UNIVERSITY OF CALIFORNIA
RIVERSIDE

Spectral Triples and Fractal Geometry

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

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June 2018

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Acknowledgments

Primero esta mi familia. Mi mama, Olga Arauza Rivera, mi papa, Eliseo Arauza, mi hermano, Fernando, y mi hermana, Michelle. Han trabajado mucho y quiero que sepan que les agradesco con todo mi corazon los sacrificios que han echo. Los quiero mucho. Estoy orgullosa de ustedes y de ser sulla.

I thank my advisor Michel L. Lapidus, who gave me his direction, patience, and kindness, as well as the members of my committee Dr. John Baez and Dr. Qi Zhang who have been very generous with their time and attention.

Sherri Moses, you were the first to show me that one could love and be excited about mathematics. Dr. Judith Clarke, I still very much want to be like you when I grow up. I thank the organizers of the MSRI-UP program and the students and leaders of MSRI-UP 2011. Thank you for believing in me. Michael Hartglass and John Rock, you have been very generous. Jessica De Silva and Camila Reyes, I called on you when I most needed support and objectivity. I thank the students of PUMP 2017 and every student who has ever let me shower them with advice. It's an absolute privilege to serve as a teacher and mentor to people like you.

Finally, I'd like to thank Mikahl Banwarth-Kuhn, Donna Blanton, Brandon Coya, Alex Sherbetchian, Edward Voskanian, and Andrew Walker. Thank you for teaching me about friendship. Thank you for giving me a place where I fit. Thank you for being my best friend for a while. It was a great gift and an honor to be let into your life. Thank you.

To my grandparents, Amelia Arauza Castro, Jesus Arauza, Aurora Rivera Espinosa,

and Jesus Rivera.

ABSTRACT OF THE DISSERTATION

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by

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University of California, Riverside, June 2018
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Fractal sets are sets that show self-similarity meaning that if one zooms in on some part of the fractal, the close up view exhibits the same patterns as the larger whole. Fractals are difficult to study using the usual tools of geometry and analysis; often classical notions from calculus cannot be meaningfully defined on fractals. The study of analysis on fractals seeks to develop analytic tools analogous to those used on "nice" spaces but that can be used on fractal sets; see [19], [34]. One can then ask if these fractal tools give results analogous to the results in the classical setting. This text contributes to a new way of thinking about fractals by developing operator algebraic tools that can provide an alternative way of studying geometry and analysis on fractals.

Work in noncommutative fractal geometry involves an operator algebraic tool kit known as a spectral triple which is constructed based on the fractal being studied. Building upon previous works, we give the construction of a spectral triple for the fractal sets known as the stretched Sierpinski gasket and the Harmonic Sierpinski gasket. We show how these spectral triples can be used to describe fractal geometric properties: Hausdorff dimension,

geodesic distance, and certain "fractal" measures. We then describe a spectral triple which

can be used to describe the standard energy on the Sierpinski gasket.

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Introduction

1.1 Motivations

Fractals, or fractal sets, are sets that show self-similarity, meaning that if one zooms in on some part of the fractal, the close up view exhibits the same patterns as the larger whole. For example, if one zooms into the middle spiral point in a sea shell, the view looks like the entire shell. The difference in the view being only in scale and not in the pattern. In this text we primarily work with the fractal sets known as the Sierpinski gasket, SG, the harmonic Sierpinski gasket, K_H, and the stretched Sierpinski gasket of parameter α, K_α. The parameter α in K_α is such that $0 < \alpha < \frac{1}{3}$ and indicates the length of the line segments labeled below.

The study of fractal geometry has roots that trace back for hundreds of years. Examples of fractal sets include those found by Wacnaw F. Sierpinski while studying set theory in the early 1900's, Giuseppe Peano in 1890 who constructed space filling curves,
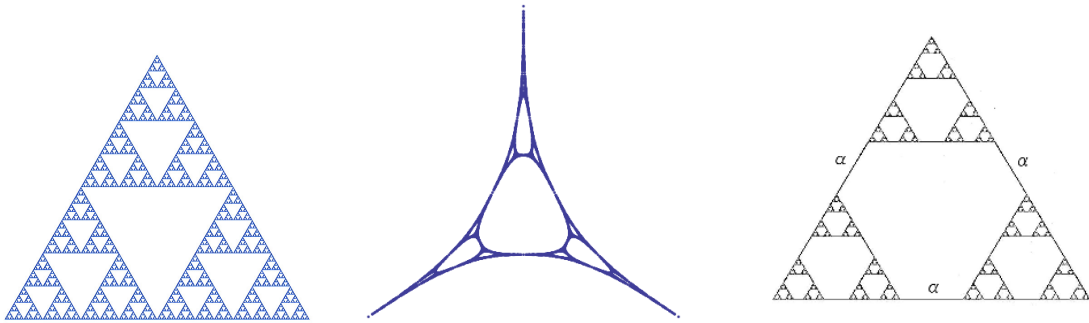
Sierpinski gasket (left), harmonic Sierpinski gasket (center) [20], stretched Sierpinski gasket of parameter α (right) [1].

Karl Weierstrass in the late 19th century with the famed nowhere differentiable curves, and the list goes on. The "monsters" of fractal geometry have been lurking beneath beds and behind closet doors for a great many years. However, the modern study of fractal sets was kicked off in the 1970's by Benoit Mandelbrot. In fact, it was Mandelbrot's groundbreaking text "The Fractal Geometry of Nature" which inspired the modern study of fractal sets. In this book, Mandelbrot puts on display the many occurrences of fractals in nature. Branching on trees, the boundaries of coast lines, blood vessels in the human body, are all examples of fractal patterns occurring in nature. This means that tools used to study abstract fractal sets may be used to study the fractal sets that nature exhibits.

There is a long tradition in mathematics of using algebra to describe and understand geometry. One can take problems from geometry, translate them into problems in algebra where solutions to the problem may be more evident, and then translate the solutions back into geometric ones. For example, instead of working with a compact Hausdorff space X, one can work with the *commutative* C*–algebra $C(X)$. One recovers the space X by considering the set of continuous nonzero ∗–homomorphisms from $C(X)$ to $\mathbb{C}$, called

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the Gelfand spectrum of $C(X)$. With a suitable topology, the Gelfand spectrum of $C(X)$ is homeomorphic to X.

The move towards noncommutativity stems from the theorem of Gelfand and Naimark which says that any C*–algebra is isometrically $*$–isomorphic to a (generally non-commutative) closed subalgebra of the bounded operators on some Hilbert space. Dropping the commutativity assumption leads to the study of noncommutative C*–algebras as "non-commutative spaces". In noncommutative geometry, operator algebraic tools are used to study spaces like compact Riemannian manifolds, and many of these tools can also be used to study fractals; see [9]. More precisely, the spectral triples of noncommutative geometry can be used to describe the fractal geometric notions of dimension, geodesic metric, and measure. In some cases, spectral triples can also be used to describe the energy forms and Laplacians from the study of analysis on fractals.

In 1990, Michel L. Lapidus outlined how one may study fractal sets by using the tools of noncommutative geometry in the papers [23] and [24]. Since then, much progress has been made in building examples of how one may describe fractal geometric notions with operator algebraic tools. Works that use noncommutative geometry to describe the geometry and analysis of fractal sets include [5], [6], [8], [14], [26].

With an eye towards being able to study heat equations on fractal sets analogous to those in Euclidean space, Jun Kigami and others began studying energy forms on post critically finite (p.c.f.) sets which would induce a Laplacian on that set. From there, the theory of analysis on fractals grew and is studied extensively. There has been much success in defining Laplacians on p.c.f. sets and studying questions, such as Weyl asymptotics,

regarding the behavior of the eigenvalues of these Laplacians. Details can be found in the paper by Kigami and Lapidus [22].

Work in noncommutative fractal geometry involves an operator algebraic tool kit known as a spectral triple which is constructed based on the fractal being studied. In 2008, E. Christensen, C. Ivan, and M. L. Lapidus [8] gave the construction of a spectral triple for various fractal sets –including the Sierpinski gasket– which can be used to describe fractal geometric properties: Hausdorff dimension, geodesic distance, and the Hausdorff measure. Christensen, Ivan, and Lapidus built a spectral triple for a circle, used this to give a spectral triple for each triangle in the Sierpinski gasket, and then defined a spectral triple for the Sierpinski gasket by taking a direct sum.

In 2014, F. Cipriani, D. Guido, T. Isola, and J-L. Sauvageot [6] gave a collection of spectral triples for the Sierpinski gasket that depend on a parameter β. Under some restrictions on β, these spectral triples recover, amongst other things, the Hausdorff measure on the Sierpinski gasket and a concept from analysis on fractals known as a Dirichlet form.

In 2015, M. L. Lapidus and J. J. Sarhad gave the construction of a spectral triple for certain length spaces and showed that their spectral triple recovers the geodesic metric on these length spaces; see [26]. This construction of a spectral triple is for length spaces, including the Sierpinski gasket and the harmonic Sierpinski gasket, made up of rectifiable C^1–curves. While one can use the spectral triple of Lapidus and Sarhad for the stretched Sierpinski gasket, the assumptions needed for their theorem on the recovery of the geodesic metric are such that the theorem does not apply to the stretched Sierpinski gasket.

1.2 Summary of Main Results

The main results of this text are contained in Chapter 5. We now state the main results concerning the stretched Sierpinski gasket, K_α, and the harmonic Sierpinski gasket, K_H.

In Chapter 3 we give the definition of a spectral triple and give the construction of a spectral triple for sets made up of C^1-curves. Using this we get a spectral triple $ST_\alpha = (C(K_\alpha), \mathscr{H}_\alpha, D_\alpha)$ for K_α and show in Chapter 5 how one can recover (1) the Hausdorff dimension, (2) the geodesic distance, and (3) the Hausdorff measure of K_α.

1. The Hausdorff dimension for the stretched Sierpinski gasket, K_α, is

$$\dim_H(K_\alpha) = \frac{\log(3)}{\log(2) - \log(1 - \alpha)}.$$

One gets a notion of dimension, $\mathfrak{d}$, from a spectral triple via the formula

$$\mathfrak{d} = \inf\{\, p > 0 : \mathrm{tr}((I + D^2)^{-p/2}) < \infty\}.$$

We show that the trace of the operators $|D_\alpha|^{-s}$ for $s > 0$ sufficiently large, gives a Dirichlet series and that calculating the dimension, $\mathfrak{d}$, induced by the spectral triple ST_α amounts to finding the minimal s value for which this series converges (i.e. finding the abscissa of convergence of this series). We then show that the dimension given by the spectral triple is the same as the Hausdorff dimension of K_α:

$$\mathfrak{d} = \dim_H(K_\alpha).$$

2. Geodesic distance on a space is determined by the length of shortest paths between points. This means that given two points x, y in K_α, the geodesic distance, denoted

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$d_{geo}(x, y)$, between them is the length of a shortest path in K_α connecting x and y. There may not be a unique shortest path between the two points, but the length of the shortest paths will be unique. An operator algebraic notion of distance will come from the definitions used in the study of metrics on state spaces found in [10], [30], [31]. For example, on the space of probability measures on a compact metric space, (X, ρ), one can define a metric by

$$\bar{\rho}(\mu, \nu) = \sup\{|\mu(f) - \nu(f)| : \text{Lip}_\rho(f) \le 1\},$$

where $\text{Lip}_\rho(f) = \sup\left\{\frac{|f(x)-f(y)|}{\rho(x,y)} : x \ne y\right\}$; see [30].

The spectral triple ST_α can be used to define a metric on K_α via the formula

$$d_{K_\alpha}(x, y) = \sup\{|f(x) - f(y)| : f \in C(K_\alpha), \|[D_\alpha, \pi(f)]\| \le 1\}.$$

We give the following result.

Theorem 1 *Let $d_{K_\alpha}(\cdot, \cdot)$ be the metric on K_α induced by the spectral triple ST_α and $d_{geo}(\cdot, \cdot)$ the geodesic distance on K_α. Then for all $x, y \in K_\alpha$,*

$$d_{K_\alpha}(x, y) = d_{geo}(x, y).$$

3. In order to formulate a notion of measure based on the spectral triple ST_α, one needs another operator algebraic tool called a Dixmier trace, $\text{Tr}_w(\cdot)$. One can use a Dixmier trace and the operator D_α from the spectral triple to create a positive linear functional on $C(K_\alpha)$. This then gives a measure on K_α. The subscript w in the notation $\text{Tr}_w(\cdot)$ indicates the dependence of the Dixmier trace on a choice of extended limit, $w : \ell^\infty \to \mathbb{C}$. We prove that the measure induced by ST_α by using the Dixmier

trace is independent of the choice of extended limit, w. Furthermore, we show that the measure induced by ST_α is the same as the $\dim_H(K_\alpha)$–dimensional Hausdorff measure on K_α:

Theorem 2 *The spectral triple ST_α recovers the $\mathfrak{d}$-dimensional Hausdorff measure, $\mathcal{H}^\mathfrak{d}$, on K_α via the formula*

$$Tr_w(\pi(f)|D_\alpha|^{-\mathfrak{d}}) = c_\mathfrak{d} \int_{K_\alpha} f \ d\mathcal{H}^\mathfrak{d}$$

for all $f \in C(K_\alpha)$, where $\mathfrak{d} = \dim_H(K_\alpha)$. Moreover,

$$c_\mathfrak{d} = \frac{2^{\mathfrak{d}+1}(2^\mathfrak{d} - 1)\zeta(\mathfrak{d})(3 + 3\alpha^\mathfrak{d})}{\mathfrak{d} \cdot \pi^\mathfrak{d}(2^\mathfrak{d}\log(2) - 3(1 - \alpha)^\mathfrak{d}\log(1 - \alpha))}.$$

This text also answers a conjecture made by Lapidus and Sarhad in [26]. We use the construction given in [26] to build a spectral triple, $ST(K_H) = (C(K_H), \mathscr{H}_H, D_{K_H})$, for the harmonic Sierpinski gasket, K_H. It was conjectured in [26] that the measure induced by $ST(K_H)$ could recover the Hausdorff measure on K_H. We show that this conjecture is false and prove that the measure induced by $ST(K_H)$ in fact recovers a measure with a certain self-affinity property.

Theorem 3 *Let $\tau : C(K_H) \to \mathbb{C}$ be given by $\tau(h) := Tr_w(\pi_H(h)|D_{K_H}|^{-\mathfrak{d}_H})$. Then*

$$\tau(h) = Tr_w(\pi_H(h)|D_{K_H}|^{-\mathfrak{d}_H}) = c \int_{K_H} h(x) \ d\mu,$$

where μ is the unique self-affine measure on K_H satisfying,

$$\int h \ d\mu = \frac{1}{3}\sum_{j=1}^{3} \int (h \circ H_j)d\mu \qquad \text{for each } f \in C(K_H).$$

1.3 Outline of Chapter Contents

- In Chapter 2 we give a review of basic results from fractal geometry and describe two of the fractals which are the focus of this text, the Sierpinski gasket and the stretched Sierpinski gasket. We also use this chapter to fix notation and definitions.

- In Chapter 3 we define spectral triples and use this definition to define notions of dimension, metric, and measure. We give examples including a spectral triple for fractal sets in $\mathbb{R}$ like the Cantor middle third set, as well as a spectral triple for the Sierpinski gasket and the stretched Sierpinski gasket. In this chapter we give a result connecting the measure induced by the spectral triple for certain Cantor like sets in $\mathbb{R}$ to the average Minkowski content of the fractal.

- Chapter 4 will introduce the subject of analysis on fractals. We define energy forms, harmonic functions, and the harmonic Sierpinski gasket. We also give a spectral triple for the harmonic Sierpinski gasket.

- Chapter 5 will hold the main results of this text. This includes a careful study of the stretched Sierpinski gasket using the spectral triple defined in Chapter 3 and a result concerning the measure induced by the spectral triple defined in Chapter 4 for the harmonic Sierpinski gasket.

- Chapter 6 will define an energy form on the Stretched Sierpinski gasket and will give the construction of a spectral triple for the classical Sierpinski gasket which can recover the standard energy form on the Sierpinski gasket.

Chapter 2

Basic Fractal Geometry

In this chapter we give the basic definitions from fractal geometry and state the standard theorems in the field. The notions of Hausdorff measure and Hausdorff dimension are introduced as these are better suited for the study of fractal geometry than the more commonly used Lebesgue measure and topological dimension. We also present the theory on iterated function systems (IFS) and self-similar (self-affine) measures. Certain IFS's can be used to create fractal sets with rich structure. The fractal sets that are the focus of this text will arise from iterated function systems. As a general reference for this chapter one should see the texts by K. Falconer [12] and the paper by J. E. Hutchinson [15].

2.1   Notions of Measure and Dimension

A key part of the study of fractal sets is the study of dimensions and measure. We wish to associate to fractal sets some notion of dimension that allows us to study the geometry of the set. In the same way that one studies line segments in 1-dimension, planes

in 2-dimensions, and cubes in 3-dimensions, we wish to study fractal sets in a dimension which allows for a meaningful study of the sets geometry. Part of the reason why we study, for example, line segments in 1-dimension is because the 1-dimensional measure (length) gives a meaningful measure of the size of a line segment. When studying fractal sets we first define measures which depend on some positive real number s, denoted $\mathcal{H}^s$. These measures have the property that for a non-empty set $F \subseteq \mathbb{R}^N$ there is a non-negative real number d such that $\mathcal{H}^s(F) = \infty$ for $s < d$ and $\mathcal{H}^s(F) = 0$ for $s > d$; see Figure 2.1. The measure $\mathcal{H}^d$ is thus the appropriate measure for studying the set F and the number d will be the dimension we associate to the set U.

Definition 2.1.0.1 *Let U be an open, non-empty, subset of $\mathbb{R}^N$ and $\delta > 0$. The **diameter** of U is defined as*

$$\mathrm{diam}(U) = \sup\left\{ |x - y| : x, y \in U \right\}.$$

If $\{U_j\}_{j \in I}$ is a countable or finite collection of open sets such that

$$F \subseteq \bigcup_{j \in I} U_j,$$

*and $0 < \mathrm{diam}(U_j) \leq \delta$ for each j, we say $\{U_j\}_{j \in I}$ is a δ-**cover** of F.*

We can now define a commonly used measure in fractal geometry. Due to the liberty in the choice of sets U_i in a δ-cover, this measure accounts for the fine scale characteristics of a set and is hence well suited for studying fractal sets.

Definition 2.1.0.2 *Let F be a subset of $\mathbb{R}^N$ and $s > 0$. For any $\delta > 0$ define*

$$\mathcal{H}_\delta^s(F) = \inf\left\{ \sum_{j=1}^{\infty} diam(U_j)^s : \{U_j\}_{j=1}^{\infty} \text{ is a } \delta\text{-cover of } F \right\}. \tag{2.1}$$

*Define the s-**dimensional Hausdorff measure** of F by*

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

Note that the value of equation 2.1 increases as δ gets smaller since the collection of possible δ-covers gets smaller. This means the limit in the definition of the Hausdorff measure exists and is possibly infinite.

Equation 2.1 also shows that $\mathcal{H}^s_\delta(F)$ is non-increasing as s increases and hence $\mathcal{H}^s(F)$ is non-increasing with s. More precisely, if $r > s$ we have for a δ cover $\{U_j\}_{j \in J}$ of F that

$$\sum_{j \in J} \operatorname{diam}(U_j)^r = \sum_{j \in J} \operatorname{diam}(U_j)^{r-s} \operatorname{diam}(U_j)^s \leq \delta^{r-s} \sum_{j \in J} \operatorname{diam}(U_j)^s$$

and hence $\mathcal{H}^r_\delta(F) \leq \delta^{r-s} \mathcal{H}^s_\delta(F)$. Letting $\delta \to 0$ we see that if for some s, $\mathcal{H}^s(F) < \infty$ then for all $r > s$, $\mathcal{H}^r(F) = 0$. This means that for most values of s the Hausdorff measure of a fixed set $F \subseteq \mathbb{R}^N$ is either 0 or infinity. The value of s at which the Hausdorff measure of a set switches from being infinite to being 0, gives us a notion of dimension that is better suited for studying fractal sets than the typical topological dimension.

Definition 2.1.0.3 *Let F be a subset of $\mathbb{R}^N$. Define the **Hausdorff dimension** of the set F as the number*

$$\dim_H(F) = \inf\{s > 0 : \mathcal{H}^s(F) = 0\} = \sup\{s > 0 : \mathcal{H}^s(F) = \infty\}.$$

We now state some scaling properties of the Hausdorff measure and dimension which will be useful in later chapters. These results can be found in Falconer's text [12].

s-Hausdorff Measure
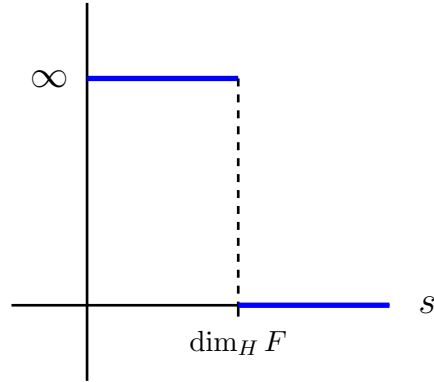
Figure 2.1: Graph of $\mathcal{H}^s(F)$ against s for a fixed set $F \subseteq \mathbb{R}^N$.

Proposition 2.1.0.4 *If $F \subseteq \mathbb{R}^N$ and $\lambda > 0$ then*

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$$

where $\lambda F = \{\lambda x : x \in F\}$.

Fractals often have the property that they are made up of smaller copies of themselves. For this reason we wish to know how the Hausdorff measure and dimension interact with maps that shrink (or contract) a space. We will use the proposition below primarily in the case when the parameters r and c are such that $c = 1$ and $0 < r < 1$.

Proposition 2.1.0.5 *Let $F \subseteq \mathbb{R}^N$ and $f : F \to \mathbb{R}^M$ be a function. Suppose there exist constants $c, r > 0$ such that*

$$|f(x) - f(y)| \leq r|x - y|^c$$

for $x, y \in F$. Then for each $s > 0$,

$$\mathcal{H}^{s/c}(f(F)) \leq r^{s/c} \mathcal{H}^s(F)$$

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and

$$\dim_H f(F) \leq (1/c) \dim_H F.$$

2.2   Iterated Function Systems

Definition 2.2.0.1 *Let F be a closed subset of $\mathbb{R}^N$. A map $f : F \rightarrow F$ is a **contraction** if there exists $0 < r < 1$, such that for all $x, y \in F$*

$$|f(x) - f(y)| \leq r|x - y|.$$

*The number r is called a **contraction ratio** of the map f. If equality holds, we say f is a **similarity** with **similarity ratio** r. An **iterated function system (IFS)** is a finite collection of contraction mappings $\{f_j\}_{j=1}^m$ from the space F to itself.*

We will use iterated function systems to create various examples of fractal sets. The following theorem gives a way of generating fractal sets as well as a way of calculating the Hausdorff dimension of those sets. These results can be found in the paper by Hutchinson [15].

Theorem 2.2.0.2 (Hutchinson [15]) *Let $\{f_j\}_{j=1}^m$ be contractions on the closed non-empty set $D \subset \mathbb{R}^N$ with contraction ratios $0 < r_j < 1$. Then there exists a unique non-empty compact set $F \subseteq \mathbb{R}^N$ that is invariant for the f_j, i.e. which satisfies*

$$F = \bigcup_{j=1}^m f_j(F).$$

Moreover, if we define the transformation S on the collection of compact sets in $\mathbb{R}^N$ by

$$S(E) = \bigcup_{j=1}^m f_j(E)$$

and write $S^k = S \circ S \circ \cdots \circ S$ for the k-th iterate of S, then

$$F = \bigcap_{k=1}^{\infty} S^k(E)$$

for any nonempty compact set E such that $f_j(E) \subseteq E$ for each j.

Given an IFS, $\{f_j\}_{j=1}^m$, we call the unique non-empty compact set $F \subseteq \mathbb{R}^N$ with the property

$$F = \bigcup_{j=1}^{m} f_j(F)$$

the **attractor** of the IFS $\{f_j\}_{j=1}^m$.

The fractal sets we consider in this text will be attractors of some IFS. Notice that the property

$$F = \bigcup_{j=1}^{m} f_j(F)$$

formalizes the idea that fractal sets are made up of smaller copies of themselves. Also, the property

$$F = \bigcap_{k=1}^{\infty} S^k(F)$$

states that the set F can be approximated by the k-th iterates of the map S in Theorem 2.2.0.2. This gives a way of constructing the set F in steps $S^1(F), S^2(F), S^3(F), \ldots$.

Consider the IFS given by $g_1(x) = \frac{1}{2}x$ and $g_2(x) = \frac{1}{2}x + \frac{1}{2}$ on $[0,1]$. The attractor of this IFS is the unit interval $[0,1]$ which we do not consider a fractal set. We would like conditions on the IFS so that the attractor of the IFS has fractal properties. The issue with $\{g_1, g_2\}$ is that the images $g_1([0,1])$ and $g_2([0,1])$ overlap at the point $\frac{1}{2}$ and hence their union gives all of $[0,1]$.

We next introduce an important condition on an IFS which ensures that the images of the contractions do not overlap "too much". A well known theorem of Hutchinson [15] will then gives us a way of calculating the Hausdorff dimension of the fractals that arise from IFS's which satisfy this condition.

Definition 2.2.0.3 *We say that the IFS $\{f_j\}_{j=1}^m$ satisfies the **open set condition** if there exists a non-empty bounded open set U such that*

$$\bigcup_{j=1}^m f_j(U) \subset U$$

where the union is disjoint.

Note that no open set U in $[0,1]$ exists so that $g_1(U)$ and $g_2(U)$ are disjoint and $\bigcup_{j=1,2} g_i(U) \subset U$; hence the IFS $\{g_1, g_2\}$ does not satisfy the open set condition.

Theorem 2.2.0.4 (Hutchinson [15]) *Let $\{f_j\}_{j=1}^m$ be similarities on the closed non-empty set $D \subset \mathbb{R}^N$ with similarity ratios $0 < r_j < 1$. Suppose further that this IFS satisfies the open set condition. If F is the invariant set satisfying*

$$F = \bigcup_{j=1}^m f_j(F)$$

then $\dim_H F = s$ where s is given by

$$\sum_{j=1}^m r_j^s = 1.$$

Moreover, for this value of s, $0 < \mathcal{H}^s(F) < \infty$.

Generally, for an IFS $\{f_j\}_{j=1}^m$ with similarity ratios $0 < r_j < 1$, the value s such that

$$\sum_{j=1}^m r_j^s = 1$$

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is called the **similarity dimension** of the IFS. Theorem 2.2.0.4 states that in the case where the contraction maps in $\{f_j\}_{j=1}^m$ satisfy the open set condition, the Hausdorff dimension is the same as the similarity dimension, s, and the s-dimensional Hausdorff measure gives a meaningful measure of the size of the attractor.

The proof of Theorem 2.2.0.2 considers the collection of non-empty compact subsets of $\mathbb{R}^N$, denoted $\mathscr{C}$, endowed with the metric given by

$$m_H(A, B) = \inf\{\delta > 0 : A \subseteq B_\delta \text{ and } B \subseteq A_\delta\}$$

where for $K \subseteq \mathbb{R}^N$, $K_\delta = \{x \in \mathbb{R}^N : |x - y| \leq \delta \text{ for some } y \in K\}$. The metric $m_H(\cdot, \cdot)$ is called the **Hausdorff metric** on $\mathscr{C}$. One can show that $(\mathscr{C}, m_H)$ is a complete metric space. Given an IFS $\{f_j\}_{j=1}^m$ one then considers the transformation $S : \mathscr{C} \to \mathscr{C}$ given in Theorem 2.2.0.2:

$$S(E) = \bigcup_{j=1}^m f_j(E)$$

for $E \in \mathscr{C}$. The map S is a contraction mapping on the compete metric space $(\mathscr{C}, m_H)$ and hence the contraction mapping principle gives the existence and uniqueness of a fixed point $F \in \mathscr{C}$ of S with

$$F = S(F) = \bigcup_{j=1}^m f_j(F).$$

In [15], this method of proof is also applied to the space of Borel regular probability measures on a complete metric space, (X, d_X). We denote this space of measures by $\mathcal{M}^1(X)$. One endows $\mathcal{M}^1(X)$ with the metric given by

$$d(\mu, \nu) = \inf \{|\mu(\phi) - \nu(\phi)| : \phi : X \to \mathbb{R}, \ \mathrm{Lip}(\phi) \leq 1\}$$

where $\mu(\phi) = \int_X \phi \, d\mu$ and $\mathrm{Lip}(\phi) = \sup\{r > 0 : |\phi(x) - \phi(y)| \leq r d_X(x, y)\}$. This makes

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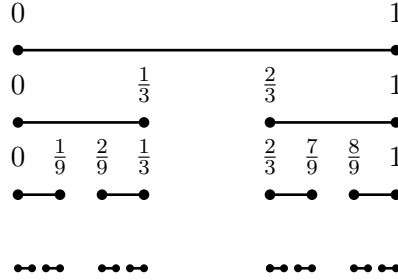
Figure 2.2: The first 4 levels in the approximations to the Cantor middle third set.

