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2017

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UNIVERSITY OF CALIFORNIA, SAN DIEGO

SAN DIEGO STATE UNIVERSITY

The Development of Mathematical Knowledge for Teaching for Quantitative
Reasoning Using Video-Based Instruction

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Mathematics and Science Education

by

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2017

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Chair

University of California, San Diego

San Diego State University

2017

DEDICATION

For Eloise Selah—

I am encouraged daily by your love and your laughter.

You are an old soul, full of promise.

I am so proud to be your father.

For Even Michael—

I am inspired daily by your creativity and your curiosity.

You are a vibrant young woman, full of fire.

I am so proud to be your father.

For Michelle Rose—

I am humbled daily by your strength and your tenacity.

You are a steadfast mother, full of love.

I am so proud to be your son.

EPIGRAPH

Let the world burn through you. Throw
the prism light, white hot, on paper.

Ray Bradbury

We can teach from our experience, but
we cannot teach experience.

Sasha Azevedo

Each fresh peak ascended teaches
something.

Sir Martin Conway

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ACKNOWLEDGEMENTS

In many ways, the single authorship on a dissertation is an illusion. Certainly, *this* dissertation is my own work, and I have personally written every single word contained in this document. Yet, not a single sentence would have been formed without the unwavering support of so many special people I have been blessed to have in my life.

First among these is my amazing wife, Jill. Every day she has contributed in some way to my progress along this dissertation journey. Over the last five years, she has tirelessly held up more than her fair share in our marriage and in raising our children. Her love has encouraged me and supported me in achieving my dream of earning a Ph.D. At least in spirit, Jill Marie Walters is second author of this study.

My two wonderfully mischievous daughters, Even and Ellie, have also contributed to my success. They lovingly left stickers and drawings on my computer to inspire me. They frequently requested walks, bike rides, and daddy's bedtime story theater, and when I could not oblige, though they were disappointed, they would say "That's ok! Maybe when you're done with your dissertation!" I plan to spend many hours in the coming years repaying them for their unwavering patience.

I also want to acknowledge several close friends who have had an impact in my life over the last five years. David Quarfoot is a friend in the purest sense of that word, and his kindness has enriched my life in ways too numerous to list. My family has been blessed to know Caleb and Karen Cleveland, and their charming and clever daughters Leura and Ella. They have been a kind and generous second family here in

San Diego, and I am thankful for all the date nights away from the dissertation they provided. Dan and Rachel Davis opened their home for many memorable board game nights, often running into the wee hours of the morning. They were a welcome escape from the reality of research. Dan and Grace Rey were always willing to watch my daughters at a moment's notice—most people are never so lucky to have such neighbors. I watched many football games/matches with Mike Pruett, who always seemed to (a) bring good luck and (b) have an endless supply of Skyline Chili—I thank him now for both! Finally, Al and Melissa Kau welcomed my family into their home every Sunday evening for fellowship and a warm meal, and for that I am grateful.

The people who make up the MSED program have shaped who I am as mathematics educator. Michael Garcia cheered me along this journey. Casey Hawthorne and John Gruver were my academic “big brothers,” always willing to guide and advise. Carren Walker generously gave several hours helping me collect data. Hayley Milbourne and Naneh Apkarian (aka Soul Blob) provided relief from writing and analyzing with their willingness to grab lunch and chat. Bridget Druken radiates wisdom, compassion, and humor, and she is always a delight to be around. To Matt Voigt, Katie Bjorkman, Lynda Wynn, Raymond LaRochelle, Brooke Ernest Downey, Mike Fredenberg, and Spencer Bagley: I appreciated your open-door policies and the conversations we shared. Deb Escamilla, *la jefecita*, cannot be thanked enough. She is the oil that makes the MSED machinery run smoothly, and I am indebted to her for the countless ways in which she kept me on track and in line.

And finally, despite her best efforts to avoid this list, I must thank Karen Foehl Palmer, who was a continual source of zany rivalry and (not so) clever practical jokes.

My dissertation would not have been possible without the time and effort of seven wonderful participants: April, Desmond, Jasper, Lily, Marshall, Sierra, and Willow. Thank you from the bottom of my heart for helping me complete this study.

I would not be where I am today without my mother, Shellie. Her sacrifices paved the way for me. My father Guy, has the heart of a lion, and he loves fiercely; he too, has sacrificed much for me. My step-dad Tony (though the connotation of that title does not convey how I feel about the man) instilled in me a desire for learning and a love of adventure. I am thankful for my brothers Cory and Jordan, for they have always been willing to make me laugh regardless of the circumstances. My wife's grandparents, Pete and Phyllis provided my family with love and support throughout my time in graduate school, and I am honored to call them Grandma and Grandpa.

Finally, I want to acknowledge my committee—Dr. Gabriele Wienhausen, Dr. Chris Halter, Dr. Bill Zahner, Dr. Susan Nickerson, and Dr. Joanne Lobato. Each of you helped shape this dissertation, and your thoughtful feedback has been appreciated. In particular, I am forever grateful for the guidance, support, and mentoring that Joanne provided throughout this process. Joanne, thank you for always being willing to listen, for helping me develop my writing skills, and for reminding me to be present in the moment. Truly, I could not have asked for a better advisor.

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- Lobato, J., Walters, C. D., Hohensee, C., Gruver, J., & Diamond, J. M. (2015). Leveraging failure in design research. *ZDM – The International Journal on Mathematics Education*, 47(6), 963–979.

ABSTRACT OF THE DISSERTATION

The Development of Mathematical Knowledge for Teaching for Quantitative Reasoning Using Video-Based Instruction

by

Charles David Walters

Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2017
San Diego State University, 2017

Professor Joanne Lobato, Chair

Quantitative reasoning (P. W. Thompson, 1990, 1994) is a powerful mathematical tool that enables students to engage in rich problem solving across the curriculum. One way to support students' quantitative reasoning is to develop prospective secondary teachers' (PSTs) mathematical knowledge for teaching (MKT; Ball, Thames, & Phelps, 2008) related to quantitative reasoning. However, this may

prove challenging, as prior to entering the classroom, PSTs often have few opportunities to develop MKT by examining and reflecting on students' thinking. Videos offer one avenue through which such opportunities are possible.

In this study, I report on the design of a mini-course for PSTs that featured a series of videos created as part of a proof-of-concept NSF-funded project. These MathTalk videos highlight the ways in which the quantitative reasoning of two high school students developed over time.

Using a mixed approach to grounded theory, I analyzed pre- and post-interviews using an extant coding scheme based on the Silverman and Thompson (2008) framework for the development of MKT. This analysis revealed a shift in participants' affect as well as three distinct shifts in their MKT around quantitative reasoning with distances, including shifts in: (a) quantitative reasoning; (b) point of view (decentering); and (c) orientation toward problem solving.

Using the four-part focusing framework (Lobato, Hohensee, & Rhodehamel, 2013), I analyzed classroom data to account for how participants' noticing was linked with the shifts in MKT. Notably, their increased noticing of aspects of MKT around quantitative reasoning with distances, which features prominently in the MathTalk videos, seemed to contribute to the emergence of the shifts in MKT.

Results from this study link elements of the learning environment to the development of specific facets of MKT around quantitative reasoning with distances. These connections suggest that vicarious experiences with two students' quantitative reasoning over time was critical for participants' development of MKT.

Chapter 1: Introduction

In the spring of 2017, I attended a workshop at the National Council of Teachers of Mathematics (NCTM) Annual Meeting and Exposition. Attendees were asked to solve the following task before discussing it in small groups:

Long-distance *Company A* charges a base rate of \$5 per month, plus 4 cents per minute that you are on the phone. Long-distance *Company B* charges a base rate of only \$2 per month, but they charge 10 cents per minute used. How much time per month would you have to talk on the phone before it would save you money to subscribe to *Company A*? (Achieve Inc., 2002, p. 149)

The six people at my table were all middle and high school teachers. Four of them produced a system of equations: $5 + .04m = c$ and $2 + .1m = c$. Two others produced graphs with intersecting lines. I had a different approach: Before a person talks one minute, Company A is going to cost $5 - 2 = 3$ dollars more. In other words, Company A has a “head-start” of 300 cents over Company B. Additionally, this lead shrinks by $10 - 4 = 6$ cents every minute that the person spends on the phone. Consequently, it would take $300 \div 6 = 50$ minutes for Company B to catch up to Company A, so if one talks on the phone for more than 50 minutes a month, he or she should choose Company A.

The six teachers sitting at my table seemed astounded by my solution. Two of them told me that they would never think to approach the task in that way, while a third interjected that she wished she had thought of it like that. All of them agreed that it would be powerful for their students to think like this.

P. W. Thompson (1990, 1994, 2011) called this way of thinking *quantitative reasoning*. It involves conceiving of measurable attributes within a mathematical

situation, assigning measures to those attributes, and using relationships between those measures to analyze the situation. Quantitative reasoning is powerful, and students who are fluent with quantitative reasoning are well-equipped to tackle many challenges across mathematical disciplines, including algebra (Ellis, 2007; Smith & Thompson, 2007), trigonometry and precalculus (Moore, 2013; Moore, Carlson, & Oehrtman, 2009; Oehrtman, Carlson, & Thompson, 2008), calculus (P. W. Thompson, 2011), and differential equations (Rasmussen, 2001).

In this chapter, I argue that despite the significant work the field has done to research and conceptualize quantitative reasoning, what seems to be missing is theory about specific ways to foster teachers' abilities to support their students' quantitative reasoning. I begin by providing an overview of the field's understanding of quantitative reasoning and mathematical knowledge for teaching (MKT). I then briefly discuss the use of videos in teacher education, before concluding with the statement of my research questions and an overview of the study.

The Quantitative Reasoning of Students and Their Teachers

The power of quantitative reasoning has not gone unnoticed. Elements of quantitative reasoning have begun to be included in standards and policy documents across all levels of education, including the Common Core State Standards, (CCSS; National Governors Association Center for Best Practices, Council of Chief State School Officers, 2012); the Mathematical Association of America's (MAA) Guidelines for Programs and Departments in Undergraduate Mathematical Sciences

(Fulton, 2003); and NCTM's Principles and Standards for School Mathematics (National Council of Teachers of Mathematics, 2000).

Meeting these standards may prove challenging, as A. G. Thompson and Thompson (1995) argued that the notion of "students' quantitative reasoning is almost an oxymoron. For the most part, students do not reason quantitatively in school mathematics" (p. 101). The research literature seems to confirm this assertion. Several studies have examined the quantitative reasoning of both K–12 and college students (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ellis, 2007, 2011; Moore et al., 2009; P. W. Thompson, 1988, 1993). Overwhelmingly, this research suggests that quantitative reasoning is difficult for students. Indeed, even when immersed in quantitatively-rich situations, students may find such reasoning challenging (e.g., Ellis, 2007; Noble, Nemirovsky, Wright, & Tierney, 2001; P. W. Thompson, 1993). For example, in a teaching experiment with fifth-grade students, P.W. Thompson (1993) found that even with targeted quantitative instruction, students experienced difficulties with comparing and combining quantities without directly connecting those quantities to numerical values and operations.

While such literature suggests that quantitative reasoning is difficult, there is a growing body of research that demonstrates students can reason quantitatively given appropriate support (Carlson et al., 2002; Carraher, Schliemann, Brizuela, & Earnest, 2006; Lobato & Siebert, 2002; Moore, 2013). In one study, elementary students identified and operated on unknown quantities during a longitudinal classroom study (Carraher et al., 2006). In another study, Lobato and Siebert (2002) demonstrated that

with quantitative instruction, students can develop sophisticated quantitative reasoning about linear functions and slope.

The implication is clear: with targeted quantitative instruction by knowledgeable teachers who themselves reason quantitatively, students can develop sophisticated quantitative reasoning. However, A. G. Thompson and Thompson (1995) noted that “it would be surprising to find many teachers teaching for quantitative reasoning since a large portion do not reason quantitatively themselves” (p. 101). Historically, quantitative reasoning has not been a point of emphasis in traditional mathematics curricula. Given that this is the context in which current math teachers were prepared, A. G. Thompson and Thompson’s premise seems reasonable. As Usiskin (2001a) asserted:

This historical argument suggests that quantitative literacy will not become mainstream in our schools until a generation of teachers has learned its mathematics with attention to quantitative literacy—a chicken-and-egg dilemma similar to that regarding the public apathy about quantitative literacy described in the case statement. (p. 85)

To remedy this dilemma, Usiskin urged the field to “engage in massive teacher training in quantitative literacy” (p. 85).

Unfortunately, the literature is sparse regarding K–12 teachers’ preparation for teaching for quantitative reasoning. For example, I found only two studies that examine prospective elementary school teachers’ burgeoning quantitative reasoning. Simon and Blume (1994) described their successful efforts to foster the understanding of area as a multiplicative relationship using a quantitative reasoning approach. Hohensee (2017) explored variations between the quantitative reasoning of elementary

students and prospective elementary teachers. At the middle school level, Sowder et al. (1998) made four recommendations for the preparation of middle school teachers, including providing opportunities for exploring and reasoning with quantities. Finally, Moore, Paoletti, and Musgrave (2013) reported that embedding problems in a polar coordinate system seemed to hold potential for fostering prospective secondary teachers' quantitative reasoning.

Despite the dearth of literature examining teachers' preparation with quantitative reasoning, there is a clear message coming from both the research literature and standards documents: Teachers need to be well-prepared to teach for quantitative reasoning (National Council of Teachers of Mathematics, 2000; Sztajn, Marrongelle, Smith, & Melton, 2012; A. G. Thompson & Thompson, 1995; Usiskin, 2001a). In recent years, there has been an increasing emphasis on teachers' MKT in teacher preparation and training (Hill, Sleep, Lewis, & Ball, 2007). While there has been very little research on MKT around quantitative reasoning, it is useful to examine what the field has learned about MKT. Doing so will help illuminate a pathway forward for synthesizing MKT with quantitative reasoning.

Mathematical Knowledge for Teaching

In 1985, Shulman introduced the construct of pedagogical content knowledge (PCK; 1986), which fuses pedagogical knowledge with specific content knowledge. This introduction illuminated new avenues for research as researchers began to analyze the mathematical work involved in teaching mathematics in classrooms (e.g., Ball & Bass, 2003; Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013; McCrory,

Floden, Ferrini-Mundy, Reckase, & Senk, 2012). This research refined the fields' notion of teacher knowledge to better reflect the specific demands of fostering and supporting the emergent and often messy mathematics of children. Consequently, conceptualizations of teachers' mathematical knowledge expanded to include PCK in addition to subject matter knowledge (SMK). This amalgam of knowledge is often referred to as MKT.

In recent years, several models of MKT have emerged in the research literature (e.g., Ball et al., 2008; Carrillo et al., 2013; McCrory et al., 2012; Silverman & Thompson, 2008; van Bommel, 2012). The vast majority of such models have two features. First, they refine MKT into subcategories. For example, perhaps the most well-known model of MKT is the Domains of MKT model (Ball et al., 2008), which refined MKT into six distinct domains. Additionally, Ball and her colleagues argued that teachers need to be able to flexibly deploy MKT in ways that support student learning (Ball, 1990; Ball & Bass, 2000, 2003; Ball et al., 2008). Their argument exemplifies the second feature many models of MKT have: these models often describe ways in which knowledge should be known or deployed so that it useful.

By carefully describing MKT, including the types of knowledge that teachers need, as well as how they need to be able to use it, the field has extended its understanding of how teacher knowledge is related to student learning (e.g., Hill, Ball, Blunk, Goffney, & Rowan, 2007; Hill, Rowan, & Ball, 2005; A. G. Thompson, 1984, 1992). Accordingly, it is widely accepted that the knowledge teachers bring to bear in

their day-to-day and moment-to-moment instruction has a significant impact on the quality of their teaching practice (An, Kulm, & Wu, 2004; Cai, 2005; Ma, 1999).

Much of the research on MKT describes or refines categories of MKT (Ball et al., 2008; e.g., Carrillo et al., 2013; Even, 1993; McCrory et al., 2012). Research of this nature can be summarized as establishing *that* MKT exists. Other research has elaborated accounts of teacher training programs designed to support and foster MKT (e.g., Manouchehri, 2009; Santagata & Guarino, 2010; Somayajulu, 2012; Walkoe, 2015; Wilson, Lee, & Hollebrands, 2011; Wilson, Sztajn, Edgington, & Confrey, 2014). However, this research on MKT has mostly avoided MKT around quantitative research. One exception is a series of studies by A. G. and P.W. Thompson that examined quantitative reasoning by students and linked it with teachers' knowledge and practice (e.g., A. G. Thompson & Thompson, 1995, 1996; P. W. Thompson, 1993).

The conceptual roots for the Silverman and Thompson (2008) framework emerged out of that series of studies. In 2008 Silverman and Thompson presented their framework for the development of MKT to spark conversations and future research about MKT development. This framework conceptualizes MKT broadly as pedagogically powerful knowledge that enables teachers to provide the kind of instruction that supports students' conceptual development, including quantitative reasoning. The details of this framework are elaborated in Chapter 2. For now, I provide a brief overview using quantitative reasoning as an example, so that I can adequately describe the proposed study.

Using the Silverman and Thompson (2008) framework, MKT around quantitative reasoning can be described as follows. Teachers have MKT for quantitative reasoning if they themselves can reason quantitatively in multiple situations, with a high level of sophistication. However, their quantitative reasoning is not sufficient by itself to be MKT. Instead, a teacher has MKT for quantitative reasoning when he or she has developed quantitative reasoning and additionally: (a) has images of the different ways that students might reason quantitatively; (b) has images of milestones for a learning trajectory for developing quantitative reasoning; (c) has images of quantitative instruction that supports students' quantitative reasoning; and (d) has images of how quantitative reasoning empowers students to engage with other mathematical ideas.

Videos of Students' Quantitative Reasoning

Silverman and Thompson (2008) suggested that MKT develops when a teacher asks herself questions such as "How might a student understand this idea?"; "What challenges might a student face in understanding this idea?"; or "What tasks or questions might I pose to help a student come to understand this idea?" Answering these questions undoubtedly requires that one has experience with the ways in which students think. However, the framework makes no mention of the role that such experience plays in the development of MKT. This seems to be a shortcoming of the framework, and any robust discussion of the processes through which MKT develops must take experience into account. For example, consider that in-service teachers have a wealth of classroom experiences they can bring to bear to help develop images of

quantitative instruction or how students develop more sophisticated quantitative reasoning over time. By contrast, prospective teachers may have little or no experience working with children and their ways of reasoning quantitatively, and thus may struggle to develop such images.

The use of videos might be one way of addressing this limitation of the Silverman and Thompson (2008) framework for MKT. For example, videos that were created by the NSF-funded Project MathTalk (www.mathtalk.org; Lobato, 2014) highlight two students' development of quantitative reasoning by presenting their authentic, unscripted efforts to make sense of the tasks they were given. The videos form a unit with a goal of establishing relationships between algebraic and geometric conceptions of parabolas. This goal is accomplished through the students' development of quantitative reasoning with distances in the plane (a detailed description of the video unit is given in Chapter 2).

These videos address four components of the framework. First, they demonstrate the development of quantitative reasoning from the perspective of the learner. Prospective teachers who view the videos might develop similar knowledge. Second, because the videos prominently highlight the reasoning of two high school students, prospective teachers can develop images of ways students reason quantitatively. Third, the videos are longitudinal in nature, meaning that they show the same two students over several lessons. The longitudinal nature of the videos may support prospective teachers' efforts to develop images of how students come to understand mathematical ideas over time. Finally, the videos feature the skillful

teaching of an experienced math educator. Consequently, prospective teachers can form images of effective instructional moves that foster quantitative reasoning with distances.

Research Questions and Overview of Study

This study examines both the nature of MKT related to quantitative reasoning and how it develops. Specifically, this study answers the following two research questions:

Research Question 1: What is the nature of the mathematical knowledge for teaching (MKT; Silverman & Thompson, 2008) that develops for prospective secondary teachers during a video-based mini-course?

Research Question 2: How do particular elements of the designed learning ecology (Cobb, Confrey, Disessa, Lehrer, & Schauble, 2003) contribute to the development of MKT (Silverman & Thompson, 2008) by prospective secondary teachers during a video-based mini-course?

To answer these research questions, I conducted a design experiment for prospective secondary teachers (PSTs). The experiment included a mini-course featuring a video unit on parabolas. Seven participants engaged in problem solving activities and watched videos of two high school students working on similar tasks.

Taking as a starting point the Silverman and Thompson (2008) framework for MKT, the mini-course was designed to promote participants' development of MKT around quantitative reasoning with distances. However, given that participants were prospective teachers, the mini-course included the extensive use of videos as a way of addressing a limitation of the framework, namely that it does not account for the role experience with students' thinking plays in the development of MKT.

To answer Research Question 1, I conducted clinical interviews (Ginsburg, 1997). This led to the identification of three shifts in MKT around quantitative reasoning as well as a shift in affect. To answer Research Question 2, I linked the shifts in MKT around quantitative reasoning to a shift in what participants noticed during the mini-course. To do so, I leveraged the focusing framework, developed by Lobato, Hohensee, and Rhodehamel (2013). This framework accounts for the socially distributed nature of mathematical noticing, and it provided me with a tool for analyzing how what participants learned was linked to what they noticed.

Significance

The significance of this study lies along several dimensions. There are few studies that that examine prospective teachers' MKT around quantitative reasoning. The potential for teacher training programs to influence prospective teachers is great, and more research that examines MKT for specific mathematical ideas and content is needed. This study contributes to that need by illuminating MKT around quantitative reasoning with distances. Specifically, in Chapter 4, I explicate four shifts in MKT participants seemed to experience, three of which relate to quantitative reasoning with distances.

In Chapter 4, I demonstrate ways that PSTs themselves reasoned quantitatively with distances in the coordinate plane. I show that quantitative reasoning is a powerful tool that enabled participants to examine relationships between algebraic and geometric conceptions of parabolas. In this way, I contribute to the body of knowledge

about quantitative reasoning by describing specific ways prospective teachers learned to reason quantitatively.

Additionally, I illuminate specific aspects of prospective teachers' MKT around quantitative reasoning. For example, participants developed capacity for taking the perspective of high school students when solving tasks. This result is nontrivial. Recall that according to the Silverman and Thompson framework for MKT, teachers develop MKT by thinking about how students might solve problems. For participants in this study, they were not able to do this until (a) they engaged with tasks that students would solve, and more importantly (b) viewed videos of high school students solving those tasks. As a result, participants could accurately predict how high school students would solve a task, despite never having seen videos of that task being solved.

Answering Research Question 2 holds both methodological and theoretical significance. I extend the focusing framework to examine not only the mathematical features of the learning environment that participants noticed, but also its pedagogical features. This methodological extension of the focusing framework demonstrates the flexibility of the framework, and provides a blueprint for future use of this method for teacher education studies.

Finally, I contribute to the field's understanding of the development of MKT. In Chapter 5, I present evidence that suggests participants' noticing of mathematics in the MathTalk videos contributed to their development of MKT around quantitative reasoning. The longitudinal nature of the videos appeared to be critically important to

the development of MKT. The videos provided participants with a near-continuous image of both how the two high school students developed quantitative reasoning, as well as how the instructor in the video fostered and supported such reasoning.

Results from Chapter 5 indicate that while the Silverman and Thompson (2008) framework for MKT holds promise, it does not completely account for MKT development by prospective teachers. Instead, experience with students' thinking and how it develops over time, seems to be a necessary component for MKT development with prospective teachers. While some training programs address this, I found no research that examined the affordances of providing opportunities for prospective teachers to have multiple exposures to the same students' thinking over time.

I end this chapter as I began it, by discussing my experiences at the NCTM workshop. As I was explaining my solution to the teachers at my table, the facilitator of the workshop overheard me. She asked me to explain again my reasoning, which I did. Her eyes lit up, and she said "You know, there are over 100 people in this workshop, and I've been doing workshops like this for years. And in all that time, you are only one of three people to approach a task that way!" Because quantitative reasoning is so powerful, the facilitator's statement should be alarming. If the field is serious about equipping students with this versatile mathematical tool, efforts must be undertaken to provide teachers with opportunities to develop the kind of knowledge necessary to teach for quantitative reasoning. By undertaking this study, I contribute to such efforts.

Chapter 2: Literature Review

This study lies in the intersection of three major themes: mathematical knowledge for teaching, quantitative reasoning, and video-based teacher education. This chapter is organized around these themes, with the first three sections devoted to reviewing literature related to each of the major themes of the study. In the fourth and final section, I elaborate my research questions using ideas developed in this chapter.

Mathematical Knowledge for Teaching

Research on the mathematical knowledge that teachers bring to bear in their day-to-day and moment-to-moment practice of teaching has resulted in the development of many different frameworks to characterize such knowledge (e.g. Ball et al., 2008; Carrillo et al., 2013; McCrory et al., 2012; Silverman & Thompson, 2008; Speer, King, & Howell, 2014; van Bommel, 2012). One prominent framework for studying teachers' mathematical knowledge is the Domains of MKT model developed by Ball and her colleagues (Ball et al., 2008).

In the next two subsections, I provide a detailed description of the Domains of MKT. I focus on this model because it is widely cited and has influenced the development of other models. Moreover, it exemplifies two features that most models of MKT have: describing MKT and what makes it useful for teachers. After discussing this model, I briefly describe other models of MKT.

Describing MKT

Ball and her colleagues performed an extensive and thorough “job analysis” of the profession of teaching mathematics (Ball & Bass, 2000, 2003; Bass, 2005). Rather

than make claims about the mathematical knowledge teachers need based on an analysis of curriculum or standards, as had been the case in the past (Hill, Sleep, et al., 2007), Ball and her colleagues examined the practice of teaching itself, including videos of teachers, transcripts of such videos, lesson plans, interviews, and student work (Ball & Bass, 2003). Their analysis revealed knowledge for teaching that was not included in prior analyses of teacher knowledge that focused only on the product of teaching, namely the mathematics that students learn (Hill, Sleep, et al., 2007).

Ball and her colleagues identified a significant area of mathematical knowledge that teachers needed beyond the mathematics embedded in school curriculum. This mathematics for teaching includes:

- interpreting, analyzing, and evaluating students' solutions and errors (Ball, Hill, & Bass, 2005);
- recognizing and fostering productive mathematical discussions (Bass, 2005);
- choosing appropriate definitions, models, and tasks, or modifying existing ones to fit the aims of the lesson and the ability of one's students (Ball & Bass, 2003);
- facility with bringing about the development of mathematical practices such as knowing what counts as a mathematical explanation (Ball & Bass, 2003; Ball et al., 2008).

Moreover, Ball and her colleagues argued that this mathematical knowledge was not represented in math curricula, nor was it present in most university math courses or teacher training programs (Ball et al., 2008).

Their model, the Domains of MKT, captured this mathematical knowledge, and it featured six domains of mathematical knowledge for teaching, across two categories (Ball et al., 2008). The first category, subject matter knowledge (SMK) is broadly defined as knowledge that is purely mathematical in nature, and it is subdivided into three domains: common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). The second category, pedagogical content knowledge (PCK), is defined as “a kind of amalgam of knowledge of content and pedagogy that is central to the knowledge needed for teaching” (Ball et al., 2008, p. 392). This second category is also subdivided into three domains: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC).

These domains of knowledge represent areas of knowledge that Ball and her colleagues argued are necessary for the work of teaching, which includes lesson planning, evaluating students’ work, creating and grading assignments, speaking with parents about classwork, navigating standards, and dealing with administrative issues related to math curriculum (Ball et al., 2008; Bass, 2005). For example, CCK represents mathematical knowledge that is used in a variety of settings, not just in teaching. Included in CCK is the mathematics that students are expected to learn. Yet teachers must know more than just the mathematics their students are expected to

learn. The model accounts for this knowledge in the domain SCK, which enables teachers to decompress their knowledge in order to make the mathematics that is to be taught “visible to and learnable by students” (Ball et al., 2008, p. 400).

Domains of knowledge within the PCK category allow teachers to leverage their knowledge of content in ways that are appropriate for their students, given particular contexts and curricula. For example, KCS allows teachers to anticipate potential and probable conceptual challenges for students, while KCT enables teachers to design or modify tasks that could help students navigate such challenges (Ball et al., 2008). By refining teacher knowledge in this way, new studies have been able to link specific domains of MKT with gains in student achievement in early elementary grades (Hill et al., 2005), contradicting earlier studies (e.g., Begle, 1979) that assumed a simplistic, curriculum-focused view of teacher knowledge.

In summary, the Domains of MKT model described the mathematical knowledge that Ball and her colleagues believed equips teachers to productively engage with the raw, complex, and often messy mathematics of their students (Ball & Bass, 2000; Ball et al., 2008). By refining teacher knowledge into six distinct domains, this model offered new ways of understanding what knowledge mathematics teachers need.

What Makes MKT Usable?

The Domains of MKT model emerged out of answers to questions about what actual mathematical work, specifically the kinds of “mathematical reasoning, insight, understanding, and skill” (Ball et al., 2008, p. 395), was involved in teaching. Such

answers not only provided insight into what teachers need to know, but also the ways in which that knowledge could be usable in teaching (Ball & Bass, 2000).

Ball and her colleagues argued that *how* mathematical knowledge is held and used may determine how useful that knowledge is for teaching (Ball et al., 2008; Hill & Ball, 2004). Teachers work with the emergent and often messy mathematics of novices, thus the mathematical knowledge that teachers leverage in their work needs to be flexible and accessible (Ball et al., 2005). Ball and Bass (2003) pointed out that the compression of knowledge, which is typically a desired hallmark of elegant and refined mathematics, may interfere with one's ability to analyze and understand the burgeoning mathematics of learners. They argued that teachers need to "be able to do something perverse: work backward from mature and compressed understanding of the content to unpack its constituent elements" (Ball & Bass, 2000, p. 98). Teachers need mathematical knowledge for teaching, but they also need to be able to unpack that knowledge in order to facilitate the learning of content (e.g., A. G. Thompson & Thompson, 1996).

Teachers must also be able to contextualize their knowledge. Ball et al. (2005) argued that teacher knowledge must be appropriately adapted for the setting in which it is deployed. For example, consider a question about dividing by zero. A teacher's response to this question will necessarily depend on the grade level of the student, yet it still must be mathematically accurate. A response to this question asked by a third grader may appeal to a division-as-sharing model, while a response to this question asked by an AP calculus student might redirect the student to consider limits.

Other Models of MKT

The job analysis undertaken by Ball and her colleagues focused primarily on elementary mathematics teachers. McCrory et al. (2012) performed a similar job analysis on algebra teaching. Their analysis produced a two-dimensional model of MKT for Algebra. Their model of MKT for Algebra (McCrory et al., 2012) built on the Domains of MKT model, which describes MKT broadly, by offering a framework for describing MKT for a specific mathematics discipline.

The first dimension described different categories of knowledge needed to teach algebra: (a) knowledge of school algebra, (b) knowledge of advanced mathematics, and (c) mathematics-for-teaching knowledge. The authors conceptualized knowledge of school algebra as common to the discipline of algebra, and in this regard, it is like the domain of CCK from the Domains of MKT model. Knowledge of advanced mathematics is knowledge that allows teachers to have a broad view of the mathematics for which algebra is a foundation. This is analogous to horizon content knowledge from the Domains of MKT model. Finally, mathematics-for-teaching knowledge is “mathematics that is useful in teaching, but is not typically taught in conventional mathematics classes” (McCrory et al., 2012, p. 598). The authors claimed this knowledge is not purely pedagogical, although its application would appeal exclusively to teachers. This category of knowledge is similar to the category of SCK in the Domains of MKT model.

The second dimension of the model of MKT for Algebra identifies the ways in which knowledge is used (McCrory et al., 2012). First, teachers need to decompress

their knowledge, a process similar to Ball and Bass's (2000) notion of unpacking knowledge. Teachers also must be able to "trim," which involves choosing tasks, definitions, activities, etc. that are appropriate for students in their class. This might mean providing students with a less sophisticated definition for a concept, but one that maintains the mathematical integrity of concept. It might also mean recognizing when students are capable of more advanced or challenging mathematics and finding ways to *trim up* the content to match students' abilities. Finally, teachers must be able "to connect and link mathematics across topics, courses, concepts, and goals" (McCrorry et al., 2012, p. 606), which the authors refer to as bridging.

Carrillo et al. (2013) also developed a model that describes MKT. They argued that the different domains of MKT identified by Ball et al. (2008) were not well-defined. Consequently, their model emerged from the perspective that all knowledge for teaching is specialized, and correspondingly is called Mathematics Teachers' Specialized Knowledge (MTSK). Like the Domains of MKT and MKT for Algebra models, MTSK refined teacher knowledge into subcategories or domains, in this case six: (a) knowledge of topics, (b) knowledge of the structure of mathematics, (c) knowledge about mathematics, (d) knowledge of features of learning mathematics, (e) knowledge of mathematics teaching, and (f) knowledge of mathematics learning standards. Like the other models, the MTSK model described the kinds of knowledge the researchers claimed is useful for teaching mathematics.

MKT Related to Quantitative Reasoning

Models of MKT like the Domains of MKT and MKT for Algebra take a large grain-sized approach to illuminating MKT. For example, the Domains of MKT broadly characterizes MKT in a way that is applicable to almost any mathematical discipline. The MKT for Algebra model, while narrowing its focus on algebra, still only provides general categories of MKT (e.g., knowledge of advanced mathematics).

The power of models such as these is their generality; however, that power comes at the cost of specificity. For example, what does MKT for quantitative reasoning look like? One could argue that teachers' quantitative reasoning is subsumed under SCK, while identifying students' nascent quantitative reasoning and supporting its development could be conceived of as KCS. However, such an argument only serves to categorize MKT around quantitative reasoning. It does not illuminate what supporting the development of quantitative reasoning looks like, nor does it illuminate how to foster such knowledge in prospective and in-service teachers.

Silverman and Thompson (2008) proposed an alternative model of MKT, one with roots deeply entrenched in quantitative reasoning. This model shows great promise for developing theory about MKT around quantitative reasoning. In the next session, I review research on quantitative reasoning and then elaborate Silverman and Thompson's model in light of that literature.

Quantitative Reasoning

Quantitative reasoning is a powerful mathematical tool, and the field has developed a large body of research that examines it. In this section, I provide an overview of some of the ways the field has conceptualized quantitative reasoning. I

then discuss some of themes around which research on quantitative reasoning has focused. I conclude by elaborating a framework for MKT that is well-suited for investigating MKT around quantitative reasoning.

Conceptions of Quantitative Reasoning

Given the power of quantitative reasoning, it is not surprising that there are several conceptions of quantitative reasoning (Mayes, Peterson, & Bonilla, 2012; Quantitative Literacy Design Team, 2001). In a meta-analysis of the construct, Mayes et al. found at least nine different ways that quantitative reasoning has been defined. These conceptualizations are often broad, and capture general ways of reasoning. For example, the CCSS (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2012) stated:

Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. (p. 6)

The National Numeracy Network (2016) referred to quantitative reasoning as a “habit of mind,” and argued that it emphasizes “higher-order reasoning and critical thinking skills needed to understand and to create sophisticated arguments supported by quantitative data.”

Such broad conceptualizations point to the importance of quantitative reasoning, but have limited practical use for research. Several math educators have elaborated conceptualizations of quantitative reasoning that focus in on specific aspects of the construct. For example, Carraher, Martinez, and Schliemann (2008)

described two types of quantities: counts (e.g., 3 books) and measures (e.g., 4.3 inches). Schwartz (1976) described numbers as both nouns (e.g., *four* plus *three* is *seven*) and adjectives (e.g., *four* children eat *three* cookies in *seven* minutes).

According to Schwartz, adjectival numbers are the basis of quantities, which can be thought of as an ordered pair of the form (*number*, *unit*), where the number is an adjective describing the amount or measure of the unit (Schwartz, 1988).

According to P. W. Thompson (e.g., 1990, 1993, 1994) a quantity is one's conception of a quality of something that can be measured. Notably, in this conceptualization, quantities do not need to have associated numerical values. Accordingly, quantitative operations are similar to numeric or arithmetic operations, but they do not necessarily result in value. Instead, the result of a quantitative operation is a quantitative relationship.

Research on Students' Quantitative Reasoning

Based on a review of the literature, P. W. Thompson's conceptualization of quantitative reasoning is one that has been readily taken up by the field. Indeed, numerous studies have drawn from his work (e.g., Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Hackenberg & Lee, 2015; Johnson, 2012; Lobato & Siebert, 2002; Mayes et al., 2012; Moore, 2012; Steffe, 1991; Ulrich, 2012).

At the elementary and middle school levels, scholars have argued for leveraging quantitative reasoning as a way of exposing students to algebra earlier in the curriculum (e.g., Ellis, 2011; Philipp & Schappelle, 1999; Smith & Thompson, 2007; P. W. Thompson, 1988). For example, Ellis demonstrated how leveraging

students' abilities to reason with quantities supported their ability to reason with functions. Other research has investigated links between students' quantitative reasoning and their abilities to generalize (e.g., Ellis, 2007; Ellis et al., 2012; Lobato, Ellis, & Munoz, 2003; Lobato & Siebert, 2002).

At the high school and college level, researchers have examined relationships between quantitative reasoning and covariational reasoning (e.g., Castillo-Garsow, 2012; Johnson, 2012; Moore, 2013). For example, Moore et al. (2013) found that by investigating covarying quantities, third year PSTs developed substantial connections between polar and Cartesian coordinate systems. Other researchers have examined links between quantitative reasoning and precalculus (e.g., Moore et al., 2009); calculus (e.g., P. W. Thompson & Silverman, 2008); and differential equations (e.g., Rasmussen, 2001).

With the abundant research on quantitative reasoning, it is surprising that there has been very little research examining MKT around quantitative reasoning. Even more surprising is that there has been little progress in developing theory about MKT around quantitative reasoning despite the emergence of a framework for MKT (Silverman & Thompson, 2008) with deep roots in the research on quantitative reasoning. I now describe this framework, as it points a way forward for investigating MKT around quantitative reasoning.

A Framework for the Development of MKT

Emerging out of research on quantitative reasoning (e.g., A. G. Thompson, Philipp, Thompson, & Boyd, 1994; A. G. Thompson & Thompson, 1995, 1996; P. W.

Thompson, 1993) Silverman and Thompson (2008) proposed a framework for the development of MKT. They argued that MKT develops through a process of a second-order reflective abstraction of a *key developmental understanding* (KDU; Simon, 2006). In the following subsections, I elaborate each element of their framework—a KDU, reflective abstraction, second-order reflective abstraction, and the components of MKT.

Key developmental understandings. Simon (2006) introduced the notion of a KDU as a conceptual advance or a “change in students’ ability to think about and/or perceive particular mathematical relationships” (p. 362). KDUs can be thought of as researcher-inferred understandings that seem to be central to the mathematical development of students.

Simon (2006) described two characteristics of KDUs. The first characteristic is that a KDU is the result of a conceptual advance by the student. Silverman and Thompson (2008) argued that KDUs are the result of reflective abstraction, “a process by which new, more advanced conceptions develop out of existing conceptions and involves abstracting properties of action coordinations to develop new cognitive structures” (p. 506). This means KDUs are not the result of empirical abstractions in which students deduce something is true based on only on observed patterns. The second characteristic is that students without a KDU do not acquire it through telling or demonstration only. Instead, the development of a KDU requires “building up of the understanding through students’ activity and reflection and usually comes about over multiple experiences” (Simon, 2006, p. 362). Consequently, it takes time for a

KDU to develop, and a KDU does not necessarily develop as the result of a particular method of teaching or instruction. Instead, KDUs develop as the result of mental actions the student takes.

These understandings are said to be *developmental* because they represent a leap in conceptual understanding on the part of the student. Accordingly, KDUs do serious work for students; they enable students to solve problems as a consequence of having the KDU, rather than as a result of being explicitly taught how to solve such problems (Silverman & Thompson, 2008). In this sense, KDUs are said to be *key* understandings. Once a student has a KDU, he or she is empowered to use the KDU as a tool to solve mathematical problems.

An example will help illustrate this construct. The videos I used in the study feature two high school students, Sasha and Keoni, making connections between geometric and algebraic conceptions of parabolas. In preparation for this study I reviewed these videos multiple times, and I identified a KDU around quantitative reasoning that the students developed over time. I now elaborate this KDU to illustrate the KDU construct.

This KDU involves reasoning with distances in the coordinate plane as quantities. Consider the point (6,9). A way to think of this point is as a particular location in the coordinate plane, whose coordinates provide instructions for how to find the point in the plane. In this case, thinking of a point in this way, one might reason “I count six tick marks along the x -axis and then from there I count nine tick

marks up.” The 6 and the 9 act as instructions for how many tick marks one must count.

Another way to think about the point is as a description of two quantities: the x -coordinate represents the distance that a particular point is from the y -axis, while the y -coordinate represents the distance that same point is from the x -axis. In this case, the point $(6,9)$ is six units from the y -axis and nine units from the x -axis.

While this second way of thinking about a coordinate pair may seem obvious, it also holds great potential for powerful reasoning with quantities in the coordinate grid. For example, consider Figure 2.1, which shows a parabola that is located in a coordinate grid with its grid lines and tick marks removed.

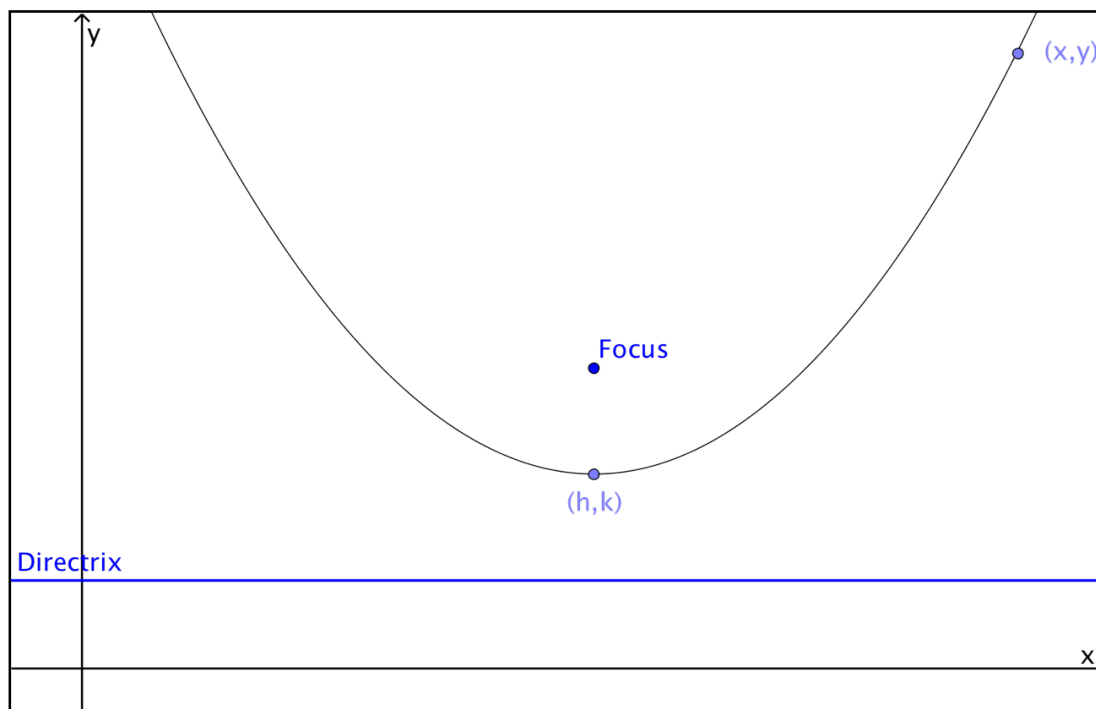


Figure 2.1. A parabola in a coordinate grid with no grid lines or tick marks.

Students with the KDU I am describing would be able to reason as follows. First, they would understand that (h, k) represents the point in the coordinate plane that is the

vertex of the parabola. It also represents that the vertex is a distance of h units to the right of the y -axis and a distance of k units above the x -axis. Students with the KDU would be able to explain that given any point on the parabola, (x, y) , the distance from that point to the directrix is given by the expression $y - k + p$, where p is the distance between the focus and the vertex (as illustrated in Figure 2.2).

For example, students could reason that the distance from the point (x, y) to the x -axis is y units. Subtracting k units represents taking away the distance from the x -axis to the vertex, which leaves the distance from the point (x, y) to the line $y = k$. Adding p represents that distance plus the distance from the vertex to the directrix. Thus, the distance from the point (x, y) to the directrix is $y - k + p$. Another way that students might reason is to say that the distance from the point (x, y) to the x -axis is y units, and to get to the directrix one needs only to subtract the distance the directrix is from the x -axis, which is $k - p$ units. Therefore, the distance is $y - (k - p)$. Knowing this distance (and other distances, through similar reasoning) allows one to leverage the Pythagorean theorem to derive the general equation of any parabola in the coordinate grid. I provide a conceptual analysis of this task in Chapter 3.

This KDU represents a conceptual advance since research suggests that students do not easily reason quantitatively (P. W. Thompson, 1988, 1993). Furthermore, studies have shown that such reasoning is difficult or absent for undergraduate students (Moore, 2013; Moore et al., 2009), PSTs (A. G. Thompson & Thompson, 1995), and even practicing teachers (P. W. Thompson, Carlson, & Silverman, 2007). Indeed, the type of understanding I have described above would not

develop as the result of telling or demonstration. Instead, such understanding must be built up over time, as a result of students continually engaging with points and their coordinates as both locations and distances, first as particular points and distances (e.g., $(6,9)$, or $p = 2$), then later in more general terms (e.g., the point (x, y) , or it is given that (h, k) is the vertex). It is through such a build-up that students can reason quantitatively with points in the plane as both locations and as distances-as-quantities.

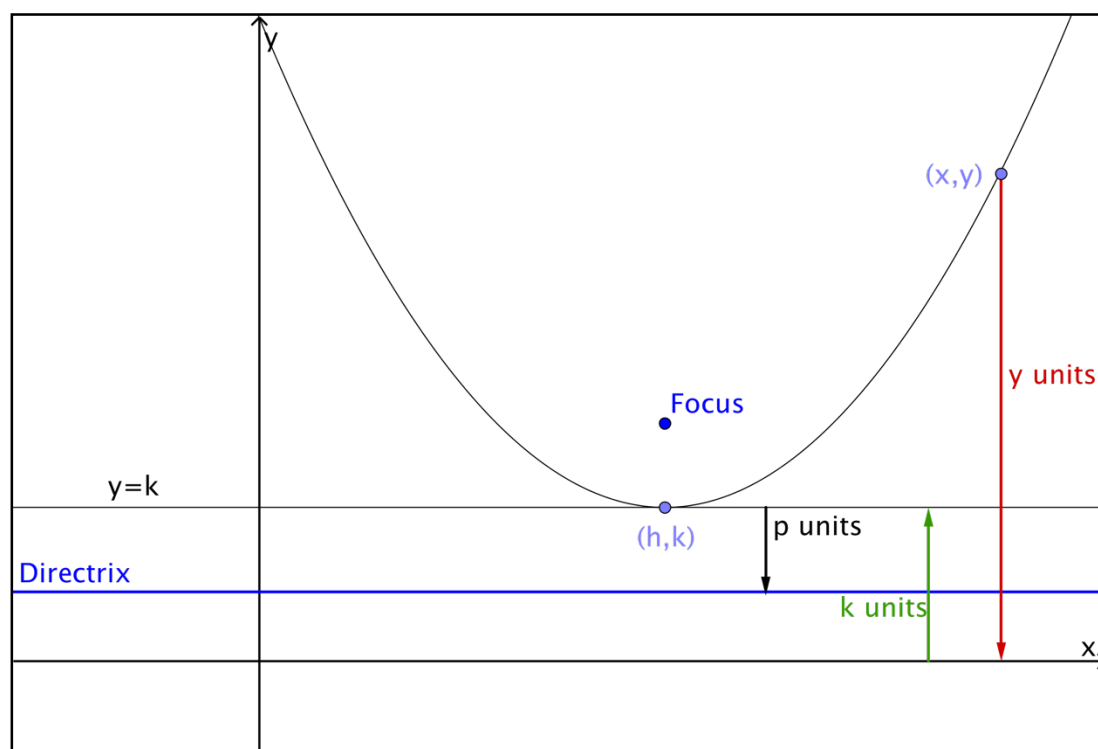


Figure 2.2. An illustration of reasoning with the KDU

Reflective abstraction. Piaget’s notion of reflective abstraction is complex and subject to multiple interpretations (e.g., Dubinsky, 2002; Simon, Tzur, Heinz, & Kinzel, 2004; Tall, 2004). For this study, I leveraged the Silverman and Thompson (2008) framework for MKT; consequently, I have chosen to operate from their conception of reflective abstraction. They described reflective abstraction as “a

process by which new, more advanced conceptions develop out of existing conceptions and involves abstracting properties of action coordinations to develop new cognitive structures” (Silverman & Thompson, 2008, p. 506).

There are two parts of reflective abstraction—reflection and abstraction. It will be useful to elaborate each of these constituent parts, as well as provide an example of reflective abstraction using the KDU around quantitative reasoning. Specifically, I show how two understandings that are necessary for the development of the KDU may develop through reflective abstraction. These two understandings are (a) how to measure distances from a point to a line; and (b) how to describe distances, both in terms of real numbers and unknown quantities.

Reflection. Reflection is characterized as a conscious mental process in which mental actions or experiences are re-presented in the mind so that they may be more closely analyzed (Battista, 1999). Reflection can be thought of as the act of excising a specific episode from one’s experiential flow so that it can be an object of mental examination. To better illuminate what I mean by *reflection*, imagine that Sasha and Keoni are engaged in mathematical activity designed to foster the development of the KDU, which involves reasoning with distances as quantities. Sasha may reflect on her experience of constructing a parabola using the geometric definition. She may re-present many aspects of that experience: images of the paper and tools that were used or the lines that were made; questions that the teacher asked; discussions she had with Keoni; initial struggles or confusion regarding measuring the distance between a point a line. The act of reflecting on this experience involves re-presenting the experience,

selecting an episode for further examination, and analyzing the result of that episode (Battista, 1999).

Abstraction. The mental process of *abstraction* was described by Battista (1999) as “the process by which the mind selects, coordinates, combines, and registers in memory a collection of mental items or acts that appear in the attentional field” (p. 429). In other words, abstraction is the process of identifying regularities in ones’ activities, selecting certain attributes from experiences related to those activities, and simultaneously suppressing other attributes (Lobato et al., 2003).

For example, when Sasha and Keoni began working on the task of constructing a parabola, they initially struggled to measure the distance between a point and a line. Not knowing that such a distance is defined along a segment from the point perpendicular to the line, Sasha tried measuring along several different segments, for example segments *b* and *c* in Figure 2.3. During this process, she may have isolated the attribute of segment length. In terms of abstraction, Sasha may have abstracted a property of segment length by identifying a regularity in her experiences.

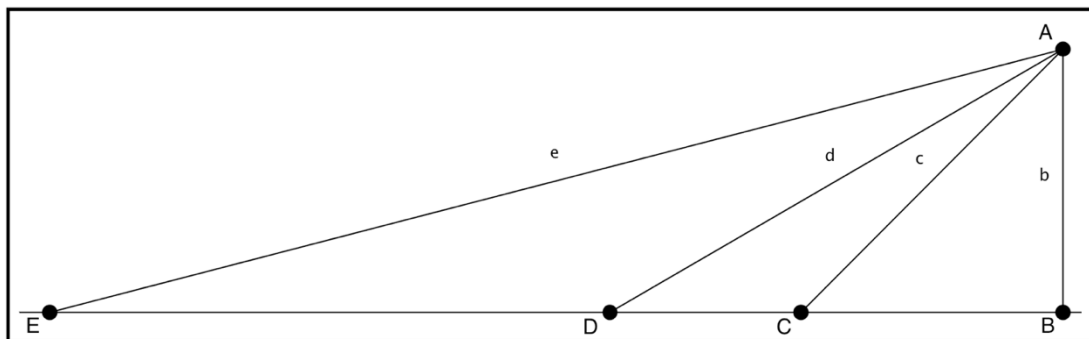


Figure 2.3. Several segments from point *A* to a line.

Reflective abstraction. Combining the acts of reflection and abstraction together, the process of reflective abstraction can be described as coordinating

multiple re-presentations of mental activity in order to draw from those a regularity that can now itself be mentally acted on. Let's again consider Sasha and her work with parabolas. First, in her attempts to measure the distance between a particular point a line, she measured along several segments. She may identify the regularity that the distance along each segment is different. Given that her goal was to find *the* distance between the point and the line, she may wonder which distance is *the* distance between the point and the line.

At this point, perhaps the instructor or a classmate tells her of the convention that distances between points and lines are measured along segments that are perpendicular to the line. In this case, her re-presentations of her actions (measuring multiple segments) and subsequent isolation of particular attributes from those experiences (the differing measurements) results in reflective abstraction. She no longer needs to measure along those segments—she can use the result of that activity as a launching point for new activity, namely understanding why the distance between a point and a line is measured along a segment that is perpendicular to the line.

Because this construct has many interpretations and can be confusing, I want to restate Silverman and Thompson's (2008) description and then unpack it using another example. They state that reflective abstraction is "a process by which new, more advanced conceptions develop out of existing conceptions and involves abstracting properties of action coordinations to develop new cognitive structures" (p. 506). Consider now Keoni and his work on a task in which he must prove that the point (4,4) is on a parabola (see Figure 2.4). To do so, he must identify that the two

segments a and b are equivalent and find the length of segment b , among many other things. His initial attempts involve counting along segment b to determine that it has a length of five units. In later tasks, he and Sasha repeat this activity with various points and parabolas, and they realize that the length of any segment from a point on the parabola to the directrix is a combination of the y -coordinate of the point on the parabola and the distance from the directrix to the x -axis.

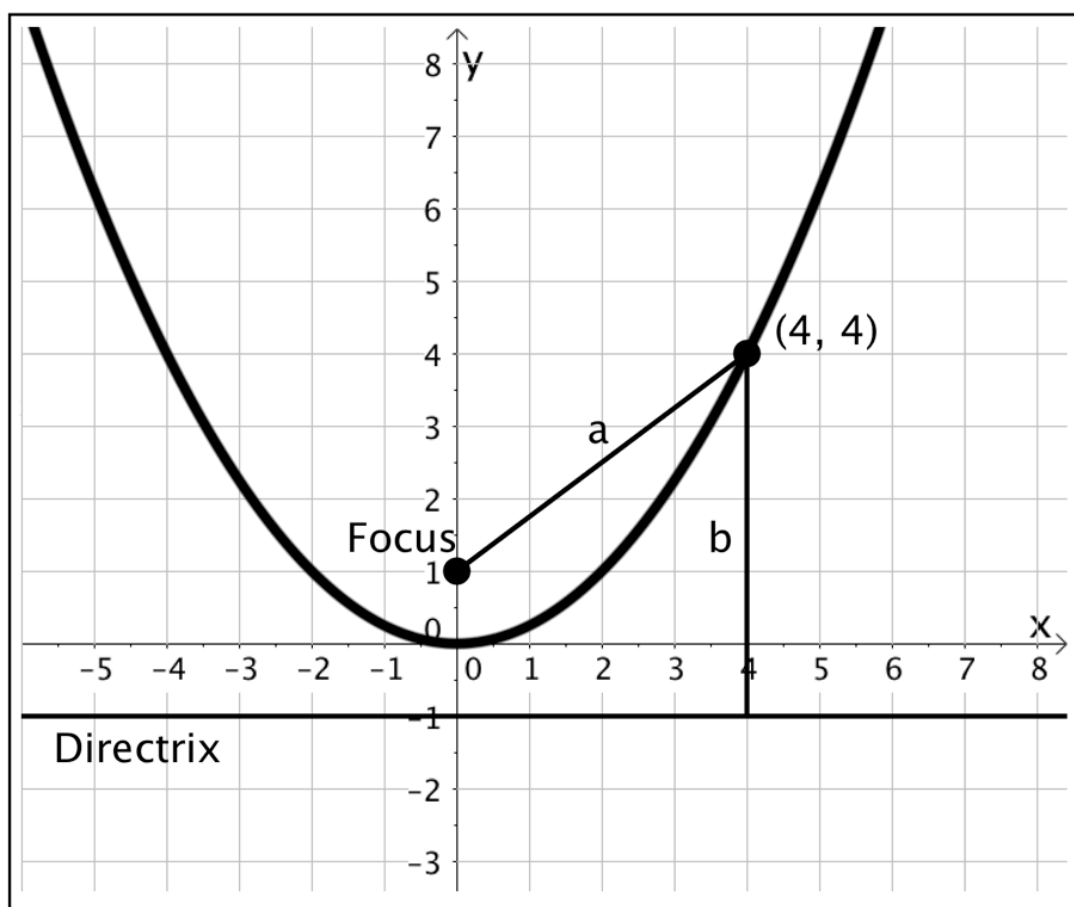


Figure 2.4. A parabola with the point $(4,4)$ labeled.

His action coordinations can be thought of as his various attempts to measure distances between a point on the parabola and the directrix. From these action coordinations, Keoni can abstract a property of his mental activity, that of combining

two quantities to find the distance of the segment from the point to the directrix. These two quantities are the distance from the point to the x -axis (which is also the y -coordinate of the point), and the distance between the x -axis and the directrix. Other attributes from these experiences (e.g., specific coordinates or parabolas, placement of the directrix, or the location of the focus) can be ignored, and the result is a new cognitive structure: Describing distances from a point to the directrix involves coordinating two quantities.

The result of this reflective abstraction is different than just being aware of the definition of a parabola. Reading a definition does not result in “new cognitive structures.” For example, suppose Keoni is trying to find the equation for any parabola in the coordinate plane (see Figure 2.5). Among other things, this task requires Keoni to find the length of segment a , which itself requires him to find the length of segment b . Previously, he had to do quite a bit of mental work to determine the length of segment b . With his new cognitive structure, Keoni can begin from the perspective that he will need to find the distance from the point to the x -axis, then find the distance between the directrix and the x -axis, and finally combine these two quantities in a way that yields the desired quantity, the length of segment b . The result of the reflective abstraction now acts as the starting point for new mathematical activity. Simply reading the definition would not provide Keoni with the knowledge needed to construct and measure segment b , but the reflective abstraction could. In this sense, reflective abstraction results in new knowledge that can be productively deployed for future mathematical activity.

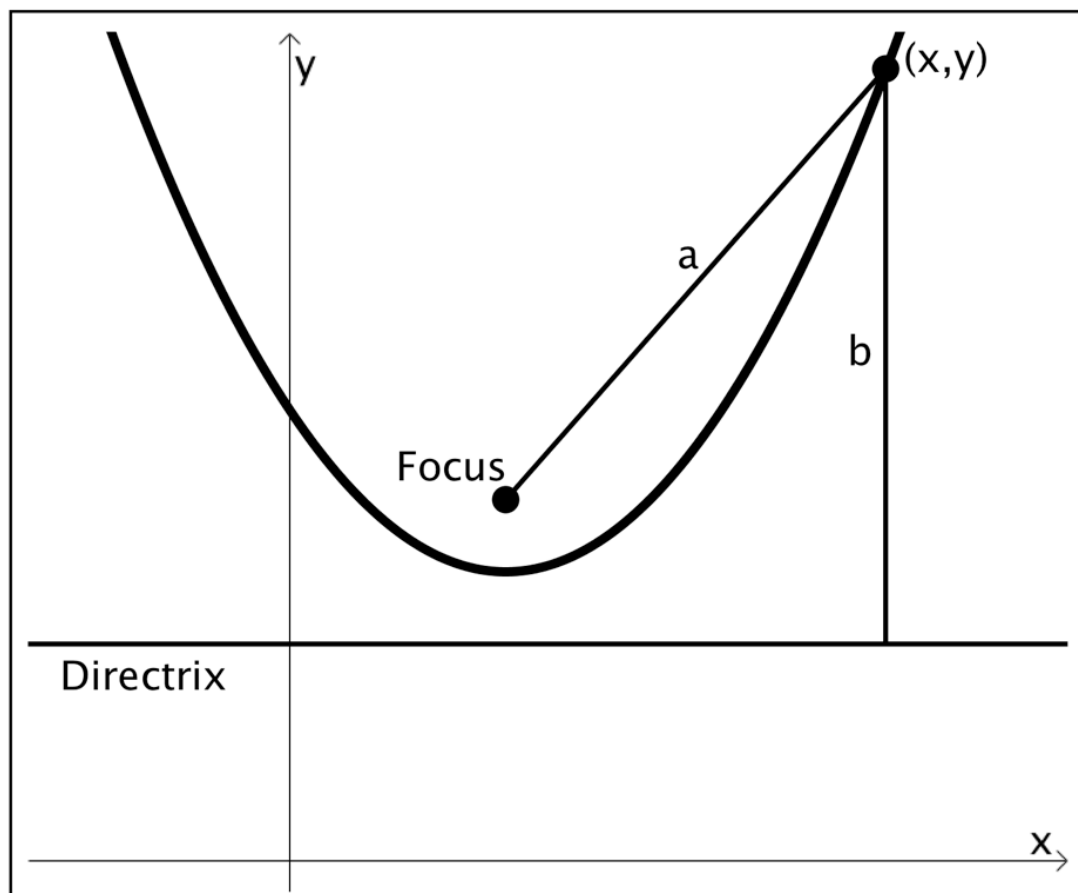


Figure 2.5. A general parabola, with segments a and b labeled.

The five components of MKT. Silverman and Thompson (2008) described MKT “as being grounded in a personally powerful understanding of particular mathematical concepts and as being created through the transformation of those concepts from an understanding having pedagogical potential to an understanding that does have pedagogical power” (p. 502). KDUs represent personally powerful understandings, and as such serve as the foundation for MKT. They are the first component of the framework for MKT. KDUs hold “pedagogical potential” in the sense that a teacher with a KDU is empowered to engage in challenging mathematical activity. However, conceptual understanding of a topic is not sufficient for effective

conceptual teaching (e.g., P. W. Thompson & Thompson, 1994). Silverman and Thompson argued that KDUs can be transformed into knowledge that has “pedagogical power” through the process of reflective abstraction.

According to Silverman and Thompson (2008),

A teacher has developed knowledge that supports conceptual teaching of a particular mathematical topic when he or she

1. Has developed a KDU within which that topic exists,
2. Has constructed models of the variety of ways students may understand the content (decentering);
3. Has an image of how someone else might come to think of the mathematical idea in a similar way;
4. Has an image of the kinds of activities and conversations about those activities that might support another person’s development of a similar understanding of the mathematical idea;
5. Has an image of how students who have come to think about the mathematical idea in the specified way are empowered to learn other, related mathematical ideas. (p. 508)

These are the five components of MKT according to Silverman and Thompson. The first component of this framework, KDUs, was elaborated in a previous section. I now elaborate each of the remaining four components of MKT.

Images of students’ thinking and understanding. To support students’ development of conceptual understanding of a mathematical idea, a teacher must first understand the variety of ways that students might understand that idea. This might involve asking questions such as “What does it mean to understand this idea?”; “How do I understand this idea?”; and “Are there other ways that someone might understand this idea?” These models should also include the different ways students may not understand the idea, including naïve conceptions and misconceptions.

Images of milestones for a learning trajectory. Having MKT that supports conceptual teaching of a mathematical idea means having a roadmap for how that idea develops over time. This involves understanding how the mathematical idea becomes more sophisticated over time and with appropriate support. A teacher may ask “What must a student understand in order to develop this mathematical idea?” Having an image of how someone might develop the mathematical idea also means having an image of the various obstacles or stumbling blocks likely to be encountered as the idea develops.

Images of instruction. Teachers must have not only an image of a learning trajectory for the mathematical idea, but they must also have images of the types of mathematical tasks and activities that promote that idea. They must be able to assess students’ progress along the trajectory and know how to pose problems and orchestrate conversations to support the generation of more sophisticated ideas.

Images of mathematical connections. Finally, teachers must have an image of how the mathematical idea fits within the larger landscape of mathematics. How does the idea develop out of other mathematics? What new mathematical ideas are students empowered to learn as a result of understanding this mathematical idea?

The last four components of this framework describe what makes a KDU pedagogically powerful. In other words, when a teacher can imagine ways in which students can come to understand content, as well as actions and activities that will foster that understanding, then that teacher has transformed a KDU into knowledge that is pedagogically powerful.

Second-order reflective abstraction. This framework shares characteristics with the Domains of MKT model (Ball et al., 2008). For example, the *images of milestones for a learning trajectory* component can be mapped into the SCK domain, while the *images of instruction* component could be mapped into both the KCT and KCS domains. The *images of students' thinking and understanding* component seems to require an “unpacking” of expert knowledge, which is one of the uses of MKT in the Domains of MKT model.

Despite these similarities, the Silverman and Thompson framework represents a departure from the Domains of MKT model (and most other models of MKT as well) in that it explicitly defines how MKT develops. Silverman and Thompson (2008) argued that KDUs are formed as the result of first-order reflective abstractions, but that MKT is the result of a second-order reflective abstraction. This process is initiated when teachers, who already have a KDU, put themselves in the position of a student who does not have that KDU. They ask themselves questions such as

“What must a student understand to create the understanding that I envision?” and “what kinds of conversations might position one to develop such understandings?” The prospective teacher must put herself in the place of a student and attempt to examine the operations that a student would need and the constraints the student would have to operate under to (logically) behave as the prospective teacher wishes a student to do. This is reflective abstraction. (Silverman & Thompson, 2008, p. 508).

Silverman and Thompson’s description of a second-order reflective abstraction has as its foundation a first-order reflective abstraction that itself resulted in a KDU.

I have developed MKT around quantitative reasoning, in part as a result of my observations of two students, Sasha and Keoni, and their mathematical activity as they

solved problems about parabolas. The videos that were used in this study were filmed as part of a grant (Lobato, 2014) that has funded my research assistantship, and they feature Sasha and Keoni solving math problems. My roles in the creation of the videos were as a collaborator during the planning for each lesson that was filmed, director for the filming of the videos, and editor during post-production. As part of my preparation for filming the videos, I solved mathematical tasks that were similar to the ones we planned to use in the videos, and I had conversations about these tasks with my advisor. This led to the refining of my own KDU around quantitative reasoning, over which MKT could be developed.

Over the course of the planning, filming, and editing of the videos, I had the opportunity to watch numerous times the high school students' development of the KDU around quantitative reasoning. Accordingly, I have been able to reflect not only on how my own understanding developed, but also on how the students' understanding developed. Consequently, this has led to my developing of MKT around quantitative reasoning. I will now elaborate my personal understanding of parts of the MKT that I developed, and in doing so illuminate how this understanding could have formed via a second-order reflective abstraction.

In filming, editing, and reviewing these videos, I noticed that the teacher asked the same kinds of questions across the ten lessons. I recorded these questions and began wondering how these questions helped the students. In doing so, I re-presented my experiences observing the instructor asking these questions, as well as the brief period of time during which the students responded to these questions. This re-

presentation of several episodes from the filming, editing, and viewing of the lessons ended with my evaluation of each episode. This is the process of reflection as I have described it above.

From these episodes, I identified regularities in the kinds of questions the teacher asked, as well as the nature of the responses given by the students. Initially the students' responses were unfocused, but over time, even though the teacher continued to ask the same kinds of questions, the students' responses evolved to the point where they were focusing on the attributes of the definition and the quantities on which the teacher wanted them to focus. My selection of these regularities in the questions asked by the teacher and the students' responses is an example of the process of abstraction, as I have described it above.

Taken together, I re-presented multiple episodes from my experiences with the teacher and her students as they talked about parabolas. From these re-presentations I selected certain attributes of those episodes, namely the questions the teacher posed as well as the students' responses to those questions. In coordinating these multiple episodes, I developed the idea that in order to get students to reason productively with distances as quantities, they must understand the following: (a) how to measure distances from a point to a line; and (b) how to describe distances, both in terms of real numbers and unknown quantities.

The first understanding may come about through conversations about distances between a point and a line. One can ask students questions such as "How many ways are there to measure this distance?"; "Do each of those ways produce the same

measurement?"; "Which way do you think makes the most sense?". A teacher may also decide to tell students that measuring along a segment from the point that is perpendicular to the line is the way the math community has agreed to make such a measurement. These supports may help students overcome the potential obstacle of measuring the distance from a point to a line along a segment that is not perpendicular.

The second understanding may come about by continually focusing students' attention on the quantities on which they are operating. During a task in which they must prove points are on a parabola, the teacher can ask questions such as "Can you use the definition of a parabola to justify that?" or "What do you think that distance (segment b in Figure 2.4) would be?" During a task in which the point $(4,4)$ is replaced with a general point (x, y) , segment b has a distance of $y + 1$ (see Figure 2.6). Asking students "Where do you see y ?" and "where do you see the 1" can help focus their attention on the quantities involved in finding that distance.

This discussion has illustrated some of the MKT that I have developed, partly as a result of my observations of Sasha and Keoni as they solved math problems. I have not explicated all of my MKT for this topic, primarily because to do so would be beyond the scope of this section. However, as can be seen, experiences in which a teacher observes or works with students can serve as the grist from which second-order reflective abstractions form, resulting in MKT.

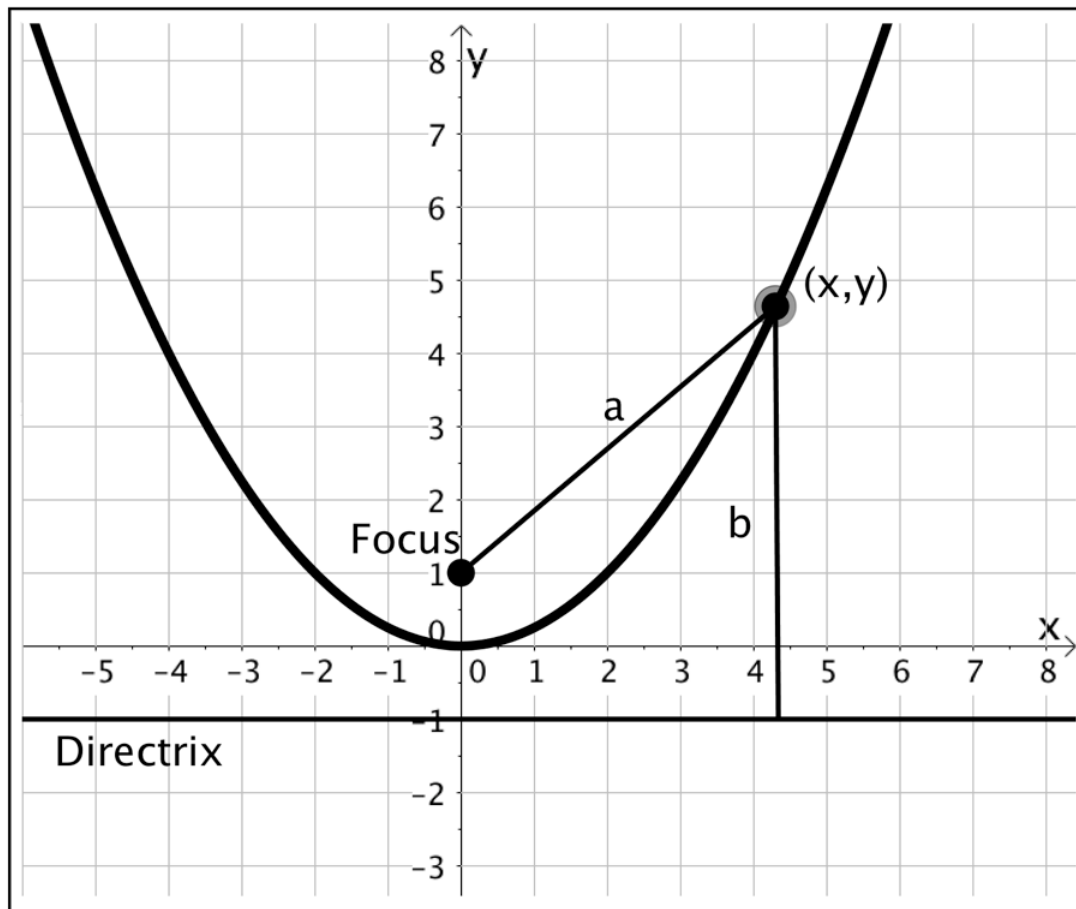


Figure 2.6. The point $(4,4)$ is replaced with a general point (x,y) .

It is important to note that Silverman and Thompson (2008) did not fully elaborate their meaning for a second-order reflective abstraction. They claimed that MKT can develop as a result of unpacking one's knowledge, considering the KDU from the perspective of a student, and engaging in some thought experiments. In other words, Silverman and Thompson seemed to suggest that the second-order reflective abstraction can come about simply through a willful act of thinking. Their framework was offered as a way of beginning to address the problem of understanding how MKT develops. Silverman and Thompson acknowledged the nascent nature of their framework, and they invited the field to contest or extend their framework.

Part of the motivation for this study was my belief that their framework could be extended in at least two ways. First, I conjectured that it would be possible that second-order reflective abstraction can form out of experiences observing or working with students who are learning the content, rather than just through an intentional thought experiment. My own experiences in developing MKT around quantitative reasoning with distances (which I just described) offer an example of how this may be possible.

Second, I conjectured that reflective abstraction does not account for all possible processes of learning that lead to the development of MKT. For example, certain discourse practices, types of mathematical tasks, and the nature of mathematical activity that PSTs engage in may also account for the development of MKT in PSTs. I go into more detail about this aspect of the study in a later section.

The Education of Mathematics Teachers

Sowder (2007) argued that preparing prospective teachers for all of the challenges they will face in the classroom in only four years is impossible. She argued that in light of this impossible task, teacher training should focus on providing opportunities for prospective teachers to learn how to be reflective learners of practice. A potential source for creating such opportunities is the use of records of practice (Borko, Koellner, Jacobs, & Seago, 2011), which include case studies, vignettes, narratives, student work, manipulatives, lesson study, and mathematical problems (Tirosh & Wood, 2008).

Borko (2004), in her review of research on teacher education, found that “a number of programs have successfully used artifacts such as instructional plans and assignments, videotapes of lessons, and samples of student work to bring teachers’ classrooms into the professional development setting” (p. 7). These kinds of records of practice provide several advantages for teacher education.

First, records of practice provide an authentic window for prospective teachers into the day-to-day activities of teaching (Borko et al., 2011). Making explicit the kinds of effective practices that teachers use can serve an important role in teacher training. Morris (2006) conducted a study with two groups of prospective teachers. Both groups were asked to analyze lessons for evidence of student thinking, link that thinking to instruction, and suggest revisions to the lesson. Prior to viewing the lesson, only one group was given guidance to attend to student thinking. The prospective teachers who were given explicit guidance were more likely to critically analyze the lesson, more likely to gather specific evidence linking student learning to instruction, and more likely to suggest revisions to the instruction (Morris, 2006). In another study, Switzer, Teuscher, and Siebert (2015) investigated prospective teachers participating in lesson study. The goal of the lesson study, which was made explicit to the prospective teachers, was to investigate productive discourse practices that elicited student thinking during class discussions. Results indicated that observing others enabled participants to reflect on, and eventually revise, their own practice. In short, research suggests that with proper support, prospective teachers benefit from teacher training that uses records of practice to bring authentic practice to life.

