)$ contains the origin in its relative interior, then the weight polytope of

v contains the origin, but it may be on the boundary. Therefore even though it would suffice for us to show that $U_C(2, \mathcal{O}_C)$ is K-polystable (it has finite automorphism group), it is still important to us that no subtorus acts trivially on V'.

We have checked all the conditions of Lemma 5.6 for the group $T \rtimes H_\Gamma$, representation V', and variety $\overline{p(\overline{T\mathcal{M}_0})}$, except for the condition that no 1-parameter subgroup of T acts trivially on V'. However, this is the statement of Lemma 7.18.

We have proved that Lemma 5.6 applies and this concludes the proof of the theorem.

Remark 8.3. Technically we have proven $\mathcal{M}_0$ contains stable points for the action on V, but these must be stable as points in $\mathcal{A}_\Gamma$, for instance because they have closed orbits and finite stabilizers.

□

We now move on to the proof of Corollary 1.2

Proof. We consider the moduli stack $\mathcal{P}ic_g^d$ which parameterizes pairs (C, ξ) where C is a smooth curve of genus g and ξ is a line bundle on C of degree d. As the curve and line bundle vary, the spaces $U_C(2, \xi)$ naturally form the fibers of a flat family $\mathcal{V}ec_{2,d,g} \to \mathcal{P}ic_g^d$. Theorem 1.1 gives a rational map $\mathcal{P}ic_g^d \dashrightarrow \mathcal{M}^{Kss}$ from the universal Picard stack to the moduli stack of K-semistable Fano varieties sending (C, ξ) to the isomorphism class of $U_C(2, \xi)$. This then descends to give a rational map of moduli spaces $Pic_g^d \dashrightarrow M^{Kps}$. As the isomorphism class of $U_C(2, \xi)$ depends only on the degree of ξ, this therefore factors through a map $M_g \dashrightarrow M^{Kps}$ by [Deb01] Lemma 1.15. Note that [Deb01] Lemma 1.15 requires the map $Pic_g^d \dashrightarrow M^{Kps}$ to be proper. However this map is proper as a map from its domain (rather, the largest open on which it is defined) to its image. To see this, we use Nagata's compactification theorem which says that this map can be factored as an open immersion followed by a projective morphism. We know by [MN68] that the fibers of the map are the same as the fibers of the map $Pic_g^d \to M_g$, i.e. Picard schemes of smooth curves. Therefore its fibers are connected and projective, and so open immersion part of the Nagata factorization has to be an isomorphism. When ξ has odd degree, the main result of [NR75] states that this map gives an isomorphism on tangent spaces, and the main result of [MN68] states that the map is injective. Together these facts imply that the map $M_g \dashrightarrow M^{Kps}$ is a dominant birational map to an irreducible component of M^{Kps}. □

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