$\mathcal{M}^1(X)$ a complete metric space. Consider $p = (p_1, p_2, \ldots, p_m)$ where $0 < p_j < 1$ and $\sum_{j=1}^{m} p_j = 1$. Let $\{f_j\}_{j=1}^{m}$ be a collection of contractions on X. Define then the mapping $(S, p) : \mathcal{M}^1(X) \to \mathcal{M}^1(X)$ given by

$$(S, p)(\mu)(E) = \sum_{j=1}^{m} p_j \mu(f_j^{-1}(E)).$$

The map (S, p) is a contraction on $\mathcal{M}^1(X)$ and hence has a unique fixed point $\mu_p \in \mathcal{M}^1(X)$. The measure, μ_p, arising from an IFS of similarities (resp. affine maps) in this manner is called the **self-similar (resp. self-affine) measure** on X with weight p.

2.3   Examples

We now give examples of fractal sets and apply the previous theorems to calculate their Hausdorff dimension. Examples include the Cantor middle third set, the classical Sierpinski gasket, and the stretched Sierpinski gasket of parameter α.

Example 4 *Our first example is the well known Cantor middle third set. This is the typical*

first example of a fractal set arising from an IFS. Consider the maps $h_j : \mathbb{R} \to \mathbb{R}$ given by

$$h_1(x) = \frac{1}{3}x, \quad h_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Applying these maps to the interval $[0,1]$ and using Theorem 2.2.0.2 gives a unique compact set C such that

$$C = \bigcup_{j=1,2} h_j(C).$$

*The set C is called the **Cantor middle third** set. Observe that iterating the map*

$$S([0,1]) = \bigcup_{j=1,2} h_j([0,1])$$

provides approximations to the Cantor middle third set. The first 4 levels in the approximation to C can be seen in Figure 2.2. Note that the open set $U = (0,1)$ has the property that $h_1(U)$ and $h_2(U)$ are disjoint and

$$h_1(U) \cup h_2(U) \subset U.$$

Thus this IFS satisfies the open set condition and hence the Hausdorff dimension of C is the solution to the equation

$$\sum_{j=1}^{2} \left(\frac{1}{3}\right)^s = 1.$$

That is $\dim_H C = \frac{\log 2}{\log 3}$. The Cantor set has many interesting properties. Among them is the fact that it is a totally disconnected set with an uncountably infinite number of points but with Lebesgue measure (i.e. length) 0.

The next examples are two of the main fractal sets treated in this text.
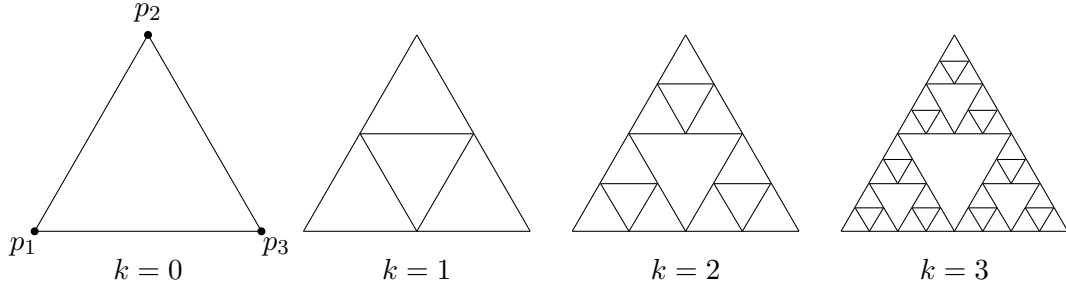
Figure 2.3: Graph approximations, SG_k for $k = 0, 1, 2, 3$, of the Sierpinski gasket.

Example 5 *Consider the IFS given by* $f_i : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f_1(x) = \frac{1}{2}(x - p_1) + p_1, \quad f_2(x) = \frac{1}{2}(x - p_2) + p_2, \quad f_3(x) = \frac{1}{2}(x - p_3) + p_3$$

where $p_1 = (0,0)$, $p_2 = (1,0)$, $p_3 = (1/2, \sqrt{3}/2)$. *Applying these similarities to the equilateral triangle* T *with vertices* p_1, p_2, p_3 *and using Theorem 2.2.0.2 this collection of similarities gives a unique non-empty compact set* $SG \subseteq \mathbb{R}^2$ *with the property*

$$SG = \bigcup_{j=1}^{3} f_j(SG).$$

We call this set the **Sierpinski gasket**.

The iterates of the map

$$S(T) = \bigcup_{j=1,2,3} f_j(T)$$

provide approximations to SG *and give graphs as in Figure 2.3. This gives us graph approximations of the Sierpinski gasket:*

$$SG_k = \bigcup_{w \in \{1,2,3\}^k} f_w(T) \qquad \text{for } k \geq 0$$

where $w \in \{1, 2, 3\}^k$ *means that* $w = w_1 w_2 \cdots w_k$ *is a word in the letters* $\{1, 2, 3\}$ *with length* $|w| = k \geq 1$, *and* $f_w = f_{w_k} \circ \cdots \circ f_{w_2} \circ f_{w_1}$. *If* $|w| = 0$, *then* $w = \emptyset$ *and* $f_\emptyset = id$.

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Notice that one can identify SG_k as a subgraph of SG_{k+1} and hence $\{SG_k\}_{k \geq 0}$ is an increasing sequence of graphs. An alternative definition for the Sierpinski gasket is as the closure of the union of these graph approximations:

$$SG = \overline{\bigcup_{k \geq 0} SG_k}.$$

Using Theorem 2.2.0.4 we have that the Hausdorff dimension of SG is the value $s > 0$ such that

$$\sum_{j=1}^{3} \left(\frac{1}{2}\right)^s = 1.$$

Solving this equation gives $\dim_H SG = s = \frac{\log 3}{\log 2}$.

Before the next example we need the following definition.

Definition 2.3.0.1 An ***affine map*** $A : \mathbb{R}^N \to \mathbb{R}^N$ is a transformation of the form

$$A(x) = T(x) + b$$

where T is a linear transformation on $\mathbb{R}^N$ (often represented as an $N \times N$ matrix) and $b \in \mathbb{R}^N$.

The example that follows is different from the previous two in that the maps in the IFS are contractive affine maps and not necessarily similarities. Affine maps are different from similarities in that they may contract the space by different amounts in different directions. This means we cannot directly apply Theorem 2.2.0.4 so we must work harder to calculate the Hausdorff dimension.

Example 6 Fix $\alpha \in (0, \frac{1}{3})$ and let $p_1, p_2, \ldots, p_6 \in \mathbb{R}^2$ be given by

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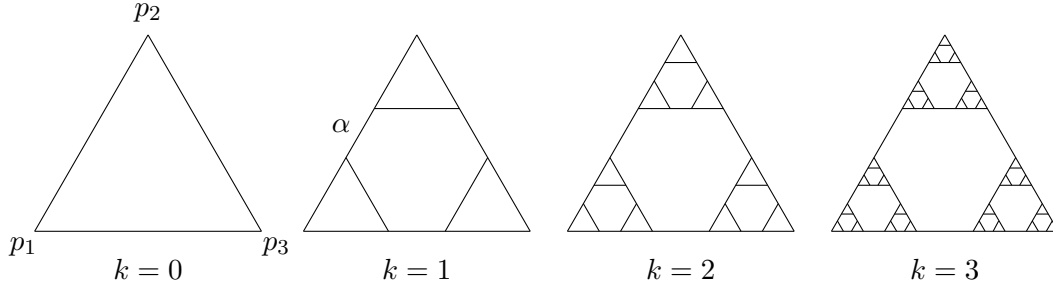
Figure 2.4: Graph approximations of the stretched Sierpinski gasket.
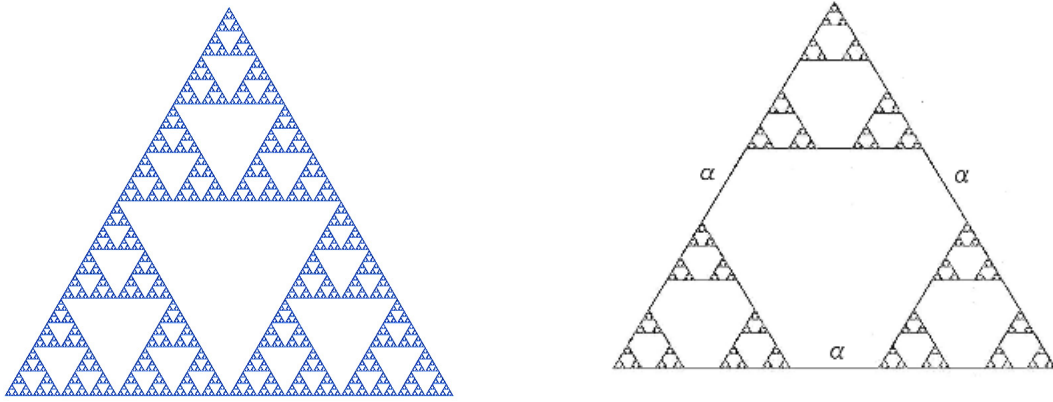
Figure 2.5: The Sierpinski gasket (left) and the stretched Sieprinski gasket (right).

$$p_1 = (0,0), \quad p_2 = \left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}\right), \quad p_3 = (1,0),$$

$$p_4 = \tfrac{p_2+p_3}{2}, \quad p_5 = \tfrac{p_1+p_3}{2}, \quad p_6 = \tfrac{p_1+p_2}{2}.$$

Let $A_1, A_2, \ldots, A_6$ be 2×2 matrices given by

$$A_1 = A_2 = A_3 = \frac{1-\alpha}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$A_4 = \frac{\alpha}{4} \begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{pmatrix}, \quad A_5 = \alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_6 = \frac{\alpha}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}.$$
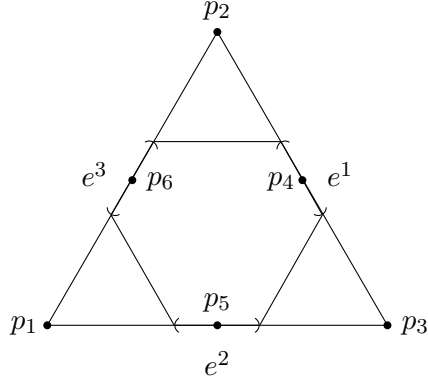
Figure 2.6: The triangle T with fixed points p_j of the maps F_j and edges e^1, e^2, e^3.

Define the maps $F_{\alpha,j} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F_{\alpha,j}(x) := A_j(x - p_j) + p_j \text{ for } j = 1, 2, \ldots, 6. \tag{2.2}$$

The maps $F_{\alpha,1}, F_{\alpha,2}, F_{\alpha,3}$ will map the equilateral triangle T with vertices p_1, p_2, p_3 to smaller triangles at each of the three corners of T. Note that $F_{\alpha,1}, F_{\alpha,2}, F_{\alpha,3}$ are contraction similarities, meaning that they are maps which shrink the space by the same ration, namely $\frac{1-\alpha}{2}$, in every directing. On the other hand, the maps $F_{\alpha,4}, F_{\alpha,5}, F_{\alpha,6}$ will map T to line segments of length α and these are contractive affine maps, meaning that they shrink the space but may do so by different ratios in different directions. We define the **stretched Sierpinski gasket** of parameter α as the unique non-empty compact set $K_\alpha \subseteq \mathbb{R}^2$ satisfying the self-affinity condition

$$K_\alpha = \bigcup_{j=1}^{6} F_{\alpha,j}(K_\alpha).$$

As with the Sierpinski gasket, an alternative definition for K_α is as the closure of the increasing union of graphs seen in Figure 2.3.

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From now on we will fix the parameter $\alpha \in (0, \frac{1}{3})$ and hence will write $F_j = F_{\alpha,j}$.

If $\alpha = 0$, then $K_\alpha = SG$. If $\alpha = 1/3$, then the geometry of the space reduces to the 1-dimensional case. Notice that K_α can be written in terms of a "discrete" part and a "continuous" part. Let W_α be the unique compact set satisfying

$$W_\alpha := \bigcup_{j=1}^{3} F_j(W_\alpha).$$

Let $J_0 = \emptyset$ and for $n \geq 1$ let

$$J_{\alpha,n} = J_n := \bigcup_{m=0}^{n-1} \bigcup_{w \in \{1,2,3\}^m} F_w \left(\bigcup_{j=1}^{3} e^j \right)$$

where $e^j = \mathrm{int}(F_{j+3}(T))$ for $j = 1, 2, 3$. Note that e^1, e^2, e^3 are the three edges in the first graph approximation of K_α which join the three triangles in K_α together. We will call the edges in J_n, the level n joining edges. Also make note of the fact that we take the level n joining edges to be open. Letting $J^* = \cup_{n \geq 1} J_n$ we see that

$$K_\alpha = \bigcup_{j=1}^{6} F_j(K_\alpha) = W_\alpha \,\dot\cup\, J^*$$

where the second union is disjoint; see [3]. The set W_α is the *discrete* part of K_α and has many properties similar to the classical Sierpinski gasket; the set J^* is the *continuous* part of K_α and is a union of shrinking intervals. This decomposition of K_α will be essential in proving results concerning the Hausdorff dimension and measure of K_α.

Chapter 3

Spectral Triples and

Noncommutative Geometry

In this chapter we focus on introducing the ideas from noncommutative geometry that are used to study fractal sets. We begin with the motivation for the field of noncommutative geometry and then define the tools of primary interest—spectral triples and the Dixmier trace. We then state some known results on how to use spectral triples to study fractal sets like the Cantor set and the Sierpinski gasket.

3.1 C^*-algebras

Noncommutative geometry begins with the observation of the duality between the category of compact Hausdorff spaces and commutative unital C^*-algebras. Let us begin with some definitions.

Definition 3.1.0.1 *An complex algebra $\mathscr{A}$ which is also a Banach space with norm $\|\cdot\|$,*

satisfying

$$\|xy\| \leq \|x\|\|y\|$$

*for all $x, y \in \mathscr{A}$ is called a **complex Banach algebra**. An **involution** on a complex Banach algebra $\mathscr{A}$ is a map $* : \mathscr{A} \to \mathscr{A}$ written $A \mapsto A^*$ such that for $a, b \in \mathbb{C}$, $S, T \in \mathscr{A}$:*

1. *$(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$,*

2. *$(ST)^* = T^*S^*$, and*

3. *$(S^*)^* = S$.*

Notice that the condition $\|xy\| \leq \|x\|\|y\|$ implies that multiplication $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is continuous in a Banach algebra. Also note that the axioms and notation for the involution operation are meant to generalize the properties of the adjoint operation on the set of bounded operators on a Hilbert space.

Definition 3.1.0.2 *Let $\mathscr{A}$ and $\mathscr{B}$ be complex Banach algebras with involution. A map $\phi : \mathscr{A} \to \mathscr{B}$ is a $*$-**homomorphism** if*

1. *ϕ is a homomorphism of algebras which preserves the multiplicative unit 1, and*

2. *ϕ has the property $\phi(A^*) = \phi(A)^*$ for all $A \in \mathscr{A}$.*

*A $*$-**isomorphism** is a $*$-homomorphism which is also an isomorphism. If $\mathscr{A} \subseteq \mathscr{B}$, we call $\mathscr{A}$ a $*$-**subalgebra** of $\mathscr{B}$ if $\mathscr{A}$ is a subalgebra of $\mathscr{B}$ closed under the $*$ operation: $A \in \mathscr{A}$ implies $A^* \in \mathscr{A}$.*

Definition 3.1.0.3 *A C^*-**algebra** is a complex Banach algebra, $\mathscr{A}$, with an involution satisfying:*

$$\|T^*T\| = \|T\|^2$$

for $T \in \mathscr{A}$.

Notice that in a C^*-algebra we have $\|T\|^2 = \|T^*T\| \leq \|T^*\|\|T\|$ and hence $\|T\| \leq \|T^*\|$. Similarly, $\|T\| \leq \|T^*\|$ so $\|T\| = \|T^*\|$ and hence the involution operation is an isometry. Also note that in a C^*-algebra $\mathscr{A}$, the norm $\|\cdot\|$ is given by

$$\|a\| = \sup_{\|x\| \leq 1} \|ax\| = \sup\left\{\|ax\| : x \in \mathscr{A},\ \|x\| \leq 1\right\}.$$

To see this note that

$$\sup_{\|x\| \leq 1} \|ax\| \leq \sup_{\|x\| \leq 1} \|a\|\|x\| \leq \|a\|$$

and if $x = a^*/\|a\|$ then $\|ax\| = \|a\|$ so indeed $\|a\| = \sup_{\|x\| \leq 1} \|ax\|$.

Example 7 *The complex numbers $\mathbb{C}$ with complex conjugation as involution is a commutative C^*-algebra. Note that the C^*-norm property $\|T^*T\| = \|T\|^2$ is the well know identity*

$$|\bar{z}z| = |z|^2$$

where $z \in \mathbb{C}$.

Example 8 *Given a compact Hausdorff space X the natural complex Banach algebra associated to this space is $C(X)$. We can make $C(X)$ into a C^*-algebra by giving $C(X)$ the supremum norm $\|\cdot\|_\infty$ and the involution $f \mapsto \bar{f}$ where $\bar{f}(x) = \overline{f(x)}$. Then $C(X)$ satisfies the C^*-norm property $\|f^*f\|_\infty = \|f\|_\infty^2$ and hence is a commutative C^*-algebra with a unit.*

Example 9 *Consider the set of bounded linear operators on a Hilbert space $\mathscr{B}(\mathscr{H}, \mathscr{H}) = \mathscr{B}(\mathscr{H})$ with the usual operator norm and the adjoint operation as involution i.e. $T \mapsto T^*$ where T^* is uniquely determined by the property $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for $x, y \in \mathscr{H}$. The space $\mathscr{B}(\mathscr{H})$ is an important example of a noncommutative C^*-algebra.*

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Example 10 *Consider the collection of bounded analytic functions on the closed unit disk D with the sup norm $\|\cdot\|_\infty$ and involution given by, $f^*(z) = \overline{f(\overline{z})}$. This is an example of a complex Banach algebra with involution, which does not satisfy the C^*-norm property. To see this consider the function $f(z) = e^{iz}$. Then $f^*(z) = \overline{e^{i\overline{z}}} = e^{-iz}$ and note that*

$$\|f(z)f^*(z)\|_\infty = \sup\{|e^{iz}e^{-iz}| : z \in D\} = \sup\{|1| : z \in D\} = 1$$

while

$$\|f(z)\|_\infty^2 = \sup\{|e^{iz}|^2 : z \in D\}$$

$$= \sup\{e^{-i\overline{z}+iz} : z \in D\}$$

$$= \sup\{e^{-2Im(z)} : z \in D\}$$

$$= \sup\{e^{-2a} : a \in [-1,1]\}$$

$$= e^2.$$

This shows that the condition $\|f^* f\|_\infty = \|f\|_\infty^2$ fails in general and hence the collection of bounded analytic functions on the closed unit disk D is not a C^*-algebra.

3.1.1 Gelfand Gymnastics

The following theorem tell us that every commutative unital C^*-algebra is of the form $C(X)$ where X is a compact Hausdorff space.

Theorem 3.1.1.1 (Gelfand Naimark Theorem) *Suppose $\mathscr{A}$ is a commutative C^*-algebra with unit $1_\mathscr{A}$. Let*

$$M(\mathscr{A}) = \{\psi : \mathscr{A} \to \mathbb{C} : \psi \text{ is bounded, linear, and multiplicative, and } \psi(1_\mathscr{A}) = 1\}.$$

For each $a \in \mathscr{A}$ define

$$\hat{a} : M(\mathscr{A}) \to \mathbb{C} \quad by \ \ \hat{a}(\psi) = \psi(a).$$

*Then $M(\mathscr{A})$ is a compact Hausdorff space in the weak *-topology and the map $a \mapsto \hat{a}$, called the **Gelfand transform**, is a $*$-isomorphism of $\mathscr{A}$ and $C(M(\mathscr{A}))$, the continuous functions on $M(\mathscr{A})$.*

This theorem is the first step in proving the duality between the category of compact Hausdorff spaces and the category of commutative unital C^*-algebras. This duality essentially gives that all topological information about a compact Hausdorff space X is algebraically stored in $C(X)$. Let us explore this duality a bit further.

Given compact Hausdorff spaces X, Y and a continuous map $f : X \to Y$ we get a map of algebras

$$Cf : C(Y) \to C(X) \text{ given by } Cf(g) = g \circ f.$$

The map Cf is a unital $*$-homomorphism and the mappings

$$X \mapsto C(X) \quad \text{and} \quad f \mapsto Cf$$

give a contravariant functor between the category of compact Hausdorff spaces and continuous maps and the category of commutative unital C^*-algebras and unital $*$-homomorphism.

Now, given unital commutative C^*-algebras $\mathscr{A}, \mathscr{B}$ and a unital $*$-homomorphism $\phi : \mathscr{A} \to \mathscr{B}$ we get a map of compact spaces

$$M\phi : M(\mathscr{B}) \to M(\mathscr{A}) \text{ given by } M\phi(\mu) = \mu \circ \phi.$$

Since the topology on $M(\mathscr{A})$ is the smallest such that each $\hat{a} : M(\mathscr{A}) \to \mathbb{C}$ is continuous, a map $f : X \to M(\mathscr{A})$ (where X is a compact Hausdorff space) is continuous if and only

if $\hat{a} \circ f : X \to \mathbb{C}$ is continuous. Then the map $M\phi$ is continuous since $\hat{a} \circ M\phi = \widehat{\phi(a)}$ is continuous. Thus we have a contravariant functor given by

$$\mathscr{A} \mapsto M(\mathscr{A}) \quad \text{and} \quad \phi \mapsto M\phi$$

between the category of commutative unital C^*-algebras and unital $*$-homomorphism and the category of compact Hausdorff spaces and continuous maps.

To fully see the connection between the category of compact Hausdorff spaces and the category of commutative unital C^*-algebras we have the following well known result which can be found in [33].

Theorem 3.1.1.2 *The map $Ev_X : X \to M(C(X))$, $Ev_X(x)(f) = f(x)$, where X is compact, is a homeomorphism of compact topological spaces.*

We now have the following commuting diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
{\scriptstyle Ev_X}\downarrow & & \downarrow{\scriptstyle Ev_Y} \\
M(C(X)) & \xrightarrow{\quad MCf \quad} & M(C(Y))
\end{array}
$$

where X and Y are compact Hausdorff spaces. The map Ev is a natural transformation between the identity functor and the functor MC on the category of compact spaces. The Gelfand Naimark theorem gives an isomorphism of $\mathscr{A}$ and $C(M(\mathscr{A}))$, so we have the diagram:

$$
\begin{array}{ccc}
\mathscr{A} & \xrightarrow{\quad \phi \quad} & B \\
{\scriptstyle \wedge}\downarrow & & \downarrow{\scriptstyle \wedge} \\
C(M(\mathscr{A})) & \xrightarrow{\quad CM\phi \quad} & C(M(\mathcal{B}))
\end{array}
$$

Thus the Gelfand transform is a natural transformation between the identity functor and the functor CM on the category of commutative unital C^*-algebras.

The duality between these categories tells us that to study compact Hausdorff topological spaces, like our fractal sets, one can study the corresponding commutative unital C^*-algebra. The next step towards the study of noncommutative geometry is to lift the condition of commutativity and consider C^*-algebras in general. For this, the more general theorem of Gelfand and Naimark is essential.

Definition 3.1.1.3 *A **representation** of a C^*-algebra $\mathscr{A}$ on a Hilbert space $\mathscr{H}$ is a $*$-homomorphism, $\phi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$. If in addition ϕ is injective we call ϕ a **faithful representation**.*

Theorem 3.1.1.4 (The Gelfand Naimark Theorem) *Every C^*-algebra has a faithful representation in a Hilbert space.*

What this theorem gives us is that any C^*-algebra is $*$-isomorphic to a closed subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. In noncommutative geometry one studies noncommutative C^*-algebras and hence "noncommutative topological spaces". This has lead to many interesting results and examples, as well as to the development of operator algebraic tools to study geometry. Noncommutative fractal geometry means to use these algebraic tools to study fractal sets.

3.1.2 Basics of Operator Algebras

Because in general C^*-algebras look like closed subalgebras of $\mathscr{B}(\mathscr{H})$, we will use many standard tools from the theory of operator algebras. Before proceeding, let us recall some basic definitions and results from operator algebras. As a reference for this section

one can see the text of Reed and Simon, [29]. Unless otherwise stated all Hilbert spaces in this text are seperable.

Definition 3.1.2.1 *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Let $T : D(T) \subseteq \mathscr{H} \to \mathscr{K}$ be a linear map with domain $D(T)$ a linear subspace of $\mathscr{H}$.*

*(a) We say T is **closed** if its graph $G(T) = \{(x, Tx) : x \in D(T)\}$ is closed in $\mathscr{H} \times \mathscr{K}$.*

*(b) If T is an unbounded operator we say T is **densely defined** if $D(T)$ is dense in $\mathscr{H}$.*

*(c) We say T_0 **extends** T, written $T \subset T_0$, when $D(T) \subseteq D(T_0)$ and $T_0 x = Tx$ for all $x \in D(T)$.*

*(d) If T is not closed but there exists an operator $\overline{T}$ with $G(\overline{T}) = \overline{G(T)}$, then we say $\overline{T}$ is the **closure** of T.*

Next we define the adjoint of an operator which is not necessarily defined on all elements of a Hilbert space. Let $T : D(T) \subseteq \mathscr{H} \to \mathscr{K}$ be a linear operator. Consider the set

$$\{y \in \mathscr{K} : \ x \mapsto \langle Tx, y \rangle \text{ is continuous for all } x \in D(T)\}$$

and note that a functional $x \mapsto \langle Tx, y \rangle$ on $D(T)$ extends to a continuous linear functional on $\mathscr{H}$ by the Hahn-Banach theorem. Therefore there is an element $z \in \mathscr{H}$ such that $\langle Tx, y \rangle = \langle x, z \rangle$ we will define the adjoint of T by the mapping $y \mapsto z$.

Definition 3.1.2.2 *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Let $T : D(T) \subseteq \mathscr{H} \to \mathscr{K}$ be a linear map with domain $D(T)$. Define the **adjoint** T^* of T as follows: Let*

$$D(T^*) = \{y \in \mathscr{K} : \ x \mapsto \langle Tx, y \rangle \text{ is continuous for all } x \in D(T)\}$$

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and let $T^*y = z$ where $z \in \mathscr{H}$ is the unique element in $\mathscr{H}$ such that $\langle Tx, y \rangle = \langle x, z \rangle$ for all $x \in \mathscr{H}$. The operator T is called **symmetric** if $T \subseteq T^*$. Equivalently, T is symmetric if and only if $\langle T\phi, \psi \rangle = \langle \phi, T\psi \rangle$ for $\phi, \psi \in D(T)$.

We have the following consequences of the above definitions.

1. If $\langle Tx, y \rangle = \langle x, z \rangle = \langle x, z' \rangle$ for all $x \in D(T)$ then as $D(T)$ is dense in $\mathscr{H}$ there exist $x_n \in D(T)$ such that $x_n \to z - z'$. Then $\langle z - z', z - z' \rangle = \lim \langle x_n, z - z' \rangle = 0$. Thus $z' = z$ and T^* is well-defined.

2. The relation $\langle Tx, y \rangle = \langle x, T^*y \rangle$ holds only when $x \in D(T)$ and $y \in D(T^*)$.

3. If T extends S then S^* extends T^*.

4. If T is densely defined, T^* is a closed linear operator.

Definition 3.1.2.3 *A densely defined operator $T : D(T) \subseteq \mathscr{H} \to \mathscr{H}$ is **self-adjoint** if $T = T^*$, that is, if and only if T is symmetric and $D(T^*) = D(T)$. A symmetric, densely-defined operator is **essentially self-adjoint** when it has a unique self-adjoint extension.*

In using operator algebras to study fractal sets, we will need operators with well behaved spectra. For this reason we will mainly use compact operators which are characterized by having the type of spectrum that will be most useful for our purposes.

Definition 3.1.2.4 *Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. Let $T : D(T) \subseteq \mathscr{H} \to \mathscr{K}$ be a linear map with domain $D(T)$.*

1. *If for $\lambda \in \mathbb{C}$, $\lambda I - T$ is a bijection with bounded inverse, we say λ is in the **resolvent set** of T, denoted $\rho(T)$. In this case we call the operator $R_\lambda(T) = (\lambda I - T)^{-1}$ the **resolvent of T at** λ.*

2. If $\lambda I - T$ is not invertible then λ is in the **spectrum** of T, denoted $sp(T) = \sigma(T)$.

   (a) If $\lambda \in sp(T)$ and $\lambda I - T$ is not one-to-one then λ is called an **eigenvalue** of T.