Second, the use of records of practice brings practice into professional development (PD) for in-service teachers, making practice an object of inquiry (Borko, Jacobs, Seago, & Mangram, 2014). One well documented example is the Cognitively Guided Instruction (CGI; Carpenter, Fennema, Franke, Levi, & Empson, 1999) professional development program. The CGI PD program trains teachers to focus on the development of student thinking in order to make effective instructional choices to foster student learning. The program provides teachers with a variety of resources (e.g., taxonomies of problems types, videos of children solving problems, and research on student thinking) designed to help teachers learn to analyze student thinking. By bringing practice (in this case analysis of student thinking) into PD, teachers were able to develop new models for instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Franke, Carpenter, Levi, & Fennema, 2001). Interestingly, the principles of the CGI PD program do not include explicit guidelines for instruction. Even so, participants in CGI PDs tend to change their instructional practices in productive, student-centered ways (Franke et al., 2001).

Finally, in using records of practice during PD, teachers engage in learning about practice in the absence of the pressures of the classroom. Teaching places many demands on a teacher, and every moment of teaching involves a flood of information that teachers must process. Examining records of practice outside of the classroom provides for authentic learning experiences in which teachers can give their full focus to the development of their practice. For example, Zhang, Lundeberg, Koehler, and Eberhardt (2011) researched how viewing video vignettes afforded opportunities for

in-service teachers to examine other teachers' practices. Some teachers reported that due to the demands of teaching, observing other teachers may not be possible without bringing records of practice into PD (Zhang et al., 2011). The researchers found that by viewing the practices of others, in-service teachers could compare their own practice to that of their colleagues, which generated new ideas for assessment, leading class discussions, and learning activities they could use in their own classrooms.

In this study, I used two types of records of practice: case studies, in the form of videos of high school students engaged in mathematical activity, and mathematical tasks. In the next two sections I review relevant literature that explores the use of these two types of records of practice, as well as elaborate the ways in which I used these artifacts.

Video Case Studies

Case studies, which include self-authored and third-person accounts of teaching practices, hold particular appeal as records of practice (Sowder, 2007). Case studies can be used to illustrate: (a) exemplary practices to show what is possible (Shulman, 1992); (b) examples of practice that are to be used "as springboards for analysis and discussion about mathematics teaching and learning, not evaluations of the videotaped teacher" (Borko et al., 2011, p. 184); and (c) unexpected or problematic classroom occurrences (Chazan & Herbst, 2012; Markovits & Even, 1999; Moore-Russo & Viglietti, 2010). Cases are adaptable tools that bring together theory and practice, providing both prospective and in-service teachers with "sites for analysis [that] are situated in practice" (Doerr & Thompson, 2004, p. 180).

Research has shown that using video cases for teacher education can yield positive effects. The CGI program utilizes videos of lessons as well as videos of interviews of children to productively influence teachers' beliefs about student thinking, as well as their knowledge of student thinking and how to make instructional decisions based on student thinking (Carpenter et al., 1989; Franke et al., 2001). Moore-Russo and Viglietti (2010) demonstrated that animated video cases can be used to help novice teachers develop more sophisticated notions of how instructional decisions can impact student learning and motivation. Wilson et al. (2011) investigated PSTs learning from video cases of students using technology to solve statistics problems. They found that the video cases afforded opportunities for PSTs to analyze, reflect on, and construct models of students' thinking.

Video case studies have become increasingly popular in mathematics education (Borko, Jacobs, Eiteljorg, & Pittman, 2008; Sherin, 2004; Sowder, 2007), and with good reason. The use of video in teacher education offers three affordances (Sherin, 2004). First, video provides viewers with a lasting record, which affords the ability to attend to different aspects or features of the clip during different viewings of the clip. Second, videos can be collected, edited, and annotated. In other words, videos are malleable in the sense that creators of video can highlight certain aspects of the recorded event while ignoring or suppressing other aspects. Third, videos provide new pathways for teachers to develop practice. Viewers of video can attend to problematic or interesting events in the video without the multitude of demands that comes with overseeing a classroom.

The widespread use of videos in teacher education is well-documented (Borko et al., 2008; Koc, Peker, & Osmanoglu, 2009; Santagata, 2009; Star & Strickland, 2008; Zhang et al., 2011). While there has been much research on the use of videos for teacher education, I have identified two ways in which my study contributes to the field's knowledge of use of video in teacher education. I elaborate each of these in arguments below.

Minimally-edited video. Much of the use of video in teacher education involves *minimally-edited* video clips (Borko et al., 2014, 2011). Minimally-edited clips typically feature one take, involve little or no panning or zooming, and include details (e.g., decorations in a classroom or off-task behavior) that may not be germane to the phenomena (e.g., techniques the teacher uses to elicit students' thinking) the presenter wishes to highlight (Erickson, 2007). Minimally-edited clips are typically used in teacher education because they are much easier to obtain and require far less time and fewer resources to prepare for educational use than do edited clips (Jaworski, 1990). Additionally, they present a record of practice that is truer to the recorded event than do edited clips.

One example of how minimally-edited video clips can be used in teacher education is video clubs (e.g., Sherin & van Es, 2005; van Es & Sherin, 2010). A video club is a type of PD in which a group of teachers meet regularly to watch and analyze videos of practice (Sztajn, Borko, & Smith, in press). In some video clubs, teachers watch videos of their own practice (e.g., Sherin & van Es, 2009), while in others the clips are pre-selected by the teacher educator (e.g., Walkoe, 2015). Research

on video clubs has shown that they support teachers' development of noticing student thinking (Sherin & van Es, 2005; Star & Strickland, 2008).

There are other examples of how researchers have used minimally-edited video clips. Santaga et al. (2007) used minimally-edited videos of lessons submitted for the TIMSS video studies to foster PSTs abilities to analyze lessons. Kersting, Givvin, Sotelo, and Stigler (2010) used minimally-edited videos of fifth and sixth grade classrooms to explore possible links between MKT and teachers' ability to analyze lessons. In another study, teachers watched minimally-edited videos of their own and their colleagues' classrooms (Kleinknecht & Schneider, 2013). The results suggested that teachers became more emotionally and intellectually engaged when watching videos of other teachers' classrooms.

The use of minimally-edited videos is not without limitations. Several researchers (e.g., Erickson, 2007; Moore-Russo & Viglietti, 2010) have noted that minimally-edited clips contain a flood of information, some of which is often irrelevant to the phenomena captured by the clip and may detract from the viewers' experience with the clip. While experts may view minimally-edited clips with sharp focus, honing in on the phenomena to be studied, novices may be lost "in a stream of continuous detail they don't know how to parse during the course of their real-time viewing in order to make sense of it" (Erickson, 2007, p. 146). Without careful guidance, viewers, especially novices, may miss the very phenomena the clip was meant to show. Indeed, research has shown that prospective, novice, and veteran

teachers struggle to attend to aspects of video that impact learning outcomes (e.g., Miller & Zhou, 2007; Star & Strickland, 2008; van Es & Sherin, 2010).

To mitigate these challenges, I used clips that have been extensively edited. These videos prominently feature only the students and their work. The background of the clips has been edited so that it is a neutral linen texture with no other visible images one might expect to see in a minimally edited clip (e.g., bookcases, a clock, other students, etc.; see Figure 2.7). While the clips do feature the voice of a teacher, she is not visible. I conjectured this feature of the videos would increase the likelihood that participants would attend to what she says, rather than how she looks, what she is wearing, or other information that could distract prospective teachers. The students' off-task behavior has been edited out of the clips, which removes another source of distractions for viewers. More details about these clips will be provided in Chapter 3.



Figure 2.7. (a) A screen shot of minimally-edited video of a classroom, and (b) a screen shot of one of the extensively-edited videos I used.

In short, the video clips I used focus on the development of mathematical concepts, and the clips have been edited in service of that. I conjectured this may prove to be fruitful for the development of MKT. In particular, by foregrounding

Sasha's and Keoni's development of mathematical concepts, I conjectured these clips could be particularly effective at supporting four components of the Silverman and Thompson (2008) framework for MKT: *development of a KDU; images of students' thinking and understanding; images of milestones for a learning trajectory; and images of instruction.*

Because the development of mathematical concepts is highlighted in the videos, the videos served as a second source for exploring the content (the first being mathematical tasks, which I elaborate in Chapter 3). Participants watched Sasha and Keoni work on tasks. They also saw some of the voiceovers that were created for the videos, which revoice ideas stated by Sasha and Keoni, and often use animations or highlighting to draw attention to important aspects of the work that Sasha and Keoni were doing.

The teacher, while not visible on camera, is a vital part of the videos. Because she is not on camera, participants could focus on the instructional moves she makes, which include questions to elicit the students' thinking, reminders to attend to certain details of the task, and responses to Sasha and Keoni's questions. Because the videos are edited to focus on the development of concepts (including quantitative reasoning), and they include the moves the teacher made to facilitate that development, I conjectured that these videos could support the construction of images of instruction.

Brief vignettes. Shulman (1992) made a strong argument to leverage the power of cases in teacher education. The field has responded, and now "the use of video cases in teacher education is quite common" (Masats & Dooly, 2011, p. 1152).

In conducting this literature review I found that video cases tended to fall into two categories. In the first category, teacher educators use multiple brief clips that show instances of the same phenomena. For example, Philipp (2008) reported on the use of multiple brief clips that showed students solving problems in a content course for prospective elementary teachers. These clips helped those teachers develop more sophisticated beliefs about mathematics and students' mathematical understanding (Philipp et al., 2007).

Other examples include the CGI program (Carpenter et al., 1989), which showcases students' thinking using brief video clips, and the Problem-Solving Cycle (PSC; Borko et al., 2008). The PSC features three phases. During the first phase participating teachers solve mathematical tasks. Participants then film their own implementation of those tasks in their classrooms. Finally, participants come together to collaboratively review short video clips from each other's lessons.

In each of these examples of brief vignettes, the term "brief" refers to the length of the clip, which is typically under ten minutes. Other researchers have used longer clips that show entire lessons (e.g., Koc et al., 2009; Morris, 2006; Santagata, 2009). Yet even clips showing entire lessons can be considered "brief" in terms of the development of students' conceptual understanding. In my review of the literature I found no examples in which in-service teachers or PSTs used multiple clips of video featuring the development of students' thinking over a longer time horizon. The videos I used do just that. The two students were filmed for over 14 hours spread across nine filming sessions over the course of several months. The videos show their

development of understanding of geometric and algebraic conceptions of parabolas. Consequently, I could show participants clips of the students at various points along their learning trajectory.

The longitudinal nature of these videos provided an opportunity to conduct research on the use of videos to help PSTs develop an understanding of particular students' realized learning trajectory. The 10 lessons show how Sasha and Keoni developed and learned to deploy quantitative reasoning. This feature afforded participants with opportunities to develop an image of one possible realized learning trajectory, which is one of the components of the Silverman and Thompson (2008) framework.

Use of Tasks in Teacher Education

Tasks are versatile tools for teacher education (Hiebert & Wearne, 1993; Sowder, 2007; Sztajn et al., in press; Watson & Mason, 2007). On the one hand, both prospective and in-service teachers can analyze student work on tasks, which acts as authentic practice of some of the work that teachers do. On the other hand, prospective and in-service teachers can engage with tasks as learners, which can promote their own content knowledge development. I briefly describe each of these ways of using tasks below.

Analyzing students' work on tasks brings authentic practice into teacher education. This type of research has been used to study the effects of such analysis on teachers' beliefs and affect (Philipp et al., 2007); their ability to develop tasks to support particular learning goals (Norton & Kastberg, 2012); and their ability to

respond to student thinking and make appropriate instructional decisions (Carpenter et al., 1989; Kazemi & Franke, 2004).

Philipp et al. (2007) found that analyzing students' work on tasks tended to change prospective teachers' beliefs about students and mathematics. For example, among prospective teachers that interviewed children as they solved tasks, the authors reported a 78% increase in those teachers who believed that children can solve tasks in novel ways prior to explicit instruction on how to solve those tasks. Norton and Kastberg (2012) reported on PSTs who wrote letters containing mathematical tasks. Students who received the letters solved the tasks and wrote about their solutions to the PSTs. The PSTs analyzed the solutions and posed new tasks in response. Through these cycles of posing problems and analyzing students' work on the problems, PSTs developed their abilities to pose more sophisticated and cognitively-demanding tasks.

Researchers have also studied prospective teachers working on tasks as learners (e.g., Guberman & Leikin, 2013; Morris, Hiebert, & Spitzer, 2009; Seago, Jacobs, Heck, Nelson, & Malzahn, 2014; Stylianides & Stylianides, 2009, 2010). Some of these research efforts have produced results that demonstrate prospective teachers often lack the kind of deep conceptual knowledge that supports MKT development (Guberman & Leikin, 2013; Morris, 2006). Guberman and Leikin (2013) reported that the use of open-ended tasks with multiple solutions seemed to support prospective teachers in developing more robust conceptions and led to spontaneous use of multiple representations on post-test items. Stylianides and Stylianides (2009) demonstrated how carefully sequenced tasks helped prospective elementary teachers

become more aware of the limitations of empirical arguments in proof and develop more sophisticated conceptions of proof.

One study has informed the design of my study (see Chapter 3 for details). Stylianides and Stylianides (2010) blended both approaches to task use in teacher education and created sequences of tasks designed to foster the learning of both content knowledge and pedagogical content knowledge. A crucial component of these sequences was the design and use of *Pedagogy-Related mathematics tasks* (P-R mathematics tasks). They outlined specific features of these tasks that afford the development of MKT, which I elaborate below.

P-R mathematics tasks. Doyle (1988) argued that “the work students do, which is defined in large measure by the tasks teachers assign, determines how they think about a curriculum domain and come to understand its meaning” (p. 167). Watson and Mason (2007) proposed that Doyle’s line of reasoning extends to both prospective and in-service teachers, namely that the tasks teachers engage with during teacher education influence what they learn. Accordingly, tasks form a crucial component of teacher education (Nipper & Sztajn, 2008). By conceptualizing MKT as applied mathematics (Bass, 2005; Usiskin, 2001b), Stylianides and Stylianides (2010) developed P-R mathematics tasks as one way to support the development of MKT.

There are three features of P-R mathematics tasks. First, P-R mathematics tasks feature a primary mathematical object that is the focus of the activity. Stylianides and Stylianides (2010) argued that the main purpose of a P-R mathematics task is to engage prospective teachers in predominantly mathematical activity. The

mathematical object may take many forms, such as the generation of a proof or generalization, defining or exploring a mathematical relationship, or constructing a mathematical object. Because P-R mathematics tasks focus primarily on a mathematical object they are inherently mathematical in nature. Consequently, P-R mathematics tasks are designed to foster the development of knowledge that is mathematical in nature.

The second feature is that P-R mathematics tasks focus on ideas that are fundamental (in the sense of Ma, 1999), serve as the building blocks for more advanced mathematics, and are crucial for students' advancement in mathematics. Examples of such ideas are generalization, functions, and fractions. These ideas are also hard to learn, as evidenced through research and practice that has shown that students struggle to understand these ideas.

The third (and defining) feature of P-R mathematics tasks is that they feature a secondary pedagogical object. By including a pedagogical component to the task, P-R mathematics tasks situate the mathematical activity in a particular pedagogical space. In doing so, P-R mathematics tasks provide PSTs with opportunities to engage in mathematical activity that approximates the mathematical activity they'll engage in as teachers. Including a pedagogical component also serves the purpose of fostering the development of MKT "from the perspective of an adult who is preparing to become a teacher of mathematics" (Stylianides & Stylianides, 2010, p. 163).

An example might help to more fully illustrate what is meant by a secondary pedagogical object. Consider a task in which PSTs are asked to respond to the

following prompt: “Prove or disprove that the area of a rectangle increases as its perimeter increases.” This is an example of a task with a primary mathematical object, namely determining the validity of the statement and justifying your conclusion. The task also features ideas, specifically proof and generalizing, that are important for teachers to know. However, this task is not a P-R mathematics task because it lacks a secondary pedagogical object.

Contrast that task with the following:

One of your 7th grade students approaches you after class and says, “So I was experimenting with some rectangles and I think I have a theorem that whenever you increase the perimeter of a rectangle, the area also increases!” How would you respond to your student?

This task features the same mathematical object and focuses on similar mathematical ideas, but it also situates the mathematical activity in the context of a teacher-student interaction. In this case, the pedagogical object is the requirement that the PSTs respond to a fictional student, which necessitates a different approach to the problem. PSTs must first decide for themselves the validity of the conjecture. But the pedagogical object of the task requires them to move beyond determining the validity of the conjecture and into a space in which they must imagine themselves as a teacher of mathematics. PSTs must decide what is a mathematically appropriate way of responding to a 7th grader: Should the response only include a confirmation or refutation of the conjecture, or should it also consider what next steps the fictional student should take? The difference is vast between simply telling a student whether their conjecture is valid and encouraging the student to write their own proof to share with others as fodder for discussion. The second task embeds the mathematical

activity in a plausible pedagogical setting, which grounds the mathematics in the domain of its application.

Stylianides and Stylianides (2010) reported on their use of P-R mathematics tasks in a college course designed for prospective elementary teachers. Participants were first given typical mathematics tasks so they could solve problems without focusing on pedagogical implications. These tasks were followed by P-R mathematics tasks, and this sequencing allowed prospective elementary teachers to reflect on their own mathematical activity from the typical mathematics tasks as they considered the various pedagogical factors embedded in the P-R mathematics tasks.

Stylianides and Stylianides (2010) argued that P-R mathematics tasks have the potential to aid the development of MKT. Like van Bommel (2012), they claimed that such tasks require a shift in perspective by prospective teachers from that of an adult engaging in mathematical activity to that of a future teacher of mathematics. Furthermore, P-R mathematics tasks serve as opportunities that promote the integration of mathematics and pedagogy, thereby situating MKT (as applied mathematics) in the domain of its application. Finally, these tasks may have contributed not only to the development of MKT, but also potential shifts in participants' beliefs about mathematics (Shilling-Traina & Stylianides, 2012).

Stylianides and Stylianides demonstrated how P-R mathematics tasks can be used in courses for prospective teachers (2010). These tasks may have potential for changing the way prospective teachers engage in content courses offered through teacher education programs. Consequently, the use of these tasks in content courses

for prospective teachers may serve as part of the solution to the problem identified by both Bass (2005) and Usiskin (2001b), namely that traditional teacher education programs do not provide prospective teachers with ample opportunities to engage in mathematical activity situated in the domain of their future profession.

Elaboration of Research Questions

I now restate my research questions and elaborate each question using ideas from the preceding literature review.

Research Question 1

My first research question follows:

What is the nature of the mathematical knowledge for teaching that develops for prospective secondary teachers during a video-based mini-course?

In answering this first research question, I explore the knowledge that seemed to develop for participants as they learned about parabolas and watched videos of students learning about parabolas. To be clear, the answer to this research question, which I present in Chapter 4, consists of accounts of what individual participants learned through their participation in this study.

For this research question I operated from the framework for the development of MKT presented by Silverman and Thompson (2008). In particular, I focused on four components from this framework: (a) the development of KDUs, (b) images of students' thinking and understanding, (c) images of milestones for a learning trajectory, and (d) images of instruction. I identified a KDU around quantitative reasoning that developed for the two students in the videos. However, because my

participants were prospective secondary teachers, they could draw upon a richer and wider mathematical background than is available to typical high school students. Nevertheless, the mathematical focus for the study was the KDU around quantitative reasoning that I identified in my review of the videos. My rationale for this is that Silverman and Thompson's (2008) framework for MKT accounts for how teachers with a particular KDU can transform that *personally* powerful knowledge into *pedagogically* powerful knowledge.

Research Question 2

My second research question follows:

How do particular elements of the designed learning ecology contribute to the development of MKT by prospective secondary teachers during a video-based mini-course?

Whereas Research Question 1 addresses the nature of the knowledge that participants developed through their involvement in the design experiment, Research Question 2 investigates the processes of learning that led to the development of that knowledge.

One possible process of learning that may lead to the development of MKT is that of reflective abstraction (Silverman & Thompson, 2008). Previously in this chapter I elaborated this construct and illustrated how my own MKT around KDU 2 may have developed as a result of a second-order reflective abstraction. However, the conditions under which this MKT developed were ideal. It is plausible to conclude that each of the following factors may have contributed in a meaningful way to the development of my own MKT:

- I have over five years of experience teaching mathematics, as well as a year of experience teaching prospective teachers.
- I have viewed each lesson at least five times, including the entirety of the filming session, and several viewings of the edited lessons.
- I helped the teacher develop several lessons for the videos, and during our time developing the lessons we had several conversations about how the students' understanding was evolving.
- It is my stated goal to investigate how MKT develops. As such, I explicitly looked for ways that knowledge was developing in the students in the videos in order to explicate the MKT I want to investigate.

It is reasonable to conclude that reflective abstraction is a viable process through which my own MKT developed. However, I expect that for prospective teachers, none of these factors play a role in the development of their MKT. Consequently, I questioned if that process would account for participants' development of MKT.

The framework offered by Silverman and Thompson (2008) posited reflective abstraction as the sole learning process responsible for the development of MKT. However, I argue this view fails to account for several factors that appeared to foster participants' development of MKT. In this study, I found that what prospective teachers noticed in the mini-course, how their attention shifted, and why their attention seemed to shift were critical factors in the development of their MKT.

There are links between noticing and reflective abstraction, in particular in the re-presenting of episodes, isolating particular attributes from those episodes, and evaluating these vis-à-vis the learner's goals (Lobato et al., 2013). However, noticing is also a social phenomenon (Goodwin, 1994), and aspects of noticing may not be captured or accounted for by reflective abstraction. Moreover, Silverman and Thompson (2008) appeared to be agnostic on the role social factors play in the development of MKT; consequently, their framework did not explicitly account for such factors.

Lobato et al. (2013) developed the focusing framework, which provides a tool for analyzing multiple aspects of students' noticing, including what students notice, discourse practices that focused students' attention, the tasks used, and the nature of mathematical activity. Briefly, there are four components of the focusing framework. *Centers of focus* (CoFs) are features or properties (physical or conceptual) that participants noticed. Their noticing was influenced by *focusing interactions*, which are discourse practices by either the teacher or other participants that had the effect of drawing attention to CoFs. *Mathematical tasks* provided opportunities for participants and me to engage in mathematical activity and discuss that activity, and it is during that activity that CoFs emerged. Finally, the *nature of that mathematical activity* may constrain or afford the emergence of CoFs.

The focusing framework provided a means to account for how certain aspects of the learning environment supported and fostered participants' learning. Utilizing the focusing framework "can raise awareness of whether or not students have selected the

features of mathematical situations that are most crucial as a foundation for particular mathematical ideas” (Lobato et al., 2013, p. 811). This quote provides insight into the explanatory power the focusing framework provided for this study, particularly as I conceptualized MKT as a form of applied mathematics, and my participants as students who were transitioning to teachers. More details about the focusing framework and its role in answering Research Question 2 is provided in Chapter 3.

A careful reader may propose another possible avenue for investigating how MKT develops. Noting that my participants were prospective teachers, and part of my study involved directing their attention to various aspects of the videos, it is reasonable to ask why I did not leverage research literature on teacher noticing. My rationale for not leveraging this research in answering Research Question 2 rests on three arguments.

First, much of the literature on teacher noticing features participants who watch instructional clips of students engaged in mathematical problem solving activity (Jacobs, Lamb, & Philipp, 2010). However, in these studies, teachers are watching clips of videos from a typical classroom in which the teacher, other students (who may be on or off task), decorations, writing on the board, and other stimuli from the classroom are captured by the video. A key element of this research is investigating what teachers notice from such a rich stream of visual and aural stimuli. The videos I used in this study stand in stark contrast to those used in other studies. The videos used in this study show only two students engaged in dialogue about math problems, their written work, and the audible voice of the teacher who is off camera. No other

features of the learning environment are present in the videos. These videos are designed to explicitly feature the development of the students' mathematical thinking. Consequently, the range of what participants could notice from these videos is far narrower than the range of what they would notice in video clips from typical classroom environments.

Second, Jacobs et al. (2010) found that when watching clips featuring students' mathematical thinking embedded in typical classrooms, teachers have difficulty attending to students' mathematical thinking. Other studies have shown that teachers tend to notice other aspects of the clips, including features of the instructor's personality, classroom management style, or instructional moves made by the teacher (Miller & Zhou, 2007; van Es & Sherin, 2010). While I hoped that participants in this study would notice specific moves made by the teacher in the videos, I aimed to bring about that noticing by explicitly drawing attention to those moves in an effort to promote the development of MKT. Unlike studies in which the question is *what* do teachers notice, for this study I hypothesized that noticing certain features would foster the development of MKT. Consequently, I engineered specific elements of the learning environment so that those features were more likely to be noticed. In short, this study was concerned not so much with *what* participants noticed as much as it is interested in *how* what participants did notice contributed to the development of their MKT.

Finally, studies on teacher noticing have shown that noticing and attending to students' mathematical thinking is challenging for teachers (Empson & Jacobs, 2008;

Jacobs et al., 2010; van Es & Sherin, 2010). In fact, Jacobs et al. found that experienced teachers who had not received explicit training on noticing student thinking tended to not notice student thinking. Moreover, it has been suggested that teachers' ability to notice students' mathematical thinking depends in part on their own MKT on the topic (Schoenfeld, 2011). Tyminski, Land, Drake, Zambach, and Simpson (2014) found that PSTs were able to successfully leverage their noticing of students' mathematical thinking, but only with sustained training in doing so. Taken together this research suggests that my participants (who were not in-service teachers, let alone experienced in-service teachers) were not yet ready or able to productively notice student thinking in the videos without assistance. In particular, participants in this study tended not to have rich MKT around quantitative reasoning (see Chapter 4 for evidence of this claim), which likely would have hindered their ability to notice students' mathematical thinking around these topics without adequate support for doing so.

In summary, the research literature suggests that prospective teachers may struggle to notice phenomena in videos that support MKT development. Moreover, this study does not investigate the range of *what* participants noticed during the mini-course. Instead, it investigates *how* what participants noticed seemed to support the development of their MKT. I now turn to elaborating the research methods used to conduct this study.

Chapter 3: Methods

The goal of this study is to build theory about how prospective math teachers (PSTs) develop mathematical knowledge for teaching (MKT; Silverman & Thompson, 2008) around quantitative reasoning with distances. In service of this goal, I answer the following research questions:

Research Question 1: What is the nature of the mathematical knowledge for teaching that develops for prospective secondary teachers during a video-based mini-course?

Research Question 2: How do particular elements of the designed learning ecology contribute to the development of MKT by prospective secondary teachers during a video-based mini-course?

To answer these research questions I conducted a design experiment (Cobb, Confrey, et al., 2003), which are used to engineer forms of learning that may not be found in traditional forms of education. In Chapter 2, I reviewed literature that argues that PSTs do not tend to develop sophisticated quantitative reasoning. Accordingly, the mathematical focus of the design experiment was MKT around quantitative reasoning.

Cobb, Confrey, et al. (2003) described a learning ecology as a “complex, interacting system involving multiple elements of different types and levels” (p. 9). Conducting a design experiment afforded me the ability to intentionally engineer particular elements of the learning ecology in an effort to bring about, study, and develop theory about desired forms of learning (Prediger, Gravemeijer, & Confrey, 2015), in this case the development of MKT around quantitative reasoning. Data collected during the experiment was analyzed to study PSTs’ learning as well as the activities, processes, environments, and relationships that supported their learning.

The result of this design experiment is local theory (Prediger et al., 2015) about the development of MKT around quantitative reasoning. In answering the first research question, I provide accounts for what individual PSTs seemed to learn during the experiment. In answering the second research question, I link participants' MKT to what was noticed during the mini-course, and then illuminate how elements of the learning environment (Cobb, Confrey, et al., 2003) contributed to participants' noticing.

In this chapter, I describe the methods I used to investigate the development of MKT around quantitative reasoning by PSTs. In broad terms, I recruited seven prospective secondary teachers and conducted pre- and post-interviews that bookended the mini-course. The data from the interviews formed the corpus of data for Research Question 1, while data from the mini-course formed the corpus of data for Research Question 2. The remainder of this chapter is organized into four sections that elaborate in detail the following: (a) participants; (b) instruction in the design experiment; (c) data collection and analysis methods for Research Question 1; and (d) data collection and analysis methods for Research Question 2.

Participants

The purpose of this study is to investigate the development of MKT in PSTs. To increase the likelihood that I could study the phenomena in question, there were two qualities I looked for in potential participants. I now briefly describe these qualities.

First, research suggests that a necessary condition for the development of MKT in PSTs is that they experience a shift in perspective from a learner of mathematics to a teacher of mathematics (van Bommel, 2012). In light of both this research and the notion of the situativity of MKT (Stylianides & Stylianides, 2010), I looked for participants who were committed to being a teacher.

Second, because I aimed to study prospective *secondary* teachers, I looked for participants who had completed most of their coursework in a mathematics degree. Certain courses, such as advanced calculus, real analysis, and algebra, tend to act as a filter for mathematics majors. These courses form the core of undergraduate degrees in mathematics, and students who initially declare an interest in pursuing a career in secondary mathematics education may change their mind as they progress through the major, in part because of the difficulty of these courses. By requiring that participants had completed most of their coursework towards a degree in mathematics, (as opposed to students who have declared an interest in secondary mathematics education but have not progressed through those core courses), I selected participants who were most likely to complete a degree leading to certification as a secondary math teacher.

Recruitment

To satisfy these requirements, I recruited seven participants who had successfully completed a capstone course for PSTs at either of two large southwestern universities. Each capstone course is a requirement for students pursuing a mathematics degree that prepares them to enter a teaching credential program at their respective university. The courses, which are designed specifically for prospective

secondary teachers, are typically taken at the end of the course study, and require advanced preparation in undergraduate mathematics.

I recruited seven participants to increase the likelihood that multiple points of view would emerge in the instructional sessions which could lead to rich idea generation. I could also divide seven participants into two smaller groups while still being able to track what individuals were saying during group discussions. Finally, seven is a small enough number that I would be able to track individual ideas, conceptions, and contributions to the entire class.

I recruited participants from lists of students who successfully completed the capstone courses in the fall semester/quarter of the 2015–2016 school year. All seven participants had finished an undergraduate degree in mathematics, and all planned to enroll in a credential program leading to certification as a secondary mathematics teacher. As an incentive for completing the study, participants were given a \$100 Visa gift card after the final interview.

Instruction in the Design Experiment

Broadly speaking, participants investigated geometric and algebraic conceptions of parabolas during the six instructional sessions. The six instructional sessions (see Appendix A) roughly followed lessons 1–8 from a unit of online video lessons hosted by Project MathTalk (www.mathtalk.org).

The use of a design experiment allowed me to engineer frequent and systematic opportunities for participants to (a) engage in mathematical activity that supported their development of MKT around quantitative reasoning and (b)

experience and reflect on high school students' mathematical thinking. To provide such opportunities, I designed three elements of the learning ecology. In the following sections I elaborate each of these elements: (a) the use of a series of videos showing high school students engaged in mathematical problem solving activities; (b) sequences of tasks designed specifically to foster participants' development of quantitative reasoning; and (c) the use of reflective activities designed specifically to foster participants' development of MKT. The typical order for each session was that participants first engaged in the tasks, then watched videos, and finally engaged in reflective activity. I present these elements out of order because the sequences of tasks draw heavily from the videos.

Videos

I used videos of two high school students engaged in mathematical problem solving. These videos are the products of the NSF-funded Project MathTalk (www.mathtalk.org; Lobato, 2014), which studies the development and use of conceptually oriented and dialogue-based videos of mathematics instruction. The purpose of the project was to create videos for online learners that present a different model of learning than the dominant talking-head or talking-hand model (Bowers, Passentino, & Connors, 2012). The videos show the development of the filmed students' quantitative reasoning as they explored relationships between algebraic and geometric conceptions of parabolas. I now describe these videos and elaborate several features of the videos that I conjectured would support the development of MKT for participants.

Two high school students, Sasha and Keoni, were filmed as they worked on various mathematics tasks that focused on relationships between geometric and algebraic conceptions of parabola. Sasha and Keoni were filmed over 14 hours, which resulted in 10 online lessons. The videos were not scripted, but they were edited in post-production to make clearer Sasha and Keoni's conceptual challenges and how their understanding developed over time. Other elements were added to the videos in post-production to aid those who view the videos online, including highlighting parts of Sasha and Keoni's work, animating ideas expressed by Sasha and Keoni, and revoicing Sasha and Keoni's stated ideas (see Figure 3.1 for a screenshot from a MathTalk video).

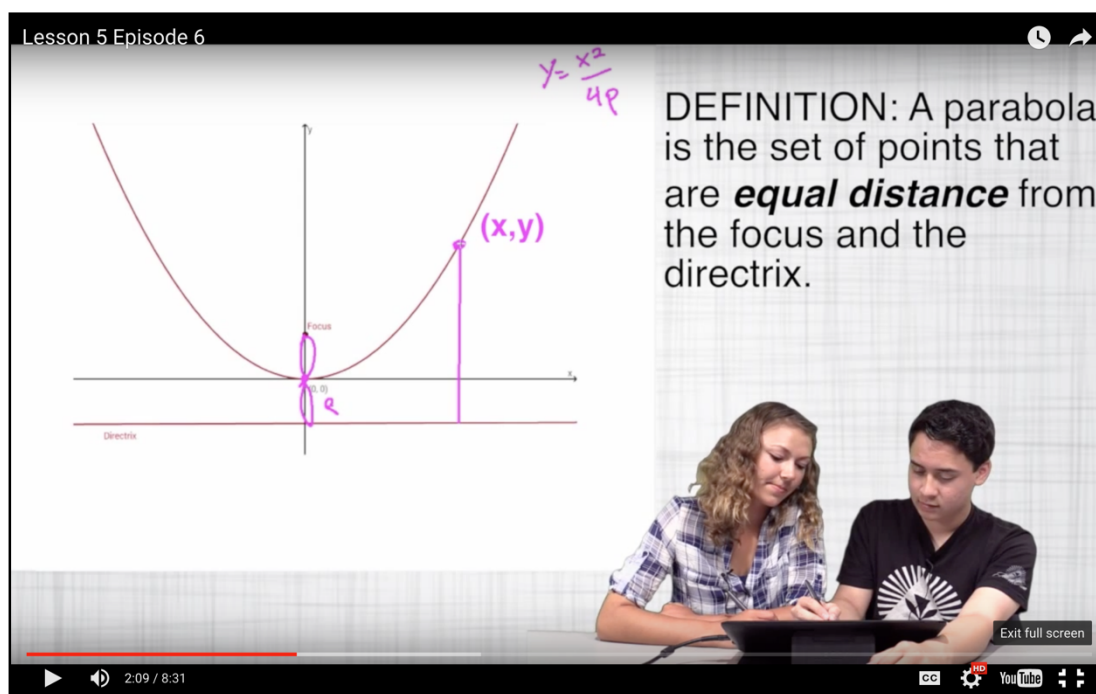


Figure 3.1. A screenshot from Project MathTalk (www.mathtalk.org).

The mathematical goals for this study differed somewhat from those of the MathTalk unit. Whereas the mathematical goals for this study were driven by the

development of MKT around quantitative reasoning, the researcher who crafted the video unit conceived of the development of student understanding in terms of conceptual learning goals (Lobato, Hohensee, Rhodehamel, & Diamond, 2012). An example of one such goal from the video unit is that students can “Conceive of a point on a Cartesian coordinate grid, not only as a location, but also as representing distances in 2-dimensional space” (see <http://www.sci.sdsu.edu/crmse/mathtalk/website/te-parab-12e2.html>). Conceptual learning goals share similarities with KDUs, but conceptual learning goals are finer grained targets of instruction. For example, the conceptual learning goal given above certainly could contribute to the KDU around quantitative reasoning with distances that was elaborated in Chapter 2.

Not all the Project MathTalk videos were used in the design experiment. Because the purpose of the study was to investigate participants’ development of MKT, I selected clips from the online video lessons that I believed would stimulate fruitful conversations and provoke thoughtful reflections. To facilitate these selections, I created a document that provides an overview for each episode of every lesson from the MathTalk parabola unit (see Appendix B for an example). I created this document as I reviewed the videos in preparation for this study, and I updated it after each instructional session based upon what had happened in that session. The document contains a table with timecodes of clips from the videos, as well as a brief description of the clip. For example, in the first lesson Sasha and Keoni created a parabola using only the geometric definition. I identified moments when Sasha and Keoni struggled

to measure the distance between a point and a line, as well as the questions that the teacher asked to help Sasha and Keoni resolve that struggle.

There were several factors that influenced which clips I showed during the instructional sessions. First, I selected clips based on the ideas that emerged during the instructional sessions, particularly during times when groups of participants were solving math tasks. By observing what participants said and did during a math task, I could make informed decisions about which clips to show participants. For example, during the first instructional session participants created a parabola from the geometric definition. I observed some participants suggesting that high school students might struggle to measure the distance from a point to a line. Accordingly, I chose to show participants a clip that showed Sasha and Keoni struggling to measure the distance between a point and a line.

I also selected clips based on my belief about how likely they were to support the development of MKT for participants. For example, I highlighted certain regularities in the videos that I conjectured would help support aspects of MKT, including the types of questions the instructor in the videos posed to Sasha and Keoni. To do so, I selected clips that featured questions from the teacher I wanted participants to notice.

In the following subsections, I describe my conjectures from before the study about how the videos might support the development of four components of MKT (as articulated in Chapter 2). I also further elaborate the factors that influenced my selection of particular clips.

Development of KDUs and images of students' thinking and

understanding. According to Silverman and Thompson (2008), the foundation of MKT is a KDU that supports the content to be taught. As discussed in Chapter 2, KDUs are characterized, in part, by the fact that the development of KDUs requires sustained mathematical activity. Much like typical instruction, the design of this experiment offered participants opportunities to engage in the kind of sustained mathematical activity that fosters the development of KDUs. These opportunities (discussed in more detail later in this chapter) came in the form of tasks designed to promote quantitative reasoning with distances.

Where this design experiment differed from typical instruction is that in addition to working on mathematics tasks, participants viewed video clips of high school students working through similar tasks. These videos provided opportunities for participants to see a near continuous image of how Sasha and Keoni came to understand powerful ideas about parabolas through the development of their own quantitative reasoning.

The first two components I targeted is the development of a KDU around quantitative reasoning with distances (which I will now simply refer to as quantitative reasoning with distances) and images of students reasoning quantitatively with distances. I selected clips that contained mathematical ideas that I conjectured were important to the development of the KDU around quantitative reasoning. These clips also provided participants with images of how Sasha and Keoni reasoned

quantitatively with distances. For example, I selected clips that showed Sasha and Keoni making sense of coordinates as distances in the coordinate plane.

Images of milestones for a learning trajectory. The third component of MKT that I targeted is images of how students may develop quantitative reasoning with distances (Silverman & Thompson, 2008). The MathTalk videos provide a coherent and cohesive authentic image of Sasha and Keoni's development of quantitative reasoning with distances. The videos are edited in ways that highlight the conceptual challenges that Sasha and Keoni faced as they learned about parabolas, as well as the ways in which they overcame those challenges.

Consequently, these videos not only provided participants with a second source of opportunities for developing their own quantitative reasoning with distances (the first being their own mathematical activity as they solved math tasks), they also embed the development of quantitative reasoning in a continuum of learning by two high school students. I selected clips that highlight key moments in Sasha and Keoni's development of quantitative reasoning, including challenges Sasha and Keoni faced, how they overcame those challenges, and advances in their understanding. This provided opportunities for participants to reflect on their own emerging understanding of the mathematics vis-à-vis their observations of what Sasha and Keoni were coming to understand.

Images of instruction. The fourth component of MKT that I targeted is images of the types of activities and conversations that support the development quantitative reasoning with distances. The mathematics tasks I used in the

instructional sessions were based on the mathematics tasks used in the videos (see below for more details on these tasks). Accordingly, I conjectured the videos would support participants coming to understand the mathematics tasks as well as how the tasks can be used to foster the development of quantitative reasoning with distances.

Moreover, the videos include the voice of the instructor, who successfully uses powerful quantitative questioning to encourage Sasha and Keoni as they develop their own mathematical understandings. I selected clips based on my belief that the clips demonstrate a powerful question, comment, or interaction between the teacher and the students. In other words, I selected clips that I thought provided participants with images of the kinds of instructional moves that can support the development of quantitative reasoning with distances.

Sequences of Tasks

Drawing from the work of Stylianides and Stylianides (2010), I used task sequences to promote the development of MKT. In each instructional session, participants worked in small groups (three or four participants in each group) on a task taken from the videos (I call these *math tasks*, whereas Stylianides and Stylianides would call them *typical mathematics tasks*). After completing the math tasks, participants were given time to reflect on their own developing understandings (see the section on reflection below). Following that, participants watched selected clips of video in which Sasha and Keoni complete a similar task. Finally, I led whole-class discussions to allow participants to talk about what they had noticed in the videos. I refer to the combined activity of watching and discussing the videos as *video tasks*.

Math tasks. The math tasks used during the instructional sessions were drawn from the MathTalk videos. Participants solved the math task before watching Sasha and Keoni do so in the videos; however, I did not use every task in the videos. For example, one lesson shows Sasha and Keoni finding coordinate values for specific points on a parabola in the coordinate grid. Sasha and Keoni needed multiple examples of finding the coordinate values before they were ready to move on to the next task which was to find a general equation for any point on the given parabola. Because participants had advanced mathematical preparation, I skipped the preliminary “find the coordinates” tasks, and gave them the task to find the general equation for a specific parabola.

As noted in Chapter 2, math tasks feature a primary mathematical object that research literature suggests is difficult for students to learn (Stylianides & Stylianides, 2010). An example of the type of task I used in the mini-course is shown in Figure 3.2 below. The solver is asked to use the geometric definition of a parabola to write an equation for the given parabola. This task features a primary mathematical object, namely writing an equation for a given graph. Moreover, this primary mathematical object is linked to research that has shown that students have difficulty recognizing relationships between graphs and equations (Knuth, 2000a, 2000b).

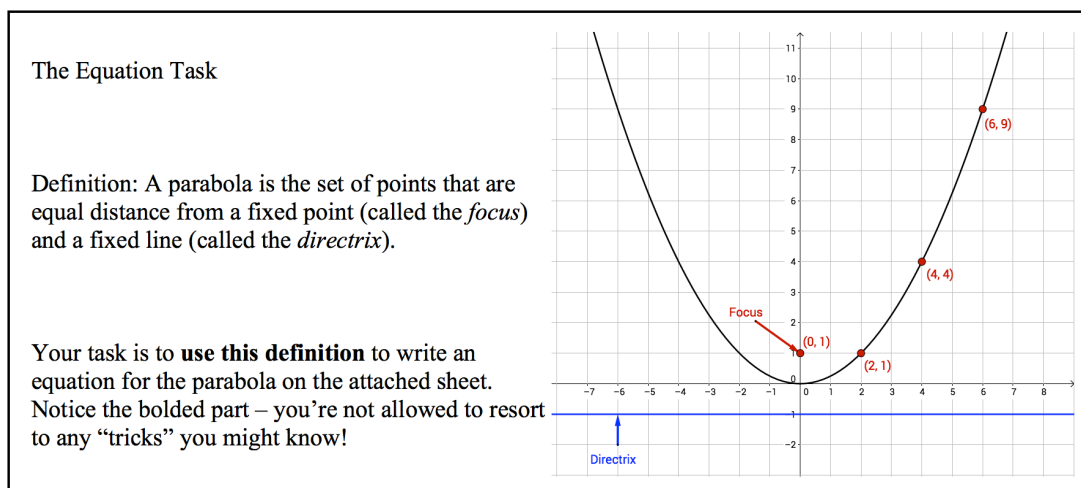


Figure 3.2: An example of what Stylianides and Stylianides (2010) would call a typical mathematics task.

Video tasks. Stylianides and Stylianides (2010) identified three features of PR-mathematics tasks: (a) a primary mathematical object; (b) a focus on important elements of MKT; and (c) a secondary, yet substantial, pedagogical object. Stylianides and Stylianides (2010) used PR-mathematics tasks with prospective elementary teachers. Of particular note is the fact that the prospective elementary teachers in that study “tended to have weak mathematical backgrounds” (Stylianides & Stylianides, 2010, p. 166). Consequently, the authors placed emphasis on the mathematical features of the PR-mathematics tasks.

I extended the notion of PR-mathematics tasks for this study, and called my version *video tasks*. First, I placed more emphasis on the pedagogical features of PR-mathematics tasks. My rationale for this modification is that participants presumably did not have weak mathematical backgrounds given they all had completed undergraduate degrees in mathematics. Consequently, the math tasks I used provided sufficient opportunities for participants to develop mathematical knowledge.

Accordingly, I used video tasks to focus on pedagogical aspects of MKT (rather than the mathematical aspects as in Stylianides and Stylianides [2010]).

Second, video tasks in the study included classroom discussions around selected clips from the videos. These discussions provided opportunities for participants to reflect on what they had seen in the videos and to begin thinking explicitly about teaching the content. During these discussions, I asked participants to talk about aspects from their work on the math tasks as well as the video clips. I conjectured that these discussions would support the development of MKT because they would provide opportunities for participants to focus on pedagogical aspects of the task, both from the perspective of a student learning the content as well as from the perspective of a teacher of that content.

Opportunities for Reflection

Before each math task, participants were given a Task Reflection Document (see Appendix C). The Task Reflection Document guided participants to think about potential pedagogical issues with the task. For example, one section of the document asked participants to list possible challenges high school students may face in completing the task and what the participant might do to help students overcome those challenges. By posing pedagogically-oriented questions, I hoped to prime participants to approach both the math task and subsequent video task from “the perspective of an adult who is preparing to become a teacher of mathematics” (Stylianides & Stylianides, 2010, p. 163), rather than the perspective of a student.

After completing the math task, participants revised their Task Reflection Document to record any changes in their thinking about the task. Finally, participants revised their Task Reflection Document after viewing and discussing the videos.

In summary, there were two different opportunities for reflection. The first was the Task Reflection Document, which captured participants' ideas and thoughts about the task and how to teach it. The second was the discussion participants engaged in after completing tasks and watching the videos. These two opportunities co-informed one another in that ideas participants wrote down on the Task Reflection Document were shared and discussed during the class discussions; likewise, ideas that emerged during class discussions influenced participants' thinking about the task, which elicited revisions to their Task Reflection Documents.

Data Collection and Analysis Methods for Research Question 1

This section provides details about the methods used for the collection and analysis of data for Research Question 1, which is restated below:

Research Question 1: What is the nature of the mathematical knowledge for teaching that develops for prospective secondary teachers during a video-based mini-course?

To answer this question, I conducted pre- and post-interviews with each of the seven participants. I now describe protocols and instruments I used in the interviews. I finish this section by elaborating the methods I used to analyze data from the interviews.

Interviews and Instruments

To better understand the nature of the MKT that develops for participants, I conducted two semi-structured clinical interviews (Ginsburg, 1997) with each

participant, one prior to the first instructional session and one after the last instructional session. Clinical interviews are an appropriate tool for this study as they promote researchers' understanding of the subjects' ways of reasoning and knowing (e.g., Bishop et al., 2014).

Semi-structured interviews are driven by the use of open-ended tasks that provide multiple points of entry for participants (Zazkis & Hazzan, 1999). This affordance allows for the fact that knowledge is an individual construction (Piaget, 1947; von Glasersfeld, 1995), and it provided me with the opportunity to study and analyze each participant's unique conceptions.

Additionally, semi-structured clinical interviews provide the researcher with flexibility to pose tailored follow-up questions to conduct hypothesis-testing (Ginsburg, 1997). While all participants saw the same set of tasks in each interview, I generated follow-up probes for individual participants based on my own emerging model of each participant's conceptions. In other words, as each interview progressed, I built hypothetical models of participants' ways of understanding, and then posed tailored questions to generate evidence that could confirm or disconfirm those models.

Data collection. Both interviews were videotaped, and participant work was captured using a Cintiq 13HD graphic pen tablet and Camtasia screen recording software. Using the Cintiq required little training for participants as it is essentially a large pad on which you write with a digital pen. The tablet consists of a screen showing exactly what is being written. The high school students in the MathTalk videos used the Cintiq capture method and had very little trouble adjusting to writing

on the tablet. Capturing participants work in this way provided several advantages over collecting pencil-and-paper work and trying to videotape the work as it develops. First, I fixed the video camera on the participant without worrying about getting a clear view of their paper. Second, I synced the recording of their work on the Cintiq with the video of the participant, providing a rich stream of data for analysis that allowed me to coordinate participants' written work with their gestures. Third, as participants responded to follow-up probes, they invariably annotated their previous work. With typical pencil-and-paper it is sometimes difficult to tell when those inscriptions appeared, but in capturing with a Cintiq I could find the exact moment those inscriptions were created to analyze those in conjunction with what the participant was saying and doing as the inscription was made.

Immediately after each interview I created a contact summary form (Miles & Huberman, 1994). These forms (see Appendix D for the form I used) served two purposes. First, a key component of the clinical interview is hypothesis testing, which requires the interviewer to be actively engaged in processing the data he or she observes as the subject completes tasks and discusses them. Consequently, taking notes during the interview is usually not a good idea, since doing so may interfere with the interviewer's ability to focus on what the subject is saying and doing. The contact summary form serves to mediate this limitation by providing a place for recording thoughts about the interview, immediately after the interview.

Second, a contact summary form is a useful tool for data analysis. I used contact summary forms to capture my immediate thoughts about participants' MKT

around quantitative reasoning that emerged during the interviews. Reviewing all contact summary forms allows the researcher to identify themes that may have emerged across several interviews, which further focuses the researcher's attention during data analysis. In this way, the forms aided in both the retrieval of data and as a tool to manage data overload (Miles & Huberman, 1994).

Pre-interview instrument. The pre-interviews lasted approximately 45–90 minutes, and provided opportunities for me to infer knowledge about parabolas participants may have already had. All participants were asked to complete three tasks, which I now elaborate (see Appendix E for the pre-interview protocol).

The Parabola? Task. The first task features a catenary curve and asks participants to state if the curve is a parabola. This task was designed to explore participants' conceptions of parabolas. Do they identify any *U*-shaped curve as a parabola? What might they use to justify their answers? Follow-up probes for this task were designed to elicit participants' existing knowledge about the geometric definition of a parabola, as well as algebraic characteristics of parabolas.

The Equation Task. For the second task, which was designed to probe participants' MKT for parabolas, I gave participants the vertex form of the equation for a parabola. I asked participants to tell me everything they know about vertex form, which provided me with initial ideas about their conceptions for this equation. Follow-up questions were designed to probe participants' images of how students may come to understand vertex form (including what prior knowledge may be necessary and

potential challenges or obstacles students may encounter) and images of instruction that may foster students' understanding of vertex form.

The final follow-up for the Equation Task was to ask participants about the various parameters in the equation. I put this probe at the end of the task because I wanted to see if participants explicitly attended to the parameters as they discussed their own conceptions of vertex form, and more importantly as they discussed what they thought is crucial for students to develop an understanding of vertex form.

The Ellipse Task. The third task featured an ellipse in a coordinate-free plane with only an x -axis and a y -axis. I designed this task because it can be solved using quantitative reasoning that is like the reasoning I described in Chapter 2 in the elaboration of the KDU around quantitative reasoning. By using an ellipse, I avoided overexposure to tasks I planned to use in the instructional sessions, while still targeting participants' existing quantitative reasoning with distances in the coordinate plane. I provide a conceptual analysis for this task later in this section.

Post-interview. The tasks used in this interview were designed to generate data that allowed me to infer shifts in participants' MKT around quantitative reasoning (see Appendix F for the post-interview protocol). I used all three tasks from the first interview, with the following modification.

The Parabola Task. For the third task, instead of using an ellipse, I gave participants a parabola in coordinate-free plane with only an x -axis and a y -axis. I asked participants to develop the equation of the parabola using the geometric definition of a parabola. The task was chosen based on my belief that it could help

reveal the extent to which participants had developed quantitative reasoning around distances. I provide a conceptual analysis of this task below.

I also asked follow-up questions that were designed to probe participants' ideas of how students may approach this task, what understanding students would need for this task, potential challenges they may face, and instructional moves the participant could take to help students successfully complete the task to develop conceptual understanding of the vertex form of the equation for a parabola. In short, the follow-up probes were designed to target the remaining two components of MKT, namely having an images of milestones for a learning trajectory for a mathematical idea, and having images of instruction that help students come to understand the idea.

Conceptual Analyses

I now present conceptual analyses for the Parabola Task and the Ellipse Task. These two tasks generated rich data in the interviews, which allowed me to reduce the interview data to focus on participants' work on these tasks.

Conceptual analysis of the Parabola Task. The purpose of this conceptual analysis is twofold. First, I want to ensure readers understand the complexity of this problem and the challenges it presented to participants. Second, I introduce and elaborate some constructs defined by P. W. Thompson (1990). To accomplish these two purposes, I will define each construct and then explain its role in solving the Parabola Task, which is reproduced below.

Before embarking on this conceptual analysis, a disclaimer is necessary. The analysis presented below is only one of many appropriate and mathematically valid

ways of solving the task. It represents the way that Sasha and Keoni solved the task in the MathTalk videos. Moreover, it represents how I envision one might solve the task if they were to watch Lessons 1–8 of the MathTalk Parabola unit and use the tools that are developed during those lessons. Nevertheless, it is only one possible way to approach the parabola task.

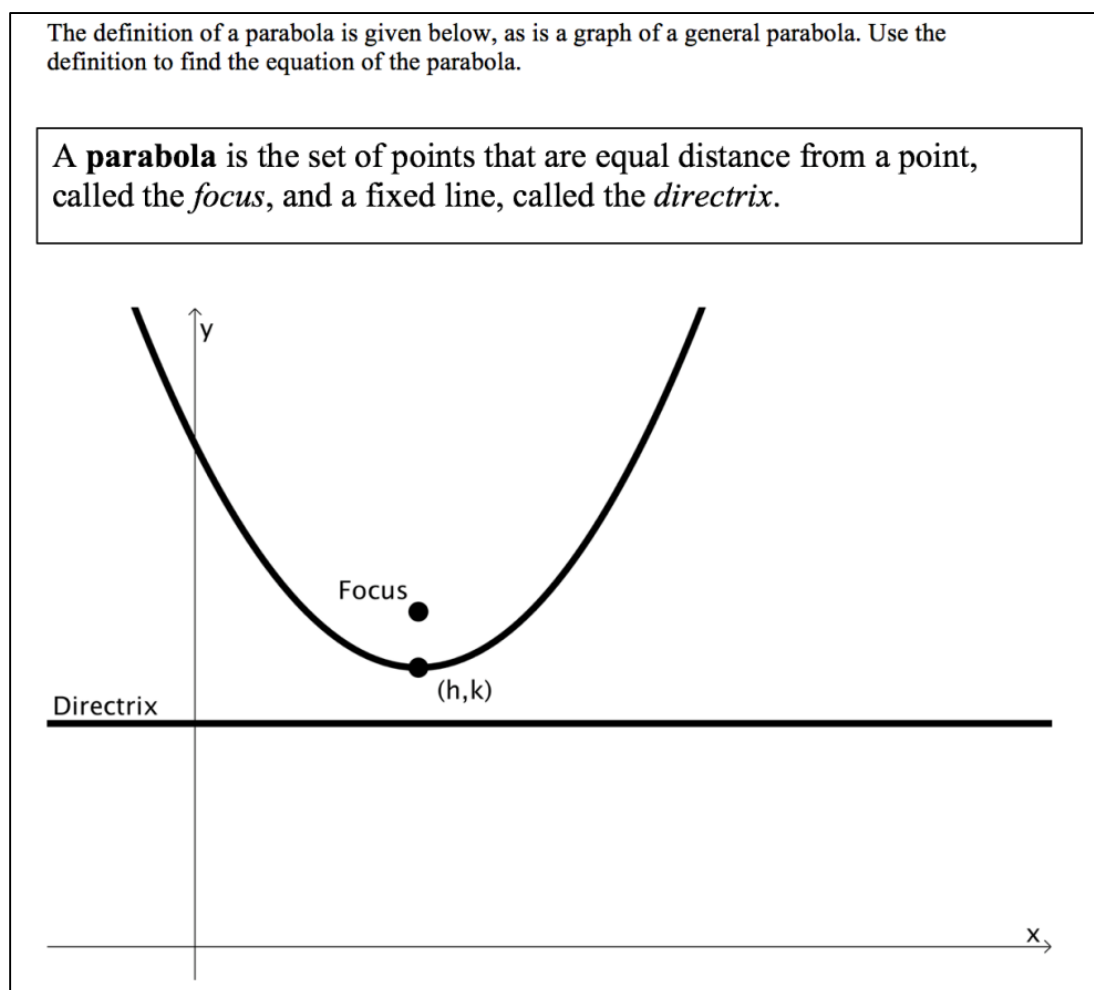


Figure 3.3. The Parabola Task from the post-interview.

Quantity and quantification. According to P. W. Thompson, a *quantity* is one's conception of a quality of something that can be measured (1994). To isolate a measurable quality and assign a measure to it is to *quantify* that quality. In the

Parabola Task, there are five qualities (see Figure 3.4 below) that when quantified are useful for solving the task:

- x , the distance from a general point on the parabola to the y -axis
- h , the distance from the vertex to the y -axis
- y , the distance from a general point on the parabola to the x -axis
- k , the distance from the vertex to the x -axis
- p , the distance from the vertex to the focus (or the vertex to the directrix)

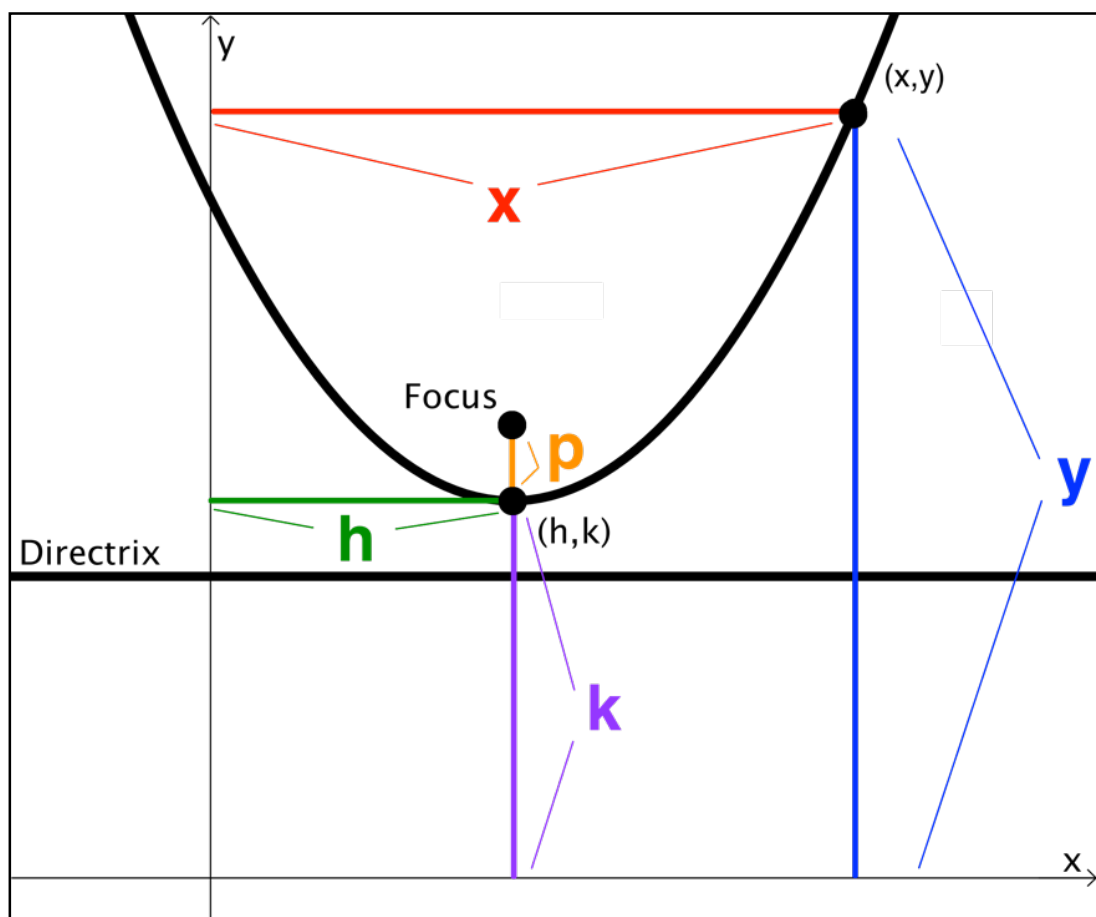


Figure 3.4. Five qualities that can be quantified in the Parabola Task.

Quantitative operation. P. W. Thompson (1990) defines a *quantitative operation* as “a conception of two quantities being taken to produce a new quantity” (p. 11). P. W. Thompson lists several quantitative operations, but for the purposes of this conceptual analysis two such operations will be elaborated. One operation is *combine additively*. This operation is analogous to the union of two sets. For example, one can additively combine the two quantities k and p to produce the new quantity $k + p$, which is the distance from the focus to the x -axis. Another operation is *compare additively*. This operation can be summarized as thinking about how much more or less one quantity is than another. For example, one can additively compare the two quantities x and h to get the quantity $x - h$, which is the quantity that describes how much longer the quantity x is compared to the quantity h . Figure 3.5 below illustrates these two examples.

Quantitative relationship. When a third quantity is realized as the result of a quantitative operation on two other quantities, a *quantitative relationship* has been formed. For example, the quantities k , p , and $k + p$ form a quantitative relationship, as do the quantities x , h , and $x - h$. *Quantitative reasoning* involves the building of multiple quantitative relationships in a given situation and leveraging those relationships to make sense of and analyze the situation.

Second-order quantitative relationship. A quantitative relationship results in the formation of new quantity, one that can itself be operated on. Consequently, when a quantity is formed as the result of a quantitative operation on two quantities, at least one of which is itself the result of a different quantitative operation, a *second-order*

quantitative relationship has been formed. As an example, consider the distance from the focus to the line segment that is labeled x in Figure 3.4 above. One could conceive of this distance as an additive comparison between the distance from that line segment to the x -axis, which is y , and the distance from the focus to the x -axis, which is $k + p$ (which is itself the result of a quantitative operation on the quantities k and p ; see Figure 3.8 below, as well as accompanying discussion). Thus, the distance from the line labeled x to the focus is $y - (k + p)$.

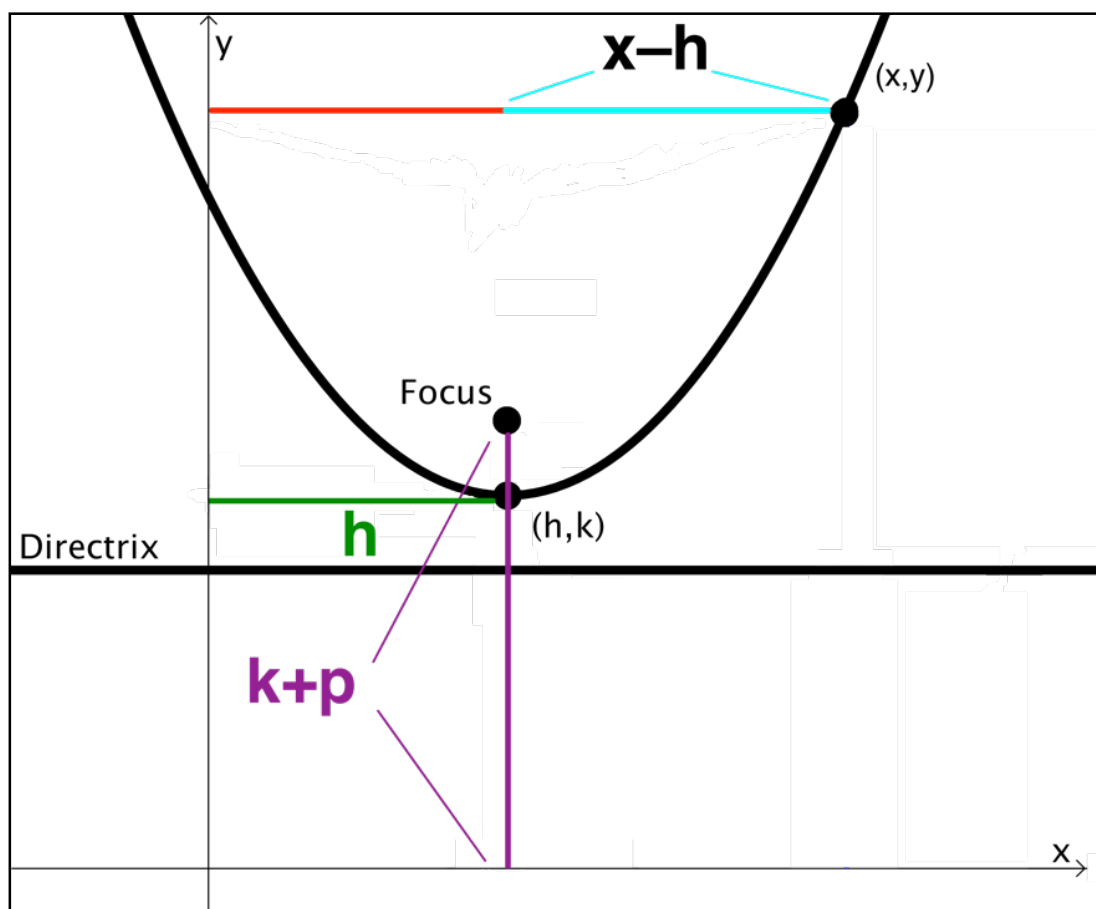


Figure 3.5. The quantities $k + p$ and $x - h$ formed via quantitative operations.

One of my goals for using the Parabola Task was to investigate participants' development of quantitative reasoning with distances, which involves quantifying

points in the plane as two separate quantities: the length in grid units of a horizontal segment from the point to the y -axis and the length in grid units of a vertical segment from the point to the x -axis. Once quantified, these quantities can be operated on to form quantitative relationships, which can then be used to find the equation of the parabola. Specifically for the Parabola Task, an individual first conceives of a point on the parabola in general terms [e.g., (x, y)]. By conceiving of a general point on the parabola, the individual can then leverage the definition of a parabola to conceive of two equivalent distances, one from (x, y) to the directrix (labeled l in Figure 3.6 below) and the other from (x, y) to the focus (labeled c in Figure 3.6 below).

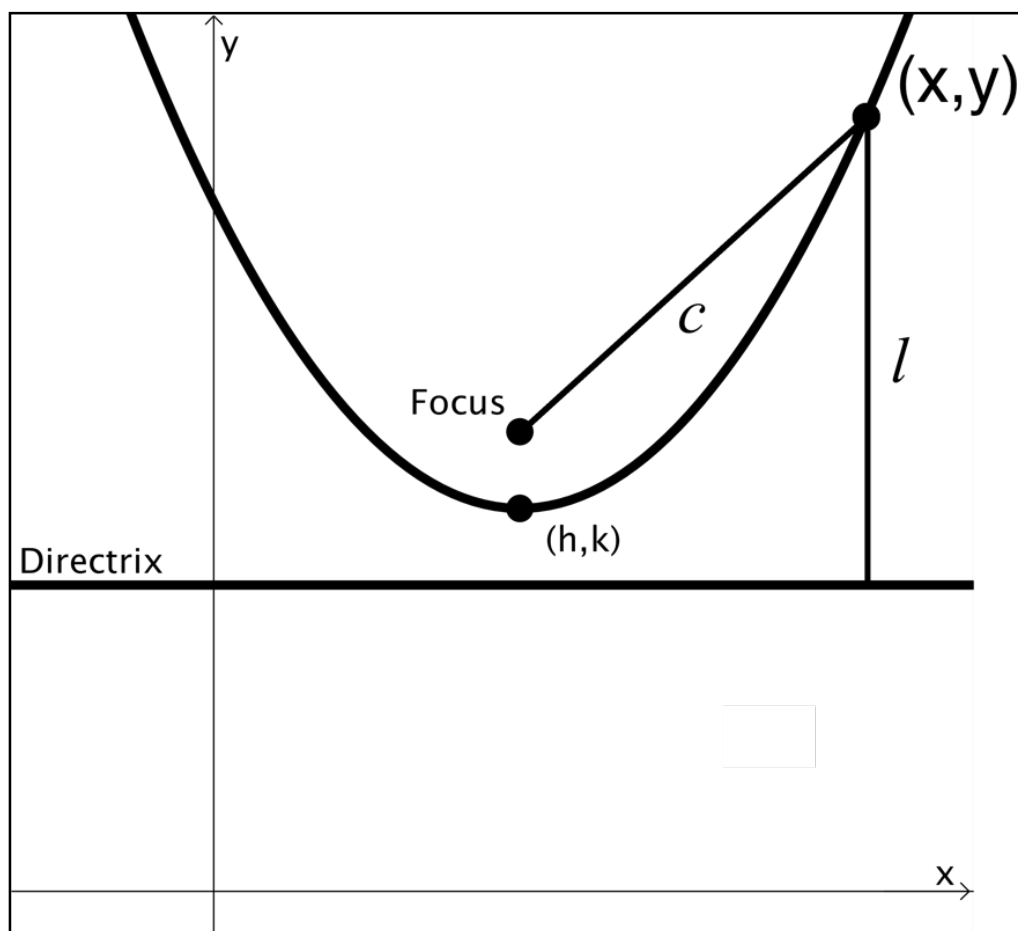


Figure 3.6. The distances c and l are equivalent by the definition of a parabola.

Once these distances have been conceived, the individual must conceive of a way to relate these distances to the five quantities discussed previously. One potential way to solve the task is to use the Pythagorean Theorem by identifying a right triangle (as shown below in Figure 3.7).

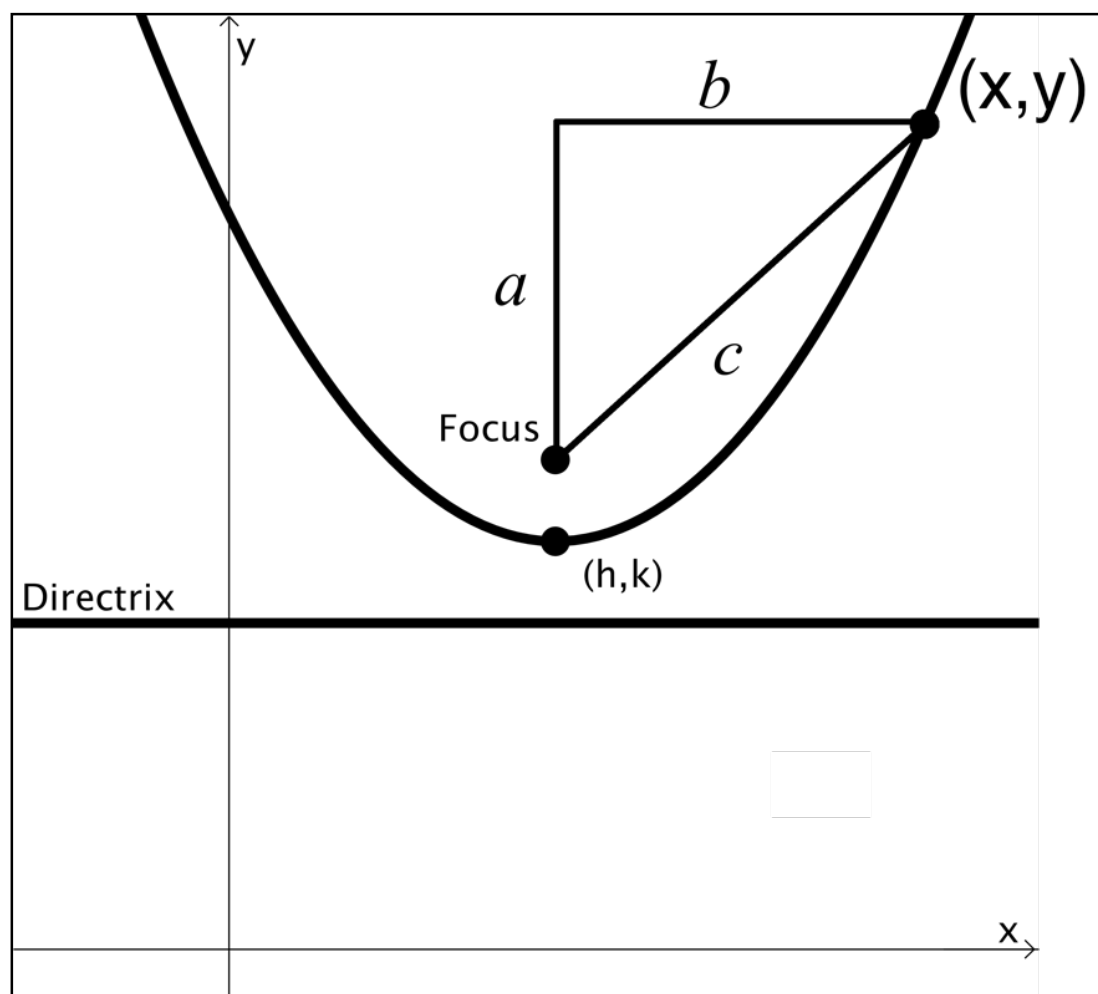


Figure 3.7. A right triangle can be used to find the equation of the parabola.

The distance b is found by forming a quantitative relationship with the quantities x and h as described above. The distance b is the amount by which x is greater than h , and thus $b = x - h$ (see Figure 3.10 below). The distance a is found by forming a second-order quantitative relationship. First, an additive combination of

k and p as described above yields the quantity $k + p$. The distance a is then formed by creating a second-order quantitative relationship with the quantities y and $k + p$ and the quantitative operation additive comparison (or takeaway). The distance a is the amount by which the quantity y exceeds the quantity $k + p$, and is given by $y - (k + p)$, as illustrated below in Figure 3.10.