   (b) If $\lambda \in sp(T)$ is not an eigenvalue and $Ran(\lambda I - T)$ is not dense, then λ is said to be in the **residual spectrum**.

Continuing with some basic definitions from operator algebras, we now define what it means for an operator to be positive and give some facts about the spectrum of a bounded operator on a Hilbert space.

Definition 3.1.2.5 Let $T \in \mathscr{B}(\mathscr{H})$. We say T is **positive** and write $T \geq 0$ if $\langle Tx, x \rangle \geq 0$ for all $x \in \mathscr{H}$.

Theorem 3.1.2.6 Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$.

1. Then $sp(T) \neq \emptyset$.

2. If T is self adjoint, $sp(T) \subseteq \mathbb{R}$ and T has no residual spectrum.

3. If T is positive then T is self adjoint and $sp(T) \subset \mathbb{R}^+$.

4. Eigenvectors corresponding to distinct eigenvalues of T are orthogonal.

We now introduce a class of operators which will be essential in the sections that follow.

Theorem 3.1.2.7 Let $\mathscr{H}$ be a Hilbert space and $T \in \mathscr{B}(\mathscr{H})$. The following are equivalent:

1. T is continuous as a map from the unit ball $(\mathscr{H})_1$ (with the weak*-topology) into $\mathscr{H}$ (with the norm topology).

2. If $x, x_1, x_2, x_3 \cdots \in \mathscr{H}$ and the x_j tend to x weakly, then $Tx_j \to Tx$ in norm.

3. Every bounded sequence (x_j) in $\mathscr{H}$ has a subsequence x_{j_k} for which (Tx_{j_k}) converges.

If T satisfies one of the above conditions, T is called a **compact operator**. We denote the collection of compact operators on $\mathscr{H}$ by $\mathcal{K}(\mathscr{H}) = \mathcal{K}$.

Example 11

1. Fix $y, z \in \mathscr{H}$ and define $Tx = \langle x, y \rangle z$. Then T is a compact operator.

2. For $x \in L^2[a, b]$, consider $(Tx)(s) = \int_a^b k(s, t) x(t) dt$ where $k(s, t)$ continuous. One can show using the Arzela Ascoli theorem that T is compact.

3. Projections (i.e. operators P with $P^2 = P$) onto finite dimensional subspaces of a Hilbert space are compact.

One can show that the set of compact operators, $\mathcal{K}$, on a Hilbert space $\mathscr{H}$ is a closed vector subspace of $\mathscr{B}(\mathscr{H})$. What's more, $\mathcal{K}$ is a two sided ideal in $\mathscr{B}(\mathscr{H})$ meaning that if $T \in \mathcal{K}$ and $B \in \mathscr{B}(\mathscr{H})$, then TB and BT are compact.

We are interested in compact operators because of the following property of their spectrum.

Theorem 3.1.2.8 *The spectrum of a compact operator consists of countably many eigenvalues, and has at most one limit point, namely 0.*

3.2 Spectral Triples

In this section we introduce the primary tool from operator algebras used to study fractal geometry–the spectral triple. We also define the Dixmier trace which will be used

to define the measure induced by a spectral triple. Examples will include spectral triples for fractals like the Cantor set in $\mathbb{R}$, for curves, and for a certain class of sets built out of curves (like the Sierpinski and stretched Sierpinski gasket).

3.2.1 Spectral Triples

We use the notation $[A, B] := AB - BA$ for the commutator of two operators A, B on a Hilbert space. Also, given a Hilbert space $\mathscr{H}$ we write $\mathscr{B}(\mathscr{H})$ for the space of bounded operators on $\mathscr{H}$.

Definition 12 *A **spectral triple** $(\mathscr{A}, \mathscr{H}, D)$ is a collection of three objects*

- *$\mathscr{A}$ a unital C^*-algebra,*

- *$\mathscr{H}$ a Hilbert space which carries a unital faithful representation $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, and*

- *an unbounded, essentially self-adjoint, operator D with domain, $Dom(D) \subseteq \mathscr{H}$, such that*

  (a) the set

  $$\{a \in \mathscr{A} : [D, \pi(a)] \text{ is densely defined and has a bounded extension to } \mathscr{H}\},$$

  is dense in $\mathscr{A}$, and

  (b) the operator $(I + D^2)^{-1}$ is compact.

The C^*-algebra $\mathscr{A}$ will often be $C(X)$, where X is a compact Hausdorff space. The operator $[D, \pi(a)]$, for $a \in \mathscr{A}$, will act like the "derivative" of the element a and the dense set in condition (a) will act like the set of C^1 functions in $C(X)$. The operator D

and its eigenvalues will be the key to recovering geometric information like dimension and measure. The condition that the operator $(I + D^2)^{-1}$ be compact ensures that the spectrum of the operator D^{-1} is made up of only eigenvalues and that the only possible accumulation point of these eigenvalues is 0. We may then consider infinite sums of these eigenvalues.

Using the three tools in a spectral triple, one can define notions of dimension, metric, and measure on a compact Hausdorff space X.

Definition 13 *Given a spectral triple* $(C(X), \mathscr{H}, D)$, *the number*

$$\mathfrak{d} = \mathfrak{d}(X) := \inf\{p > 0 : \operatorname{tr}((I + D^2)^{-p/2}) < \infty\}$$

*is the **spectral dimension** (or **metric dimension**) of the space X.*

Note that condition (b) in the definition of a spectral triple is needed so that the trace in the definition of spectral dimension has a possibility of being finite. A priori there is no reason why the spectral dimension $\mathfrak{d}$ should be finite.

We next define a notion of distance induced by a spectral triple. The definition will look familiar to those who know of metrics on state spaces. For more on this, see the works of Marc Rieffel in [30], [31], [32] and of Alain Connes in [10].

Definition 14 *Given a spectral triple* $(C(X), \mathscr{H}, D)$, *define the **spectral distance** by*

$$d_X(x, y) = \sup\{|f(x) - f(y)| : f \in C(X), \|[D, \pi(f)]\| \leq 1\},$$

for $x, y \in X$.

Using a spectral triple and another notion from noncommutative geometry we can define a notion of measure. For this we must introduce the concept of the Dixmier trace.

3.2.2 The Dixmier Trace

As a reference for the following discussion, one can see the text *Noncommutative Geometry* by Alain Connes [9]. This text of Connes serves as the standard reference for noncommutative geometry. To define the Dixmier trace we will need extensions of the usual limiting operation. For this we use extended limits. Extended limits are extensions to l^∞ of the usual limit functional acting on c, the space of convergent sequences. By Hahn Banach the classical limit on c extends to l^∞, denoted Lim, and $|Lim(x)| \le \|x\|_\infty$ for all $x \in l^\infty$.

Definition 3.2.2.1 *A positive linear functional ϕ on a von Neumann algebra $\mathcal{N}$ is a **state** if $\phi(1) = 1$.*

Extended limits are states on l^∞ since $Lim(1) = 1$ and are characterized by the fact that they vanish on c_0. In other words, a state ϕ on l^∞ vanishes on c_0 if and only if ϕ is an extension of the classical limit to l^∞ (i.e. $\phi = Lim$). Note that every state on l^∞ is continuous:

$$|\phi(x)| \le |\phi(1 \cdot \|x\|_\infty)| \le \|x\|_\infty$$

where $x = \{x_n\}_{n=1}^\infty \in l^\infty$. This means it is enough for a state to vanish on sequences with finitely many non-zero entries in order for the state to be an extended limit.

Definition 3.2.2.2 *Let w be a state on the von Neumann algebra l^∞. Then w is called an **extended limit** if it vanishes on every sequence with finitely many non-zero entries in l^∞.*

We will need our extended limits to satisfy a certain dilation property. The discrete dilation semigroup $\sigma_k : l^\infty \to l^\infty$ for $k \in \mathbb{N}$ acts by the formula

$$\sigma_k(x) = (x_0, x_0, \ldots, x_0, x_1, x_1, \ldots, x_1, \ldots)$$

where $x \in l^\infty$ and each x_j appears k times. We will use 2-dilation invariant extended limits. That is extended limits, $w : l^\infty \to \mathbb{R}$, which satisfy

$$w(\sigma_2(x)) = w(x).$$

The fact that dilation invariant extended limits exist, follows from a dilation invariant version of the Hahn Banach theorem. The proof of this version of the Hahn Banach Theorem can be found in the text [11] by Edwards, Theorem 3.3.1.

Theorem 3.2.2.3 (Invariant Hahn Banach Theorem) *Let X be a linear space and G be a commutative semigroup. Given*

(a) an action $g : x \to g(x)$ of G on X

(b) a G-invariant subspace Y of X

(c) a convex homogeneous functional $p : X \to \mathbb{R}$ such that $p \circ g \leq p$ for every $g \in G$.

(d) a G invariant linear functional $w : Y \to \mathbb{R}$ such that $w \leq p$.

then there exists a G invariant extension $w : X \to \mathbb{R}$ such that $w \leq p$.

Corollary 3.2.2.4 *Dilation invariant extended limits exist on l^∞.*

The space in the definition that follows is an ideal in the set of compact operators and will serve as the domain of the Dixmier trace. For a compact operator T, denote by $\mu_j(T)$ the eigenvalues of $|T|$ ordered so that $0 \leq \mu_{j+1}(T) \leq \mu_j(T)$ for $j \in \mathbb{N}$.

Definition 3.2.2.5 *Let $w : l^\infty \to \mathbb{R}$ be a linear functional which vanishes on c_0 and satisfies for $x \in l^\infty$, $w(\sigma_2(x)) = w(x)$. Define*

$$\mathcal{L}^{(1,\infty)} = \{T \in \mathcal{K} : \|T\|_{(1,\infty)} := \sup_N \frac{1}{\log(1+N)} \sum_{j=1}^N \mu_j(T) < \infty\}.$$

The **Dixmier trace** of $T \in \mathcal{L}^{(1,\infty)}$ where $T \geq 0$, is given by

$$Tr_w(T) = w \left\{ \frac{1}{\log(1+N)} \sum_{j=1}^{N} \mu_j(T) \right\}.$$

Define Tr_w for self-adjoint operators and then for arbitrary operators by linearity.

The sequence

$$\left\{ \frac{1}{\log(1+N)} \sum_{j=1}^{N} \mu_j(T) \right\}_{N=1}^{\infty}$$

does not always converge as $N \to \infty$, so $Tr_w(T)$ may depend on the extended limit w. In most applications we can show independence of $Tr_w(T)$ from w. Much like the usual operator trace, the Dixmier trace has various useful properties.

Proposition 3.2.2.6 *[9]*

1. $Tr_w(\cdot)$ *is a positive linear functional on the ideal of operators T for which $\mu_j(T) = O(n^{-1})$.*

2. $Tr_w(ST) = Tr_w(TS)$ *for all compact operators T with $\mu_j(T) = O(n^{-1})$ and $S \in \mathcal{B}(\mathcal{H})$.*

3. $Tr_w(\cdot)$ *vanishes on compact operators T with $\mu_j(T) = O(n^{-\alpha})$ for $\alpha > 1$. i.e. $Tr_w(T) = 0$ if $n\mu_n \to 0$ as $n \to \infty$.*

A result of Connes is that for a suitable choice of spectral triple, the map $Tr_w(\pi(f)|D|^{-\partial})$ is a non-trivial positive linear functional on $C(X)$ and hence induces a measure; see [9]. This is how we will use a spectral triple to induce a measure on a fractal set. Now that we have all the necessary tools, we can begin to explore how to use these to study fractal geometry. See [28] for more on the theory of singular traces such as the Dixmier trace. The following

theorem of Alain Connes in [9] is often used to compute the Dixmier trace as the residue of a certain series. We will make use of this theorem in the sections that follow.

Theorem 15

For $T \geq 0$, $T \in \mathcal{L}^{(1,\infty)}$, the following two conditions are equivalent:

1.

$$(s - 1) \sum_{n=0}^{\infty} \mu_n(T)^s \to L \text{ as } s \to 1^+;$$

2.

$$\frac{1}{\log(N + 1)} \sum_{n=1}^{N} \mu_n(T) \to L \text{ as } N \to \infty.$$

3.2.3   Spectral Triple for Cantor Sets

First we give an example of a spectral triple for Cantor type sets in $\mathbb{R}$ and show how one can use a spectral triple to recover the Hausdorff measure on these sets. We also make a connection between a constant arising from the measure induced by a spectral triple and the average Minkowski content of Cantor type sets.

We now introduce some basic terminology and results from the study of fractal strings. A reference for the theory of fractal strings is [27].

It is known that a bounded open subset of $\mathbb{R}$ can be written as a union of countably many open intervals with lengths $\ell_1, \ell_2, \ell_3, \ldots$. We order the lengths ℓ_j so that they are non-increasing and counted according to multiplicity. In the literature on fractal strings, one allows for bounded open subset of $\mathbb{R}$ which are the union of *finitely* many open intervals. For our purposes we will exclude this case and define fractal strings as follows.

Definition 3.2.3.1

1. A **fractal string** $\mathcal{L}$ is a bounded open subset of $\mathbb{R}$ which can be written as a union of a countably infinite number of open intervals. We denote the length of these intervals by $\{\ell_j\}_{j=1}^{\infty}$ where we have ordered the lengths so that they are non-increasing and counted according to multiplicity.

2. Let $\epsilon > 0$. For a fractal string $\mathcal{L}$, define the **dimension of** $\mathcal{L}$ by

$$D_{\mathcal{L}} = \inf\{\alpha \geq 0 : V(\epsilon) = O(\epsilon^{1-\alpha}) \text{ as } \epsilon \to 0\}$$

where

$$V(\epsilon) = vol_1\{x \in \mathcal{L} : d(x, \partial\mathcal{L}) < \epsilon\}$$

and vol_1 refers to one dimensional Lebesgue measure in $\mathbb{R}$.

3. For a fractal string $\mathcal{L}$ we define the corresponding **geometric zeta function** of $\mathcal{L}$ to be

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} \ell_j^s$$

where $s \in \mathbb{C}$ and $Re(s) > D_{\mathcal{L}}$.

We will work with examples of fractal stings that have a rich self-similar structure.

Definition 3.2.3.2 Given a closed interval I of length L, a **self-similar string** $\mathcal{L}$ is constructed as follows. Let $N \geq 2$ and $\phi_j : I \to I$ for $j = 1, 2, \ldots N$ be contraction similarities with similarity ratios $0 < r_j < 1$. We name the ϕ_j so that the r_j are non-increasing. Assume that

$$\sum_{j=1}^{N} r_j < 1$$

and that the $\phi_j(I)$ do not overlap except possibly at endpoints.

One can subdivide I into the pieces $\phi_j(I)$ with the remaining pieces being the first pieces of the string with lengths $\ell_k = Lg_k$ for $k = 1, 2, \ldots K$. Here the g_k are the lengths of the gaps between the pieces $\phi_j(I)$. We say that a self similar fractal string is **lattice** if there is some $0 < r < 1$ with $r_j = r^{n_j}$ for some $n_j \in \mathbb{N}$ and for each $j = 1, 2, \ldots, N$.

A concrete example of a lattice self-similar fractal string is the complement of the Cantor middle third set in the interval $[0, 1]$. More generally one can consider a totally disconnected subset of $[0, 1]$ with no isolated points, and it's fractal string will be the complement of K in $[0, 1]$.

Let K be the totally disconnected subset of $[0, 1]$ with no isolated points, associated to a fractal string, $\mathcal{L}$. Let $\mathscr{H} = l^2(D)$ where D is the set of endpoints of the intervals $I_j = (b_j^-, b_j^+)$ of a fractal string and $F := 2P - 1$ where P is projection onto the subspace

$$P\mathscr{H} = \{\phi \in \mathscr{H} : \phi(b_j^-) = \phi(b_j^+), \ b_j^\pm \in D\}.$$

Proposition 3.2.3.3 (3. ϵ Proposition 21) *[9]*

(a) The pair $(\mathscr{H}, F)$ is a Fredholm module over $C(K)$.

(b) The eigenvalues of the operator $|dx| = ||[F, x]||$ where $x \in C(K)$ is the embedding of K in $\mathbb{R}$, are the lengths $l_j = L(I_j)$ of the intervals I_j, each with multiplicity 2.

Define for a compact operator T,

$$\zeta_T(s) = tr(T^s) = \sum_{j=1}^{\infty} \mu_j(T)^s$$

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where the $\mu_i(T)$ are the characteristic values of T (i.e. the eigenvalues of $|T| = (T^*T)^{1/2}$) and $s \in \mathbb{C}$. Since the above result gives that the eigenvalues of $|dx|$ have multiplicity 2 we have

$$\zeta_{|dx|}(s) = \sum_{j=1}^{\infty} \mu_j(|dx|)^s = 2\sum_{j=1}^{\infty} l_j^s = 2\zeta_{\mathcal{L}}(s)$$

where $\zeta_{\mathcal{L}}$ is the fractal zeta function for the set K.

Proposition 3.2.3.4 (Connes [9]) *Let K be a totally disconnected subset of $[0,1]$ with no isolated points and $\mathcal{L}$ the associated fractal string. Then for $f \in C(K)$,*

$$Tr_w(f|dx|^D) = c \int_K f \, d\mathcal{H}^D$$

where $\mathcal{H}^D$ is D dimensional Hausdorff measure, D is the Minkowski dimension of K, and c is some fixed constant not depending on f.

It can be shown that a lattice self-similar string does not have a Minkowski content. In this case one can consider the **average Minkowski content** of $\mathcal{L}$ given by

$$\mathcal{M}_{av} = \lim_{T \to \infty} \frac{1}{\log T} \int_{1/T}^{1} \epsilon^{-(1-D_{\mathcal{L}})} V(\epsilon) \frac{d\epsilon}{\epsilon}.$$

One has the following result.

Theorem 3.2.3.5 (Theorem 8.30 [27]) *Let $\mathcal{L}$ be a lattice self-similar string of total length L, with scaling ratios $r_1 = r^{k_1}, \ldots, r_N = r^{k_N}$ and gaps $g_1, \ldots, g_K$. Then the average Minkowski content of $\mathcal{L}$ exists and is given by the finite positive number*

$$\mathcal{M}_{av} = \frac{2^{1-D} \sum_{j=1}^{K} (g_i L))^D}{D(1-D)\log(r^{-1}) \sum_{j=1}^{N} k_j r^{k_j D}} = \frac{2^{1-D}}{D(1-D)} res(\zeta_{\mathcal{L}}(s); D).$$

Using these results we make the following connection between the geometric object, Minkowski content, and a constant that comes from our operator algebraic measure.

Proposition 3.2.3.6 *In the case of a lattice fractal string which is not Minkowski measurable, the average Minkowski content is equal to the Dixmier trace of the operator $|dx|^{-D}$ up to a constant depending on the dimension of the string.*

Proof. We have

$$\zeta_{|dx|}(s) = 2\zeta_{\mathcal{L}}(s)$$

and in the case of a lattice fractal string this gives

$$\mathcal{M}_{av} = \frac{2^{1-D}}{D(1-D)} res(\zeta_{\mathcal{L}}(s); D) = \frac{2^{1-D}}{2(1-D)} \mathrm{Tr}_w(|dx|^{-D}).$$

∎

Example 16 *Cantor String*

Using the above results we have

$$\mathrm{tr}(|dx|^s) = 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{ns} (2^{n-1}) = \sum_{n=1}^{\infty} \left(\frac{2}{3^s}\right)^n = \frac{2}{3^s - 2}$$

and $D = \frac{\log(2)}{\log(3)} = \log_3(2)$. Then

$$Tr_w(|dx|^D) = \frac{1}{D} res(\zeta(s), D) = \frac{1}{D}\left(\frac{2}{3^D \log(3)}\right) = \frac{1}{\log(2)}.$$

On the other hand

$$\mathcal{M}_{av} = \frac{2^{1-D}\left(\frac{1}{3^D}\right)}{D(1-D)\log(3)\left(\frac{2}{3^D}\right)}) = \frac{2^{1-D}}{2D(1-D)\log(3)} = \frac{2^{1-D}}{2(1-D)\log(2)}.$$

Example 17 *Fibonacci String*

Denote by F_n the Fibonacci numbers so $F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$ and let $\phi = \frac{1+\sqrt{5}}{2}$, the Golden ratio. Using the above results we have

$$\mathrm{tr}(|dx|^s) = 2\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{ns} F_n = \frac{2}{1 - 2^{-s} - 4^{-s}}$$

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and $D = \frac{\log \phi}{\log 2} = \log_2 \phi$. Then

$$\frac{1}{D} res(\zeta(s), D) = \frac{2}{(\phi^{-1} + 2\phi^{-2}) \log \phi} = \frac{2\phi^2}{(\phi + 2) \log \phi} = \frac{2(2 + \phi)}{5 \log \phi}.$$

On the other hand

$$\mathcal{M}_{av} = \frac{2^{1-D}(2 + \phi)}{5(1 - D) \log \phi}.$$

3.2.4 Spectral Triples for some Fractal Sets Built on Curve

The following constructions and results are due to Christensen, Ivan, and Lapidus in [8] and Lapidus and Sarhad in [26]. These papers provide a way of building a spectral triple for a variety of spaces, which in some cases recovers notions of dimension, metric, and measure. A key example in these papers is the Sierpinski gasket and the Harmonic Sierpinski gasket which we will define in a later section.

We assemble a spectral triple for fractals like the Sierpinski gasket by defining a triple for the basic pieces in the gasket and then collecting these triples (i.e. taking a direct sum) to build an triple for the entire set. One way to approach this is to take the triangles in the gasket, build a triple for them and then collect the triples. Another approach is to first think of the Sierpinski gasket as the increasing union of graphs. One then builds a triple for each edge in the approximating graphs and then collects the triples. Following this strategy, we will first define spectral triples for curves.

Definition 18 *Let X be a compact Hausdorff space, $\ell > 0$, and $R : [0, \ell] \to X$ a continuous injective map. Then a spectral triple for the R-curve is*

- $C(X)$;

- $\mathscr{H}_\ell := L^2\left([-\ell, \ell], (2\ell)^{-1}m\right)$ where $(2\ell)^{-1}m$ is the normalized Lebesgue measure and the representation is given by $\pi_\ell : C(X) \to \mathscr{B}(\mathscr{H}_\ell)$,

$$\pi_\ell(f)h(x) := f(R(|x|))h(x);$$

- $D_\ell := D + \frac{\pi}{2\ell}I$ where D is the closure of the operator $-i\frac{d}{dx}$ restricted to the linear span of the set $\{\phi_k^\ell = e^{i\pi kx/\ell} : k \in \mathbb{Z}\}$. That is, $D = \overline{-i\frac{d}{dx}|_{span(\phi_k^\ell)}}$.

Note that $\{\phi_k^\ell(x) = e^{i\pi kx/\ell}\}_{k \in \mathbb{Z}}$ is an orthonormal basis for $\mathscr{H}_\ell$ and that these are eigenfunctions of the operator $-i\frac{d}{dx}$, with eigenvalues $\{\frac{\pi k}{\ell} : k \in \mathbb{Z}\}$. We consider functions in $\mathscr{H}_\ell$ as restrictions of 2ℓ-periodic functions on $\mathbb{R}$ and hence the operator D_ℓ has periodic boundary conditions. The translation in the definition of the operator D_ℓ is needed in order to ensure that 0 is not an eigenvalue of the operator. This allows us to talk about the eigenvalues of the operator $|D_\ell|^{-1}$.

The eigenvalues of D_ℓ are

$$\sigma(D_\ell) = \left\{\frac{(2k+1)\pi}{2\ell} : k \in \mathbb{Z}\right\}$$

and the operator D_ℓ can be defined for $f \in L^2[-\ell, \ell]$ by

$$D_\ell f = \sum_{k \in \mathbb{Z}} \frac{(2k+1)\pi}{2\ell}\langle f, \phi_k^\ell \rangle \phi_k^\ell,$$

where we say that f is in the domain of D_ℓ, written $\mathrm{dom}(D_\ell)$, if

$$\|D_\ell f\|_2^2 = \sum_{k \in \mathbb{Z}} \left|\frac{(2k+1)\pi}{2\ell}\right|^2 |\langle f, \phi_k^\ell \rangle|^2 < \infty.$$

We think of functions in $C(X)$ as functions in $L^2([-\ell, \ell])$ by working with $f(R(|x|)) \in L^2([-\ell, \ell])$ rather than $f \in C(X)$. In particular, note that we care about the a.e. *equivalence class* of the function $f(R(|x|))$ in $L^2([-\ell, \ell])$. It is also important to note that for

functions $f(R(|x|)) \in C^1([-\ell, \ell])$ and $g \in C^1([-\ell, \ell]) \subseteq L^2([-\ell, \ell])$ we have

$$[D_\ell, \pi_\ell(f)]g = \pi_\ell\left(-i\frac{df}{dx}\right)g = \pi_\ell(Df)g.$$

This shows that the operator $[D_\ell, \pi_\ell(f)]$ is densely defined and extends to the bounded

operator $\pi_\ell(Df)$ on $L^2([-\ell, \ell])$. Proposition 4.1 in [8] shows that the set in condition (a)

of the definition of a spectral triple, is dense in $C(X)$. It follows that the above is indeed a

spectral triple for the R-curve.

The following lemma was stated in [8].

Lemma 19 *Let $f : [-\ell, \ell] \to \mathbb{C}$ be a continuous function. Then the following are equivalent:*

1. *$[D_\ell, \pi_\ell(f)]$ is densely defined and bounded.*

2. *$f \in Dom(D)$ and Df is essentially bounded.*

3. *There exists a measurable, essentially bounded function $g : [-\ell, \ell] \to \mathbb{C}$ such that*

$$\int_{-\ell}^{\ell} g(t) \ dt = 0 \ \text{and for all } x \in [-\ell, \ell]: \quad f(x) = f(0) + \int_0^x g(t) \ dt.$$

If the conditions above are satisfied then $g(x) = (iDf)(x)$ almost everywhere.

Using curve spectral triples, Christensen, Ivan, and Lapidus constructed a spectral

triple for the classical Sierpinski gasket that recovers the Hausdorff dimension, the geodesic

metric, and the $\log_2 3$-dimensional Hausdorff measure. Later, Lapidus and Sarhad used the

spectral triple for an R-curve to build a spectral triple for compact length spaces $X \subseteq \mathbb{R}^N$

satisfying the axioms below. We write $L(\gamma)$ for the length of the path γ parameterized by

arclength.

Axiom 1. $X = \overline{\mathcal{R}}$ where $\mathcal{R} = \bigcup_{j=1}^{\infty} R_j$ and each R_j is a C^1 rectifiable curve such that $L(R_j) \to 0$ as $j \to \infty$.

Axiom 2. There is a dense set $\mathcal{B} \subset X$ which is such that for each $p \in \mathcal{B}$ and $q \in X$ one of the minimizing geodesics from p to q is given by a countable (or finite) concatenation of the R_j's.

Given a compact length space $X = \overline{\bigcup_{j=1}^{\infty} R_j}$ satisfying Axioms 1 and 2, we can consider the **direct sum spectral triple** given by

$$ST(X) = \left(C(X), \; \bigoplus_{j \geq 1} \mathcal{H}_{R_j}, \; D = \bigoplus_{j \geq 1} D_{R_j} \right)$$

where $\mathcal{H}_{R_j}$ and D_{R_j} are the Hilbert space and unbounded operator in the spectral triple for the curve R_j.

Notice that the two Axioms imply that $\mathcal{B}$ is a subset of the set of endpoints of the R_j. It follows that the set of endpoints of the R_j's is dense in X. Proposition 1 in [26] states that for a compact length space X satisfying Axiom 1, the direct sum spectral triples does indeed give a spectral triple for X and the operator D in that spectral triple has eigenvalues

$$\sigma(D) = \bigcup_{j \geq 0} \left\{ \frac{(2k+1)\pi}{2\ell_j} : k \in \mathbb{Z} \right\},$$

where $\ell_j := L(R_j)$. Furthermore, in Theorem 2 of [26] Lapidus and Sarhad prove that for a compact length space X with Axioms 1 and 2, the spectral distance induced by the direct sum spectral triple and the geodesic distance on X are the same:

$$d_X(x, y) = d_{geo}(x, y) \quad \text{for } x, y \in X.$$

This result can be used to show that the direct sum spectral triple for the classical Sierpinski gasket and for the harmonic Sierpinski gasket recovers geodesic distance. If one takes for the curves R_j the edges of the triangles and the joining edges in the stretched Sierpinski gasket, K_α, then Axiom 2 is not satisfied and hence the theorem of Lapidus and Sarhad does not give that the spectral metric is the same as the geodesic metric on K_α. We will prove the recovery of the geodesic distance on K_α in Chapter 5. In addition, we will show that the direct sum spectral triple for K_α recovers the Hausdorff dimension and Hausdorff measure on K_α.