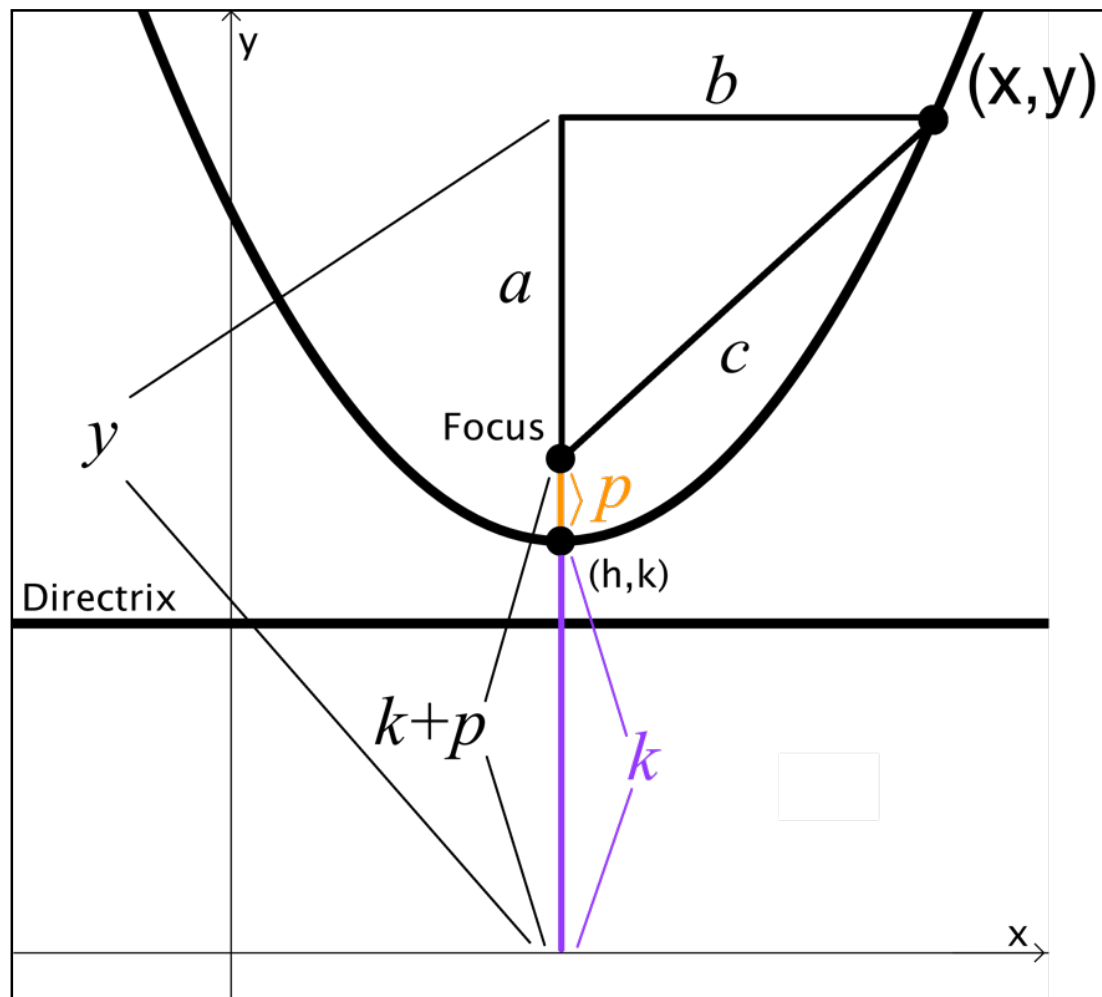


Figure 3.8. The quantity $a = y - (k + p)$ is a quantitative relationship between the quantities y and $k + p$.

Finding the quantity c can be accomplished by finding the quantity l in Figure 3.6 above, which, in turn, can be found by establishing a second-order quantitative

relationship. First the quantity $k - p$ is formed, which is the amount by which the quantity k exceeds the quantity p . This is also the distance from the directrix to the x -axis. The second-order relationship is the difference between y and $k - p$, which is $y - (k - p)$; this is the distance from the general point on the parabola to the directrix. These quantities are illustrated below in Figure 3.9.

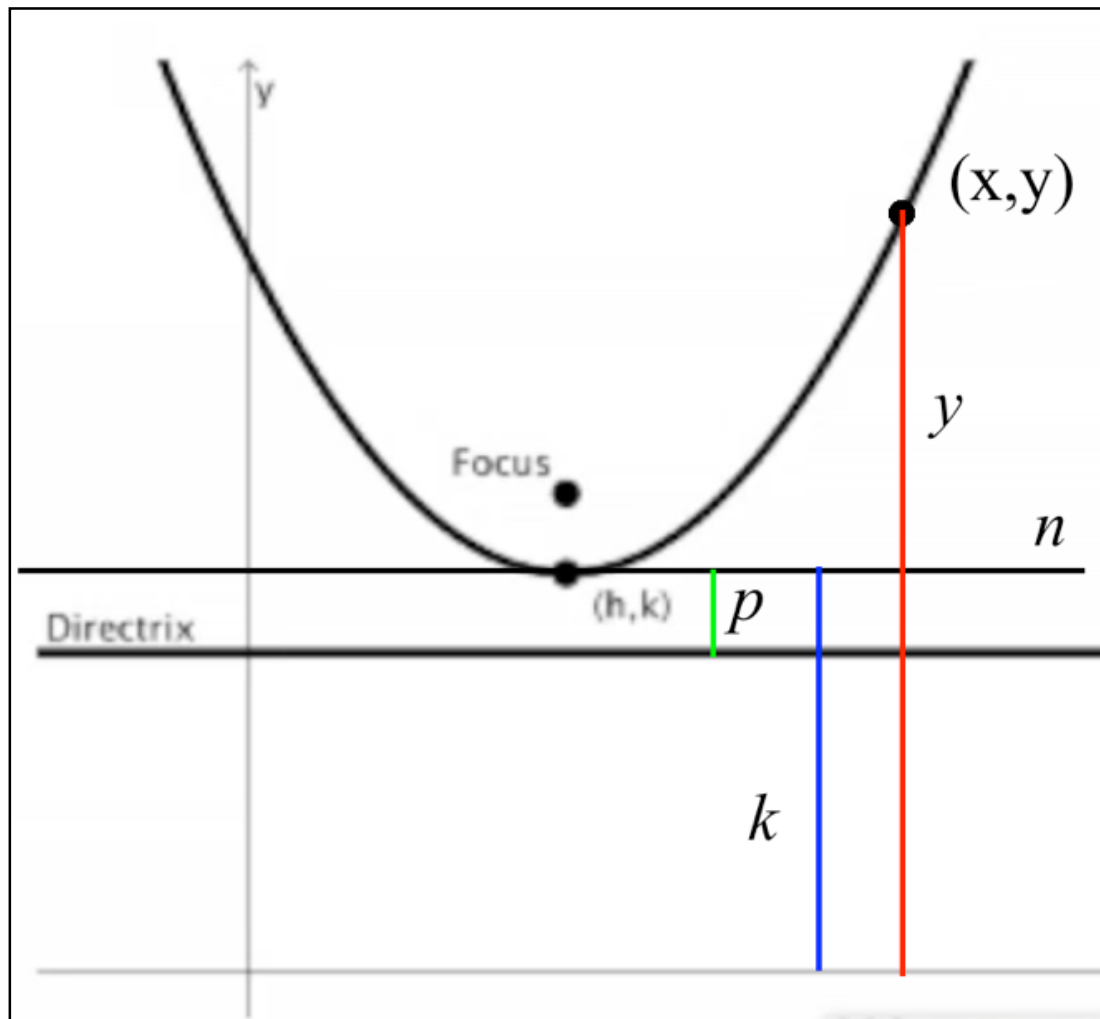


Figure 3.9. The quantities y , k , and p can be used to find the quantity $y - (k - p)$.

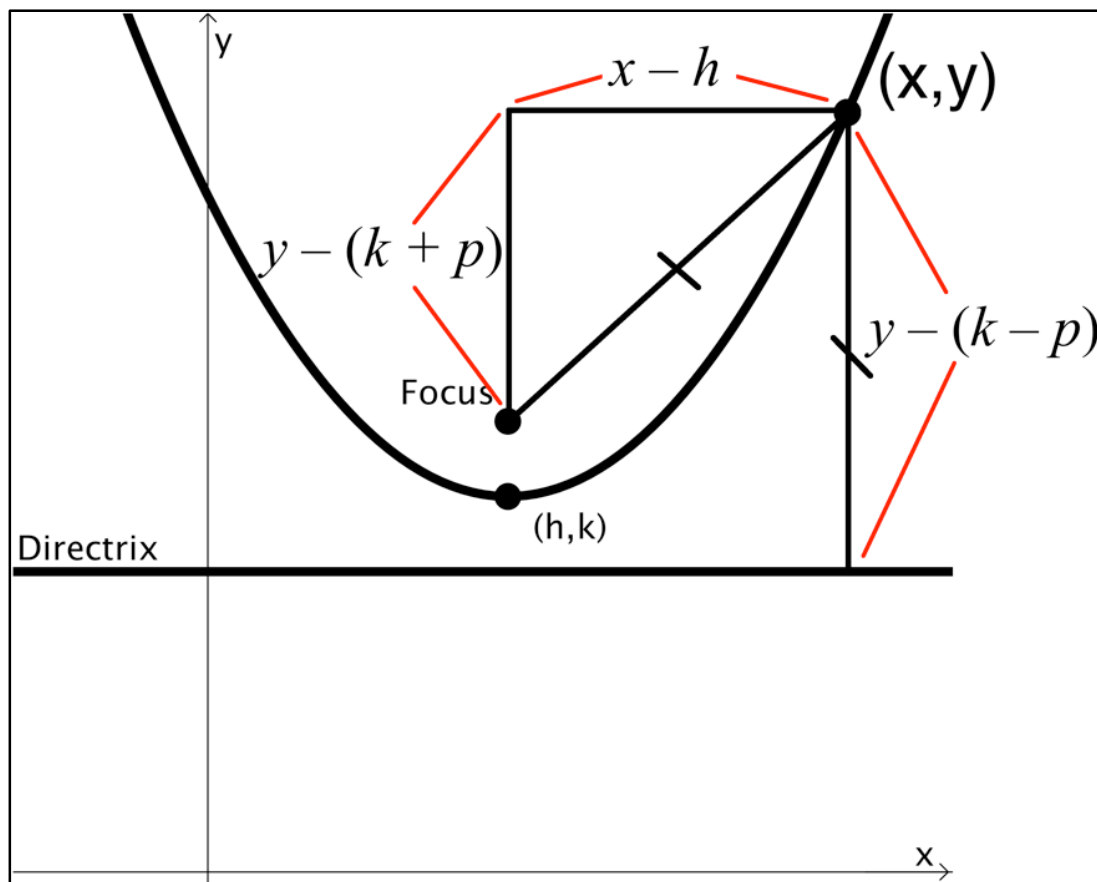


Figure 3.10. The quantities $x - h$, $y - (k + p)$, and $y - (k - p)$.

Once these distances have been quantified, the individual can set up the following equations:

$$a^2 + b^2 = c^2$$

$$(y - [k + p])^2 + (x - h)^2 = (y - [k - p])^2$$

The second equation can be solved for y to yield the equation for the general parabola,

which is $y = \frac{(x-h)^2}{4p} + k$.

Conceptual analysis of the Ellipse Task from the pre-interview. In the pre-interview, I gave participants the Ellipse Task, which is provided in Figure 3.11.

The definition of an ellipse is given below, as is a graph of a general ellipse. Use the definition to find the area of the two right triangles in the picture.

An **ellipse** is the set of points in the plane, the sum of whose distances r_1 and r_2 from two fixed points F_1 and F_2 , called the *foci* (which themselves are separated by a distance of $2c$), is a positive constant given by $2a$

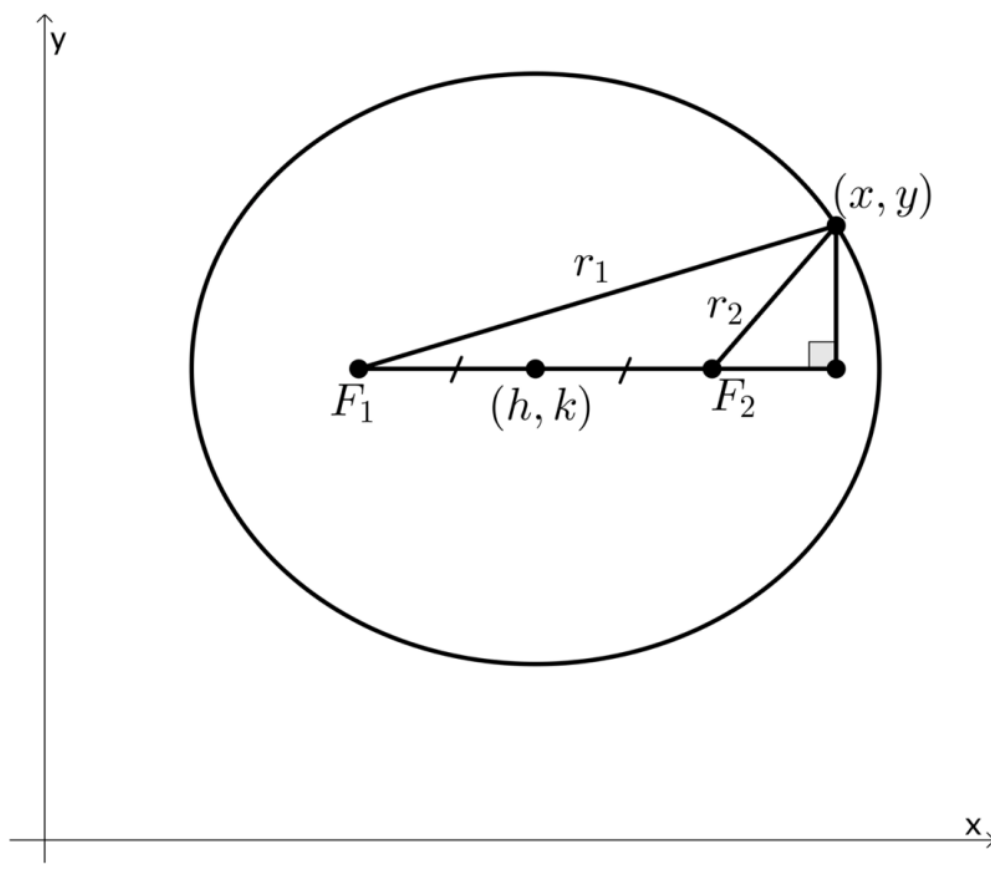


Figure 3.11. The Ellipse Task from the pre-interview.

I chose this task for two reasons. First, I wanted a task that would not spoil much of what was to come in the mini-course. I believed that giving participants the Parabola Task in the pre-interview might be problematic because a major emphasis of the mini-course is to create machinery that would be helpful in solving a task like the Parabola

Task. However, I also wanted a task that could be successfully solved using quantitative reasoning that is similar to the reasoning used in the Parabola Task in the post-interview so I could make pre-post comparisons. Moreover, solving both the Parabola Task and The Ellipse Task can be accomplished by leveraging well-known triangle formulae: the area formula in the Ellipse Task and the Pythagorean Theorem in the Parabola Task. To illustrate how similar quantitative reasoning can be used to solve this task, I now will present a brief conceptual analysis.

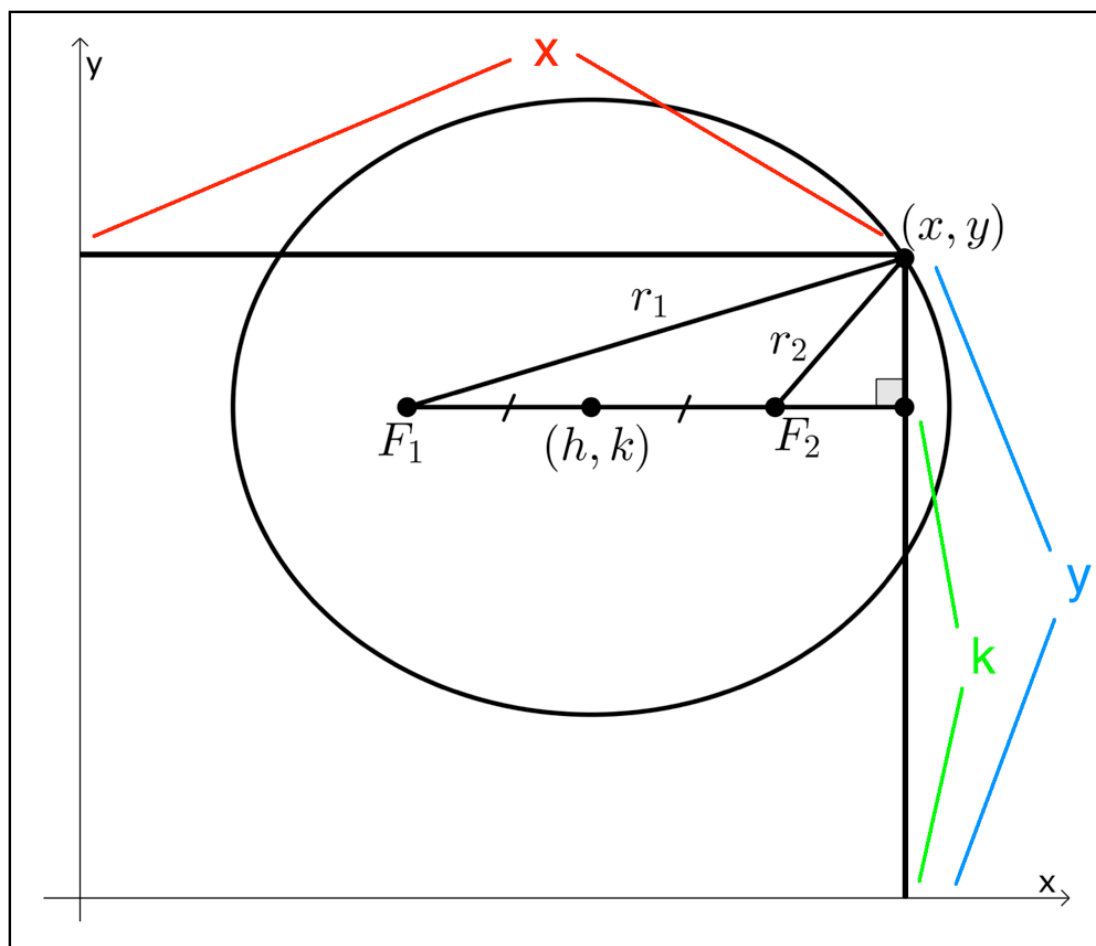


Figure 3.12. The quantities x , y , and k .

To find the area of the two right triangles in the diagram an individual may first quantify the coordinates of (x, y) as two separate distances: the distance y from the point to the x -axis and the distance x from the point to the y -axis. The vertex of the right angle of the right triangles is the point (x, k) , and similarly its coordinates can be quantified as the distance k from the point to the x -axis and the distance x from the point to the y -axis. The vertical leg of the right triangles can then be found by forming the quantitative relationship $y - k$ (labeled in Figure 3.14 below) with the quantities y and k and the additive comparison quantitative operation. The quantities x , y , and k are shown above in Figure 3.12.

The horizontal legs of each of the right triangles can be found using the quantities x , h , and the given distance c , which is the distance from (h, k) to either of the points F_1 or F_2 . To find the distance from F_2 to the point (x, k) , one may quantify the x -coordinate of the point (h, k) as the distance h from the point to the y -axis. Forming the quantitative relationship $h + c$ with the quantities h and c and the additive combination quantitative operation yields the distance from F_2 to the y -axis. The individual can now form the second order quantitative relationship $x - (h + c)$, which is the distance from F_2 to the point (x, k) , and is also the length of the horizontal leg of the smaller right triangle. Finding this distance is illustrated below in Figure 3.13, and the quantity is labeled in Figure 3.14 below.

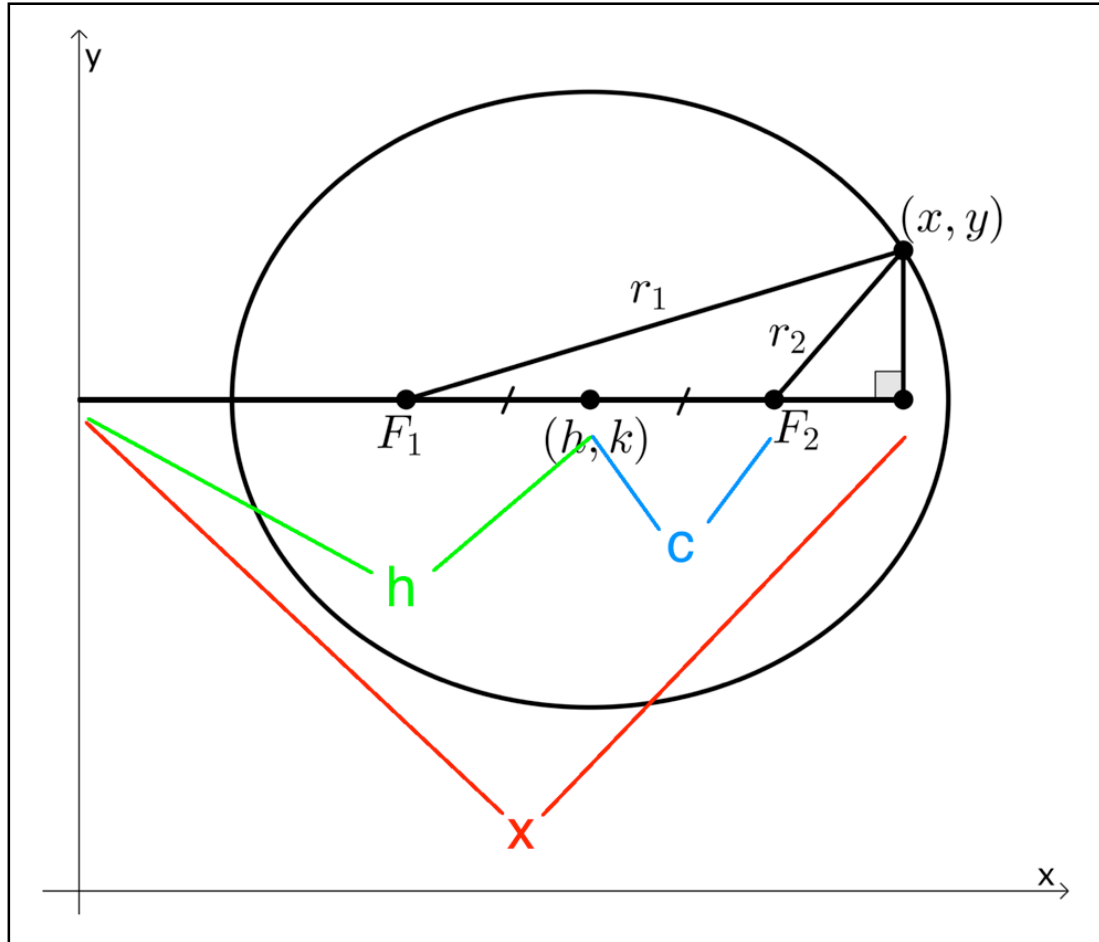


Figure 3.13. The quantities x , h , and c , which are needed to find the distance of the horizontal leg of the smaller right triangle.

Finally, to find the length of the horizontal leg of the larger right triangle, one may recognize that the distance from F_1 to the point (x, k) differs from the distance from F_2 to the point (x, k) by a distance of $2c$. One can then form the second-order quantitative relationship $x - (h - c)$ with the quantities $x - (h + c)$ and $2c$ and the additive comparison quantitative operation.

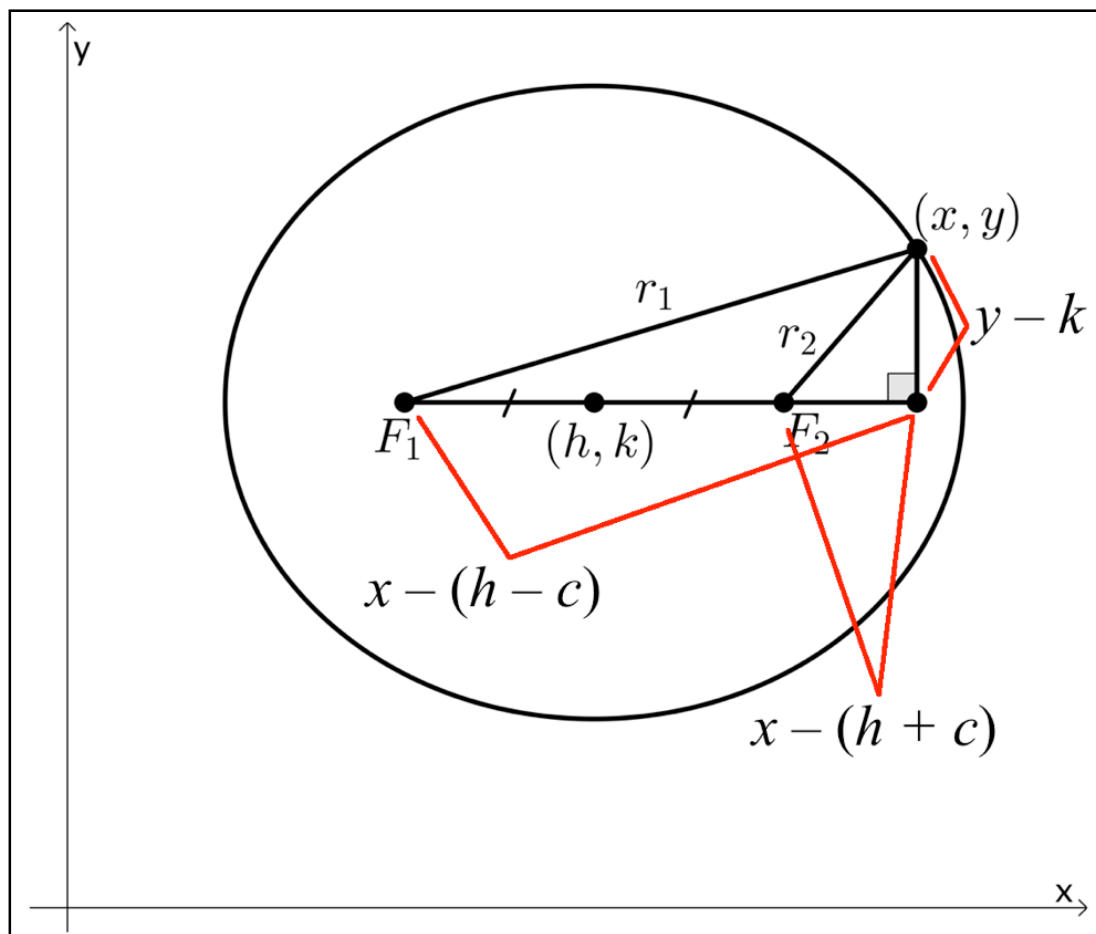


Figure 3.14. The quantities $y - k$, $x - (h - c)$, and $x - (h + c)$.

This quantitative reasoning results in the lengths of the legs of two right triangles (shown in Figure 3.14 below): $y - k$ is the length of the vertical leg of both triangles, $x - (h + c)$ is the length of the horizontal leg of the smaller right triangle, and $x - (h - c)$ is the length of the horizontal leg of the larger right triangle. The areas of these triangles are then easily found using the formula for the area of a right triangle, and are shown below.

$$\text{Area of smaller triangle:} \quad \frac{1}{2}(y - k)(x - [h + c])$$

$$\text{Area of larger triangle:} \quad \frac{1}{2}(y - k)(x - [h - c])$$

As can be seen from this conceptual analysis, the reasoning used to find the distances in the Parabola Task from the post-interview is like the reasoning used to find the distances in the Ellipse Task in the pre-interview. Finding the distance $x - h$ in the Parabola Task is like finding $y - k$ in the Ellipse Task, and finding the second-order quantitative relationships $y - (k + p)$ and $y - (k - p)$ in the Parabola Task is similar to finding the second-order quantitative relationships $x - (h + c)$ and $x - (h - c)$ in the Ellipse Task.

Data Analysis

Analysis of the interview data proceeded in two phases. In the first phase I analyzed data from the post-interviews to identify the nature of participants' MKT. This resulted in several categories that described the nature of MKT around quantitative reasoning that participants potentially developed. The second phase consisted of analyzing data from the pre-interview in search of evidence consistent with these categories in an effort to determine whether or not participants who showed evidence of MKT around quantitative reasoning in the post-interview had also showed evidence of such in the pre-interview. This allowed me to make claims about the nature of MKT that developed during the instructional sessions. These categories are briefly discussed in the following subsection, and then again in greater detail in Chapter 4.

Analysis of post-interview data. To facilitate the reduction of data, I created descriptive accounts (following Miles & Huberman, 1994) after I watched the video recording for each post-interview. Descriptive accounts provide a factual account of

the interview with little or no inferences made on the part of the researcher. By making such accounts, the researcher focuses on what actually happened in the interview before trying to make sense of what happened. I used the descriptive accounts to identify or refine themes from the data.

In particular, creating the descriptive accounts revealed that participants' engagement with the Parabola Task generated rich data around quantitative reasoning. While other tasks from the post-interview contained evidence of participants' conceptions of parabolas and some evidence of their quantitative reasoning, episodes from the interviews in which participants solved the Parabola Task seemed like a productive place to start my analysis of the post-interview data. Those episodes formed the reduced data set for the post-interview.

I transcribed and annotated the reduced data set to facilitate analysis. This data was analyzed using open coding (Strauss, 1987) from grounded theory. During open coding, data are given labels and grouped into conceptually similar categories. Keeping in mind that the answer to Research Question 1 is a description of the nature of MKT that developed for participants, I began coding data as it pertained to MKT around quantitative reasoning.

As categories emerged they were evaluated against the corpus of data (including data from other tasks) as well as against other categories using the constant comparative method (Glaser & Strauss, 1967). This method requires that segments of data continually be revisited throughout the analysis of data. In doing so, I reexamined meanings initially assigned to data and revised or updated these meanings as new

insights emerged during my analysis. This method increases the precision and accuracy of the development of categories, and it guards against bias (Corbin & Strauss, 1990).

Analysis of the reduced data set revealed four categories. Three of these categories described different aspects of MKT around quantitative reasoning that participants seemed to develop: (a) sophisticated quantitative reasoning with distances; (b) conceptual orientations toward problem solving; and (c) taking the perspective of high school students. The fourth category described productive or positive dispositions toward high school students that was not necessarily associated with quantitative reasoning. These four categories are described in greater detail in Chapter 4.

Analysis of pre-interview data. Each category that emerged in the analysis of the post-interview data described different aspects of MKT that some participants seemed to demonstrate. However, data from the post-interview alone was not sufficient to make claims that participants developed that MKT from engaging in the instructional sessions. Instead, I searched for evidence from the pre-interview to determine if participants had demonstrated similar MKT prior to their engagement in the mini-course.

The emergence of the four categories of MKT from the post-interview provided direction in how to reduce the data from the pre-interviews. Specifically, I focused my analysis on data from participants' work on the Ellipse Task, as that task was most similar to the Parabola Task. I created descriptive accounts for the segments

of data featuring the Ellipse Task and transcribed relevant sections of the data that directly related to the four categories of MKT from the post-interview.

Analysis of these data revealed three additional categories (a) unsophisticated or unproductive quantitative reasoning; (b) numerical or calculational orientations toward problem solving; and (c) unproductive or negative dispositions toward high school students. I made comparisons between these categories and the four that emerged in the post-interview to identify potential development of MKT. These comparisons resulted in descriptions of four shifts in MKT that participants seemed to experience as a result of their engagement in the instructional sessions: (a) a shift in quantitative reasoning; (b) a shift in orientation toward problem solving; (c) a shift in point of view; and (d) a shift in affect toward high school students. Elaboration of and evidence for these shifts are provided in Chapter 4.

Data Collection and Analysis Methods for Research Question 2

This section provides details about methods to collect and analyze data for Research Question 2, which is restated below:

Research Question 2: How do particular elements of the designed learning ecology contribute to the development of MKT by prospective secondary teachers during a video-based mini-course?

As I noted in my elaboration of this research question in Chapter 2, I extended the focusing framework developed by Lobato et al. (2013) to use as a tool for answering this question. As this framework will be featured prominently in Chapter 5, I elaborate each of the four components of the framework in this section. I will then describe how

I collected and reduced data for this question. I will end this chapter by outlining the methods of analysis used to answer this question.

The Focusing Framework

While Silverman and Thompson (2008) posited that MKT is the result of a second-order reflective abstraction, I argued in Chapter 2 that this process of learning will not adequately account for the development of MKT by PSTs. Lobato et al. (2013) explained how certain mathematical features of a learning environment may serve as the foundation for more productive mathematical ideas than do other features. I extend this perspective by arguing that certain features of a learning environment may serve as the foundation for more productive MKT than do other features.

This is not to say that the construct of reflective abstraction does not provide explanatory power. Instead, Lobato et al. (2013) observed that there is a relationship between noticing and reflective abstraction that has implications for how and what one learns. As Lobato et al. argued,

By connecting noticing to reflective abstraction, we posit that what one notices mathematically can serve as the rootstock upon which one constructs ways to reason in new situations. Several different shoots (specific ways of reasoning) may be supported, but each is constrained by the rootstock (what one notices). (p. 812)

Lobato et al. (2013) defined noticing “as selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for one’s attention” (p. 809). To account for the impact noticing has on learning, Lobato et al. developed the focusing framework, which is comprised of four interrelated components: (a) centers of focus; (b) focusing interactions; (c)

mathematical tasks; and (d) the nature of mathematical activity. I discuss each of these briefly below, and provide more elaboration in Chapter 5.

Centers of focus. Centers of focus (CoFs) are “properties, features, regularities, or conceptual objects that students notice” (Lobato et al., 2013, p. 814) in information-dense situations. Lobato et al. conceived of CoFs as a way to identify mathematical features in the learning environment that attracted students’ attention. For this study, I adapted this construct to capture a wider range of other information, notably pedagogical features of the learning ecology that appeared to attract PSTs’ attention as they developed MKT.

Focusing interactions. Focusing interactions can be described as the discourse practices (gesture, talk, use of diagrams or drawings, etc.) of teachers or participants that direct others’ attention to mathematical or pedagogical features of the mini-course (Lobato et al., 2013). As teachers or participants direct the attention of others to mathematical and pedagogical features of the class, those features are more likely to be noticed by others, giving rise to CoFs.

Features of tasks. In this study, tasks (including both math and video tasks) formed what Lobato et al. (2013) described as “the backdrop for discourse practices because these are the situations that students and teachers discuss” (p. 814). The rationale for including tasks in the focusing framework is that tasks influence what students notice and learn. Lobato et al. described how the affordances and constraints of various mathematical tasks can impact what students notice and learn.

The nature of mathematical activity. The final component of the focusing framework is the ways in which participation in the classroom is organized and regulated by class-established norms that seem to influence what participants notice. Lobato et al. (2013) posited that these norms “regulate who is allowed to talk and what types of contributions they can make” (p. 814), and consequently can shape or limit what participants notice in the learning environment.

Data Collection

Collection of data for Research Question 2 occurred over six instructional sessions, each lasting approximately two hours. Two cameras were used to capture participants’ and my own activity. When participants worked in small groups, the video cameras captured the conversations and activity that occurred in each of the small groups. During discussions with the entire class, one of those cameras captured the discussion as well as inscriptions that were made during the presentations of participants’ work. Finally, written work produced by participants was collected, including their work on mathematics tasks and Task Reflection Documents.

Another doctoral student, Carren Walker, attended each session as an observer (Steffe & Thompson, 2000), and she operated a video camera. She recorded whole class interactions as well as group work, and she took detailed field notes. After each session, Carren shared what she noticed about participants’ thinking and their experiences during the session. These debriefing sessions provided opportunities for me to reflect on insights, challenges, and issues that emerged during the session, and

informed the revising of plans for subsequent sessions (Cobb, McClain, & Gravemeijer, 2003; Steffe & Thompson, 2000).

Data Reduction

Reduction of the classroom data was informed by the analysis of the pre- and post-interviews, which resulted in the generation of four pre-post shifts in MKT. To reduce this data, I reviewed the classroom videos and created descriptive accounts (following Miles & Huberman, 1994) of each of the instructional sessions. I then located segments of data that seemed of particular importance for each shift in MKT around quantitative reasoning with distances.

Data Analysis

Segments of data that were related to a shift in MKT around quantitative reasoning with distances were analyzed in an effort to understand how that MKT developed. Specifically I leveraged the four components from the focusing framework to account for the impact that participants' noticing had on their learning. Consequently, I sought evidence of what participants noticed, which discourse practices appeared to influence their noticing, and how the types of tasks used and the nature of mathematical activity influenced their noticing (Lobato et al., 2013).

Following Lobato et al. (2013), I conducted a total of four analytic passes of the data. I briefly describe each of these below, and then elaborate each pass in Chapter 5.

Identifying CoFs. In the first analytic pass, I identified two centers of focus: *MKT around mathematics not in the MathTalk videos* and *MKT around quantitative*

reasoning with distances. This involved fracturing the data (Strauss & Corbin, 1990) into manageable chunks and analyzing these to identify what participants seemed to be noticing. I analyzed participants' verbal utterances and coordinated these with both the written inscriptions and the gestures participants made during the mini-course. In the focusing framework, CoFs are essentially sites for analysis for potential shifts in students' learning (Lobato et al., 2013), and they served a similar purpose for this study.

Identifying focusing interactions. Once the CoFs were identified, it became clear that there was a significant shift in what participants were noticing. This shift occurred around Session 3, so I looked at data from Sessions 1–4 for evidence of focusing interactions that seemed to play a role in precipitating the shift in CoFs.

Lobato et al. (2013) provided an extant coding scheme from which I began my analysis. They identified and defined three focusing interactions: (a) highlighting (Goodwin, 1994); (b) quantitative dialogue; and (c) renaming (Goodwin, 1994). Two of these, highlighting and quantitative dialogue, contributed to the shift in CoFs.

Identifying features of tasks that contributed to the shift in CoFs. Lobato et al. (2013) analyzed specific moments in the data when there seemed to be shifts in CoFs. I extended the focusing framework by examining the role that both math and video tasks played in the shift in CoFs. In this study, the video tasks and the Task Reflection Document seemed to foster participants noticing of MKT around quantitative reasoning with distances.

Identifying the nature of mathematical activity. The main goal of analysis for this component was to describe “the rules for engagement in mathematical activity that appeared to govern students’ and teachers’ roles and that seemed to influence the number and nature of the candidates for centers of focus” (Lobato et al., 2013, p. 823). I looked at differences in the participatory roles participants seemed to take on in Session 1 and Session 2 versus in Session 3 and later. These differences helped account for the shift in CoFs.

Chapter 4: Shifts in Mathematical Knowledge for Teaching

In this chapter I answer Research Question 1, which is restated below:

Research Question 1: What is the nature of the mathematical knowledge for teaching that develops for prospective secondary teachers during a video-based mini-course?

The goal for this study was to examine the development of MKT around quantitative reasoning, which requires first the identification of MKT that participants appeared to develop through their engagement with the mini-course. As discussed in Chapter 3, I conducted pre- and post-interviews with each participant. Analysis of the interview data revealed four distinct shifts in MKT that participants seemed to experience.

Initially, one of my hypotheses was that participants would develop more sophisticated quantitative reasoning with distances in the coordinate plane. However, participants entered the study with a wide range of background knowledge. Thus, several participants demonstrated more sophisticated quantitative reasoning in the pre-interview than I had anticipated. Consequently, while some participants experienced major shifts in the nature of their quantitative reasoning from pre- to post-interviews, others demonstrated more minor development because of their differentially better starting point with respect to quantitative reasoning.

Even though some participants seemed to not develop more sophisticated quantitative reasoning, it did appear that they developed MKT around quantitative reasoning. For example, some participants developed the ability to think about quantitative situations from the perspective of high school students, while others developed a more productive conceptual orientation toward solving problems

involving quantities. Additionally, many participants appeared to experience a shift in MKT that was not specifically related to quantitative reasoning. Given the wide range of the nature of MKT that seemed to develop, my goal in answering Research Question 1 was to capture major shifts in MKT from pre- to post-interviews, as well as capture at least one shift per participant.

In this chapter, I provide evidence to support the claim that there were four shifts in MKT. First, some of the prospective teachers shifted from numerical to quantitative reasoning or from less to more sophisticated quantitative reasoning with distances in the coordinate plane (as described in the KDU presented in Chapter 2). Second, several participants shifted from attending only to their own ways of understanding mathematical situations to being able to decenter (Teuscher, Moore, & Carlson, 2016) and make sense of how others (specifically students) understand the situations. Third, most of the participants seemed to shift from a more calculational orientation toward teaching and learning to a more conceptual orientation (A. G. Thompson et al., 1994). Finally, all but one of the preservice teachers experienced a shift in affect, from demonstrating a deficit and teacher-oriented perspective regarding student understanding to exhibiting a growth-oriented and student-centered perspective regarding student understanding. Table 4.1 below provides an overview of which shifts each participant experienced (the names of all participants have been changed to gender-preserving pseudonyms).

Table 4.1. Overview of shifts in MKT that each participant seemed to experience. The checks denote participants who experienced each shift, while the circles represent that a participant likely decentered prior to the post-interview.

Participant	Shift in Quantitative Reasoning	Shift in Point of View (Decentering)	Shift in Orientation	Shift in Affect
Willow	✓	O	✓	✓
Desmond	✓		✓	✓
Sierra	✓	O	✓	✓
Marshall		✓		✓
April		✓	✓	
Jasper		✓	✓	✓
Lily		O		✓

The remainder of this chapter is divided into five sections. In the first section I present evidence to support my claim that three participants experienced a shift in their quantitative reasoning. In the second section, I provide a brief overview of the construct of decentering, and then I provide evidence that suggests three participants seemed to be able to decenter in the course of their problem-solving activities during the post interview; in other words, they shifted their point of view. The third section begins with a review of literature related to calculational and conceptual orientations, which is followed by evidence that suggests that five participants experienced a shift in their own orientation toward problem solving. I then describe how six of the seven participants seemed to shift their affect toward one that was more student-centered and growth-oriented. I conclude the chapter with a summary and brief discussion. I have reproduced both the Ellipse and Parabola Tasks below for the convenience of the reader.

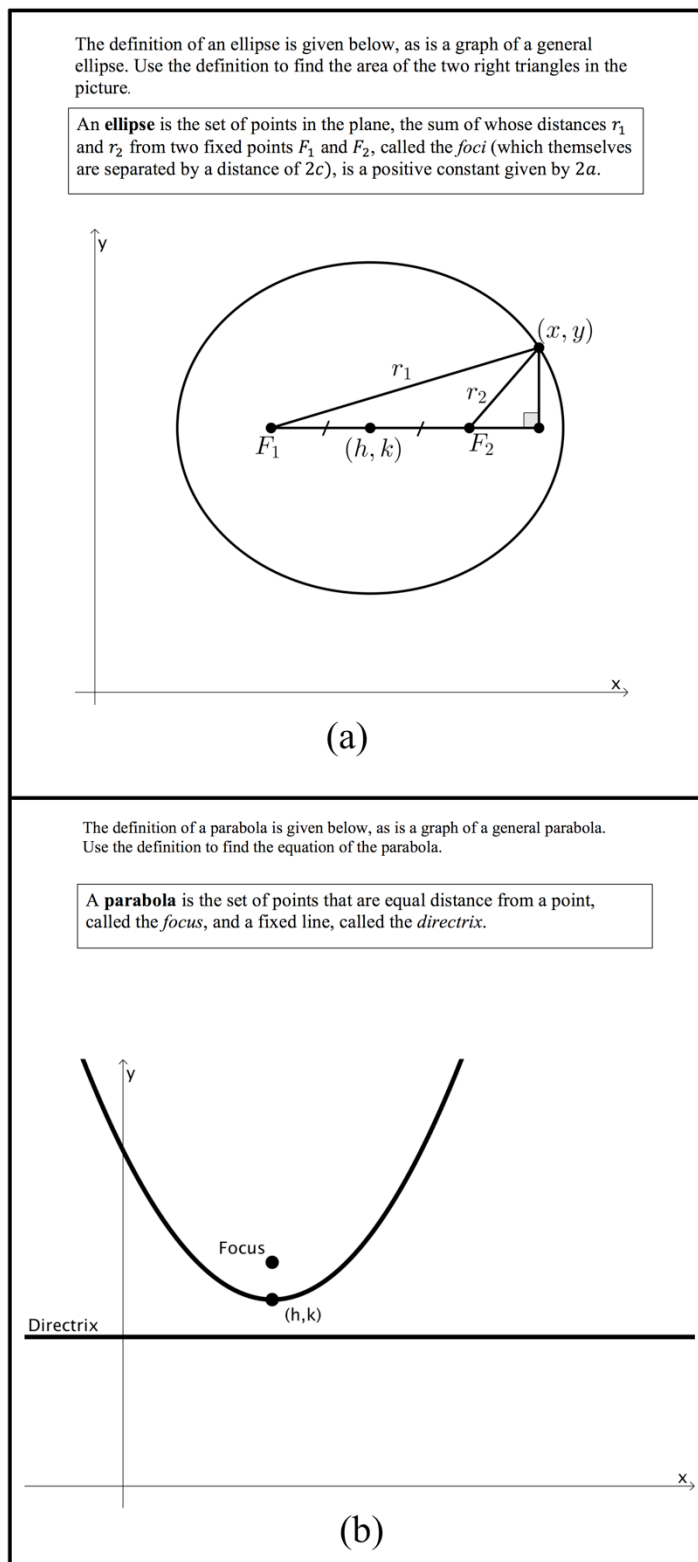


Figure 4.1. (a) The Ellipse Task from the pre-interview and (b) The Parabola Task from the post-interview.

A Shift in Quantitative Reasoning

In Chapter 3, I elaborated several elements of quantitative reasoning (P. W. Thompson, 1994). Briefly, these elements include isolating a measurable quality (quantifying); one's conception of that quality that can be measured along with its measure (quantity); combining or comparing two quantities to create a third (quantitative operation); and realizing a third quantity through a quantitative operation (quantitative relationship). Elements of quantitative reasoning that build upon other elements (e.g., quantitative relationships) can be thought of as more sophisticated than the elements they are built upon (e.g., quantities). For example, suppose in the pre-interview a participant did not quantify the x -coordinate of the point (h, k) , but was able to say that the point $(4,3)$ was four units from the y -axis. If in the post-interview he or she quantified the x -coordinate of the point (h, k) as a distance of h units from the y -axis and then formed quantitative relationships with that quantity, I considered this reasoning to be more sophisticated than the reasoning exhibited in the pre-interview.

Accordingly, one way to think of a shift in quantitative reasoning is to consider which quantities a participant quantified and how those quantities were used to solve a problem. I considered a participant to have experienced a shift in quantitative reasoning if the participant seemed to demonstrate more sophisticated quantitative reasoning in the post-interview than he or she did in the pre-interview. Another way that this shift manifested was when an individual appeared to reason numerically (or pre-quantitatively) in the pre-interview and then reasoned quantitatively in the post-

interview (see Figure 4.2 below for examples of these two types of reasoning). Finally, it should be noted that my claims are based on the nature and type of reasoning participants demonstrated; in other words, my claims are not about what participants were capable of, rather my claims are about what they demonstrated in the interviews.

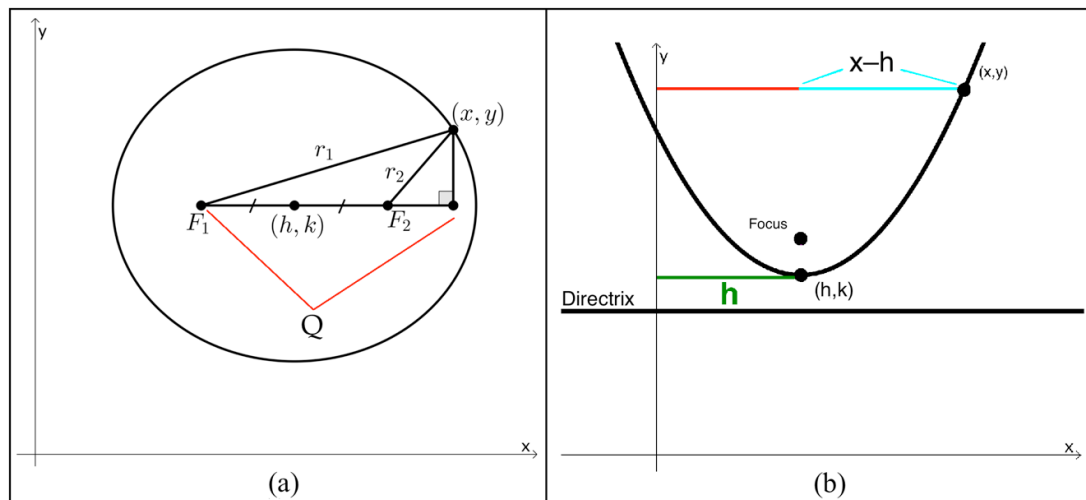


Figure 4.2. (a) A participant reasoning pre-quantitatively may claim that the segment labeled Q has a length of $x - F_1$. (b) A participant reasoning quantitatively may form the quantitative relationship $x - h$ by first quantifying and later operating on the quantities x and h .

The remainder of this section is devoted to providing evidence that suggests three participants, Willow, Desmond, and Sierra, experienced shifts in their quantitative reasoning. Three other participants, Marshall, April, and Jasper experienced less dramatic shifts in quantitative reasoning because they came into the study with more sophisticated prior knowledge related to quantitative reasoning. Moreover, these participants demonstrated much stronger quantitative reasoning in the pre-interview than Willow, Desmond, or Sierra. Additionally, while the shifts in quantitative reasoning by Marshall, April, and Jasper were not as dramatic, all three of these participants demonstrated an ability to shift their point of view, which is related

to quantitative reasoning and which I discuss in the next section. Lily demonstrated sophisticated quantitative reasoning with distances in both the pre- and post-interviews.

Finally, the structure of this section varies slightly from the presentation of the other three shifts in MKT in the remaining sections in this chapter. While each participant seemed to develop capacity for quantitative reasoning, their starting and ending conceptions are different. Consequently, I develop an argument for each participant by analyzing data and providing evidence from the pre- and post-interviews. In subsequent sections I discuss a prototypical participant in detail and then briefly summarize how other participants also experienced the shift.

Willow

In this section I present evidence to support the claim that Willow experienced a shift in her quantitative reasoning, from reasoning about points only as a locations or values to quantifying coordinates of points as distances and forming quantitative relationships with those quantities. In the pre-interview, Willow did not demonstrate quantitative reasoning with distances in the plane. Instead, she appeared to struggle to quantify coordinates, and she only demonstrated conceptions of points as locations in the plane or as values on which she could operate. These two conceptions—points as locations and points as values—represent the totality of Willow’s demonstrated reasoning with points in the plane during the pre-interview. While the points-as-locations conception is productive, it alone is not sufficient for forming quantitative relationships with coordinates (as distances). By contrast, in the post-interview Willow

seemed to demonstrate sophisticated quantitative reasoning. Not only did she quantify coordinates as distances, she also formed quantitative relationships with those quantities as she solved the Parabola Task. I now present four claims about Willow's reasoning, two of which are supported by evidence from the pre-interview and two of which are supported by evidence from the post-interview.

Points as locations. Willow began the Ellipse Task (see Figure 4.1 above) in the pre-interview by writing down the formula for the area of triangle ($A = \frac{1}{2} b * h$), which she later erased. She also labeled two segments on the diagram as $2c$. She then said that the “whole thing” (meaning the segment from the point F_1 to the point F_2) would be $4c$, and she labeled the segment as such. This early work can be seen below in Figure 4.3.

Willow then seemed to conceive of points as locations in the plane. She pointed to specific places in the coordinate plane to identify points and provided coordinates for points in the plane that she had located. For example, in the following exchange, Willow pointed to two different points in the plane and named one of them.

- Willow: [Draws a vertical line segment from the x -axis to the vertex of the right angle, which can be seen in Figure 4.3 below] This is also an x, y [points to the vertex of the right angle], right, technically?
- Interviewer: Um, can you say more?
- Willow: Because this [points to the vertex of the right angle] is a point on the graph. This is an x, y [underlines the printed (x, y)], this could be x -one, er, x -two, y -two [labels the vertex of the right triangle (x_2, y_2)].

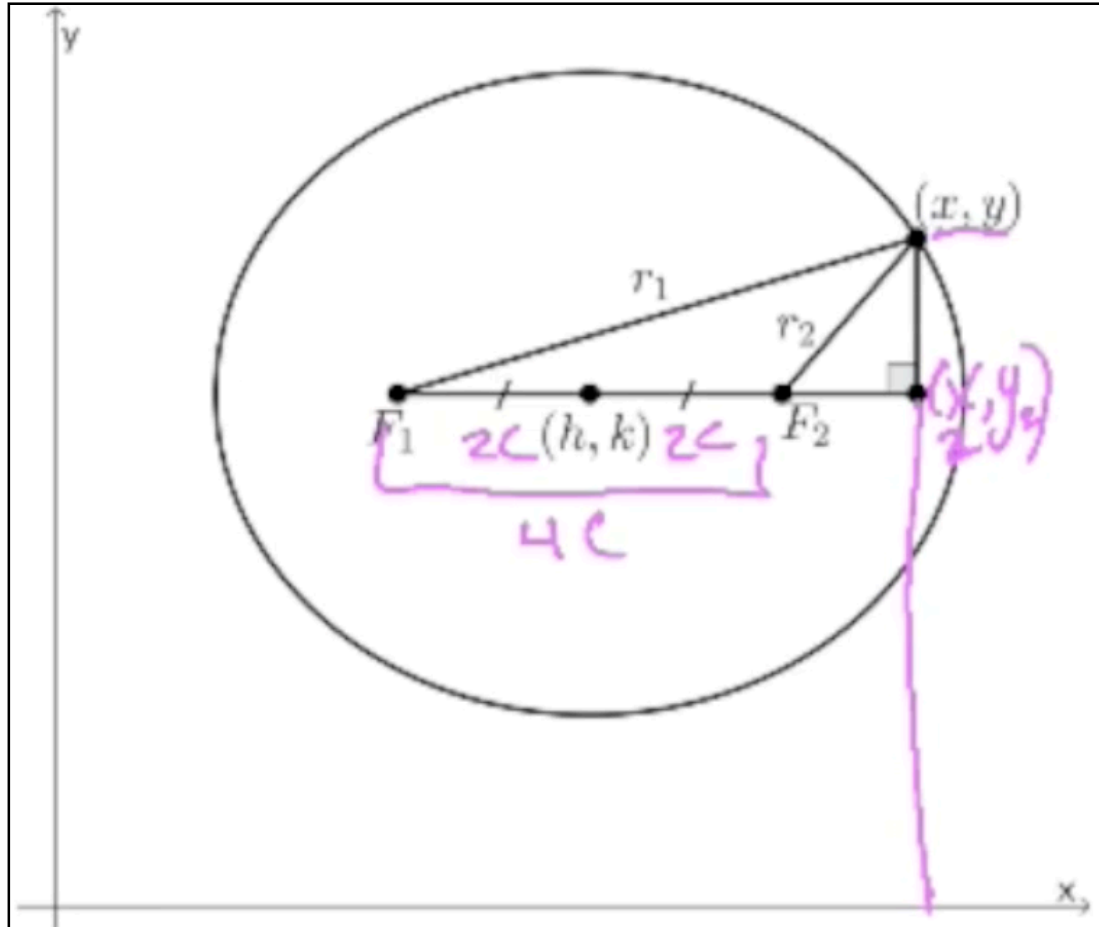


Figure 4.3. Willow's initial work on the Ellipse Task, including her labeling of the right angle as (x_2, y_2) .

Willow pointed to the vertex of the right angle and said it was a “point on the graph.” Moreover, she labeled it by giving it a name with specific coordinates, (x_2, y_2) . This can be taken as evidence that Willow conceived of that point as a location in the plane. Right after she said the vertex of the right angle was “a point on the graph,” she underlined the point that was labeled (x, y) and said it was “an x, y ,” which could also indicate she was locating that point and conceiving of it as a location in the plane. In total, there were five additional instances in which Willow pointed to

and named or labeled points in the plane. These instances suggest that Willow conceived of points as locations in the plane.

A careful reader may be thinking “Is it not legitimate to think of a point as a location on a graph?” Indeed, such a conception is valid and productive. However, Willow’s conceptions of points appeared limited in the pre-interview, which potentially constrained her ability to reason quantitatively with distances in the coordinate plane.

In fact, it appeared that in the pre-interview Willow did not quantify the coordinates as also representing distances. Consider that at the beginning of the previous exchange, she drew a vertical line segment from the x -axis to the vertex of the right angle. This action is consistent with the types of actions taken by individuals when they conceive of the y -coordinate of a point as representing some distance above the x -axis. However, she did not label, or name, or use that segment in any way for the rest of the interview. Moreover, Willow labeled the vertex of the right angle (x_2, y_2) , even though it was vertically aligned with the point (x, y) . This would make $x_2 = x$. If Willow had conceived of the x -coordinates as distances from the y -axis, she might have more readily recognized that the two x -coordinates should be the same. Willow did not seem to attend to this quality of the vertex of the right angle, which suggests she did not see the two x -coordinates for those points as being equivalent. Taken together, this serves as evidence that Willow did not quantify these coordinates as distances.

Points as values. In the pre-interview, Willow also demonstrated a conception of points as values. In her attempt to find the height of the right triangle, Willow appeared to want to find the difference between two points, as opposed to the difference between the y -values of two points:

- Willow: My height would be this x -one y -one [circles (x, y)]—this point, and this is the difference [gestures with her pen from (x_2, y_2) to (x, y)] between these two points, so it would be, this height would actually be—is it okay if I do that? Because I don't know how else to label this side, actually.
- Interviewer: Yeah, continue along that path.
- Willow: So going off the regular, it's one-half times, then I'm going to say, I'm not sure, but this would be the difference x -one y -one minus x -two y -two [writes $A = \frac{1}{2} ()((x_1, y_1) - (x_2, y_2))$, as seen in Figure 4.4 below].

The image shows a rectangular box containing a handwritten equation in purple ink. The equation is $\frac{1}{2} ((x_1, y_1) - (x_2, y_2))$. The parentheses are slightly messy, and the ink is somewhat faded.

Figure 4.4. Willow's equation using the two points instead of the y -coordinates of the two points.

Willow's statement that the height would be the difference between the two points suggests that she thought she could operate on the two points to find the height. Her written inscription served as more evidence that she conceived as points as values that she could operate on.

That Willow appeared to conceive of points as values on which she could operate could be problematic, as the notation that she used is not mathematically acceptable, and points are typically not operated on using arithmetic in the way that

she was attempting. For high school students, such a conception might not be as unexpected, but Willow is a prospective secondary teacher, and it would be expected that she would be more careful in her algebraic notation.

In summary, in the pre-interview, Willow did not seem to demonstrate quantitative reasoning that empowered her to solve the Ellipse Task. Willow's points-as-locations conception was productive but not sufficient for solving the task. Willow's points-as-values conception helped her isolate the length of the vertical side of the right triangle, yet her notation was problematic. Neither conception involved quantifying coordinates of points as distances. Throughout the entire pre-interview Willow did not seem to form any quantitative relationships.

Quantifying coordinates as distances. From the pre- to the post-interview, Willow seemed to experience a shift in her ability to reason quantitatively. In the post-interview, she was able to quantify coordinates of points as distances—something she did not demonstrate in the pre-interview. In particular, she quantified the coordinates h , x , k , and y as distances, where (x, y) was a point chosen by Willow to represent any point on the parabola and (h, k) is the vertex of the parabola.

Willow began the Parabola Task (see Figure 4.1b above) by drawing a horizontal line segment starting from the right of the parabola, through the focus to the y -axis (see Figure 4.5). She drew a vertical line segment from the parabola to the directrix. Next, she drew a vertical segment from the focus to the vertex, and then she drew a diagonal line from the vertex to the parabola. Finally, Willow drew two dotted lines emanating from the vertex, and labeled these as h and k .

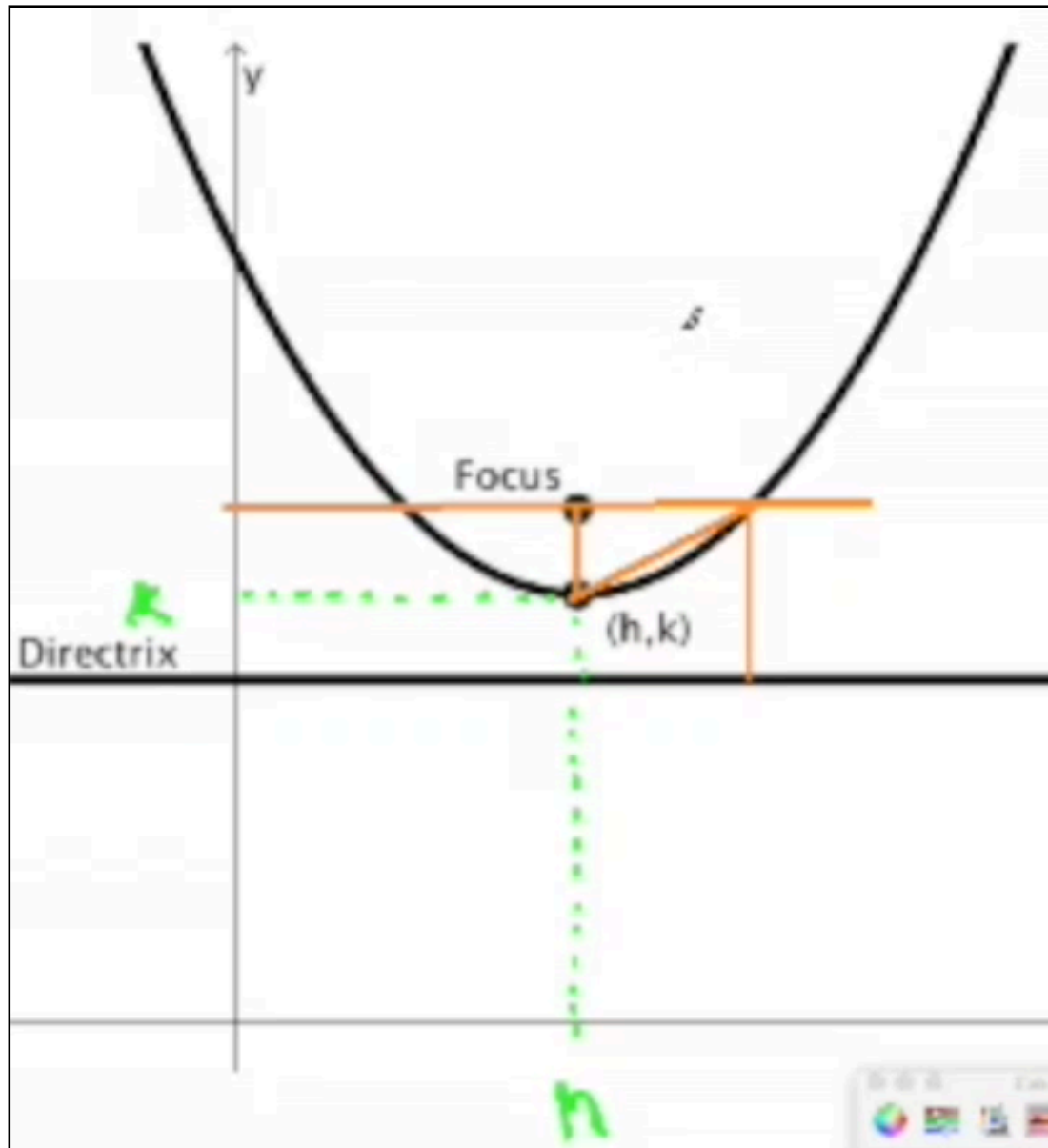


Figure 4.5. Willow's preliminary work on the Parabola Task.

Willow then drew another dotted line from the directrix to the x -axis and labeled it x . She labeled as y the intersection of her first horizontal line segment and the y -axis. Finally, she colored in light blue a horizontal segment from the focus to the right side of the parabola and labeled it $x - h$ (as shown in Figure 4.6). Similarly, she colored in light blue the segment from the focus to the vertex and labeled it $y - k$.

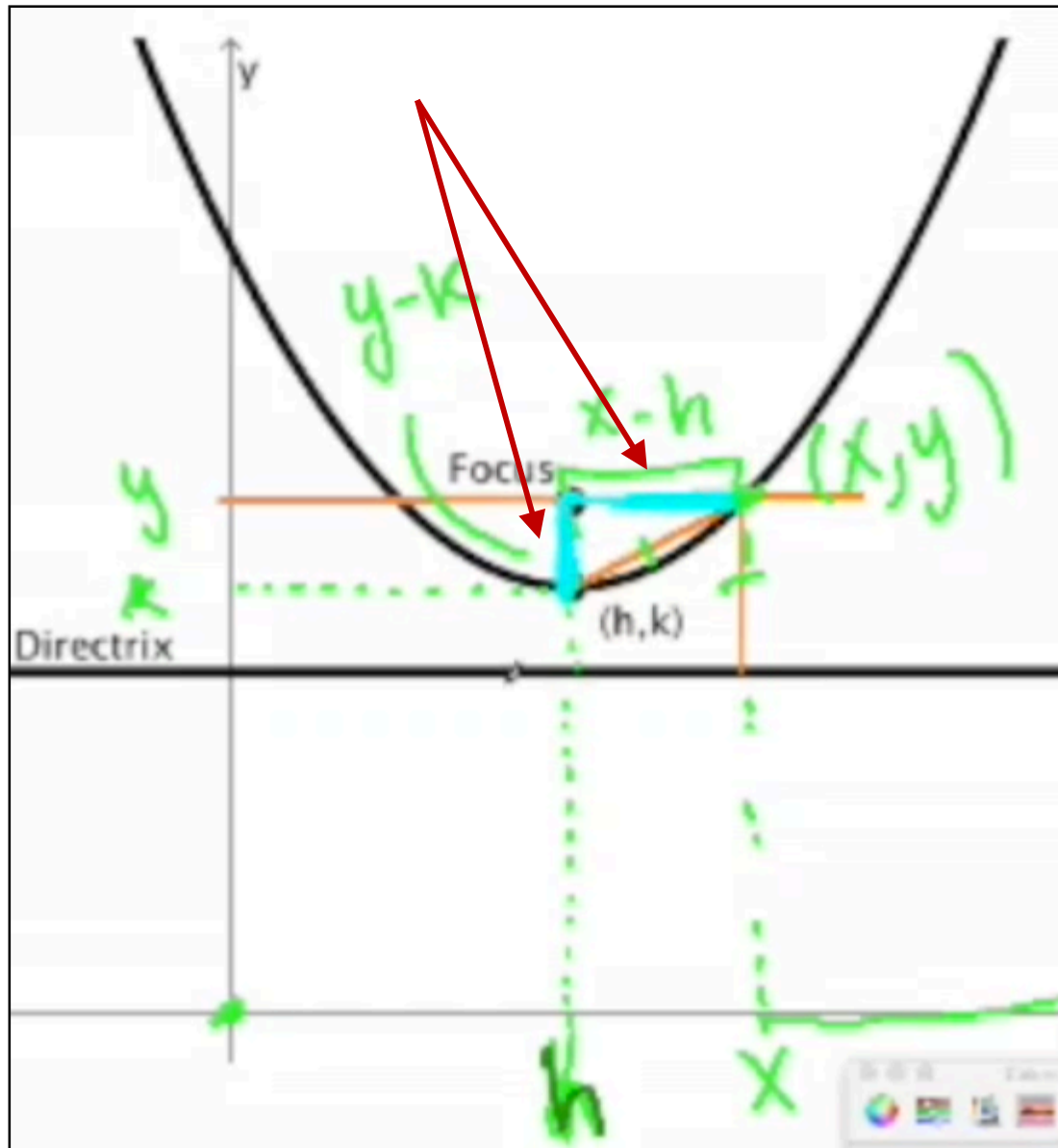


Figure 4.6. The author's arrows indicate Willow's segments that she labeled $x - h$ and $y - k$.

I hypothesized that Willow had formed the quantitative relationships $x - h$ and $y - k$; however, I had no evidence that she had quantified the associated coordinates x , y , h , and k as quantities. To help determine if she had quantified those coordinates I posed the following question:

- Interviewer: If you were to color in x , like here you colored in x minus h [points to the horizontal segment Willow colored before], what would you color x in as?
- Willow: X —[draws a light blue vertical segment from (x, y) to the x -axis, see Figure 4.7 below]—and then I guess this would be h [draws a green vertical segment from the vertex to the x -axis, see Figure 4.7 below].

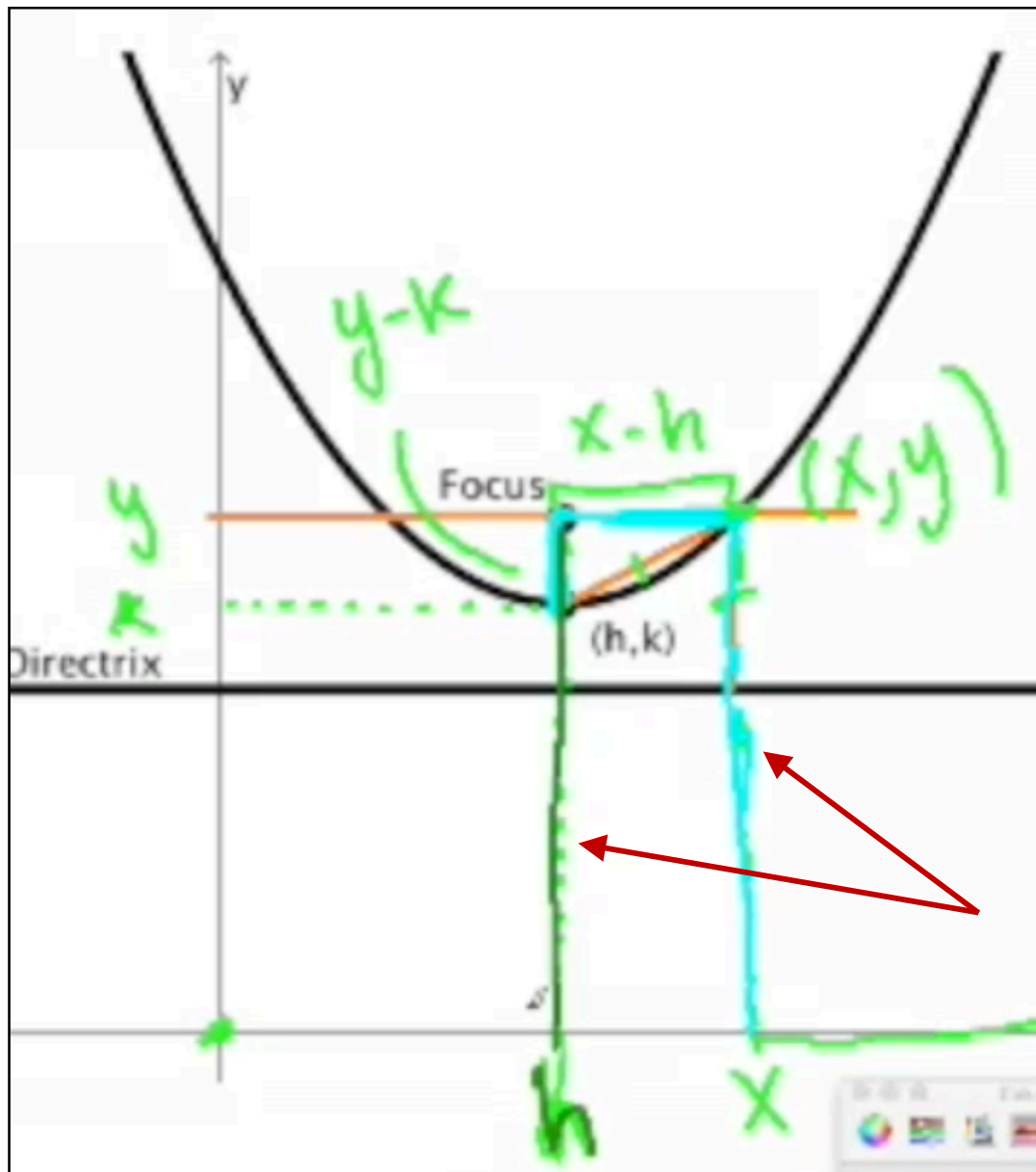


Figure 4.7. The author's arrows indicate the two vertical segments Willow drew, one to represent x and the other to represent h .

Willow seemed to conceive of both x and h as vertical segments. However, before I could respond, she changed her mind and drew in new segments for both x and h .

Willow: No, no, that's just the line [erases the two segments she just drew]. This is the distance to h [draws a green horizontal segment along the x -axis from roughly the y -axis to a point roughly below (x, y) , see Figure 4.8 below] and this would be the distance to x [draws a light blue horizontal segment along the x -axis, see Figure 4.8 below], and this is x minus h [gestures a circle around the part of the light blue segment that extends past the green line].

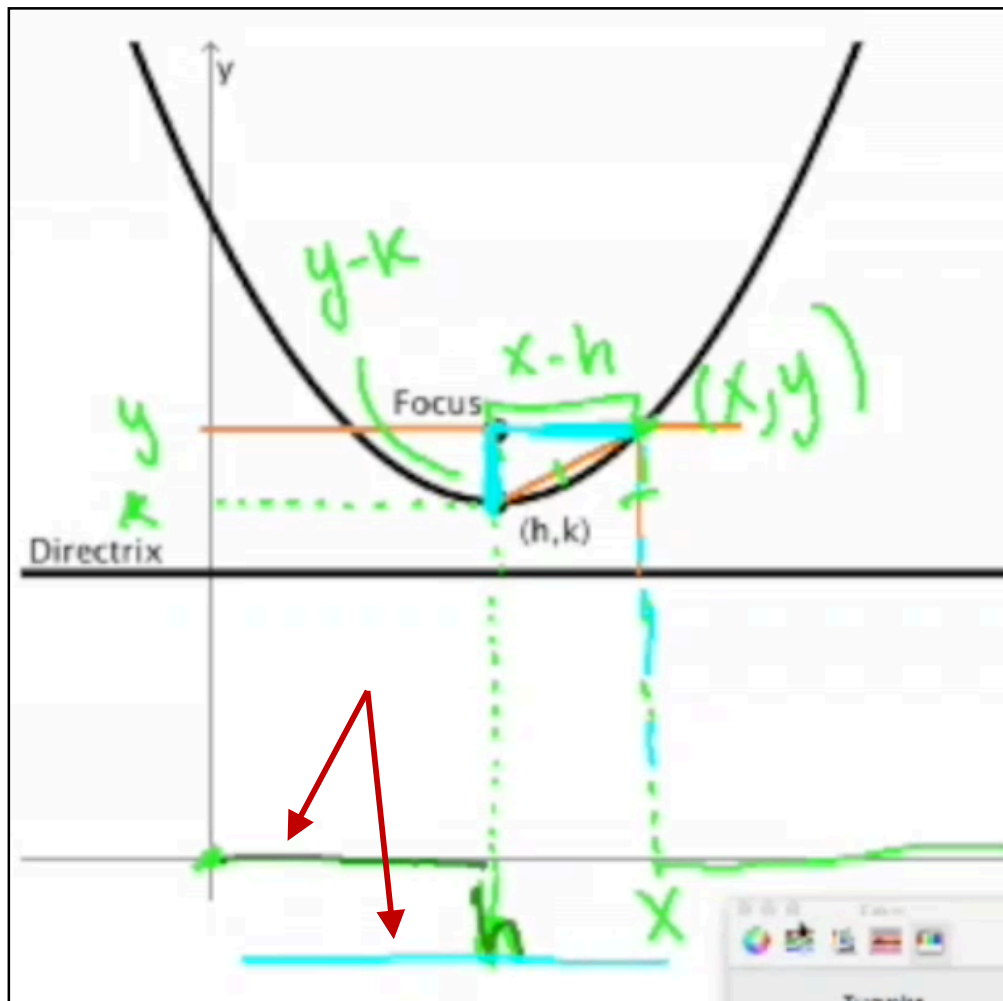


Figure 4.8. The author's arrows indicate Willow's new segments representing x and h .

- Interviewer: So, x is from where to where?
 Willow: Um, it's from the origin to x [traces over the light blue segment she had drawn to indicate x]. And then this distance is x minus h [draws a light blue horizontal segment and labels it $x - h$, see Figure 4.9 below].
 Interviewer: And then where is the distance h ? Just to make sure I'm absolutely clear.
 Willow: Right here, the darker line [gestures back and forth with her pen along the shorter green segment that is drawn on the x -axis].

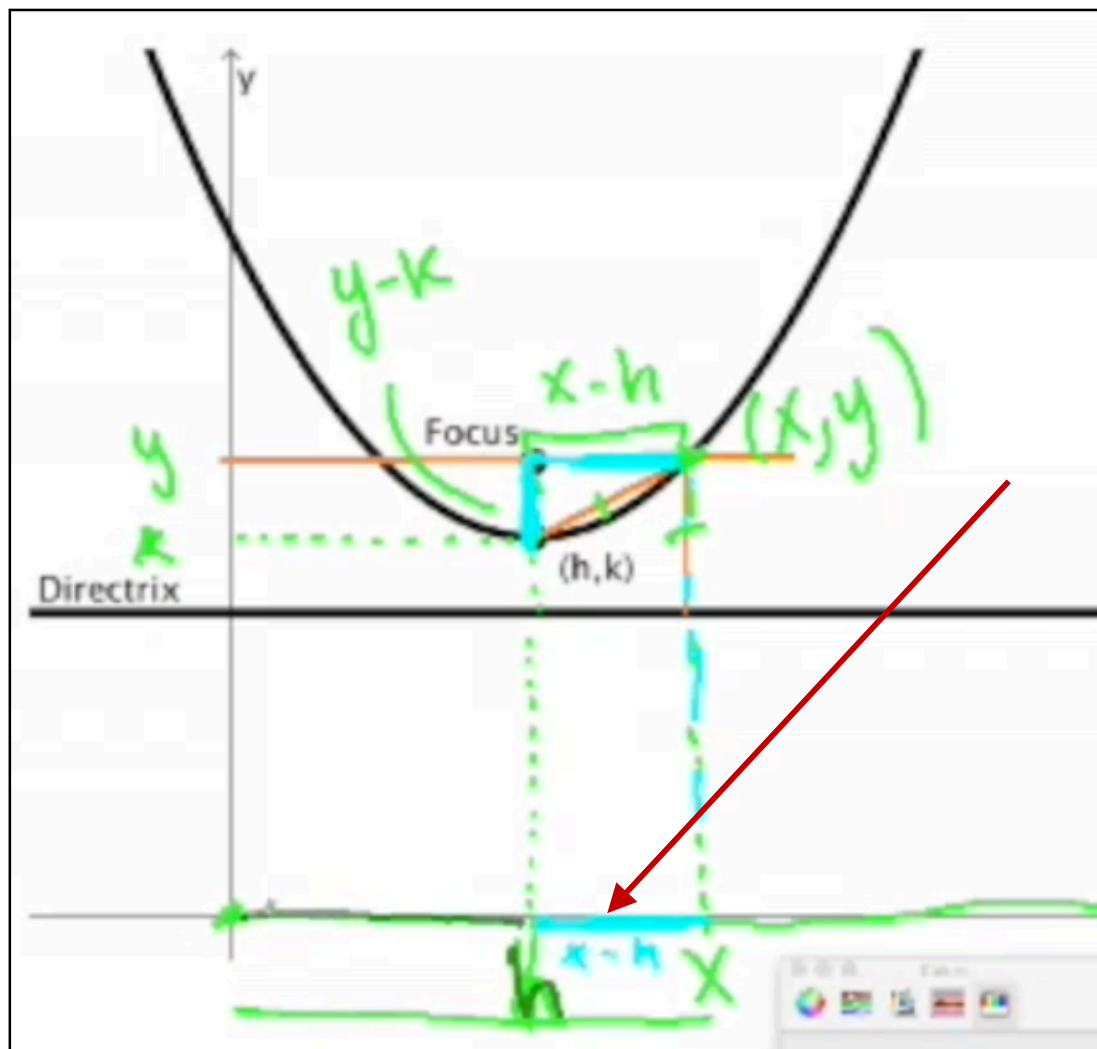


Figure 4.9. The author's arrow indicates where Willow drew a segment and labeled it $x - h$.

Even though Willow initially marked out vertical segments, she corrected herself by erasing those segments and replaced them with horizontal segments along the x -axis. As she drew those segments, she said “This is the **distance to h** and this would be **the distance to x** ” (emphasis added). She was careful to make her segments so that the endpoints were close to the origin and the places on the x -axis she had marked h and x , respectively, which suggests she drew the segments with an intent to mark off specific distances. Moreover, she referred to these segments as “distances.” Taken together, her drawings and her use of the phrase “distance to” suggest that Willow had quantified the x -coordinates of the points (x, y) and (h, k) as distances from the y -axis. Shortly after this, Willow seemed to quantify y and k as distances.

Willow: And then similarly, with y . If I make k this dark color, this would be the distance k [draws a dark green vertical segment from the x -axis to the point labeled k on the y -axis, see Figure 4.10 below], and this would be the distance y [draws a green vertical segment next to the one she just drew, see Figure 4.10 below], and then this distance here [draws in a vertical segment in light blue, see Figure 4.10 below] is y minus k .

Willow quantified the y -coordinates of the points (x, y) and (h, k) appropriately as distances. She drew vertical segments near the y -axis emanating from the x -axis and said, “If I make k this dark color, this would be **the distance k** , and this would be **the distance y** ” (emphasis added). Again, Willow seemed to carefully draw the segments so that the endpoints were at the x -axis and then at points roughly aligned with where y and k were marked along the y -axis. Her continued use of the phrase “the distance” suggests she had conceived of the segments she drew as

distances in the plane. Additionally, Willow linked those segments with coordinates (e.g., the dark green segments in Figure 4.10 below were drawn as Willow said they represented the distances h and k). Taken together, this evidence suggests that Willow shifted from interpreting points in the plane only as locations, to quantifying the coordinates of points as distances.



Figure 4.10. The author's arrows indicate where Willow drew line segments to indicate the distances y , k , and $y - k$.

Forming quantitative relationships. Not only did Willow quantify coordinates as distances, she also formed quantitative relationships. For example, Willow seemed to form the quantitative relationship $x - h$ with the quantities x and h . First, Willow appeared to quantify x and h as distances, as just discussed. She also

drew a segment (see Figure 4.9 above) and stated, “this distance is x minus h .” Her use of this phrase suggests that she had quantified the segment as a distance having a measure of $x - h$. While I do not have evidence to make claims about which quantitative operation she may have used, she did use the word “minus,” which is consistent with the quantitative operation of either take-away or additive comparison. She segmented the x -axis into two distinct segments, one with distance h and the other with distance $x - h$. Underneath these, she drew a third segment having a distance of x . Willow organized the two quantities x and h in a way that helped her justify her earlier claim that one segment had a distance of $x - h$. Because Willow quantified x , h , and $x - h$ as distances, and she seemed to operate on x and h to form the quantity $x - h$, it is reasonable to conclude that she formed a quantitative relationship with those quantities.

Willow similarly formed the quantitative relationship $y - k$ with the quantities y and k . In Figure 4.10 above, Willow colored two segments, labeling one y and one k . She stated, “This would be **the distance k** and this would be **the distance y** , and then **this distance here is y minus k** ” (emphasis added). As before, her segmenting of the y -axis into two segments k and $y - k$ alongside a single segment y indicates Willow may have conceived of the segment y as comprised of the two segments $y - k$ and k . As before, I have no evidence of which quantitative operation Willow may have used, but this segmentation suggests she did perform a quantitative operation on two quantities. Taken together, this evidence supports my claim that Willow formed the quantitative relationship $y - k$ with the quantities y , and k .

In summary, Willow seemed to experience a shift in her quantitative reasoning from the pre-interview to the post-interview. In the pre-interview, she did not quantify coordinates as distances; instead her conceptions about points were limited to points-as-locations and points-as-values. Indeed, even when she did potentially link a coordinate to a distance from either the x -axis or the y -axis (when she drew a vertical segment from the x -axis to the vertex of the right angle, as shown in Figure 4.3 above), she never made use of the distance, nor did she explicitly state that it was a distance. By contrast, Willow demonstrated quantitative reasoning in the post-interview that was productive for her. She could quantify coordinates as distances and use those quantities to form new quantities. Indeed, Willow's reasoning in the post-interview closely matched the kinds of reasoning that Sasha and Keoni employed, and her solution for the Parabola Task was similar to Sasha and Keoni's, despite having never seen the videos in which Sasha and Keoni solve that task.

Desmond

Like Willow, Desmond did not seem to quantify coordinates as distances in the pre-interview; instead he demonstrated a conception of coordinates as numerical values with which he could operate arithmetically. Unlike Willow, Desmond also seemed to conflate labels of points with coordinates. In short, Desmond's reasoning with coordinates and points was limited in the pre-interview, and to the extent that his reasoning was productive, it would best be characterized as numerical reasoning rather than quantitative reasoning.

In the post-interview, Desmond seemed to exhibit more sophisticated quantitative reasoning by quantifying unknown coordinates as distances in the plane and forming quantitative relationships with those quantities, which suggests he experienced a shift in his quantitative reasoning. I present a careful analysis of Desmond here because (a) his initial conceptions about coordinates and points differed from Willow's points-as-location conception and (b) the quantitative reasoning he demonstrated in the post-interview varied from that which Willow demonstrated.

Coordinates as values. In the pre-interview, Desmond seemed to struggle to quantify coordinates of points as distances. These struggles manifested in two distinct ways. First, Desmond appeared to think of coordinates as values rather than distances. For example, consider the following exchange from the pre-interview in which Desmond explained how he viewed coordinates. Desmond had done some preliminary work on the Ellipse Task, including labeling the vertical leg of the right triangle as $x - h$, replicating the triangle in a blank area of the paper, and labeling the vertical leg of that triangle $x - h$ and the horizontal leg of that triangle $y - F_1$. This initial work by Desmond is shown below in Figure 4.11. In the following transcript, notice how Desmond points to the coordinates x and h , changes those coordinates to numerical values, and then calculates the arithmetic difference of those values.

- Interviewer: So, can you talk me through this x minus h part [points to the $x - h$ Desmond had written, annotated with a circle in Figure 4.11 below]?
- Desmond: Well, um, at least I'm assuming this x [points to the x in (x, y) printed on the paper, annotated with a circle in Figure 4.11 below] is at a higher value than this h [points to the center (h, k) printed on the paper, annotated with a circle in Figure 4.11 below].

You can probably plug it in with—I'll just make it really different [writes a 10 above the x and a 5 above the h in the expression $x - h$ circled in Figure 4.11 below; these inscriptions are visible in Figure 4.12 below]. You know you have ten minus five, the remaining distance [points to the bracket he drew near the vertical leg of the right triangle, indicated by an annotated arrow in Figure 4.11 below] should be five essentially, like I could call this.

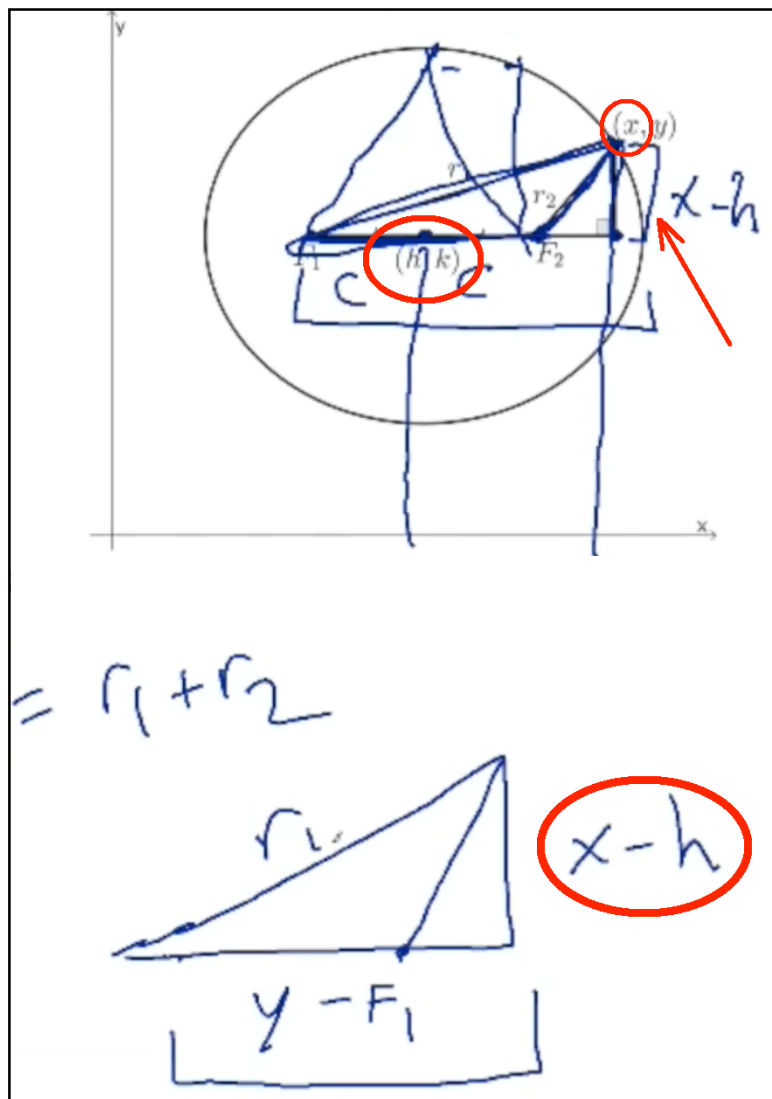


Figure 4.11. Desmond's work and inscriptions to begin the Ellipse Task, as well as author's annotations to facilitate analysis.

Desmond responded to my question about $x - h$ by talking about the values of x and h . When he said the x was a higher value than the h , he did not point out distances or make gestures to indicate lengths; instead he pointed to the written labels on the paper. He also spontaneously changed the coordinates x and h to specific numerical values—10 and 5 respectively. His action is consistent with my interpretation that Desmond conceived of the coordinates as values rather than distances. Indeed, when I asked him where the 10 would be, he marked where he thought 10 would be on the x -axis, as shown in Figure 4.12 below. Even with a more pointed question about “tracing” the 10 with his finger, Desmond did not indicate that he conceived of the 10 as a distance:

- Interviewer: So in this graph, where would the ten be? Can you trace it with your finger?
- Desmond: Yeah, the ten would be— [labels the x -axis with a 10, annotated with a square in Figure 4.12 below]
- Interviewer: Ok, and then the five?
- Desmond: [Labels the x -axis with a 5, annotated with a square in Figure 4.12 below]

Eventually, Desmond realized that he was using the x -coordinates to determine the length of the vertical leg of the right triangle and said he “flip-flopped” the x and y coordinates. He replaced the $x - h$ with $y - k$, and made marks along the y -axis with labels of 10 and 5 (these inscriptions are visible in Figure 4.12 below). Shortly after that, the following exchange took place:

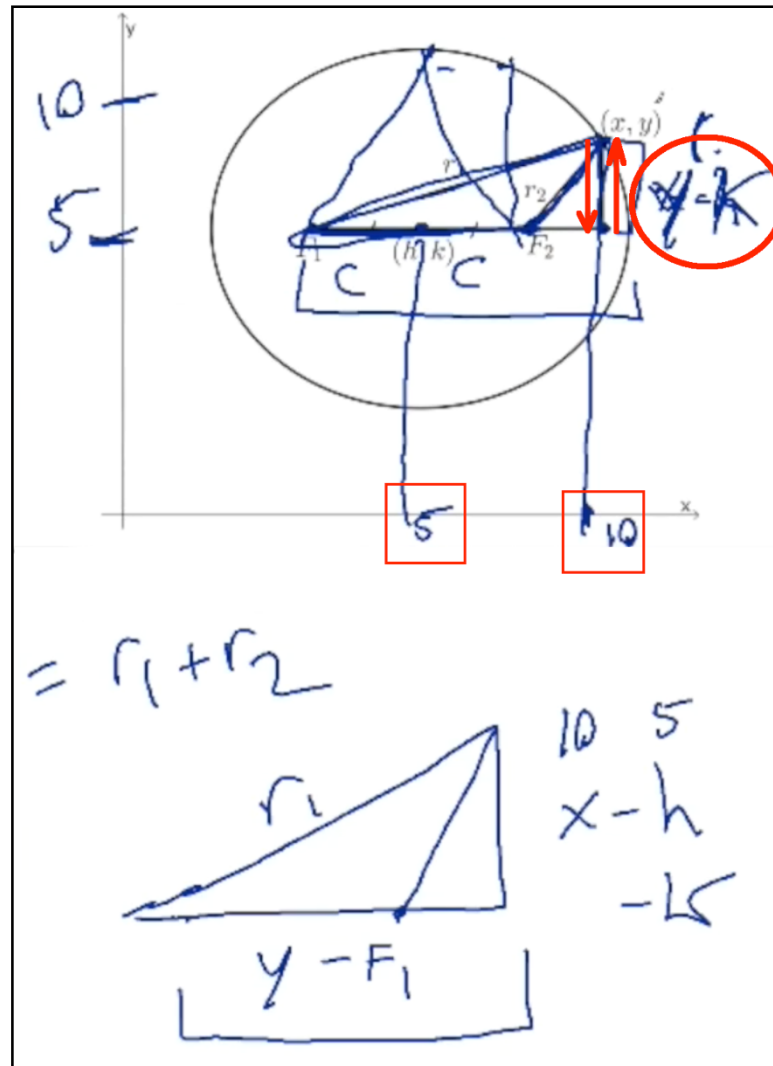


Figure 4.12. Desmond's corrected labels, along with author's arrow annotations of gestures and prior labels.

Interviewer: So, this distance [gestures back and forth along the vertical leg of the right triangle, annotated with arrows in Figure 4.12 above] you said is y minus k . And can you just show me the y ? Where do you see the y ? The distance of y ?

Desmond: Uh, **it's more of a value** [emphasis added; points to the printed y -coordinate of (x, y) , annotated with a circle in Figure 4.13 below], since it matches up with here [gestures with his pen horizontally from (x, y) to the y -axis, annotated with an arrow in Figure 4.13 below; then he marks the y -axis, labels it 10, then scratches out the previous mark he made].

And, you know, you basically have the top [makes a mark above the y -coordinate in (x, y) , annotated with a rectangle in Figure 4.13 below] and the bottom [makes a mark below the y in the expression $y - k$ annotated with a rectangle in Figure 4.13 below] and what's left over is the middle [draws a bracket between the two marks he made].

Interviewer:

And so the k is?

Desmond:

Right there [emphasis added; points to the k in the printed (h, k) on the paper, annotated with a circle in Figure 4.13 below].

Interviewer:

Right there.

Desmond:

Which is perpendicular to that line [gestures along the horizontal base of the triangle].

Interviewer:

Ok, so when you subtract them you get?

Desmond:

This value [emphasis added; draws a 5 next to the vertical leg].

Interviewer:

That value there. Great, I think I understand what you're saying.

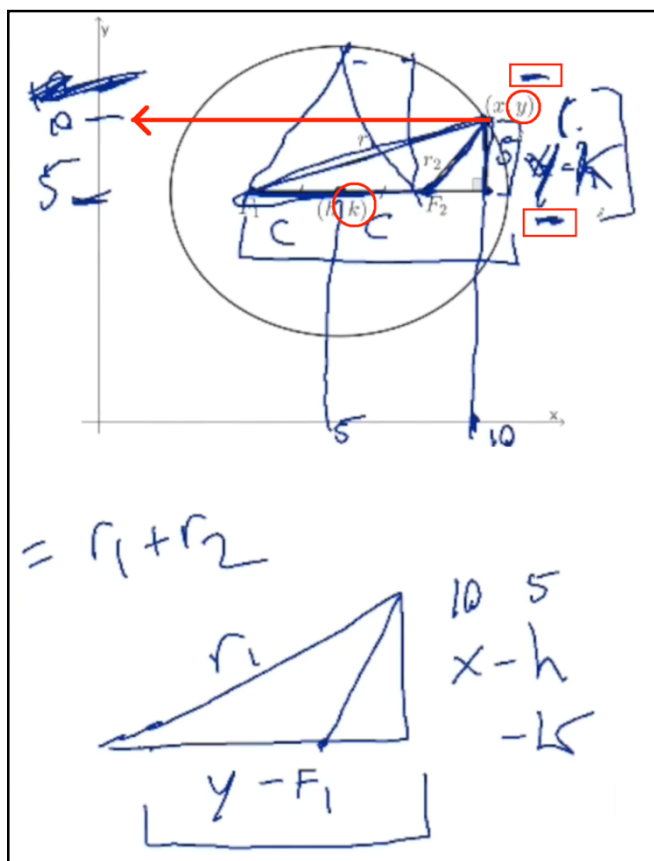


Figure 4.13. Author's annotations of Desmond's work in the Ellipse Task.

In this exchange, there are three pieces of evidence that suggest Desmond conceived of coordinates as values. First, I asked him to show me where the y was on the graph, and even used the phrase “the distance of y .” Desmond’s response, “Uh, it’s more of a value,” seemed to indirectly refute the notion that y indicates a distance, and is instead a value. Later I asked Desmond where the k was, and he pointed to the printed k in the label for the center (h, k) . This suggests that Desmond did not see k as a distance, but rather only as a value, namely the one printed on the paper. On the other hand, Desmond could have taken my question literally and showed me where the label k was printed on the paper. Finally, despite my many uses of the word “distance,” Desmond did not refer to quantities in the plane as distances. Instead he referred to the vertical leg as “what’s left over,” “the middle,” and “this value.”

Even though there is a plausible alternative interpretation that Desmond took my question of “Where is k ” literally, the totality of Desmond’s actions in this event are consistent with my claim that he conceived of coordinates as values. The strongest evidence of this claim is that in response to my direct query to locate y as a distance, Desmond dismissed the notion of y as a distance and instead said, “It’s more of a value.”

Conflating labels of points with coordinates. Desmond also seemed to conflate the labels of points with coordinates when he attempted to find lengths for the horizontal legs of the right triangles. Just prior to the two exchanges detailed above, Desmond attempted to find the length of the horizontal leg of the large right triangle.

In his statement below, Desmond treated the label for a point in the plane (F_1) as a value.

Desmond: This thing would be [draws a bracket from F_1 to the vertex of the right angle; this can be seen in Figure 4.13 above] y minus F -one [writes $y - F_1$ under the triangle he drew, this can be seen in Figure 4.13 above], er, yeah. It would be this whole thing [draws a bracket under the $y - F_1$ he just wrote, this can be seen in Figure 4.13 above].

As mentioned before, Desmond later realized he had used x coordinates for vertical lengths and y coordinates for horizontal lengths, and so he corrected some of his earlier work. Eventually he corrected the expression $y - F_1$ he had written by changing the y to an x so that the expression was $x - F_1$. However, when he did this, he did *not* change the F_1 in the expression. This stands in contrast with his earlier work in which he “flip-flopped” the coordinates; once he recognized his error, he changed the $x - h$ to a $y - k$. The fact that he did not change the F_1 could indicate that he took the label to be a value he could operate with, instead of conceiving of it as a point for which the coordinates indicate distances in the plane.

Later, Desmond wrote the expression $x - F_2$ to describe the horizontal leg of the smaller right triangle, while saying “I guess we could also say our x -value minus F_2 .” Desmond’s use of the labels of points in the plane as values can be taken as evidence that he conceived of F_1 and F_2 as values that can be operated on, much like the x and y coordinates. In other words, Desmond seemed to treat the point F_1 and F_2 like how he treated the coordinates y , x , k , and h , namely as values.

Quantifying coordinates as distances in the plane is a prerequisite for forming quantitative relationships with those quantities. Even though Desmond used terms such as $y - k$ and $x - F_2$, I found no evidence that he quantified those coordinates as distances. Instead, his reasoning seemed driven by his numerical conceptions of coordinates and labels of points. Because Desmond did not quantify coordinates as distances in the pre-interview, it is reasonable to conclude that Desmond did not form quantitative relationships with distances in the plane. In contrast to this, Desmond did form quantitative relationships with distances in the post-interview, which represents a shift in Desmond's quantitative reasoning from the pre-interview to the post-interview.

Before proceeding with analysis from Desmond's work in the post-interview, it must be noted that as I observed Desmond as he worked on the Parabola Task in the post-interview it became obvious to me that he was rushing through the task. Consequently, I did not think that his work provided enough evidence for me to make claims about his quantitative reasoning, so I asked him to also complete the Ellipse Task from the pre-interview. His work in the post-interview on both the Parabola Task and the Ellipse Task did provide enough evidence for me to make claims about his quantitative reasoning. In the following paragraphs, when I use evidence from his work on the Ellipse Task, it is taken from his work on this task in the post-interview, not the pre-interview.

Forming quantitative relationships. To begin the Parabola Task, Desmond drew in several lines and segments and labeled them (e.g., $y - k$ and $x - h$, as shown below in Figure 4.14). He did much of this work in silence. Eventually, I asked

- Interviewer: What is x ?
- Desmond: The whole thing [puts his left forefinger on the intersection of the y -axis and the directrix, and his right forefinger further out on the directrix; annotated with upward pointing fingers in Figure 4.15 below].
- Interviewer: That's x . And where is h ?
- Desmond: h is [keeps his forefingers spread and moves them slightly above the paper] the point there [points to a point on the directrix; annotated with a square in Figure 4.15 below].
- Interviewer: And then the x minus h is?
- Desmond: Like if this was eight [points to the directrix, annotated with a circle in Figure 4.15 below] and this was five [points to a different point on the directrix, annotated with a square in Figure 4.15 below] it would be [using his left forefinger and the pen, he makes a span gesture from the two points he just pointed to on the directrix] the distance of three.

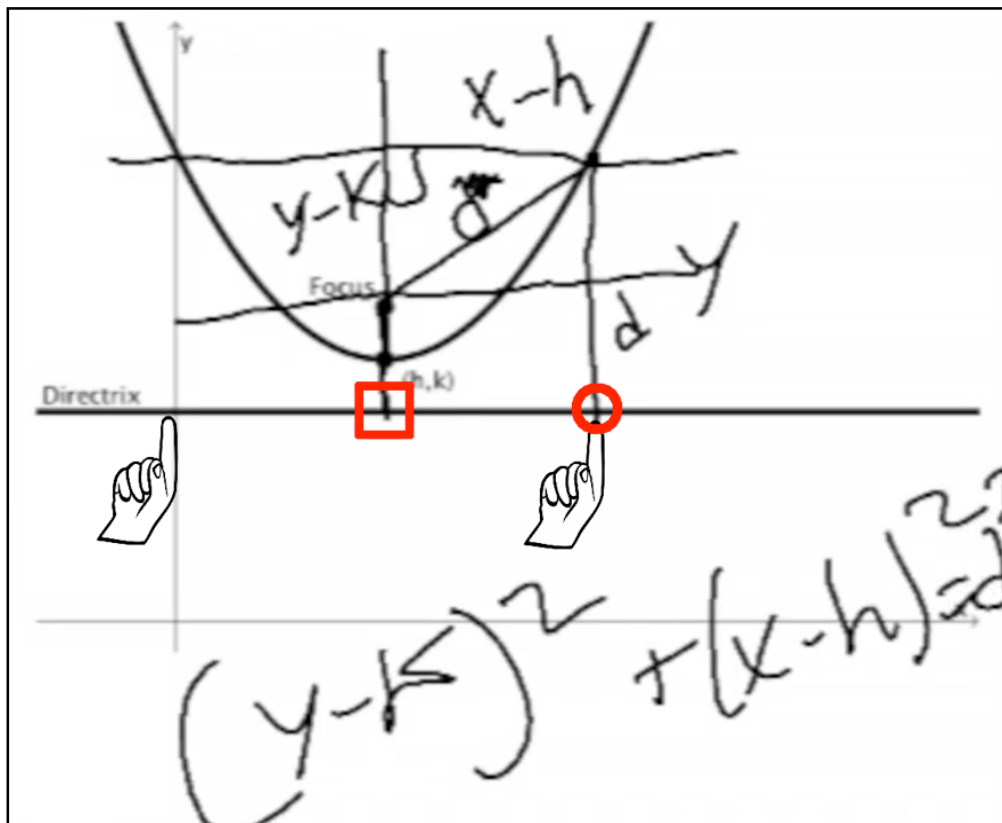


Figure 4.15. Author's annotations showing Desmond's gestures and the points he indicated.

In this exchange, Desmond formed the quantitative relationship $x - h$ with the quantities x and h . First, Desmond seemed to quantify the coordinate x as a distance. When I asked him “What is x ?” he responded by saying “The whole thing” and making a gesture with his two forefingers to indicate a span or a length along the x -axis. This behavior is consistent with quantifying a coordinate as a distance in the plane in that Desmond was using two fingers to simultaneously point out two points in the plane, thereby indicating a measurable segment in the plane (the span).

Desmond also quantified the coordinate h as a distance in the plane. In response to my question “Where is h ,” Desmond initially kept a spanning gesture and hovered over his paper, but he eventually pointed to a point on the x -axis vertically aligned with the vertex (annotated with a box in Figure 4.15 above). This could be taken as evidence that he did not quantify the coordinate h , and instead merely conceived of it as a value or a location. However, I think there is an alternate interpretation. Unlike in the pre-interview, Desmond did not point to the printed letter on the paper. Instead, he gestured to a point on the directrix that was vertically aligned with the vertex. In other words, he gestured to a point that had a distance of h from the y -axis. Moreover, I think that Desmond was answering my question literally. I asked him *where* h was, and he showed me where it was, rather than showing me *what* it was, as he did when he showed a distance of x , which was in response to my question of *what* x was. My questions were not consistent. Moreover, given Desmond’s previous propensity in the pre-interview for pointing to printed labels, the fact that he did *not* point to a printed label and had just quantified the coordinate x , supports an

alternative interpretation that he could have quantified h as a distance, but my questioning was not sufficient to elicit that quantification.

When I asked Desmond about $x - h$, he changed the coordinates to numbers, as he did in the pre-interview. However, this time he used the change to illustrate a “distance,” a word he did not seem to want to use in the pre-interview, despite my repeated use of it. Desmond’s description of $x - h$ as a “distance” can be taken as evidence that he had quantified that expression as a distance, which is also evidence that he potentially quantified the coordinate h as a distance. In summary, Desmond quantified the coordinate x , potentially quantified the coordinate h , and quantified the expression $x - h$ as a distance.

Finally, Desmond seemed to perform a quantitative operation on the quantities x and h . His first response to my question “Where do you see x minus h ,” was to talk about “ x plus h ” and to create a spanning gesture with his two fingers. He then moved his fingers closer together when he said, “ x minus h goes to here.” This movement can be taken as evidence that Desmond was perhaps performing the quantitative operation “take away” by indicating that a part of the segment could be removed to obtain a shorter segment, $x - h$.

Taken together, this evidence is consistent with an interpretation that Desmond formed a quantitative relationship with x , h , and $x - h$. However, his replacement of variable with values may indicate a lingering coordinates-as-values conception, although I found no other disconfirming evidence that supported that conclusion. Moreover, coordinates can take on specific values, and Desmond could have been

replacing the unknowns with numeric values as a way of providing an example for how he was thinking about $x - h$. Because he later seemed to form another quantitative relationship (which I discuss below), it is reasonable to conclude that Desmond did form the quantitative relationship $x - h$ after quantifying the coordinates x and h as distances.

Desmond also formed a quantitative relationship as he worked on the Ellipse Task in the post-interview. Consider this exchange, in which Desmond seemed to quantify the coordinates y and k and form the quantitative relationship $y - k$.

- Interviewer: So, where do you see the y minus k ?
- Desmond: The distance of y is this whole line [gestures with his pen vertically from (x, y) to the x -axis, as annotated below in Figure 4.16].
- Interviewer: Uh huh.
- Desmond: The distance of k is that line [draws a vertical segment and labels it k ; annotated with an oval in Figure 4.16 below].
- Interviewer: Ok. So then the y minus k is from where to where?
- Desmond: Here [points to the $y - k$ he wrote on his page earlier, annotated with a rectangle in Figure 4.16 below].
- Interviewer: Ok.
- Desmond: And that was [gestures with his pen vertically up and down along the vertical leg of the right triangle]—
- Interviewer: That distance?
- Desmond: That distance.

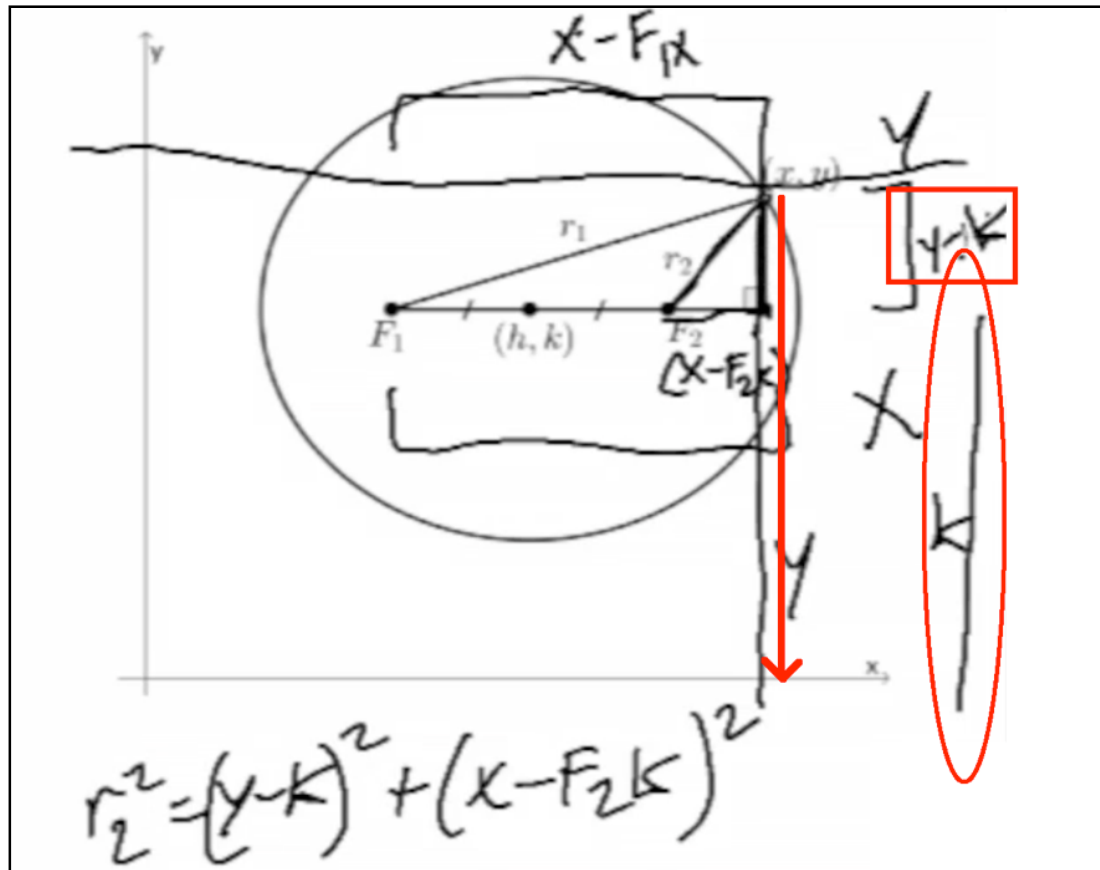


Figure 4.16. Desmond's work on the Ellipse Task and the author's annotations of Desmond's gestures and drawings.

The previous transcript evidence begins with me asking Desmond where he sees $y - k$. Instead of answering that directly (perhaps by taking the question literally, as he seemed to earlier in the post-interview), Desmond spontaneously quantified y and k , which is noteworthy considering his work on this task in the pre-interview. Desmond's gesture along a vertical line from (x, y) to the x -axis while saying "The distance of y is this whole line" suggests that he quantified the coordinate y as a distance. This stands in contrast to the pre-interview in which he contradicted the notion of y being a distance by saying "Uh, it's more of a value." After he quantified y , he drew a vertical segment to the x -axis while saying, "The distance of k is that

line.” By inscribing the line, Desmond made clear that he was not referencing a value or a printed label, as he did in the pre-interview. Instead, Desmond seemed to be marking off a distance in the plane with a measure of k ; in other words, Desmond quantified the coordinate k as a distance.

At the end of this exchange I made two moves that could have been leading. First, I said to Desmond “The $y - k$ would be where to where?” I also finished his statement when he gestured along the segment: Desmond started saying “And that was,” and I interrupted and said, “That distance.” These two moves may have affected what Desmond said and did afterward. Even so, he did gesture along the vertical leg of the right triangle, which could indicate he had quantified that segment as a distance. Desmond seemed to quantify the coordinates y and k as distance, and used similar gestures to indicate $y - k$ as a distance, and these actions are consistent with actions taken when forming quantitative relationships.

Sierra

Unlike Desmond and Willow, who both seemed to demonstrate very little quantitative reasoning in the pre-interview, Sierra was able to quantify segments in the plane as distances in the pre-interview. However, she did not quantify coordinates as distances, nor did she seem to form quantitative relationships. From the pre- to post-interviews, Sierra apparently experienced a shift in the sophistication of her quantitative reasoning. In the post-interview, not only did Sierra quantify coordinates as distances, but she also appeared to form first and second-order quantitative relationships.

Coordinates as values used to calculate distances. In the pre-interview, Sierra seemed to conceive of segments in the plane as distances. To start the task, she drew a bracket along the two segments from F_1 to F_2 , labeled each of those two segments c , traced the smaller triangle, and drew circles next to the legs of that triangle (it was unclear if those were labels). She did all of this silently, and her initial work is visible in Figure 4.17 below. In the following exchange, notice how Sierra formed the expression $y - k$ and discussed it as a “distance” or a “length.” She even used gestures to indicate which segment in the plane had a length of $y - k$. However, she did not appear to conceive of either coordinate y or k as distances in the plane. Instead she demonstrated a conception of the endpoints of the segment as values, which allowed her to calculate the segment’s length. Consequently, Sierra did not seem to form the quantitative relationship $y - k$.

- Sierra: This distance right here [gestures with her pen along the vertical leg of the right triangle; annotated with a circle in Figure 4.17 below] is this point [gestures with her pen toward the point labeled (x, y) ; annotated 1 in Figure 4.17 below] minus this point [gestures with her pen toward the vertex of the right angle; annotated 2 in Figure 4.17 below], which is some y minus k [writes the expression $y - k$ next to the line segment].
- Interviewer: And where do you see the k ?
- Sierra: Right here [points with her pen to the printed label k in the point (h, k) ; annotated with a square in Figure 4.17 below].
- Interviewer: Ok.
- Sierra: This is h . And over here is my x [as she says this, she makes two marks on the x -axis and labels them h and x ; these can be seen in Figure 4.17 below].
- Interviewer: Ok.
- Sierra: So up here is my y and right here is my k [as she says this, she makes two marks on the y -axis and

- labels them y and k ; these can be seen in Figure 4.17 below].
- Interviewer: Ok. And so just so that I am crystal clear, how is this y minus k ?
- Sierra: Because it's the distance.
- Interviewer: Mmhmm.
- Sierra: Yeah, distance is, like the highest number [gestures in a circle around the point annotated with a 1 in Figure 4.17 below] minus the lowest number [gestures in a circle around the point annotated with a 2 in Figure 4.17 below] will give you the total distance here [gestures with her pen back and forth between those two points].
- Interviewer: Ok. And so where do you see the y ?
- Sierra: Right here [points with her pen to the printed label y in the point (x, y) ; annotated with a square in Figure 4.17 below].

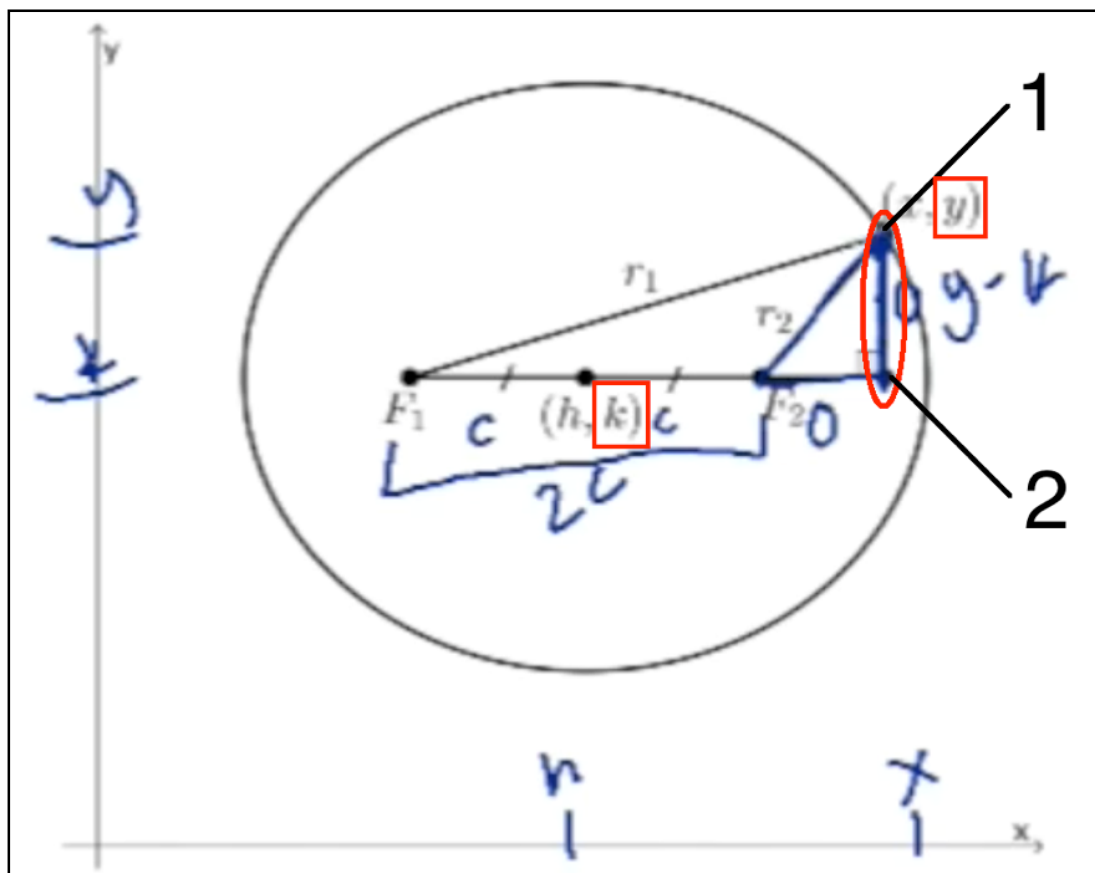


Figure 4.17. Sierra's initial work, along with the author's annotations of Sierra's gestures.

Sierra conceived of the vertical leg of the right triangle as a distance. She gestured back and forth along the leg and called it a distance, both in response to my initial question and later when she said, “the highest number minus the lower number will give you the total distance here.” However, quantifying the leg of a triangle as a distance does not mean that Sierra had formed a quantitative relationship with coordinates quantified as distances in the plane. In particular, she did not quantify the coordinates y and k as distances. When I asked her where she saw these, she pointed to the printed labels for each. She also said that $y - k$ is a distance because it is the “highest number minus the lowest number,” which suggests Sierra conceived of the coordinates as values, but not necessarily as distances. Indeed, I asked her twice more where she saw the y and the k , and both times she pointed to the printed label or the point itself.

This was typical of the kind of reasoning Sierra seemed to demonstrate in the pre-interview. Even though she appeared to have a conception of “distance” that was linked with segments in the plane, at no point in the pre-interview did she explicitly link coordinates to distances. Instead, she conceived of the coordinates of the points as values that allowed her to calculate the distances of segments in the plane.

Forming quantitative relationships. I now present two episodes from Sierra’s work in the post-interview in which she appeared to quantify coordinates as distances and form quantitative relationships with these distances. Early in her work on the Parabola Task, Sierra had made several inscriptions on her paper and labeled some of the distances. She also conjectured that Sasha and Keoni would have given

the height of the directrix above the x -axis a specific value, for example 3. She then labeled the segment from (x, y) to the directrix as $y - 3$, which can be seen in Figure 4.18 below. In this exchange, Sierra explained how she thought about $x - h$.

- Sierra: Since this is h units on the x -axis [gestures along the directrix from where the directrix intersects the y -axis to a point roughly below the vertex; annotated with an arrow labeled 1 in Figure 4.18 below].
- Interviewer: Yeah.
- Sierra: So it's, this is the same thing, this would be h units [gestures in a circle around the point annotated with a circle in Figure 4.18 below].
- Interviewer: Mmmhmm.
- Sierra: And if I say this whole thing is x [gestures horizontally from the y -axis to (x, y) , annotated as 2 with an arrow in Figure 4.18 below], this distance [gestures with her pen along the horizontal leg of the right triangle] is x minus h .

Sierra seemed to quantify the coordinate x as a distance when she called it “the whole thing” as she gestured along a horizontal segment. Additionally, her use of the phrase “the whole thing” suggested that Sierra viewed x as being something in the plane other than a value. Similarly, she also appeared to quantify h , as evidenced by her gesture along a horizontal segment while saying it would be “ h units.” Sierra then said, “**If** I say this whole thing is x , this distance is $x - h$ ” (emphasis added). The use of the word “if” suggests that the forming of $x - h$ depended on quantifying x ; in other words, $x - h$ is the result of a relationship that includes the quantity x . This evidence suggests that Sierra formed the quantitative relationship $x - h$ with the quantities x and h .

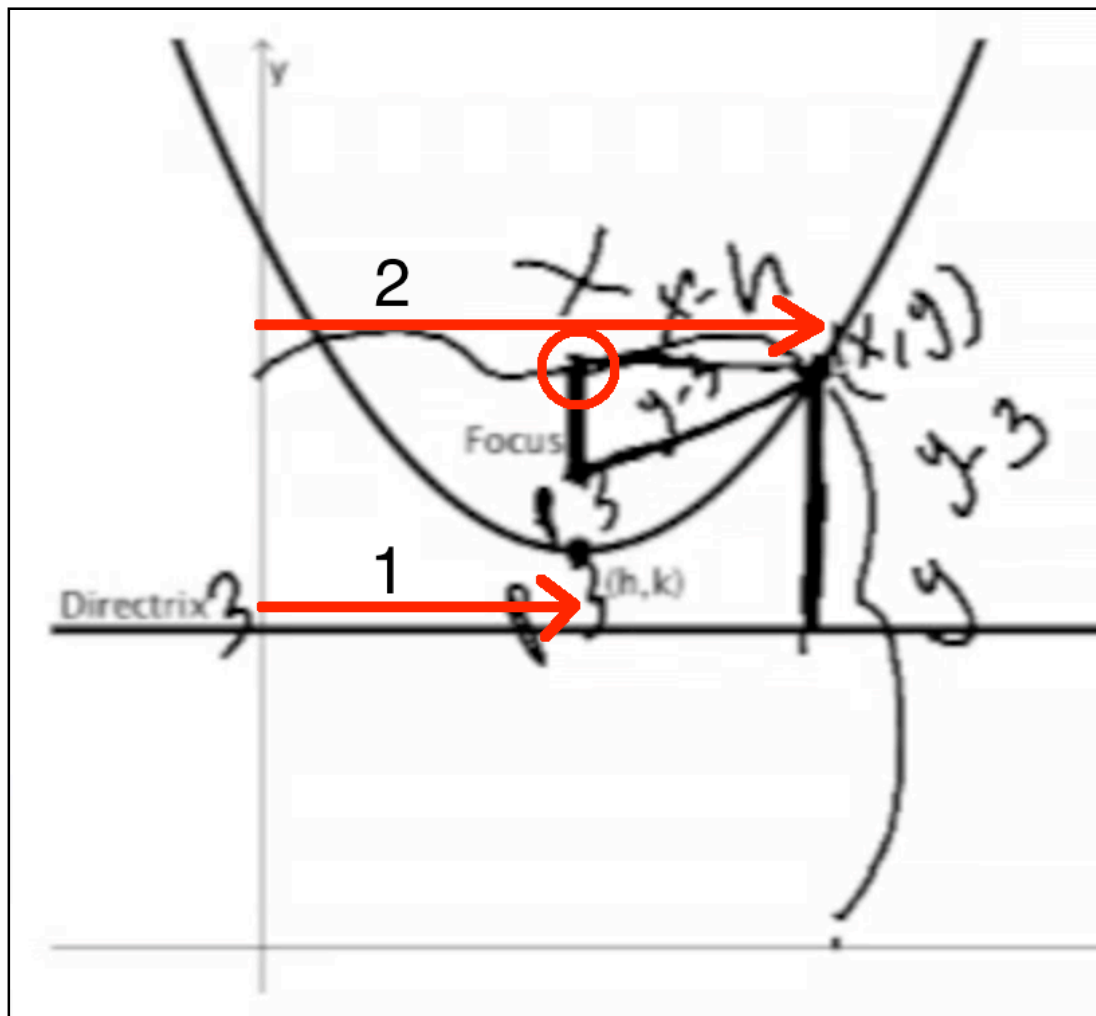


Figure 4.18. Sierra's work and author's annotations of her gestures.

Later during her work on the Parabola Task (approximately ten minutes after the previous exchange), Sierra seemed to form the second-order quantitative relationship $y - k + p$ with the quantities $y - k$ and p . The quantity $y - k + p$ is one way of conceiving of the length of the segment extending from the point (x, y) to the directrix (which is comprised of both the blue and red segments in Figure 4.19b below).

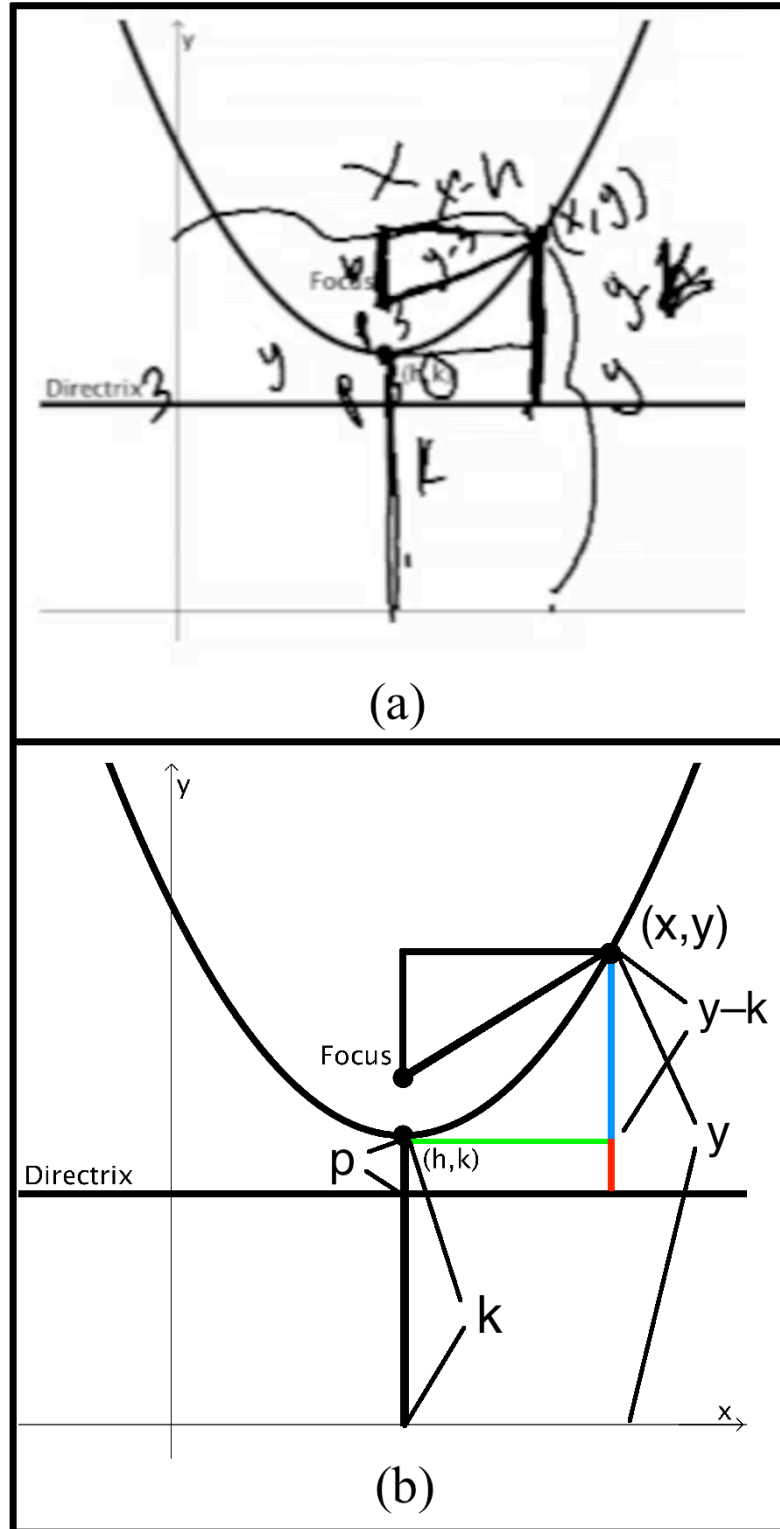


Figure 4.19. (a) Inscriptions Sierra made as she seemed to form a second order quantitative relationship and (b) Author's recreation of the same.

First, Sierra seemed to form the quantitative relationship $y - k$ in a manner consistent with how she formed the quantitative relationship $x - h$ in the previous episode. She quantified the coordinates y and k , using language and gestures in ways similar to when she quantified x and h . For example, she said “This is k all the way” while gesturing along the segment labeled k in Figure 4.19b above. She then said she wanted “this distance” while gesturing along the segment from the point (x, y) to the directrix, and “not this distance” while gesturing along the blue segment I have labeled $y - k$ in Figure 4.19b above. This suggests that she had quantified that segment as a distance, and is consistent with her earlier actions in which she formed the quantitative relationship $x - h$. In other words, Sierra formed the quantitative relationship $y - k$.

Sierra said she did not want the distance $y - k$, so I asked her if she could modify it. Sierra then seemed to form a second-order quantitative relationship by combining the quantities $y - k$ and p .

- Interviewer: So, could you modify it?
 Sierra: Modify this [gestures around the expression $y - k$ she had written]?
 Interviewer: Yeah.
 Sierra: So, it would be y minus k plus p [modifies her expression so that it is $y - (k + p)$]. This is, wait. No, no, y minus k without the parentheses, plus p [amends her expression so that it is $y - k + p$].
 Interviewer: Ok so can you show me that, the distances involved there, and explain how you got to that
 Sierra: So it would be y , this whole thing [gestures along from the point (x, y) to the x -axis]—
 Interviewer: Uh huh
 Sierra: —minus k [makes a span gesture with her middle finger and thumb; middle finger on the vertex (h, k) and thumb directly below on the x -axis]—
 Interviewer: Yeah

Sierra: —plus this distance [draws a small bracket near the segment that I have colored red in Figure 4.19b above], because I want this whole thing [gestures with her pen along the segment comprised of the blue and red segments in Figure 4.19b above].

Sierra originally modified the expression $y - k$ by writing $y - (k + p)$; however, without being prompted, she seemed to notice that something was not right about the expression. She said “No, no, y minus k without the parentheses, plus p .” Her use of the phrase “ y minus k without the parentheses” suggests that Sierra had formed a quantitative relationship $y - k$ with the quantities y and k . This was confirmed when I asked her to show me how she came up with her expression, and she first pointed out a segment y (“So it would be y , this whole thing”) “minus k .” She then went on to explain that because she wanted a certain segment (the one comprised of both the blue and red segments in Figure 4.19b above), she would want to add the distance of p (“plus this distance, because I want this whole thing”). In other words, Sierra seemed to form the second-order quantitative relationship $y - k + p$ with the quantities $y - k$ and p .

In summary, three participants experienced a shift in quantitative reasoning. Willow appeared to initially conceive of points as locations, and both she and Desmond seemed to conceive of points as values. Like Willow and Desmond, Sierra demonstrated a conception of coordinates as values, but her conception included using those values to calculate distances. All three participants shifted in their quantitative reasoning, with both Willow and Desmond not only quantifying coordinates as distances, but also forming quantitative relationships. Sierra’s quantitative reasoning

in the post-interview went even further, as she seemed to form second-order quantitative relationships.

Of the four remaining participants, Lily exhibited strong quantitative reasoning in both pre- and post-interviews but little noticeable difference in the level of sophistication of her reasoning. The other three participants, Marshall, April, and Jasper, also demonstrated some quantitative reasoning in the pre-interview, so their shifts were not as pronounced as those of Willow, Desmond, and Sierra. Instead, Marshall, April, and Jasper experienced shifts in their ability to adopt the quantitative reasoning of high school students. This shift, which I call a shift in point of view, is the subject of the next section of this chapter.

A Shift in Point of View (Decentering)

This section is divided into three parts. First, I discuss the construct of decentering and elaborate its use as an analytical tool for answering my first research question. Next, I carefully analyze Marshall's work on the Parabola Task in the post-interview to demonstrate what a shift in point of view looked like for my participants. Jasper's and April's shifts in point of view were like Marshall's, so in many ways Marshall's shift is representative of the group. Nevertheless, I conclude this section discussing more briefly the work of those participants to further elaborate this category.

Decentering

The construct of decentering was originally developed by Piaget to describe a child's actions as he or she considers thoughts, feelings, or perspectives that are

different from his or her own (Teuscher et al., 2016). Piaget and Inhelder (1967) extended this idea to mathematics by arguing that through decentering, a child is able to take on the visual perspective of another without physically changing locations (e.g., “imagining” how another individual perceives an array of geometric objects). In simplest terms, then, decentering is the process by which an individual foregoes his or her own understanding of a situation in order to attempt to understand the situation from the perspective of another individual (Teuscher et al., 2016).

Teuscher et al. (2016) argued that decentering is a key component in the formation of MKT. In particular, they linked decentering to the notions of first and second-order models. According to Steffe and Olive (2010), first-order models are “the models an individual constructs to organize, comprehend, and control his or her experience, i.e., their own mathematical knowledge” (p. 16). In other words, first-order models are the ways in which individuals understand mathematical situations. By contrast, a second-order model is an image of another’s ways of operating in and understanding of a mathematical situation (Teuscher et al., 2016). Teuscher et al. argued that decentering is necessary for the creation of second-order models; essentially one must first cast aside his or her own understanding in order to understand the situation from another’s perspective.

According to Silverman and Thompson (2008), reflecting on the ways in which students understand and come to understand mathematical ideas is necessary for the development of MKT. Consequently, the construct of decentering offers one way to think about how a teacher develops MKT: by turning away from his or her own

ways of operating in order to create second-order models of students' mathematical understandings (Teuscher et al., 2016).

During the post-interview, I wanted to explore the degree to which participants could take on the point of view of Sasha and Keoni. My protocol included a follow-up question to the Parabola Task, for which I asked participants how they thought Sasha and Keoni would solve the same task. Sasha and Keoni's method for solving the Parabola Task (which can be viewed in Lesson 9 of the parabola unit, currently available at <http://cpucips.sdsu.edu/website/parabolas-lesson-9.html>) closely follows the conceptual analysis of the Parabola Task that I elaborated in Chapter 3. It must be stressed that even though participants viewed several clips of videos that featured Sasha and Keoni working on problems dealing with parabolas, I did not show participants any clips from the lesson in which Sasha and Keoni solved the Parabola Task.

Three participants, Marshall, Jasper, and April, provide the strongest evidence for decentering because they changed their solution in response to the follow-up prompt to describe how Sasha and Keoni might solve the task. For the rest of this section I present evidence that supports my claim that those participants demonstrated an ability to decenter and consider a mathematical situation from the perspective of high school students. Because participants had no experience with Sasha and Keoni's ways of thinking prior to engaging in the mini-course, I did not ask them how they thought Sasha and Keoni would solve tasks in the pre-interview. Consequently, evidence for this claim comes exclusively from the post-interview.

Marshall

In this subsection, I first present evidence from Marshall's initial attempts to solve the Parabola Task (see Figure 4.1 above). This evidence will be used to help establish the claim that while his initial attempts included two different methods, each leading to a correct equation, those attempts were qualitatively different than Sasha and Keoni's method. I will then present evidence from Marshall's final attempt to solve the Parabola Task, which came about in response to my follow-up prompt to talk about how he thought Sasha and Keoni would solve the task. Marshall's final attempt was similar to Sasha and Keoni's method, which suggests that Marshall was able to decenter by turning away from his own understanding of the mathematical situation in order to elaborate his understanding of how Sasha and Keoni would understand the task.

Marshall's first two attempts were dissimilar from Sasha and Keoni's method. Marshall's initial solution of the Parabola Task featured a written equation which Marshall justified by leveraging his prior knowledge of the effect that translating a base parabola has on its equation. In this excerpt, which happened right at the start of Marshall's engagement with the Parabola Task, Marshall seemed to have little difficulty in expressing his understanding of the task and how it could be solved:

You have this distance here [draws a segment from the focus to the vertex, then another from the focus to the directrix, these are indicated by annotated arrows in Figure 4.20 below], that's going to show up and be p . You have a horizontal shift of h , a vertical shift of k , so it's going to look like, something like y equals x minus h squared over four p plus k [writes the equation $y = \frac{(x-h)^2}{4p} + k$].

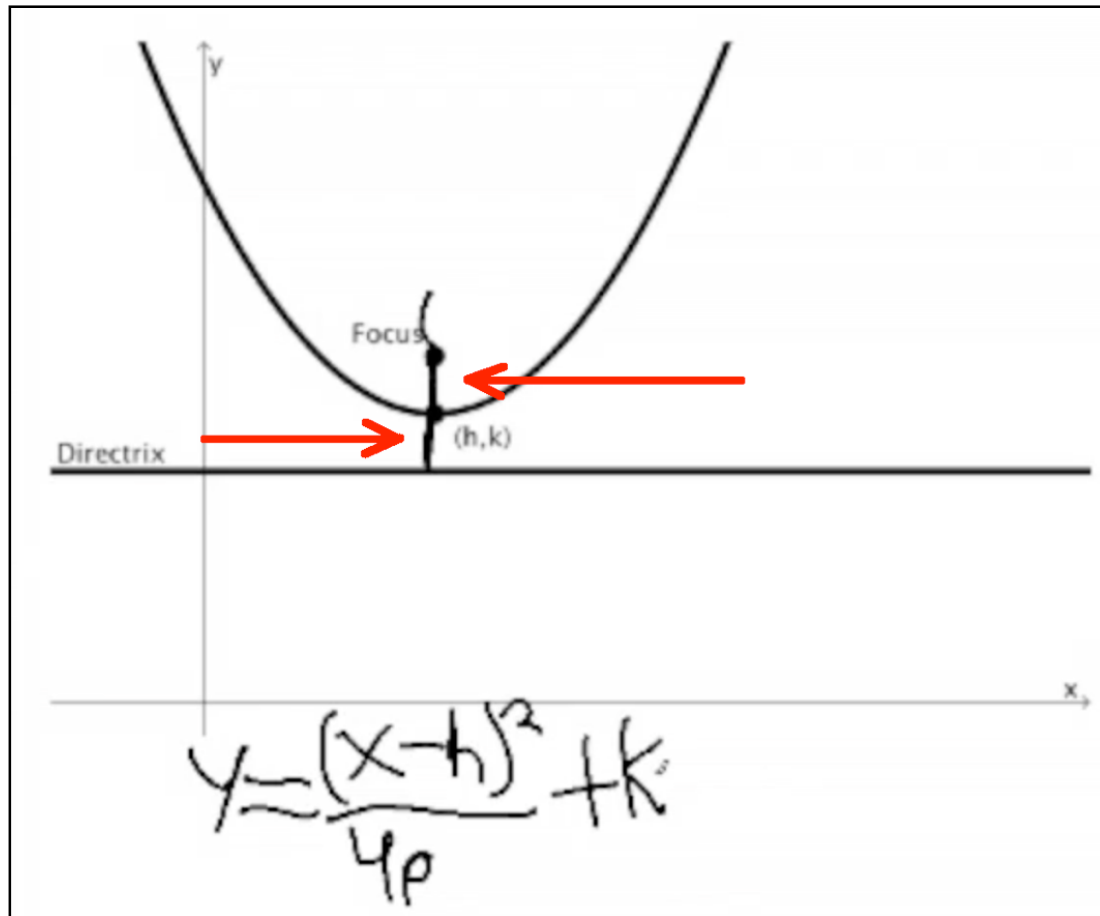


Figure 4.20. Marshall's drawings indicating the distance p (annotated with arrows) and the equation he wrote.

After quantifying the distance between the focus and the vertex as the distance p , which had been discussed at length during the mini-course, Marshall immediately noted that the parabola was shifted from having its vertex at the origin to having its vertex at (h, k) . After stating the two shifts he saw in the graph, Marshall wrote the equation for the parabola. A bit later, the following exchange took place in which Marshall admitted to not using the definition to solve the task.

Marshall: So, uh, I think that is the equation of the parabola, just based on the fact that I'm remembering this sort of general equation, applying these horizontal and vertical shifts to it, and that's the result that I get.

- Interviewer: So, could you point out where you used the definition to help you find that?
- Marshall: Um, I mean, I guess I didn't really use it directly, I just relied a lot on my prior knowledge and the idea that the definition gets you to this [points to the equation he wrote], yeah so. If you want to go just from, you could just start with the definition and like use the distance formula to do it, that might be a way that uses the definition more.

This first attempt by Marshall illustrates one of the ways that he demonstrated his understanding of the task. His prior knowledge about the general equation of a parabola as well as horizontal and vertical shifts seemed to enable him to develop an equation for the parabola.

Marshall's initial solution was qualitatively different from Sasha and Keoni's method. Sasha and Keoni's method for solving the Parabola Task involved their quantifying of several coordinates as distances and forming multiple quantitative relationships. They also leveraged the definition as well as the creation of a right triangle in order to use the Pythagorean Theorem to generate an equation. Although Marshall's method yields a correct equation, it involved quantifying only one distance (p , the distance from the focus to the vertex), did not leverage the definition, did not involve a right triangle in any way that was evident in the data, and made no use of the Pythagorean Theorem.

At the end of the previous excerpt, Marshall mentioned he could leverage the definition and the distance formula, which I encouraged him to try. He proceeded to solve the Parabola Task a different way, which I will now describe. He began by labeling the focus (x_1, y_1) and the directrix $y = y_2$. He then labeled a point on the

parabola (x, y) and said it was an “arbitrary point” (see Figure 4.21 below). Although he did begin by invoking the definition of a parabola, Marshall quickly turned to the distance formula and a series of algebraic manipulations:

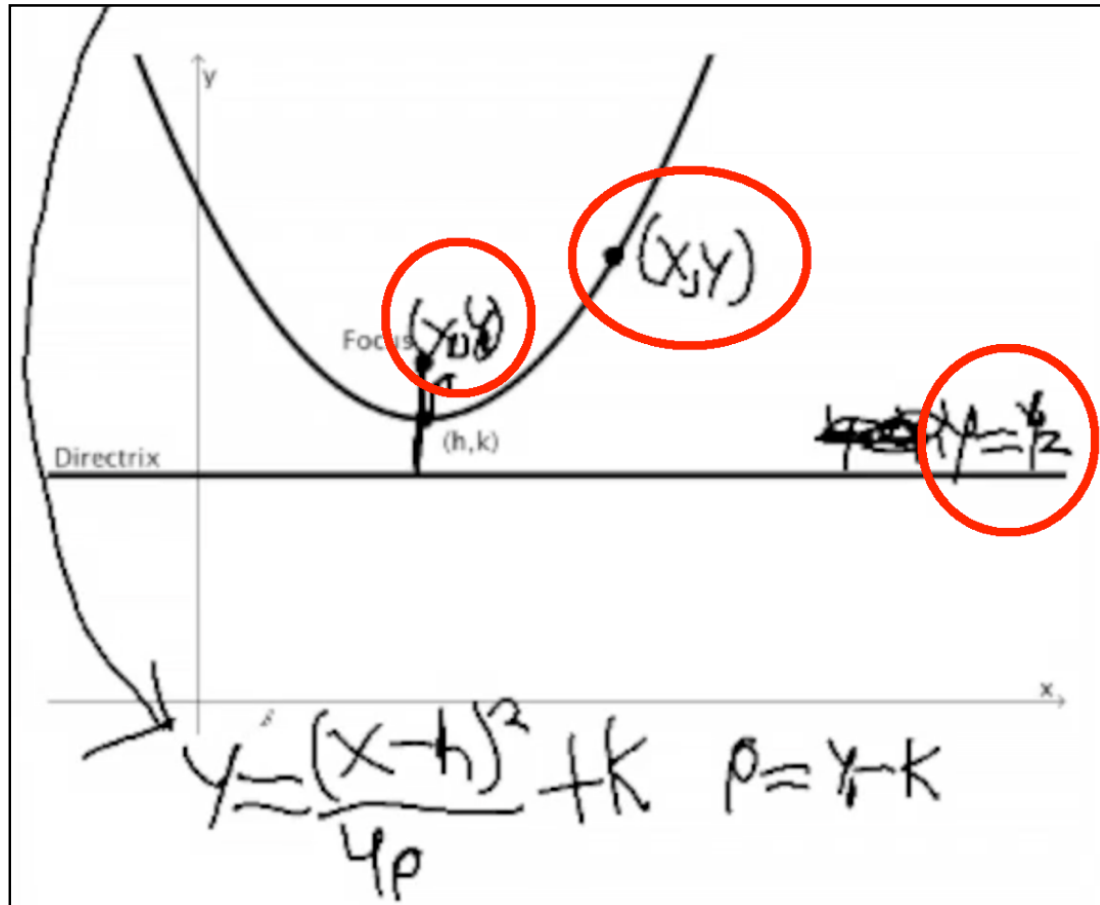


Figure 4.21. Marshall's inscriptions at the beginning of his second attempt, with labels annotated with circles.

Marshall:

You can use the definition to realize these two distances [draws two segments from (x, y) , one to the directrix and one to the focus, then marks them with tick marks; these are visible in Figure 4.22 below] have to be equal for it to be on the parabola. So then use the distance formula, of uh [trails off as he writes the equation $\sqrt{(x - x_1)^2 + (y - y_1)^2} = \sqrt{(y - y_2)^2}$; this can be seen in Figure 4.22 below]. That, and uh, I think you would eventually get to, if you square both sides and solve for just y you'll get

to somewhere that looks something like this equation [draws an arrow to the equation he wrote earlier, $y = \frac{(x-h)^2}{4p} + k$].

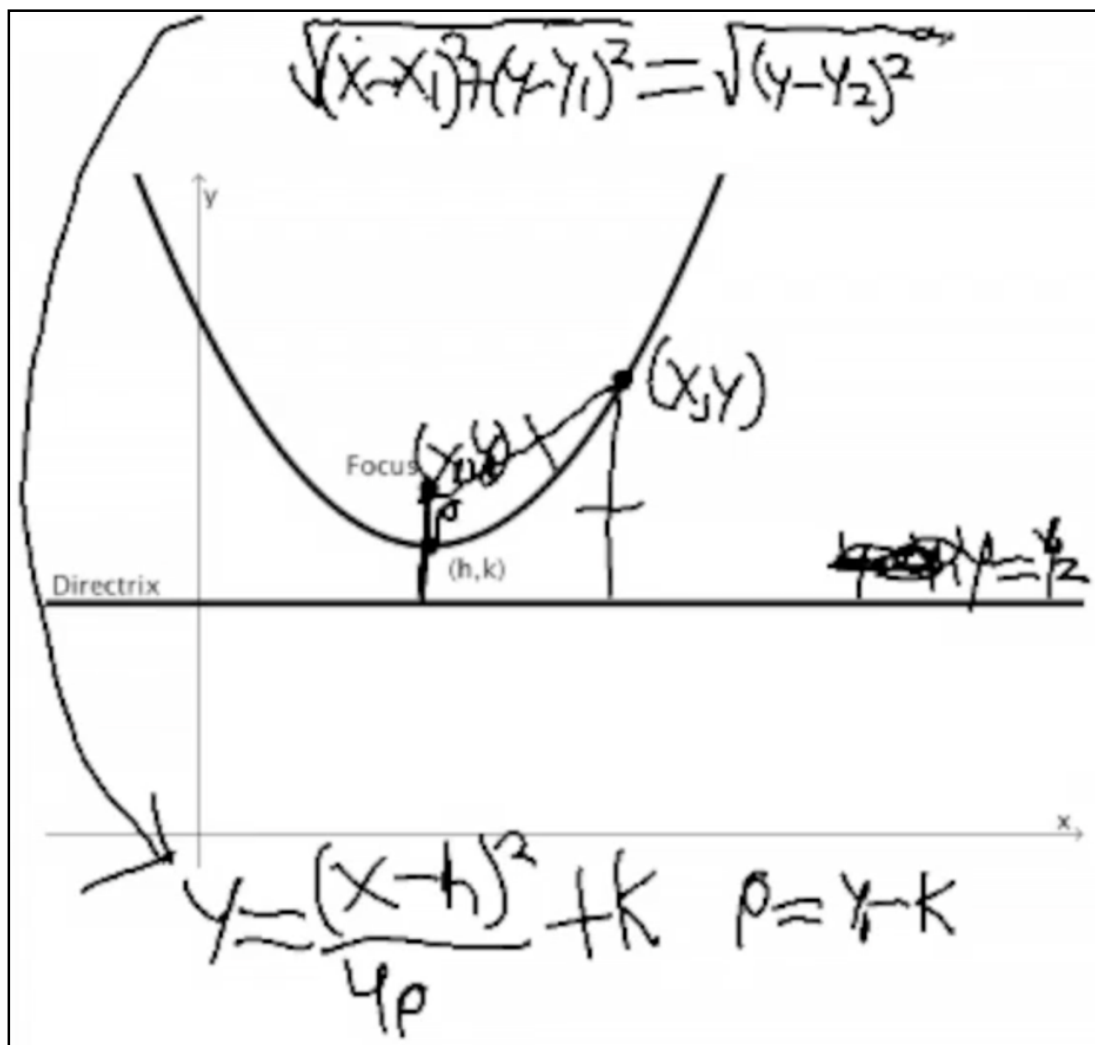


Figure 4.22. Marshall's work for his second attempt on the Parabola Task.