It was conjectured in [26] that the Hausdorff measure on the harmonic Sierpinski gasket K_H with the geodesic metric can be recovered by the direct sum spectral triple via the Dixmier trace. In Chapter 5 we will show that the Dixmier trace recovers the standard self-affine measure on the harmonic Sierpinski gasket but does not recover the Hausdorff measure on K_H. The following chapter will give an introduction to analysis on fractals and will define the Harmonic Sierpinski gasket.

Chapter 4

Analysis on Fractals

We would like to study analysis and differential equations on fractals. In Euclidean space, it is of great interest to study heat equations which involve the classical Euclidean Laplace operator. Since we cannot use the classical Laplacian on fractal spaces, we wishes to define an operator which acts like a Laplacians on fractals and study the analogues heat equations and their solutions. Because of the relationship between the Laplace operator and harmonic functions, it is essential to define and understand harmonic functions in order to define and understand a Laplace operator on fractal sets.

In this chapter we define harmonic functions and Laplace operators on the Sierpinski gasket. We will see how harmonic functions can be used as a smoothing change of coordinates. In fact we can use harmonic functions to define a homeomorphism which smooths the corners in SG and gives a space with the property that any two points in the space can be connected by a C^1-path. The fractal space on which we do our analysis is the Sierpinski gasket. The Sierpinski gasket is a well understood fractal in $\mathbb{R}^2$ and comes
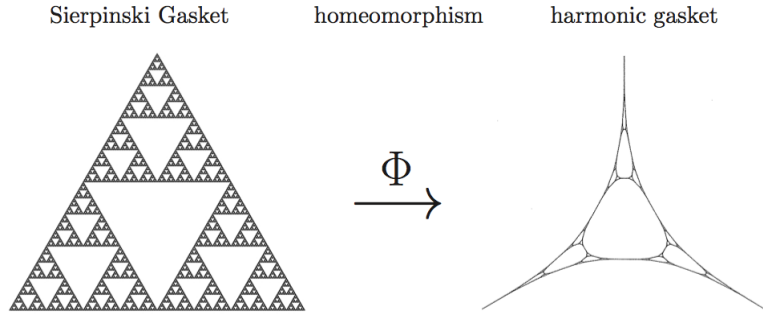
Figure 4.1: The Sierpinski gasket and Harmonic Sierpinski gasket; [26].

with the rich structure induced by its corresponding iterated function system. It is also important that the Sierpinski gasket is "just connected enough" to give nice results. More precisely, fractals which are post critically finite (p.c.f) spaces are well understood from the point of view of analysis on fractals, see [22]. The Sierpinski gasket is the simplest example of a p.c.f. space. References for this section include the text by Kigami [19] and the text by Strichartz [34].

4.1 Energy on the Sierpinski Gasket

Recall that $p_1 = (0,0)$, $p_2 = (1,0)$, $p_3 = (1/2, \sqrt{3}/2)$ are the vertices of an equilateral triangle. Let $V_0 = \{p_1, p_2, p_3\}$ and for $n \geq 1$ define

$$V_n = \bigcup_{w \in \{1,2,3\}^n} f_w(V_0).$$

These are the vertices in the level n approximation to the Sierpinski gasket, SG_n. Let

$$V^* = \bigcup_{n \geq 0} V_n.$$

Definition 20 *Given $f, g : V_n \to \mathbb{R}$ define the **energy** on SG_n by*

$$E_n(f, g) := \sum_{x \sim_n y} (f(x) - f(y))(g(x) - g(y)),$$

where $x \sim_n y$ means $x, y \in F_w(V_0)$ for some $w \in \{1, 2, 3\}^n$. That is, x and y are connected by an n-edge in SG_n. In the sum, we count each pair x, y with $x \sim_n y$ exactly once.

We will focus on the case when $f = g$ so

$$E_n(f) := E_n(f, f) = \sum_{x \sim_n y} (f(x) - f(y))^2.$$

Given $f : V_n \to \mathbb{R}$ we can extend f to V_{n+1} in many ways. If we extend so that $E_{n+1}(f)$ is as small as possible, the extension is called the harmonic extension of f to V_{n+1}.

Definition 21 *A function $f : V_n \to \mathbb{R}$ is **harmonic** if given its values at V_0 it minimizes $E_k(f)$ for each $k = 1, 2, \ldots, n$.*

A calculation shows that for a harmonic function $f : V_{n+1} \to \mathbb{R}$ we have

$$E_n(f) = \frac{5}{3} E_{n+1}(f).$$

Definition 22 *Given $f : V_n \to \mathbb{R}$ define the **renormalized energy** on SG_n by*

$$\mathcal{E}_0(f) := E_0(f) \quad \text{and} \quad \mathcal{E}_n(f) := \left(\frac{5}{3}\right)^n E_n(f) \text{ for } n \geq 1.$$

So long as f is extended harmonically, the quantity $\mathcal{E}_n(f)$ is constant as n increases. Otherwise, $\mathcal{E}_n(f)$ increases as n increases. This means the limit

$$\mathcal{E}(f) := \lim_{n \to \infty} \mathcal{E}_n(f)$$

exists (but is possibly infinite) for $f : V^* \to \mathbb{R}$.

$$SG \xrightarrow{\;f_j\;} SG$$
$$\downarrow{\scriptstyle\Phi} \qquad\qquad \downarrow{\scriptstyle\Phi}$$
$$K_H \xrightarrow{\;H_j\;} K_H$$

Figure 4.2: Relation between the homeomorphism Φ and the contractions H_j and f_j; [20].

We can now define a Laplacian on the Sierpinski gasket and relate this Laplacian to the harmonic functions.

Definition 4.1.0.1 *Let μ be a regular probability measure on SG, $f \in dom\, \mathcal{E}$, and $u \in C(SG)$. If*

$$\mathcal{E}(f,g) = -\int_{SG} ug \; d\mu$$

*for all $g \in dom\, \mathcal{E}$ then we say $f \in dom\Delta_\mu$ and $\Delta_\mu f = u$. We call the operator Δ_μ the **Laplacian** on SG with respect to the measure μ.*

Theorem 4.1.0.2 (Kigami [19]) *If h is harmonic, then $h \in dom\Delta_\mu$ and $\Delta_\mu h = 0$. Conversely, if $u \in dom\Delta_\mu$ and $\Delta_\mu u = 0$ then u is harmonic.*

Given a function $h : V_0 \to \mathbb{R}$, there is a simple rule for extending h to the set V_1 so that the extension to V_1 will minimize E_1. If $x \in V_1 \setminus V_0$, we can define $h(x)$ by $h(x) = \frac{2}{5}(p_i + p_j) + \frac{1}{5}p_k$ where $p_i, p_j \in V_0$ are distinct and are each connected to x by an edge in SG_1 and $p_k \in V_0 \setminus \{p_i, p_j\}$. This extension rule applies more generally to a function $h : V_n \to \mathbb{R}$. One can extend harmonically to V_{n+1} by defining $h(x)$ for $x \in V_{n+1} \setminus V_n$ by $h(x) = \frac{2}{5}(a + b) + \frac{1}{5}c$ where $a, b \in V_n$ are distinct and are each connected to x by an edge in SG_{n+1} and $c \in F_w(V_0) \setminus \{a, b\}$ where w is the word corresponding to the triangle in SG_n of which a, b, c are vertices.

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This $\frac{2}{5}$-$\frac{1}{5}$ rule can be used to uniquely extend a function $h : V_0 \to \mathbb{R}$ to a *harmonic* function $h : SG \to \mathbb{R}$. This means that each $v \in \mathbb{R}^3$, uniquely determines a harmonic function h_v on SG. See [34] and [19] for more on the space of harmonic functions. We make a few remarks about the energy form $\mathcal{E}$ and the harmonic extension process.

- The harmonic extension procedure is linear: $h_{\lambda v + \beta u} = \lambda h_v + \beta h_u$, where $\lambda, \beta \in \mathbb{R}$ and $u, v \in \mathbb{R}^3$.

- For $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^3$, the energy form, $\mathcal{E}$, satisfies:

$$\mathcal{E}(h_{\lambda v}) = \mathcal{E}(\lambda h_v) = \lim_{m \to \infty} \left(\frac{5}{3}\right)^m \sum_{x \sim_m y} (\lambda h_v(x) - \lambda h_v(y))^2 = \lambda^2 \mathcal{E}(h_v)$$

  where h_v is the unique harmonic function on SG with values v_1, v_2, v_3 on p_1, p_2, p_3, respectively.

- The map $\mathcal{E}(h, g) := \frac{1}{2}(\mathcal{E}(h + g) - \mathcal{E}(h) - \mathcal{E}(g))$ is bilinear.

- From the definition and the harmonic extension procedure, one can see that $\mathcal{E}(h_v)$ is non-negative for all $v \in \mathbb{R}^3$ and vanishes on constant vectors $v = (a, a, a) \in \mathbb{R}^3$.

- The energy form, $\mathcal{E}$, induces a positive semidefinite quadratic form on $\mathbb{R}^3$ by taking the map

$$v = (v_1, v_2, v_3) \mapsto h_v \mapsto \mathcal{E}(h_v).$$

The set V^* is dense in SG and hence a uniformly continuous function on V^* can be uniquely extended to a function on all of SG. One can show that harmonic functions on V^*, and in fact functions for which the limit in $\mathcal{E}(f)$ is finite, are uniformly continuous on V^*; see [19], [34]. This gives a way of extending a harmonic function $h : V^* \to \mathbb{R}$ to all of SG.

4.2 The Harmonic Sierpinski Gasket

For each $j = 1, 2, 3$, consider the function $h_j : SG \to \mathbb{R}$, where $h_j(p_k) = \delta_j(k)$ for $k = 1, 2, 3$ and h_j is extended harmonically to V^* and by continuity to the Sierpinski gasket, SG.

Define $\Phi : SG \to \mathbb{R}^3$ by

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right).$$

We define the **harmonic Sierpinski gasket** by $K_H := \Phi(SG)$; see Figure 4.1. It was shown by Kigami in [20] that Φ is a homeomorphism between SG and K_H when endowing these spaces with the topology induced by the restriction of the Euclidean metric.

We can also define K_H in terms of contraction maps, as was done for the classical Sierpinski gasket. Let $Z = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ and let

$$P = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

be the orthogonal projection of $\mathbb{R}^3$ onto Z. Let $q_j = \frac{3P(b_j)}{\sqrt{6}}$ for $j = 1, 2, 3$ where $\{b_1, b_2, b_3\}$ is the standard basis for $\mathbb{R}^3$. Choose $q'_j \in \mathbb{R}^3$ such that $\{q_j, q'_j\}$ is an orthonormal basis for Z. For $j = 1, 2, 3$, define $M_j : Z \to Z$ by

$$M_j(q_j) = \frac{3}{5}q_j \quad \text{and} \quad M_j(q'_j) = \frac{1}{5}q'_j$$

and let $H_j : Z \to Z$ be given by

$$H_j(x) = M_j(x - q_j) + q_j. \tag{4.1}$$

The maps H_j are contractive affine maps and K_H is the unique non-empty compact set such that

$$K_H = \bigcup_{n=1}^{3} H_j(K_H).$$

The two equivalent ways of defining the harmonic Sierpinski gasket are connected via the relation $\Phi \circ f_j = H_j \circ \Phi$ (for $j = 1, 2, 3$) or the commutative square in Figure 4.2; see [20].

Chapter 5

The Connections

5.1 A Spectral Triple for the Harmonic Sierpinski Gasket

First we define curves which correspond to the edges in the graphs which approximate SG and then get curves for the edges in K_H via the homeomorphism Φ. This will allow us to use the direct sum spectral triple to study K_H. Let R_j for $j \geq 1$ be the continuous injective functions which map to the edges in the graphs SG_n:

$R_j : [0,1] \to \mathbb{R}^2$ for $j = 1,2,3$ be the edges in the graph SG_0,

$R_j : [0, 2^{-1}] \to \mathbb{R}^2$ for $j = 4,5,\ldots,12$ be the edges in the graph SG_1,

$R_j : [0, 2^{-2}] \to \mathbb{R}^2$ for $j = 13,14,\ldots,39$ be the edges in the graph SG_2,

and so on. The curves we use to build spectral triples are parameterized by arc length and the sets $R_j([0, 2^{-k}])$ are precisely the edges in the graph approximations of SG. For

simplicity we write R_j for $R_j([0, 2^{-k}])$. One can show that

$$SG = \overline{\bigcup_{j \geq 1} R_j}$$

(see [26]).

Applying the map $\Phi : SG \to K_H$, we get curves $\Phi(R_j)$. Set $\ell_j = L(\Phi(R_j))$ and after a reparameterization we have curves

$$\{\Phi(R_j) : [0, \ell_j] \to K_H\}_{j=1}^{\infty}.$$

Again one can show that

$$K_H = \overline{\bigcup_{j \geq 1} \Phi(R_j)}$$

(see [26]). The direct sum of the spectral triples for the curves $\Phi(R_j)$ gives a spectral triple for the harmonic Sierpinski gasket (by applying Proposition 1 in [26])

$$S(K_H) = \left(C(K_H), \ \bigoplus_{j \geq 1} \mathscr{H}_{\ell_j}, \ D_{K_H} := \bigoplus_{j \geq 1} D_{\ell_j} \right)$$

where $\ell_j = L(\Phi(R_j))$ and the representation is given by $\pi_H = \bigoplus_j \pi_{\ell_j}$.

We begin by showing that the spectral dimension $\mathfrak{d}_H = \mathfrak{d}(K_H)$ of K_H is finite. A direct computation or an application of Proposition 1 in [26] gives that

$$\mathfrak{d}_H = \inf \left\{ s > 1 : \sum_{j \geq 1} \ell_j^s < \infty \right\},$$

from which it follows that $\mathfrak{d}_H \geq 1$; however, one must show that $\mathfrak{d}_H < \infty$. In [21] Kigami obtains bounds for the lengths $\ell_j = L(\Phi(R_j))$. We will use these results to prove the lemma that follows.

Lemma 23 *For $s \in \mathbb{R}$, the sum*

$$\sum_{j \geq 1} \ell_j^s$$

where $\ell_j = L(\Phi(R_j))$ *converges for* $s > \frac{\log 3}{\log 5 - \log 3} \approx 2.151$. *In particular,*

$$1 \leq \mathfrak{d}_H \leq \frac{\log 3}{\log 5 - \log 3}.$$

Proof. Let $p, q \in \Phi(f_w(V_0))$ for some word w of length $|w| = m$ and let $\Phi(R_j)$ be the curve in K_H which connects p and q. By Lemma 5.6 in [21],

$$\frac{2}{5} \ \mathrm{diam}(\Phi(f_w(T))) \leq L(\Phi(R_j)) \leq 2 \ \mathrm{diam}(\Phi(f_w(T))).$$

Note that

$$\mathrm{diam}(\Phi(f_w(T))) = \sup\{|\Phi(f_w(x)) - \Phi(f_w(y))| : x, y \in T\}$$

$$= \sup\{|H_w(\Phi(x)) - H_w(\Phi(y))| : x, y \in T\}$$

$$\leq \left(\frac{3}{5}\right)^m \sup\{|\Phi(x) - \Phi(y)| : x, y \in T\},$$

so $\mathrm{diam}(\Phi(f_w(T))) \leq c \left(\frac{3}{5}\right)^m$, where c is some constant not depending on w. Then

$$\sum_{j=1}^{\infty} \ell_j^s \leq \sum_{m=0}^{\infty} 3^{m+1} 2c \left(\frac{3}{5}\right)^{ms} \tag{5.1}$$

$$= 6c \sum_{m=0}^{\infty} \left(\frac{3^{s+1}}{5^s}\right)^m \tag{5.2}$$

$$= 6c \frac{5^s}{5^s - 3^{s+1}}, \tag{5.3}$$

where we have assumed that $s > \frac{\log 3}{\log 5 - \log 3}$ in equality (5.3). From this and Proposition 1 in [26] we have that

$$1 \leq \mathfrak{d}_h \leq \frac{\log 3}{\log 5 - \log 3}.$$

■

According to the previously mentioned results of Alain Connes in [9], the map $\mathrm{Tr}_w(\pi_H(\cdot)|D_{K_H}|^{-\mathfrak{d}_H})$ is then a positive linear functional on $C(K_H)$.

For $n \geq 0$ and $k \in \{1, 2, \ldots, 3^n\}$, write $\Delta_{n,k}$ for the 3^n triangles in SG_n and for $j \in \{1, 2, 3\}$ write $x_{n,k,j}$ for the midpoints of the edges of these triangles. For $n \geq 0$, define a positive linear functional $\psi_n : C(SG) \to \mathbb{C}$ of norm 1, by

$$\psi_n(f) = \frac{1}{3^{n+1}} \sum_{k=1}^{3^n} \sum_{j=1}^{3} f(x_{n,k,j}).$$

In Proposition 8.6 of [8] it was shown that the sequence $\{\psi_n\}$ converges in the weak-* topology on the dual of $C(SG)$ to the positive linear functional ψ given by

$$\psi(f) := \int_{SG} f \, d\mathcal{H} \, ,$$

where $\mathcal{H}$ is the $\frac{\log 3}{\log 2}$-dimensional Hausdorff probability measure on SG.

Recall that the map $\Phi : SG \to K_H$ is a homeomorphism when we give SG and K_H the topology induced by the Euclidean metric in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. In SG the Euclidean metric and the geodesic metric are equivalent, but in K_H this is not the case [21]. However, one can say that the geodesic metric on K_H, denoted $d_{geo}(\cdot, \cdot)$, satisfies $|\cdot| \leq d_{geo}(\cdot, \cdot)$ where $|\cdot|$ is the Euclidean metric. Then $\Phi : (SG, |\cdot|) \to (K_H, d_{geo})$ is still a bijection and Φ^{-1} is a continuous map. From here forward, we will endow the harmonic Sierpinski gasket with the geodesic metric.

Lemma 24 *If $h \in C(K_H)$, then $h \circ \Phi \in C(SG)$.*

Proof. Let $\epsilon > 0$ and $h \in C(K_H)$. Then there is a $\delta > 0$ such that $d_{geo}(\phi(x), \phi(y)) < \delta$ implies $|h \circ \Phi(x) - h \circ \Phi(y)| < \epsilon$. Since the perimeter of the "triangles", $\Phi(\Delta_{n,j})$, goes to zero as n grows, we can choose an n_0 large enough so that the perimeter of $\Phi(\Delta_{n_0,j})$ is small enough and $d_{geo}(\phi(x), \phi(y)) < \delta$ for x, y in the portion of SG contained within $\Delta_{n_0,j}$. It follows that $h \circ \Phi$ is continuous from $(SG, |\cdot|)$ to $\mathbb{R}$. ∎

Let $\tilde{\psi}_n$ be the positive linear functional on $C(K_H)$ given by

$$\tilde{\psi}_n(h) = \frac{1}{3^{n+1}} \sum_{k=1}^{3^n} \sum_{j=1}^{3} h(\Phi(x_{n,k,j})) = \psi_n(h \circ \Phi),$$

where $h \in C(K_H)$.

Proposition 25 *The sequence $\{\tilde{\psi}_n\}$ converges in the weak-*topology on the dual of $C(K_H)$*

to the positive linear functional given by

$$\tilde{\psi}(h) := \int_{SG} h \circ \Phi(x) \, d\mathcal{H}(x) = \int_{K_H} h(y) \, d(\mathcal{H} \circ \Phi^{-1})(y),$$

where $\mathcal{H}$ is the $\log_2 3$-Hausdorff probability measure on SG and $h \in C(K_H)$. Also, $\tilde{\psi}$ has

the property

$$\tilde{\psi}(h) = \frac{1}{3} \sum_{j=1}^{3} \tilde{\psi}(h \circ H_j),$$

where $h \in C(K_H)$ and H_j for $j = 1, 2, 3$ are the affine maps which determine K_H.

Proof. That $\tilde{\psi}_n \to \tilde{\psi}$ follows from the fact that $\psi_n \to \psi$ and that, according to

Lemma 24, $h \circ \Phi \in C(SG)$ whenever $h \in C(K_H)$.

To see that $\tilde{\psi}$ satisfies the stated property, note that the condition $\tilde{\psi}(h) = \frac{1}{3} \sum_{j=1}^{3} \tilde{\psi}(h \circ$

$H_j)$ is the same as

$$\int_{SG} h \circ \Phi \, d\mathcal{H} = \frac{1}{3} \sum_{j=1}^{3} \int_{SG} h \circ H_j \circ \Phi \, d\mathcal{H}$$

and since $H_j \circ \Phi = \Phi \circ f_j$, where f_j are the similarities defining SG, this condition is the

same as

$$\int_{SG} h \circ \Phi \, d\mathcal{H} = \frac{1}{3} \sum_{j=1}^{3} \int_{SG} h \circ \Phi \circ f_j \, d\mathcal{H}.$$

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This condition holds since $h \circ \Phi \in C(SG)$ whenever $h \in C(K_H)$ and since $\mathcal{H}$ is the unique self-similar measure on SG satisfying,

$$\int_{SG} g \, d\mathcal{H} = \frac{1}{3} \sum_{j=1}^{3} \int_{SG} g \circ f_j \, d\mathcal{H} \qquad \text{for all } g \in C(SG).$$

∎

We can now use this spectral triple to recover the standard self-affine measure on K_H. Self-affine measures such as this are described by Hutchinson in [15].

Proposition 26 *Let $\tau : C(K_H) \to \mathbb{C}$ be given by $\tau(h) := Tr_w(\pi_H(h)|D_{K_H}|^{-\partial_H})$. Then*

$$\tau(h) = Tr_w(\pi_H(h)|D_{K_H}|^{-\partial_H}) = c \int_{K_H} h(x) \, d\mu,$$

where μ is the unique self-affine measure on K_H satisfying,

$$\int h \, d\mu = \frac{1}{3} \sum_{j=1}^{3} \int (h \circ H_j) d\mu \qquad \text{for each } f \in C(K_H).$$

Proof. Let $h \in C(K_H)$ and $\epsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that for any $k \in \{1, 2, \ldots, 3^{n_0}\}$ and x, y inside or on the "triangle", $\Phi(\Delta_{n_0,k})$, we have $|h(x) - h(y)| < \epsilon$. Let $n > n_0$ and define

$$v_{n_0,k}^n(h) = \frac{1}{3^{(n-n_0)+1}} \sum_{i_k} \sum_{j=1}^{3} h(\Phi(x_{n,i_k,j})),$$

where the $x_{n,i_k,j}$ are the midpoints of the edges in the triangles in the n-th step construction of the gasket, SG_n, which are contained in or on the border of $\Delta_{n_0,k}$. Note the dependence of i_k on k and that the number of terms in the sum $\sum_{i_k}$ is precisely 3^{n-n_0}. Denote by $\Phi(SG_{n_0,k})$ the image under Φ of the portion of SG in $\Delta_{n_0,k}$, and $I_{n_0,k}$ and $h_{n_0,k}$ for the restrictions of the functions $I = 1$ and h on K_H to $\Phi(SG_{n_0,k})$.

Notice that

$$\left| v_{n_0,k}^n(h) I_{n_0,k} - h_{n_0,k} \right| = \left| \frac{1}{3^{(n-n_0)+1}} \sum_{i_k} \sum_{j=1}^{3} h(\Phi(x_{n,i_k,j})) I_{n_0,k} - \frac{1}{3^{(n-n_0)+1}} \sum_{i_k} \sum_{j=1}^{3} h_{n_0,k} \right|$$

$$\leq \frac{1}{3^{(n-n_0)+1}} \sum_{i_k} \sum_{j=1}^{3} \left| h(\Phi(x_{n,i_k,j})) I_{n_0,k} - h_{n_0,k} \right|$$

$$< \frac{1}{3^{(n-n_0)+1}} \sum_{i_k} \sum_{j=1}^{3} \epsilon$$

$$= \epsilon,$$

so we have the inequalities,

$$-\epsilon I_{n_0,k} \leq v_{n_0,k}^n(h) I_{n_0,k} - h_{n_0,k} \leq \epsilon I_{n_0,k}.$$

Now, for each space $\Phi(SG_{n_0,k})$, we can define a spectral triple by deleting all summands in $S(K_H)$ which correspond to an edge not in $\Phi(SG_{n_0,k})$. For such a triple we get the corresponding functional $\tau_{n_0,k}$. By the linearity of the Dixmier trace and the fact that as operators $\pi_H(h) = \sum_{k=1}^{3^{n_0}} \pi_{n_0,k}(h)$ (where $\pi_{n_0,k}$ are the representations corresponding to the triples for $\Phi(SG_{n_0,k})$), we have

$$\tau(h) = \sum_{k=1}^{3^{n_0}} \tau_{n_0,k}(h_{n_0,k}) \text{ and } \tau_{n_0,k}(I_{n_0,k}) = 3^{-n_0} \tau(I).$$

As τ is a positive linear functional and hence preserves order, we have

$$-\epsilon \, \tau_{n_0,k}(I_{n_0,k}) \leq v_{n_0,k}^n(h) \tau_{n_0,k}(I_{n_0,k}) - \tau_{n_0,k}(h_{n_0,k}) \leq \epsilon \, \tau_{n_0,k}(I_{n_0,k})$$

and hence

$$-\epsilon \, 3^{-n_0} \tau(I) \leq v_{n_0,k}^n(h) 3^{-n_0} \tau(I) - \tau_{n_0,k}(h_{n_0,k}) \leq \epsilon \, 3^{-n_0} \tau(I).$$

Summing over $k \in \{1, 2, \ldots, 3^{n_0}\}$, we get

$$-\epsilon \, \tau(I) \leq 3^{-n_0} \sum_{k=1}^{3^{n_0}} v_{n_0,k}^n(h) \tau(I) - \tau(h) \leq \epsilon \, \tau(I)$$

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and using the fact that $\tilde{\psi}_n = 3^{-n_0} \sum_{k=1}^{3^{n_0}} v_{n_0,k}^n$, we find

$$-\epsilon\ \tau(I) \le \tilde{\psi}_n(h)\tau(I) - \tau(h) \le \epsilon\ \tau(I).$$

Letting $n \to \infty$ we have

$$-\epsilon\ \tau(I) \le \tilde{\psi}(h)\tau(I) - \tau(h) \le \epsilon\ \tau(I)$$

and hence $\left|\tau(I)\tilde{\psi}(h) - \tau(h)\right| < \tau(I)\epsilon$. This gives

$$\mathrm{Tr}_w(\pi_H(h)|D_{K_H}|^{-\partial_H}) = c \int_{K_H} h(x)\ d\mu\ ,$$

where $c = \tau(I)$. ∎

It was previously conjectured that the Dixmier trace on K_H would recover the Hausdorff measure on (K_H, d_{geo}); however, the Dixmier trace recovers the self-affine measure of weights $1/3$ and it can be shown that this self-affine measure is not the same as the Hausdorff measure on K_H, [18]. Briefly, the value of μ, the self-affine measure of weights $1/3$, on sets of the form $H_w(K_H)$ where $H_w = H_{w_1 w_2 \ldots w_k}$, is given by $\mu(H_w(K_H)) = \left(\frac{1}{3}\right)^{|w|} \mu(K_H) = \left(\frac{1}{3}\right)^{|w|}$. This means the value of μ on a set like $H_w(K_H)$ is completely determined by the *length* of the word w. It was shown by Kajino in Proposition 6.4 of [17] that there exist positive constants c_1, c_2 such that

$$c_1\|M_w\|^d \le \mathcal{H}^d(H_w(K_H)) \le c_2\|M_w\|^d,$$

where d is the Hausdorff dimension of (K_H, d_{geo}), $M_w = M_{w_1} \cdots M_{w_k}$, and the M_{w_i} are the matrices in the definition of the maps $H_j : K_H \to K_H$ which determine K_H. Changing the word w can drastically change the norm of the matrices M_w and hence the value of $\mathcal{H}^d(H_w(K_H))$. With a bit more work, these facts show that the self-affine measure is not

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the same as the Hausdorff measure and hence this construction of a spectral triple for K_H cannot recover the Hausdorff measure.

It would be interesting to see what kind of spectral triple on K_H would recover the Hausdorff measure.

5.2 Spectral Triple for the Stretched Sierpinski Gasket, K_α

In this section we will consider the direct sum curve triple for the stretched Sierpinski gasket and show that it recovers the Hausdorff dimension, the geodesic metric, and the Hausdorff measure on K_α. These results are of interest since the space K_α is a self-affine space as opposed to a self-similar space. In general, self-affine spaces are more difficult to study than their structure rich self-similar sisters.

Let us introduce some notation. The notation will be similar to that used for the Sierpinski gasket, but will include a superscript s to indicate that we are working with the *stretched* Sierpinski gasket. Let $p_1 = (0,0)$, $p_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $p_3 = (1,0)$ as before. Define,

$$V_0^s := \{p_1, p_2, p_3\} \quad \text{and for } n \geq 1, \text{ let} \quad V_n^s := \bigcup_{w \in \{1,2,3\}^n} F_w(\{p_1, p_2, p_3\}),$$

where $w = w_1 w_2 \cdots w_n \in \{1,2,3\}^n$, $F_w = F_{w_n} \circ \cdots \circ F_{w_2} \circ F_{w_1}$, and the F_j's are as in (2.2). These are the vertices of the triangles in the approximations of the stretched Sierpinski gasket. Let $V^{s*} := \bigcup_{n \geq 0} V_n^s$, the set of all vertices of the triangles in K_α.

It will be important to distinguish between the two different types of edges in the graph approximations of K_α, namely the triangle edges and the edges joining the triangles. For $x, y \in \mathbb{R}^2$, the symbols $[x \to y]$ or $(x \to y)$ refer to the line segment in $\mathbb{R}^2$ connecting x

and y which include or exclude the points x and y, respectively. Define

$$T_0 := \{[p_j \rightarrow p_i] : i, j = 1, 2, 3 \text{ and } j \neq i\}$$

(the edges in the outer triangle) and for $n \geq 1$,

$$T_n := \{[x \rightarrow y] : \exists \, w \in \{1, 2, 3\}^n \text{ such that } x, y \in F_w(V_0^s)\}$$

(edges in the triangles at level n). The set T_n is the collection of triangle edges in the n-th level approximation of the stretched Sierpinski gasket. Recall the notation for the joining edges in K_α: $J_0 = \emptyset$ and for $n \geq 1$

$$J_n = \bigcup_{m=0}^{n-1} \bigcup_{w \in \{1,2,3\}^m} F_w \left(\bigcup_{i=1}^{3} e_i \right),$$

where e_1, e_2, e_3 are the three initial joining edges. Also, $J^* = \cup_{n \geq 1} J_n$.

We would like to distinguish between the collection of *points* in K_α which lie in the sets J_n and the collection of *edges* that make up the set J_n. Write $\mathcal{J}_n$ for the collection of joining edges at stage n, which include the endpoints:

$$\mathcal{J}_n = \bigcup_{m=0}^{n-1} \bigcup_{w \in \{1,2,3\}^m} \{F_w(\overline{e_i}) : i = 1, 2, 3\} \qquad \text{for } n \geq 1$$

and $\mathcal{J}^* = \cup_{n \geq 1} \mathcal{J}_n$. Finally, define $\mathcal{E}_n := T_n \cup \mathcal{J}_n$.

For each $\epsilon = [\epsilon^- \rightarrow \epsilon^+] \in \mathcal{E}_n$, where $\epsilon^+, \epsilon^- \in \mathbb{R}^2$ denote the endpoints of the edge ϵ, define $R_\epsilon^s : [0, L(\epsilon)] \rightarrow \mathbb{R}^2$ by

$$R_\epsilon^s(t) = \frac{1}{L(\epsilon)} \left(\epsilon^+ t + (L(\epsilon) - t)\epsilon^- \right),$$

where $L(\epsilon)$ denotes the length of the edge ϵ.

It was shown in [1] that

$$K_\alpha = \overline{\bigcup_{n \geq 0} \bigcup_{\epsilon \in \mathscr{E}_n} R_\epsilon^s([0, L(\epsilon)])},$$

where the closure is taken with respect to the Euclidean metric. It was also shown in [1] that the Euclidean metric, the effective resistance metric, and the geodesic metric on K_α are all equivalent. This means K_α satisfies Axiom 1, where the curves are the R_ϵ^s corresponding to the edges in the sets $\mathscr{E}_n$. It follows from the results in [26] mentioned previously that the direct sum of the R_ϵ^s curve triples is a spectral triple for K_α. Denote this spectral triple by

$$S(K_\alpha) = (C(K_\alpha), \ \mathscr{H}_\alpha, \ D_\alpha),$$

with representation $\pi_\alpha : C(K_\alpha) \to \mathcal{B}(\mathscr{H}_\alpha)$.

5.2.1   Recovery of the Hausdorff Dimension and Geodesic Metric on K_α

It was shown in [1] that the Hausdorff dimension of the stretched Sierpinski gasket of parameter α is

$$d_\alpha := \frac{\log(3)}{\log(2) - \log(1 - \alpha)}.$$

We begin the section by showing that the spectral triple $S(K_\alpha)$ recovers the Hausdorff dimension of K_α. First let us enumerate the edges in the set $\mathscr{E} := \bigcup_{n \geq 0} \mathscr{E}_n$ and write $\mathscr{E} = \{\epsilon_1, \epsilon_2, \dots\}$. To simplify notation we write $R_j^s = R_{\epsilon_j}^s$.

Proposition 27 *For $p > 1$ and each fixed $j \geq 1$,*

$$tr(|D_j|^{-p}) = \beta_p l_j^p,$$

where D_j is the operator in the spectral triple for the edge R_j^s, $L(R_j^s) = l_j$, and $\beta_p = \dfrac{2^{p+1}(1 - 2^{-p})\zeta(p)}{\pi^p}$. Furthermore, for $p > 1$,

$$tr(|D_\alpha|^{-p}) = \beta_p \sum_{j=1}^{\infty} l_j^p,$$

where D_α is the operator in the spectral triple for K_α. If $p > d_\alpha$, we have

$$tr(|D_\alpha|^{-p}) = \frac{\beta_p 2^p (3 + 3\alpha^p)}{2^p - 3(1 - \alpha)^p}.$$

Proof. Recall that the eigenvalues of the operator D_α are given by

$$\bigcup_{j \geq 1} \left\{ \frac{(2k + 1)\pi}{2l_j} : k \in \mathbb{Z} \right\}.$$

The values $L(R_j^s) = l_j$ are in the set

$$\bigcup_{n \geq 0} \left\{ \left(\frac{1 - \alpha}{2} \right)^n, \quad \alpha \left(\frac{1 - \alpha}{2} \right)^n \right\},$$

with multiplicity 3^{n+1} for each length like $\left(\frac{1-\alpha}{2} \right)^n$ or like $\alpha \left(\frac{1-\alpha}{2} \right)^n$. Assuming $p > 1$, we

have

$$tr(|D_j|^{-p}) = \sum_{k \in \mathbb{Z}} \left| \frac{(2k + 1)\pi}{2l_j} \right|^{-p} = \frac{2^{p+1} l_j^p}{\pi^p} \sum_{k=0}^{\infty} \frac{1}{|2k + 1|^p} = \frac{2^{p+1} l_j^p}{\pi^p} (1 - 2^{-p})\zeta(p) = \beta_p l_j^p$$

and

$$tr(|D_\alpha|^{-p}) = \beta_p \sum_{j=1}^{\infty} l_j^p \tag{5.4}$$

$$= \beta_p \left(\sum_{n=0}^{\infty} 3^{n+1} \left(\frac{1 - \alpha}{2} \right)^{np} + \sum_{m=0}^{\infty} 3^{m+1} \alpha^p \left(\frac{1 - \alpha}{2} \right)^{mp} \right) \tag{5.5}$$

$$= \beta_p \left(3 \sum_{n=0}^{\infty} \left(3 \left(\frac{1 - \alpha}{2} \right)^p \right)^n + 3\alpha^p \sum_{m=0}^{\infty} \left(3 \left(\frac{1 - \alpha}{2} \right)^p \right)^m \right) \tag{5.6}$$

$$= \beta_p (3 + 3\alpha^p) \frac{2^p}{2^p - 3(1 - \alpha)^p}, \tag{5.7}$$

where (5.7) requires the further assumption that $p > d_\alpha$. ∎

Corollary 28 *The spectral dimension, $\mathfrak{d}(K_\alpha)$, induced by the spectral triple $S(K_\alpha)$ for K_α*

is equal to $d_\alpha = \frac{\log(3)}{\log(2)-\log(1-\alpha)}$, the Hausdorff dimension of the stretched Sierpinski gasket

of parameter α.

Proof. An application of the limit comparison test will show that computing the abscissa

of convergence of the series $\text{tr}((1 + D_\alpha^2)^{-p/2})$ is the same as computing the abscissa of

convergence of the series $\text{tr}(|D_\alpha|^{-p})$. It was shown in [1] that the Hausdorff dimension

of K_H is given by $d_\alpha = \frac{\log(3)}{\log(2)-\log(1-\alpha)}$. In Proposition 27 we found that the abscissa of

convergence of the series $\text{tr}(|D_\alpha|^{-p})$ is d_α. It follows that

$$\mathfrak{d}(K_\alpha) = \frac{\log(3)}{\log(2) - \log(1 - \alpha)}.$$

∎

Thus the spectral triple $S(K_\alpha)$ recovers the Hausdorff dimension of K_α. Next we

recover the geodesic metric on K_α by using the spectral metric induced by $S(K_\alpha)$.

Recall that $K_\alpha = W_\alpha \cup J^*$ and since the set V^{s*} is dense in W_α, the set $V^{s*} \cup J^*$

is dense in K_α.

Proposition 29 *For any $p \in V^{s*} \cup J^*$ and any $q \in K_\alpha$, there is a path of minimal length*

from p to q which is a concatenation of (finite or countably many) triangle edges, joining

edges, or segments of joining edges at the start or end of the path (possibly both).

Proof. (Case $p \in V^{s*}$ and $q \in W_\alpha$)

Let $p \in V^{s*}$ and $q \in W_\alpha$. Let m be the smallest integer such that p and q are in different

m cells, $F_w(W_\alpha)$ and $F_v(W_\alpha)$, where $|w| = |v| = m$. Suppose further that p is an m-vertex

for an edge in $F_w(W_\alpha)$ (and not just an endpoint for some further approximation). We

are considering the space K_α with the metric d_{geo} which is equivalent to the Euclidean metric. By the Hopf–Rinow theorem we know that K_α has minimizing geodesics. Let γ be a minimal path between p and q. Then there is a vertex v_1 of an edge in $F_v(W_\alpha)$ such that γ passes through v_1. The portion of γ which connects p and v_1 must look like one of

$$\epsilon_j, \qquad \epsilon_j * \epsilon_t, \qquad \epsilon_j * \epsilon_t * \epsilon_{j'}, \qquad \epsilon_t, \qquad \epsilon_t * \epsilon_j, \qquad \epsilon_t * \epsilon_j * \epsilon_{t'},$$

where $\epsilon_j, \epsilon_{j'} \in \mathcal{J}_m$, $\epsilon_t, \epsilon_{t'} \in T_m$, and $\epsilon * \beta$ denotes the concatenation of the edges ϵ and β. Note that here, $\epsilon * \beta$ means that we first travel along the edge ϵ and then along the edge β. Note that if p and v_1 can be joined by a path with one or two edges, then that path is unique of minimal length. There may be more than one path with three edges connecting p and v_1 and these will look like $\epsilon_j * \epsilon_t * \epsilon_{j'}$ and $\epsilon_t * \epsilon_j * \epsilon_{t'}$. In this case we take the shorter of the two paths, namely $\epsilon_j * \epsilon_t * \epsilon_{j'}$, which will be the unique path of minimal length. No path between p and v_1 with four or more edges will be minimal.

Write γ_1 for the concatenation of the edges connecting p and v_1. Repeating this argument with v_1 and q, we get a unique path γ_2 of minimal length from v_1 to some vertex v_2 in an m' cell $F_{w'}(W_\alpha)$, where $|w'| = m' > m$ and $q \in F_{w'}(W_\alpha)$. In this way, we get a countable concatenation of paths γ_i whose lengths go to zero since the lengths of the edges making them up go to zero. Then we must have that $\gamma = \overline{\cup_{i \geq 1}\gamma_i}$ is the path of minimal length from p to q.

If p is not an m-vertex in $F_w(W_\alpha)$, then choose an m-vertex, u, which lies on a shortest path between p and q and apply the previous argument on u, p and u, q. We then reverse the path (which will consist of finitely many edges since p is a vertex) between u and p and concatenate with the path between u and q to get a path from p to q.

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(Case $p \in V^{s*}$ and $q \in J^*$)

If $q \in \epsilon_j$ for some $\epsilon_j \in \mathcal{J}^*$, then apply the argument above to get a path, γ, from p to one

of the endpoints ϵ_j^- or ϵ_j^+ (whichever is closest to p and hence yields the shortest path γ).

Concatenate γ with the line segment between ϵ_j^- or ϵ_j^+ and q. This again yields a unique

shortest path between p and q.

(Case $p \in J^*$ and $q \in K_\alpha$)

Suppose now that $p \in \epsilon_j$ for some $\epsilon_j \in \mathcal{J}^*$ and $q \in K_\alpha$. Without loss of generality, assume

ϵ_j^- is closer to q than ϵ_j^+. Apply the above argument to get a path between ϵ_j^- and q and

concatenate with the line segment connecting p to ϵ_j^-. This concludes the proof. ■

In the following lemma we use the notation

$$\mathrm{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : x \neq y \in \mathbb{R}\right\}$$

for the Lipschitz seminorm of a function $f : \mathbb{R} \to \mathbb{R}^n$.

Lemma 30 *Let $r : [0, \ell] \to X \subseteq \mathbb{R}^n$ be a curve and consider its spectral triple $(C(X), \mathcal{H}_\ell, D_\ell)$.*

For $f \in C(X)$, we have that $[D_\ell, \pi_\ell(f)]$ is bounded if and only if $f \circ r(|x|)$ is Lipschitz if

and only if $f \circ r(|x|)$ is differentiable almost everywhere.

Proof. It is well known that a function $h : [0, \ell] \to \mathbb{R}^n$ is Lipschitz if and only if it is

differentiable almost everywhere. Hence, it only remains to prove the first equivalence.

Suppose $f \in C(X)$ and $[D_\ell, \pi_\ell(f)]$ is bounded. For $g = 1 \in L^2[-\ell, \ell]$,

$$\|[D_\ell, \pi_\ell(f)]g\|_2 \leq \|[D_\ell, \pi_\ell(f)]\|\|g\|_2 = \|[D_\ell, \pi_\ell(f)]\| < \infty$$

and since

$$[D_\ell, \pi_\ell(f)]g = D_\ell \pi_\ell(f)g - \pi_\ell(f)D_\ell(g) = D_\ell(f \circ r(|x|)),$$

it follows that $f \circ r(|x|) \in \text{Dom}(D_\ell)$. By Lemma 19, there exists a bounded measurable function g such that

$$|f \circ r(|x|) - f \circ r(|y|)| = \left| \int_{|y|}^{|x|} g(t)dt \right| \le \int_{|y|}^{|x|} |g(t)|dt \le \|g\|_\infty ||x| - |y||.$$

This shows that $f \circ r(|x|)$ is Lipschitz and $\text{Lip}(f) \le \|g\|_\infty = \|Df\|_\infty$.

Suppose now that $f \circ r(|x|)$ is Lipschitz and hence is differentiable almost everywhere. Then

$$[D_\ell, \pi_\ell(f)]g = \pi_\ell(Df)g$$

for $g \in C^1[-\ell, \ell]$. Thus $[D_\ell, \pi_\ell(f)]$ is densely defined and can be extended to the bounded operator $\pi_\ell(Df)$ on $L^2[-\ell, \ell]$. ∎

Definition 31 *Let $\text{Lip}_g(\cdot)$ be the Lipschitz seminorm for the compact metric space (K_α, d_{geo}) given by*

$$\text{Lip}_g(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d_{geo}(x, y)} : x, y \in K_\alpha, x \ne y \right\}.$$

Note that since the geodesic metric and the Euclidean metric on K_α are equivalent, if f is Lipschitz with respect to the Euclidean metric (i.e. $\text{Lip}(f) < \infty$), then f is Lipschitz with respect to the geodesic metric (i.e. $\text{Lip}_g(f) < \infty$) and conversely.

Proposition 32 *For any function $f \in C(K_\alpha)$ such that $\|[D_\alpha, \pi_\alpha(f)]\| < \infty$,*

$$\|D_\alpha f\|_{\infty, K_\alpha} = \text{Lip}_g(f).$$

Proof. By Lemma 30, if $\|[D_\alpha, \pi_\alpha(f)]\| < \infty$ then f is Lipschitz and differentiable almost everywhere. Since $K = \overline{\cup_{j\geq 1} R_j^s}$,

$$
\begin{aligned}
\|D_\alpha f\|_{\infty,K_\alpha} &= \sup_j \{\|D_j f\|_{\infty,R_j^s}\} \\
&= \sup_j \left\{ \left\| -i\frac{\partial f}{\partial x} \right\|_{\infty,R_j^s} \right\} \\
&= \sup_j \left\{ \sup_{p,q \in R_j^s} \left\{ \frac{|f(p) - f(q)|}{d_{geo}(p,q)} \right\} \right\} \\
&\leq \operatorname{Lip}_g(f).
\end{aligned}
$$

For the reverse inequality first suppose $p \in V^{s*} \cup J^*$ and $q \in K_\alpha$. Then by Proposition 29 there is a minimizing geodesic between p and q made of R_j^s curves (i.e. triangle or joining edges) and segments of joining edges. Suppose first that the geodesic consists only of complete R_j^s curves. Let (p_k, p_{k+1}) track the endpoints of these R_j^s, so $p = p_1$ and $\lim_{k\to\infty} p_k = q$. Then

$$
\begin{aligned}
|f(p) - f(p_k)| &\leq \sum_{j=1}^{k-1} |f(p_j) - f(p_{j+1})| \\
&\leq \sum_{j=1}^{k-1} d_{geo}(p_j, p_{j+1}) \|D_j f\|_{\infty,R_j} \\
&\leq \|D_\alpha f\|_{\infty,K_\alpha} \sum_{j=1}^{k-1} d_{geo}(p_j, p_{j+1}) \\
&= \|D_\alpha f\|_{\infty,K_\alpha} d_{geo}(p, p_k)
\end{aligned}
$$

and by continuity of f and $d_{geo}(p,x)$,

$$
\frac{|f(p) - f(q)|}{d_{geo}(p,q)} \leq \|D_\alpha f\|_{\infty,K_\alpha}.
$$

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Now suppose the minimizing geodesic between p and q looks like $\gamma_1 * \{R^s_{j_k}\} * \gamma_2$, where the γ_1, γ_2 are segments of joining edges (which can be empty or and entire edge) and are being concatenated with either finitely many edges, $\{R^s_{j_k}\}_{k \in \{1,2,\dots,N\}}$, or infinitely many edges, $\{R^s_{j_k}\}_{k \in \{1,2,\dots\}}$. We will allow for γ_1 to be an entire edge but will assume that it is nonempty. On the other hand, we may assume that γ_2 is not an entire edge, since otherwise it would be included as one of the $R^s_{j_k}$ curves, but γ_2 may be empty. In fact, if the number of curves $R^s_{j_k}$ is countably infinite then $\gamma_2 = \emptyset$.

Set $p_0 = p \in \gamma_1$ and for $k \geq 1$, let (p_k, p_{k+1}) track the endpoints of the $R^s_{j_k}$ curves. Let $R^s_{j_0}$ be the edge on which p lies (if γ_1 is an entire edge, then $R^s_{j_0} = \gamma_1$) and note that p_1 is an endpoint of both $R^s_{j_0}$ and $R^s_{j_1}$. If $\gamma_2 \neq \emptyset$, then the number of $R^s_{j_k}$ curves is finite; let p_{N+1} be the endpoint which connects the last edge, $R^s_{j_N}$, to γ_2. Let $R^s_{j_{N+1}}$ be the edge with endpoint p_{N+1} and containing q, and set $p_k = q$ for $k \geq N + 1$. As before,

$$|f(p) - f(p_k + 1)| \leq \|D_\alpha f\|_{\infty, K_\alpha} d_{geo}(p, p_k),$$

where we have used the fact that for $k \geq 0$, the points p_k, p_{k+1} are on the same edge $R^s_{j_k}$. Again by continuity,

$$\frac{|f(p) - f(q)|}{d_{geo}(p, q)} \leq \|D_\alpha f\|_{\infty, K_\alpha}.$$

If $\gamma_2 = \emptyset$, then repeat the above arguments with the sequence

$$\{p_0 = p\} \cup \{p_k : p_k \text{ endpoints of the edges } R^s_{j_k}, k \geq 1\},$$

to get the same estimate:

$$\frac{|f(p) - f(q)|}{d_{geo}(p, q)} \leq \|D_\alpha f\|_{\infty, K_\alpha}, \tag{5.8}$$

which holds for any $p \in V^{s*} \cup J^*$ and $q \in K_\alpha$.

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Now let p, q be arbitrary points in K_α. Let γ be a minimizing geodesic between p and q. Since the set $V^{s*} \cup J^*$ is dense in K_α, the path γ intersects $V^{s*} \cup J^*$ at some point r. Let γ_1 be a minimizing geodesic between p and r and γ_2 a minimizing geodesic between r and q, where γ_1, γ_2 are made up of R_j^s edges and portions of R_j^s edges. The lengths of γ_1 and γ_2 must be the same as the lengths of the portions of γ connecting p and r, and r and q. Let γ_2 be tracked by points $\{r_i\}$, where these points are endpoints of some R_j^s edges or possibly points on a joining edge (as in the paths described above). Define γ_{1i} to be the path obtained by concatenating the first i parts of γ_2 with γ_1 at the point r. Using the estimate (5.8) on the points $r_i \in V^{s*} \cup J^*$ and $p \in K_\alpha$ gives

$$\frac{|f(p) - f(r_i)|}{d_{geo}(p, r_i)} \leq \|D_\alpha f\|_{\infty, K_\alpha}$$

and by continuity

$$\frac{|f(p) - f(q)|}{d_{geo}(p, q)} \leq \|D_\alpha f\|_{\infty, K_\alpha}.$$

It now follows that $\mathrm{Lip}_g(f) = \|D_\alpha f\|_{\infty, K_\alpha}$, as desired. ∎

Theorem 33 *Let $d_{K_\alpha}(\cdot, \cdot)$ be the metric on K_α induced by the spectral triple $S(K_\alpha)$. Then for all $x, y \in K_\alpha$,*

$$d_{K_\alpha}(x, y) = d_{geo}(x, y).$$

Proof. The proof here relies on Proposition 32 and is the same as the proof of Theorem 2 in [26]. We recreate it here, for the sake of completeness.

Let $p, q \in K_\alpha$ and $f \in C(K_\alpha)$ such that $\|[D_\alpha, \pi_\alpha(f)]\| \leq 1$. By Lemma 19, $f \in \mathrm{Dom}(D_\alpha)$ and since $[D_\alpha, \pi_\alpha(f)]$ is bounded it must be the operator $\pi_\alpha(Df)$. Using

that representations of C^*-algebras are isometries, we deduce that

$$\|D_\alpha f\|_\infty = \|\pi_\alpha(Df)\| = \|[D_\alpha, \pi_\alpha(f)]\| \leq 1.$$

By Proposition 32, $\text{Lip}_g(f) = \|D_\alpha f\|_\infty \leq 1$ and hence

$$\frac{|f(p) - f(q)|}{d_{geo}(p,q)} \leq 1;$$

so that $|f(p) - f(q)| \leq d_{geo}(p,q)$. This gives that $d_{K_\alpha}(p,q) \leq d_{geo}(p,q)$. For the reverse inequality consider the continuous function $h(x) = d_{geo}(x,q)$. Note that $\text{Lip}_g(h) = 1$ and hence, by Lemma 19 and Lemma 30, $\|[D_\alpha, \pi_\alpha(h)]\| \leq 1$. Now since

$$|h(p) - h(q)| = |0 - d_{geo}(p,q)| = d_{geo}(p,q),$$

we have $d_{K_\alpha}(p,q) \geq d_{geo}(p,q)$ and hence $d_{K_\alpha}(p,q) = d_{geo}(p,q)$. ∎

5.2.2 Recovery of the Hausdorff Measure on K_α

In this section we show that the d_α-dimensional Hausdorff measure, $\mathcal{H}^{d_\alpha}$, is the unique self-affine measure satisfying

$$\mathcal{H}^{d_\alpha}(A) = \frac{1}{3}\sum_{i=1}^{3}\mathcal{H}^{d_\alpha}(F_i^{-1}(A))$$

for any Borel set $A \subseteq K_\alpha$. We then show that the measure defined by the Dixmier trace is the same as the d_α-dimensional Hausdorff measure. Denote the Hausdorff dimension of a metric space (X,d) by $\dim_H(X)$.

Recall that K_α can be written in terms of its discrete part and its continuous part

$$K_\alpha = W_\alpha \cup J^*,$$

where this union is disjoint. Notice that for $A \subseteq K_\alpha$ it holds that $\dim_H(A \cap J^*) \leq \dim_H(J^*) = 1$ and $d_\alpha > 1$ so $\mathcal{H}^{d_\alpha}(A \cap J^*) = 0$. This means

$$\mathcal{H}^{d_\alpha}(A) = \mathcal{H}^{d_\alpha}(A \cap W_\alpha) + \mathcal{H}^{d_\alpha}(A \cap J^*) = \mathcal{H}^{d_\alpha}(A \cap W_\alpha),$$

which shows that the d_α-Hausdorff measure on K_α is the same as the d_α-Hausdorff measure on W_α.

The following is an easy consequence of the work in [1] and [2]. We give a proof, for the sake of completeness.

Proposition 34 *The d_α-dimensional Hausdorff measure on K_α satisfies the condition,*

$$\mathcal{H}^{d_\alpha}(A) = \frac{1}{3} \sum_{i=1}^{3} \mathcal{H}^{d_\alpha}(F_i^{-1}(A))$$

for any Borel set $A \subseteq K_\alpha$.

Proof. Let $A \subseteq K_\alpha$. Then there exist sets $A_1, A_2, A_3 \subseteq K_\alpha$ such that

$$A = F_1(A_1) \cup F_2(A_2) \cup F_3(A_3) \cup J \tag{5.9}$$

where $J \subseteq J^*$ and the unions are disjoint. Then $\mathcal{H}^{d_\alpha}(A) = \sum_{j=1}^{3} \mathcal{H}^{d_\alpha}(F_j(A_j))$. Note that since the maps F_j, for $j = 1, 2, 3$, are similarities of parameter $\dfrac{1-\alpha}{2}$, for $U \subseteq K_\alpha$, it holds that

$$\mathcal{H}^{d_\alpha}(F_j(U)) = \left(\frac{1-\alpha}{2}\right)^{d_\alpha} \mathcal{H}^{d_\alpha}(U), \qquad j = 1, 2, 3$$

and since

$$\left(\frac{1-\alpha}{2}\right)^{d_\alpha} = \left(\frac{1-\alpha}{2}\right)^{\frac{\log(3)}{\log\left(\frac{2}{1-\alpha}\right)}} = \left(\frac{1-\alpha}{2}\right)^{\log_{\frac{2}{1-\alpha}}(3)} = 3^{-1}$$

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we have $\mathcal{H}^{d_\alpha}(F_j(U)) = \frac{1}{3}\mathcal{H}^{d_\alpha}(U)$ for $j = 1, 2, 3$. Note that $F_j^{-1}(A) = A_j$ since the union in (5.9) is disjoint and the F_j, for $j = 1, 2, 3$, are injective. It then follows that

$$\frac{1}{3}\sum_{j=1}^{3}\mathcal{H}^{d_\alpha}(F_j^{-1}(A)) = \frac{1}{3}\sum_{j=1}^{3}\mathcal{H}^{d_\alpha}(A_j) = \sum_{j=1}^{3}\mathcal{H}^{d_\alpha}(F_j(A)) = \mathcal{H}^{d_\alpha}(A),$$

as was to be shown. ∎

For $n \geq 1$ define the maps $\psi_{\alpha,n} : C(K_\alpha) \to \mathbb{R}$ by

$$\psi_{\alpha,n}(f) = 2^{-1}3^{-n}\sum_{\epsilon \in \mathcal{J}_n \setminus \mathcal{J}_{n-1}}\sum_{s \in \{+,-\}}f(\epsilon^s),$$

where ϵ^-, ϵ^+ are the endpoints of the edge $\epsilon \in \mathcal{J}_n \setminus \mathcal{J}_{n-1}$.

We will need the following notation. For $n > n_0 \geq 0$, let

$$\mathcal{S}_{n_0,h}^{n} = \{\epsilon \in \mathcal{J}_n \setminus \mathcal{J}_{n-1} : \epsilon \subset \Delta_{n_0,h}\},$$

where $\Delta_{n_0,h}$ is a triangle in the n_0-th step in the construction of the gasket and these triangles have been enumerated clockwise by $h \in \{1, 2, \ldots, 3^{n_0}\}$. Let $e_{n_0,h}^{\pm}$ denote the two endpoints of edges in $\mathcal{J}_{n_0}$ which also lie in $\Delta_{n_0,h}$. Note that the points $e_{n_0,h}^{\pm}$ do not belong to the same edge in $\mathcal{J}_{n_0}$. We use the notation with superscript $\pm$ for convenience and not to indicate that these are the "right" and "left" endpoints of an edge, as is the case with the notation $\epsilon^{\pm}$. See Figure 5.1.