I asked Marshall where he thought the h , k , or p would show up in his second equation. Marshall responded by performing several algebraic calculations to describe relationships between the unknowns he identified (e.g., x_1 , y_1 , y_2 , and others) that would yield h , k , and p . For example, he noted that $2p$ would be the distance between

the focus and the directrix; so $= \frac{y_1 - y_2}{2}$. Marshall's equations describing the

relationships he elaborated can be seen in Figure 4.23 below.

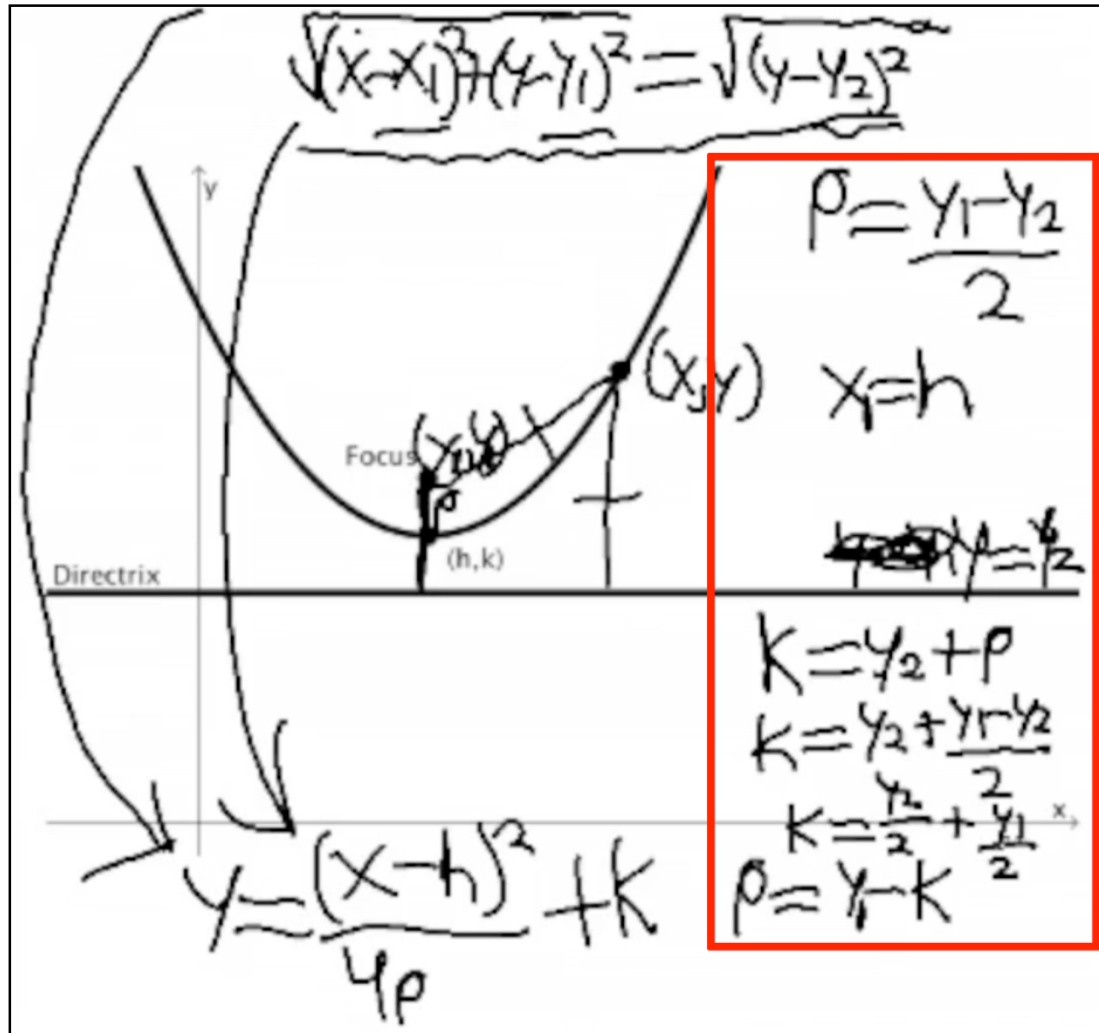


Figure 4.23. The annotated red box indicates the many equations Marshall used to describe relationships between the unknowns he identified and the parameters of the parabola h , k , and p .

Marshall's second attempt was mathematically accurate and produced a correct solution; however, like his first attempt, it was qualitatively different from Sasha and Keoni's. For example, Marshall did not quantify coordinates as distances nor did he seem to form quantitative relationships in order to develop his equation. Also, despite

leveraging the definition in a way that was like how Sasha and Keoni found two segments that had the same length emanating from an arbitrary point, Marshall did not build upon the definition or those segments. He did not form a right triangle nor did he use or mention the Pythagorean Theorem. Instead, he used the distance formula. Although the distance formula is mathematically equivalent to the Pythagorean Theorem when dealing with right triangles in the coordinate plane, it utilizes more complex notation and may be more difficult for high school students to understand or use. Consequently, high school students might not think that the two formulas are the same.

Finally, unlike Sasha and Keoni, who developed their equation by carefully analyzing the geometry of the situation, Marshall developed his equation by relying almost exclusively on algebra. Moreover, the connections he made between his algebraic equation and the geometry of the parabola were made through a series of complex algebraic maneuvers, and Marshall never seemed to genuinely connect the geometry of the parabola to his algebra.

Marshall decentered. The evidence presented previously demonstrates that Marshall had developed mathematical content knowledge that enabled him to think flexibly about the Parabola Task and approach it from different angles. Whether that knowledge developed as a result of his participation in the mini-course is not a claim I am making. However, the goal for the mini-course was for participants to develop MKT, not just mathematical content knowledge. I told Marshall that Sasha and Keoni

had solved the task in a lesson we did not watch in the mini-course, and I asked Marshall how he *thought* Sasha and Keoni would solve the Parabola Task.

I first present evidence that suggests that Marshall solved the task in a way that was similar to Sasha and Keoni's method. Then I present transcript evidence to show that Marshall had several ideas about what Sasha and Keoni would figure out and what they would struggle with as they solved the task. Taken together, this evidence supports the claim that Marshall decentered to take on the perspective of Sasha and Keoni.

Marshall's third solution was similar to Sasha and Keoni's method. Marshall began his third attempt by stating that Sasha and Keoni would pick an arbitrary point on the parabola and label it (x, y) , which he then did. He drew two segments from (x, y) , one to the directrix and the other to the focus and said, "They would be able to construct their triangle, since they used the Pythagorean Theorem for all of these." As he said this, he drew a right triangle on the graph. This initial work can be seen in Figure 4.24 below.

Comparing this initial work to Sasha and Keoni's solution (see Figure 4.25 below), there is already evidence that suggests Marshall decentered. For example, Marshall drew segments from (x, y) which he later used in forming a right triangle, which is similar to what Sasha and Keoni did. Moreover, despite having never seen the videos in which Sasha and Keoni solve the Parabola Task, Marshall justified drawing and using a right triangle by noting that Sasha and Keoni would attempt to

use the Pythagorean Theorem, which is how Sasha and Keoni approached the Parabola Task.

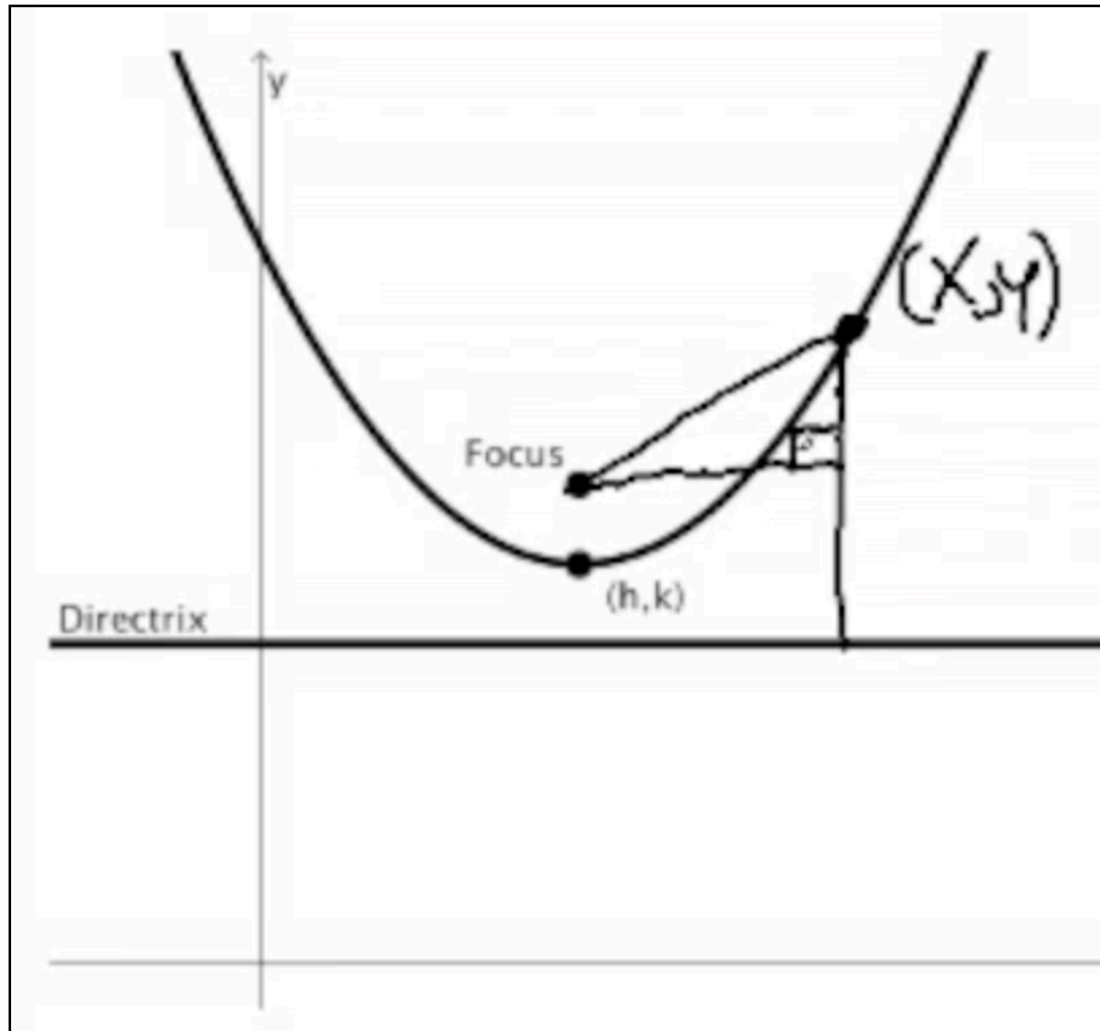


Figure 4.24. Marshall's initial work for his third attempt. He has drawn a right triangle, which was missing from his first two attempts.

In fact, his drawing of a right triangle and mentioning of the Pythagorean Theorem is notable because he did neither of these in either of his two first attempts. This suggests that Marshall's initial ideas about the task did not necessarily include right triangles and the Pythagorean Theorem, but as he thought about how high school

students (Sasha and Keoni) might solve the task, he was able to think about the task from their perspective. In other words, Marshall appeared to decenter.

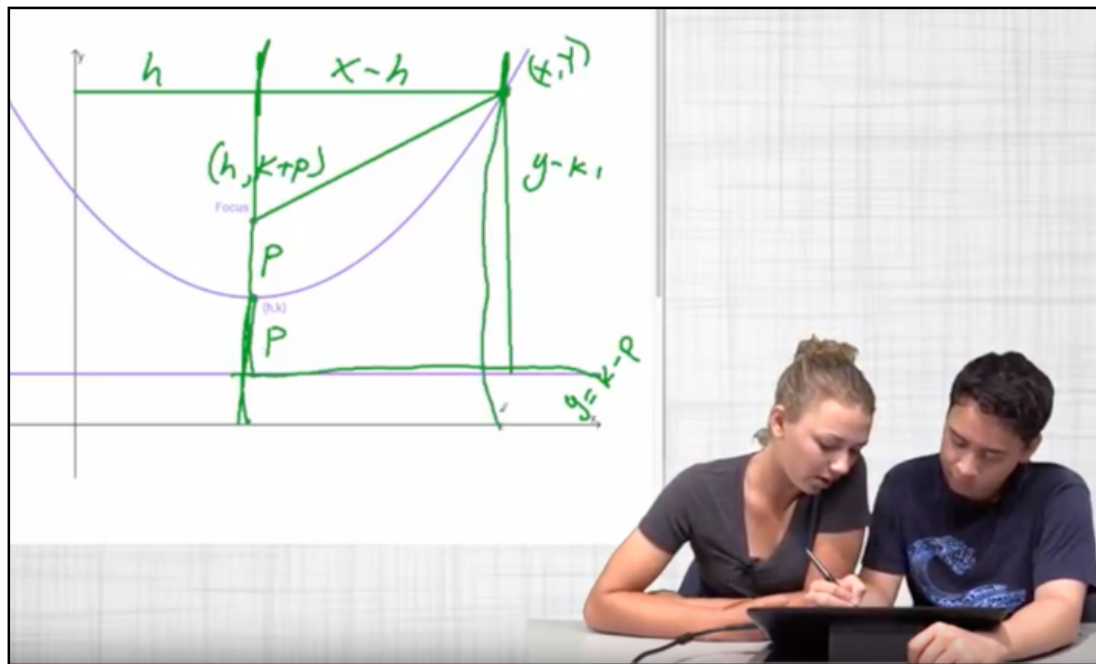


Figure 4.25. A screenshot from Lesson 9 of the MathTalk videos illustrating Sasha and Keoni's initial work on the Parabola Task.

Marshall continued his third attempt by explaining how he thought Sasha and Keoni would find the length of each side of the right triangle. His explanations leveraged the kind of quantitative reasoning Sasha and Keoni employed, including quantifying coordinates as distances and forming quantitative relationships with those quantities. For example, he talked about how Sasha and Keoni would find the vertical leg of the right triangle:

- Marshall: As they [Sasha and Keoni] look at that distance they would have the y minus k to get up to the vertex, and then they need to take away a little more, they need to take away p .
- Interviewer: Ok, so I'm going to ask you to just show me that on your diagram, and if you want a new one, I can get you a new one.

- Marshall: So we have y [draws a line segment off to the side, starting at a point that is roughly the same height above the x -axis as the point (x, y) , down to the x -axis and labels it y , recreated with a red segment in Figure 4.26b below] and then take away k from it [draws a bracket/line down part of the line he just drew and labels it k , recreated in blue in Figure 4.26b below].
- Interviewer: Ok. Where k is?
- Marshall: Oh, it doesn't look like it in the drawing, but it's where the vertex is [revises his diagram by drawing a short horizontal segment; recreated in green in Figure 4.26b below]
- Interviewer: Ok, so if that new line is there, then you would?
- Marshall: And then we have this little more, it's p [labels a segment p , see Figure 4.26 below]. So, to get up to y right here [makes a dot at the top of the line labeled y], we have this whole piece y [motions with his pen up and down along that line, which is the red segment in Figure 4.26b below], we take away k , and we take away p , and that gives us that side of the triangle.

This episode serves as evidence to support the claim that Marshall took on the perspective of Sasha and Keoni. At the beginning of the episode he talked about what “they” would need to do, where the “they” was referencing Sasha and Keoni. As he explained the quantitative reasoning that yields $y - k - p$, he seemed to be taking Sasha and Keoni’s perspective through his repeated use of the word “we.” His explanations regarding how he thought Sasha and Keoni would find the lengths of each side of the right triangle were accurate, and captured the spirit of Sasha and Keoni’s method. These explanations can be taken as more evidence that Marshall decentered in an effort to think about and solve the problem from Sasha and Keoni’s perspective.

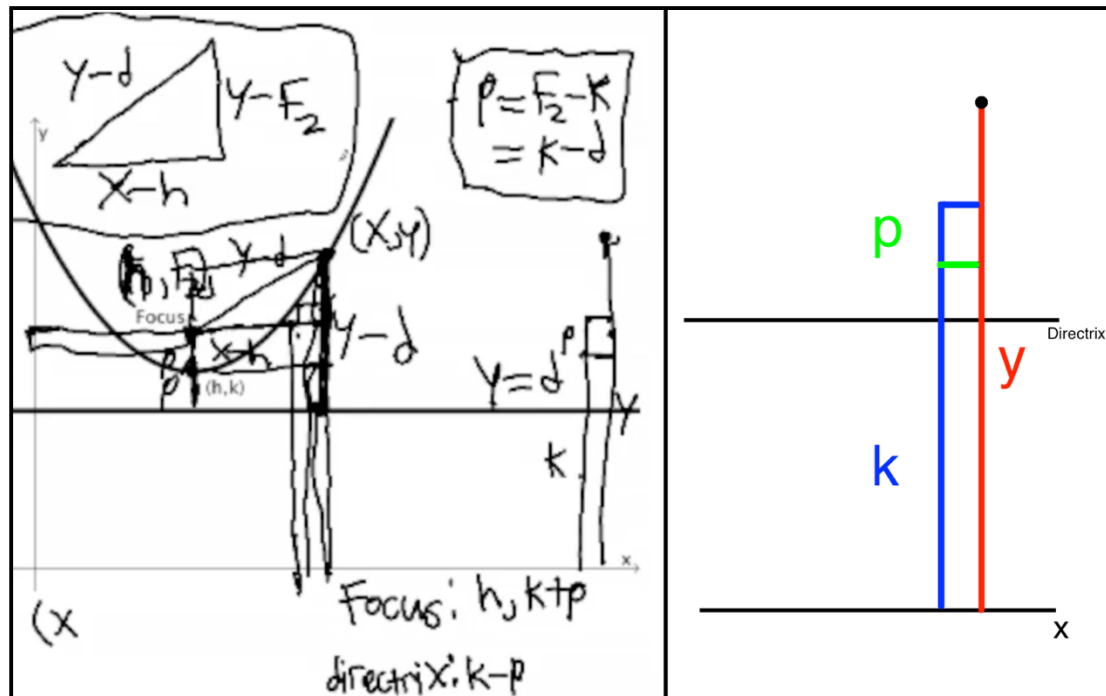


Figure 4.26. (a) At left, the diagram Marshall used to explain how he thought Sasha and Keoni would think about $y - k - p$, and (b) at right, the author's recreation of Marshall's inscriptions.

Marshall finished his third attempt by stating that Sasha and Keoni would find the lengths of the sides of the right triangle to be $x - h$, $y - k - p$, and $y - k + p$, and he explained that Sasha and Keoni would use the Pythagorean Theorem to set up an equation. Marshall's third solution is qualitatively different from his first two solutions. He did not seem to demonstrate much quantitative reasoning with coordinates as distances in his first two solutions, instead relying on his own prior knowledge about geometric transformations. He also leveraged the distance formula along with sophisticated algebraic notation schemes in order to solve the task by explicitly using the geometric definition. These solutions stand in contrast to his solution given from Sasha and Keoni's perspective, which featured Marshall's demonstration of quantitative reasoning with coordinates as distances, the formation of

quantitative relationships, and the use of a right triangle along with the Pythagorean Theorem. Indeed, Marshall's third solution serves as strong evidence that Marshall decentered and took on the perspective of Sasha and Keoni in producing a new solution.

Marshall's ideas about Sasha and Keoni and their understanding of the task.

Marshall did not just solve the task in a way that was similar to Sasha and Keoni's method. He also expressed several ideas about what Sasha and Keoni might understand and what they might have difficulty understanding. For example, Marshall said that Sasha and Keoni "would try to build on what they did before," but that "when it comes to labeling distances they [Sasha and Keoni] would run into some trouble." He also expected that Sasha and Keoni would "think of p right away" which would help them figure out that " k plus p is the y -value of the focus, and the directrix is at k minus p ." As mentioned before, Marshall noted that Sasha and Keoni would use the Pythagorean Theorem. He also stated multiple times that Sasha and Keoni liked to draw their triangle "above it," meaning the horizontal leg of the right triangle extending leftward from (x, y) , instead of extending rightward from the focus (these two ways are highlighted in Figure 4.27 below), and Marshall even drew a triangle "above it" to illustrate what he meant (visible in Figure 4.26a above, and highlighted in green in Figure 4.27 below).

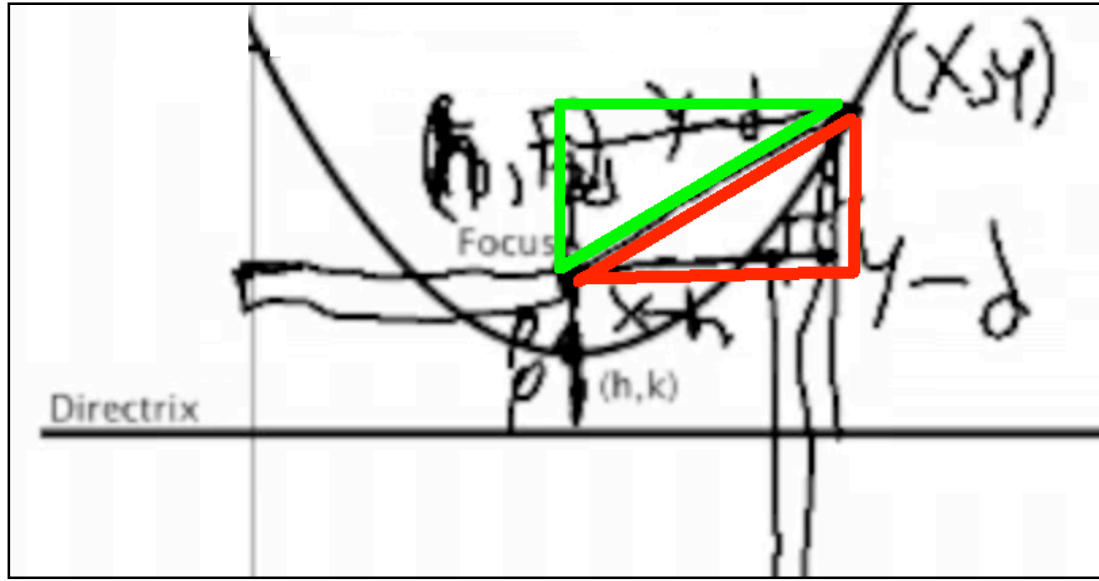


Figure 4.27. The red annotated triangle is how Marshall drew the triangle, while the green annotated triangle is how Sasha and Keoni drew theirs, and is what Marshall meant when he said they would draw theirs “above it.” The original triangles can be seen in Figure 4.26a above.

These examples are indicative of the kinds of comments Marshall made as he worked through his third solution. I provide these to support my overarching claim that Marshall turned away from his own mathematical understanding of the Parabola Task in order to solve the task from the perspective of two high school students. These statements add texture to Marshall’s solution as they suggest Marshall did not just offer a different solution. Instead, Marshall appeared to thoughtfully consider the task from Sasha and Keoni’s perspective by considering what they might or might not do, as well as what they may or may not struggle with. These statements, along with his solution, can be taken as evidence that Marshall decentered by shifting his point of view from his own mathematical understanding to the mathematical understandings he thought Sasha and Keoni would have.

April and Jasper

In this section I briefly discuss the work in the post-interview of two more participants, April and Jasper. Their initial solutions were qualitatively different from Sasha and Keoni's solution. Then, like Marshall, April and Jasper changed their solutions after I asked them how they thought Sasha and Keoni would solve the task. Their new solutions were similar to Sasha and Keoni's and both provided evidence of decentering.

April's initial attempts were dissimilar from Sasha and Keoni's method.

Like Marshall, April's initial attempts to solve the Parabola Task involved using transformations and the distance formula. For example, April spent several minutes working in silence producing several diagrams in which she appeared to be applying transformations or shifts of the parabola (see Figure 4.28 below). When I asked her about this, she said her plan was to place a parabola with its vertex at the origin and then apply a series of shifts to help her determine the y -coordinate of the focus. In response, I asked her what she planned to do with that coordinate, and she said she ultimately wanted to use the distance formula to derive the equation for the parabola. After some time in which she appeared to reason quantitatively with some coordinates as distances (e.g., quantifying y_0 , k , and a to form the quantitative relationship $y_0 - (k + a)$, where a is the distance from the vertex to the focus), April wrote the equation $\sqrt{(x_0 - h)^2 + (y_0 - (k + a))^2} = \sqrt{(x_0 - x_0)^2 + (y_0 - (k - a))^2}$.

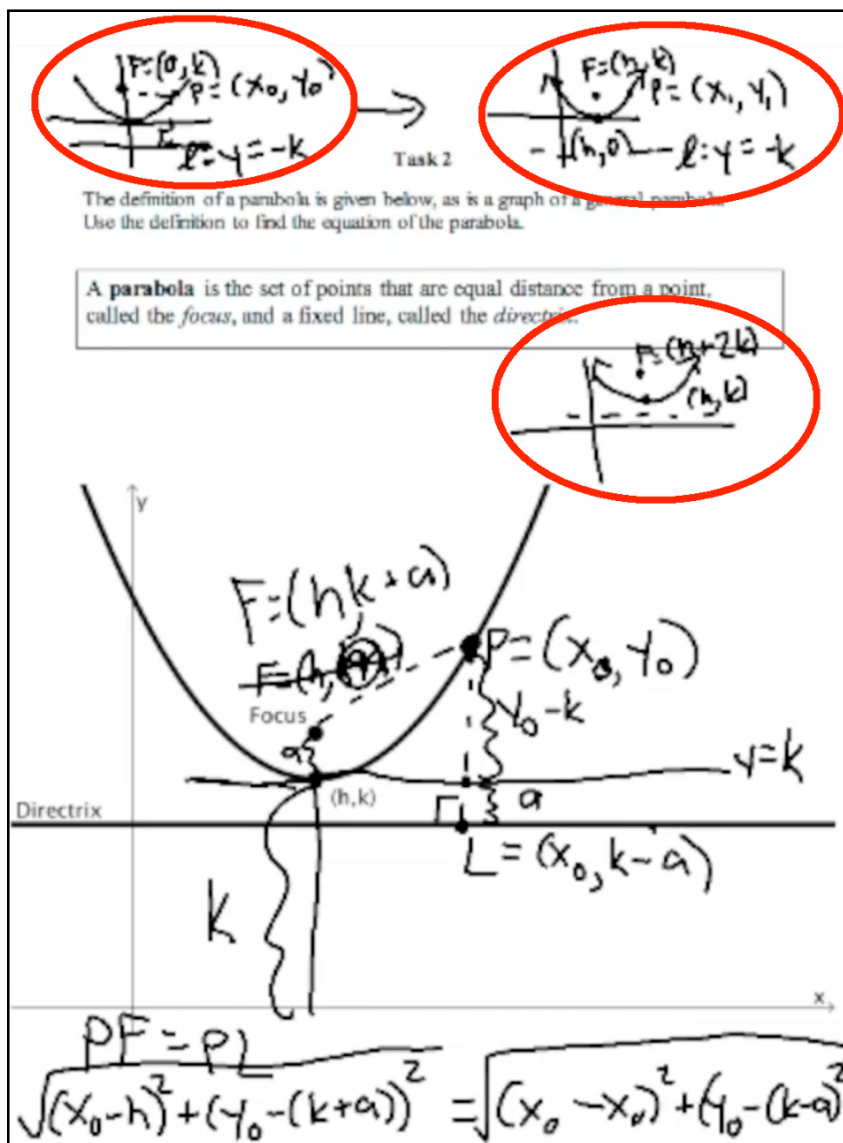


Figure 4.28. April's initial work on the Parabola Task, including transformations of the parabola (annotated with ovals), seen at the top of the page, and her final equation, seen at the bottom of the page.

To the extent that April demonstrated quantitative reasoning, it seemed to be in service of finding coordinates of points so she could use the distance formula. For example, consider this quote from April which confirms this claim:

What I want to do is, use the distance formula. And I want to know what this coordinate is [circles the m , which she had written to stand in for the y -coordinate of the focus; see Figure 4.29 below], well the

coordinate of the focus, and I want to know what my y -point is for here [points to the focus] and for here [points to the point labeled L ; see Figure 4.29 below], that way I can just use the distance formula.

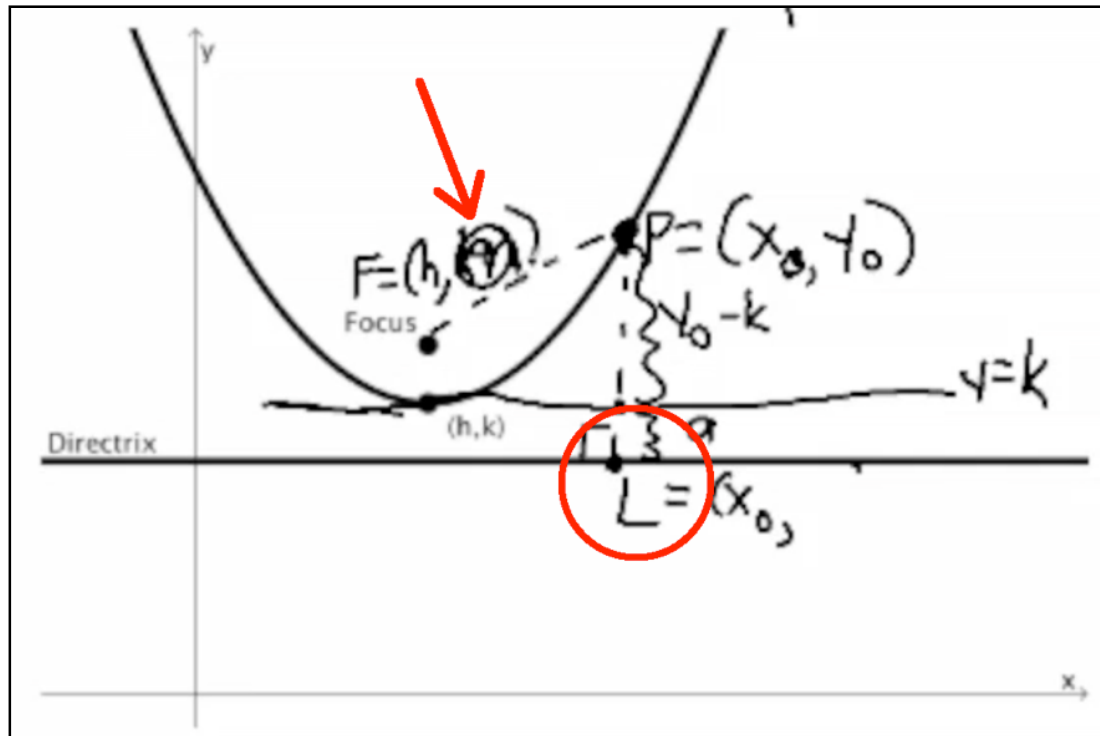


Figure 4.29. April circled the y -coordinate of the focus (annotated with an arrow) and indicated the point labeled L (annotated with a circle).

This approach is qualitatively different than Sasha and Keoni's approach. Sasha and Keoni wanted to find the lengths of the sides of a triangle in service of using the Pythagorean Theorem, whereas April did not draw a triangle nor did she refer to a triangle during her initial attempts. April's quantitative reasoning appeared to be in service of finding the values of the coordinates of certain points, which were then used in the distance formula. By contrast, Sasha and Keoni's quantitative reasoning was in service of finding lengths of sides of triangles to be used in the Pythagorean Theorem.

April decentered. In response to the follow-up question about how Sasha and Keoni would solve the Parabola Task, April appeared to take on their perspective in

solving the task. She started by saying that Sasha and Keoni would label an arbitrary point on the parabola (x, y) , and then draw lines from that point to the directrix and the focus, which she then did (see Figure 4.30 below).

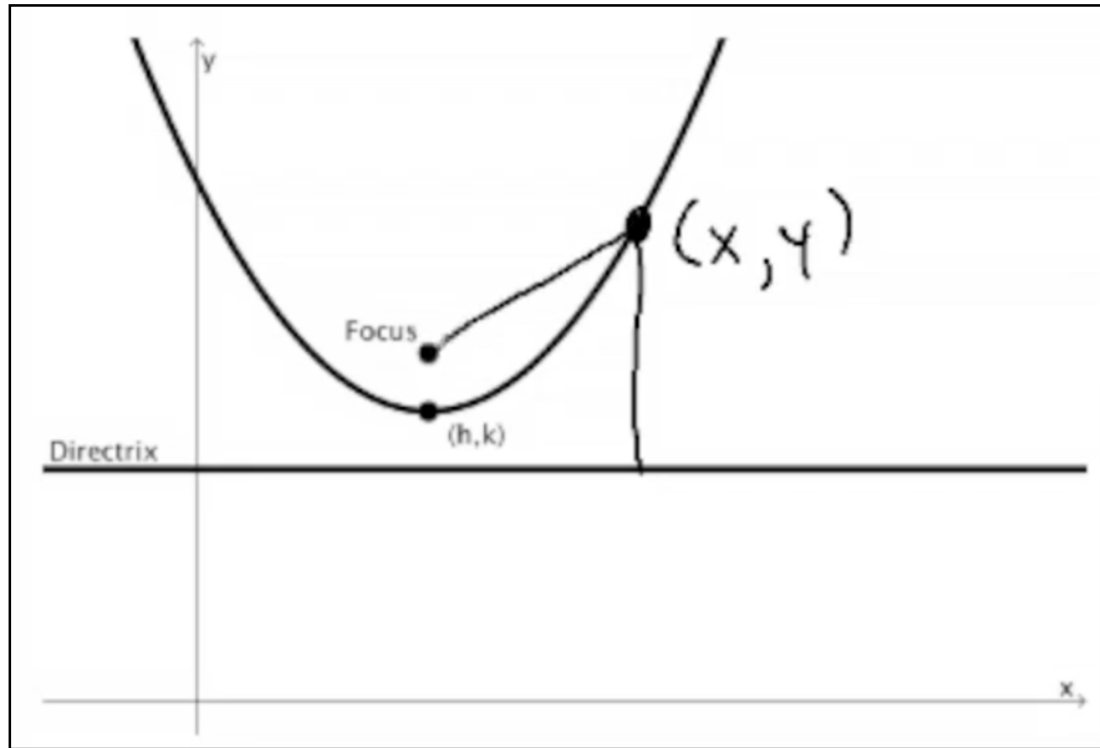


Figure 4.30. April's inscriptions indicating how she thought Sasha and Keoni would start the Parabola Task. Note that the segments are not dashed, and there are no subscripts for the coordinates.

April later noted that Sasha and Keoni would attend to the distance from the focus to the directrix and label it with a variable (she chose z , unlike Sasha and Keoni who used p). She claimed that Sasha and Keoni would be able to determine that the distance from the x -axis to the vertex was k and would then say that the focus was at $(h, k + z)$. She continued by saying that Sasha and Keoni would draw a horizontal line from the x -axis through the vertex to help them find the height of the directrix, which she labeled $k - z$ (see Figure 4.31 below). She claimed that Sasha and Keoni

would then find the distance from (x, y) to the directrix, which would be $y - (k - z)$. Finally, April stated that Sasha and Keoni would draw a triangle and use the Pythagorean Theorem once they found the lengths of the sides of the triangle. She then sketched a line from the y -axis through (x, y) , and drew a line segment from the focus to that line to form a triangle and drew a bracket for one side (see Figure 4.31 below). At this point, April seemed fatigued with the problem; so I did not push her any further.

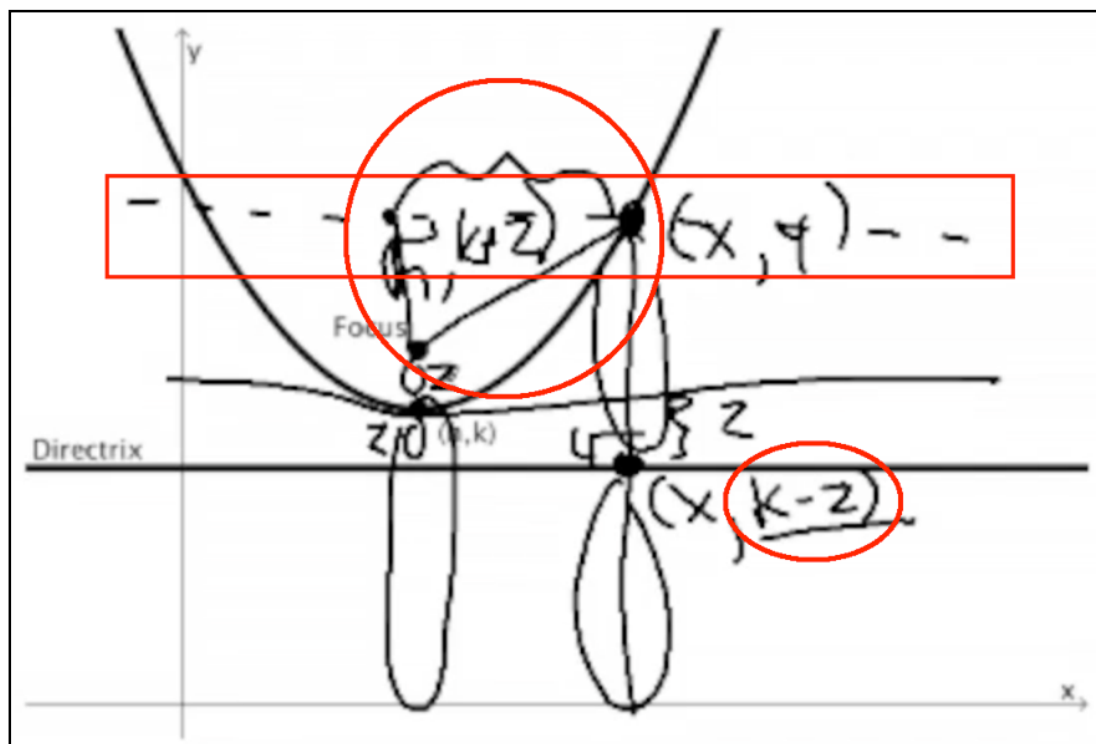


Figure 4.31. April found the distance $k - z$ (annotated with an oval), and later sketched a dotted line (annotated with a rectangle) and formed a triangle (annotated with a circle).

April's description of how Sasha and Keoni would solve the task serves as evidence of my claim that she decentered. First, there are differences between her initial attempt and her final attempt. For example, April initially used a notation

scheme for her arbitrary point that used subscripts (she used (x_0, y_0) as her arbitrary point); by contrast, in her final attempt she used (x, y) as the arbitrary point, which is exactly what Sasha and Keoni used throughout the lessons, as well as in their own solution of the Parabola Task. April also used dashed line segments for the segments from the arbitrary point to the focus and directrix (see Figure 4.29 above) in her initial work, but used solid line segments for those segments when describing how Sasha and Keoni would solve the task. Sasha and Keoni used solid line segments for much of their work throughout the parabola unit, (see Figure 4.32 below as one example). Her use of solid lines, rather than dashed lines, is consistent with taking on the point of view of Sasha and Keoni. These qualitative differences are notable because the lack of subscripts and the use of solid segments match what Sasha and Keoni did in their own attempt. In other words, April changed her approach in ways that matched how Sasha and Keoni approached the task.

Second, throughout the entire mini-course, April seemed to strongly prefer a method of finding coordinates of points to use with the distance formula (see Chapter 5 for evidence). This method was how she initially approached the Parabola Task in the post-interview. However, when I asked her to describe how Sasha and Keoni would solve the task, she appeared to think about the task from their perspective. As evidence of this claim, consider that her solution initially did not involve a right triangle; yet in her description of Sasha and Keoni's method, she not only made a point that they would draw a triangle, she also drew one herself. Indeed, her final attempt was driven by a search for the lengths of the legs of a right triangle, whereas

her initial attempt was driven by a search for the values of coordinates of certain points to be used in the distance formula. Moreover, she explicitly said in her initial attempt that she was trying to use the distance formula, which contrasts with her statement in her final attempt that Sasha and Keoni would use the Pythagorean Theorem.

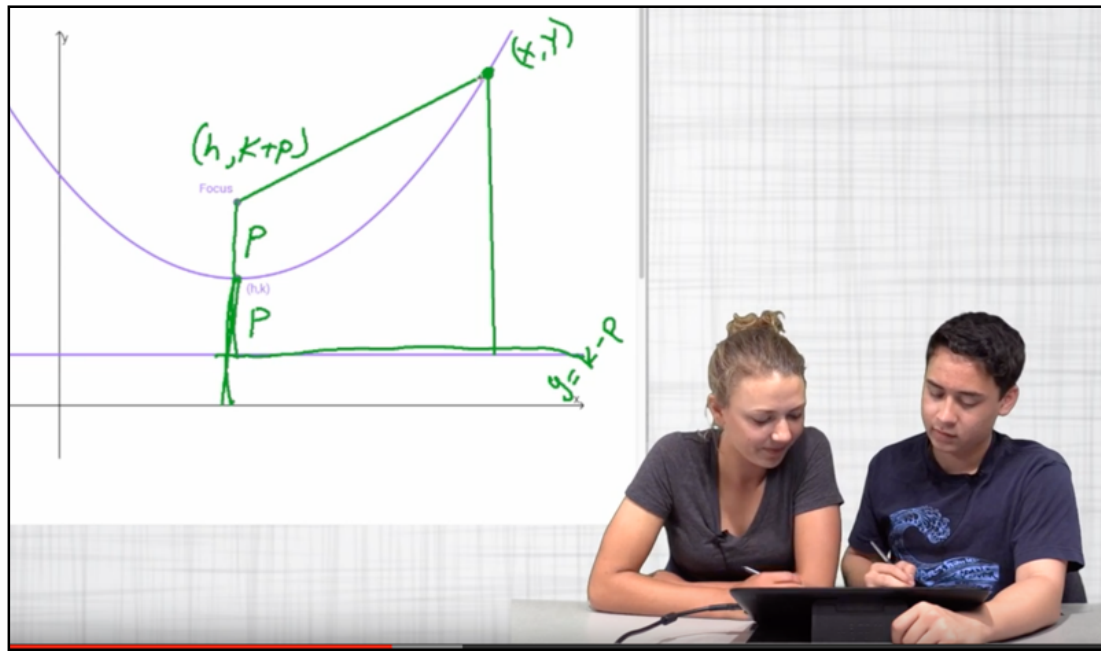


Figure 4.32. Sasha and Keoni's work typically featured solid line segments, as seen in this screenshot from Lesson 9, Episode 6.

April's statements about how she thought Sasha and Keoni would solve the Parabola Task were qualitatively different from her own solution attempts to solve the task. The evidence I have provided from her attempts to solve the task suggests that April decentered and took on the perspective of Sasha and Keoni. I now turn to Jasper, who like Marshall and April, seemed to decenter by taking on the perspective of Sasha and Keoni to produce a solution method that was qualitatively different from his initial solution attempts.

Jasper's initial attempt was dissimilar from Sasha and Keoni's method.

Like Marshall's initial attempt, Jasper's first solution for the Parabola Task seemed to leverage transformations in the plane. Specifically, Jasper wrote the equation $y =$

$\frac{(x-h)^2}{4(y_f-k)} + k$ (where y_f is the y -coordinate of the focus; see Figure 4.33 below).

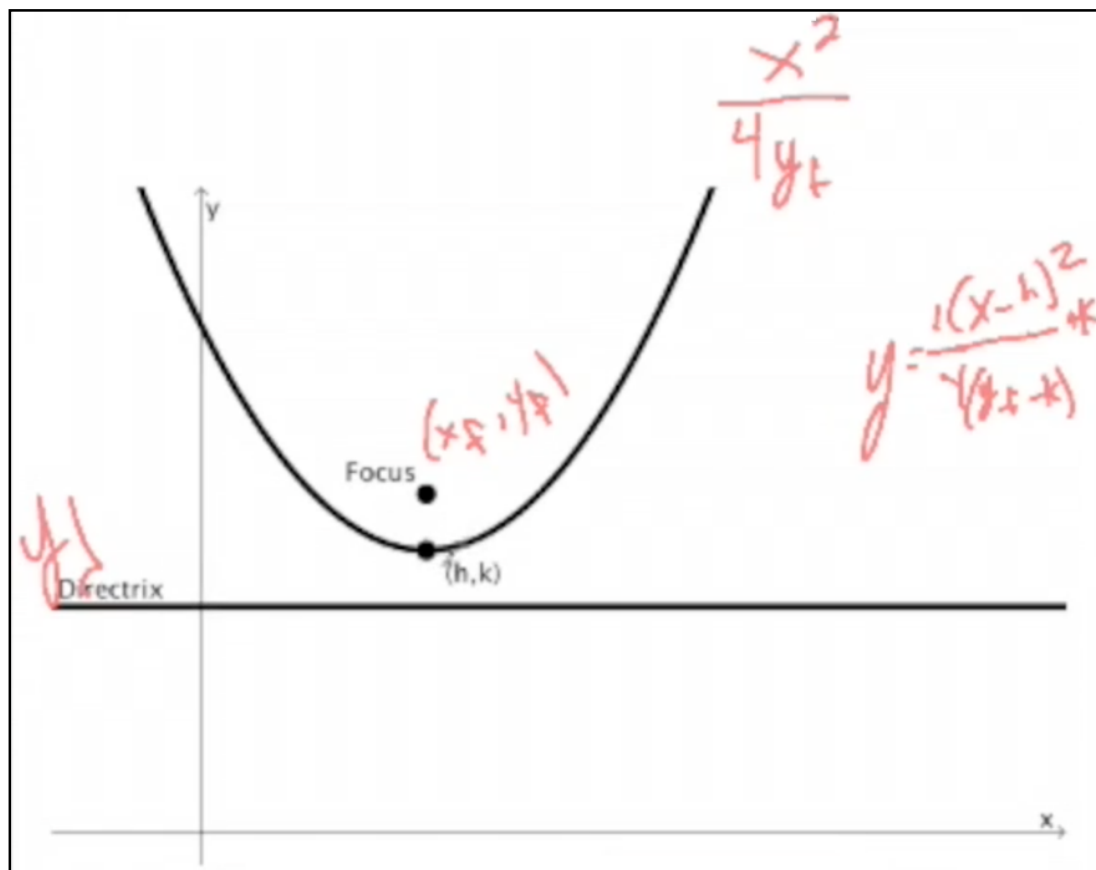


Figure 4.33. Two equations from Jasper's initial work on the Parabola Task.

He explained how he arrived at that solution by stating that “the general one will be x squared over four y -eff [writes $\frac{x^2}{4y_f}$],” which would be the equation if the parabola “crossed at the origin.” I asked him to clarify, and he said he meant that the vertex would be at the origin. He then described some transformations of the parabola:

If I remember correctly, that we do some shifts, because we shifted over h times, then it would be x minus whatever the shift is, which is h , squared. And then down here [points to where he wrote $y_f - k$] will be the shift up and down, which in this case, will be, instead of the origin is zero, in this case it would be k , that would be the shift.

Jasper decentered. After I asked Jasper how he thought Sasha and Keoni might solve the task, he produced a solution that was similar to the students' approach. Jasper began by placing an arbitrary point and drawing segments from that point to the directrix and to the focus (see Figure 4.34 below). He then said "They [Sasha and Keoni] would do the triangle," and as he said this he drew a triangle (see Figure 4.34 below). He said that Sasha and Keoni would say that the arbitrary point would be (x, y) , and he labeled the point as such.

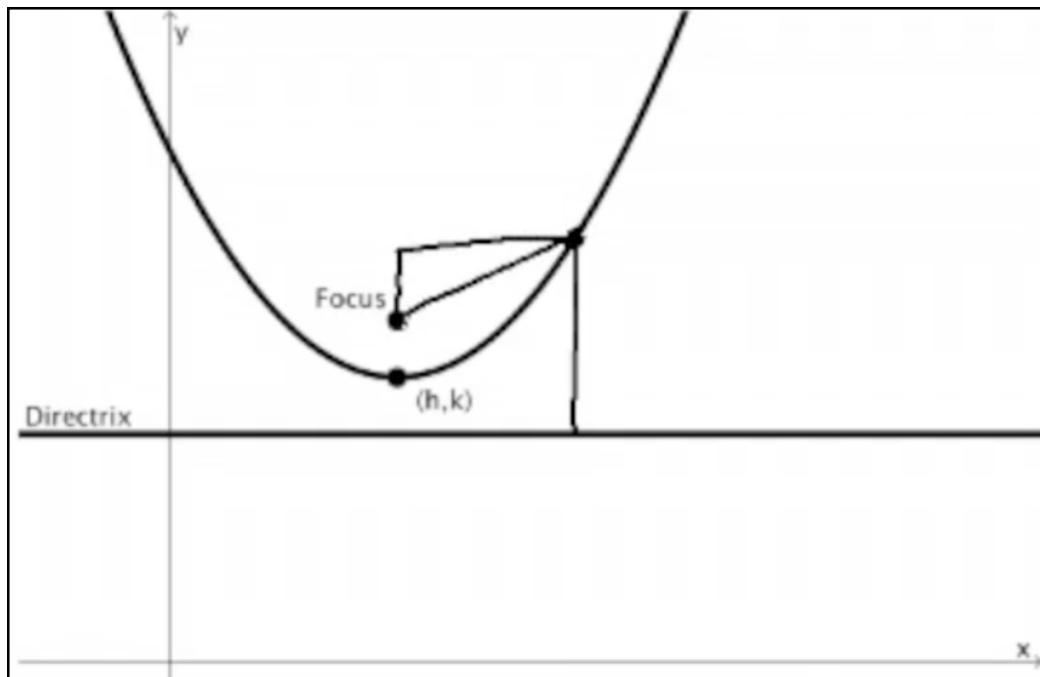


Figure 4.34. Jasper's second attempt featured segments drawn from an arbitrary point and a right triangle.

Jasper then discussed how Sasha and Keoni would find the lengths of each side of the right triangle he drew. He began by stating they would find the length of the

horizontal leg by finding the distance from (x, y) to the y -axis as x and then the distance from the focus to the y -axis as h . He concluded by saying that Sasha and Keoni would say the horizontal leg would be $x - h$. He used similar quantitative reasoning to find the distances of the other two sides of the right triangle. Jasper did state that he thought Sasha and Keoni would have difficulty with finding the length of the hypotenuse because they would have trouble finding the equation for the directrix. Eventually he decided they would say that the directrix was at $y = d$, so that the hypotenuse would have a length of $y - d$. His equation can be seen in Figure 4.35 below.

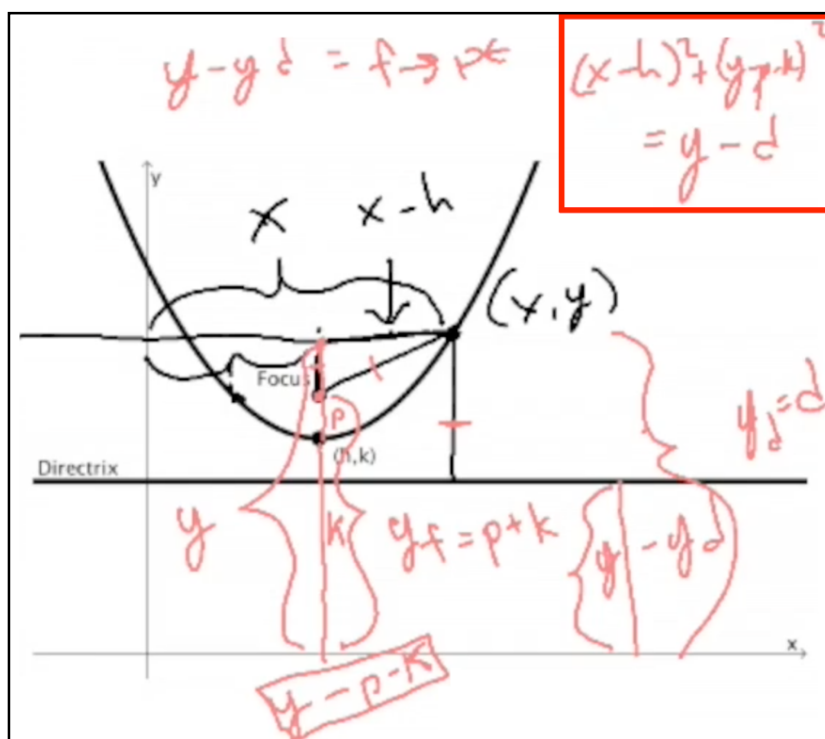


Figure 4.35. Jasper's work for his second attempt, including his final equation (annotated with a rectangle).

Jasper's descriptions of how he thought Sasha and Keoni would solve the task can be taken as evidence that Jasper decentered. First, there are several notable

differences between his two attempts. For example, in Jasper's initial attempt he did not draw an arbitrary point, nor did he draw segments from that point in an effort to leverage the definition of a parabola. In contrast, he did both of these things in his second attempt. In his initial attempt, he did not tend to reason quantitatively with coordinates as distances, nor did he seem to form quantitative relationships. However, in his second attempt he did reason quantitatively. Jasper also did not draw a triangle or mention the Pythagorean Theorem in his initial attempt; instead he talked about shifts and "general" parabolas. By contrast, in his second attempt Jasper did draw a triangle, found the lengths of the sides of the triangle, and used the Pythagorean Theorem to generate an equation for the parabola. These differences suggest that Jasper was able to turn away from his own initial understandings of the task. Moreover, his second attempt, as described above, was similar to Sasha and Keoni's method. This suggests that not only did Jasper set aside his own mathematical understandings, but that he also took on the perspective of Sasha and Keoni. In other words, Jasper appeared to decenter.

In summary, three participants (Marshall, April, and Jasper) seemed to be able to shift their point of view to that of Sasha and Keoni's to solve the Parabola Task like high school students. Three other participants, Willow, Sierra, and Lily, produced initial solutions to the Parabola Task that were similar to Sasha and Keoni's solution. When I asked them how they thought Sasha and Keoni would solve the task, all three replied that Sasha and Keoni would produce solutions similar to theirs. Willow even said, "I would assume they went at it the same way I did, because I was going the way

Sasha and Keoni might approach this, because that is how I learned about it.” In other words, Willow, Sierra, and Lily may have decentered during the mini-course as they looked to the videos with Sasha and Keoni to deepen their understanding of the mathematics. If this is the case, the shift to decentering may not have been visible in the post-interview, because it had occurred previously during instruction. As for Desmond, he had few ideas about what Sasha and Keoni would do with the task. He did state they would use the " $a^2 + b^2 = c^2$ triangle like I did” However, there was not sufficient data to suggest that Desmond had clear ideas about how Sasha and Keoni would solve the task.

I now turn toward the third shift in MKT around quantitative reasoning, which was a shift in orientation toward problem solving. All three participants who experienced a shift in quantitative reasoning (Willow, Sierra, and Desmond), as well as two participants who experienced a shift in point of view (Jasper and April) seemed to shift from a calculational orientation in the pre-interview to a conceptual orientation in the post-interview.

A Shift in Orientation

This section is divided into three parts. First, I discuss the constructs of calculational and conceptual orientations. Second, I provide evidence in support of my claim that one of the participants, Sierra, experienced a shift in her orientation toward problem solving from one that was calculational in nature in the pre-interview to one that was conceptual in nature in the post-interview. Finally, I briefly discuss four other participants' shifts in orientation.

Calculational and Conceptual Orientations

A. G. Thompson et al. (1994) characterized two different orientations that mathematics teachers may exhibit in their teaching: *calculational* and *conceptual*. A calculational orientation is characterized by teaching that seems to focus on (a) calculations and procedures, (b) finding numerical answers or performing arithmetic operations; (c) performing calculations whenever an opportunity to do so arises, regardless of the context or purpose of the task; and (d) speaking in terms of numbers or numerical operations. By contrast, a teacher with a conceptual orientation tends to have: (a) images of different ways students understand an idea; (b) images of how those understandings develop; (c) images of instruction that promote that development; and (d) expectations that students are intellectually engaged in mathematical activity (Silverman & Thompson, 2008; A. G. Thompson et al., 1994).

A. G. Thompson et al. (1994) argued that orientations influence classroom discourse and in turn influence student learning. For example, in a two-part series, A. W. and P. W. Thompson (A. G. Thompson & Thompson, 1996; P. W. Thompson & Thompson, 1994) explored how talking about rates from different orientations seemed to influence the learning of one student, Ann. They reported on a four-day teaching experiment for which the goal was for Ann to develop a conceptual understanding of speed.

Bill, the teacher for the first two days and the subject of the first part of the series, appeared to have a deep understanding of speed, but his calculational orientation seemed to be problematic for Ann's own learning (P. W. Thompson &

Thompson, 1994). For example, one of the tasks in the teaching experiment was to figure out the speed an animal traveled if it went 100 feet in seven seconds. Bill continually tried to focus Ann's attention on the operation needed to produce the answer (division). For Bill, a measurement induced a partition and vice versa, so he saw Ann's initial successful attempts to find time given a speed as being indicative that she could find speed given time. Bill's focus was on the arithmetic operation of division, particularly the fact that division was an appropriate operation in either context (find speed given time and vice versa). However, Ann did not see the two contexts as requiring the same operation; indeed, Ann did not seem to see the necessity that as distance accumulates, so does time at a rate that is proportional to the rate that distance is accumulating. Bill's apparent calculational orientation prevented him from being able to attend to Ann's conceptual difficulties, which eventually led to a breakdown in the teaching experiment.

In the second part of the series, A.G. Thompson and Thompson (1996) reported on Pat's attempts to help Ann come to understand speed during the third day of the teaching experiment. The authors highlighted differences between Pat's and Bill's orientations, and argued that Pat's conceptual orientation helped Ann come to better understand speed. For example, Pat believed that Ann needed to see speed as a simultaneous accumulation of both distance and time. To help Ann come to understand speed in this way, he swept his index fingers simultaneously along both a distance line and a time line to represent an animal's motion. Moreover, his language

also reflected his desire to have Ann come to understand division not only as an operation, but also as a tool for both measuring and partitioning.

During the analysis of the pre- and post-interviews I noticed that the constructs of calculational and conceptual orientations seemed useful for characterizing how participants oriented themselves to the interview tasks. Moreover, several participants approached the Ellipse Task (see Figure 4.1a above) with an orientation toward the task that could be characterized as calculational, yet seemed to approach the Parabola Task (see Figure 4.1b above) with an orientation that could be characterized as conceptual. In other words, some participants experienced a shift in their orientations.

The calculational and conceptual constructs elaborated by A. G. Thompson et al. (1994) are useful for this analysis for two reasons. First, the characterizations of these orientations provided by the authors include descriptions of actions and language that can be applied to problem solving activities (e.g., a focus on numbers, values, and numerical operations). Second, and more importantly, participants in this study were prospective teachers. Furthermore, the purpose of this study is to investigate MKT development by PSTs. The Silverman and Thompson (2008) framework for MKT development takes as a foundation for MKT rich conceptual understanding of a mathematical situation; consequently, a conceptual orientation toward problem solving can be thought of as a necessary prerequisite for a conceptual orientation toward teaching. Accordingly, I use the constructs of calculational and conceptual orientations primarily to characterize my participants' orientation toward solving mathematical tasks. However, because some of the follow-up probes in both the pre- and post-

interview dealt with pedagogical ideas, I also apply these two orientations to the participants' images of instruction.

Sierra

I now present evidence to support the claim that Sierra experienced a shift in her orientation toward problem solving and in her images of instruction. I present four pairs of episodes, with one episode from the pre-interview and one episode from the post-interview comprising a single pair. These paired events are similar in terms of what Sierra was trying to accomplish mathematically or in terms of her reflections on mathematics instruction. Presenting the episodes in this manner allows for contrasts in her orientation to be analyzed.

Before presenting the analysis, I acknowledge that a careful reader might start to wonder how this analysis differs from the analysis in the Shifts in Quantitative Reasoning section. In that section, the analysis targeted ways in which participants reasoned quantitatively with distances. I examined how they seemed to conceive of coordinates and whether they formed quantitative relationships. In the following analysis, quantitative reasoning (or a lack thereof) features prominently, but it serves as the backdrop for the analysis. Instead, I investigate participants' images of instruction and approach to solving problems that feature quantitative reasoning.

Emphasis on numeric values versus quantities. In the pre-interview, Sierra seemed to emphasize finding and working with numeric values. For example, consider that she appeared to treat points as numeric values to be operated on. Early in her attempt to solve the Ellipse Task, Sierra stated that the length of the vertical leg of the

right triangle was $y - k$ (see Figure 4.17 in the Shift in Quantitative Reasoning section). In response to my question about where she saw $y - k$, Sierra referred to the point (x, y) and/or the coordinate y (it was not always clear if she meant the point or the coordinate) as “the highest number,” “the higher value,” and “the top height.” When I asked her to show me y as the higher number, she replied “My higher number? This is the point” while pointing to (x, y) . She also referred to the vertex of the right angle and/or k (again, it was often unclear to which she was referring) as “the lowest number” and “the bottom height.” These statements are consistent with a calculational orientation toward problem solving.

By contrast, in the post-interview Sierra seemed to emphasize finding and working with quantities. In the post-interview, Sierra declared that the length of a horizontal leg of a right triangle was $x - h$. Instead of emphasizing the numeric values, as she did in the pre-interview, Sierra emphasized the quantities that the coordinates represented. For example, she gestured along a horizontal segment and said “the whole thing” would be x , indicating the quantity of distance with her gesture. She characterized another segment by using a similar gesture and claiming it would be “ h units.” As elaborated in the Shifts in Quantitative Reasoning section, Sierra was quantified coordinates as distances, which suggests that she emphasized quantities and reasoning with quantities in the post-interview. This behavior is consistent with a conceptual orientation toward problem solving.

Arithmetic versus quantitative operations. In the pre-interview, Sierra seemed to focus on arithmetic operations, which is consistent with a calculational

orientation toward problem solving. For example, initially Sierra formed a quantitative relationship, namely $y - k$ (this episode is analyzed in detail in the Shifts in Quantitative Reasoning section—I briefly recap it here). However, when I asked her to explain how she saw the $y - k$, she said “this distance right here is this point minus this point,” and later said the “distance is like the highest number minus the lowest number.”

This language suggests that Sierra did not form a quantitative relationship since she did not quantify y or k as distances. Moreover, rather than performing a quantitative operation such as take away or compare (see the conceptual analysis for the interview tasks in Chapter 3) Sierra seemed to be performing the arithmetic operation of subtraction with unknown values without explicit links to a quantitative meaning for subtraction. She also said it did not have to be that way (i.e. highest minus lowest), because she could find “ k minus y , which would be negative” and then take the absolute value.

In the post-interview, Sierra appeared to not only quantify coordinates as distances, she also appeared to form quantitative relationships. For example, in the Shifts in Quantitative Reasoning subsection above, I presented an exchange in which Sierra formed the quantitative relationship $y - k + p$. I have reproduced that exchange below and added emphasis throughout. Notice that instead of speaking of values to be subtracted and added, Sierra used spanning gestures to indicate distances as she talked about each algebraic term.

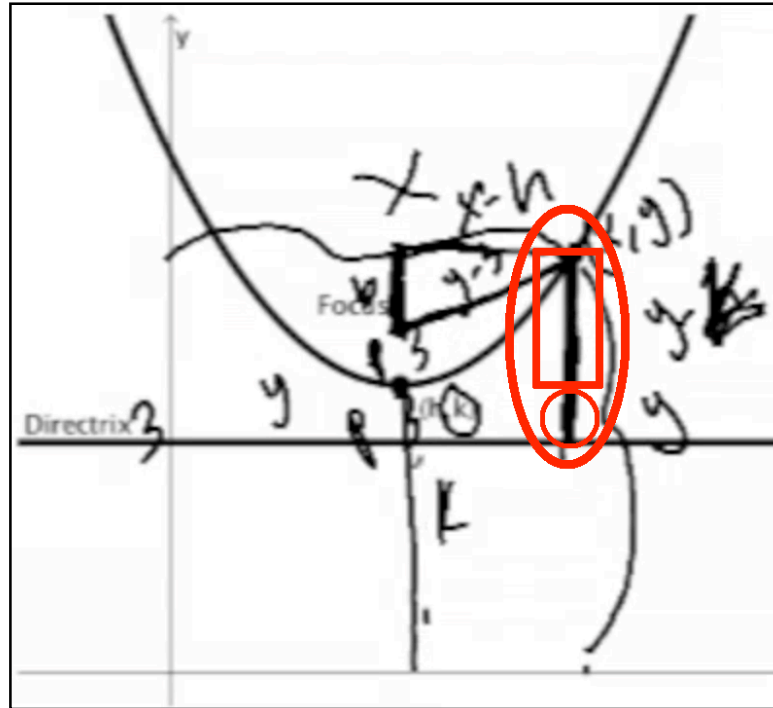


Figure 4.37. Author's annotation of Sierra's gestures and drawing of a line segment.

The spanning gestures Sierra used suggests that as she talked about subtracting and adding, she linked those operations with quantities (i.e., distances in the coordinate plane) and not numeric values. In other words, her spanning gestures suggest she was taking away or combining distances, which are quantitative operations.

Calculating versus sense making. One of the characteristics of a calculational orientation is performing calculations whenever an opportunity presents itself to do so (A. G. Thompson et al., 1994). During the pre-interview, Sierra seemed to exhibit this characteristic as she solved the Ellipse Task. For example, during the pre-interview, Sierra appeared to have the goal to use the Pythagorean Theorem to find the length one side of the triangle. As demonstrated in the conceptual analysis for the Ellipse Task presented in Chapter 3, the Pythagorean Theorem is not necessary. Yet Sierra's

calculational orientation seemingly influenced her goal for the task so that she got so caught up in a series of calculations that she lost track of what she was trying to find:

I know this [points to the $y - k$ she had written], I know that [points to the label r_2 printed on the graph], so I can find this [gestures along the horizontal leg of the smaller right triangle]. Which would be with the Pythagorean Theorem, y minus k squared plus b will give me r -two squared [writes the equation $(y - k)^2 + b^2 = r_2^2$, see Figure 4.38 below]. So I want to solve for this [points to the b in the equation she just wrote], I would subtract this whole thing [gestures in a circle around the $(y - k)^2$ in the equation she just wrote, then writes a new equation $b^2 = r_2^2 - (y - k)^2$, see Figure 4.38 below.], so then I would just square root everything [draws square root symbols over both sides of the equation] so these two would cancel [draws a segment through the exponent 2 and the square root symbol on the left side of the equation, annotated with a circle in Figure 4.38 below]. And I'm left with this, as my b , as this length [gestures back and forth along the horizontal leg of the right triangle; then there is 17 seconds of silence in which Sierra gestures with her pen across the task statement as if she is reading it]. Oh, ok! Find the area, I'm like "What am I trying to find?"

$$(y-k)^2 + b^2 = r_2^2$$

$$\sqrt{b^2} = \sqrt{r_2^2 - (y-k)^2}$$

Figure 4.38. Sierra's equations for finding b .

Sierra's pause at the end of this episode suggests that she had lost track of what the goal for the task was. Indeed, all her work leading up to this moment seemed to be

in service of using the Pythagorean Theorem to find the length of the leg of the right triangle. As just mentioned, the Pythagorean Theorem is not needed to solve the task, as the legs of the triangles can be found through quantitative reasoning. Not only did Sierra not employ quantitative reasoning to find the length of the horizontal leg of the smaller right triangle, her focus on calculating its length using the Pythagorean Theorem obscured the main goal of the task.

During the post-interview, Sierra seemed to forgo calculating in order to make more sense of the mathematical situation. This is in contrast to her work during the pre-interview where she calculated just for the sake of doing so.

As evidence of Sierra's emphasis on sense-making, I present an episode from the post-interview in which Sierra seemed to turn away from her calculational orientation toward one that is more conceptual. Leading up to this episode, Sierra had made some progress in solving the Parabola Task, as discussed in the Shifts in Quantitative Reasoning section. She had created a right triangle; set the equation for the directrix to be $y = 3$; and represented the length of the hypotenuse of the right triangle as $y - 3$, the length of the horizontal side of the triangle as $x - h$, and the length of the vertical length as b (see Figure 4.39). I then asked Sierra if she could express b in terms of y . During her response, there is a moment when she appeared to catch herself wanting to give in to her inclination to perform a series of calculations, but instead, she appeared to step back and try to make sense of the situation in terms of quantities:

Interviewer: Now can you find b in terms of y ?
Sierra: b in terms of y ?

- Interviewer: Yeah, b in terms of y and the other distances. Because what I see is you have used p here, you've identified k , you've identified h and you've identified y , and then you introduced this b as this distance, and I'm wondering if you can get b —
- Sierra: In terms of y ?
- Interviewer: —in terms of everything else you have identified.
- Sierra: Oh. [For the next 50 seconds, Sierra is silent. She looks at her paper, and moves her pen around it as she looks] So this whole distance would also be y [gestures along the segment marked b , all the way down to the x -axis. This is followed by another 1 minute and 15 seconds of silence, only interrupted when I encouraged Sierra by telling her she was doing well]. **I mean, I could solve it over here** [gestures with her pen in a wide circle around the equation she had written], **but I don't think that's what you're looking for. I'm thinking there has to be something here** [gestures with her pen in circles around the segment from (x, y) to the directrix] **that can tell me what this distance** [gestures along the vertical leg of the triangle labeled b] **will be the same as like a portion of this distance** [gestures along the segment from (x, y) to the directrix; emphasis added throughout].

The long pauses in this episode seemed to be moments where Sierra was trying to analyze the situation to see how she could answer my question about finding b in terms of y . Eventually she went on to find b in terms of y , which is outlined in more detail in the Shifts in Quantitative Reasoning Section.

Two things from this episode stand out in contrast to how she approached the Ellipse task in the pre-interview. First, in the pre-interview, I never explicitly asked Sierra to solve for b , yet she did so without prompting. Given her inclination to solve for b in the pre-interview, I expected her to do something similar in response to my question to “find b in terms of y .” However, instead of solving for b using algebraic

manipulations as she did in the pre-interview, she instead seemed to begin to carefully analyze the mathematical situation (as evidenced by the long pauses).

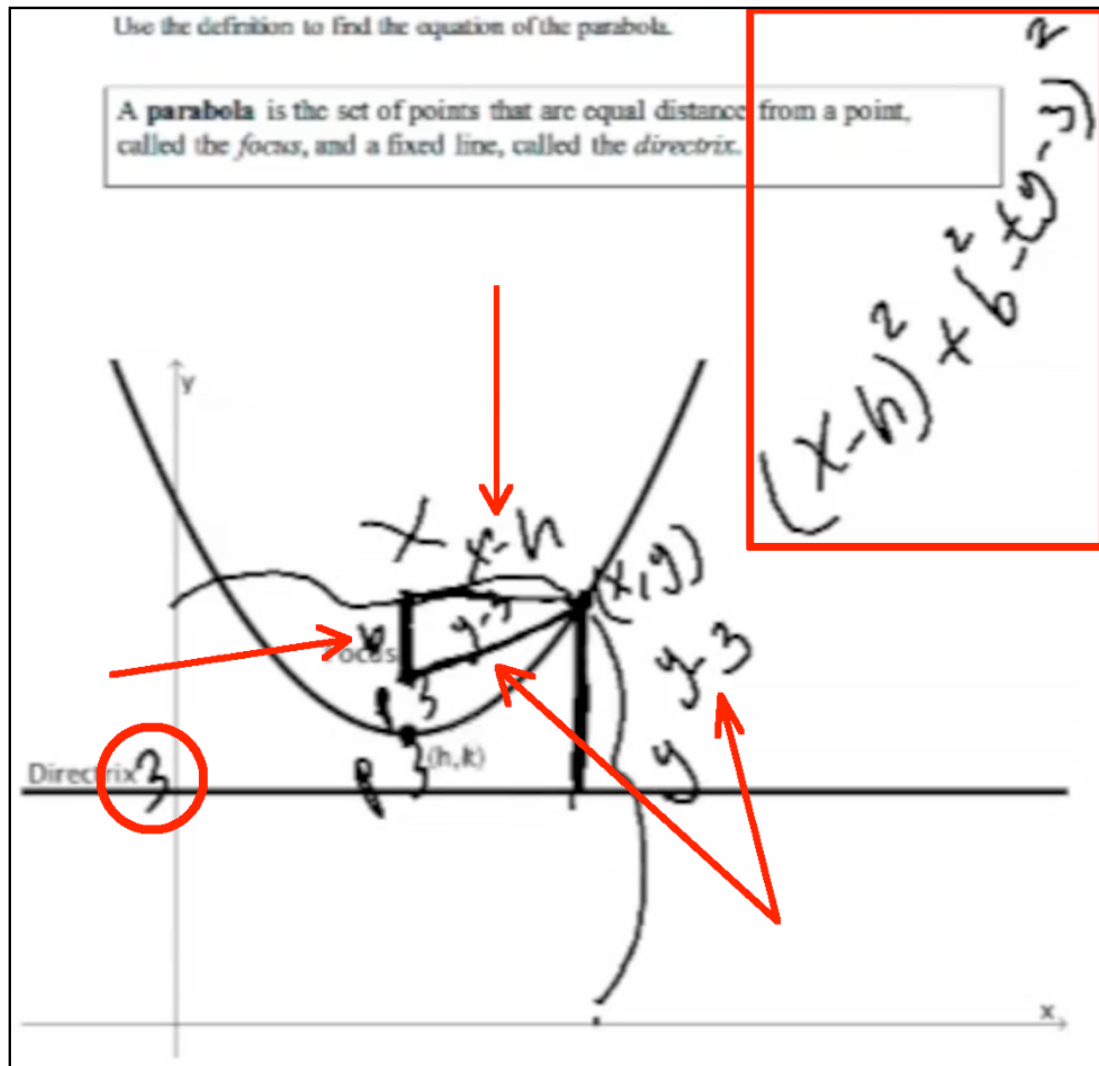


Figure 4.39. Sierra's initial work on the Parabola Task included declaring the equation of the directrix to be $y = 3$ (annotated with a circle), finding the lengths of the sides of a right triangle (annotated with arrows), and finding an equation for the parabola (annotated with a rectangle).

Second, she admitted that she could solve for b (see the emphasized text above), but insisted that she could also find a way to relate one distance given the other. This suggests that Sierra's orientation toward problem solving from the pre- to

the post-interview had begun to shift. That she appeared to analyze the situation and then assert that she should be able to find the distance using quantities already found (and then did so—see the Shifts in Quantitative Reasoning section) is consistent with a conceptual orientation toward problem solving.

Telling students to substitute numerical values versus posing quantitative questions. From the pre-interview to the post-interview, Sierra's images of instruction related to quantities changed. In the pre-interview, Sierra seemed to think that substituting numerical values for unknowns would help students; however, she did not link those values with quantities. By contrast in the post-interview, Sierra appeared to have developed an image of instruction that included quantitative questions.

Part of my interview protocols included asking participants what challenges they thought high school students would face in solving the tasks, as well as what the participants would do or say as teachers to help students overcome those challenges. In the pre-interview in response to this question, Sierra made the claim that students might struggle to attend to how all points along a horizontal line above the x-axis have the same height above the x-axis.

Probably relating this point [the center (h, k) , annotated with an oval in Figure 4.40 below] throughout this [gesturing across the horizontal segment from F_1 to the vertex of the right angle of the right triangle, annotated with an arrow in Figure 4.40 below], knowing that this [pointing to the vertex of the right angle of the right triangle, annotated with a square in Figure 4.40 below] is the same height of k .

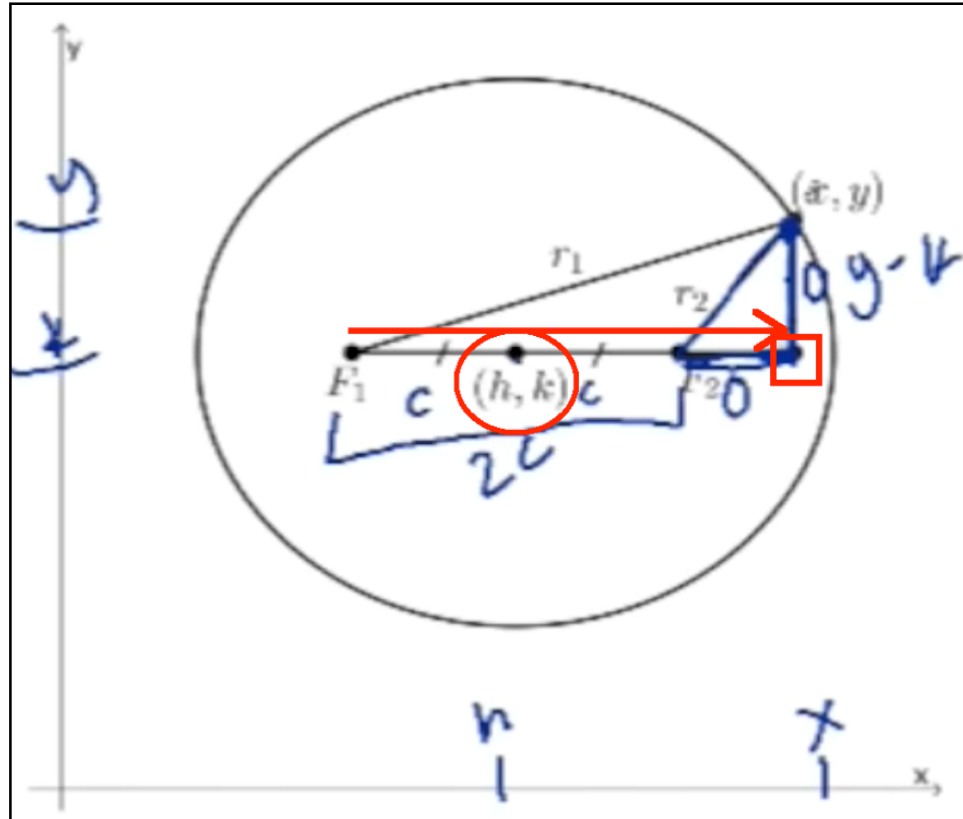


Figure 4.40. Sierra pointed to the center (annotated with an oval), then gestured along the line segment (annotated with an arrow) and finally pointed to the vertex of the right angle of the right triangle (annotated with a square).

Sierra seemed to have in mind that students might struggle with recognizing that points along a segment that is parallel to the x -axis will have the same height above the x -axis. While this idea about students' struggles is not necessarily indicative of a calculational orientation toward problem solving, the way that she said she would help students could be taken as evidence of a calculational orientation. Consider her response to my question about how she would help students come to understand that the points would have the same height:

Sierra: I would probably like, let's say this was what, like five, k was five. I would tell them to just graph y equals five, which would be this whole line right here [draws a line from the y -axis through the

- segment annotated with an arrow in Figure 4.40 above], like that, and label it y equals five. So that's my height. So they can actually see that the value of y throughout this whole line is five. That's probably something I would do.
- Interviewer: So, would you do that before the task, or in the middle of the task you would say "Ok, let's pretend that k is five?"
- Sierra: Probably when I start seeing that they're not getting anywhere, like they're just like "I don't know what to do!" Just as a hint. Like "So, guys," I would ask them, "Like what is my value? Let's say my k is five," and then I would just be like, "So I want you to graph that line." And then I would be like "What do you see? What's my value of y at F -one? F -two? And at my corner point?"

Sierra's image of instruction to help high school students seemed to emphasize numerical values rather than linking those values to quantities. She said she would change the unknown k to a known value, five. She could have said that she would want students to understand that the value five would mean the point is five units above the x -axis, which would be more consistent with a conceptual orientation. In other words, she could have changed the unknown to five, and then emphasized that all points on the line would have a distance of five units from the x -axis, which would reflect a conceptual orientation. Instead, she said "They could actually see the **value of y** (emphasis added) throughout this whole line is five." Her image of instruction included a need for students to see a value of y , rather than to see y as a distance from the x -axis.

In the post-interview, Sierra talked about moves the instructor in the MathTalk videos had made, and how those helped her solve the Parabola Task. For example, she

had this to say about how the teacher would help students attend to distances in the plane:

I could remember her [the teacher] saying “Oh well, like, you have what this whole distance is [gestures with her pen from (x, y) to the directrix], but what is this distance [gestures along the same segment as before, but stops roughly at the same height as the focus]? And, just remembering that will help me like, “Well I need to find what that distance is,” and also like bringing in a new variable I can find it in terms of the actual variables I already have, just to make it easier.

Sierra’s image of instruction for helping someone (in this case, herself) find distances seemed to have changed from the pre-interview. Instead of trying to replace unknowns with knowns and focusing on values that are not explicitly linked with quantities, Sierra appeared to be saying that trying to find relationships between distances was helpful. Sierra’s statement about how the teacher helped Sasha and Keoni can be taken as evidence that Sierra’s images of instruction have shifted to have a greater emphasis on quantities and relationships between quantities (“You have what this whole distance is, but what is this distance?”). This emphasis on quantities and relationships is consistent with a conceptual orientation toward problem solving. Sierra’s images of instruction seemingly shifted from helping students by providing numerical values in lieu of quantities to posing quantitative-related questions.

Other Participants’ Shifts in Orientation

In total, five participants (April, Desmond, Jasper, Sierra, and Willow) experienced a shift in their orientation toward problem solving. In this section, I describe the ways in which other participants’ orientations and shifts were similar to Sierra’s as I described it above.

In the pre-interview April seemed to emphasize values and numerical operations, both in how she solved the problem and in her images of instruction. For example, she spent several minutes finding coordinates to use with the distance formula so she could find the length of one leg of the triangle. Later, she claimed that high school students would need to understand how to find coordinates to use with the distance formula, and how to calculate with the distance formula. Like Sierra, April also claimed that providing students with known values would help them better understand and solve the task. In the post-interview, April's orientation seemed to shift to one that was more conceptual, especially when she talked about how she thought Sasha and Keoni would solve the task. She reasoned quantitatively, and though her images of instruction included providing students with known values, she stated clearly that such moves would be in service of helping students generalize.

Throughout the pre-interview, Desmond emphasized numerical values over unknowns. Unlike Sierra, Desmond also appeared to conflate points and labels for values that he could operate with. For example, he found the length of one leg of a triangle to be $x - F_1$, where F_1 was the label for a point in the graph. His images of instruction also focused on how students might struggle to differentiate between "fixed distances and distances that aren't fixed," which seemed to be in reference to his predominant strategy of replacing unknowns in the problem with known values (e.g., he replaced $x - h$ with $10 - 5$). In the post-interview, Desmond did not need to replace unknowns with known values, nor did he operate with points or labels as values. Instead, he tried to make sense of quantities and relationships between

quantities. Moreover, he made claims about how high school students might struggle because “most people like to put it in their calculator and plug and chug,” and then argued that high school students need to be able to see all the points on the graph and “pull the variables out of the picture.”

During the pre-interview, there were several moments where Jasper calculated when the opportunity presented itself. For example, like Sierra, he used the Pythagorean Theorem to find the length of a leg of a right triangle. Moreover, he expanded the term $(y - k)^2$ in his attempt to find the length of that leg, which was more evidence of his tendency to calculate when the opportunity presented itself. Like Sierra, Jasper had images of instruction that included giving students known values to help them solve the problem, and he said that the goal of the task was to calculate the lengths of the bases and heights of the triangle to find the area. By contrast, in the post-interview, he seemed to emphasize quantitative operations over numerical operations. Like Sierra, Jasper also focused his efforts on making sense of the Parabola Task by talking about distances as quantities and analyzing relationships among quantities. Jasper’s images of instruction also included helping students “see” distances and finding one distance in relation to another.

Finally, Willow’s shift in quantitative reasoning seemed to include elements of a shift in orientation. In the pre-interview, she oriented toward numeric values and arithmetic operations. For example, her description of finding the height by finding the difference in the two points included an expression in which she tried to subtract the two points (see Figure 4.4 in the Shifts in Quantitative Reasoning section above). She

also talked about “giving” students the equation for an ellipse and “explaining it.” By contrast, in the post-interview, Willow reasoned quantitatively, which included forming quantitative relationships and using quantitative operations. She also mentioned how she would try to do what the teacher in the videos did to help Sasha and Keoni “ease into” using unknowns as distances by first introducing known values.

Two other participants, Marshall and Lily, seemed to have more conceptual orientations in the pre-interview. Neither participant fixated on numeric values or arithmetic operations. Additionally, neither participant discussed instructional moves that seemed more oriented to finding or using values (or substituting such values for unknowns), or on arithmetic operations. Consequently, I was not able to find enough data in the interviews to support a claim that these two participants experienced a shift in their orientation. Instead, their ideas about instruction were aligned along a different dimension, one about affect and mindset. I now turn to the final shift in MKT, which emerged as a result of trying to capture the shift around their ideas about students, learning, and teaching that Marshall and Lily seemed to experience

A Shift in Affect

In this section I briefly discuss some of work the field has done to characterize teacher affect, which is comprised of emotions, attitudes, and beliefs (McLeod, 1992; Philipp, 2007). I do this in order to elaborate some ideas about affect that informed the analysis of the pre-interview and post-interview data. The protocols used in the interviews did not include items explicitly designed to measure or probe participants’ affect. However, I did pose three follow-up questions for both the Ellipse Task and the

Parabola Task that elicited traces of participants' affect: (a) What do you think high school students would need to know in order to understand and solve this task? (b) What challenges do you think high school students would encounter as they tried to understand and solve this task? and (c) What would you do as a teacher to help students overcome those challenges and understand and solve this task? By comparing participants' responses to these questions across the pre- and post-interviews, I was able to identify shifts in participants' affect toward students and learning.

Teacher Affect

There are several dimensions along which one may measure teachers' and prospective teachers' affect toward students, learning, and knowledge. For example, Rheinberg (1983) reported on three different evaluation norms that reflect teachers' beliefs about learning. These norms were a social reference norms (evaluation relative to peers); criterion-oriented norms (evaluation relative to fixed standards); and individual reference norms (evaluation relative to self). Dweck's (2006) work on mindsets expanded and elaborated Rheinberg's work, and others have demonstrated that growth mindsets (i.e., a belief that intelligence can be developed, as opposed to fixed mindsets which indicate a belief that intelligence is fixed or pre-determined) are correlated with higher achievement (as reported in Dweck, 2010).

In her review of affect and teacher education, Richardson (1996) reported on conceptions held by prospective teachers on students and learning that reflect a positivistic view of learning. In essence, prospective teachers tend to view students as receivers of knowledge, learning as memorization and/or routinization of procedures,

and knowledge as being either correct or incorrect. In many ways, this view of teaching and learning reflects a teacher-centered orientation as outlined by Civil (1992), which includes conceptions of teachers as the primary source of knowledge and the linearity of knowledge and learning (McDiarmid, 1990; A. G. Thompson, 1984). Student-centered orientations emphasize more interpretive conceptions of knowledge and learning, and they tend to view students as more active agents in the creation of knowledge (A. G. Thompson, 1984).

Other researchers have characterized teacher affect by contrasting the beliefs, attitudes, and emotions of teachers who take traditional instructional approaches to those who take inquiry-oriented approaches (e.g., Stipek, Givvin, Salmon, & MacGyvers, 2001). Similar dimensions exist, including a reflective or internal learning orientation versus an authoritarian or external learning orientation (Richardson, 1996; Vermunt & Vermetten, 2004), and a dimension that measures the degree to which teachers and students regulate learning and learning activities (Vermunt & Verloop, 1999; Vermunt & Vermetten, 2004).

One thread that seems to be woven throughout this literature is that affect toward student thinking and learning are aligned along a spectrum. On one end of the spectrum, there is a pervasive deficit perspective. Teachers who take on social or criterion-based evaluation norms tend to measure students' learning against other students or against a fixed set of standards (Rheinberg, 1983). This fixed mindset (Dweck, 2006) leads to notions that students either "get it" or they do not. Teacher-centered orientations are fueled by beliefs that teachers are the gatekeepers of

knowledge and that students learn by watching and mimicking what the teacher does (Civil, 1992; A. G. Thompson, 1984), which in turn leads to an emphasis on correct answers. On the other end of the spectrum, teachers view learning as a continual process (Rheinberg, 1983), students as agents of their own learning (A. G. Thompson, 1984), and knowledge or intelligence as something that can be developed through hard work (Stipek et al., 2001).

I now present evidence that participants appeared to experience a shift in their affect from the pre-interview to the post-interview. Specifically, in the pre-interview, participants tended to talk about students, learning, and teaching in ways that were consistent with the deficit perspective, whereas in the post-interview participants expressed views of learning and student think more consistent with a growth mindset perspective. I first present a more detailed analysis of one participant's shift and then discuss general trends among the remaining participants. Lily was not accounted for in any of the three prior shifts (with the exception that she was able to take on the perspective of Sasha and Keoni, potentially from decentering during the mini-course). Lily entered the study with extensive prior knowledge and already seemed oriented toward conceptual teaching. Consequently, I wanted to highlight her contributions to the mini-course and how she appeared to grow through her participation in this study. Of most importance, her shift in affect is representative of general trends in how participants experienced this shift.

Lily

Lily's responses to the three follow-up questions during the Ellipse Task (see Figure 4.1 above) in the pre-interview were consistent with a deficit perspective of student learning and thinking. She tended to talk about students' understanding in terms of what students would not think about or not know. For example, she said that students "wouldn't think of h is actually starting from the zero value or from the y -axis to this point (the center of the ellipse)." She later elaborated on this by saying:

If they are finding this distance [gestures along the vertical leg of the right triangle] they wouldn't thought of to borrow those information [circles the center (h, k)] in order to fit in this point [points to the vertex of the right angle of the right triangle], and they wouldn't think about they are actually in the same height. That's where they might be confused about.

Lily seemed to think that the difficulty in solving the Ellipse Task stems from students' inability to think about or know how to find the distance along the legs of the triangles. It is true that finding distances in the Ellipse Task may be challenging for students, but her comments suggest she viewed students as either "getting it" or not. Moreover, her ideas about how to help students overcome this challenge can be taken as more evidence that Lily had a deficit perspective of student understanding, and that for Lily, teachers are the source of knowledge. For example, she said

I would just jot down the dots [points to the dotted line she drew from the y -axis to the point (x, y) , see Figure 4.41 below] and also let them know that any point on the x - y coordinates, it would be starting from the coordinate [gestures along the y -axis, then rightward along the x -axis, which I believe indicates she meant to say axis] to that point [gestures vertically from the x -axis to the center of the ellipse (h, k)], and make sure that this is x and y [points to the coordinates of the point (h, k)].

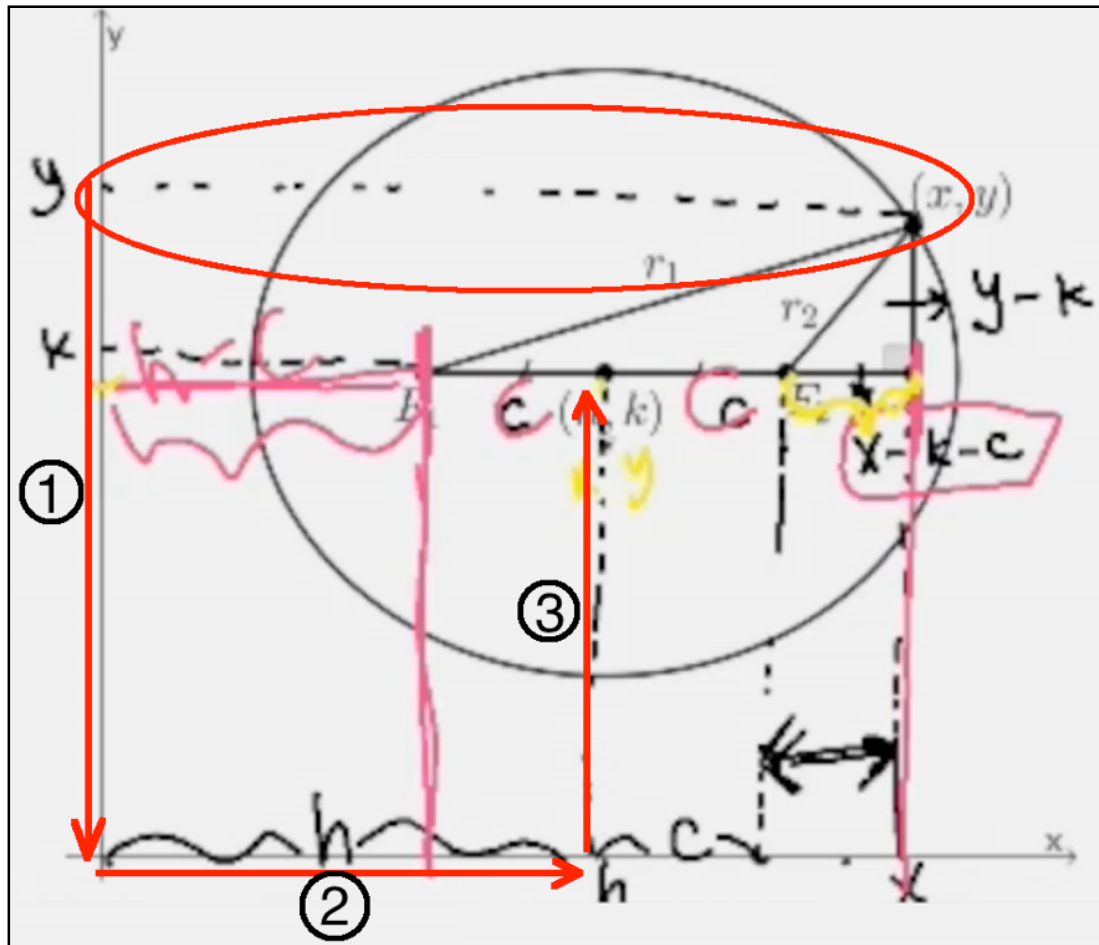


Figure 4.41. Author's annotations of Lily's work, including the dotted line she drew (indicated by an oval), and her gestures (1) along the y -axis, (2) along the x -axis, and (3) from the x -axis to the center of the ellipse.

In this excerpt, Lily seemed to suggest that she would do some of the work for the students. She said she would “just jot down the dots” and “let them know” how they should view the x and y coordinates. These statements are consistent with a teacher-centered view of learning and instruction, and serve as more evidence that Lily appeared to have a deficit perspective of student thinking. Her statements suggest that she believed students lacked some skill or knowledge (e.g., jotting down dots or seeing that a point has a fixed distance from each of the axes) and that as a teacher she

could provide the skill or knowledge by showing the students. In other words, she seemed to say that the challenge would be a deficit in student understanding that she could fix by showing and telling.

In the post-interview, Lily's comments about students' thinking and understanding suggested she had experienced a shift away from a deficit perspective. Even though Lily still thought that finding distances or lengths in the plane would be a significant challenge for students, she now talked about this challenge in terms of what students would need to know or understand to overcome it:

They [students] need to be able to distinguish what does the point actually mean. So for example, if I know this is h and k , then I need to know that k is the height from the origin to that point, so it's the distance.

Lily also said students must be able to “change the meaning of the point, convert it into the distance, the meaning of distance from the x -axis or the y -axis to that point.” There is evidence in these statements of a shift away from a deficit perspective. For example, no longer did Lily talk about students' knowledge and understanding by stating what they will not know, understand, or remember. Instead, she talked about what they'll need to know or do to come to an understanding about distance. The difference is subtle, yet important. Lily appeared to shift away from thinking about students *not* having knowledge, to shifting toward thinking about how students can come to understand a mathematical idea. In other words, in the pre-interview Lily focused on what students would not know (a deficit perspective), whereas in the post-interview Lily focused on what students need to come to understand (a growth mindset).

This shift is also apparent in how Lily talked about how she would help students. She no longer mentioned showing or telling students how to do something or “letting them know.” Instead, she talked about a “process of improving their understanding.” For example, in describing how to help students tackle the Parabola Task, Lily had the following to say:

I would let them [the students] start with [a parabola at] the origin so they don't have to deal with that much information, but they can still generate the equation by choosing a random point from the parabola. And then, I would move parabola, I wouldn't move the parabola right above the axis that much. I would either move up and down – so that's one action – or left or right. So it's a process of improving their understanding.

This excerpt serves as evidence of Lily's shift away from a fixed, deficit-perspective. Lily talked about providing students with a task they can access (“they don't have to deal with that much information, but they can still generate the equation”) and learn from as they build up understanding (“it's a process of improving their understanding”).

The previous excerpt points to another characteristic of how Lily talked about students in the post-interview, namely she appeared to express care for their understanding. For example, Lily said she would start with having students work with problems in which the parabola was on the grid, like Sasha and Keoni did in the MathTalk videos. She explained “Just like Sasha and Keoni did, they liked to circle things out to look for distance. It helps them see the relationship.” Lily also seemed to think that telling students wouldn't help them, another shift from the pre-interview. In talking about how she might help students with the task she had this to say:

I would say “Ok, this is the point to the directrix”—well, I can’t think of the words right now, but I wouldn’t tell them. I would want them to see the point to the directrix is the same as the point to the focus”

Lily did not quite know how to help students with using the definition of a parabola, but she was sure that she did not just want tell students how to use it. This and other statements represent a shift in affect for Lily, from seeing her role as a teacher as someone who shows (e.g., “jotting down the dots”) or tells (“I would just let them know...”) to seeing her role as a guide or facilitator.

In summary, during the pre-interview Lily seemed to have a deficit perspective of students’ thinking and understanding, and a teacher-oriented perspective with regard to her role in students’ learning. By contrast, in the post-interview Lily’s ideas were more consistent with a growth mindset for students’ thinking and understanding, and she seemed to view her role as a guide for students, rather than as the source of knowledge.

Other Participants

Five other participants (Desmond, Jasper, Marshall, Sierra, and Willow) also seemed to experience a shift in their affect. In broad terms, the shifts they experienced were similar to Lily’s, which I briefly describe in this section. In the pre-interview participants appeared to have a narrow, deficit perspective view of students’ thinking and understanding and they tended to view the role of a teacher instruction as showing and telling students what to do.

For example, Marshall said that students needed to “remember” the area formula for a triangle, since “if they didn’t remember, it’s, you almost don’t know

where to begin.” Willow echoed this by stating “They don’t know that formula, they wouldn’t even have anywhere to really start.” Willow also seemed to think students would struggle with the task since ellipses are not “covered” until later in high school. Desmond said that students would experience “word overload” and “constant overload” when trying to solve the task. Jasper seemed to think that students would need actual values to solve the task. He said “I don’t know if high school students would think of it this way [meaning the task as given with unknowns]. A high school student wants numbers, they want actual values.” According to Sierra, students would forget how to use the Pythagorean Theorem by confusing the hypotenuse for a leg. She said “They [students] don’t know that the c is on the other side [of the equation], like that it equals c squared. They use the hypotenuse over here [the other side of the equation] with the other leg.”

Like Lily, the other five participants appeared to have a teacher-oriented perspective. For instance, Desmond, Jasper, and Sierra all seemed to think that providing students with values instead of unknowns would be a good way to help them solve the Ellipse Task. Jasper succinctly stated the prevailing idea when he said, “I mean, that’s a way to help them [students] understand this, is to give them values.” Unlike Lily’s suggestion in the post interview to link those values with quantities, participants in the pre-interview tended to offer this suggestion by only considering values in the absence of any context or without an underlying goal of helping students quantify.

Desmond, Jasper, Marshall, and Willow appeared to have in mind a step-by-step instructional method for helping students with the task. Desmond talked about showing students “extensive, detailed problems on the board,” and Marshall said he would “guide them” by “breaking it down into simpler steps” and giving students “a worksheet with a list of steps of what needs to be done.” To put it succinctly, these participants seemed to think of instruction in terms of providing clear information and step-by-step procedures for students to follow, which suggests they believed that students need teachers to fill in the deficits in knowledge students may have.

In the post-interview, participants appeared to shift away from a deficit perspective of students’ knowledge and learning. Like Lily, the other five participants seemed to have more ideas about what students will already understand and be able to do. Instead of emphasizing what students will not know or remember, they discussed what they must understand in order to solve the task. For example, Willow said that students might struggle with algebra, but “that’s kind of not the main goal of being able to do the algebra, because that’s really fixable. But understanding how the triangle, like how we get these distances for the triangle is key.” Instead of focusing on “being able to do the algebra,” Willow emphasized *how* to find the distances or lengths of the sides of the triangle. Jasper, Marshall, and Sierra all made similar claims about students needing to come to understand how to find distances in the coordinate plane.

Compared to the pre-interview, participants also had different ideas about their role as a teacher. For example, instead of providing step-by-step worksheets,

Marshall talked about leveraging students' prior knowledge and building on that. He said he would ask them questions such as "What did we use in the previous tasks?" and "What have you done in the past that you can use again?" Sierra said that she would use a sequence of tasks like those used in the videos, which she seemed to think would help students "build up to generalizing." Jasper said he would ask students to "explain what each term was—What is x ? What is h ?" Finally, Desmond abandoned his idea that solving a task was of the utmost importance. Instead, he would ask students "What does it look like? Don't try to solve, but what should this be like." Desmond seemed to say that he would want students to analyze the task before jumping in and trying to solve it; this is very different from his pre-interview ideas about giving students values to make the problem easier for them to solve.

The lone participant who did not seem to experience a shift in affect was April. I did not include her in this shift because her statements about students and instruction in the pre-interview did not strongly reflect a deficit perspective. Instead, they were more consistent with a growth-oriented perspective. For example, she talked about asking guiding questions to help students rather than telling them directly, or prompting them by asking them what they already knew. Her statements about student understanding reflected a view that students may "have trouble" or "struggle," but not that students lacked knowledge or understanding. This affect seemed magnified in the post-interview, and it more closely matched the other participants' affect from the post-interviews. However, it did not appear that April's affect had shifted dramatically

from the pre-interview to the post-interview, specifically because her affect in the pre-interview already seemed to be more growth-oriented and student-centered.

Summary and Brief Discussion

In posing Research Question 1, I wanted to investigate the *nature* of MKT that developed. By examining shifts in MKT, I identified different facets of MKT related to quantitative reasoning that seemed to develop, as well as a shift in participants' affect. The shift in quantitative reasoning accounts for the mathematical knowledge participants developed, knowledge that is foundational to developing MKT. The shift in point in view explicitly accounts for how participants were able to solve a task using quantitative reasoning that was qualitatively and mathematically similar to the reasoning used by high school students in the MathTalk videos. The shift in orientation accounts for how participants' orientation toward solving problems related to quantitative reasoning changed from one that emphasized calculations and procedures in the pre-interview to one that emphasized quantities and quantitative operations. Finally, the shift in affect describes a broader aspect of MKT, and seems to be related to van Bommel's (2012) finding that MKT development requires a shift in a prospective teacher's identity from a learner of mathematics to a teacher of mathematics.

In total three of the four shifts (the shifts in quantitative reasoning, point of view, and orientation) account for different ways participants developed MKT around quantitative reasoning. Additionally, though only three of the seven participants seemed to experience strong shifts in quantitative reasoning, of the other four

participants, three of those appeared to develop the ability to decenter (shift in POV) and two of those shifted in orientation. In other words, six of the seven participants experienced one or more shifts in MKT related to quantitative reasoning.

In Chapter 2, I described the framework for MKT presented by Silverman and Thompson (2008), which includes five components of MKT: (a) KDU around the content, in this case quantitative reasoning; (b) images of students' thinking and understanding; (c) images of milestones for a learning trajectory for how the content develops over time; (d) images of instruction; and (e) images of mathematical connections. My study was designed with this framework in mind; however, in hindsight my interview protocols did not align completely with this framework and were not sensitive enough to capture MKT development across each of these dimensions. That said, the shifts I identified in this chapter do have a degree of alignment with the framework, which I now briefly discuss.

The shift in quantitative reasoning corresponds to the development of a KDU, which is the first component of MKT. In this chapter I do not make claims that participants developed or had a KDU around quantitative reasoning, as that construct describes a more complex and larger web of knowledge for which I did not analyze. Nonetheless, the shift in quantitative reasoning seems related to, and necessary for, the development of a KDU around quantitative reasoning.

The second shift, which is a shift in point of view, is related to the construct of decentering. Teuscher et al. (2016) described decentering as the ability to put aside one's own mathematical understanding to better understand another's way of

understanding a mathematical situation. They argued that decentering is a necessary requirement for the development of MKT, and Silverman and Thompson (2008) used the construct to describe how teachers build images or models of how students understand the content, which is the second component of MKT.

In presenting their framework, Silverman and Thompson (2008) said, “A teacher has knowledge that supports *conceptual* [emphasis added] teaching of a particular mathematical topic when he or she [has developed the five components of MKT]” (p. 508). I emphasized the qualifier *conceptual* because the framework attempts to account for how teachers develop MKT for teaching that is different than traditional teaching in which calculations and procedures are emphasized over understanding and concept development. Consequently, it can be argued that having a *conceptual* orientation is a prerequisite for developing MKT for *conceptual* teaching. In this way, the third shift, a shift in orientation, is linked with the framework for MKT development. In Chapter 6 I discuss these results in more detail.

Having identified the nature of MKT participants seemed to develop during the mini-course, I now turn to answering Research Question 2. In the next Chapter, I describe how what participants noticed appeared to be linked with the shifts in MKT around quantitative reasoning. I then explore how features of the mini-course contributed to participants’ noticing.

Chapter 5: Development of MKT Around Quantitative Reasoning with Distances

In this chapter, I argue that shifts in participants' MKT around quantitative reasoning with distances appear to be linked to elements of the learning environment. In Chapter 4, I elaborated four shifts that described the nature of the MKT participants seemed to develop during the mini-course. Three of these shifts were associated with quantitative reasoning with distances, while the fourth was related to participants' affect. This fourth shift was unexpected, as I had not accounted for such a shift, either in the planning or execution of the design of the study. Consequently, there was little data from the mini-course that suggested how this shift developed. Considering this, I restate Research Question 2 below:

Research Question 2: How do particular elements of the designed learning ecology contribute to the development of MKT by prospective secondary teachers during a video-based mini-course?

The mini-course provided participants with a flood of information emanating from the math tasks they completed, the conversations in which they engaged, and the MathTalk videos they watched. Accordingly, participants' attention could be drawn to any of the myriad mathematical or pedagogical features of these elements of the mini-course. As defined by Lobato et al. (2013) in their presentation of the focusing framework, *noticing* is “selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for one's attention” (p. 809). To answer Research Question 2, I extend this definition to include pedagogical features or regularities (e.g., instructional moves to foster learning or students' mathematical struggles), in addition to mathematical ones.

In this chapter, I leverage the focusing framework to account for how participants' shifts in MKT around quantitative reasoning with distances seemed to develop. This chapter is organized around the four components of that framework. First, I investigate the *centers of focus*, which describe features of the mini-course that participants noticed, and I discuss how a shift between those centers of focus seemed related to the development of participants' MKT. Next, I examine how two *focusing interactions* may have contributed to that shift in noticing. These discourse practices had the effect of directing participants' attention toward one of the centers of focus. Finally, I provide accounts for how *features of mathematical tasks* supported participants' noticing and changes in the *nature of mathematical activity* contributed to the shift in what participants noticed. I conclude the chapter with a summary of the results and a brief discussion about the role the MathTalk videos seemed to play in the development of participants' MKT.

Centers of Focus

Lobato et al. (2013) defined centers of focus as “the properties, features, regularities, or conceptual objects that students notice” (p. 814). In other words, centers of focus account for what seems to hold individuals' attention. Two caveats must be made about centers of focus. First, researchers do not have direct access to the process of noticing; consequently, centers of focus are inferred based on participants' verbal utterances, gestures, and written responses. Second, centers of focus capture what individuals notice; therefore, it is likely to observe multiple centers of focus in each learning situation.

Conception and Methods

The centers of focus that Lobato, Rhodehamel, et al. (2012) and Lobato et al. (2013) described were mathematical features or regularities in a classroom setting that seemed to capture students' attention. I extend this conception of centers of focus to include other features or regularities of a teacher training course. While participants in this study attended to mathematical features, they also tended to notice pedagogical features of the mini-course. Such features included the reasoning of Sasha and Keoni, the tasks or questions posed by the instructor in the videos, and the questions and prompts I posed to participants during the instructional sessions.

To identify centers of focus that seemed to emerge in the mini-course, I analyzed two sources of data. The first source of data was approximately 24 hours of video from the six instructional sessions. I used two video cameras to capture each two-hour session; one camera was fixed on one of the two small groups, while the other was operated by the observer, who focused the camera on the other small group or the whole class as appropriate. The written work and reflections participants completed during the mini-course served as the second data source.

Reducing the corpus of data was crucial given the amount of video data I collected. To reduce data, I watched all videos from the instructional sessions and created descriptive accounts. I flagged episodes in the descriptive accounts that appeared related to the shifts in MKT around quantitative reasoning with distances. The first analytic pass of the data focused on those episodes.

My first pass through the data was informed by the framework for MKT (Silverman & Thompson, 2008). I thought it would be fruitful to look for instances in the data where participants seemed to notice components of MKT since I had designed the mini-course with this framework in mind. Accordingly, I applied the framework to the content (quantitative reasoning with distances) and the context (a video-based mini-course) to develop the following a priori codes: (a) mathematics associated with quantitative reasoning with distances; (b) ways Sasha and Keoni reasoned quantitatively with distances; (c) milestones for a learning trajectory for developing quantitative reasoning with distances; and (d) the video instructor's quantitative instructional moves. These codes correspond with the first four components from the framework for the development of MKT proposed by Silverman and Thompson (see Chapter 2 for elaboration of the framework). The code that corresponds with the fifth component (mathematical connections) was not used because as I had conjectured prior to conducting the study, participants did not attend to the connections students could make as a result of developing quantitative reasoning with distances.

Following Lobato et al. (2013), I considered the verbal utterances, gestures, and written work of each participant during each instructional session and labeled data using the a priori codes to describe what the participant seemed to be noticing. Using a mixed approach (Miles & Huberman, 1994) to grounded theory (Strauss & Corbin, 1990) allowed me to use a priori codes while remaining open to the emergence of new codes. Indeed, as I analyzed data, new codes emerged because the four a priori codes did not account for all centers of focus. As a new code emerged, I reevaluated

previously coded data to determine if the new code was a more appropriate label.

Using this constant comparative method (Glaser & Strauss, 1967) provided for greater precision and accuracy in developing codes to describe centers of focus.

In total, four additional codes emerged that seemed related to components of the Silverman and Thompson (2008) framework for MKT. However, these centers of focus were not related to quantitative reasoning with distances. Instead, they were related to mathematics that does not appear in the MathTalk videos (such as the distance formula, law of sines, quadratic equations, etc.). In other words, these centers of focus emerged from participants' own mathematical knowledge they brought into the mini-course. These centers of focus were: (a) mathematics not in the MathTalk videos; (b) ways of reasoning Sasha and Keoni did not use; (c) milestones for learning trajectories of mathematics not addressed in the MathTalk videos; and (d) the video instructor's moves related to mathematics not in the MathTalk videos. All eight codes are summarized below in Table 5.1. They are grouped into two macro-codes: (a) MKT around quantitative reasoning with distances, and (b) MKT around mathematics not in the MathTalk videos.

Finally, I note that these codes capture what participants seemed to notice during the mini-course. In presenting data with corresponding codes, I am not evaluating or judging the data. In other words, my coding of the data was not attaching an evaluative label such as "correct" or "appropriate."

From micro to macro centers of focus. Initial analysis of the classroom data resulted in the eight centers of focus (CoFs) described in Table 5.1 below. To account

for participants' development of MKT, I looked for links between individuals' shifts in MKT and shifts from one CoF to another. However, a major issue immediately emerged with this approach: the eight CoFs did not directly correlate with the shifts in MKT I had identified in the interview data (as presented in Chapter 4).

Two factors may account for this, each associated with an aspect of the design of my study. First, in hindsight, my interview protocols lacked the sensitivity to capture participants' explicit knowledge about the various components of the Silverman and Thompson framework for MKT. My tasks and follow-ups did not adequately address these components, (e.g., milestones for a learning trajectory for developing quantitative reasoning with distances). Consequently, the shifts in MKT identified in Chapter 4 are not closely aligned with components of the MKT framework.

Second, when designing the mini-course, I did not anticipate the specific shifts in MKT that participants seemed to experience. Additionally, the mini-course was designed with the framework for MKT in mind. Consequently, it is not surprising that the CoFs that did emerge were linked with the components of the framework, and not the shifts in MKT from Chapter 4.

Table 5.1. Description and examples of eight codes that emerged during the initial analysis of the classroom data. These codes are grouped into two macro codes: (a) MKT around quantitative reasoning (QR) with distances, and (b) MKT around mathematics not in the MathTalk videos.

	Codes	Descriptions of Codes	Examples
MKT around QR with distances	Math associated with QR with distances	Participant attends to distances, quantities, or quantitative operations in own mathematical activity.	While solving a math task, participant draws vertical segment from point on parabola to x-axis and says, "This distance would be y."
	Ways Sasha and Keoni reasoned quantitatively with distances	Participant attends to distances, quantities, or quantitative operations in Sasha and Keoni's mathematical activity.	A participant says she thinks Sasha and Keoni will "draw the circles and count the distance down."
	Milestones for a LT for developing QR with distances	Participant notices some developmental milestone or ordering of such for developing QR.	A participant explains that Sasha and Keoni began finding distances using a grid and counting, but later they could find distances without a grid or counting.
	Video instructor's quantitative instructional moves	Participant attends to a question, comment, task, or challenge related to the development of QR, posed by Joanne to Sasha and Keoni.	A participant notices that Joanne asked Sasha and Keoni "Where do you see the y?"
MKT around mathematics not in the MathTalk videos	Math not in the MathTalk videos	Participant attends to a mathematical idea in own mathematical activity that does not appear in the MathTalk.org videos (e.g., the distance formula, law of sines, solving quadratic equations).	A participant tries to create a parabola from the geometric definition by constructing a coordinate grid and using the distance formula.
	Ways of reasoning Sasha and Keoni did not use.	Participant attends to mathematics not in the videos while noticing Sasha and Keoni's mathematical activity.	A participant asks, "Why don't Sasha and Keoni use the Law of Sines [for finding length of side of triangle]?"
	Milestones for LTs not addressed in videos	Participant attends to mathematics not in the videos while noticing potential/hypothetical developmental milestone or ordering of such for developing QR.	A participant says that Sasha and Keoni should learn how to solve quadratics in order to complete one of the tasks in the videos.
	Video instructor's moves related to math not in the videos	Participant attends mathematics not in the videos while noticing a question, comment, task, or challenge that was not posed by Joanne to Sasha and Keoni.	After seeing a video in which Joanne let Sasha and Keoni explore whether or not the focus can be on the directrix, a participant remarks that his "gut reaction would be to step in and say it's wrong, to try something

One additional issue complicated linking specific shifts in MKT with shifts between CoFs. The eight CoFs capture micro-level noticing. This made it difficult to identify significant shifts in CoFs that I could link to the three shifts in MKT related to quantitative reasoning with distances present in the interview data. For example, in Chapter 4, I presented evidence that suggested Willow had experienced a shift in quantitative reasoning. One might think that I would see evidence of Willow shifting from the CoF *mathematics not in the MathTalk videos* to the CoF *mathematics associated with quantitative reasoning with distances*. Indeed, this shift did occur; however, this shift in CoFs alone did not account for Willow's shift in quantitative reasoning. For example, at several points in the mini-course, Willow seemed to notice the CoF *the video instructor's quantitative instructional moves*. Consider this exchange in Session 5 in which Willow points out the quantitative questions the instructor in the videos posed:

- Willow: She [the instructor] went from the broad, like the general parabola—right?—to the grid, and then back to the general. It's like now they [Sasha and Keoni] can visualize the units without actually—like if she had kept the graph there, they still probably would have been stuck.
- Jasper: I think she's having them explain every single variable they're using. Like y , p , x . “What do you mean by y ?” “Now what do you mean by p ?” “Now what about x ?” Explain each variable.
- Willow: And “What do you mean by y plus p ?” “Which one is p ?” “Which one is y ?” “And which one is y minus p ?”
- Lily: And she keeps asking “Where is the p ?” “Where is the y plus p ?”
- Willow: Every time!
- Lily: “Where is the y minus p ?”

Willow: And even when they were using numbers, they were asked that.

Willow and other members of her group seemed to notice the quantitative questions the instructor posed to Sasha and Keoni. Later, in Session 6, Willow said that the questions the instructor asked “were the kind of questions you could ask yourself in the future.” Taken together, this suggests that the teacher’s instructional moves were salient for Willow, and it is reasonable to conclude that noticing these moves contributed to Willow’s shift in quantitative reasoning. In other words, Willow’s shift in quantitative reasoning cannot be attributed solely to a shift in CoFs that are purely mathematical (i.e., from *mathematics not in the MathTalk videos* to *mathematics associated with quantitative reasoning with distances*). Consequently, the grain size for the CoFs presented in Table 5.1 was too fine to account for shifts in MKT around quantitative reasoning with distances.

Settling on a productive grain size appears to be part of the process of identifying CoFs (J. Lobato, personal communication, May 3, 2017). Considering the preceding discussion, I examined CoFs on a macro-level by collapsing the four CoFs that dealt with MKT around quantitative reasoning with distances into one macro code and the four CoFs that dealt with MKT around mathematics not in the MathTalk videos into another macro code. This resulted in two macro-level CoFs: *MKT around quantitative reasoning with distances* and *MKT around mathematics not in the MathTalk videos*. These codes are at the far left of Table 5.1.

Once these macro-level CoFs were established, I counted instances when each participant seemed to notice either of these CoFs. I did this for each participant across

the six instructional sessions. Those counts are provided in Table 5.2 below. As a reminder, I could only infer what participants noticed based on their visual or audible contributions to the group and classroom conversations, or their written inscriptions. These counts only include instances for which I could reasonably infer what a participant noticed. Consequently, the counts almost certainly do not account for all instances in which each participant noticed either CoF.

To get a better idea of how these CoFs may have played a role in the development of each of the shifts in MKT around quantitative reasoning identified in Chapter 4, I also counted instances of the CoFs by shift in MKT around quantitative reasoning with distances. For example, I counted all instances for each participant who experienced a shift in quantitative reasoning (Desmond, Willow, and Sierra). These counts are also provided in Table 5.2 below. To remind the reader of which participants experienced each shift I use the following symbols: (a) * denotes participants who experienced a shift in quantitative reasoning (QR); (b) † denotes participants who experienced a shift in point of view (POV); and (c) ‡ denotes participants who experienced a shift in orientation toward problem solving (Orientation).

Centers of Focus: Results

Table 5.2 provides an overview of what participants seemed to notice. For example, looking across the row for April one can see that she initially noticed MKT around mathematics not in the videos more frequently than she noticed MKT around quantitative reasoning with distances. However, in Session 4, those counts were equal,

and after Session 4, April noticed MKT around quantitative reasoning with distances more frequently than she did MKT around mathematics not in the videos. Not all participants followed this pattern. For example, only in Session 1 did Lily seem to notice MKT around mathematics not in the videos more frequently than she did MKT around quantitative reasoning with distances. One pattern that held for all participants (except Desmond) was that from Session 4 onward, participants tended to notice MKT around quantitative reasoning with distances as frequently as or more frequently than they tended to notice MKT around mathematics not in the videos.

In looking at aggregate totals for shifts in MKT, one can also see that there appeared to be a shift in CoFs between Session 2 and Session 3. For example, in looking at the “Total: QR” row, in Sessions 1 and 2, participants in this category noticed MKT around mathematics not in the videos far more frequently than they did MKT around quantitative reasoning with distances. But this trend reversed itself starting in Session 3 and continued throughout the remainder of the mini-course. Similar trends can be seen in the “Total: POV” and “Total: Orientation” rows.

Center of focus: MKT around mathematics not in the MathTalk videos. In this section, I present qualitative evidence to support the claim that in Session 1 and Session 2 participants noticed MKT around mathematics not in the MathTalk videos. I present two episodes as examples of how this CoF manifested during the first two instructional sessions.

Table 5.2. Counts of instances of each center of focus, by participant, across the six instructional sessions. The bottom part of the table includes count totals for each shift in MKT around quantitative reasoning with distances.

Center of Focus	Session 1		Session 2		Session 3		Session 4		Session 5		Session 6	
	MKT around QR with distances	MKT around math not in the videos	MKT around QR with distances	MKT around math not in the videos	MKT around QR with distances	MKT around math not in the videos	MKT around QR with distances	MKT around math not in the videos	MKT around QR with distances	MKT around math not in the videos	MKT around QR with distances	MKT around math not in the videos
Jasper † †	3	4	7	8	5	3	10	8	9	3	3	3
Marshall †	6	8	13	12	10	6	8	8	8	6	4	1
April † †	2	18	12	17	6	9	9	9	7	5	3	1
Desmond * † †	0	18	4	8	2	5	0	3	2	3	1	0
Lily	0	2	11	5	10	1	6	6	6	0	2	1
Willow * † †	4	1	3	10	5	7	7	2	9	1	2	1
Sierra * † †	2	10	8	4	14	1	8	4	5	5	2	0
Totals	17	61	58	64	52	32	48	40	46	23	17	7
Total: QR *	6	29	15	22	21	13	15	9	16	9	5	1
Total: POV †	11	30	32	37	21	18	27	25	24	14	10	5
Total: Orientation †	11	51	34	47	32	25	34	26	32	17	10	5

In the first episode, participants seemed determined to utilize a coordinate grid to create a parabola, even though a grid is not needed, and potentially obfuscates quantitative aspects of the task. This episode took place in Session 1, during which participants were creating a parabola using its geometric definition:

A parabola is the set of all points equidistant from a fixed point (the focus) and a fixed line (the directrix).

April, Desmond, and Sierra seemed fixated on the idea that to create a parabola, they needed to first construct a coordinate grid, and then use coordinates to either find the equation of a parabola (such as $y = x^2$) they could place on the grid or to place points that they could test with the distance formula.

April argued that “it would be easier to put it [the parabola] on a coordinate system” so the group could leverage the distance formula. When I asked the group to talk about their method, Desmond explained that they “went back to the equation,” presumably referring to the distance formula. At one point, I issued the following challenge to the group: “Without using a coordinate grid, create a parabola for a given fixed point and fixed line.” However, their fixation with the grid continued even after this challenge. As can be seen in Figure 5.1a below, the group labeled the points (2,0) and (0,0). When I asked the group to explain their solution, April told me that the directrix could be the x -axis and the segment connecting the two labeled points could be their y -axis. Finally, it appears they used a grid-like system to place points on the wider parabola, and in Figure 5.1b below, I have superimposed a grid on their work, highlighting the degree to which their inscriptions fit a grid-like system.

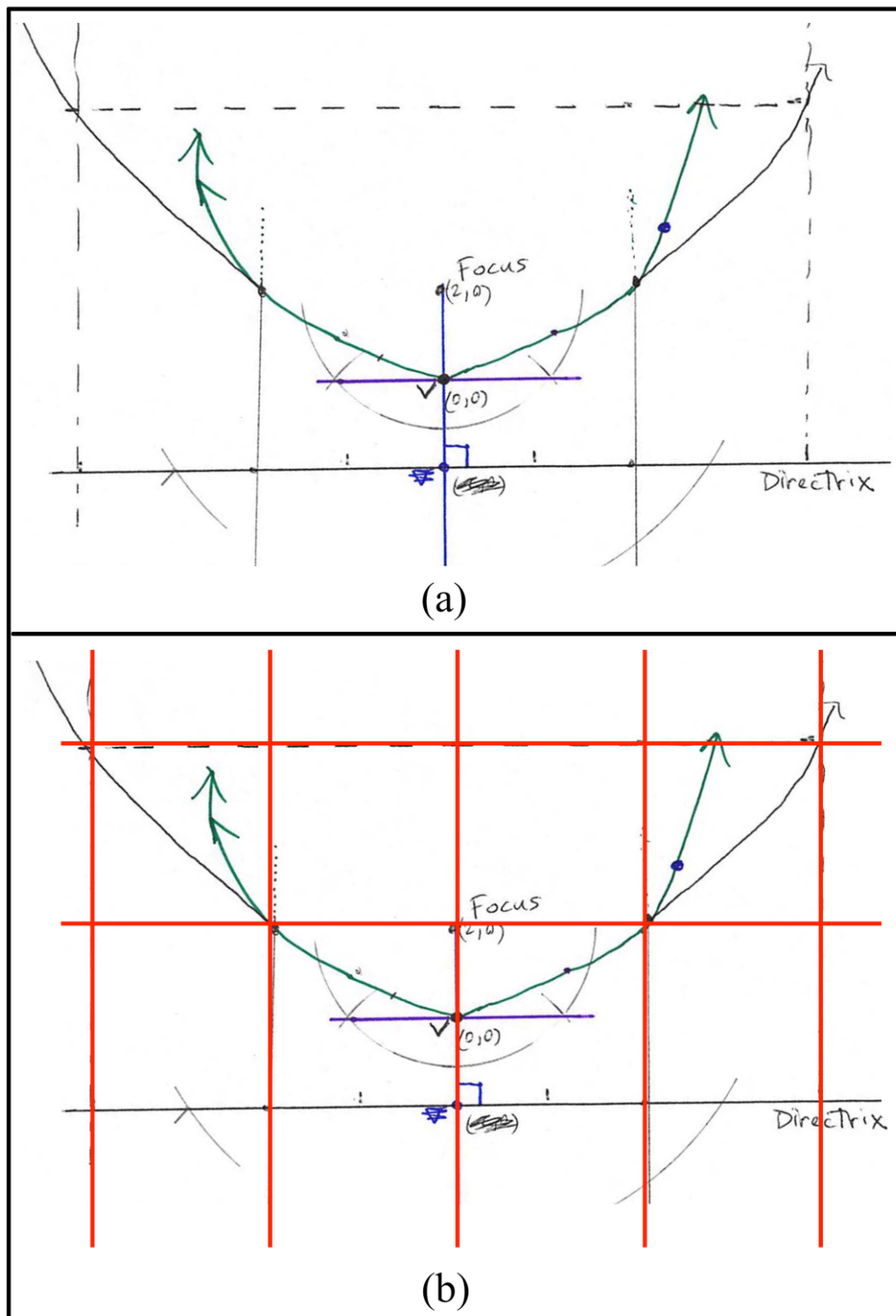


Figure 5.1. (a) Original inscriptions by Desmond, Sierra, and April. The focus and directrix were drawn by the author during the first instructional session. (b) Author's annotations highlighting the grid-like nature of the participants' inscriptions.

The use of a grid with coordinates does not appear in Lesson 1 of the MathTalk videos, in which Sasha and Keoni create a parabola from the definition. While constructing a grid and using the distance formula (as April suggested) would be a viable mathematical approach for the task, such an approach would foreground calculations and numerical operations at the expense of quantities and quantitative reasoning. Recall that quantifying means recognizing that an object has a certain quality that can be measured, and then assigning a measure to it (P. W. Thompson, 1994). One of the primary features of the task is that it provides opportunities for solvers to explore distances as quantities in the absence of fixed numerical systems (such as a coordinate grid).

For example, when placing or testing points, solvers of the task need to determine if a point is “equidistant” from both the focus and the directrix. Solvers can use several different tools (e.g., string, wire, compass, or ruler) to measure the two distances, and the choice of tool is not important. What matters is that approaching the task without a grid allows solvers to attend to distances as quantities, which can be imagined in the absence of numerical systems. If participants first construct a grid in an attempt to use a known equation for a parabola (e.g., $y = x^2$) or use the distance formula, this potentially reduces the number of opportunities to quantify, and instead the task serves as an opportunity to calculate with coordinates as values.

Participants also tended to notice MKT around mathematics not in the videos even while watching or discussing the videos. At times, particularly during the first two instructional sessions, participants attended to the videos through a lens of the

mathematics and instructional moves not present in the videos. In other words, participants saw “deficiencies” in the mathematics and instructional moves they noticed in the videos. As an example, I present the following episode in which some participants noticed that the instructor in the videos, along with Sasha and Keoni, seemed to ignore or be unaware of the fact that when solving an equation involving square roots, there are often two solutions, one positive and one negative.

In the second instructional session, participants were tasked with creating a method for finding specific x -values when given the y -value of a point on a parabola (see Figure 5.2). After they completed this task, they watched videos in which Sasha and Keoni solved a similar task. During the class discussion about the videos, Marshall reported that his group (which also included Jasper, Sierra, and Desmond) noticed that Sasha and Keoni solved the equation $x^2 = 4y$ for x to get $x = \sqrt{4y}$. Marshall continued by saying his group noticed that “it really should be plus-or-minus square root of four y ,” and that because Sasha and Keoni did not include the negative, it “only described the right-half of the parabola.”

April noticed the “plus-or-minus” as well, arguing that the instructor did not want to bring up “plus-or-minus” square roots. When I asked the class if there was anything they would want to tell Sasha and Keoni, April said she thought if you drew a horizontal line through the parabola, Sasha and Keoni would see there are two x 's, and that then they would “know to put plus-or-minus”. She continued, saying she was “really curious why the teacher did not want to bring that up yet.”

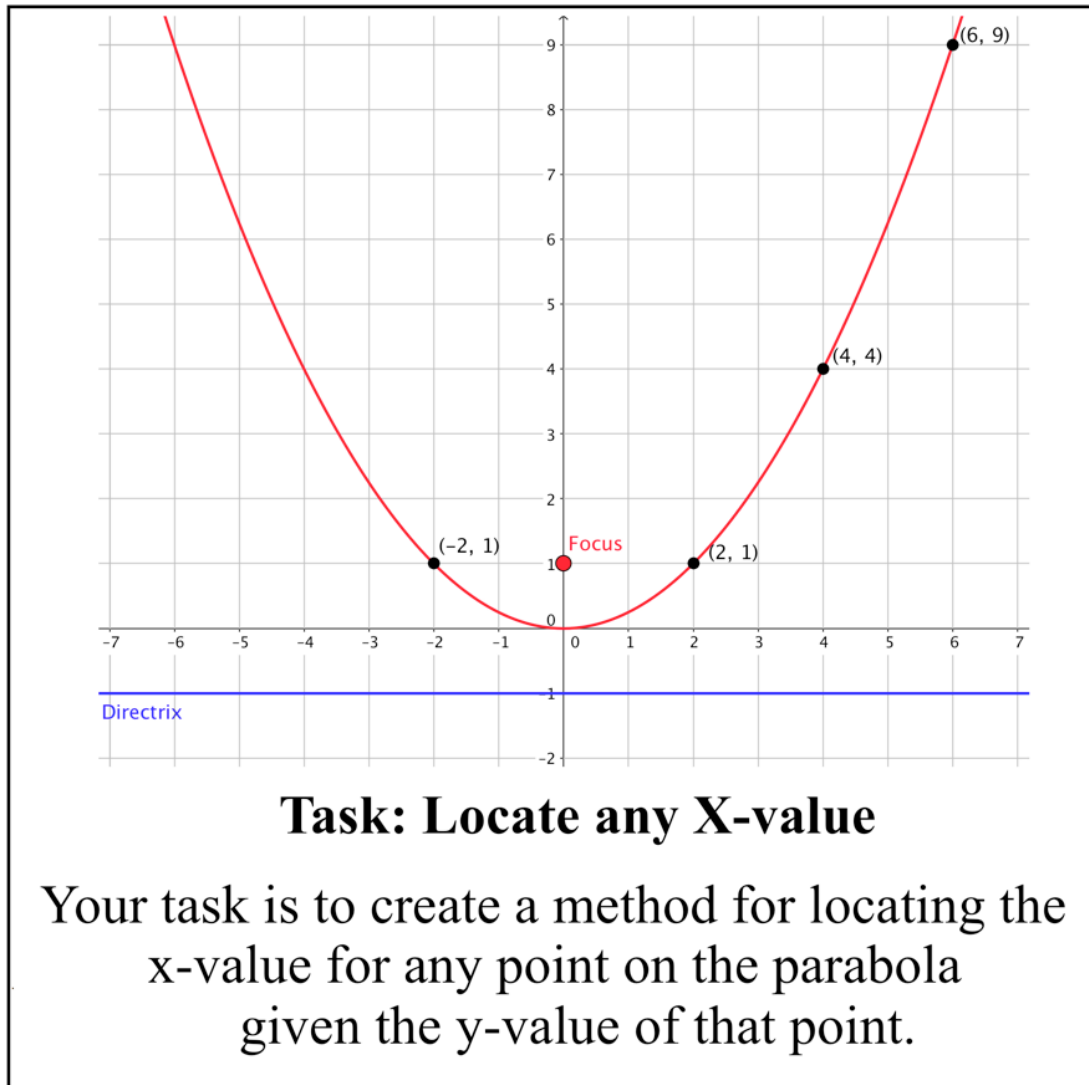


Figure 5.2. The math task for Session 2.

Finally, Willow offered her thoughts by observing that Sasha and Keoni “didn’t even suggest to put [plus-or-minus].” She conjectured that perhaps Sasha and Keoni were thinking since the parabola was symmetrical, they could “just flip it,” presumably referring to finding the values for the x -coordinates for one side of the parabola and then copying or mirroring that side to find corresponding values for the other side. Willow also wondered about the instructor, stating “would it be too much

to explain to them that it [the square root] is the absolute value, it's the distance from zero that we're looking at." Not only did Willow notice that Sasha and Keoni did not use "plus-or-minus," but her comment "would it be too much to explain to them..." suggests that Willow had ideas or expectations about instruction that were not realized in the videos.

These questions and concerns about the "plus-or-minus" are not necessarily incorrect or mathematically invalid. Indeed, while conducting the study, I asked the instructor from the videos if she intentionally ignored the left side of the parabola and the "plus-or-minus" issue with square roots. She responded that there were many valid learning trajectories an instructor could target, but that she had decided not to foreground either solving square root equations or the left side of the parabola because either might have waylaid her goal of helping Sasha and Keoni create and interpret expressions and equations quantitatively in terms of distances (J. Lobato, personal communication, June 2016).

As Lobato et al. (2013) contended, what one notices necessarily constrains what one is able to learn. In this case, by attending to the mathematics and instructional moves not in the MathTalk videos, participants potentially ignored the nascent quantitative reasoning exhibited by Sasha and Keoni and the quantitative questions posed by the instructor in the videos.

Center of focus: MKT around quantitative reasoning with distances.

Session 3 brought about a dramatic shift in what participants noticed. From Session 3 onward, MKT around quantitative reasoning with distances was noticed more

frequently than was MKT around mathematics not in the videos. As an example of how this CoF manifested, consider this episode from Session 3. The groups were tasked with finding the y -value given the x -value for any point on a specific parabola (see Figure 5.3 for the given task).

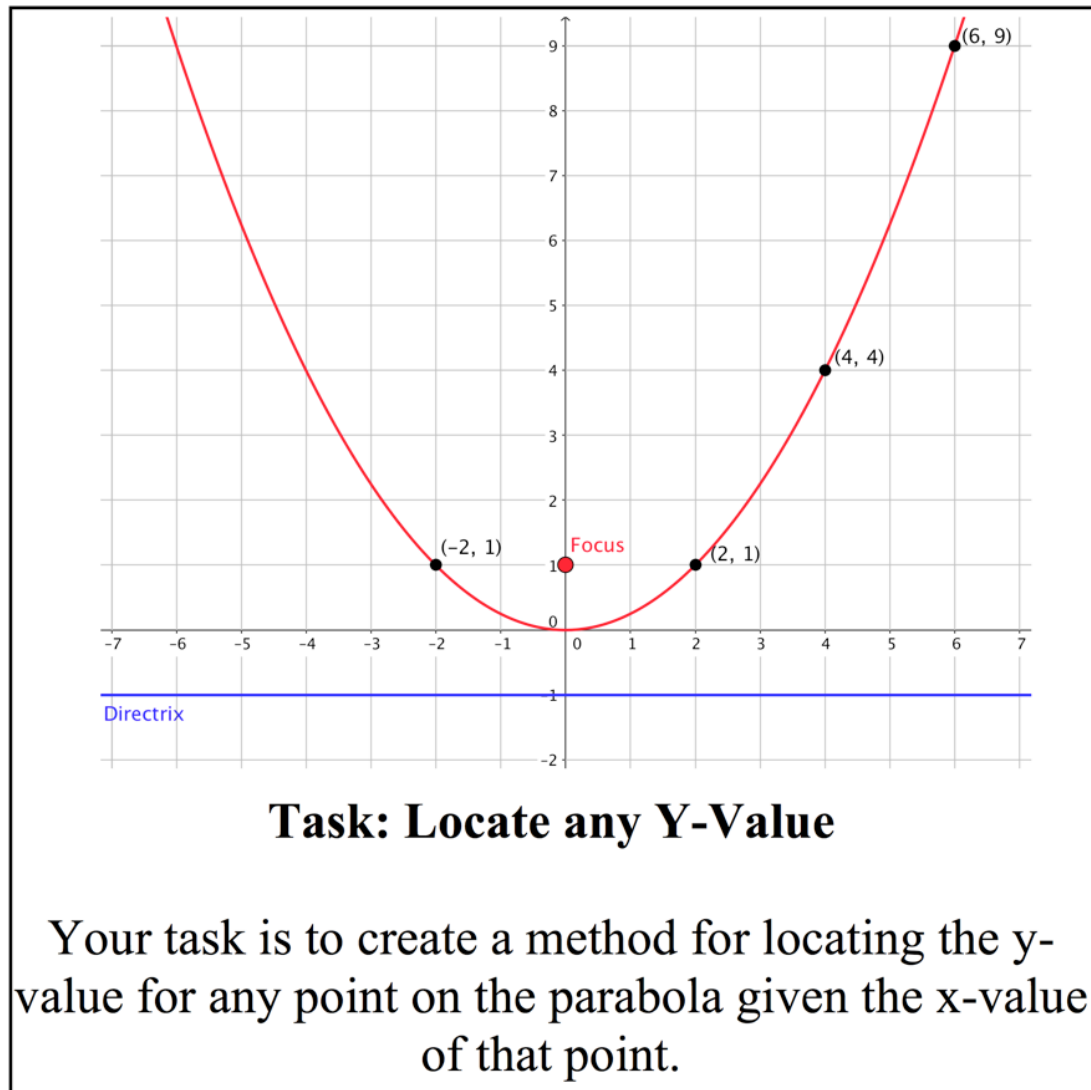


Figure 5.3. The math task for Session 3.

The group consisting of Desmond, Willow, Sierra, and Lily had been working on the task for about five minutes, and their plan seemed to be to inscribe triangles on

the graph in service of using the Pythagorean Theorem to derive an equation. They had just drawn a triangle on the graph (in dashed lines, seen below in Figure 5.4) with vertices at $(4,4)$, $(0,1)$, and $(4,1)$. In the following exchange, notice how both Sierra and Lily use language and gestures that are consistent with indicating distances in the plane.

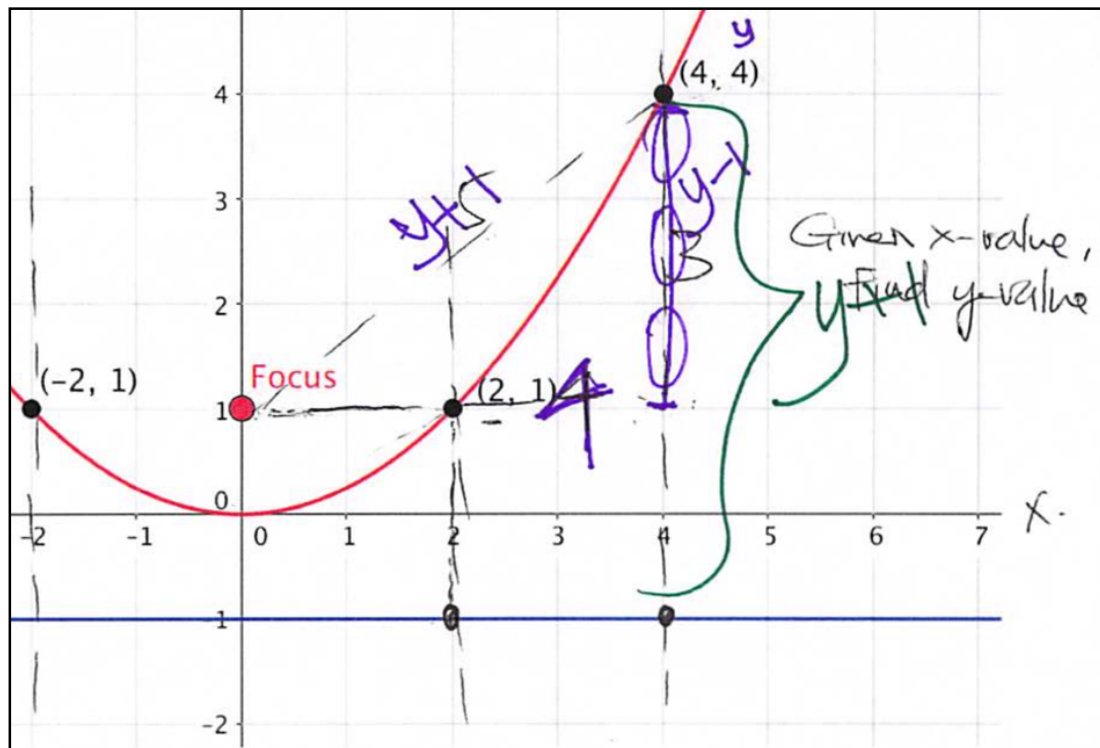


Figure 5.4. The dashed triangle can be seen in the group's work. The purple circles and green bracket were added after the exchange detailed in the given transcript while the group further discussed the task.

- Lily: We can still think about it with the Pythagorean Theorem.
- Sierra: So, we don't know this? [points to Willow's paper, possibly at a vertical segment Willow had labeled with a question mark; paper is partially blocked by Willow, so it is not clear exactly what she was pointing at]
- Lily: We don't.

- Sierra: We know this whole thing [gestures downward with pen from (4,4), exact gesture was not completely captured by the camera] is five—
- Willow: —Is y plus one.
- Sierra: Which for that one is five, right?
- Willow: Well, we know it's five, but if we don't know.
- Sierra: But how would they [Sasha and Keoni] do it? They would count, one, two, three [gestures with her pen in small circles on her paper, one circle each time she counts]—I mean, it's three, never mind.
- Lily: It's three.
- Sierra: Yeah, three. My bad.
- Lily: So this is three.
- Sierra: They would just count.
- Lily: And this is one, two, three, four, because we know where is x , right? So this is four, so this would be five. So that means from here [makes a spanning gesture with thumb on focus and forefinger on (4,4)] focus to the point is five, and then point to here [lifts her spanning gesture and moves it so forefinger is on (4,4) and thumb is on directrix] is five too.

In this episode, there is evidence that participants appeared to notice MKT around quantitative reasoning with distances. Sierra appeared to recall Sasha and Keoni's method for identifying distances in the coordinate plane when she said, "They would just count, one, two, three." Even though she never said who "they" was, as she said this, she made circle gestures with her pen, which seemed to mimic the circles that Sasha and Keoni draw to find distances in the plane during Lessons 2 and 3. Moreover, in Session 2, after watching Sasha and Keoni solve a task in which they drew circles and counted, Sierra wrote in a reflection that high school students might solve the task by "counting the distance from the point to the directrix." This evidence supports the claim that Sierra was noticing Sasha and Keoni's method for identifying distances.

Willow and Lily also noticed MKT around quantitative reasoning with distances. Willow seemed to complete Sierra's sentence "This whole thing is"—"y plus one," presumably referring to the distance between a general point (x, y) on the parabola and the directrix. As Sierra was saying this, she was gesturing along a segment in the coordinate plane, and Willow was looking at that paper. Lily's spanning gesture between two sets of points is consistent with gestures one would make when reasoning quantitatively with distances, as is her use of the phrase "from here to here."

During the rest of the time they solved the task they continued to notice the distances between points, as evidenced by their utterances, inscriptions, and gestures. For example, later Lily drew circles on her paper from $(4,4)$ down to $(4,1)$, which can be seen in Figure 5.4 above. These circles look like the circles Sasha and Keoni used to count distances in the videos.

Relationships Between Shift in CoFs and Shifts in MKT

There is evidence linking each of the three shifts in MKT related to quantitative reasoning with distances (from the interviews) to a shift in participants' CoFs (during the mini-course). Table 5.3 contains evidence of this linkage for each category of shift in MKT around quantitative reasoning. For each shift, I've provided a brief example of a single participant's noticing from either Session 1 or Session 2, and then two brief examples of that same participant's noticing from Session 3 (which is when the shift in CoFs seemed to occur) or later.

Table 5.3. Evidence from before and after the shift in centers of focus from one participant from each shift in MKT around quantitative reasoning with distances.

Shift in MKT (from the interviews)	Sessions 1 and 2	Sessions 3 through 6
Quantitative Reasoning	Willow suggests to the class that to find a distance, you could use the distance formula to set up a quadratic, which would be “pretty simple” to solve for y .	As Sierra gestures along a vertical segment and says “This whole thing,” Willow finishes the sentence with “is y plus one.”
Point of View	Marshall remarks that Sasha and Keoni “never really figured out” what his small group did, which was to find an equation for the directrix.	On his reflection document, Marshall writes that high school students need to “understand distance in the coordinate plane” in order to find equation for any parabola with vertex at the origin.
Orientation	April writes on her reflection document that to find the x -value given the y -value of a point on a parabola, high school students “might use an algebraic approach or plug the unknowns into a standard equation.”	Marshall notices that Sasha and Keoni have not used the distance formula, instead they’ve only used the Pythagorean Theorem. He conjectures that to start the task, Sasha and Keoni will probably draw triangles. As he says this, he gestures with his pen along the sides of a triangle. April said the instructor in the videos had Sasha and Keoni look at the graph when the parabola shifted right so they “see it visually so it made sense.”

To give the reader a sense for how the shifts in MKT around quantitative reasoning with distances are linked to the shift in CoFs, I first provide an in-depth analysis of three vignettes that serve as evidence of a link between the shift in quantitative reasoning and the shift in CoFs. I then briefly describe how the other two shifts in MKT (the shift in point of view and the shift in orientation) are linked with the shift in CoFs.

Shift in quantitative reasoning. I now present and analyze three vignettes, each featuring Willow. The first vignette comes from Session 2 before the shift in CoFs. The other two vignettes come from Sessions 3 and 4, which is after the shift in CoFs. These vignettes provide evidence for my claim that Willow's shift in quantitative reasoning was linked to the shift in CoFs.

Vignette 1. In Session 2, Willow's group (which consisted of Willow, April, and Lily) seemed determined to use the distance formula to solve the math task for the session (see Figure 5.2 above for the task). The group worked together for seventeen minutes, and for the final fourteen minutes the group's work revolved around the distance formula.

However, Willow had reservations about the approach, as she expressed several times. For example, when I checked in on the group, Willow said, "I guess the distance formula is the distance between two points, so you'd apply it between these two points, but **I still don't get how that finds the x -value** (emphasis added)." Later she asked her group "What would be another way, other than using the distance formula, do you think?" Her group talked very little about this question, with April

mentioning briefly that one could “maybe do something with triangles” and then “use the sine function.” Except for these two brief comments, Willow’s concerns were never addressed by her group.

Despite these reservations, when Willow presented the group’s work to the class, she talked about using the distance formula to “eventually solve for a quadratic, which would be pretty simple.” However, she did not present a detailed solution, and provided very little evidence that she understood the method the group had developed. In fact, when another participant (Marshall, who was not in her group) asked her to clarify, April (who had argued for and explained the use of the distance formula during the small group portion of the class) stepped in to answer for Willow.

In this vignette, Willow seemed to struggle to engage with the math task. She expressed doubt about using the distance formula, even saying at one point that she was “not convinced that the math is there.” Other members of her group seemed to engage more productively with the task, but their noticing of MKT around mathematics not in the video (in this case, the distance formula, which never appeared in any of the parabola unit lessons) was counterproductive for Willow’s development of MKT around quantitative reasoning with distances. Her attempts to redirect the group’s attention toward a different solution went largely unnoticed. Unsurprisingly, when she presented the group’s work in front of the class, she seemed unable to answer Marshall’s question, and looked relieved to have April step in for her.

Vignette 2. As shown in Table 5.2, Willow’s noticing shifted from noticing MKT around mathematics not in the videos to noticing MKT around quantitative

reasoning with distances. This shift seems to have contributed to Willow’s shift in quantitative reasoning as evidenced in her post-interview. For example, in Session 3 as Willow’s group solved the task (see Figure 5.3 above), she and Sierra talked about the distance from a point to the directrix. Sierra noticed the specific measure of the segment from the point (4,4) to the directrix (which was the line $y = -1$), while Willow seemed to notice a generalized quantity $y + 1$.

- Sierra: We know this whole thing [gestures downward with pen from a point in the plane, exact gesture was not completely captured by the camera] is five—
 Willow: —Is y plus one.
 Sierra: Which for that one is five, right?
 Willow: Well, we know it’s five, but if we don’t know.
 Sierra: But how would they [Sasha and Keoni] do it? They would count, one, two, three [gestures with her pen in small circles on her paper, one circle each time she counts]—I mean, it’s three, never mind.