Proposition 35 *Let $\mathcal{H}^{d_\alpha}$ be the d_α-dimensional Hausdorff probability measure on K_α and $\psi_\alpha : C(K_\alpha) \to \mathbb{R}$ given by*

$$\psi_\alpha(f) = \int_{K_\alpha} f(x)\ d\mathcal{H}^{d_\alpha}.$$

Then the sequence $\{\psi_{\alpha,n}\}$ converges to ψ_α in the weak-* topology on the dual space of $C(K_\alpha)$.
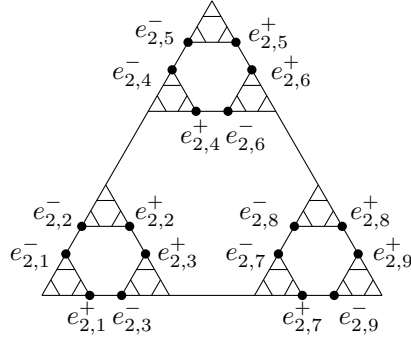
Figure 5.1: Example of edges $e^{\pm}_{n_0,h}$ for $n_0 = 2$ and $1 \leq h \leq 9$.

Proof. Let $\varepsilon > 0$. Since f is uniformly continuous on K_α, there exists an $n_0 \in \mathbb{N}$ such that for all $h \in \{1, \ldots, 3^{n_0}\}$ and any two points x, y inside or on the triangle $\Delta_{n_0,h}$, we have $|f(x) - f(y)| < \varepsilon$. Let $n > n_0$ and define $u^n_{n_0,h} : C(K_\alpha) \to \mathbb{R}$ by

$$u^n_{n_0,h}(f) = \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}_{n_0,h}} \sum_{s \in \{+,-\}} f(\epsilon^s).$$

Then

$$\left| u^n_{n_0,h}(f) - \frac{f(e^-_{n_0,h}) + f(e^+_{n_0,h})}{2} \right|$$

$$= \left| \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}^n_{n_0,h}} \sum_{s \in \{+,-\}} f(\epsilon^s) - \frac{1}{2} \sum_{s \in \{+,-\}} f(e^s_{n_0,h}) \right|$$

$$= \left| \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}^n_{n_0,h}} \sum_{s \in \{+,-\}} f(\epsilon^s) - \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}^n_{n_0,h}} \sum_{s \in \{+,-\}} f(e^s_{n_0,h}) \right|$$

$$\leq \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}^n_{n_0,h}} \sum_{s \in \{+,-\}} \left| f(\epsilon^s) - f(e^s_{n_0,h}) \right|$$

$$< \varepsilon.$$

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Notice that $\psi_{\alpha,n}(f) = 3^{-n_0} \sum_{h=1}^{3^{n_0}} u_{n_0,h}^n(f)$; using the above estimate,

$$|\psi_{\alpha,n}(f) - \psi_{\alpha,n_0}(f)| = \left| 3^{-n_0} \sum_{h=1}^{3^{n_0}} \left(u_{n_0,h}^n(f) - \frac{1}{2} \sum_{s\in\{+,-\}} f(e_{n_0,h}^s) \right) \right|$$

$$\leq 3^{-n_0} \sum_{h=1}^{3^{n_0}} \left| u_{n_0,h}^n(f) - \frac{1}{2} \sum_{s\in\{+,-\}} f(e_{n_0,h}^s) \right|$$

$$< \varepsilon.$$

Thus the sequence $\{\psi_{\alpha,n}\}_{n\geq 1}$ converges to a functional ψ_α in the weak-*topology on the dual of $C(K_\alpha)$. We now show that ψ_α is self-affine and hence must induce the d_α-dimensional Hausdorff measure, which is the unique measure on SG with the self-affinity (really, self-similarity) property. Consider the desired equality:

$$\psi_\alpha(f) = \frac{1}{3} \sum_{j=1}^{3} \psi_\alpha(f \circ F_j) \qquad \text{for } f \in C(K_\alpha). \tag{5.10}$$

Indeed, notice that

$$\frac{1}{3} \sum_{j=1}^{3} \psi_{\alpha,n}(f \circ F_j) = \frac{1}{3} \sum_{j=1}^{3} 3^{-n} \cdot 2^{-1} \sum_{\epsilon \in \mathcal{J}_n \setminus \mathcal{J}_{n-1}} \sum_{s\in\{+,-\}} f(F_j(\epsilon^s))$$

$$= \frac{1}{3^{n+1} \cdot 2} \sum_{\epsilon \in \mathcal{J}_{n+1} \setminus \mathcal{J}_n} \sum_{s\in\{+,-\}} f(\epsilon^s)$$

$$= \psi_{\alpha,n+1}(f)$$

and letting $n \to \infty$ shows that (5.10) holds. Thus, $\psi_{\alpha,n} \to \psi_\alpha$, where $\psi_\alpha(f) = \int_{K_\alpha} f(x) \, d\mathcal{H}^{d_\alpha}$.

∎

Lemma 36 *For the spectral triple $S(K_\alpha)$ of dimension $\mathfrak{d} = \mathfrak{d}(K_\alpha) = d_\alpha$,*

$$Tr_w(|D_\alpha|^{-\mathfrak{d}}) = \frac{2^{\mathfrak{d}+1}(2^{\mathfrak{d}} - 1)\zeta(\mathfrak{d})(3 + 3\alpha^{\mathfrak{d}})}{\mathfrak{d} \cdot \pi^{\mathfrak{d}}(2^{\mathfrak{d}}\log(2) - 3(1-\alpha)^{\mathfrak{d}}\log(1-\alpha))}.$$

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Proof. Using Theorem 15 and Proposition 27, as well as the fact that $\mathfrak{d} > 1$,

$$
\begin{aligned}
Tr_w(|D_\alpha|^{-\mathfrak{d}}) &= \lim_{s \to 1+} (s-1) \mathrm{tr}(|D_\alpha|^{-\mathfrak{d}s}) \\
&= \lim_{s \to 1+} (s-1) \frac{2^{\mathfrak{d}s+1}(1 - 2^{-\mathfrak{d}s})\zeta(\mathfrak{d}s)}{\pi^{\mathfrak{d}s}} \frac{2^{\mathfrak{d}s}(3 + 3\alpha^{\mathfrak{d}s})}{2^{\mathfrak{d}s} - 3(1-\alpha)^{\mathfrak{d}s}} \\
&= \frac{2^{\mathfrak{d}+1}(2^{\mathfrak{d}} - 1)\zeta(\mathfrak{d})(3 + 3\alpha^{\mathfrak{d}})}{\pi^{\mathfrak{d}}} \lim_{s \to 1+} (s-1) \frac{1}{2^{\mathfrak{d}s} - 3(1-\alpha)^{\mathfrak{d}s}} \\
&= \frac{2^{\mathfrak{d}+1}(2^{\mathfrak{d}} - 1)\zeta(\mathfrak{d})(3 + 3\alpha^{\mathfrak{d}})}{\pi^{\mathfrak{d}}} \lim_{s \to 1+} \frac{1}{\mathfrak{d}2^{\mathfrak{d}s}\log(2) - \mathfrak{d}3(1-\alpha)^{\mathfrak{d}s}\log(1-\alpha)} \\
&= \frac{2^{\mathfrak{d}+1}(2^{\mathfrak{d}} - 1)\zeta(\mathfrak{d})(3 + 3\alpha^{\mathfrak{d}})}{\mathfrak{d} \cdot \pi^{\mathfrak{d}}(2^{\mathfrak{d}}\log(2) - 3(1-\alpha)^{\mathfrak{d}}\log(1-\alpha))}.
\end{aligned}
$$

∎

The spectral dimension $\mathfrak{d} = \mathfrak{d}(K_\alpha)$ is the same as the Hausdorff dimension d_α. In what follows we will write $\mathfrak{d}$ in order to showcase how our operator algebraic tools recover fractal geometric data like the Hausdorff measure on K_α.

Theorem 37 *The spectral triple $S(K_\alpha)$ recovers the $\mathfrak{d}$-dimensional Hausdorff measure, $\mathcal{H}^{\mathfrak{d}}$, on K_α via the formula*

$$
Tr_w(\pi_\alpha(f)|D_\alpha|^{-\mathfrak{d}}) = c_{\mathfrak{d}} \int_{K_\alpha} f \, d\mathcal{H}^{\mathfrak{d}}
$$

for all $f \in C(K_\alpha)$. Moreover,

$$
c_{\mathfrak{d}} = \frac{2^{\mathfrak{d}+1}(2^{\mathfrak{d}} - 1)\zeta(\mathfrak{d})(3 + 3\alpha^{\mathfrak{d}})}{\mathfrak{d} \cdot \pi^{\mathfrak{d}}(2^{\mathfrak{d}}\log(2) - 3(1-\alpha)^{\mathfrak{d}}\log(1-\alpha))}.
$$

Proof. Let $\varepsilon > 0$ and $f \in C(K_\alpha)$. Let $n_0 \in \mathbb{N}$ such that for all $h \in \{1, 2, \ldots, 3^{n_0}\}$ and any two points x, y inside or on the triangle $\Delta_{n_0, h}$, we have $|f(x) - f(y)| < \varepsilon$. Choose $n > n_0$ and define $u^n_{n_0, h} : C(K_\alpha) \to \mathbb{R}$, as in the proof of Proposition 35:

$$
u^n_{n_0, h}(f) = \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}_{n_0, h}} \sum_{s \in \{+, -\}} f(\epsilon^s).
$$

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Denote by $K_{n_0,h}$ the portion of K_α contained in the triangle $\Delta_{n_0,h}$. Let $I_{n_0,h} = 1$ in $C(K_{n_0,h})$, $f_{n_0,h} = f|_{K_{n_0,h}}$ in $C(K_{n_0,h})$, and $f_{J_{n_0}} = f|_{J_{n_0}}$ in $C(\overline{J}_{n_0})$. Then

$$
\left| u_{n_0,h}^n(f) I_{n_0,h}(x) - f_{n_0,h}(x) \right| = \left| \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}_{n_0,h}} \sum_{s \in \{+,-\}} f(\epsilon^s) I_{n_0,h}(x) - f_{n_0,h}(x) \right|
$$

$$
= \left| \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}_{n_0,h}} \sum_{s \in \{+,-\}} (f(\epsilon^s) I_{n_0,h}(x) - f_{n_0,h}(x)) \right|
$$

$$
\leq \frac{1}{2 \cdot 3^{(n-n_0)}} \sum_{\epsilon \in \mathcal{S}_{n_0,h}} \sum_{s \in \{+,-\}} |f(\epsilon^s) I_{n_0,h}(x) - f_{n_0,h}(x)|
$$

$$
< \varepsilon;
$$

so

$$
(u_{n_0,h}^n(f) - \varepsilon) I_{n_0,h} < f_{n_0,h} < (u_{n_0,h}^n(f) + \varepsilon) I_{n_0,h}. \tag{5.11}
$$

Next note that one can define a spectral triple for $K_{n_0,h}$ and one for $\overline{J}_{n_0}$ by deleting the summands from the spectral triple for K_α which correspond to edges outside of $K_{n_0,h}$ or outside of $\overline{J}_{n_0}$, respectively. The argument that this construction does indeed gives a spectral triple for $K_{n_0,h}$ is the same as that for the spectral triple for K_α. In the case of $\overline{J}_{n_0}$, that this deletion of summands still gives a spectral triple follows from Proposition 5.1 in [8]. Denote by $\mathrm{Tr}_w(\pi_{n_0,h}(f_{n_0,h})|D_{n_0,h}|^{-\partial})$ and $\mathrm{Tr}_w(\pi_{J_{n_0}}(f_{J_{n_0}})|D_{J_{n_0}}|^{-\partial})$ the positive linear functionals respectively associated to these triples. Using the fact that

$$
\sigma(D_\alpha) = \bigcup_{h=1}^{3^{n_0}} \sigma(D_{n_0,h}) \cup \sigma(D_{J_{n_0}})
$$

and that as operators $\pi_\alpha(f) = \oplus_{h=1}^{3^{n_0}} \pi_{n_0,h}(f_{n_0,h}) \oplus \pi_{J_{n_0}}(f_{J_{n_0}})$, we have

$$
\mathrm{Tr}_w(\pi_\alpha(f)|D_\alpha|^{-\partial}) = \sum_{h=1}^{3^{n_0}} \mathrm{Tr}_w(\pi_{n_0,h}(f_{n_0,h})|D_{n_0,h}|^{-\partial}) + \mathrm{Tr}_w(\pi_{J_{n_0}}(f_{J_{n_0}})|D_{J_{n_0}}|^{-\partial}).
$$

Next we show that $\mathrm{Tr}_w(\pi_{J_{n_0}}(f_{J_{n_0}})|D_{J_{n_0}}|^{-\partial}) = 0$. Note that

$$\lim_{s\to 1^+}(s-1)\mathrm{tr}(|D_{J_{n_0}}|^{-\partial s}) = \lim_{s\to 1^+}(s-1)\sum_{j=1}^{n_0}\beta_{\partial s}\,3^j\alpha^{\partial s}\left(\frac{1-\alpha}{2}\right)^{\partial s(j-1)} = 0$$

since

$$\sum_{j=1}^{n_0}\beta_{\partial s}3^j\alpha^{\partial s}\left(\frac{1-\alpha}{2}\right)^{\partial s(j-1)}$$

converges as $s \to 1^+$ and hence

$$\mathrm{Tr}_w(|D_{J_{n_0}}|^{-\partial}) = \lim_{s\to 1^+}(s-1)\mathrm{tr}(|D_{J_{n_0}}|^{-\partial s}) = 0.$$

For a continuous function $f_{J_{n_0}}$ on the closed set $\overline{J}_{n_0}$, there is an M such that $|f_{J_{n_0}}| \leq M$.

Since $\mathrm{Tr}_w(\pi_{J_{n_0}}(\cdot)|D_{J_{n_0}}|^{-\partial})$ is a positive linear functional on $C(\overline{J}_{n_0})$, we know that

$$\mathrm{Tr}_w(\pi_{J_{n_0}}(f_{J_{n_0}})|D_{J_{n_0}}|^{-\partial}) \leq \mathrm{Tr}_w(M|D_{J_{n_0}}|^{-\partial}) = M\mathrm{Tr}_w(|D_{J_{n_0}}|^{-\partial}).$$

It follows that $\mathrm{Tr}_w(\pi_{J_{n_0}}(f_{J_{n_0}})|D_{J_{n_0}}|^{-\partial}) = 0$ and

$$\mathrm{Tr}_w(\pi_\alpha(f)|D_\alpha|^{-\partial}) = \sum_{h=1}^{3^{n_0}}\mathrm{Tr}_w(\pi_{n_0,h}(f_{n_0,h})|D_{n_0,h}|^{-\partial}).$$

Also note that

$$\mathrm{Tr}_w(\pi_{n_0,h}(I_{n_0,h})|D_{n_0,h}|^{-\partial}) = 3^{-n_0}\mathrm{Tr}_w(\pi_\alpha(I)|D_\alpha|^{-\partial}) = 3^{-n_0}\mathrm{Tr}_w(|D_\alpha|^{-\partial}).$$

Using the inequalities (5.11), the fact that $\mathrm{Tr}_w(\pi_\alpha(\cdot)|D_\alpha|^{-\partial})$ is a positive linear functional

on $C(K_\alpha)$, and summing, gives

$$\sum_{h=1}^{3^{n_0}}(u_{n_0,h}^n(f)-\varepsilon)(3^{-n_0}\mathrm{Tr}_w(|D_\alpha|^{-\partial})) \leq \mathrm{Tr}_w(\pi_\alpha(f)|D_\alpha|^{-\partial}) \leq \sum_{h=1}^{3^{n_0}}(u_{n_0,h}^n(f)+\varepsilon)(3^{-n_0}\mathrm{Tr}_w(|D_\alpha|^{-\partial})),$$

which is the same as

$$\left|\mathrm{Tr}_w(\pi_\alpha(f)|D_\alpha|^{-\partial}) - 3^{-n_0}\mathrm{Tr}_w(|D_\alpha|^{-\partial})\sum_{h=1}^{3^{n_0}}u_{n_0,h}^n(f)\right| \leq 3^{-n_0}\mathrm{Tr}_w(|D_\alpha|^{-\partial})\varepsilon. \qquad (5.12)$$

In Proposition 35 we showed that the functionals $\psi_{\alpha,n}$ converge to the functional ψ_α in the weak-* topology on the dual of $C(K_\alpha)$ and that for $n \geq n_0$ we can write $\psi_{\alpha,n}(f) = 3^{-n_0} \sum_{h=1}^{3^{n_0}} u_{n_0,h}^n(f)$; so (after possibly choosing n_0 to be larger)

$$\left| \psi_\alpha(f) - 3^{-n_0} \sum_{h=1}^{3^{n_0}} u_{n_0,h}^n(f) \right| < \varepsilon$$

and multiplying by $\mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})$,

$$\left| \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\psi_\alpha(f) - \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})3^{-n_0} \sum_{h=1}^{3^{n_0}} u_{n_0,h}^n(f) \right| \leq \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\varepsilon. \qquad (5.13)$$

Note by Lemma 36 that the value $\mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})$ is positive so the inequality above is preserved. Then using the estimates (5.12) and (5.13),

$$\left| \mathrm{Tr}_w(\pi_\alpha(f)|D_\alpha|^{-\mathfrak{d}}) - \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\psi_\alpha(f) \right|$$

$$\leq \left| \mathrm{Tr}_w(\pi_\alpha(f)|D_\alpha|^{-\mathfrak{d}}) - \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\psi_{\alpha,n}(f) \right| + \left| \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\psi_{\alpha,n}(f) - \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\psi_\alpha(f) \right|$$

$$\leq 3^{-n_0} \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\varepsilon + \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\varepsilon$$

$$= (3^{-n_0} + 1)\, \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\varepsilon,$$

from which it follows that

$$\mathrm{Tr}_w(\pi_\alpha(f)|D_\alpha|^{-\mathfrak{d}}) = \mathrm{Tr}_w(|D_\alpha|^{-\mathfrak{d}})\psi_\alpha(f).$$

Using Lemma 36 we have

$$\mathrm{Tr}_w(\pi_\alpha(f)|D_\alpha|^{-\mathfrak{d}}) = c_\mathfrak{d} \int_{K_\alpha} f \, d\mathcal{H}^\mathfrak{d},$$

as desired. ∎

Chapter 6

Energy Form on K_α

In this chapter we introduce the energy form on K_α constructed by Alonso-Ruiz and Freiberg in [2]. We show how one can use a spectral triple to recover the energy on SG by following the construction given by Cipriani, Guido, Isola, and Sauvageot in [6], but using a different operator in the spectral triple.

Recall the sets T_n which contain the triangle edges in the n-th level approximation of the stretched Sierpinski gasket

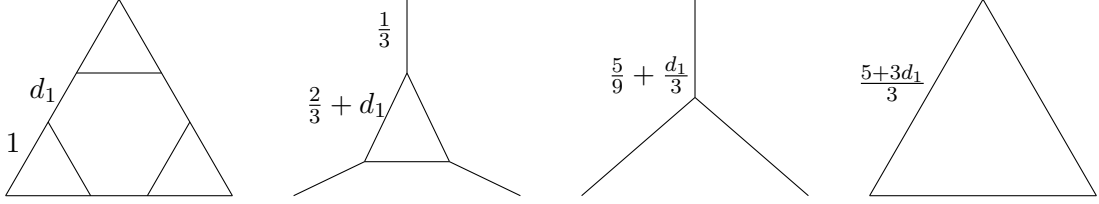$$T_0 = \{[p_j \to p_i] : i, j = 1, 2, 3 \text{ and } j \neq i\}$$

(the edges in the outer triangle) and for $n \geq 1$,

$$T_n = \{[x \to y] : \exists \, w \in \{1, 2, 3\}^n \text{ such that } x, y \in F_w(V_0^s)\}$$

(edges in the triangles at level n). We also have the joining edges in K_α: $J_0 = \emptyset$ and for $n \geq 1$

$$J_n = \bigcup_{m=0}^{n-1} \bigcup_{w \in \{1,2,3\}^m} F_w\left(\bigcup_{i=1}^{3} e_i\right),$$

Figure 6.1: Renormalization constants for stage $n = 1$.

where e_1, e_2, e_3 are the three initial joining edges. Also, $J^* = \cup_{n \geq 1} J_n$.

Again, we distinguish between the collection of *points* in K_α which lie in the sets J_n and the collection of *edges* that make up the set J_n by writing $\mathcal{J}_n$ for the collection of joining edges at stage n, which include the endpoints:

$$\mathcal{J}_n = \bigcup_{m=0}^{n-1} \bigcup_{w \in \{1,2,3\}^m} \{F_w(\overline{e_i}) : i = 1, 2, 3\} \qquad \text{for } n \geq 1$$

and $\mathcal{J}^* = \cup_{n \geq 1} \mathcal{J}_n$. Finally, recall that $\mathscr{E}_n = T_n \cup \mathcal{J}_n$.

6.1 Defining an Energy Form on K_α

We now construct a quadratic form on the stretched Sierpinski gasket. This quadratic form was given by Alonso Ruiz and Freiberg in [2] and is constructed in a similar way to $\mathcal{E}$ on SG. First one defines quadratic forms for functions on sets that approximate K_α, then one renormalizes these quadratic forms and takes a limit to construct a quadratic form for functions on K_α.

Let $U_n = V_n^s \cup J_n$ for $n \geq 0$ and $U_* = \bigcup_{n \geq 0} U_n$. We will use these sets to approximate K_α.

Definition 6.1.0.1 *Let $\mathcal{D}_0 := \{u : U_0 \to \mathbb{R}\}$ and for $n \geq 1$ let*

$$\mathcal{D}_n := \{u : U_n \to \mathbb{R} \mid u|_e \in H^1(e, dx) \text{ for all } e \in \mathcal{J}_n\}$$

where $H^1(e, dx)$ is the Sobolev space of functions on the interval corresponding to e and dx is Lebesgue measure. Define

$$E_n^d(u) = \sum_{x \sim_n y} (u(x) - u(y))^2 \quad and \quad E_n^c(u) = \int_{J_n} |\nabla u|^2 dx = \sum_{\epsilon \in \mathcal{J}_n} \int_0^{L(\epsilon)} |(u \circ R_\epsilon^s(t))'|^2 dt$$

where $x \sim_n y$ means that x and y are in V_n^s and are connected via an edge in T_n. Set $E_n(u) = E_n^d(u) + E_n^c(u)$.

In the notation $E_n^d(\cdot)$ and $E_n^c(\cdot)$, the superscripts d and c label the discrete and continuous parts of the quadratic form $E_n(\cdot)$. Once again we define harmonic functions, h, as functions which minimize the value of $E_n(h)$.

Definition 6.1.0.2 *Let $h : U_n \to \mathbb{R}$. If given the values of h on U_0, $E_k(h)$ is minimized for each $k = 1, 2, \ldots, n$, then we say h is a **harmonic function**.*

The following proposition given in [2], gives a simple rule by which one can extend functions on U_n harmonically to U_{n+1}. This rule is analogous to the $\frac{2}{5}$-$\frac{1}{5}$ rule we described for harmonic functions on SG.

Proposition 6.1.0.3 ([2]) *Let $d_0 = 0$ and $d_n = \alpha(\frac{1-\alpha}{2})^{n-1}$ for any $n \in \mathbb{N}$. For any function $u \in \mathcal{D}_n$,*

$$\inf\{E_{n+1}(v) \mid v \in \mathcal{D}_{n+1} \text{ and } v|_{U_n} = u\}$$

is attained by a unique function $\tilde{u} \in \mathcal{D}_{n+1}$ defined on each $p_{iwj} = F_{wi}(p_j) \in W_{n+1}$ by

$$\tilde{u}(p_{iwj}) = \frac{2 + 3d_n}{5 + 3d_n} u(p_{iwi}) + \frac{2}{5 + 3\alpha} u(p_{jwj}) + \frac{1}{5 + 3\alpha} u(p_{kwk})$$

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for $wi \in \{1,2,3\}^{n+1}$, $\{i,j,k\} = \{1,2,3\}$, and linear interpolation on $J_{n+1} \setminus J_n$.

We would like to have a quadratic form on K_α which is invariant under harmonic extension in the way that $\mathcal{E}(\cdot)$ is. For this, one uses the maps $E_n(\cdot)$ and computes renormalization constants using techniques for manipulating electrical networks. Specifically, one uses the $\Delta - Y$ transform. See 6.2 for an example of how we confirm that the constants r_n^d and r_n^c defined below give the desired renormalization. Here the superscripts are to distinguish between the *discrete* renormalization constants and the *continuous* ones.

Define for $n \geq 1$ the constants

$$r_n^d := \frac{3}{5 + 3d_n} \quad \text{and} \quad r_n^c := r_n^d d_n = \frac{3d_n}{5 + 3d_n}.$$

The renormalization constants are given by $\rho_0^d = 1$ and for $n \geq 1$ let

$$\rho_n^d := \prod_{i=1}^{n} r_i^d \quad \text{and} \quad \rho_n^c := \rho_{n-1}^d r_n^c = \left(\prod_{i=1}^{n} \frac{3}{5+3d_i} \right) d_n.$$

Definition 6.1.0.4 *For each $n \geq 1$, define $\mathcal{E}_n^{s,d} : \mathcal{D}_n \to \mathbb{R}$ and $\mathcal{E}_n^{s,c} : \mathcal{D}_n \to \mathbb{R}$ by*

$$\mathcal{E}_n^{s,d}(u) := \frac{1}{\rho_n^d} E_n^d(u) \text{ and } \mathcal{E}_n^{s,c}(u) := \sum_{k=1}^{n} \frac{1}{\rho_k^c} E_{k^-}^c(u),$$
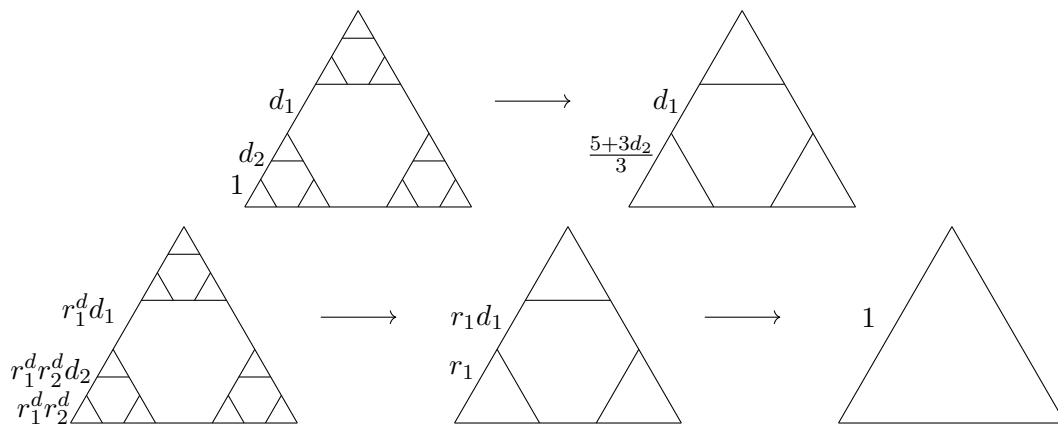
where $E_{k^-}^c(u) := \sum_{\epsilon \in \mathcal{J}_k \setminus \mathcal{J}_{k-1}} \int_0^{L(\epsilon)} |(u \circ R_\epsilon^s(t))'| g^2 dt$. Let $\mathcal{E}_n^s(u) := \mathcal{E}_n^{s,d}(u) + \mathcal{E}_n^{s,c}(u)$.

The proposition below was given in [2] and shows that when u is extended harmonically, the value of $\mathcal{E}_n^s(u)$ stay constant as n increases.

Proposition 6.1.0.5 ([2]) *Let $u_n : U_n \to \mathbb{R}$ be the harmonic extension to level $n \geq 1$ of a function $u_0 : U_0 \to \mathbb{R}$, then*

$$\mathcal{E}_0^s(u_0) = \mathcal{E}_n^s(u_n).$$

Figure 6.2: Renormalization constants for stage $n = 2$.

Definition 6.1.0.6 *Define the space*

$$\mathcal{D}_* := \{u : U_* \to \mathbb{R} \mid u|_{U_n} \in \mathcal{D}_n, \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n \to \infty} \mathcal{E}_n^s(u|_{U_n}) < \infty\},$$

and

$$\mathcal{E}^s(u) := \lim_{n \to \infty} \mathcal{E}_n^s(u|_{U_n})$$

for $u \in \mathcal{D}_*$.