First, Willow seemed to finish Sierra’s sentence “We know this whole thing...” with the phrase “...is y plus one.” Even though Willow was not gesturing as she said this, she appeared to be watching Sierra’s gesture. Second, Willow’s comment “but if we don’t know” suggests that she was generalizing the distance for any point in the plane. Taken together, this suggests that Willow was quantifying the distance from the point to the directrix as $y + 1$.

Vignette 3. In Session 4, Willow and her group (April, Jasper, and Sierra) had worked on developing an equation for any parabola with its vertex. The group’s solution was the following equation (I’ve recreated April’s graph below in Figure 5.5 to serve as a reference for April’s notation).

$$\sqrt{(x_0 - 0)^2 + (y_0 - y_F)^2} = \sqrt{(x_0 - x_0)^2 + (y_0 - (-y_F))^2}$$

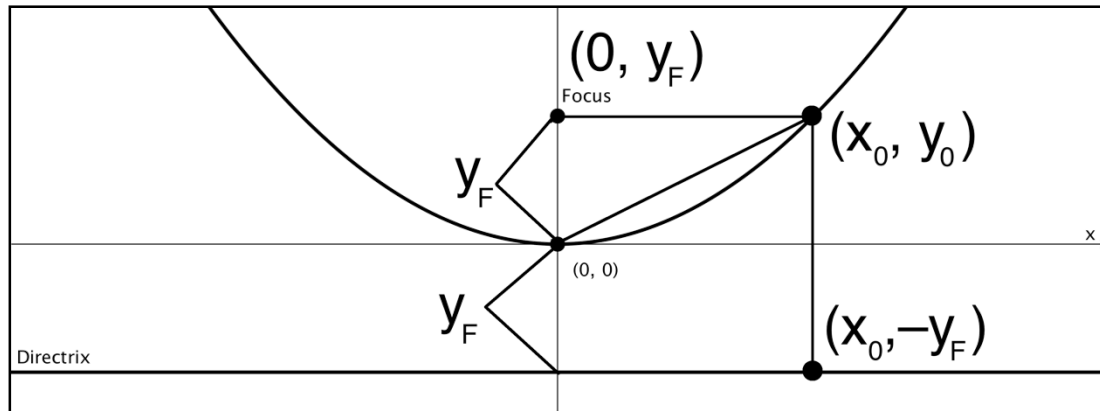


Figure 5.5. Author's recreation of April's graph.

After approximately seven minutes, I stopped the group and asked them to point out various distances in the coordinate plane. I asked Willow to show me the distance $y_0 + y_F$. She initially said, "It would be y -naught [points to the arbitrary point] plus this [points to the focus]." Jasper said, "It would be here [places both forefingers on the directrix], the distance from the directrix [moves one finger to the arbitrary point]." As he said this, Willow said "Oh!" and then Jasper drew a bracket from the directrix to the arbitrary point. After Jasper drew the bracket, Willow said "So this is y -eff [motions from the directrix to the x -axis], and then this is y -naught [motions from the x -axis to the arbitrary point]." Willow then drew two brackets where she had gestured and labeled them y_0 and y_F (shown in Figure 5.6).

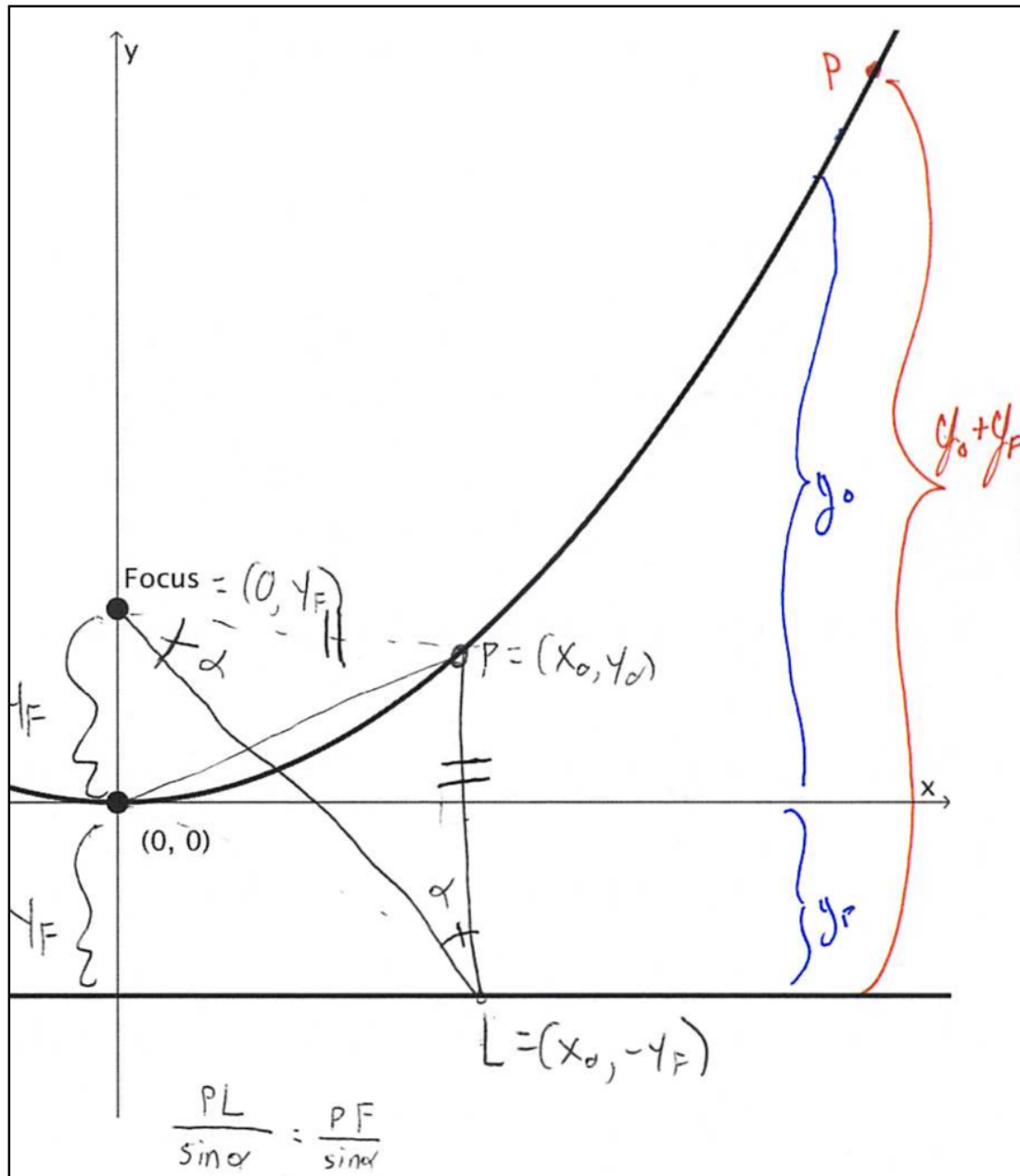


Figure 5.6 April's original inscriptions, along with the bracket (in orange) and label that Jasper drew, and the brackets (in blue) and labels that Willow drew.

Willow's initial response was to gesture to specific points, which suggests that Willow's quantitative reasoning with distances was still developing. However, this episode captures what appeared to be a generative moment for Willow. As she attended to Jasper's gesture along an imagined vertical segment from the directrix to

an arbitrary point on the parabola, Willow exclaimed “Oh!” She then drew brackets and labeled them, suggesting she had quantified two imagined segments as having specific measures of y_0 and y_F . It could be argued that Willow copied April’s earlier inscription (at the far left of Figure 5.6); however, there was no pre-existing inscription detailing y_0 for Willow to copy. Instead, she decomposed Jasper’s larger bracket representing the distance $y_0 + y_F$ into two distinct quantities.

Discussion. Taken together, these three vignettes serve as evidence that Willow’s shift in quantitative reasoning (as inferred from her post-interview) was linked with a shift in CoFs during the mini-course. As discussed in Chapter 4, Willow entered the mini-course not demonstrating any quantitative reasoning in the pre-interview. Moreover, during the first two instructional sessions, Willow did not demonstrate quantitative reasoning with distances.

In Session 3, there appeared to be a shift in what she noticed, which coincides with the first instances in the data in which Willow seemed to reason quantitatively. In the second and third vignettes, Willow’s burgeoning quantitative reasoning with distances seemed connected to her noticing of other participants’ gestures indicating distances. Notably, in the third vignette, Jasper’s gesturing along an imagined segment (with measure $y_0 + y_F$) helped Willow quantify two imagined segments (with measures y_0 and y_F) that combined formed Jasper’s imagined segment.

Shift in point of view. The shift in CoFs from MKT around mathematics not in the videos to MKT around quantitative reasoning with distances seemed to involve increased scrutiny of the videos by the participants. In Session 1 and Session 2,

participants did not tend to spontaneously talk about the videos unless I prompted them. For example, in Session 2, no one in Marshall's group (comprised of Marshall, Sierra, Jasper, and Desmond) mentioned the videos as they solved the task (see Figure 5.2 for a reminder of the task). After watching videos from Lesson 2, Marshall said he noticed that Sasha and Keoni "never really figured out" what his small group did, which was to find an equation for the directrix. Marshall noticed ways of reasoning that Sasha and Keoni did not use, and specifically evaluated what Sasha and Keoni did (or did not do) against what he and his group had done. This suggests that Marshall was not decentering.

Starting in Session 3, there was evidence that participants' noticing of MKT around quantitative reasoning tended to include references to Sasha and Keoni or the instructor in the videos. By turning their attention to what actually happened in the videos (versus interpreting the videos through the lens of their more sophisticated mathematics), participants who experienced a shift in point of view likely began to develop images of how Sasha and Keoni reasoned quantitatively.

For example, in Session 4 Marshall argued that he and his classmates "hadn't really seen them [Sasha and Keoni] use distance formula, they use Pythagorean Theorem, which is really the same thing, so I think we'll see them draw some triangles." Contrast this statement with the one from Session 2. In Session 2, Marshall seemed to evaluate Sasha and Keoni's reasoning against the reasoning of math majors (namely Marshall and his group). By Session 4, Marshall's statement is more nuanced. Even though he compared the reasoning exhibited by some participants (i.e., the

distance formula) to Sasha and Keoni's reasoning (i.e., the Pythagorean Theorem), Marshall appeared to value Sasha and Keoni's reasoning as valid and productive. As I discussed in Chapter 4, statements like these suggest that Marshall was decentering by taking the point of view of Sasha and Keoni.

Shift in orientation. The videos provide images of instruction that are consistent with a conceptual orientation toward instruction. The instructor in the videos is a veteran educator who carefully crafted tasks and prompts to elicit Sasha's and Keoni's thinking. In Chapter 4, I outlined characteristics of a conceptual orientation toward teaching (A. G. Thompson et al., 1994), which included (a) images of different ways students understand an idea; (b) images of how those understandings develop; (c) images of instruction that promote that development; and (d) expectations that students are intellectually engaged in mathematical activity.

The instructor in the video exhibited these characteristics. For example, in Lesson 2, Episode 1 of the MathTalk parabola unit, she asks Sasha and Keoni "Now is the origin on the parabola? Can you use the definition of a parabola to justify or explain why it's on the parabola?" In Lesson 2, Episode 2, in response to Sasha and Keoni's confusion she says, "Why don't we take stock of where we are at? What is it that you're trying to figure out?" In Lesson 3, Episode 1, the instructor tells Keoni to write down his idea and then asks Sasha what she thought of the idea. In Lesson 5, Episode 1, the instructor tells Sasha and Keoni "First, I want you to try to make sense of what you're seeing."

Now, consider how April's noticing changed. In the first two sessions, she tended to notice MKT around mathematics not in the MathTalk videos. For example, in Session 2, she conjectured that students would "plug the unknowns into a standard equation" to find x -values given y -values. By Session 3, she seemed to be noticing MKT around quantitative reasoning. Her images of instruction (as evidenced by her verbal utterances and written inscriptions) from Session 3 and beyond began to sound more like the kinds of questions and prompts made by the instructor in the videos. For example, in Session 3 April said she liked how the instructor asked Sasha and Keoni to "prove things on the parabola" because it helped them "better understand the definition." In Session 4, April wondered if the instructor in the videos would pose questions to Sasha and Keoni to help them "understand the length" of segments in the coordinate plane. Her shift from noticing MKT around mathematics not in the MathTalk videos to noticing MKT around quantitative reasoning with distances coincided with a shift in her orientation.

Focusing Interactions

According to Lobato, Rhodehamel, et al. (2012), "*focusing interactions* refer to the discursive practices (conceived broadly to include gesture, diagrams, and talk) that can give rise to particular centers of focus" (p. 440). Focusing interactions help account for the role that both the teacher (in this case, the author) and participants played in the emergence of and shift between CoFs. Additionally, focusing interactions help account for how what participants seemed to notice is socially organized.

To infer focusing interactions, I began with an a priori coding scheme taken from the research literature (Goodwin, 1994; Lobato et al., 2013; Lobato, Rhodehamel, et al., 2012). The extant codes include *highlighting*, *quantitative dialogue*, and *renaming*. Renaming did not seem to play a role in my data, and consequently does not appear in the following results. Instead, I provide an example of it here for the reader. As an example of renaming, Lobato et al. (2013) reported on how in one seventh-grade class, a teacher called the “arms” of a visually growing figural pattern “the growth.” This seemed to have the effect of redirecting students’ attention away from a relationship between the step number in the pattern and the number of objects, back toward additive reasoning.

Using a grounded theory approach (Strauss & Corbin, 1990), I looked for new focusing interactions; however, the a priori coding scheme was sufficient for the data. In fact, two of these codes seemed to coincide with the shift in CoFs: quantitative dialogue and highlighting. I elaborate each of these below and provide evidence that suggests these focusing interactions contributed to the shift in CoFs in Session 3.

Quantitative Dialogue

Quantitative dialogue is defined as “verbal communication that focuses attention on quantities as measurable attributes of objects (following Thompson, 1994)” (Lobato, Rhodehamel, et al., 2012, p. 832). One key difference between this study and other studies that have leveraged the focusing framework (e.g., Lobato et al., 2013; Lobato, Rhodehamel, et al., 2012) is that this study focuses on MKT and not just mathematical knowledge. Not surprisingly, throughout the mini-course, I and my

participants often talked about pedagogical aspects of MKT around quantitative reasoning with distances. Accordingly, I extend the notion of quantitative dialogue to include verbal communication that focuses attention on any of the following: (a) quantities as measurable attributes of objects; (b) ways students reason quantitatively; (c) milestones for a learning trajectory for quantitative reasoning; or (d) quantitative instructional moves.

Quantitative dialogue appeared to play a pivotal role in the shift in CoFs. As evidence of this claim, I first present data from late in Session 2 in which I directed participants' attention to the instructional moves made by the teacher in the video. This event is the first instance in the data in which quantitative dialogue appears in connection with the MathTalk videos. I then present evidence from Session 3 in which two participants, Willow and Sierra, seemed to continually redirect their group's attention to reexamining a math task by considering it from Sasha and Keoni's perspective. Finally, I present evidence from Session 4, in which both April and Willow appear to direct the group's attention to both instructional moves the instructor in the videos might make and ways that Sasha and Keoni might approach a task.

During Session 1 and much of Session 2, I had observed that participants were not attending to MKT around quantitative reasoning with distances during the math tasks. Moreover, based on the conversations the class had about the MathTalk videos, it seemed that participants were noticing neither the quantitative reasoning of Sasha and Keoni nor the instructor's quantitative instructional moves, even though the MathTalk videos have several features that foreground these aspects of MKT around

quantitative reasoning with distances (see Chapter 3 for a broader discussion of these features).

Toward the end of Session 2, the class had already watched and discussed videos from Lesson 2 of the MathTalk parabola unit. The class was watching a clip from Episode 1 of Lesson 3 (currently viewable at <http://cpucips.sdsu.edu/website/parabolas-l3-p1.html>), in which Sasha and Keoni begin generalizing a method for finding the x -value given the y -value for any point on a parabola (the same task as given in Figure 5.2 above). At one point in the episode, Sasha and Keoni seemed to be on the cusp of quantifying the unknown coordinate y as a distance from a point to the x -axis. The instructor in the video tries to get Sasha and Keoni to point out the distance y (transcript is from 4:28–4:40 of the video):

Instructor:	Show me y , show me the distance y on there.
Sasha:	What?
Instructor:	You said it was one more than y , where is y ?

I paused the video and directed participants' attention to the instructional move the teacher had just made by posing the following question:

Just talk briefly for like thirty seconds in your group, what is [the instructor] trying to get them to show her? She said, "Show me the y " and they're sort of [gestures haphazardly with hands to indicate confusion], yeah. So just talk in your group, what do you think it is that [the instructor] is trying to get Sasha and Keoni to show her?

By asking this question, I hoped that participants would attend to both the instructor's quantitative question and the quantity y , which was the distance of a vertical segment from a point on the parabola to the x -axis.

In response to this prompt, Marshall said the instructor wanted Sasha and Keoni to label as y the vertical segment from the point to the x -axis. Jasper said:

She [Sasha] drew the unit for the directrix [makes a small span gesture with thumb and forefinger, see Figure 5.7a below]. Now they [the instructor] want to see where y , they want her to draw just the y [makes a larger span gesture with thumb and forefinger, see Figure 5.7b below].

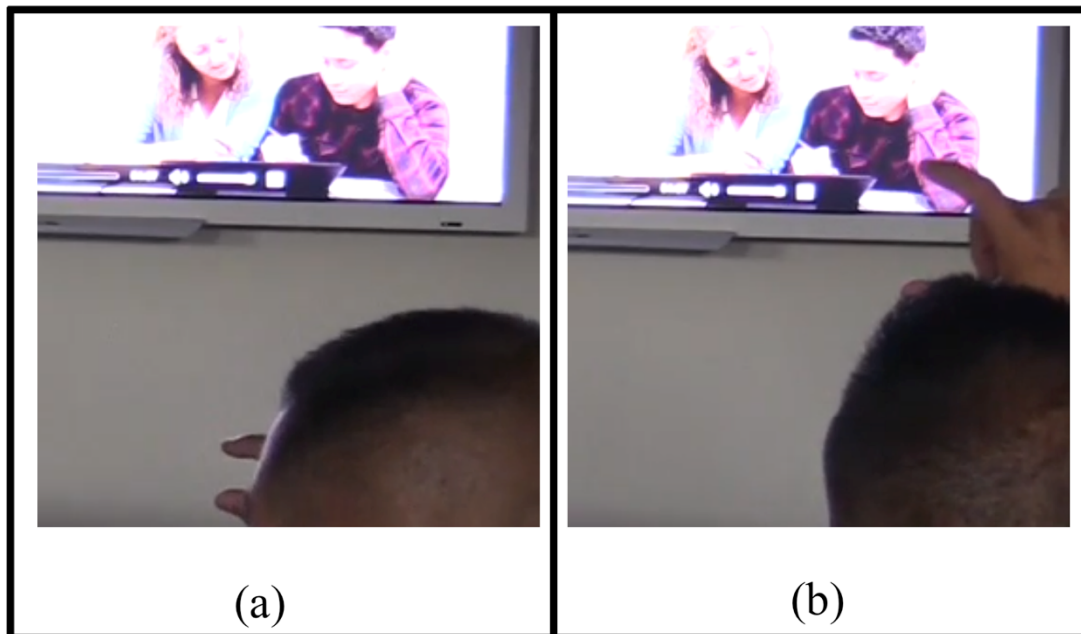


Figure 5.7. (a) Jasper’s gesture indicating the unit. (b) Jasper’s gesture indicating y . Sierra said, “Yeah, this would be y [while seeming to swipe in a line with her pen to possibly indicate a distance; however, her gesture was mostly blocked by other participants].”

In the one minute during which participants discussed my prompt, four different participants noticed MKT around quantitative reasoning with distances, including ways Sasha and Keoni were reasoning quantitatively and quantitative instructional moves the teacher made. I took these reactions to my question about the instructor’s motivation as evidence that my prompt served as quantitative dialogue.

This focusing interaction seemed important, as it came at the end of Session 2, and the shift in CoFs occurred in Session 3. It also served as a catalyst for more quantitative dialogue by participants in subsequent sessions, as I now discuss.

Participants' own quantitative dialogue appeared to have the effect of sustaining the shift in centers of focus from Session 3 onward. For example, in Session 3, Willow and Sierra seemed to try to redirect their group's attention to examining the task (see Figure 5.3 for the task) by considering what Sasha and Keoni would do. The group discussed solving the equation they had derived the day before (which was $x = \sqrt{4y}$) to isolate y . Willow said, "That's what I would do, but what I don't know is what a high schooler would think." This led to a discussion about using the Pythagorean Theorem, which included the transcript (reproduced below) I analyzed in *Vignette 2*.

- Sierra: We know this whole thing [gestures downward with pen from a point in the plane, exact gesture was not completely captured by the camera] is five—
- Willow: —Is y plus one.
- Sierra: Which for that one is five, right?
- Willow: Well, we know it's five, but if we don't know.
- Sierra: But how would they [Sasha and Keoni] do it? They would count, one, two, three [gestures with her pen in small circles on her paper, one circle each time she counts]—I mean, it's three, never mind.

The group's attention seemed directed toward MKT around quantitative reasoning with distances as a consequence of Willow's and Sierra's quantitative dialogue. Sierra and Willow both quantified segments, with Sierra quantifying a segment as having a length of five and Willow generalizing the point to quantify the segment as having a length of $y + 1$. Moreover, Sierra attended to ways that Sasha

and Keoni reasoned quantitatively with distances when she said, “They would just count, one, two, three.”

As one final example, I present evidence from Session 4 that this focusing interaction seemed to play a role in keeping participants’ attention on MKT around quantitative reasoning with distances. In *Vignette 3* in the previous section, I described how one group (April, Jasper, Willow, and Sierra) worked to develop an equation for any parabola with its vertex at the origin. Just after the events described in that vignette, April asked two questions. The first was about the possible moves the instructor in the videos might make, and the second was about challenges Sasha and Keoni might face. These questions acted as quantitative dialogue because they had the effect of directing participants’ attention to MKT around quantitative reasoning with distances. Throughout the following transcript, the pronoun *they* refers to Sasha and Keoni.

- April: **I wonder if she’s [the instructor in the video] going to do something with, how he [the author] was asking us what is y naught plus y sub- f . I wonder if she asks them questions like that to understand the length** (emphasis added), like that and that [points to the two brackets Willow had drawn, which are shown in Figure 5.6 above], and what the distance formula is showing.
- Willow: Well, is there another way to come up with that other than the distance formula?
- April: Um, you could probably use triangles, like right triangles.
- Willow: Yeah, I was thinking that, but I don’t know. They’ll probably try, go with triangles again, that’s what they’ve been doing.
- Jasper: Yeah, that’s what we were doing.
- Willow: They’ll probably start with that.
- April: Oh, you drew a triangle?

- Jasper: Yeah. (incomprehensible)
[38 seconds of transcript omitted in which Jasper talks to April about his notation]
- April: **What other challenges do you think they'll have** (emphasis added)?
- Willow: The distance between the points. I don't know if they've even—
- Sierra: —have they said the distance formula?
- Willow: No.
- April: I think that they've used it slightly because they've used the Pythagorean Theorem which is pretty much the same thing [as April says this, Willow demonstratively shakes her head “no” from side to side].
- Sierra: Yeah, I don't know if they have connected it that way.
- April: Yeah.
- Willow: I don't think they were thinking about it as the distance formula.
- Sierra: They were just thinking of triangles and the Pythagorean Theorem.

In this exchange, April's two questions (“I wonder if she's going to do something...” and “What other challenges do you think they'll have?”) seemed to direct her group's attention on MKT around quantitative reasoning with distances. Consider that in response to these questions, the group began discussing instructional moves (e.g., asking questions about distances and quantitative relationships) and students ways of understanding (e.g., Willow stating Sasha and Keoni might have difficulty with “the distance between the points”). This can be taken as evidence that these questions served as quantitative dialogue.

Before moving to the next section, I offer one final note about quantitative dialogue. Utterances that serve as quantitative dialogue may not themselves be quantitative in nature. For example, when April asked, “What other challenges do you

think they'll have," there was nothing in her words that would be considered quantitative. Moreover, there is no agency or intentionality implied for data that was coded as quantitative dialogue. April may or may not have intended for her group to turn their attention to MKT around quantitative reasoning with distances. Whatever her intention was, the question ultimately seemed to have the effect of directing the group's attention to MKT around quantitative reasoning with distances, and for that reason it was coded as quantitative dialogue.

Highlighting

Highlighting, is defined as the visible acting upon an external phenomena such as gesturing, labeling, or annotating (Lobato et al., 2013). In the mini-course, highlighting played a vital role in the shift in CoFs. As an example of the role highlighting played, I present an episode from Session 3 in which Marshall presented his small group's work to the class (for a reminder of the task for this session, see Figure 5.3). This episode seems significant, as it came during the middle of Session 3, which is when the shift in centers of focus occurred.

In presenting his group's work, Marshall said that "by looking at the graph, you can also use right triangles." He put his graph under the document camera (as shown in Figure 5.8a below) and proceeded to draw a right triangle followed by several brackets to indicate different distances. Marshall's highlighting took the form of labeling, gesturing, and annotating his graph in ways that were linked with distances as quantities (in the following transcript, I have added emphasis throughout to facilitate analysis).

Marshall: So, let's say they give us x equals negative three, so we have a point right here [draws the point seen in the upper left of Figure 5.8b below], and we draw our right triangle, it's going to look like that [draws a right triangle, seen Figure 5.8c below]. And because they give us x equals negative three, we can label this side, it's three units long [**labels the horizontal leg of the right triangle 3**, seen in Figure 5.8d below]. Now we have this length right here [**draws a vertical segment from the point to the directrix**, seen in Figure 5.8d below]—so we don't know what y is, we want to find it. I'm just going to call it y right now. So, we know this [**gestures with his pen vertically, down from the point to the x-axis**] is going to be y [stops his gesture momentarily], and then plus one down here [**continues gesture from x-axis to the directrix**]. So, this is one unit [**labels the segment from the x-axis to the directrix with a 1**, seen in Figure 5.8e below]. So, this length right here [**draws a bracket next to the vertical segment and labels it $y + 1$** , seen in Figure 5.8e below] is y plus one, that means that the hypotenuse of the triangle is y plus one [**labels the hypotenuse $y + 1$** , seen in Figure 5.8f below]

Author: And how do we know that?

Marshall: Because of the definition of a parabola. We have y plus one is the distance [**gestures vertically, down from the point to the directrix along the segment he drew**] from the point [**places pen back on the point and moves it down**] to the directrix [**finishes the gesture with his pen at the directrix**]. So, it must be equal to the distance from the point to the focus [**gestures with his pen along the hypotenuse**]. Now this last side right here [**gestures with his pen along the shorter vertical leg of the right triangle**], the focus is one unit above the x -axis, so we know this length is going to be y minus one, because there is this one extra unit down here [**gestures back and forth with his pen from the right angle of the right triangle to the x-axis, then labels the vertical leg $y - 1$** , seen in Figure 5.8f below]

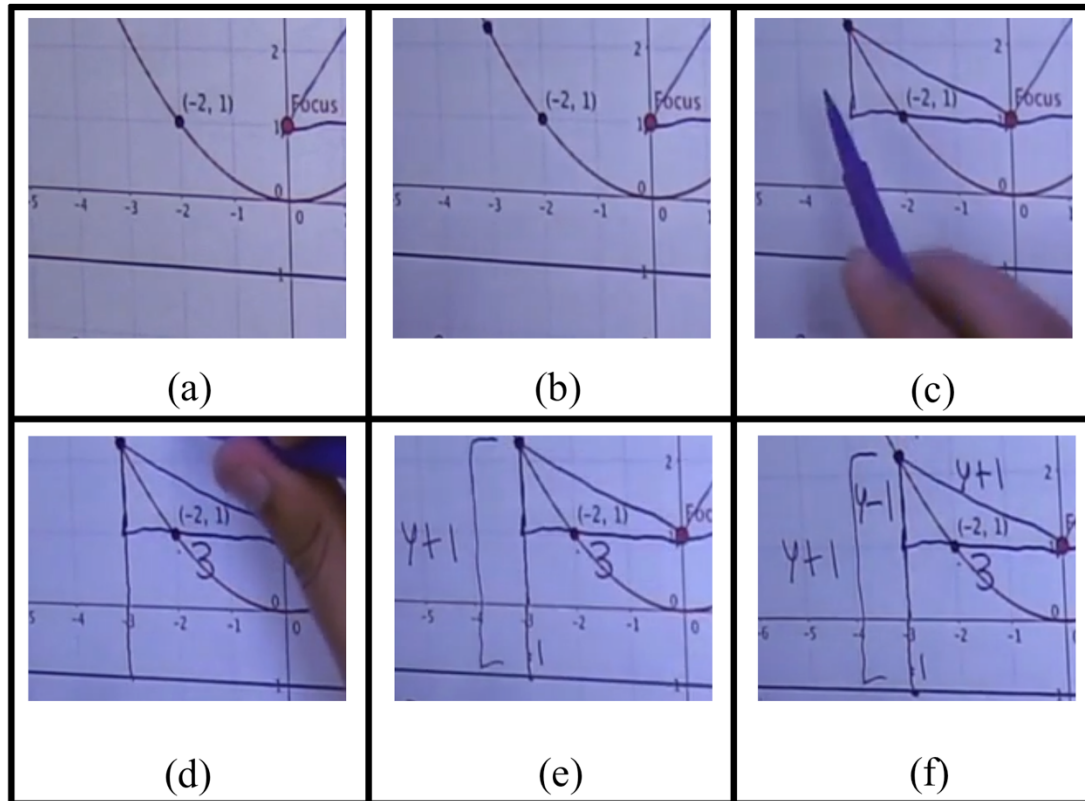


Figure 5.8. A series of pictures depicting the evolution of Marshall's example in which he highlights several quantitative features of a solution.

Throughout this event, Marshall repeatedly linked his quantitative language (e.g., length, distance, unit) with physical segments in the plane by gesturing or annotating his graph, as indicated with the emphasized text in the transcript. In other words, Marshall's gestures and annotations *highlighted* quantitative aspects of his work.

Indeed, highlighting seemed to serve as a catalyst for shifting participants' noticing from MKT around math not in the videos to MKT around quantitative reasoning with distances. For example, in the previous section I presented evidence from two vignettes in which Willow's attention appeared to be directed by other participants' gestures and inscriptions. Notably, in the third vignette (which came in

Session 4) Willow exclaimed “Oh!” as Jasper first made a spanning gesture along an imagined segment and then drew in a large bracket to indicate a distance of $y_0 + y_F$. Jasper’s highlighting seemed to make salient for Willow his quantifying of an imagined segment having a length of $y_0 + y_F$. In turn, this helped Willow decompose that quantity into two separate quantities, y_0 and y_F .

Features of Tasks

In this section, I present the result of the third analytic pass of the data, which is an account of how features of the tasks used in the mini-course potentially contributed to the shifts in the centers of focus. As described in Chapter 3, the mini-course was driven by a variety of tasks, which included math tasks, video tasks, and reflective activity. These tasks served as the context in which I and the participants interacted. It was in the solving of math tasks and discussing of both the math tasks and the videos that participants’ ideas were made visible to the class. Both the Task Reflection Document and the video tasks (which included both watching and discussing the videos), had features that seemed to contribute to the social organization of participants’ noticing. I now briefly discuss how each of these may have contributed to the shift in centers of focus.

The Task Reflection Document (which can be seen in Appendix C) was given to participants in each of the first five sessions. The document consisted of three questions: (a) What do high school students need to know in order to complete this task? (b) How might high school students complete this task? and (c) What challenges might high school students encounter and how might you help them overcome those

challenges? Participants recorded their thoughts about each math task by answering these three questions at three different times: prior to solving the task, after solving the task, and then again after watching the MathTalk videos.

This document seemed to support noticing MKT around quantitative reasoning with distances. The questions guided participants to consider tasks through a pedagogical lens. The document also provided participants with ways of talking about the task through a pedagogical lens. For example, April's question "What other challenges do you think they'll have?" was like one of the questions from the Task Reflection Document. Other participants posed similar questions, notably from Session 3 onward, which suggests the Task Reflection Document played an important role in drawing their attention to MKT around quantitative reasoning with distances.

The video tasks had three features that appeared to support participants' noticing of MKT around quantitative reasoning with distances. First, as I discussed in Chapter 3, the videos were edited in a way that foregrounds the interactions between Sasha, Keoni, and the instructor. These interactions became objects of inquiry during the mini-course. Participants wondered aloud about the instructor's motivation for particular questions, how Sasha and Keoni might solve a task or overcome a challenge, and what new tasks the instructor might pose or what Sasha and Keoni might learn about in the next video. Entire conversations sprung up during the small group time that revolved around the videos. These conversations seemed to generate rich ideas about MKT around quantitative reasoning with distances.

Second, I established a protocol for discussing the videos, which included questions like “What did you notice in the videos?” and “Did you want to tell Sasha and Keoni anything?” Consequently, our discussions often revolved around components of MKT. Initially those components were for MKT around mathematics not in the videos. Eventually this shifted so that our conversations were targeted toward MKT around quantitative reasoning with distances.

Finally, as I discussed in Chapter 3, the videos feature the same two students over the course of several lessons. Consequently, the videos highlight the development of their quantitative reasoning with distances. The longitudinal nature of these videos seemed to play an important role in the shift in CoFs. Initially in Session 1 and Session 2, participants had little experience with students’ ways of reasoning quantitatively with distances. This manifested as statements like Marshall’s on his Session 2 Task Reflection Document. Marshall conjectured that to find x -values given y -values (see Figure 5.2 for the task), high school students would “rewrite the definition using the distance formula.” He continued, stating, “How do I know? Experience in high school classrooms (kids know the distance formula). Guessing.”

The longitudinal nature of the videos seemed to afford opportunities to develop images of ways Sasha and Keoni reasoned quantitatively as well as milestones along a learning trajectory for quantitative reasoning. Contrast Marshall’s statement above with one he made in Session 3. Again, Marshall made a conjecture about high school students, this time about how they might find y -values given x -values (see Figure 5.3 for the task). On his Task Reflection Document, he wrote that to solve that task, high

school students would “use the def. of parabola and x -value given to create a right triangle and use Pythagorean Theorem to find the third side, then use that to get the y -value.” Again, Marshall justified his answer, this time by calling to mind the videos. He wrote, “How do I know? Videos from last session and from doing the previous task.”

His conjecture in Session 2 did not match what Sasha and Keoni did to solve the task, but his conjecture in Session 3 did align with how Sasha and Keoni approached the task. His conjecture in Session 3 can be taken as evidence that Marshall had begun to develop images of ways that Sasha and Keoni reason quantitatively and had started to identify milestones in their learning trajectory. The difference in his two conjectures is striking, and serves as evidence that the longitudinal nature of the videos afforded the shift in CoF

The Nature of Mathematical Activity

Lobato et al. (2013) describe the nature of mathematical activity by as “the participatory organization that establishes roles governing students’ and teachers’ actions” (p. 814). In this section, I present the results of the fourth analytic pass of the data, which is an account of how the social organization of participation in the mini-course played a role in the shift in centers of focus. Before presenting this result, I note that unlike other studies that reported using the focusing framework (e.g., Lobato et al., 2013; Lobato, Rhodehamel, et al., 2012) this study featured only one class whereas the other studies featured multiple classes. Those studies could draw contrasts between differences in the nature of mathematical activity in each class. Instead, in this study I

looked at subtle differences between the mathematical activity in Session 1 and Session 2 and the mathematical activity in Session 3 onward.

In Session 1 and Session 2, participants seemed to expect that their role during the time in which they solved math tasks, was to approach the task like a mathematics major. Mentions of pedagogical facets of MKT (e.g., students' ways of understanding, milestones or challenges along a learning trajectory, and instructional moves) were largely missing from their conversations. During these sessions, I would often check in with the groups and attempted to redirect participants' attention to pedagogical ideas. This usually resulted in some discussion about MKT around quantitative reasoning with distances. However, these check-ins did not result in sustained focus on MKT, as groups tended to revert to discussing only mathematical aspects of the task once I left the group to work on its own.

In Session 3, there appeared to be a shift in participants' expectations regarding their role. Instead of just approaching the math tasks like a math major might, participants expanded the range of what they discussed to include pedagogical aspects of the task. They began taking on the role of future teachers by asking each other how students might approach a task, and wondering aloud about challenges Sasha and Keoni might face. Participants no longer seemed content to only discuss the task in front of them and instead they expected that the task would be discussed through the lens of what had happened and what might happen in the MathTalk videos.

These videos were designed and edited in ways that highlight Sasha and Keoni's reasoning and ways of thinking. Their quantitative reasoning with distances, and the instructor's moves that supported that reasoning, permeates the videos. Consequently, when participants more frequently included Sasha and Keoni's reasoning in their conversations about math tasks, the effect was an increase in noticing MKT around quantitative reasoning.

Summary and Discussion

In this chapter, I presented the results from my analysis of the classroom data. The analysis consisted of four analytic passes through the data. The first pass yielded evidence of the emergence of, and shifts between, two centers of focus: *MKT around mathematics not in the MathTalk videos* and *MKT around quantitative reasoning with distances*. The second pass resulted in descriptions of two focusing interactions, highlighting and quantitative dialogue, that seemed to contribute to the shift in CoFs. The third and fourth analytic passes illuminated how both features of the tasks from the mini-course and the nature of mathematical activity contributed to and sustained the shift in CoFs. By leveraging the four components of the focusing framework, I provided a plausible account for how key elements of the learning ecology appeared to contribute to the development of MKT by participants.

The MathTalk videos are woven throughout the analyses for each of the four components of the focusing framework. This suggests that these videos played an important role in participants' development of MKT.

The center of focus *MKT around quantitative reasoning with distances* was not exclusively linked to videos, and indeed there are several instances in the data where participants seemed to notice this CoF without noticing the videos. However, the bulk of instances that were coded with this CoF were instances in which participant were noticing features or events from the videos.

Additionally, the focusing interaction *quantitative dialogue* often took the form of prompts or questions by either myself or participants about features or events from the videos. When I checked in with groups I often asked them what they thought Sasha and Keoni might do for the task, and this seemed to direct their attention back to *MKT around quantitative reasoning with distances*.

The video tasks provided opportunities for participants to engage in conversations about quantitative reasoning with distances. Because the videos so prominently feature Sasha and Keoni's quantitative reasoning, the video tasks not only provided participants with opportunities to hone their own quantitative reasoning skills, but also with opportunities to consider pedagogical implications for quantitative reasoning.

Finally, starting in Session 3, participants appeared to appropriate questions I asked, indicating their expectations about their role in the mini-course had changed. Participants began to push each other to talk about and engage with the task vis-à-vis what they thought might happen in the videos with Sasha, Keoni, and the instructor.

Chapter 6: Conclusion

The findings of this dissertation study contribute to the field by illuminating the development of MKT around quantitative reasoning. In Chapter 4, I examined three facets of MKT around quantitative reasoning and one shift in affect. In Chapter 5, I linked the development of that MKT around quantitative reasoning with distances to shifts in what participants noticed during the mini-course. These results are both theoretically and methodologically significant. In this chapter I review these results and elaborate their significance. I then turn to describing some limitations of the study. I conclude the chapter by discussing areas for future research.

Summary of Findings

The goal for this dissertation study was to develop theory about MKT around quantitative reasoning. My inquiry was guided by two research questions. The first question examined the nature of MKT that participants seemed to develop during the mini-course. Answering this question provided insight into specific ways that MKT around quantitative reasoning manifest in prospective secondary teachers. The second research question explored how that MKT developed. In this section, I summarize the results from answering each of these research questions.

Answering Research Question 1

In answering Research Question 1, I identified four shifts in MKT, and three of these were related to quantitative reasoning with distances. I now summarize each of these shifts, starting first with the shift in affect.

Affect. Participants who experienced this shift tended to demonstrate in the pre-interview teacher-centered views of instruction. Additionally, in the pre-interviews, these participants talked about students' thinking and understanding from a deficit perspective. They said students "wouldn't understand" or "wouldn't remember," and consequently students either "got it" or they didn't. Such comments and attitudes are consistent with a fixed mindset (Dweck, 2006). Research has shown that fixed mindsets are counterproductive for teaching and are correlated with lower student achievement (Dweck, 2010).

During the post-interview, participants' affect appeared to have shifted. Their views of instruction seemed student-centered, and they talked with more nuance about student thinking and understanding. They discussed "process[es] of improving their [students'] understanding," and ways "to help them [students] see the relationship." These statements and attitudes are consistent with a growth mindset (Dweck, 2006), which has become an area of interest and emphasis for reform-oriented education (Boaler, 2013; Dweck, 2010; Yeager & Dweck, 2012).

Shifts in MKT around quantitative reasoning with distances. The three other shifts in MKT that I identified in Chapter 4 relate to quantitative reasoning with distances. I briefly summarize these shifts below.

Quantitative Reasoning. The first shift captured how participants who entered the study with limited quantitative reasoning with distances appeared to develop more sophisticated ways of reasoning quantitatively with distances. In the pre-interview, three participants struggled to reason quantitatively, and these struggles seemed to

stem from their limited conceptions of points (as locations, or as values with which they could calculate). Their inability to reason quantitatively with distances limited their progress with the Ellipse Task.

In the post-interview, all participants reasoned quantitatively with distances. Notably, the three participants who seemed to struggle in the pre-interview, were able to demonstrate more sophisticated quantitative reasoning in the post-interview. This included quantifying coordinates as distances and forming quantitative relationships with those quantities. This quantitative reasoning seemed to serve as a powerful tool that enabled these participants to make significantly more progress on the Parabola Task than they did on the Ellipse Task.

Point of View. The second shift in MKT related to quantitative reasoning was a shift in point of view, or the ability to decenter. In the post-interview, three participants changed their solution for the Parabola Task to one that was qualitatively similar to the Sasha and Keoni's solution in Lesson 9 of the MathTalk videos. They were able to do this despite having never seen the videos in which that solution is developed. Moreover, these participants made accurate predictions about how Sasha and Keoni would think about the problem, what they would do to solve the problem, and what challenges they might face. In total, six of the seven participants seemed to decenter. Three participants changed their solutions in the post-interview, while three others offered initial solutions that were similar to Sasha and Keoni's solution.

Orientation. The final shift in MKT around quantitative reasoning was a shift in orientation. In the pre-interview, five of the seven participants appeared to have an

orientation consistent with the calculational orientation described by A. G. Thompson et al. (1994). Broadly speaking, these participants tended to: (a) emphasize arithmetic operations; (b) foreground calculations; and (c) hold images of instructions that focused on calculating with numerical values.

By contrast, in the post-interview these participants seemed have developed an orientation that was more consistent with the conceptual orientation described by A. G. Thompson et al. (1994). They tended to: (a) emphasize quantitative operations over arithmetic ones; (b) foreground sense-making instead of calculations; and (c) hold images of instruction focused on reasoning quantitatively.

Answering Research Question 2

As I have summarized, there were four shifts in participants' MKT. To answer Research Question 2, I investigated how three of these shifts developed. Because the MathTalk videos played a central role in the mini-course, investigating what participants noticed in the videos and subsequent dialogue around the videos was crucial in explaining those shifts. Consequently, I used the focusing framework (Lobato et al., 2013; Lobato, Rhodehamel, et al., 2012) to develop theory about how MKT seemed to develop during the mini-course. By leveraging the four parts of the focusing framework in my analysis, I was able to link what participants learned with what they noticed in the mini-course.

Centers of focus. I identified two centers of focus (CoFs) that seemed to capture participants' attention during the mini-course. The first was *MKT around mathematics not in the MathTalk videos*. This CoF emerged early in Session 1 and

appeared to be the dominant CoF through Session 2. This CoF accounts for how participants initially noticed mathematics they brought with them into the mini-course. Surprisingly, participants even tended to view the MathTalk videos through the lens of their own math major knowledge. Given that all participants planned to enroll in a teacher credential program, I did not expect participants to view the videos almost exclusively through that lens. This had the effect of participants noticing what *did not* occur in the videos, such as Sasha and Keoni failing to use “plus-or-minus” when solving an equation involving square roots or the instructor not prompting Sasha and Keoni to use the distance formula.

The second CoF, *MKT around quantitative reasoning with distances*, also emerged in Session 1. However, it was not until Session 3 that this CoF seemed to play a more substantial role in what participants noticed. This shift in CoFs brought about an increase in participants’ noticing of quantitative reasoning with distances, as well as the ways in which Sasha and Keoni reasoned quantitatively and the quantitative instructional moves made by the teacher in the videos.

The shift from the CoF *MKT around mathematics not in the MathTalk videos* to the CoF *MKT around quantitative reasoning with distances* seemed to play a role in the development of participants’ MKT. In Chapter 5, I demonstrated how the shift in CoFs coincided with the first instances in the data of Willow reasoning quantitatively as an example of how the shift appeared to contribute to the development of participants’ quantitative reasoning. I also linked the shift in CoFs to participants’ efforts to decenter by establishing how Marshall’s emerging understanding of Sasha

and Keoni's ways of reasoning coincided with the shift in CoFs. Finally, by illuminating how April's images of instruction were informed by her noticing of the instructor in the video, I presented a plausible explanation for how the shift in CoFs contributed to participants' shift in orientation.

Focusing interactions, tasks, and the nature of mathematical activity. Part of the rationale for leveraging the focusing framework in my analysis is that it provided explanatory power for the ways in which participants' noticing was socially organized. Following Lobato et al. (2013), I examined how participants' noticing emerged "through the interplay of a set of discourse practices called focusing interactions and features of mathematical tasks during engagement in particular types of mathematical activity" (p. 814). I now briefly discuss results related to each of these factors.

Focusing interactions. Two focusing interactions, *quantitative dialogue* and *highlighting*, contributed to the shift in CoFs. For example, late in Session 2, just before the shift in CoFs, I asked participants to consider an exchange between Sasha and Keoni and the instructor in the videos. This prompt acted as quantitative dialogue because it had the effect of directing participants' attention to MKT around quantitative reasoning. Moreover, it seemed to provoke a change in participants' own language, as in Session 3 and later they started using quantitative dialogue as they worked in groups and discussed the videos.

Features of tasks. The features of two tasks used in the mini-course seemed to foster the shift in CoFs. Even though participants were planning to enroll in a teacher

credential program in the fall following the study, in Session 1 and Session 2 they appeared to approach the math tasks more as a student of mathematics than as a teacher of mathematics (Stylianides & Stylianides, 2010; van Bommel, 2012). The Task Reflection Document (see Appendix C) guided participants to consider the math tasks through a pedagogical lens. This seemed to affect their noticing, and by Session 3, they were asking questions of each other that were similar to the questions on the Task Reflection Document.

The video tasks had several features that contributed to the shift in CoFs. For example, the editing of the videos highlight the interactions between Sasha, Keoni, and the instructor, and specifically the nature of the quantitative reasoning that Sasha and Keoni developed as well as the instructional moves that seemed to support that development. This allowed participants to hone in on different facets of MKT around quantitative reasoning. Additionally, the conversations I led around the videos were rich and generative; participants' ideas about Sasha and Keoni, the instructor, and the content emerged and were discussed. Finally, the longitudinal nature of the videos provided multiple opportunities for participants to reflect on Sasha and Keoni's developing quantitative reasoning.

The nature of mathematical activity. In Chapter 5, I elaborated how changes in roles that participants took on seemed to contribute to the shift in CoFs. Their mathematical activity in Session 1 and Session 2 was guided by implicit assumptions that their role was to engage in the math tasks like a math major. This meant that it was acceptable to contribute to the group mathematical ideas that were beyond the

scope of typical high schoolers (e.g., using the law of sines to find the length of one side of a right triangle, even when the length of two of the sides were given).

In Session 3 the roles that participants assumed changed so that they started to approach the task more like a teacher of mathematics. For the most part they abandoned the more complex mathematical ideas that were not as appropriate for high schoolers, and when they did discuss such ideas they often hedged (e.g., Jasper suggested using limits for exploring a parameter change, but then quickly added that he did not think that would be appropriate for Sasha and Keoni). This change in participants' roles seemed to elicit an increase in noticing MKT around quantitative reasoning, and thus contributed to the shift in CoFs.

In summary, there were three elements of the learning environment that gave rise to the emergence of and shift between the two CoFs. Analysis presented in Chapter 5 explored how each of these influenced participants' noticing and subsequent MKT development. I want to take a step back and examine the mini-course holistically. Some readers may wonder if the videos alone would be sufficient for prospective teachers; in other words, would it work to just post the videos online for PSTs to learn from on their own outside of a formal setting. I believe the answer to such a challenge is a resounding "No!" The videos served for me, as a teacher educator, a tool box for helping participants develop MKT. I carefully selected clips to show based on my second-order models of participants' emerging quantitative reasoning and MKT around quantitative reasoning. Participants also played important roles in others' development of MKT—for example, Jasper's moment with Willow in

which he deliberately drew brackets to indicate his quantification of the distance $y_0 + y_f$ was a powerful moment in Willow's development of quantitative reasoning. Additionally, the supports for reflection that I designed into the mini-course were likely crucial components of the course that fostered participants' development of MKT. My protocol that guided conversations about the videos and the Task Reflection Document served as models for how participants should engage with both the math tasks and viewing the videos. In turn, participants seemed to appropriate some of these reflective prompts in their own conversations around the tasks, signaling a potential transformation from math student to math teacher.

Significance

Having summarized the results of this dissertation study, I now return to the significance arguments I introduced in Chapter 1. In this section, I first elaborate several ways in which the study is theoretically significant. I then discuss how the findings from Chapter 5 have methodological significance.

Theoretical Significance

The findings reported in this study contribute to the field's understanding of quantitative reasoning in several ways. First, three participants initially struggled to reason quantitatively with distances. Notably, in the pre-interview Willow, Desmond, and Sierra did not demonstrate quantitative reasoning at a level of sophistication one might expect of senior mathematics majors. This finding contributes to the narrative that quantitative reasoning is hard for students across grade levels; that all three participants seemed to develop more sophisticated quantitative reasoning through their

engagement in the mini-course is significant. This study contributes to the field's understanding of prospective teachers' quantitative reasoning.

Second, I explicated two facets of MKT around quantitative reasoning: decentering and orientation. Decentering has recently been introduced as one way of thinking about MKT development, and this study contributes to the fields' understanding of how decentering may specifically aid in the development of MKT around quantitative reasoning. For example, decentering to reason quantitatively as a high school student might allow teachers to better anticipate students' conceptual challenges with quantifying and forming quantitative relationships.

Additionally, by comparing two participants, Willow and Marshall, this study demonstrates that decentering is not developmental. Willow seemed to decenter during the mini-course, suggesting that her shift in quantitative reasoning and her ability to decenter developed concurrently. Marshall entered the study with more sophisticated quantitative reasoning, and was also able to decenter. This suggests that we do not need to wait until prospective teachers have developed sophisticated mathematical knowledge before providing opportunities for them to engage with students' mathematical thinking. Instead, prospective teachers like Willow may benefit from learning the math content alongside virtual students such as Sasha and Keoni.

Finally, while the notion of a conceptual orientation is not new, this study contributes to the fields' understanding of the construct by demonstrating that a mini-course designed around conceptually-oriented longitudinal videos can be effective in helping prospective teachers develop such an orientation. Moreover, the evidence

presented in Chapter 4 showed that all three participants who experienced a shift in quantitative reasoning also experienced a shift in their orientation. This suggests that these two constructs may co-inform one another as they develop. Of course, one might argue that the two constructs (quantitative reasoning and conceptual orientation) are equivalent; however, quantitative reasoning is a tool that teachers can deploy regardless of their orientation (as evidenced by April in the pre-interview).

Additionally, findings from this study extend the Silverman and Thompson (2008) framework for the development of MKT. As I have discussed elsewhere, the framework assumes that MKT develops as teachers ask themselves questions about the content, instruction, and how students will approach the content. Results from this study do not contradict that hypothesis; however, the findings from Chapter 5 suggest that hypothesis alone is not sufficient for explaining the development of MKT.

Instead, two factors seemed to play a critical role in the development of participants' MKT around quantitative reasoning with distances. First, the use of the focusing framework in Chapter 5 highlighted the role that socially-organized noticing seemed to play in participants' development of MKT. Their reflections on quantitative reasoning, instruction for quantitative reasoning, and how Sasha and Keoni reasoned quantitatively did not occur in isolation. Instead, these reflections played out in a complex social environment. Ideas were shared and assumptions were challenged (e.g., Willow disagreeing with April's claim that Sasha and Keoni had used the distance formula), and this interplay and exchange of ideas between participants seemed generative for their MKT.

Prior to conducting the study, I conjectured that while experienced educators may have the ability to develop MKT simply by thinking about the content, instruction for that content, and how students might respond, such a thought experiment would likely not be sufficient for prospective teachers. My hypothesis was that experience with students' thinking and ways of reasoning is necessary for MKT development. The results from Chapter 5 are consistent with this hypothesis, which suggests that the MathTalk videos (due to their longitudinal nature) were an important tool that supported participants' emerging MKT.

Methodological Significance

This study also holds methodological significance. The focusing framework was originally created to examine mathematical learning. My conceptualization of MKT is that it is fundamentally mathematical in nature (Bass, 2005; Stylianides & Stylianides, 2010), yet it is pedagogical in the sense that its domain of application is the mathematics classroom. Accordingly, I extended the focusing framework to include several pedagogical considerations, which allowed me to examine the development of MKT.

To extend the framework, I first modified the definition of noticing given by Lobato et al. (2013) to include the noticing of pedagogical features in a learning environment. Doing so allowed me to account for both mathematical and pedagogical features and regularities in the learning environment that seemed to capture participants' attention. Accordingly, I modified the definitions of both focusing

interactions and features of tasks so that they too helped account for pedagogical features that participants noticed.

The focusing framework is an analytical tool that allows researchers to account for how noticing as a socially-situated phenomenon influences learning. By modifying the framework to include pedagogical features, I provided an account of how noticing influenced the development of MKT around quantitative reasoning with distances. Consequently, this study serves as a blueprint for how researchers may make similar modifications to the focusing framework to explore MKT development for other mathematical content.

Study Limitations

One limitation of this study was the use of clinical interviews as a tool for exploring participants' MKT around quantitative reasoning. The interviews captured some facets of participants' individual MKT around quantitative reasoning. However, I wonder about the fit between the knowledge that seemed to be captured in those interviews and the knowledge participants deployed during the mini-course. This concern stems from my view of the situated nature of MKT (Brown, Collins, & Duguid, 1989; Stylianides & Stylianides, 2010).

For example, in both the pre-and post-interviews, Desmond seemed reserved, and he tended to rush through problems. My sense of Desmond based solely on those interviews is that he might not have learned much in the mini-course. However, throughout my analysis of the classroom data I was continually surprised at his insight and the contributions he made to his groups as they worked on math tasks. Almost

none of that emerged in the post-interview. It is plausible that as Desmond interacted in small groups of his peers, he “felt” more like a teacher and could more readily deploy his teacher knowledge, whereas in the clinical setting of a post-interview he “felt” more like a student who was taking a test. I have no data to support this interpretation; instead, I offer it as a plausible explanation for why Desmond’s post-interview might not have adequately captured his MKT around quantitative reasoning.

A second limitation is that the study included only seven participants. My goal for designing the study was to develop local theory (Prediger et al., 2015) about the development of MKT around quantitative reasoning. In Chapter 3, I outlined my rationale for limiting the number of participants, and I elaborated the selection criteria I used to ensure a representative sample (Corbin & Strauss, 1990). Consequently, while I am confident that I have succeeded in “producing theory that is ‘conceptually dense’” (Strauss & Corbin, 1994, p. 278), this has come potentially at the expense of producing theory that is more broadly generalizable that might result from a study of more prospective secondary teachers.

I briefly discussed in Chapter 5 how the instruments used in the interviews did not seem to capture participants’ knowledge about the components of the Silverman and Thompson (2008) framework for MKT. In hindsight, more care should have been taken to design tasks and questions that would provide opportunities for participants to be explicit about ways students understand quantitative reasoning, milestones along a learning trajectory for quantitative reasoning, and instructional moves that support students’ quantitative reasoning.

Additionally, I did not anticipate the shift in affect. This is reflected in both the interview protocols I used as well as the design of the mini-course. Consequently, I was unable to link the shift in affect to specific features or regularities in the course. This is unfortunate since affect seems to play a large role in the efficacy of instruction (McLeod, 1992; Philipp, 2007; Philipp et al., 2007) and being able to explicitly link the shift in affect to the features of the mini-course could be informative.

I argued in Chapter 2 and Chapter 3 that the MathTalk videos have several features that (a) make them unlike other videos that have been used in teacher training courses or professional development and/or (b) foster the development of MKT. These features include professional-level editing, instruction grounded by research on children's thinking, and a longitudinal treatment that highlights the evolving thinking and reasoning of the same pair of students over several hours of lessons.

If, as I have concluded, participants' engagement with the MathTalk videos was critical to the development of their MKT around quantitative reasoning, then there is a practical limitation of this study to consider. The features that supported MKT development come at steep cost, both in money and time. The equipment used to create and edit the videos is expensive. It took months of planning and filming to capture the lessons on film, and then several more months of editing to create the final product that is hosted on the MathTalk website (www.mathtalk.org). Consequently, replicating this study to examine MKT around other mathematical content would likely require significant investment in resources and human capital.

Future Research and Implications

Usiskin (2001a) talked of the “chicken-and-egg dilemma” that describes how students will not learn to reason quantitatively until they have teachers who themselves reason quantitatively. With the numerous recommendations to improve students’ quantitative reasoning, more research on prospective teachers’ MKT around quantitative reasoning is needed.

Saying that researchers should examine MKT around quantitative reasoning requires elaboration in specificity along two dimensions. While larger grain-sized models of MKT for broad mathematical disciplines (e.g., algebra) contribute to the field’s understanding of MKT as a construct, the field seems to be missing descriptions of specific facets of MKT around particular mathematical ideas. Much like the field has developed fine-grained learning progressions and trajectories for mathematical knowledge (Lobato & Walters, in press), development of similar fine-grained accounts of MKT is needed.

This study acts as one such account by elaborating three facets of MKT around quantitative reasoning with distances. However, I think it is plausible that there are other facets of MKT around quantitative reasoning with distances. For example, due in part to the limitations with my interview protocol discussed previously, I did not capture participants’ MKT that might help them make in-the-moment instructional moves. Such MKT certainly played a role in the video instructor’s ability to respond to Sasha and Keoni’s unexpected ways of reasoning (e.g., placing the focus on the

directrix in Lesson 1). More research could be conducted to elaborate additional facets of MKT around quantitative reasoning with distances.

The other dimension of specificity deals with the nature of quantitative reasoning. There are several conceptualizations of quantitative reasoning, but for this study I took a view of quantitative reasoning consistent with P. W. Thompson's (e.g., 1990, 1994, 2011). Even within this conceptualization, there are several threads one could follow. There is quantitative reasoning: (a) with distances; (b) related to speed (e.g., P. W. Thompson, 1994); (c) with functional relationships (e.g., Ellis, 2011); (d) with ratio and proportions (e.g., Lobato & Siebert, 2002); (e) with angle measures (e.g., Moore, 2013), and many others. Each of these threads can, and should, be objects of inquiry for developing MKT.

Finally, I end by reflecting on the MathTalk videos. There is so much pedagogical power in the MathTalk videos, and I am excited at the prospect of leveraging these videos as a mathematics teacher educator. The longitudinal nature of these videos seemed to be particularly powerful for participants. Participants began conjecturing about Sasha and Keoni's relationship, what they were like in high school, and if they went on to major in mathematics in college. Participants got to know Sasha and Keoni, and began treating Sasha and Keoni like their own students. Simply put, the MathTalk videos allowed participants to become vicarious teachers by taking on the role of an educator with the students in the videos.

Appendix A: Plans for Six Instructional Sessions in the Mini-Course

Session 1 Plan: Constructing a parabola from the definition

5 minutes – Introduction. THANK YOU FOR PARTICIPATING! I am a former teacher, PhD student, interested in working with math teachers. I hope that by participating you'll leave with some new understandings about parabolas and ideas about how to help students come to similar understandings. [Me]

15 minutes – REFLECT – Check out this task (Construct a parabola), but don't work on it yet. I want you to write your thoughts about the task on this *Task Reflection* document. [Individual]

5 minutes – Introduce first task: "Construct a Parabola." [Me]

50 minutes – Task: Construct a Parabola. As you work, feel free to revise your *Task Reflection* document if you have any new insights. [Groups]

10 minutes – REFLECT – Discussion about the task, share out some solution methods. [Whole Class]

35 minutes – Watch parts of Lesson 1 with Sasha & Keoni. Discuss as we watch. As you watch, and as we discuss what we are seeing, feel free to revise your *Task Reflection* document if you have any new insights. [Whole Class]

Task:**The Parabola Task**

Definition: A parabola is the set of points that are equal distance from a fixed point (called the *focus*) and a fixed line (called the *directrix*).

Your task is to use this definition to create a parabola. I have many tools that you can use. You can request tools that you think might be useful, and if I have it I will give it to you (and if I don't have it, we'll try to figure out how to get one quickly!).

Session 2 Plan: Create an Equation for a Specific Parabola

6:00 – 6:20 – Welcome, Task Reflection document. Task will be put on side TVs,

TRD will be at each desk. [Individual]

6:20 – 6:40 – Task Time. Participants will work together on the sequence of tasks I’ve given them. – [Groups]

6:40 – 6:50 – Discussion. Share out insights from group work. [Whole Class]

6:50 – 6:10 – Watch videos of Sasha and Keoni. See list below. [Whole Class]

Lesson 2 [14:40]

- E1: 2:08 – 3:35 [1:27]
- E2: 0:10 – 5:00 [4:50]
- E3: 0:00 – 4:33 [4:33]
- E4: 0:22 – 3:48 [3:26]
- E5: 3:32 – 3:56 [0:24]

6:10 – 6:20 – Discussion. Let participants “catch up” mathematically.

What do you think Sasha and Keoni will do next?

6:20 – 6:45 – Watch videos of Sasha and Keoni. See list below. [Whole Class]

Lesson 3 [17:10]

- E1: 0:45 – 10:30 [9:45]
- E2: 0:18 – 2:45 [2:27]
- E5: 0:00 – 4:58 [4:58]

What are some of the go-to moves Joanne seems to make? Why?

6:45 – 7:00 – Discussion. [Whole Class]

Task: Use the definition of a parabola to create a method for locating the x -value for any point on the parabola given the y -value of that point.

Session 3 Plan: Create an Equation for a Parabola with Vertex at the Origin

3:00 – 3:15 – Welcome, Task Reflection document. Task will be put on side TVs,

TRD will be at each desk. [Individual]

3:15 – 3:35 – Task Time. Participants will work together on the sequence of tasks I've given them. – [Groups]

3:35 – 3:45 – Discussion. Share out insights from group work. [Whole Class]

3:45 – 4:00 – Watch videos of Sasha and Keoni. See list below. [Whole Class]

Lesson 4 [13:38]

- E2: 0:13 – 1:55 and then 2:57 – 3:24 [2:09] (didn't show—Based on what groups were saying I decided to show these instead of E2)
 - E1 all
 - E3 all
- E4: 0:00 – 2:10 [2:10]
- E6: 1:33 – 9:33 and then 11:22 – 12:41 [9:19]

4:00 – 4:10 – Discussion. Use Post-Video Protocol. [Whole Class]

4:10 – 4:20 – Groups discuss and fill out “Recapping the First 4 Lessons” sheet.

[Groups]

4:30 – 5:00 – Discussion. [Whole Class]

Task: Your task is to create a method for locating the y -value for any point on the parabola given the x -value of that point.

Recapping the First 4 Video Lessons

So far we have seen parts of 4 lessons with Sasha and Keoni. We are about half-way through the mini-course – so let's recap what has happened so far.

1. Summarize what has happened so far with Sasha, Keoni, and Joanne.
2. What have Sasha and Keoni learned? What is your evidence for your claims?
3. What challenges have Sasha and Keoni faced?
4. What did Joanne do to help Sasha and Keoni overcome those challenges?

Session 4 Plan: Develop an Equation for any Parabola Whose Vertex is (0,0)

3:00 – 3:20 – Discussion. [Whole Class]

3:20 – 3:35 – Task Reflection document. Task will be put on side TVs, TRD will be at each desk. [Individual]

3:35 – 4:00 – Task Time. Participants will work together on the task. – [Groups]

4:00 – 4:10 – Discussion. Share out insights from group work. [Whole Class]

4:10 – 4:40 – Watch videos of Sasha and Keoni. See list below. [Whole Class]

Consider sprinkling in discussion after each video!

Lesson 5 [27:59]

- E1: 0:33 – 1:20 and then 1:33 – 6:06 [5:20]
 - Ask: Why did Joanne ask about the points?
- E2: 0:27 – 2:08 and then 2:08 – 3:29 and then 3:29 – 3:50 and then 4:50 – 5:20 [4:53]
- E4: 0:15 – 4:10 [3:55] – INTRODUCE by saying Joanne asked them to predict for focus at (0,3)
- E5: 0:20 – 4:57 and then 5:47 – 6:23 [6:13]
- E6: 0:32 – 4:22 and then 4:22 – 6:05 and then 6:05 – 8:06 [7:34]

4:40 – 5:00 – Discussion. [Whole Class]

Task: Your task is to develop an equation for any parabola whose vertex is at the origin (0,0).

Session 5 Plan: Explain a Parameter Change

3:00 – 3:15 – Task Reflection document. Task will be put on side TVs, TRD will be at each desk. [Individual]

3:15 – 3:25 – Quick recap of what happened last time, then E6: 0:32 – 4:22 and then 6:05 – 8:06 [5:51]

3:25 – 3:50 – Task Time. Participants will work together on the task. – [Groups]

3:50 – 4:10 – Discussion. Share out insights from group work. [Whole Class]

4:10 – 4:40 – Watch videos of Sasha and Keoni. See list below. [Whole Class]

Consider sprinkling in discussion after each video!

Lesson 6 [16:53]

- E1: 0:47 – 2:25 and then 3:20 – 4:11 and then 4:41 – 8:07 and then 8:27 – 11:42 [9:10]

Tell class that Sasha and Keoni have done this for two other parabolas, using similar methods. Also, Joanne planned in advance to make sure that Sasha and Keoni found “comparison points” for each parabola – points sharing x -values, points sharing y -values, and special points.

- E5: 0:25 – 1:12 and then 1:46 – 2:44 and then 4:55 – 10:53 [7:43]

Lesson 7 [11:29]

- E2: 0:20 – 3:42 and then 5:01 – 5:37 [3:58]
- E3: 0:21 – 2:32 [2:11]
- E6: 0:18 – 5:38 [5:20]

4:40 – 5:00 – Discussion. Use Post-Video Protocol. [Whole Class]

Task: Your task is to:

1. Figure out what effect changing the value of p in the equation $y = \frac{x^2}{4p}$ has on the graph of a parabola.
2. Explain why this relationship exists.

Session 6: Final Session

4:15 – 5:00 – Watch videos of Sasha and Keoni. [Whole Class]

Consider sprinkling in discussion after each video!

Lesson 8 [24:30]

- E1: 0:31 – 2:16 and then 4:29 – 6:51 [4:06]
- E2: 0:20 – 7:00 and then 7:38 – 9:36 [8:38]
- E4: 0:20 – 1:37 and then 2:06 – 5:41 [4:52]
- E5: 0:18 – 3:27 and then 4:24 – 9:09 [7:54]

5:00 – 5:45 – Final Discussion

5:45 – 5:55 – Interview sign up

5:55 – 6:15 – Hand out Final Reflection; collect at post-interview if not enough time.

Final Discussion

Be sure to ask for attribution: For example, “Does your thinking about this build off of someone else’s ideas? Can you talk about that?”

(These questions were read aloud to participants, who then discussed these in groups)

1. Is there a difference between how you approached and solved these tasks and how HS students would approach and solve these tasks?
2. What understandings did Sasha and Keoni develop as a result of lesson 6 and 7 in which they generated three explanations for how and why changing p affects the graph of the parabola?
 - a. Now, imagine that instead of this instructional sequence, Joanne instead had them experiment with sliders and then Sasha and Keoni stated that “as p increases, the parabola gets wider.” Joanne replied, “Exactly! You can remember this by thinking ‘If I have more pea soup, I’ll need a wider bowl!’”
 - b. What understandings would Sasha and Keoni have developed in this scenario?
3. Consider the following statement:

“Joanne taught with purpose. Every task she posed, every question she asked, and every prompt she gave Sasha and Keoni was in anticipation of, or in response to, a challenge Sasha and Keoni faced.”

How do you react to this statement?

Final Reflection

(You may use the back if you need more room)

1. What did Sasha and Keoni learn during the 8 lessons we watched?
2. What did you learn over these six instructional sessions?
3. The mathematical goal of these sessions was to develop deeper understanding of the relationships between geometric and algebraic conceptions of parabolas. What connections did you make?
4. I also had some pedagogical goals for these sessions. What insights did you gain about how to teach this specific content to high school students? Some things you might think about: planning, instructional actions, tasks, student conceptions and understanding, challenges to learning, practices, etc.

Appendix B: Parabola Unit Episodes of Note

Lesson 1

E1:MS	1:45	Struggle with focus on directrix begins. Resolved by S @5:45
	7:50	K tries to put focus on directrix again. This leads to...
	8:15	... issues with measuring distance from point to line . Resolved by K @10:00
E2:Ex	1:03	Focus NOT on directrix . First find vertex @1:03.
	2:07	Next point tested prompted by Joanne “can you estimate where you think the next point will be?”
	2:33	Joanne: “What has to be – can you point to where the two things in the definition would be? Where’s the distance from his finger to the line? And where’s the distance from his finger to the focus? Are those two equal?”
	4:15	How many points are on a parabola?
	4:50	Dotdotdotdotdot...
	5:25	...then Joanne “Pick one – is that the same distance to your line as it is to the focus?”
	5:40	A new directrix and focus
	6:15	Special points!
	8:00	Point tested , not on parabola, but leads to...
	9:00	Begin solution with parallel lines and compass – this starts out when...
	9:00	Point is wrong, and Joanne says “adapt it, change it a little so that it works”
	11:30	Another point using parallel lines and compass
	12:30	Another point using parallel lines and compass
	13:00	Parabola is sketched
E3:Ref		Explanation (about 1 minute)
E4:RR	1:30	Prediction is made regarding shape of new parabola (focus is closer this time, S predicts it will be steeper – V-ish instead of U-ish)

EX = Episode X

MS = Making Sense

Ex = Explaining

Ref = Reflecting

RR = Repeating Your Reasoning

Lesson 2

E1:MS	1:21	Joanne: What do you think (2,1) means?
	2:30	Origin is on parabola
	2:43	Joanne: Now is the origin on the parabola? Can you use the definition of a parabola to justify or explain why it's on the parabola?
	3:11	Joanne: Now can you look at those points... (2,1) and use the definition of the parabola to explain why those meet the definition of a parabola, why they are on the parabola?
E2:Ex	0:19	(4,4) on parabola – Joanne: Use the definition of a parabola to justify why it's on the parabola
	2:00	Joanne: Can you mark down everything you do know? I heard you say something about a 3 and a 4 – maybe you can start by labelling where those are
	2:55	Joanne: Why don't we just take stock of where we are at? What is it that you're trying to figure out? ... Can you summarize what you know so far
	3:40	Joanne: Is there any math formulas or methods that might be useful here?...now if you could find a right triangle then the PythThm might help...
	4:30	first use of PythThm
E3:Ex	0:00	$y = 5$ what is x ? First appearance of parallel line
	3:15	Joanne: Now can you write the coordinate pair for the point that is on the parabola?
E4:RR	0:00	$y = 7$ what is x ? Another parallel line
	3:00	Joanne: Now are you starting to notice anything that you did similar to find the x value when y was 5 versus when y was 7?
	3:15	S – We do the Pythagorean theorem each time
	3:45	Joanne: Can you say the definition in your own words?
E5:RR	0:00	$y = 10$ what is x ? Another parallel line

Lesson 3

E1:MS	0:58	Joanne: Let's pick any point not on a grid, and maybe you can make the coordinate pair for the y , yeah, just put y for the y -value
	1:05	Joanne: Now I'm trying to see if you can generalize what you did for, like how would you find the x ? What's the method?
	1:20	K – well we would find the distance between our point and the directrix
	1:25	Joanne: Ok what do you think that distance would be?
	1:30 3:05 4:03 4:07	Initial struggle with generalizing y as distance (and $y + 1$). Joanne shows their previous work K seems to resolve the issue Joanne: Why don't you write that down, and we'll see what Sasha thinks about that Joanne: Show me the distance y on that
	8:45	Joanne: What are you trying to find Keoni? K struggles with quantity/distance $y - 1$. Resolved by S&K through 11:25
	9:55	S begins another particular example to help K see quantity/distance $y - 1$
E2:Ex	0:26	Joanne: Now can you use that information to find the x -value?
	0:40	K is still not ready to use variables, so he uses particular $y = 7$
	1:04	Joanne: This is helpful, so that's what you would do if you had a specific y value, what if I told you we're just going to work with y so that it stands for any y value? Can you still use the Pythagorean Theorem?
	1:22	S writes first equation with the quantity/distances
	3:00	S&K solve equation
E3:R	0:30	K has confusion about how one might know what y is. (That is, he hasn't thought of choice of y being <i>his</i>). Not really resolved in this part, but S seems to get it
	2:30	Joanne: So what is b ? Is it a point? Is it a location? Is it a distance?... Can you show me?
E4:RR	0:40	S&K find x value when y is 3.5, first by using the triangle method, then using the equation.
	4:00	Joanne: Do you know the coordinate of that point?
	4:35	Joanne: Do you know the coordinates of any other points?
	5:30	Clarification of the $x/b/b$ squared confusion

Lesson 4

E1:MS	2:00	S&K have drawn a triangle and found the distance 5, but they seem confused about what to do next.
	2:30	Usage of equation – Joanne: I think that’s cheating!
E2:Ex	0:00	$x = 5$ what is y ? Long way (reasoning with quantity/distance)
	1:15	Not much doubt about quantity/distance!
	3:10	Joanne: Why do you think that works?
E3:RR	0:30	$x = 5$ what is y ? using equation
	1:26	$x = 10$ what is y ? using equation
	1:50	Joanne: You pick the b value this time! $x = 437$ what is y ? using equation
	3:00	Joanne: It sounds to me like you have sort of a method
E4:Ex	0:25	Joanne: can you explain a sort of general method of what you are doing? Because when b was 5, 10, 437 you had the same set of steps each time
	0:40	Joanne: How did you find your y ?
	1:00	Joanne: Stop did you square both sides on all? (generalizing move)
	1:40	Joanne: I just want you to use b , solve for b .
E5:R	0:15	Joanne: What do you think about them [the two equations]? How are they alike how are they different? When would you use one versus the other?
	1:30	Joanne: Tell me what the y value of the parabola is when b is 2.5
	3:45	Joanne: So I want you to compare these two equations. What’s alike and what’s different?
E6:Ex	2:00	Joanne: I want to know if you can derive that equation directly by using what you know about parabolas
	2:15	Joanne: Can you mark something that you think would stand for a general point on the parabola?...And if it’s a general point what would you call it?
	2:30	Sasha responds with “Point X ”
	2:40	Joanne: What would the coordinates be?
	2:45	Sasha: $x y$
	2:50	K: it’s kind of hard to tell In fact, at 3:40 he moves to a specific point to say why (x, y) makes sense – I think it’s because he’s providing rationale for the <i>order</i> of x and y
	4:00	Joanne: I think you making a really important point that that x corresponds to that x -axis – Keoni: Yeah – Joanne: You say yeah, what sense do you make of that?
	6:40	Joanne asks how they know two distances are equal (by the definition)
	7:27	Joanne: Can you kind of point to where the – is the $y - 1$ that entire axis? (gets Sasha to draw it in explicitly)

	9:00	Joanne: So can you use what you have here to derive either one of those equations?
	11:40	Joanne: Is there a way to get from one equation to the other?