The following is a useful characterization of certain quadratic forms on $\mathbb{R}^3$. This result was first stated and used in [6] without proof. We give a proof here for completeness.

Proposition 6.1.0.7 *Up to a positive constant, there exists a unique nontrivial positive semidefinite quadratic form $Q : \mathbb{R}^3 \to \mathbb{R}$ such that Q vanishes on constant vectors $(a, a, a) \in \mathbb{R}^3$ and is invariant under permutation of the components of $v = (v_1, v_2, v_3) \in \mathbb{R}^3$.*

Proof. Let q be a quadratic form satisfying the stated conditions. Since q is a quadratic

form it is given by a quadratic polynomial

$$q(v) = a_1 v_1^2 + a_2 v_2^2 + a_3 v_3^2 + a_{1,2} v_1 v_2 + a_{1,3} v_1 v_3 + a_{2,3} v_2 v_3.$$

Evaluating at $(1,0,0), (0,1,0), (0,0,1)$ and using the fact that q is invariant under permutation of the components of a vector, gives $a := a_1 = a_2 = a_3$. Note that since q is positive semidefinite and nontrivial, $a > 0$. Now evaluate at $(1,1,0), (1,0,1), (0,1,1)$ and again by the invariance property,

$$2a + a_{1,2} = 2a + a_{1,3} = 2a + a_{2,3}$$

so that $a_{1,2} = a_{1,3} = a_{2,3}$. Evaluation at $(1,1,1)$ gives $3a + 3a_{1,2} = 0$ and hence $q(v) = av_1^2 + av_2^2 + av_3^2 - av_1 v_2 - av_1 v_3 - av_2 v_3$. It now follow that any form with the stated properties must be a constant multiple of

$$v \longmapsto v_1^2 + v_2^2 + v_3^2 - v_1 v_2 - v_1 v_3 - v_2 v_3.$$

∎

The result above is more general and holds for $\mathbb{R}^n$. The proof of the more general result is the same as the $n = 3$ case, so we omit it.

Proposition 6.1.0.8 *Up to a positive constant, there exists a unique nontrivial positive semidefinite quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$ such that Q vanishes on constant vectors $(a, a, \ldots, a) \in \mathbb{R}^n$ and is invariant under permutation of the components of $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$.*

We have seen that $\mathcal{E}$ induces a positive semidefinite quadratic form which vanishes on constants. Note that if one permutes the values of a function, h, on V_0 and extends harmonically, one simply permutes the values of h on each set of vertices V_m. Since the

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sum in $\mathcal{E}(h)$ ranges over all points in V_m, a permutation of the points in V_0 simply switches the order of the terms in the sum $\sum_{x\sim_m y}(h(x) - h(y))^2$. Hence the value of $\mathcal{E}(h)$ does not change.

6.2 Recovering the Energy on the Sierpinski Gasket

In this section we give the construction of a spectral triple on SG which will recover the resistance form $\mathcal{E}(\cdot)$ on SG. We begin by setting up the motivation for the choice of operator D in the spectral triple. In what follows we fix $a \in (0, 1)$.

6.2.1 Building a Spectral Triple

Definition 6.2.1.1 *Define for $f \in L^2([-\pi, \pi], \frac{1}{2\pi}m)$, the norm*

$$\|f\|_{(a,2)} = \left(\|f\|_2^2 + \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} dx \, dy \right)^{1/2}.$$

The space

$$H^{a,2}([-\pi, \pi]) = \{f \in L^2([-\pi, \pi]) : \|f\|_{(a,2)} < \infty\}$$

*is an $(a, 2)$-**Sobolev space**.*

Consider the space

$$\operatorname{dom}(\mathcal{E}) = \{u : U_* \to \mathbb{R} \mid u|_{U_n} \in \mathcal{D}_n, \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} \mathcal{E}_n^s(u|_{U_n}) < \infty\},$$

equipped with the norm

$$\|f\|_{\mathcal{E}} := \left(\sum_{x \in U_0} f(x)^2 + \mathcal{E}(f) \right)^{1/2}.$$

Place the Sierpinski gasket in $\mathbb{R}^2$ so that the bottom edge of SG lies on $[-\pi, \pi]$. The following trace and extension results of Jonsonn from [16] will be essential in proving the recovery of the energy $\mathcal{E}$ on SG by our soon to be defined spectral triple. These results also help motivate the definition of the operator D in our spectral triple for SG.

Theorem 6.2.1.2 (Jonsson) *Let SG be the Sierpinski gasket, $d = \ln 3/\ln 2$, $\beta = \ln 5/\ln 4$, and $\alpha_0 = \beta - (d-1)/2$. The restriction map*

$$r : \mathrm{dom}(\mathcal{E}) \to B_{\alpha_0}^{2,2}([-\pi, \pi]) = H^{\alpha_0,2}([-\pi, \pi])$$

is a bounded linear operator. That is, there exists a constant $C > 0$ such that

$$\|f|_{[-\pi,\pi]}\|_{(\alpha_0,2)} \leq C\|f\|_{\mathcal{E}}.$$

In [16], Jonsonn also constructs an extension operator and gives the following theorem.

Theorem 6.2.1.3 (Jonsson) *Let SG be the Sierpinski gasket, $d = \ln 3/\ln 2$, $\beta = \ln 5/\ln 4$, and $\alpha_0 = \beta - (d-1)/2$. Then there is a bounded linear operator E from $B_{\alpha_0}^{2,2}[-\pi, \pi] = H^{\alpha_0,2}[-\pi, \pi]$ to $\mathrm{dom}(\mathcal{E})$ which is an extension operator in the sense that the pointwise restriction to $[-\pi, \pi]$ of the continuous function $Ef \in \mathrm{dom}(\mathcal{E})$ is f.*

As was pointed out by Jonsonn in [16], the number β is a smoothness index. The drop of smoothness when taking the trace of $\mathrm{dom}(\mathcal{E})$ to $[-\pi, \pi]$ is $\beta - \alpha_0 = (d-1)/2$. This matches the situation in $\mathbb{R}^n$ where the trace of $B_\beta^{2,2}(\mathbb{R}^n)$ is $B_{\alpha_0}^{2,2}(\mathbb{R}^n)$ for $\alpha_0 = \beta - (n-m)/2$.

We would like the operator D in our spectral triple to be made up of operators T_a which have the property that

$$v = (v_1, v_2, v_3) \to \|T_a h_v\|_2^2,$$

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where h_v is the harmonic function on SG defined by v on V_0, is the unique quadratic form Q on $\mathbb{R}^3$ satisfying the properties:

- positive semidefinite,

- invariant under permutations of the components of $v \in \mathbb{R}^3$, and

- vanishes on the constant vectors, $v = (v_0, v_0, v_0)$.

This will help show that our spectral triple can recover the energy of functions f that are finitely harmonic, meaning that, for some $m \geq 0$, given the values of f on V_m, f minimizes the values of $E_j(f)$ for $j \geq m$.

In order to recover the energy of an arbitrary function in $\mathrm{dom}(\mathcal{E})$ we need the result of Jonsson stated above and we need for our operator T_a to be such that

$$f \to (\|f\|_2^2 + \|T_a \tilde{f}\|_2^2)^{1/2}$$

is a seminorm on $H^{a,2}([-\pi, \pi])$ equivalent to $\| \cdot \|_{(a,2)}$. Here the function $\tilde{f}$ is a suitable "version" of the function f with domain $[-\pi, \pi]$ rather than SG. This will give a key inequality:

$$\|T_a f\|_2^2 \leq C \, \mathcal{E}(f)$$

for all $f \in \mathrm{dom}(\mathcal{E})$.

Let $[-\pi, \pi]^2 = [-\pi, \pi] \times [-\pi, \pi]$. If we look at the quantity

$$[f]_{H^{a,2}} := \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} dx \, dy$$

from the definition of the seminorm $\| \cdot \|_{(a,2)}$, we see that it is the $L^2([-\pi, \pi]^2, \frac{1}{4\pi^2} m)$ norm of the function

$$\frac{f(x) - f(y)}{|x - y|^{1/2+a}}.$$

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This motivates the following definition.

Definition 6.2.1.4 *Let $L^2[-\pi, \pi]$ be the space of square integrable functions with normalized Lebesgue measure: $\frac{1}{2\pi}m$. Define $T_a : \text{dom}(T_a) \subseteq L^2[-\pi, \pi] \to L^2([-\pi, \pi]^2)$ by*

$$(T_a f)(x, y) = \frac{f(x) - f(y)}{|x - y|^{1/2+a}}$$

where

$$\text{dom}(T_a) = \left\{ f \in L^2[-\pi, \pi] : \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} dx \, dy < \infty \right\}.$$

We will also make use of known results concerning Sobolev spaces and fractional powers of the Laplace operator.

Proposition 6.2.1.5 *Let $\mathbb{T}$ denote the torus, $f \in L^2(\mathbb{T})$, and $a \in (0, 1)$. The following are equivalent definitions for the **fractional Laplacian** $(-\Delta)^a$:*

1. *$(-\Delta)^a$ is given by the expression*

$$(-\Delta)^a u = \mathscr{F}^{-1}(|\xi|^{2a} \mathscr{F} u)$$

2. *$(-\Delta)^a$ is the unique operator such that $\langle (-\Delta)^a f, \phi \rangle = E_a(f, \phi)$ for all $\phi \in H^a(\mathbb{T})$, the fractional Sobolev space. The quadratic form, E_a, is given by*

$$E_a(f, g) := \frac{c_a}{4\pi^2} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{(f(x) - f(y))(\overline{g(x)} - \overline{g(y)})}{|x - y|^{1+2a}} \, dx \, dy,$$

where $c_a = \frac{2^{2a} \Gamma(\frac{1+2a}{2})}{2\pi^{1/2} |\Gamma(-a)|}$.

We will need to define a $C([-\pi, \pi])$-bimodule structure on the space $L^2([-\pi, \pi]^2)$. Define the left and right actions of $C([-\pi, \pi])$ on $L^2([-\pi, \pi]^2)$ by

$$(f\psi)(x, y) = f(x)\psi(x, y) \text{ and } (\psi f)(x, y) = \psi(x, y)f(y)$$

where $\psi \in L^2([-\pi, \pi]^2)$ and $\psi \in C([-\pi, \pi])$. For $f \in C[-\pi, \pi]$ define $f^*(x) = \overline{f(-x)}$.

Notice that

$$\langle (-\Delta)^a f, f \rangle = E_a(f, f) = c_a \|T_a f\|_2^2 = c_a \langle T_a f, T_a f \rangle = c_a \langle T_a^* T_a f, f \rangle$$

so that

$$\left(c_a^{1/2} T_a \right)^* \left(c_a^{1/2} T_a \right) = c_a T_a^* T_a = (-\Delta)^a.$$

This shows that $c_a^{1/2} T_a$ is acting as a sort of square root for $(-\Delta)^a$. Define $(-\Delta)^{a/2}$ by

$(-\Delta)^{a/2}(f) = \mathscr{F}^{-1}(|\xi|^a \mathscr{F}(f))$. One can quickly see that $((-\Delta)^{a/2})^2 = (-\Delta)^a$. It is known

that $(-\Delta)^{a/2}$ is self-adjoint, so it follows that

$$\langle (-\Delta)^{a/2}(f), (-\Delta)^{a/2}(g) \rangle = \langle (-\Delta)^a f, g \rangle = c_a \langle T_a f, T_a g \rangle$$

so in particular

$$\|(-\Delta)^{a/2} f\|_2^2 = c_a \|T_a f\|_2^2.$$

Notice that for $e_k(x) = e^{ikx}, e_j(x) = e^{ijx}$, we have

$$\langle (-\Delta)^{a/2} e_k, (-\Delta)^{a/2} e_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-\Delta)^{a/2} e_k(x) \overline{(-\Delta)^{a/2} e_j(x)} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} |k|^a e_k(x) |j|^a e_{-j}(x) dx$$

$$= |k|^{2a} \delta_{k,j}.$$

Also notice that T_a satisfies the product rule:

$$T_a(fg)(x, y) = \frac{f(x)g(x) - f(y)g(y)}{|x - y|^{1/2+a}}$$

$$= \frac{f(x)(g(x) - g(y))}{|x - y|^{1/2+a}} + \frac{(f(x) - f(y))g(y)}{|x - y|^{1/2+a}}$$

$$= f(x)(T_a g)(x, y) + (T_a f)(x, y)g(y).$$

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We can now begin to construct a spectral triple for the stretched Sierpinski gasket.

Definition 6.2.1.6 *Define* $M_a : \operatorname{dom}(T_a^*) \oplus L^2([-\pi, \pi]) \to \operatorname{dom}(T_a^*) \oplus L^2([-\pi, \pi])$ *by*

$$
M_a := \begin{pmatrix} 0 & T_a \\ T_a^* & 0 \end{pmatrix}.
$$

Given a finite length word w *define* $D_w := 2^{|w|} M_a$ *where* $|w|$ *is the length of the word* w.

Define $D_a := \bigoplus_{w \in W} D_w$ *where* $W := \bigcup_{m \geq 0} \{1, 2, 3\}^m$.

The operator D_a will be the operator in our spectral triple and

$$
\mathscr{H} := \bigoplus_{w \in W} L^2([-\pi, \pi]^2) \oplus L^2([-\pi, \pi])
$$

will be the Hilbert space. We now describe a representation of $C(SG)$ on $\mathscr{H}$.

Let $R : [-\pi, \pi] \to T$ be the isometry mapping the interval to the initial triangle in SG. For $w \in W$ define $\pi_w : C(SG) \to L^2([-\pi, \pi]^2) \oplus L^2([-\pi, \pi])$ by

$$
\pi_w(f)(h) := (f \circ F_w \circ R)h
$$

and define $\pi_a : C(SG) \to \mathscr{B}(\mathscr{H})$ by $\pi_a := \bigoplus_{w \in W} \pi_w$.

Definition 6.2.1.7 *Define the spectral triple for the Sierpinski gasket by* $ST(SG) := (C(SG), \mathscr{H}, D_a)$.

Proposition 6.2.1.8 *Let* $e_k = e^{ikx}$ *and recall that* $c_a = \frac{2^{2a} \Gamma(\frac{1+2a}{2})}{2\pi^{1/2} |\Gamma(-a)|}$ *for* $a \in (0, 1)$.

(a) *We have* $E_a(e_k, e_j) = |k|^{2a} \delta_{k,j}$.

(b) *The collection* $\{e'_k := c_a^{1/2} |k|^{-a} T_a e_k : k \neq 0\}$ *is an orthonormal basis for the range of* T_a.

(c) The following holds

$$T_a^*((T_a e_k) e_j) = (2c_a)^{-1}(|k|^{2a} + |k+j|^{2a} - |j|^{2a}) e_{k+j}.$$

(d) Let $M_{a,f} : C(\mathbb{T}) \to L^2(\mathbb{T} \times \mathbb{T})$ denote the multiplication operator given by $M_{a,f} g = (T_a f) g$. Then

$$\text{tr}((T_a T_a^*)^{-s/4} M_{a,f} M_{a,f}^* (T_a T_a^*)^{-s/4}) \leq 2c_a^{s/2-1} \zeta(sa) E_a(f) = \text{tr}((T_a^* T_a)^{-s/4} M_{a,f}^* M_{a,f} (T_a^* T_a)^{-s/4})$$

Proof. (a) We know $E_a(f, g) = \langle (-\Delta)^a f, g \rangle$ and so

$$E_a(e_k, e_j) = \langle (-\Delta)^{a/2} e_k, (-\Delta)^{a/2} e_j \rangle$$

$$= |k|^{2a} \delta_{k,j}.$$

(b) Notice

$$\langle e_k', e_j' \rangle = c_a |k|^{-a} |j|^{-a} \langle T_a e_k, T_a e_j \rangle$$

$$= c_a |k|^{-2a} \frac{1}{c_a} |k|^{2a} \delta_{k,j}$$

$$= \delta_{k,j}$$

and hence $\{e_k'\}_{k \in \mathbb{Z} \setminus \{0\}}$ is an orthonormal set. Recall that the set of finite linear combinations of the functions $\{e_k\}$ is a dense subset of $C([-\pi, \pi])$ and hence $L^2[-\pi, \pi]$. We would like the set of finite linear combinations of the functions $\{T_a e_k\}$ to be dense in the range of T_a. Suppose $T_a f \in L^2([-\pi, \pi]^2)$. Since $f \in L^2[-\pi, \pi]$ there is a sequence $f_n = \sum_{j=1}^{N_n} a_{n,j} e_{n,j}$

where $a_{n,j} \in \mathbb{C}$, $e_{n,j} \in \{e_k\}$, and $f_n \to f$. Then

$$
\begin{aligned}
T_a f_n &= \sum_{j=1}^{N_n} a_{n,j} T_a e_{n,j} \\
&= \sum_{j=1}^{N_n} a_{n,j} \frac{e_{n,j}(x) - e_{n,j}(y)}{|x-y|^{1/2+a}} \\
&= \frac{1}{|x-y|^{1/2+a}} \left(\sum_{j=1}^{N_n} a_{n,j} e_{n,j}(x) - \sum_{j=1}^{N_n} a_{n,j} e_{n,j}(y) \right).
\end{aligned}
$$

The last term tends to $\frac{f(x)-f(y)}{|x-y|^{1/2+a}}$ as n tends to infinity. Thus, the algebra of finite linear combinations of functions $\{T_a e_k\}$ is dense in $L^2([-\pi, \pi]^2)$ and hence the collection $\{e'_k\}$ is an orthonormal basis for the range of T_a.

(c) We will check that $\langle T_a e_p, (T_a e_k) e_j \rangle = (2c_a)^{-1}(|k+j|^{2a} + |k|^{2a} - |j|^{2a})\delta_{p,k+j}$. First notice that for $f \in \mathrm{dom}(T_a)$,

$$
\langle (T_a f)e_n, (T_a f)e_n \rangle = \int (T_a f)e_n \overline{(T_a f)e_n} = \int (T_a f)e_n \overline{(T_a f)}e_{-n} = \langle T_a f, T_a f \rangle
$$

and similarly $\langle e_n(T_a f), e_n(T_a f) \rangle = \langle T_a f, T_a f \rangle$. Additionally, since T_a satisfies the product rule and $e_{k+j} = e_{j+k}$,

$$
T_a(e_{k+j})(x,y) = e_j(x)(T_a e_k)(x,y) + (T_a e_j)(x,y)e_k(y)
$$

$$
= e_k(x)(T_a e_j)(x,y) + (T_a e_k)(x,y)e_j(y)
$$

If $p = k + j$, then by the polarization identity and the above facts we have,

$$
\begin{aligned}
\langle T_a e_{k+j}, (T_a e_k)e_j \rangle &= \frac{1}{2} \left(\langle T_a e_{k+j}, T_a e_{k+j} \rangle + \langle (T_a e_k)e_j, (T_a e_k)e_j \rangle - \langle e_k(T_a e_j), e_k(T_a e_j) \rangle \right) \\
&= \frac{1}{2} \left(\langle T_a e_{k+j}, T_a e_{k+j} \rangle + \langle T_a e_k, T_a e_k \rangle - \langle T_a e_j, T_a e_j \rangle \right) \\
&= \frac{1}{2} \left(c_a^{-1}|k+j|^{2a} + c_a^{-1}|k|^{2a} - c_a^{-1}|j|^{2a} \right) \\
&= \frac{1}{2} c_a^{-1} \left(|k+j|^{2a} + |k|^{2a} - |j|^{2a} \right).
\end{aligned}
$$

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If $p \neq k + j$ then

$$\langle T_a e_{k+j}, (T_a e_k) e_j \rangle = \iint \frac{(e_p(x) - e_p(y))}{|x - y|^{1/2 + a}} \cdot \overline{\frac{e_k(x) e_j(y) - e_k(y) e_j(y)}{|x - y|^{1/2 + a}}} \, dx dy$$

$$= \iint \frac{e_p(x) \overline{e_k(x) e_j(y)}}{|x - y|^{1/2 + a}} - \frac{e_p(x) \overline{e_k(y) e_j(y)}}{|x - y|^{1/2 + a}} - \frac{e_p(y) \overline{e_k(x) e_j(y)}}{|x - y|^{1/2 + a}} + \frac{e_p(y) \overline{e_k(y) e_j(y)}}{|x - y|^{1/2 + a}} \, dx dy$$

$$= 0.$$

(d) Recall that $E_a(f) = \|(-\Delta)^{a/2} f\|_2^2$ so that

$$\|M_{a,f}^*(T_a e_k)\|_2^2 = \sum_{n=1}^{\infty} |\langle e_n, M_{a,f}^*(T_a e_k) \rangle|^2$$

$$= \sum_{n=1}^{\infty} |\langle M_{a,f} e_n, T_a e_k \rangle|^2$$

$$= \sum_{n=1}^{\infty} |\langle (T_a f) e_n, T_a e_k \rangle|^2$$

$$= \sum_{n=1}^{\infty} |\langle (T_a f), (T_a e_k) e_{-n} \rangle|^2$$

$$= \frac{1}{4} |c_a|^{-2} \sum_{n=1}^{\infty} (|k|^{2a} + |n - k|^{2a} - |n|^{2a})^2 |\langle f, e_{k-n} \rangle|^2$$

$$= \frac{1}{4} |c_a|^{-2} \sum_{p=1}^{\infty} (|k|^{2a} + |p|^{2a} - |p - k|^{2a})^2 |\langle f, e_p \rangle|^2$$

$$\leq |c_a|^{-2} |k|^{2a} \sum_{p=1}^{\infty} |p|^{2a} |\langle f, e_p \rangle|^2$$

$$= |c_a|^{-2} |k|^{2a} E_a(f)$$

where we have used the fact that

$$|p|^{2a} + |k|^{2a} - |k - p|^{2a} \leq 2|k|^a |p|^a.$$

For a proof of this fact see [6]. Next, since

$$(T_a T_a^*) T_a e_k = T_a (T_a^* T_a) e_k = \frac{1}{c_a} |k|^{2a} T_a e_k$$

the functions $T_a e_k$ are eigenfunctions for the operator $T_a T_a^*$ with eigenvalue $\frac{1}{c_a}|k|^{2a}$. Using this fact we have,

$$
\begin{aligned}
\mathrm{tr}((T_a T_a^*)^{-s/4} M_{a,f} M_{a,f}^* (T_a T_a^*)^{-s/4}) &= \sum_{k\in\mathbb{Z}} \langle (T_a T_a^*)^{-s/4} e_k' \ , \ M_{a,f} M_{a,f}^* (T_a T_a^*)^{-s/4} e_k' \rangle \\
&= \sum_{k\in\mathbb{Z}} c_a |k|^{-2a} \langle (T_a T_a^*)^{-s/4}) T_a e_k \ , \ M_{a,f} M_{a,f}^* (T_a T_a^*)^{-s/4} T_a e_k \rangle \\
&= \sum_{k\in\mathbb{Z}} c_a |k|^{-2a} |k|^{-sa} c_a^{s/2} \langle T_a e_k \ , \ M_{a,f} M_{a,f}^* T_a e_k \rangle \\
&= \sum_{k\in\mathbb{Z}} c_a^{1+s/2} |k|^{-a(2+s)} \langle T_a e_k \ , \ M_{a,f} M_{a,f}^* T_a e_k \rangle \\
&= \sum_{k\in\mathbb{Z}} c_a^{1+s/2} |k|^{-a(2+s)} \| M_{a,f}^* T_a e_k \|_2^2 \\
&\le \sum_{k\in\mathbb{Z}} c_a^{s/2-1} |k|^{-sa} E_a(f) \\
&= 2 c_a^{s/2-1} \zeta(sa) E_a(f).
\end{aligned}
$$

For the next needed inequality we have

$$
\begin{aligned}
\mathrm{tr}((T_a^* T_a)^{-s/4} M_{a,f}^* M_{a,f} (T_a^* T_a)^{-s/4}) &= \sum_{k\in\mathbb{Z}} \langle (T_a^* T_a)^{-s/4}) e_k \ , \ M_{a,f}^* M_{a,f} (T_a^* T_a)^{-s/4} e_k \rangle \\
&= \sum_{k\in\mathbb{Z}} c_a^{s/2} |k|^{-sa} \langle e_k \ , \ M_{a,f}^* M_{a,f} e_k \rangle \\
&= \sum_{k\in\mathbb{Z}} c_a^{s/2} |k|^{-sa} \| M_{a,f} e_k \|_2^2 \\
&= \sum_{k\in\mathbb{Z}} c_a^{s/2} |k|^{-sa} \| (T_a f) e_k \|_2^2 \\
&= \sum_{k\in\mathbb{Z}} c_a^{s/2-1} |k|^{-sa} E_a(f) \\
&= 2 c_a^{s/2-1} \zeta(sa) E_a(f).
\end{aligned}
$$

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It now follows that

$$\operatorname{tr}((T_a T_a^*)^{-s/4} M_{a,f} M_{a,f}^* (T_a T_a^*)^{-s/4}) \le 2 c_a^{s/2-1} \zeta(sa) E_a(f) = \operatorname{tr}((T_a^* T_a)^{-s/4} M_{a,f}^* M_{a,f} (T_a^* T_a)^{-s/4})$$

as was needed. ∎

To show that $ST(SG) = (C(SG), \mathscr{H}, D_a)$ is in fact a spectral triple we need to show that the set

$$\mathscr{A}_a = \{f \in C(SG) : \|[D_a, \pi(f)]\| < \infty\}$$

is dense in $C(SG)$. We will show that on the initial triangle $\Delta_0 = T$ in SG, the set

$$\mathscr{A}_0 = \{f \in C(\Delta_0) : \|[D_\emptyset, \pi_\emptyset(f)]\| < \infty\}$$

is dense in $C(\Delta_0)$.

Consider the map $M_{a,f} : C[-\pi, \pi] \to L^2([-\pi, \pi]^2)$ given by $M_{a,f}(g) = T_a(f)g$ for $f \in C[-\pi, \pi]$.

Let $R : [-\pi, \pi] \to \Delta_0$ be the map taking the interval to the triangle Δ_0. We will identify the functions $f \in C(\Delta_0)$ with the functions

$$f \circ R : [-\pi, \pi] \to \Delta_0 \to \mathbb{C}.$$

For simplicity, when $f \in SG$ we will write $M_{a,f}$ instead of $M_{a, f \circ R}$.

Proposition 6.2.1.9 *The set $\mathscr{A}_0$ is the same as the set*

$$\{f \in C(\Delta_0) : p_a(f) < \infty\}$$

where

$$p_a(f) = \left(\frac{1}{2\pi} \sup_{y \in [-\pi, \pi]} \int_{-\pi}^{\pi} |(T_a f)(x, y)|^2 \, dx \right)^{1/2} = \left(\frac{1}{2\pi} \sup_{y \in [-\pi, \pi]} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} \, dx \right)^{1/2}.$$

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Proof. Let $f \in C[-\pi, \pi]$. Consider the map $M_{a,f} : C[-\pi, \pi] \to L^2([-\pi, \pi]^2)$ given by $M_{a,f}(g) = T_a(f)g$ for $g \in C[-\pi, \pi]$. Notice that if f is such that $p_a(f) < \infty$, then

$$\|M_{a,f}g\|_2^2 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} |g(y)|^2 dx\, dy$$

$$\leq \frac{1}{2\pi} \|g\|_2^2 \sup_{y \in [-\pi, \pi]} \int_{-\pi}^{\pi} |(T_a f)(x, y)|^2\, dx$$

$$= \|g\|_2^2\, p_a(f)^2$$

$$< \infty$$

and hence we can extend the operator $M_{a,f}$ to all of $L^2[-\pi, \pi]$ by continuity since $C[-\pi, \pi]$ is dense in $L^2[-\pi, \pi]$. We now show that

$$\|M_{a,f}\|^2 = (p_a(f))^2.$$

The above gives that $\|M_{a,f}\| \leq p_a(f)$. For the reverse let

$$S_\epsilon = \left\{ y \in [-\pi, \pi] : \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} dx > p_a(f)^2 - \epsilon \right\}.$$

Note that the function

$$F(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} dx = \frac{1}{2\pi} \int_{-\pi}^{y} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} dx + \frac{1}{2\pi} \int_{y}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+2a}} dx$$

is continuous since the assumption that $p_a(f) < \infty$ gives that $\frac{|f(x)-f(y)|^2}{|x-y|^{1+2a}}$ is in L^1. Then $\sup_{y \in [-\pi, \pi]} F(y) = \text{ess sup } F(y)$, so S_ϵ has positive measure and

$$\|M_{a,f}\, \chi_{S_\epsilon}\|_2^2 = \frac{1}{2\pi} \int_{S_\epsilon} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x) - f(y)|^2}{|x - y|^{1+2a}} dx\, dy$$

$$> (p_a(f)^2 - \epsilon) \frac{1}{2\pi} \int_{S_\epsilon} 1\, dx$$

$$= (p_a(f)^2 - \epsilon) \|\chi_{S_\epsilon}\|_2^2.$$

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Hence $\|M_{a,f}\| \geq (p_a(f) - \epsilon)$ and so $\|M_{a,f}\| = p_a(f)$.

We now show that $\|[D_\emptyset, \pi_\emptyset(f)]\| = \|M_{a,f}\|$ by considering the quadratic form corresponding to $[D_a, \pi_\emptyset(f)]$. Recall, the domain of $D_\emptyset$ is $\text{dom}(T_a^*) \oplus L^2([-\pi, \pi])$. Then for $\psi_1 \oplus h_1, \psi_2 \oplus h_2 \in \text{dom}(T_a^*) \oplus L^2([-\pi, \pi])$ we have

$$\langle \psi_1 \oplus h_1, [D_\emptyset, \pi_\emptyset(f)]\psi_2 \oplus h_2 \rangle = \langle \psi_1 \oplus h_1, D_\emptyset \pi_\emptyset(f)(\psi_2 \oplus h_2) - \pi_\emptyset(f)D_\emptyset(\psi_2 \oplus h_2) \rangle$$

$$= \langle \psi_1 \oplus h_1, D_\emptyset \pi_\emptyset(f)(\psi_2 \oplus h_2) \rangle - \langle \psi_1 \oplus h_1, \pi_\emptyset(f)D_\emptyset(\psi_2 \oplus h_2) \rangle$$

$$= \langle D_\emptyset(\psi_1 \oplus h_1), \pi_\emptyset(f)(\psi_2 \oplus h_2) \rangle - \langle \pi_\emptyset(f^*)(\psi_1 \oplus h_1), D_\emptyset(\psi_2 \oplus h_2) \rangle$$

$$= \langle T_a h_1 \oplus T_a^* \psi_1, f\psi_2 \oplus fh_2 \rangle - \langle f^*\psi_1 \oplus f^*h_1, T_a h_2 \oplus T_a^* \psi_2 \rangle$$

$$= \langle T_a h_1, f\psi_2 \rangle + \langle T_a^* \psi_1, fh_2 \rangle - \langle f^*\psi_1, T_a h_2 \rangle - \langle f^*h_1, T_a^* \psi_2 \rangle$$

$$= \langle T_a h_1, f\psi_2 \rangle + \langle \psi_1, T_a(fh_2) \rangle - \langle f^*\psi_1, T_a h_2 \rangle - \langle T_a(f^*h_1), \psi_2 \rangle.$$

Notice that by the product rule for T_a we have

$$\langle \psi_1, T_a(fh_2) \rangle = \langle \psi_1, T_a(f)h_2 \rangle + \langle \psi_1, fT_a(h_2) \rangle$$

and

$$\langle T_a(f^*h_1), \psi_2 \rangle = \langle T_a(f^*)h_1, \psi_2 \rangle + \langle f^*T_a h_1, \psi_2 \rangle$$

and hence

$$\langle \psi_1 \oplus h_1, [D_\emptyset, \pi_\emptyset(f)]\psi_2 \oplus h_2 \rangle = \langle T_a h_1, f\psi_2 \rangle + \langle \psi_1, T_a(fh_2) \rangle - \langle f^*\psi_1, T_a h_2 \rangle - \langle T_a(f^*h_1), \psi_2 \rangle.$$

$$= \langle \psi_1, T_a(f)h_2 \rangle - \langle T_a(f^*)h_1, \psi_2 \rangle$$

$$= \langle \psi_1, T_a(f)h_2 \rangle + \langle h_1, -T_a^*(f^*)\psi_2 \rangle$$

$$= \langle \psi_1 \oplus h_1, T_a(f)h_2 \oplus -T_a^*(f^*)\psi_2 \rangle.$$

This means that

$$[D_\emptyset, \pi_\emptyset(f)] = \begin{pmatrix} 0 & M_{a,f} \\ -M^*_{a,f^*} & 0 \end{pmatrix}$$

and hence,

$$\|[D_\emptyset, \pi_\emptyset(f)]\| = \|M_{a,f}\| = p_a(f),$$

from which the result follows. ∎

Lemma 6.2.1.10 *For $a \in (0,1)$ and $f \in C[-\pi, \pi]$ with $p_a(f) < \infty$ we have for every $\epsilon > 0$, there exists $c_{\epsilon,a} > 0$ with $p_a(f) \leq c_{\epsilon,a}\|f\|_{0,a+\epsilon}$ where*

$$\|f\|_{0,a} = \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^a}.$$

Proof. If f is such that $\|f\|_{0,a+\epsilon} < \infty$ then $|f(x) - f(y)| \leq \|f\|_{0,a+\epsilon}|x-y|^{a+\epsilon}$ for all $x, y \in [-\pi, \pi]$. This gives

$$(p_a(f))^2 = \sup_{y \in [-\pi,\pi]} \int_{-\pi}^{\pi} |T_a(f)(x,y)|^2 dx$$

$$\leq \sup_{y \in [-\pi,\pi]} \int_{-\pi}^{\pi} \|f\|^2_{0,a+\epsilon}|x-y|^{2\epsilon-1} dx$$

$$= \|f\|^2_{0,a+\epsilon} \sup_{y \in [-\pi,\pi]} \left. \frac{|x-y|^{2\epsilon}}{2\epsilon} \right|_{-\pi}^{\pi}$$

$$\leq \|f\|^2_{0,a+\epsilon} \frac{\pi^{2\epsilon}}{\epsilon}.$$

∎

Theorem 6.2.1.11 *The triple $ST(SG)$ is a spectral triple for the Sierpinski gasket.*

Proof. From Proposition 6.2.1.8 and Lemma 6.2.1.10 we see that the algebra

$$\mathscr{A}_a = \{f \in C(SG) : \|[D_a, \pi(f)]\| < \infty\}$$

is dense in $C[-\pi, \pi]$. Also since $T_a^* T_a = \frac{1}{c_a}\Delta^a$ and $E_a(e_k, e_j) = c_a\langle T_a e_k, T_a e_j\rangle = |k|^{2a}\delta_{j,k}$ we have that

$$\sigma(T_a) = \left\{\frac{1}{c_a}|k|^{2a} : k \in \mathbb{Z}\right\}.$$

From this it follows that $ST(SG)$ is a spectral triple. ∎

6.2.2 Recovering $\mathcal{E}$ on SG

Proposition 6.2.2.1 *Let* $s < a^{-1}$, $\alpha_0 = \beta - \frac{d-1}{2} = \frac{\log(5)}{\log(4)} - \frac{\frac{\log(3)}{\log(2)}-1}{2} = \frac{\log(10/3)}{\log(4)}$. *Then:*

1. $tr(|D_\emptyset|^{-s/2}|[D_\emptyset, \pi(f)]|^2 |D_\emptyset|^{-s/2})$ *is finite if and only if* $f \in H^{a,2}([-\pi, \pi])$.

2. *If* $tr(|D_\emptyset|^{-s/2}|[D_\emptyset, \pi(f)]|^2 |D_\emptyset|^{-s/2})$ *is finite for all* f *with finite energy on* SG, *then*

   $a \leq \alpha_0$.

Proof. (1.) Note that

$$|[D_\emptyset, \pi(f)]|^2 = [D_\emptyset, \pi(f)][D_\emptyset, \pi(f)]^*$$

$$= \begin{pmatrix} 0 & M_{a,f} \\ -M_{a,f^*}^* & 0 \end{pmatrix}\begin{pmatrix} 0 & -M_{a,f^*} \\ M_{a,f}^* & 0 \end{pmatrix}$$

$$= \begin{pmatrix} M_{a,f}M_{a,f}^* & 0 \\ 0 & M_{a,f^*}^* M_{a,f^*} \end{pmatrix}$$

and

$$|D_\emptyset|^{-s/2} = \begin{pmatrix} (T_a T_a^*)^{-s/4} & 0 \\ 0 & (T_a^* T_a)^{-s/4} \end{pmatrix}$$

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so that

$$|D_\emptyset|^{-s/2}|[D_\emptyset, \pi(f)]|^2|D_\emptyset|^{-s/2}$$

$$= \begin{pmatrix} (T_a T_a^*)^{-s/4} M_{a,f} M_{a,f}^* (T_a T_a^*)^{-s/4} & 0 \\ 0 & (T_a^* T_a)^{-s/4} M_{a,f^*}^* M_{a,f^*} (T_a^* T_a)^{-s/4} \end{pmatrix}$$

and

$$\mathrm{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset, \pi(f)]|^2|D_\emptyset|^{-s/2})$$

$$= \mathrm{tr}((T_a T_a^*)^{-s/4} M_{a,f} M_{a,f}^* (T_a T_a^*)^{-s/4}) + \mathrm{tr}((T_a^* T_a)^{-s/4} M_{a,f^*}^* M_{a,f^*} (T_a^* T_a)^{-s/4}).$$

Using Proposition 6.2.1.8. we have that

$$2 c_a^{s/2-1} \zeta(sa) E_a(f) \leq \mathrm{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset, \pi(f)]|^2|D_\emptyset|^{-s/2}) \leq 4 c_a^{s/2-1} \zeta(sa) E_a(f).$$

(2.) Assume $a > \alpha_0$. Then $H^{a,2} \subset H^{\alpha_0,2}$ and we can pick $g \in H^{\alpha_0,2} \setminus H^{a,2}$. By Theorem 6.2.1.3. we have a function f on SG such that $f|_{[-\pi,\pi]} = g$ and by part (1) we have $\mathrm{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset, \pi(f)]|^2|D_\emptyset|^{-s/2}) = \infty$, from which the result follows. ∎

Theorem 6.2.2.2 *Let $a \in (0, \alpha_0]$, $\delta_D = \max\{a^{-1}, d_E\}$, with $d_E = \frac{\log(12/5)}{\log 2}$. Then:*

1. *For any f with finite energy, $s > \delta_D$, $|D_a|^{-s/2}|[D_a, \pi(f)]|^2|D_a|^{-s/2}$ is a trace class operator.*

2. *For $a \in (d_E^{-1}, \alpha_0]$, so that $\delta_D = d_E$, and f with finite energy, the functional*

$$Z_{D_a,f}(s) = \mathrm{tr}(|D_a|^{-s/2}|[D_a, \pi(f)]|^2|D_a|^{-s/2}),$$

defined for $\mathrm{Re}(s) > d_E$, has abscissa of convergence d_E, where it has a simple pole, and there exists a constant N such that,

$$\lim_{s \to d_E} (s - d_E) Z_{D_a,f}(s) = N\mathcal{E}(f).$$

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Proof. (1.) We have by Proposition 6.2.1.8.

$$\mathrm{tr}(|D_a|^{-s/2}|[D_a, \pi(f)]|^2|D_a|^{-s/2}) = \sum_{w \in W} \mathrm{tr}(|D_w|^{-s/2}|[D_w, \pi_w(f)]|^2|D_w|^{-s/2})$$

$$= \sum_{w \in W} 2^{|w|(2-s)}\mathrm{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset, \pi_\emptyset(f \circ F_w)]|^2|D_\emptyset|^{-s/2})$$

$$\leq \sum_{w \in W} 2^{|w|(2-s)}4c_a^{s/2-1}\zeta(sa)E_a(f \circ F_w \circ R).$$

In the above expression we see the term $E_a(f \circ F_w \circ R)$. The function $f \circ F_w \circ R$ can be written $f \circ F_w \circ R = \pi_\emptyset(f \circ F_w)(1)$ or just $f \circ F_w \circ R = \pi_\emptyset(f \circ F_w)$ for simplicity. We will use this notation for the remainder. Notice that the norm on $H^{a,2}[-\pi, \pi]$ can be written $\|f\|_{H^{a,2}}^2 = \|f\|_2^2 + E_a(f)$. According to Theorem 6.2.1.2. (or the generalization by Hino and Kumagai) there is a constant $C_{1,a} > 0$ such that

$$\|T_a\pi_\emptyset(f \circ F_w)\|_2^2 = E_a(\pi_\emptyset(f \circ F_w)) \leq C_{1,a}\mathcal{E}(f)$$

for all $f \in \mathrm{dom}(\mathcal{E})$. Summing over all $w \in \{1, 2, 3\}^n$ (words in the symbols 1, 2, 3 of length n) we get

$$\sum_{|w|=n} \|T_a(\pi_\emptyset(f \circ F_w))\|_2^2 \leq C_{1,a} \sum_{|w|=n} \mathcal{E}(f \circ F_w) = C_{1,a}\left(\frac{3}{5}\right)^n \mathcal{E}(f).$$

This then gives for $s > \delta_D$,

$$\mathrm{tr}(|D_a|^{-s/2}|[D_a, \pi(f)]|^2|D_a|^{-s/2}) \leq \sum_{n=0}^{\infty} \sum_{|w|=n} 2^{n(2-s)}4c_a^{s/2-1}\zeta(sa)E_a(f \circ F_w \circ R) \tag{6.1}$$

$$\leq \sum_{n=0}^{\infty} C_{1,a}\left(2^{(2-s)}\frac{3}{5}\right)^n 4c_a^{s/2-1}\zeta(sa)\mathcal{E}(f) \tag{6.2}$$

$$= 4C_{1,a}c_a^{s/2-1}\zeta(sa)\mathcal{E}(f) \sum_{n=0}^{\infty} \left(2^{(2-s)}\frac{3}{5}\right)^n \tag{6.3}$$

$$= \frac{20C_{1,a}c_a^{s/2-1}\zeta(sa)\mathcal{E}(f)}{5 - 3 \cdot 2^{2-s}}. \tag{6.4}$$

Note that in the above calculation we see the terms $\zeta(sa)$ and $\sum_{n=0}^{\infty}\left(2^{(2-s)}\frac{3}{5}\right)^n$ which require that $s > \delta_D = \max\{a^{-1}, d_E\}$ in order to converge. If the function f has finite energy, then this shows that $\mathrm{tr}(|D_a|^{-s/2}|[D_a, \pi(f)]|^2|D_a|^{-s/2}) < \infty$ and hence $|D_a|^{-s/2}|[D_a, \pi(f)]|^2|D_a|^{-s/2}$ is trace class for $s > \delta_D$.

(2.) We have shown in Proposition 6.1.0.7. that, up to a constant, there is only one positive semi-definite quadratic form on $\mathbb{C}^3$ which is invariant under permutations and vanishes on constants (in the sense explained in Proposition 6.1.0.7.). We now show that the map

$$Q_a : v \in \mathbb{C}^3 \longrightarrow h_v \longrightarrow \|T_a(\pi_\emptyset(h_v))\|_2^2 = E_a(h_v)$$

where h_v is the harmonic function on SG determined by the values of $v = (v_1, v_2, v_3)$ on the vertices of Δ_0, satisfies these conditions. First note that by definition this map is a positive semidefinite quadratic form. Next, notice that permuting the components of $v = (v_1, v_2, v_3) \in \mathbb{C}^3$ permutes the values of h_v by possibly changing on which edge of the triangle Δ_0 the value occurs. Since in $\|T_a(\pi_\emptyset(h_v))\|_2^2$ we are integrating over all of Δ_0 this does not change the value of the integral. Finally, if $v = (v_0, v_0, v_0)$ is constant then the harmonic function h_v is constant and hence $T_a(\pi_\emptyset(h_v)) = 0$. It now follows by Proposition 6.1.0.7 and the discussion following it that $\mathcal{E}(\cdot)$, that there exists a constant $K_0 > 0$ such that

$$E_a(h_v) = \|T_a(\pi_\emptyset(h_v))\|_2^2 = K_0\, \mathcal{E}(h_v).$$

Next recall that in the proof of Proposition 6.2.1.8. we obtained the expression

$$\|M_{a,f}^* T_a e_k\|_2^2 = \frac{1}{4} c_a^{-2} \sum_p (|k|^{2a} + |p|^{2a} - |p - k|^{2a})^2 |\langle \pi_\emptyset(f), e_p\rangle|^2$$

from which we can see that $v \mapsto \|M_{a,h_v}^* T_a e_k\|_2^2$ is a non-degenerate positive semidefinite quadratic form. We can also see that if f is constant then $|\langle \pi_\emptyset(f), e_p \rangle|^2 = 0$ and hence $\|M_{a,f}^* T_a e_k\|_2^2 = 0$. Again since we are integrating over all of Δ_0 we have that permuting the entries in the vector $v \in \mathbb{C}$ will not change the value of $\|M_{a,h_v}^* T_a e_k\|_2^2$. Thus, for $k \neq 0$ there is a constant $A_k > 0$ such that

$$\|M_{a,h_v}^* T_a e_k\|_2^2 = A_k \mathcal{E}(h_v)$$

and we know $0 < A_k < K_0 c_a^{-2} |k|^{2a}$ for $k \neq 0$. Let

$$A(s) = \sum_{k \in \mathbb{Z} \setminus \{0\}} A_k c_a^2 |k|^{-(s+2)a}$$

and note that

$$\mathcal{E}(h_v) A(s) \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} c_a^2 |k|^{-(s+2)a} \|M_{a,h_v}^* T_a e_k\|_2^2 \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} K_0 |k|^{-(s+2)a} |k|^{2a} \mathcal{E}(h_v) \leq 2K_0 \zeta(sa) \mathcal{E}(h_v)$$

so that for h_v non-constant, $A(s) \leq \zeta(sa)$ and hence $A(s)$ is an analytic function for $s > a^{-1}$.

Recall, in the proof of Proposition 6.2.1.8. we obtained the expressions

$$\mathrm{tr}((T_a T_a^*)^{-s/4} M_{a,f} M_{a,f}^* (T_a T_a^*)^{-s/4}) = \sum_{k \in \mathbb{Z}} c_a^{1+s/2} |k|^{-a(2+s)} \|M_{a,f}^* T_a e_k\|_2^2$$

and

$$\mathrm{tr}((T_a^* T_a)^{-s/4} M_{a,f^*}^* M_{a,f^*} (T_a^* T_a)^{-s/4}) = 2c_a^{s/2-1} \zeta(sa) E_a(f^*) = 2c_a^{s/2-1} \zeta(sa) E_a(f).$$

Now for functions f which are harmonic on and after the m-th level construction of SG we

have

$$Z_{D_a,f}(s) = \text{tr}(|D_a|^{-s/2}|[D_a, \pi(f)]|^2|D_a|^{-s/2})$$

$$= \sum_{w \in W} \text{tr}(|D_w|^{-s/2}|[D_w, \pi_w(f)]|^2|D_w|^{-s/2})$$

$$= \sum_{w \in W} 2^{|w|(2-s)}\text{tr}(|D_\emptyset|^{-s/2}|[D_\emptyset, \pi_\emptyset(f \circ F_w)]|^2|D_\emptyset|^{-s/2})$$

$$= \sum_{w \in W} 2^{|w|(2-s)}\Big[\text{tr}((T_a T_a^*)^{-s/4}M_{a,f\circ F_w}M_{a,f\circ F_w}^*(T_a T_a^*)^{-s/4})$$

$$+ \text{tr}((T_a^* T_a)^{-s/4}M_{a,(f\circ F_w)^*}^*M_{a,(f\circ F_w)^*}(T_a^* T_a)^{-s/4})\Big]$$

$$= \sum_{w \in W} 2^{|w|(2-s)}\left(\sum_{k \in \mathbb{Z}} c_a^{1+s/2}|k|^{-a(2+s)}\|M_{a,f\circ F_w}^* T_a e_k\|_2^2 + 2c_a^{s/2-1}\zeta(sa)E_a(\pi_\emptyset(f \circ F_w))\right).$$

Now for $s > d_E$ and a function f which is m-harmonic, we have that $f \circ F_w$ is harmonic if $|w| \geq m$. Then we have

$$\sum_{w \in W,\ |w|\geq m} \text{tr}(|D_w|^{-s/2}|[D_w, \pi_w(f)]|^2|D_w|^{-s/2})$$

$$= \sum_{w \in W,\ |w|\geq m} 2^{|w|(2-s)}\left(\sum_{k \in \mathbb{Z}} c_a^{1+s/2}|k|^{-a(2+s)}\|M_{a,f\circ F_w}^* T_a e_k\|_2^2 + 2c_a^{s/2-1}\zeta(sa)E_a(\pi_\emptyset(f \circ F_w))\right)$$

$$= \sum_{w \in W,\ |w|\geq m} 2^{|w|(2-s)}\left(\sum_{k \in \mathbb{Z}} c_a^{1+s/2}|k|^{-a(2+s)}A_k\mathcal{E}(f \circ F_w) + 2K_0 c_a^{s/2-1}\zeta(sa)\mathcal{E}(f \circ F_w)\right)$$

$$= \sum_{n=m}^{\infty}\sum_{|w|=n} 2^{n(2-s)}\left(c_a^{s/2-1}A(s) + 2K_0 c_a^{s/2-1}\zeta(sa)\right)\mathcal{E}(f \circ F_w)$$

$$= \left(c_a^{s/2-1}A(s) + 2K_0 c_a^{s/2-1}\zeta(sa)\right)\sum_{n=m}^{\infty}\left(\frac{3}{5}\right)^n 2^{n(2-s)}\mathcal{E}(f)$$

$$= \left(c_a^{s/2-1}A(s) + 2K_0 c_a^{s/2-1}\zeta(sa)\right)\left(\frac{3}{5}\cdot 2^{2-s}\right)^m \frac{5}{5 - 3\cdot 2^{2-s}}\,\mathcal{E}(f).$$

Note that the sum

$$\sum_{w\in W,\ |w|<m} \mathrm{tr}(|D_w|^{-s/2}|[D_w,\pi_w(f)]|^2|D_w|^{-s/2})$$

is finite for $s > d_E$ since the sum is over a finite number of terms and the operators $|D_w|^{-s/2}|[D_w,\pi_w(f)]|^2|D_w|^{-s/2}$ are trace class. Thus we know

$$\lim_{s\to d_E^+}(s-d_E)\sum_{w\in W,\ |w|<m} \mathrm{tr}(|D_w|^{-s/2}|[D_w,\pi_w(f)]|^2|D_w|^{-s/2}) = 0.$$

In what follows, it is also helpful to note that $\lim_{s\to d_E^+}\frac{3}{5}\cdot 2^{2-s}=1$. Using these facts and splitting the sum in $Z_{D_a,f}(s)$ we get

$$
\begin{aligned}
\lim_{s\to d_E^+}(s-d_E)Z_{D_a,f}(s) &= \lim_{s\to d_E^+}(s-d_E)\sum_{w\in W,\ |w|\geq m} \mathrm{tr}(|D_w|^{-s/2}|[D_w,\pi_w(f)]|^2|D_w|^{-s/2})\\
&= \lim_{s\to d_E^+}(s-d_E)\left(c_a^{s/2-1}A(s)+2K_0c_a^{s/2-1}\zeta(sa)\right)\left(\frac{3}{5}\cdot 2^{2-s}\right)^m \frac{5}{5-3\cdot 2^{2-s}}\,\mathcal{E}(f)\\
&= \left(c_a^{d_E/2-1}A(d_E)+2K_0c_a^{d_E/2-1}\zeta(d_E a)\right)\left(\frac{3}{5}\cdot 2^{2-d_E}\right)^m \mathcal{E}(f)\lim_{s\to d_E^+}\frac{5(s-d_E)}{5-3\cdot 2^{2-s}}\\
&= \frac{c_a^{d_E/2-1}\left(A(d_E)+2K_0\zeta(d_E a)\right)}{\log 2}\,\mathcal{E}(f)
\end{aligned}
$$

We have thus far shown that

$$\mathrm{Tr}_w(|D_a|^{-s/2}|[D_a,\pi(f)]|^2|D_a|^{-s/2}) = \lim_{s\to d_E^+}(s-d_E)Z_{D_a,f}(s) = N\,\mathcal{E}(f)$$

where

$$N = \frac{c_a^{d_E/2-1}\left(A(d_E)+2K_0\zeta(d_E a)\right)}{\log 2}$$

for functions f which are finitely harmonic, meaning that they are harmonic after some m-th level in the construction of SG.

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We now prove the result for general functions f with finite energy (i.e. $\mathcal{E}(f) < \infty$). First note that for $s > d_E$, the map $N_s : \text{dom}(\mathcal{E}) \to [0, \infty)$ given by $N_s(f) = \sqrt{Z_{D_a,f}(s)}$ is a seminorm on $\text{dom}(\mathcal{E})$. This follows easily from the various expressions we have obtained for $Z_{D_a,f}(s)$ as well as $\|M^*_{a,f\circ F_w} T_a e_k\|_2^2$ and $E_a(f)$.

Let $f \in \text{dom}(\mathcal{E})$ and g be harmonic on some m-th level in the construction of SG. We now have that

$$|(s - d_E)^{1/2} N_s(f) - N^{1/2}\mathcal{E}^{1/2}(f)|$$

$$\leq (s - d_E)^{1/2}|N_s(f) - N_s(g)| + |(s - d_E)^{1/2}N_s(g) - N^{1/2}\mathcal{E}^{1/2}(g)| + N^{1/2}|\mathcal{E}^{1/2}(g) - \mathcal{E}^{1/2}(f)|$$

$$\leq (s - d_E)^{1/2}N_s(f - g) + |(s - d_E)^{1/2}N_s(g) - N^{1/2}\mathcal{E}^{1/2}(g)| + N^{1/2}\mathcal{E}^{1/2}(f - g)$$

$$\leq \left(\left(4C_{1,a}c_a^{s/2-1}\zeta(sa)\frac{5(s - d_E)}{5 - 3 \cdot 2^{2-s}}\right)^{1/2} + N^{1/2}\right)\mathcal{E}^{1/2}(f - g) + |(s - d_E)^{1/2}N_s(g) - N^{1/2}\mathcal{E}^{1/2}(g)|$$

where in the second inequality we have used the reverse triangle inequality, in the third inequality we used (6.4). Taking a limit we get,

$$\lim_{s \to d_E^+} \left(\left(4C_{1,a}c_a^{s/2-1}\zeta(sa)\frac{5(s - d_E)}{5 - 3 \cdot 2^{2-s}}\right)^{1/2} + N^{1/2}\right)\mathcal{E}^{1/2}(f - g)$$

$$= \left(\left(4C_{1,a}c_a^{d_E/2-1}\zeta(d_E a)\frac{1}{\log 2}\right)^{1/2} + N^{1/2}\right)\mathcal{E}^{1/2}(f - g)$$

and

$$\lim_{s \to d_E^+} |(s - d_E)^{1/2}N_s(g) - N^{1/2}\mathcal{E}^{1/2}(g)| \longrightarrow 0$$

since g is finitely harmonic and for such functions we know $\lim_{s \to d_E^+}(s - d_E)Z_{D_a,f}(s) = N\,\mathcal{E}(f)$. Using the fact that finitely harmonic functions are dense in $\text{dom}(\mathcal{E})$ we see that $\mathcal{E}^{1/2}(f - g)$ can be made arbitrarily small and hence we have that

$$\lim_{s \to d_E^+} |(s - d_E)^{1/2}N_s(f) - N^{1/2}\mathcal{E}^{1/2}(f)| = 0.$$

It now follows that

$$\mathrm{Tr}_w(|D_a|^{-s/2}|[D_a, \pi(f)]|^2|D_a|^{-s/2}) = \lim_{s \to d_E}(s - d_E)Z_{D_a,f}(s) = N \; \mathcal{E}(f)$$

for all $f \in \mathrm{dom}(\mathcal{E})$. ∎

Chapter 7

Conclusion

The results in this text are intended to further develop the intersection between fractal geometry, noncommutative geometry, and analysis on fractals. We have proved that the conjecture in [26] regarding the recovery of the Hausdorff measure with respect to the geodesic distance on the harmonic gasket by the Dixmier trace is false. The Dixmier trace on the harmonic gasket can recover the standard self-affine measure on the harmonic gasket, but this measure is not the same as the Hausdorff measure with respect to the geodesic distance. We have also shown that a spectral triple built on the edges of the stretched Sierpinski gasket can be used to recover the Hausdorff dimension, the geodesic metric, and the Hausdorff measure. This is especially interesting since the stretched Sierpinski gasket is a self-affine space and not a self-similar space.

In the future we will consider the question of constructing a spectral triple on the harmonic gasket which will recover the Hausdorff measure. There are results concerning the asymptotics of the Laplacian on the harmonic gasket with respect to the Hausdorff measure

(see [17]) and it may prove useful to find a connection between this fractal analysis and the operator algebraic tools that come with spectral triples. Also, there are some results on the use of spectral triples to recover energy forms on fractal sets like the Sierpinski gasket; see [6]. One can construct an energy form on the stretched Sierpinski gasket and it would be interesting to assemble a spectral triple that recovers the energy on the stretched Sierpinski gasket.

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