Lesson 5

E1:MS	0:42	Joanne: Your goal for today is to create an equation for any parabola that goes through the origin, but we're going to start by finding the equation of the blue parabola
	0:50	Joanne: First I want you to try to make sense of what you're seeing – tell me everything you notice about the two parabolas
	1:30	Joanne: Do you think the equation for the blue parabola is going to be the same as the equation for the red parabola
	1:45	Keoni: I don't see why not But Sasha has her doubts...
	2:20	Joanne: Let's start by finding some points
	2:50	Joanne: Do you see any connection between the points you just found and the focus for either one?
	4:20	Joanne: Is there any relationship between that point (2,1) and the equation?
	5:10	Joanne: Now what was the other point on the blue parabola? Does that fit in the equation for the red one?
	5:50	Keoni resolves his doubts about the equation
E2:Ex	1:25	Joanne: When you say this is your y , can you just motion to me what you're saying is the distance that's y ?
	1:45	Joanne: Is the whole thing y or $y + 2$? Where is the y then?
E3:R	1:30	Both students explain their conjectures about the equation
	1:45	Joanne: So are you saying there is a connection with the focus? What is that connection?
E4:RR	1:00	No concerns with the general point (x, y) at this point!
E5:MS	0:30	Joanne: what do you think the equation is depending on? Why do you think that? What else are you noticing about the equations? How are they alike? How are they different?
	1:20	Joanne: And before Sasha you said something about the focus and the directrix – do you think they're related to the equation at all?
	2:15	Joanne: Are you seeing any relationship between anything involving the focus or the directrix or both of them and those equations?
	3:00	Joanne: Is there any relationship between those distances and the equations. You can make a conjecture – we can try some things out.
	3:20	Joanne: Let's make two predictions – what if the distance between the focus and the origin was four?
	4:20	Joanne: Did that pattern work for the other ones?
E6:Ex	1:50	Joanne: Now label anything else you know, now that you know the distance from the origin to directrix is p – what else do you know?
	3:00	Joanne: Well let's stop for just one second, I want to make sure – maybe in a different color you could outline each one of these. Where is the y ? The distance of y ? Ok now where is the distance of p ? Ok, and then $y + p$? OK now on the other side, where's the y ? So if $y - p$, I think of you have some total distance y , and you're

		taking off a distance of p ? Ok, and then what part represents the distance $y - p$?
E7:R	0:17	Joanne: Let's just end by thinking about what this means, what this equation gives us
	1:55	Joanne: Now can you use that to generate some points on the parabola?
	2:10	Keoni seems to have some idea about special points already, as evidenced by his desire to choose one instead of picking a value and using the equation as Joanne asked
	2:50	Sasha picks a point way out at $x = 6$, but Keoni says we don't know if it's on the same parabola
	4:00	Joanne: Can you use Sasha's method to find some other points that are visible to us?
	4:50	Joanne: Does it look like a parabola?

Lesson 6

E1:MS	3:40	Joanne: So you found a point on the parabola – how did you find it?
	4:50	Joanne: Can you find another point without plugging in to the equation – just by using what you know about the geometry of parabolas?
	5:30	Keoni puts the focus at $(0,1)$ even though the p value is one-fourth
	8:25	Joanne: Now can you use this information, the focus and the directrix, to find another point on the parabola without plugging directly into the equation?
	9:50	Joanne: Now what are a couple of different ways you could check that point? Using the definition does it fit the definition? What is that distance from the point to the focus? And from that point to the directrix? Why is it one-half? Does it also fit into your equation?
	12:20	Joanne: Why don't you just go ahead and finish plotting enough points so that you can fill in your parabola – you can get the shape of it?
	12:46	Joanne: Now I want you to make a prediction! We're going to now plot, we're going to graph the parabola where p is one-half – can you just show what you think the graph will look like of the parabola when p is one-half? Do you think, compared to this one - ? Just sketch it in.
E2:Ex	2:00	Joanne: Could you also have used the parabola definition to [find the point]?
	2:20	Joanne: What is the distance from the point to the focus?
	3:20	Joanne: What's the p value – what's the distance from the origin to the focus?
	5:40	Joanne: Go ahead and find some more points on the parabola
	6:30	Joanne: You know, I'm kind of interested – in the first parabola you have a point $(2,4)$ – what do you think the x value would need to be for the y value to be 4 on this new parabola?
E3:R	0:20	Joanne: Now I would like you to tell me what else you noticed – like comparing different points in the two parabolas -
	0:40	S: When p changes the focus and directrix change
	1:30	Joanne: Let's start with the special points – what do you notice about that? What's the same and what is different?
E4:RR	1:05	Joanne: Why is that your directrix? And why was your focus one away from your origin?
	4:30	Joanne: Now before you did it with $y = 4$, why don't you do one for this parabola so we can get a nice comparison?
E5:MS	0:20	Joanne: Can you compare? K: They get wider! Joanne: How would you state what you've figured out about when you change the value of p , what it does to the parabola?
	1:00	Joanne: Do you have any evidence for that?

	1:45	Joanne: Now let's compare points so we can see what we notice about these parabolas. So we can compare all three of these parabolas across some nice points of comparison. What would be some points that we could nicely compare?
	2:20	S: They get wider as they go. Joanne: What does that mean?
	2:28	Joanne: So complete the sentence The x -value gets greater as the...
	3:15	Joanne: are there any other comparable points?
	4:00	Joanne: Ok, so as p is increasing and x is 2, what is happening to the y values?
	4:55	Joanne: How about our special points?
	5:30	Joanne: What do you notice about that? Can you say something about the x value compared to the y value for a single [special] point?
	5:45	Sasha: It's half – our y value is half of the x value
	6:00	Joanne: Uh huh – why do you think that is? Can you relate it to the p value? And the focus?
	7:20	S&K are having trouble putting into words the relationship between x and y coordinates for the special point in terms of p , so Joanne: Let me try something here (puts up a generic parabola with unknown p) – Where is the special point? And can you tell me the coordinate pair for either the special point, the focus, or both of them?
	8:20	Sasha and Keoni correctly label $(2p, p)$ so Joanne: Show me all the other places where you can see p on this graph
	9:00	Joanne: Show me where you see $2p$ Where else? Where else? Where else? Where else? Where else? Can you do a vertical distance that's $2p$? What is $2p$ what does it mean?

Lesson 7

E1:MS	0:55	Joanne: What did you discover about how changing the value of p affects the parabola?
	1:20	Joanne has Keoni write down his two statements about the relationship so he and Sasha can refer back to it later
E2:Ex	0:30	Joanne: What do you notice about these points?
	1:00	Joanne: Can you use these three points to explain why when you increase p the parabola gets wider?
	1:30	Joanne uses the written statement from Keoni to refocus the students
	2:00	Joanne: You've already made the claim, now I want you to look at these three points and see if you can explain what's going on
	2:20	Joanne: And what's happening to the y values from the red to the blue to the green? And why does that mean it's going to get wider?
	3:20	Joanne: As it decreases it goes down – ok – why does that force the parabola to get wider?
	3:30	Sasha – because your x goes out and y is up, so it's a shorter y but the same amount of x , which causes it to go out
E3:R	0:20	Joanne: I want you to turn to that equation at the top – in this case what x value were we considering? Plug that value in for x . Now does that help you make an argument for when p is increasing what's happening to y ?
	1:40	Joanne: Now what does that help you see? Either the last thing we did or with the algebra in general what does it help you understand with what's going with this relationship between increasing p and the shape of the graph?
E4:RR	1:00	Joanne: Now can you use these points to again give us an even deeper understanding of why when you increase the value of p the parabola gets wider?
	3:50	Joanne: Now can you use that general equation and put 4 in for y and see what's happening when you increase p
	6:30	Joanne: Now what happens as p increases? How do you know?
E5:MS	1:10	Joanne: So first remind me how did you get the special points?
	1:40	Joanne: Why is the x value double the y value? can you use the definition of a parabola to think about that?
	2:30	Joanne: Now why does that mean the x value is going to be double the y value?
	5:00	S&K play with applet
E6:Ex	0:45	Joanne: And what are those two distances? Can you draw that in? And where is p ?
	1:30	Joanne: Can you use this general representation to explain why as you increase p you're going to get another parabola that's wider than the one you just had?
	1:44	Sasha: For every y you're going to double the x

	2:45	Joanne: And why is that x is double the y forcing the parabola to get wider?
	2:50	Sasha: It's going out wider faster
	4:30	Joanne: So you've expressed the special points as $(2p, p)$ can you use that to express again why that means that as you increase p the next parabola is going to be wider than the one that you had.

Lesson 8

E1:MS	1:05	Joanne: What did you figure out?
	6:00	Joanne: Why do you think the focus changes but not the directrix?
	6:25	Joanne: Can you see the p value for this parabola? Can you label it?
E2:Ex	1:40	Joanne: Can you summarize what you've got so far?
	2:20	Joanne: Keoni – you drew some lines, tell me about those.
	2:30	Joanne: What do you know about those two [lines]?
	3:05	Joanne: Let's see if you can figure out what the distances will be
	3:25	Joanne: Can you point to where the distance x is? Ah! But your Pythagorean Theorem only uses that short line!
	3:40	Joanne: Ok, I heard $x - 7$ as a question – Sasha what do you think?
	4:00	Joanne: So the distance $x - 7$ can you just show me with your hand?
E3:R	1:07	Joanne: Why do you think that's a minus 7 instead of a plus 7?
E4:RR	0:33	Joanne: I want you to imagine the applet again, and instead of letting h equal 7, I want you to in your mind let h equal -3 . The vertex will be? Now make a prediction what do you think the equation will be?
	3:15	Joanne: You've figured something out already! What is it, Sasha? Can you record that somewhere on there, what you just figured out?
	3:40	Joanne: So what would the expression be for that entire horizontal distance?
	4:20	Joanne: And you figured out why the sign will be a plus instead of a minus.
	5:40	Joanne: What have you concluded about the look of the equation?
E5:Ex	0:30	Joanne: What's your guess as to what happens? Sasha: We're going to have like y squareds or something!
	1:50	Joanne: Keoni can you put a dot where you think the focus is? And then I see a dot below – is that where you think the directrix is? Sasha what do you think?
	2:20	Joanne: What could you do to check?
	3:05	Joanne: What else do you notice? How else can we check and make sure the focus and directrix are correct?
	5:00	Joanne: I see a $y - 5$ can you show me how you got that?
	5:45	Joanne: So before you do the algebra, how does this differ from your base parabola?
	8:50	Joanne: Compare this to the others – what do you notice?
E6:Ex	4:00	Joanne: Let's talk through each of these. So you started with $y - 1$ then changed it to $y + 1$. How come?
	4:16	Joanne: So where is the distance y ? And then the 1? So the distance $y + 1$ is the distance between what and what? Ah, and you needed that because that helps you do what?

	5:30	Joanne: And that five breaks down into 3 and a two – the two is the distance from what to what?
E7:R	0:20	Joanne: You pick an h and a k value

Appendix C: Task Reflection Document

What do high school students need to know in order to complete this task?
How do you know? 1

Pre-Task	Post-Task	Post-Videos

Task: _____

Name: _____

2

How might high school students complete this task?
How do you know?

Pre-Task	Post-Task	Post-Videos

Task: _____

Name: _____

What challenges do you think high school students will encounter as they complete this task?
What understandings or mathematical tools are necessary to overcome those challenges?
As a teacher, what might you do or say to help high school students overcome those challenges?
How do you know?

Pre-Task	Post-Task	Post-Videos

Task: _____

Name: _____

Appendix D: Interview Contact Summary Form

Location: _____
Interview Date: _____

Today's Date: _____

1. What were the main issues or themes that struck you in this interview?

2. Summarize the information you got (or failed to get) on each of the target questions you had for this contact.

Question: Quantitative Reasoning

Information:

Image of a learning trajectory?

Images of instruction?

3. Anything else that struck you as salient, interesting, illuminating, or important in this interview?

4. What new (or remaining) questions have come up as a result of this interview? What changes to the protocol or tasks might you make in future interviews?

Appendix E: Pre-Interview Protocol

Start Camtasia!

At the start of the interview read to the student the following:

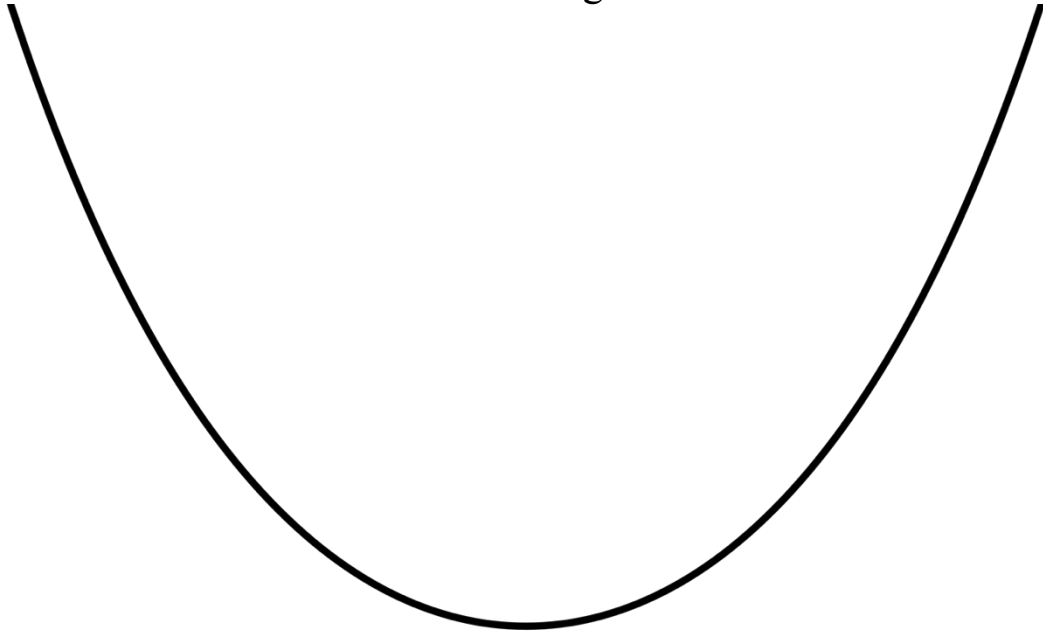
Thank you for agreeing to participate in this study! This interview is part of my dissertation study, and I want to reassure you that your confidentiality is very important to me. Even though I'm videotaping this interview, as I mentioned in the consent form, I will not use any identifying information in any published results.

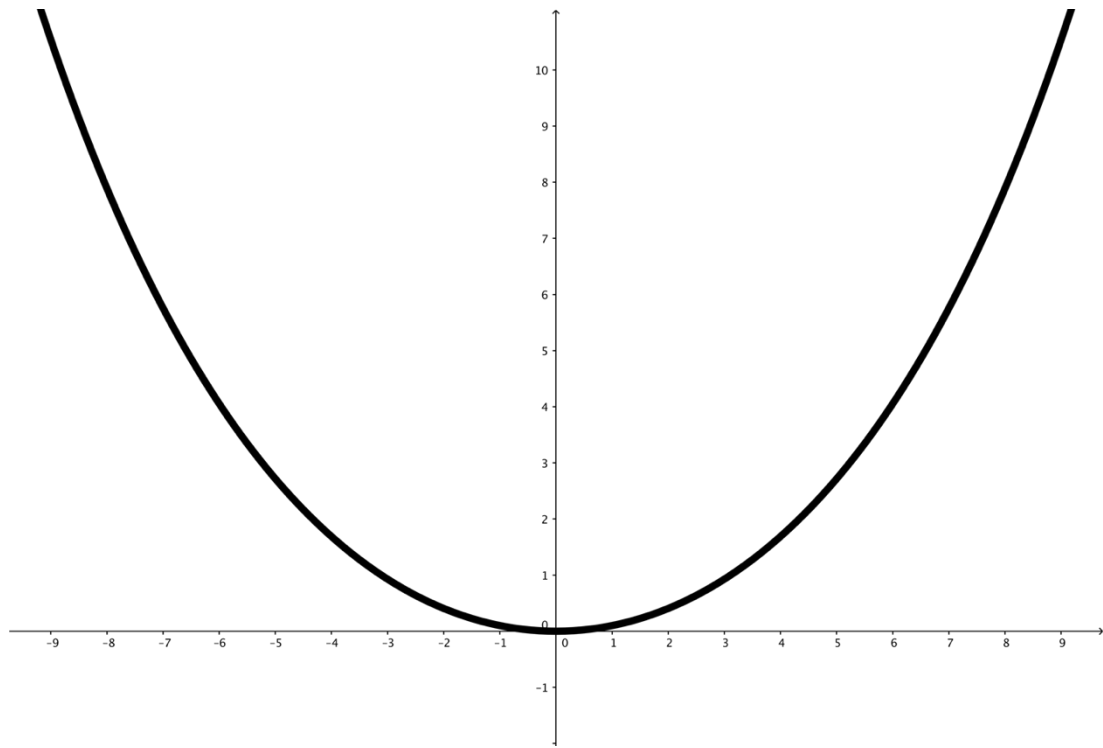
The purpose of the interview is to help me better understand your thinking and how you solve problems. I am not trying to evaluate your work as good or bad or right or wrong. All I want to know is how you are thinking. Please try your best to think out loud as you solve each problem.

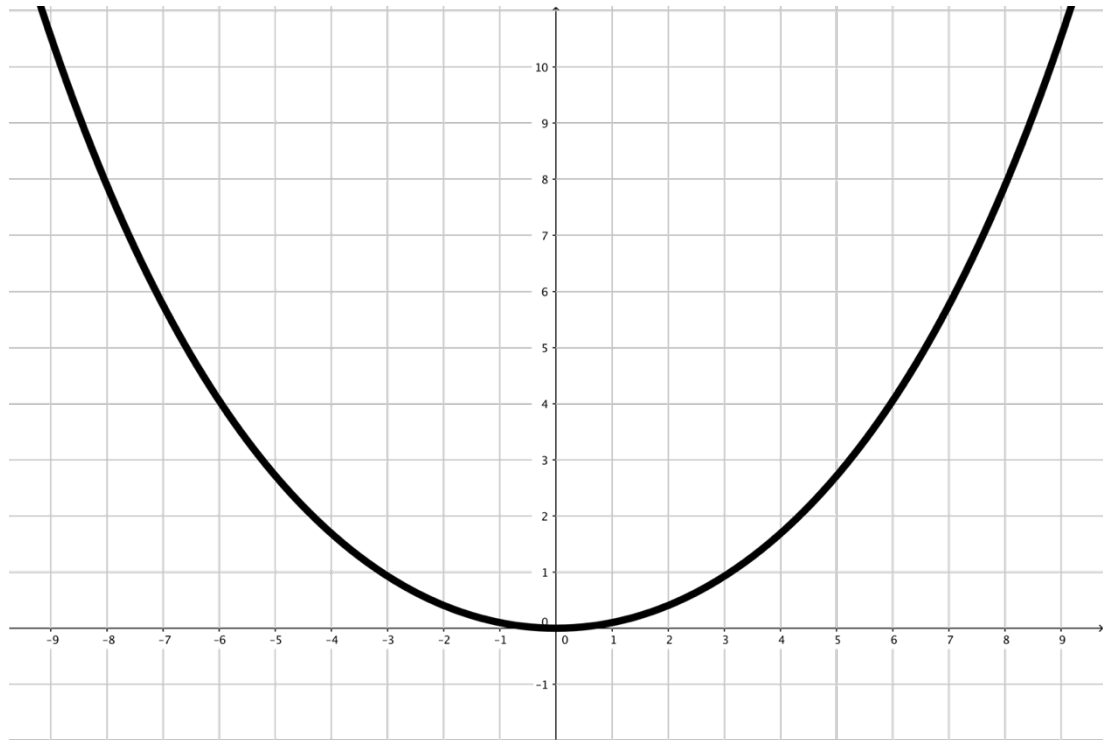
I may stop you in the middle of a task to ask questions. This is a normal part of these kinds of interviews, and it doesn't mean I think what you are doing is wrong or bad. Instead, it means I'm trying to make sure I understand all of your thinking.

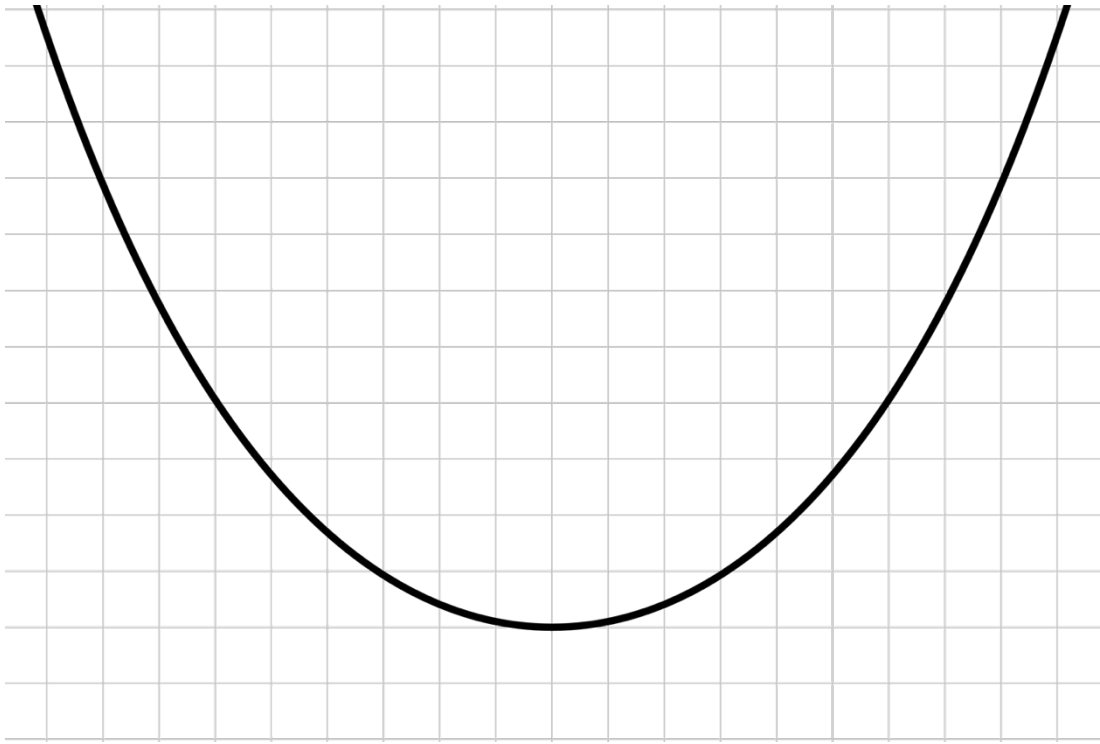
Task 1

Look at the curve below. Is it a parabola? Explain your reasoning.









Interviewer Protocol for Task 1: The “Parabola?” Task

Before handing the participant the task sheet ask the following question: Tell me everything you know about parabolas.

Hand the participant the task sheet. After the participant makes a claim about the curve, ask them to justify their claim.

Use the half-pages if the participant says they'd like the curve to be in a grid, or with axes, or both.

Follow-up Probes

- ❖ Is there a way you can tell if a given curve is a parabola? What characteristics do you look for? What do you think a high school student would say about this curve?

- ❖ Is there a definition for parabolas? How might that help you decide if the curve is a parabola/prove to me that the curve is a parabola?

- ❖ One of my former students told me that any U-Shaped curve is a parabola. I think he'd probably argue that this curve is a parabola. How would you respond?

- ❖ You mentioned something about an equation. What do you think the equation of this curve is? How would that help you determine if this was a parabola?

Task 2

Below is what is known as the *vertex form* of a parabola. What do you know about this?

$$y = a(x - h)^2 + k$$

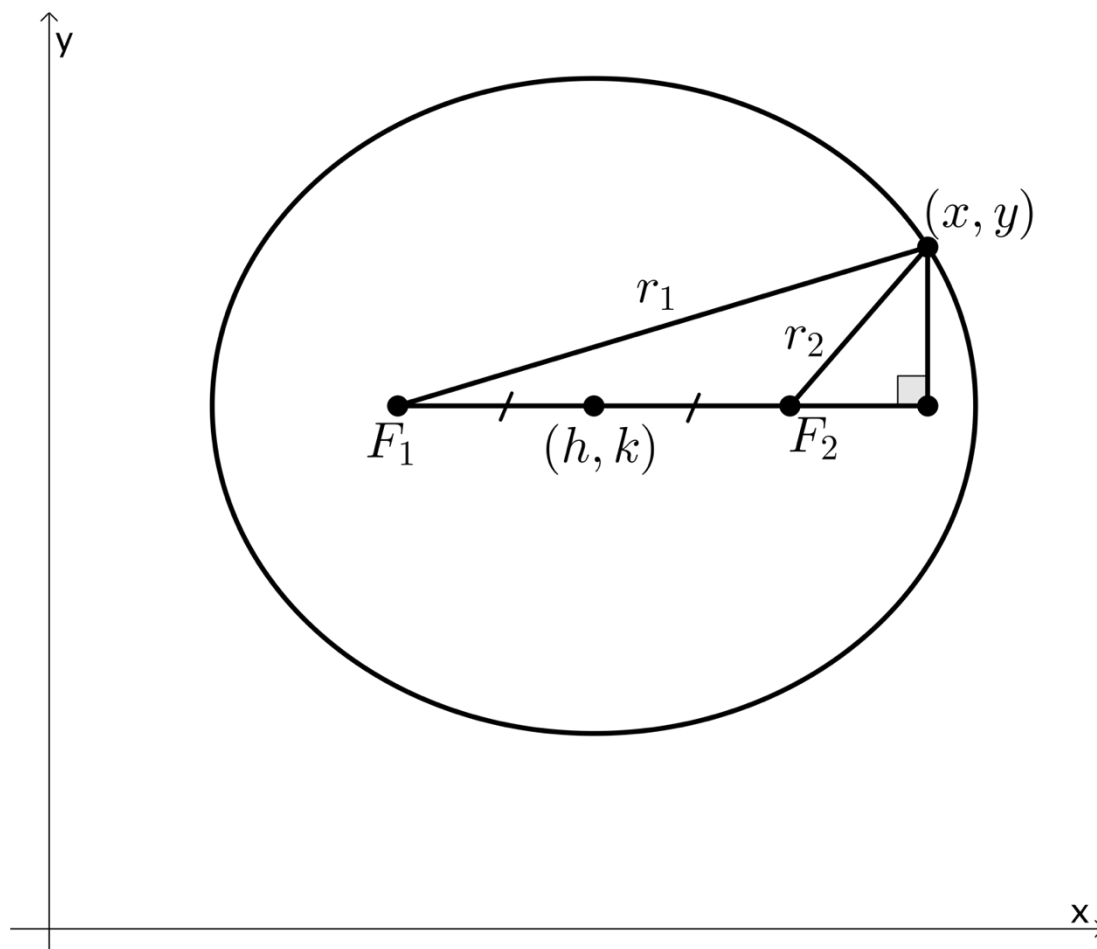
Interviewer Protocol for Task 2: The Equation Task

- ❖ What do you think high school students need to know in order to understand vertex form?
- ❖ What do you think high school students need to know in order to develop vertex form?
- ❖ How might high school students develop vertex form?
- ❖ Can you imagine other ways high school students might develop vertex form?
- ❖ What challenges do you think high school students will have in deriving this equation?
- ❖ What might you do or say to help them overcome those challenges?
- ❖ There are quite a few parameters there! What do you know about each of them (a, h, k) ?
 - Where does a come from? What about h or k ?
 - What does a “do” for the parabola? What about h or k ?

Task 3

The definition of an ellipse is given below, as is a graph of a general ellipse. Use the definition to find the area of the two right triangles in the picture.

An **ellipse** is the set of points in the plane, the sum of whose distances r_1 and r_2 from two fixed points F_1 and F_2 , called the *foci* (which themselves are separated by a distance of $2c$), is a positive constant given by $2a$.



Interviewer Protocol for Task 3: The Ellipse Task

- ❖ As participants label distances ask (for example) “Can you show me where you see c ?”

- ❖ If participants struggle, ask:
 - Have you used all of the information in the definition (perhaps they haven't located distances of c , for example)

 - If I told you the coordinates of the point labeled (x, y) could you find the areas of the triangle?
 - How did that help?

 - Could you now find the areas with the point given as (x, y)

- ❖ What do you think high school students need to know in order to solve this problem?

- ❖ What do you think would be most challenging for high school students who solve this problem?
 - What other challenges might they encounter?

 - What might you do or say to help them overcome those challenges?

Appendix F: Post-Interview Protocol

At the start of the interview read to the student the following:

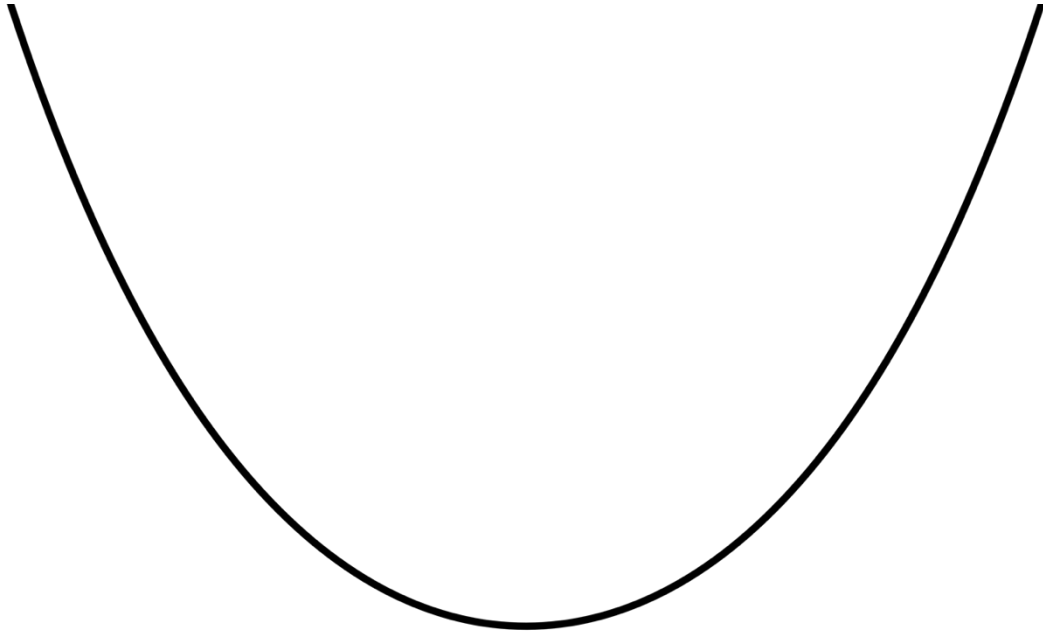
Thank you again for agreeing to participate in this study, and thank you for all of the hard work you put in during the sessions! As you know, this interview is part of my dissertation study, and I want to reassure you that your confidentiality is very important to me. Even though I'm videotaping this interview, as I mentioned in the consent form, I will not use any identifying information in any published results.

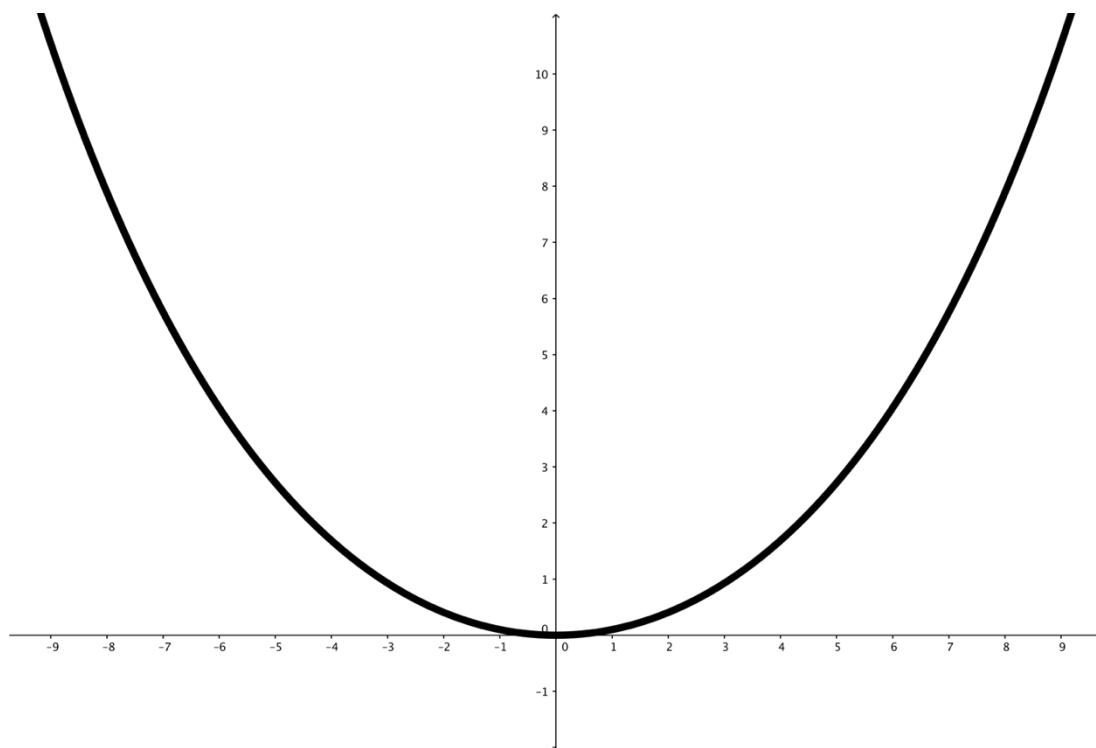
The purpose of the interview is to help me better understand your thinking and how you solve problems. I am not trying to evaluate your work as good or bad or right or wrong. All I want to know is how you are thinking. Please try your best to think out loud as you solve each problem.

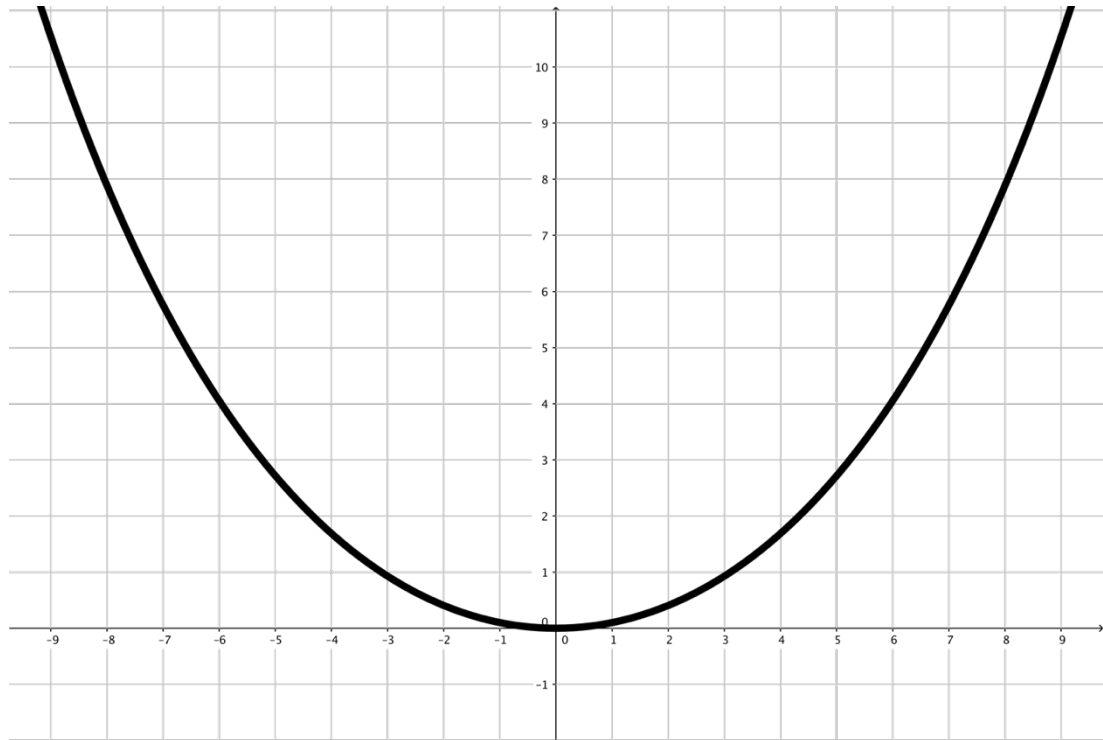
I may stop you in the middle of a task to ask questions. This is a normal part of these kinds of interviews, and it doesn't mean I think what you are doing is wrong or bad. Instead, it means I'm trying to make sure I understand all of the wonderful thinking you are doing.

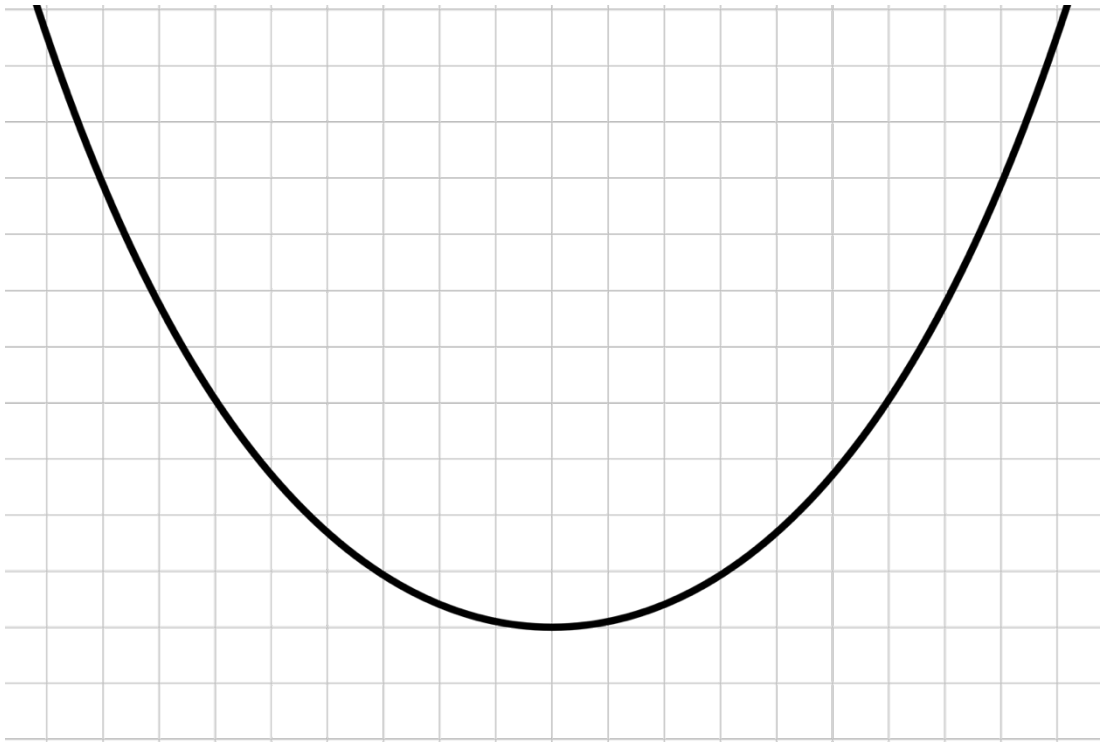
Task 1

Look at the curve below. Is it a parabola? Explain your reasoning.









Interviewer Protocol for Task 1: The “Parabola?” Task

Before handing the participant the task sheet ask the following question:

- ❖ Tell me everything you know about parabolas.

Hand the participant the task sheet. After the participant makes a claim about the curve, ask them to justify their claim.

Each half-page is labeled – use them if the participant says they’d like the curve to be in a grid, or with axes, or both.

Follow-up Probes

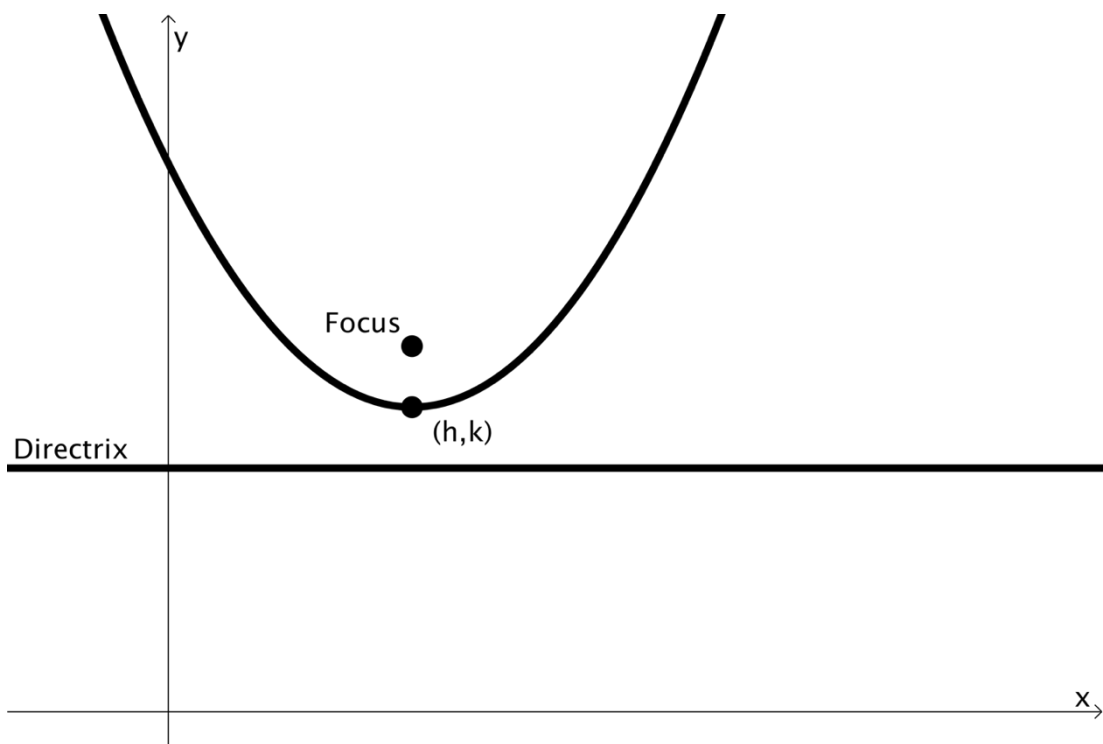
- ❖ Is there a way you can tell if a given curve is a parabola? What characteristics do you look for? What do you think a high school student would say about this curve?
- ❖ Is there a definition for parabolas? How might that help you decide if the curve is a parabola/prove to me that the curve is a parabola?

You mentioned something about an equation. What do you think the equation of this curve is? How would that help you determine if this was a parabola?

Task 2

The definition of a parabola is given below, as is a graph of a general parabola. Use the definition to find the equation of the parabola.

A **parabola** is the set of points that are equal distance from a point, called the *focus*, and a fixed line, called the *directrix*.



Interviewer Protocol for Task 2: The Parabola Task

- ❖ As participants label distances ask (for example) “Can you show me where you see y – *and what about the p ?*”

- ❖ If participants struggle, ask:
 - If I told you the coordinates of the vertex could you find the equation?
 - How did that help?

 - Could you now find the equation of the general parabola?

- ❖ What do you think high school students need to know in order to solve this problem?

- ❖ What do you think would be most challenging for high school students who solve this problem?
 - What other challenges might they encounter?

 - What might you do or say to help them overcome those challenges?

Task 3

Below is what is known as the *vertex form* of a parabola. What do you know about this?

$$y = a(x - h)^2 + k$$

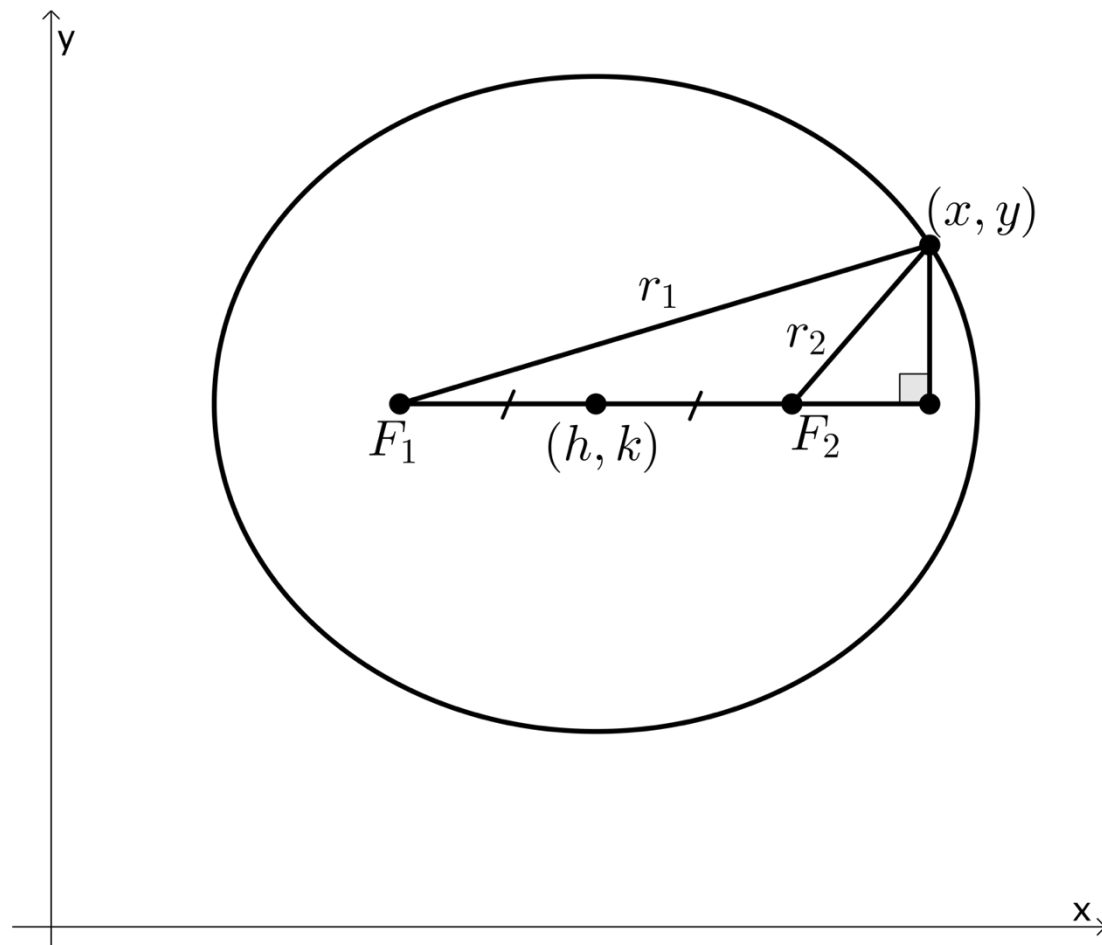
Protocol for Task 3: Equation Task

- ❖ What do you think high school students need to know in order to develop and understand vertex form?
- ❖ How might high school students develop vertex form?
- ❖ What challenges do you think high school students will have in deriving this equation?
- ❖ What might you do or say to help them overcome those challenges?
- ❖ There are quite a few parameters there! What do you know about each of them (a, h, k) ?
 - Where does a come from? What about h or k ?
 - What does a “do” for the parabola? What about h or k ?

Task 4: The Ellipse Task (Revisited)

The definition of an ellipse is given below, as is a graph of a general ellipse. Use the definition to find the area of the two right triangles in the picture.

An **ellipse** is the set of points in the plane, the sum of whose distances r_1 and r_2 from two fixed points F_1 and F_2 , called the *foci* (which themselves are separated by a distance of $2c$), is a positive constant given by $2a$.



Interviewer Protocol for Task 4: The Ellipse Task

- ❖ As participants label distances ask (for example) “Can you show me where you see c ?”

- ❖ If participants struggle, ask:
 - Have you used all of the information in the definition (perhaps they haven't located distances of c , for example)

 - If I told you the coordinates of the point labeled (x, y) could you find the areas of the triangle?
 - How did that help?

 - Could you now find the areas with the point given as (x, y)

- ❖ What do you think high school students need to know in order to solve this problem?

- ❖ What do you think would be most challenging for high school students who solve this problem?
 - What other challenges might they encounter?

 - What might you do or say to help them overcome those challenges?

(Look at their final reflection first)

What are you taking from this mini-class? What will you use from this class?

At the very end of the last session we talked about how Joanne taught with purpose. What enabled her to do so?

We've filled out the task reflection document 5 times now. What did you notice about the document and about your answers?

- How did your answers change for each task? Over the course of this mini-class?
- Was it useful to reflect on the task at three different times? Why?

How did Sasha and Keoni come to understand so much about parabolas?

- What were the understandings that developed over time that enabled them to solve the complex problems Joanne gave them?

How would you describe Joanne's instruction?

- What specific moves did Joanne make while planning?
- While specific moves did Joanne make while teaching?

REFERENCES

- Achieve Inc. (2002). *Foundations for success: Mathematics expectations for the middle grades*.
- An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, mathematics teachers in China and the US. *Journal of Mathematics Teacher Education*, 7(2), 145–172.
- Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, 21(2), 132–144.
- Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83–104). Westport, CT: Ablex Publishing.
- Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3–14). Edmonton, AB: CMESG/GDEDM.
- Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(1), 14–17, 20–22, 43–46.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.
- Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin of the American Mathematical Society*, 42(4), 417–430.
- Battista, M. T. (1999). The mathematical miseducation of America's youth: Ignoring research and scientific study in education. *Phi Delta Kappan*, 80(6), 425–433.
- Begle, E. (1979). Critical variables in mathematics education: Findings from a survey of empirical research. *Mathematical Association of America/National Council of Teachers of Mathematics, Washington, DC*.
- Bishop, J., Lamb, L. C., Philipp, R. A., Whitacre, I., Schappelle, B., & Lewis, M. (2014). Obstacles and affordances for integer reasoning from the analysis of children's thinking and the history of mathematics. *Journal for Research in Mathematics Education*, 45(1), 19–61.
- Boaler, J. (2013). Ability and mathematics: The mindset revolution that is reshaping education. In *Forum* (Vol. 55, pp. 143–152). Symposium Journals. Retrieved

from http://www.youcubed.org/wp-content/uploads/14_Boaler_FORUM_55_1_web.pdf

- Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher*, 33(8), 3–15.
- Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M. E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education*, 24(2), 417–436.
- Borko, H., Jacobs, J., Seago, N. M., & Mangram, C. (2014). Facilitating video-based professional development: planning and orchestrating productive discussions. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction* (pp. 259–281). Springer International Publishing. https://doi.org/10.1007/978-3-319-04993-9_16
- Borko, H., Koellner, K., Jacobs, J., & Seago, N. M. (2011). Using video representations of teaching in practice-based professional development programs. *ZDM*, 43(1), 175–187.
- Bowers, J., Passentino, G., & Connors, C. (2012). What is the complement to a procedural video? *Journal of Computers in Mathematics and Science Teaching*, 31(3), 213–248.
- Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32–41.
- Cai, J. (2005). US and Chinese teachers' constructing, knowing, and evaluating representations to teach mathematics. *Mathematical Thinking and Learning*, 7(2), 135–169.
- Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352–378.
- Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (1999). *Children's mathematics: Cognitively guided instruction*. Portsmouth, NH: Heinemann.
- Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C.-P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, 26(4), 499–531.
- Carraher, D. W., Martinez, M. V., & Schliemann, A. D. (2008). Early algebra and mathematical generalization. *ZDM*, 40(1), 3–22.

- Carraher, D. W., Schliemann, A. D., Brizuela, B. M., & Earnest, D. (2006). Arithmetic and algebra in early mathematics education. *Journal for Research in Mathematics Education*, 37(2), 87–115.
- Carrillo, J., Climent, N., Contreras, L. C., & Muñoz-Catalán, M. C. (2013). Determining specialised knowledge for mathematics teaching. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of the CERME* (Vol. 8, pp. 2985–2994). Antalya, Turkey: ERME.
- Castillo-Garsow, C. (2012). Continuous quantitative reasoning. In R. L. Mayes, R. Bonilla, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 39–53). Laramie, Wyoming: University of Wyoming Press.
- Chazan, D., & Herbst, P. (2012). Animations of classroom interaction: Expanding the boundaries of video records of practice. *Teachers College Record*, 114(3), 1–34.
- Civil, M. (1992). Prospective elementary teachers' thinking about teaching mathematics. Presented at the Annual Meeting of the American Educational Research Association, San Francisco. Retrieved from <http://eric.ed.gov/?id=ED353135>
- Cobb, P., Confrey, J., Disessa, A. A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32(1), 9–13.
- Cobb, P., McClain, K., & Gravemeijer, K. (2003). Learning about statistical covariation. *Cognition and Instruction*, 21(1), 1–78. https://doi.org/10.1207/S1532690XCI2101_1
- Corbin, J. M., & Strauss, A. (1990). Grounded theory research: Procedures, canons, and evaluative criteria. *Qualitative Sociology*, 13(1), 3–21.
- Doerr, H. M., & Thompson, T. (2004). Understanding teacher educators and their pre-service teachers through multi-media case studies of practice. *Journal of Mathematics Teacher Education*, 7(3), 175–201.
- Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist*, 23(2), 167.
- Dubinsky, E. (2002). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95–126). Springer Netherlands. https://doi.org/10.1007/0-306-47203-1_7

- Dweck, C. S. (2006). *Mindset: The new psychology of success*. New York, NY: Random House.
- Dweck, C. S. (2010). Mind-sets and equitable education. *Principal Leadership*, 10(5), 26–29.
- Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, 25(4), 439–478.
- Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 215–238). Heidelberg: Springer.
- Ellis, A. B., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: the case of the Jactus. In *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 93–112). Laramie, Wyoming: University of Wyoming Press. Retrieved from http://www.uwyo.edu/wisdome/_files/documents/ellis_et_al.pdf
- Empson, S. B., & Jacobs, V. R. (2008). Learning to listen to children's mathematics. In D. Tirosh & T. Wood (Eds.), *Tools and processes in mathematics teacher education* (pp. 257–281). Rotterdam, NL: Sense Publishers.
- Erickson, F. (2007). Ways of seeing video: Toward a phenomenology of viewing minimally edited footage. In R. Goldman, R. Pea, B. Barron, & S. J. Derry (Eds.), *Video research in the learning sciences* (pp. 145–155). Mahwah, NJ: Erlbaum.
- Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, 24(2), 94–116.
- Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal*, 38(3), 653–689. <https://doi.org/10.3102/00028312038003653>
- Fulton, J. D. (2003). Guidelines for programs and departments in undergraduate mathematical sciences. *MAA Guidelines Task Force*. Retrieved from <http://www.maa.org/sites/default/files/pdf/guidelines/Dept-Guidelines-Feb2003.pdf>

- Ginsburg, H. (1997). *Entering the child's mind*. Cambridge, UK: Cambridge University Press.
- Glaser, B. G., & Strauss, A. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Hawthorne, NY: Aldine Publishing Company.
- Goodwin, C. (1994). Professional vision. *American Anthropologist*, 96(3), 606–633.
- Guberman, R., & Leikin, R. (2013). Interesting and difficult mathematical problems: changing teachers' views by employing multiple-solution tasks. *Journal of Mathematics Teacher Education*, 16(1), 33–56.
<https://doi.org/10.1007/s10857-012-9210-7>
- Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, 46(2), 196–243.
- Hiebert, J., & Wearne, D. (1993). Instructional tasks, classroom discourse, and students' learning in second-grade arithmetic. *American Educational Research Journal*, 30(2), 393–425. <https://doi.org/10.3102/00028312030002393>
- Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education*, 35(5), 330–351.
- Hill, H. C., Ball, D. L., Blunk, M., Goffney, I. M., & Rowan, B. (2007). Validating the ecological assumption: The relationship of measure scores to classroom teaching and student learning. *Measurement*, 5(2–3), 107–118.
- Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371–406.
- Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 111–156). Reston, VA: National Council of Teachers of Mathematics.
- Hohensee, C. (2017). Preparing elementary prospective teachers to teach early algebra. *Journal of Mathematics Teacher Education*, 20(3), 231–257.
- Jacobs, V. R., Lamb, L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, 41(2), 169–202.

- Jaworski, B. (1990). Video as a tool for teachers' professional development. *British Journal of In-Service Education*, 16(1), 60–65.
<https://doi.org/10.1080/0305763900160112>
- Johnson, H. L. (2012). Reasoning about quantities involved in rate of change as varying simultaneously and independently. In R. L. Mayes, R. Bonilla, L. L. Hatfield, & S. Belbase (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 39–53). Laramie, Wyoming: University of Wyoming Press.
- Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education*, 7(3), 203–235.
<https://doi.org/10.1023/B:JMTE.0000033084.26326.19>
- Kersting, N. B., Givvin, K. B., Sotelo, F. L., & Stigler, J. W. (2010). Teachers' analyses of classroom video predict student learning of mathematics: Further explorations of a novel measure of teacher knowledge. *Journal of Teacher Education*, 61(1–2), 172–181.
- Kleinknecht, M., & Schneider, J. (2013). What do teachers think and feel when analyzing videos of themselves and other teachers teaching? *Teaching and Teacher Education*, 33, 13–23. <https://doi.org/10.1016/j.tate.2013.02.002>
- Knuth, E. J. (2000a). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education*, 31(4), 500–508.
- Knuth, E. J. (2000b). Understanding connections between equations and graphs. *The Mathematics Teacher*, 93(1), 48–53.
- Koc, Y., Peker, D., & Osmanoglu, A. (2009). Supporting teacher professional development through online video case study discussions: An assemblage of preservice and inservice teachers and the case teacher. *Teaching and Teacher Education*, 25(8), 1158–1168. <https://doi.org/10.1016/j.tate.2009.02.020>
- Lobato, J. (2014). *Re-imagining video-based online learning*. National Science Foundation Discovery Research K–12; Award #1416789.
- Lobato, J., Ellis, A. B., & Munoz, R. (2003). How “focusing phenomena” in the instructional environment support individual students' generalizations. *Mathematical Thinking and Learning*, 5(1), 1–36.
- Lobato, J., Hohensee, C., & Rhodehamel, B. (2013). Students' mathematical noticing. *Journal of Research in Mathematics Education*, 44(5), 809–850.

- Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning*, *14*(2), 85–119.
- Lobato, J., Rhodehamel, B., & Hohensee, C. (2012). “Noticing” as an alternative transfer of learning process. *Journal of the Learning Sciences*, *21*(3), 433–482. <https://doi.org/10.1080/10508406.2012.682189>
- Lobato, J., & Siebert, D. (2002). Quantitative reasoning in a reconceived view of transfer. *The Journal of Mathematical Behavior*, *21*(1), 87–116.
- Lobato, J., & Walters, C. D. (in press). A taxonomy of approaches to learning trajectories and progressions. In J. Cai (Ed.), *First compendium for research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers’ understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Manouchehri, A. (2009). Motivating growth of mathematics knowledge for teaching: A case for secondary mathematics teacher education. *The Mathematics Educator*, *18*(2), 3–10.
- Markovits, Z., & Even, R. (1999). The decimal point situation: a close look at the use of mathematics-classroom-situations in teacher education. *Teaching and Teacher Education*, *15*(6), 653–665. [https://doi.org/10.1016/S0742-051X\(99\)00020-7](https://doi.org/10.1016/S0742-051X(99)00020-7)
- Masats, D., & Dooly, M. (2011). Rethinking the use of video in teacher education: A holistic approach. *Teaching and Teacher Education*, *27*(7), 1151–1162. <https://doi.org/10.1016/j.tate.2011.04.004>
- Mayes, R. L., Peterson, F. I., & Bonilla, R. (2012). Quantitative reasoning in context. In R. L. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 7–38). Laramie, Wyoming: University of Wyoming Press.
- McCrorry, R., Floden, R., Ferrini-Mundy, J., Reckase, M., & Senk, S. (2012). Knowledge of algebra for teaching: A framework of knowing and doing. *Journal for Research in Mathematics Education*, *43*(5), 584–615.

- McDiarmid, G. W. (1990). Challenging prospective teachers' beliefs during early field experience: A quixotic undertaking? *Journal of Teacher Education*, 41(3), 12–20.
- McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 575–596). New York, NY: Macmillan.
- Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis* (2nd ed.). Thousand Oaks, CA: Sage Publication, Inc.
- Miller, K., & Zhou, X. (2007). Learning from classroom video: What makes it compelling and what makes it hard. In R. Goldman-Segal & R. Pea (Eds.), *Video research in the learning sciences* (pp. 321–334). Hillsdale, NJ: Lawrence Erlbaum Associates.
- Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. In R. L. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 75–92). Laramie, Wyoming: University of Wyoming Press.
- Moore, K. C. (2013). Making sense by measuring arcs: A teaching experiment in angle measure. *Educational Studies in Mathematics*, 83(2), 225–245.
- Moore, K. C., Carlson, M. P., & Oehrtman, M. (2009). The role of quantitative reasoning in solving applied precalculus problems. In *Proceedings of the Twelfth Conference on Research in Undergraduate Mathematics Education*. Raleigh, NC. Retrieved from http://rume.org/crume2009/Moore1_LONG.pdf
- Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *The Journal of Mathematical Behavior*, 32(3), 461–473.
- Moore-Russo, D., & Viglietti, J. M. (2010). Teachers' reactions to animations as representations of geometry instruction. *ZDM*, 43(1), 161–173. <https://doi.org/10.1007/s11858-010-0293-2>
- Morris, A. K. (2006). Assessing pre-service teachers' skills for analyzing teaching. *Journal of Mathematics Teacher Education*, 9(5), 471–505.
- Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education*, 40(5), 491–529.

- National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- National Governors Association Center for Best Practices, Council of Chief State School Officers. (2012). *Common Core State Standards (Mathematics)*. Washington, D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf
- Nipper, K., & Sztajn, P. (2008). Expanding the instructional triangle: conceptualizing mathematics teacher development. *Journal of Mathematics Teacher Education*, 11(4), 333–341.
- Noble, T., Nemirovsky, R., Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education*, 32(1), 85–108.
- Norton, A., & Kastberg, S. (2012). Learning to pose cognitively demanding tasks through letter writing. *Journal of Mathematics Teacher Education*, 15(2), 109–130. <https://doi.org/10.1007/s10857-011-9193-9>
- Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27–42). Washington, DC: Mathematical Association of America.
- Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 257–315). Reston, VA: National Council of Teachers of Mathematics.
- Philipp, R. A. (2008). Motivating prospective elementary school teachers to learn mathematics by focusing upon children's mathematical thinking. *Issues in Teacher Education*, 17(2), 7–26.
- Philipp, R. A., Ambrose, R., Lamb, L. L., Sowder, J., Schappelle, B. P., Sowder, L., ... Chauvot, J. (2007). Effects of early field experiences on the mathematical content knowledge and beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education*, 38(5), 438–476.
- Philipp, R. A., & Schappelle, B. P. (1999). Algebra as generalized arithmetic: Starting with the known for a change. *The Mathematics Teacher*, 92(4), 310.

- Piaget, J. (1947). *The psychology of intelligence*. New York, NY: Routledge.
- Piaget, J., & Inhelder, B. (1967). *The child's conception of space*. New York, NY: Norton.
- Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: An overview on achievements and challenges. *ZDM*, 47(6), 877–891.
- Quantitative Literacy Design Team. (2001). The case for quantitative literacy. In L. A. Steen (Ed.), *Mathematics and democracy: The case for quantitative literacy* (pp. 1–22). Princeton, NJ: The National Council on Education and the Disciplines.
- Rasmussen, C. L. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *The Journal of Mathematical Behavior*, 20(1), 55–87.
- Rheinberg, F. (1983). Achievement evaluation: A fundamental difference and its motivational consequences. *Studies in Educational Evaluation*, 9(2), 185–194.
- Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula (Ed.), *Handbook of research on teacher education* (2nd ed., pp. 102–119). New York, NY: Macmillan.
- Santagata, R. (2009). Designing video-based professional development for mathematics teachers in low-performing schools. *Journal of Teacher Education*, 60(1), 38–51.
- Santagata, R., & Guarino, J. (2010). Using video to teach future teachers to learn from teaching. *ZDM*, 43(1), 133–145. <https://doi.org/10.1007/s11858-010-0292-3>
- Santagata, R., Zannoni, C., & Stigler, J. W. (2007). The role of lesson analysis in pre-service teacher education: an empirical investigation of teacher learning from a virtual video-based field experience. *Journal of Mathematics Teacher Education*, 10(2), 123–140. <https://doi.org/10.1007/s10857-007-9029-9>
- Schoenfeld, A. H. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM*, 43(4), 457–469.
- Schwartz, J. L. (1976). *Semantic aspects of quantity*. Unpublished manuscript, MIT, Cambridge.
- Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in*

the middle grades (pp. 41–52). Reston, VA: National Council of Teachers of Mathematics.

- Seago, N. M., Jacobs, J. K., Heck, D. J., Nelson, C. L., & Malzahn, K. A. (2014). Impacting teachers' understanding of geometric similarity: results from field testing of the Learning and Teaching Geometry professional development materials. *Professional Development in Education, 40*(4), 627–653.
- Sherin, M. G. (2004). New perspectives on the role of video in teacher education. In J. Brophy (Ed.), *Using video in teacher education* (Vol. 10, pp. 1–27). Emerald Group Publishing Limited.
- Sherin, M. G., & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education, 13*(3), 475–491.
- Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education, 60*(1), 20–37.
- Shilling-Traina, L. N., & Stylianides, G. J. (2012). Impacting prospective teachers' beliefs about mathematics. *ZDM, 45*(3), 393–407.
<https://doi.org/10.1007/s11858-012-0461-7>
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.
- Shulman, L. S. (1992). Toward a pedagogy of cases. In J. H. Shulman (Ed.), *Case methods in teacher education* (pp. 1–29). New York, NY: Teachers College Press.
- Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*(6), 499–511.
- Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning, 8*(4), 359–371.
- Simon, M. A., & Blume, G. W. (1994). Building and understanding multiplicative relationships: A study of prospective elementary teachers. *Journal for Research in Mathematics Education, 25*(5), 472–494.
- Simon, M. A., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education, 35*(5), 305–329.
<https://doi.org/10.2307/30034818>

- Smith, J., & Thompson, P. W. (2007). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). Erlbaum.
- Somayajulu, R. B. (2012). *Building pre-service teacher's mathematical knowledge for teaching of high school geometry* (Doctoral dissertation). The Ohio State University, Columbus, OH.
- Sowder, J. (2007). The mathematical education and development of teachers. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 157–223). Reston, VA: National Council of Teachers of Mathematics.
- Sowder, J., Armstrong, B., Lamon, S., Simon, M., Sowder, L., & Thompson, A. (1998). Educating teachers to teach multiplicative structures in the middle grades. *Journal of Mathematics Teacher Education*, 1(2), 127–155.
- Speer, N. M., King, K. D., & Howell, H. (2014). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education*, 18(2), 105–122. <https://doi.org/10.1007/s10857-014-9277-4>
- Star, J. R., & Strickland, S. K. (2008). Learning to observe: using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education*, 11(2), 107–125.
- Steffe, L. P. (1991). The learning paradox: A plausible counterexample. In *Epistemological foundations of mathematical experience* (pp. 26–44). Springer. Retrieved from http://link.springer.com/chapter/10.1007/978-1-4612-3178-3_3
- Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York, NY: Springer.
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Mahwah, NJ: Lawrence Erlbaum Associates.
- Stipek, D. J., Givvin, K. B., Salmon, J. M., & MacGyvers, V. L. (2001). Teachers' beliefs and practices related to mathematics instruction. *Teaching and Teacher Education*, 17(2), 213–226.
- Strauss, A. (1987). *Qualitative analysis for social scientists*. Cambridge, UK: Cambridge University Press.

- Strauss, A., & Corbin, J. (1990). *Basics of qualitative research* (Vol. 15). Newbury Park, CA: Sage.
- Strauss, A., & Corbin, J. M. (1994). Grounded theory methodology: An overview. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research*. Thousand Oaks, CA: Sage Publication.
- Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education*, 40(3), 314–352.
- Stylianides, G. J., & Stylianides, A. J. (2010). Mathematics for teaching: A form of applied mathematics. *Teaching and Teacher Education*, 26(2), 161–172.
- Switzer, J. M., Teuscher, D., & Siebert, D. (2015). See it, try it, and reflect on it (STaR): Using video to scaffold and support preservice teachers in the reflective process of developing participation questioning discourse. In E. Ortlieb, L. E. Shanahan, & M. B. McVee (Eds.), *Video research in disciplinary literacies (Literacy research, practice and evaluation, Volume 6)* (pp. 3–20). Emerald Group Publishing Limited.
- Sztajn, P., Borko, H., & Smith, T. (in press). Research on mathematics professional development. In J. Cai (Ed.), *Compendium for research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.
- Sztajn, P., Marrongelle, K. A., Smith, P., & Melton, B. L. (2012). *Supporting implementation of the Common Core State Standards for Mathematics: recommendations for professional development*. Raleigh, NC: Friday Institute for Educational Innovation at North Carolina State University.
- Tall, D. (2004). Thinking through three worlds of mathematics. In *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 281–288).
- Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education*, 19(5), 433–456.
- The National Numeracy Network (NNN). (2016). What is numeracy/QL/QR? Retrieved June 1, 2017, from <http://serc.carleton.edu/nnn/resources/index.html>
- Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, 15(2), 105–127.

- Thompson, A. G. (1992). *Teachers' beliefs and conceptions: A synthesis of the research*. Macmillan Publishing Co, Inc.
- Thompson, A. G., Philipp, R. A., Thompson, P. W., & Boyd, B. A. (1994). Computational and conceptual orientations in teaching mathematics. In D. Aichele & Coxford (Eds.), *Professional development of teachers of mathematics* (pp. 79–92). Reston, VA: National Council of Teachers of Mathematics.
- Thompson, A. G., & Thompson, P. W. (1995). A cognitive perspective on the mathematical preparation of teachers: The case of algebra. In C. B. Lacampagne, W. Blair, & J. Kaput (Eds.), *The algebra initiative colloquium* (Vol. 1, pp. 95–116). Washington, DC: US Department of Education.
- Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, 27(1), 2–24.
- Thompson, P. W. (1988). Quantitative concepts as a foundation for algebra. In M. Behr & M. Wheeler (Eds.), *Proceedings of the annual conference of the International Group for the Psychology of Mathematics Education* (pp. 163–170). DeKalb, IL: Northern Illinois University.
- Thompson, P. W. (1990). *A theoretical model of quantity-based reasoning in arithmetic and algebra*. Unpublished manuscript, Center for Research in Mathematics & Science Education. Retrieved from <http://pat-thompson.net/PDFversions/1990TheoryQuant.pdf>
- Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, 25(3), 165–208.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In J. Confrey & G. Harel (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). Albany, NY: SUNY Press.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (Vol. 1, pp. 33–57). Laramie, Wyoming: University of Wyoming Press.
- Thompson, P. W., Carlson, M. P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education*, 10(4–6), 415–432. <https://doi.org/10.1007/s10857-007-9054-8>

- Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (Vol. 73, pp. 43–52).
- Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education*, 25(3), 279–303.
- Tirosh, D., & Wood, T. (Eds.). (2008). *International handbook of mathematics teacher education: Vol. 2. Tools and processes in mathematics teacher education*. Amsterdam, The Netherlands: Sense Publishers.
- Tyminski, A. M., Land, T. J., Drake, C., Zambak, V. S., & Simpson, A. (2014). Preservice elementary mathematics teachers' emerging ability to write problems to build on children's mathematics. In J.-J. Lo, K. R. Leatham, & L. R. V. Zoest (Eds.), *Research trends in mathematics teacher education* (pp. 193–218). Springer International Publishing. https://doi.org/10.1007/978-3-319-02562-9_11
- Ulrich, C. (2012). The addition and subtraction of signed quantities. In R. L. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context* (Vol. 2, pp. 75–92). Laramie, Wyoming: University of Wyoming Press.
- Usiskin, Z. (2001a). Quantitative literacy for the next generation. In L. A. Steen (Ed.), *Mathematics and democracy: The case for quantitative literacy* (pp. 79–86). Princeton, NJ: The National Council on Education and the Disciplines.
- Usiskin, Z. (2001b). Teachers' mathematics: A collection of content deserving to be a field. *The Mathematics Educator*, 6(1), 86–98.
- van Bommel, J. (2012). *Improving teaching, improving learning, improving as a teacher: Mathematical knowledge for teaching as an object of learning* (Doctoral dissertation). Karlstad University, Faculty of Technology and Science, Mathematics, Karlstad, Sweden. Retrieved from <http://www.diva-portal.org/smash/record.jsf?pid=diva2:531248>
- van Es, E. A., & Sherin, M. G. (2010). The influence of video clubs on teachers' thinking and practice. *Journal of Mathematics Teacher Education*, 13(2), 155–176.
- Vermunt, J. D., & Verloop, N. (1999). Congruence and friction between learning and teaching. *Learning and Instruction*, 9(3), 257–280.

- Vermunt, J. D., & Vermetten, Y. J. (2004). Patterns in student learning: Relationships between learning strategies, conceptions of learning, and learning orientations. *Educational Psychology Review*, *16*(4), 359–384.
- von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.
- Walkoe, J. (2015). Exploring teacher noticing of student algebraic thinking in a video club. *Journal of Mathematics Teacher Education*, *18*(6), 523–550. <https://doi.org/10.1007/s10857-014-9289-0>
- Watson, A., & Mason, J. (2007). Taken-as-shared: a review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education*, *10*(4–6), 205–215. <https://doi.org/10.1007/s10857-007-9059-3>
- Wilson, P. H., Lee, H. S., & Hollebrands, K. F. (2011). Understanding prospective mathematics teachers' processes for making sense of students' work with technology. *Journal for Research in Mathematics Education*, *42*(1), 39–64.
- Wilson, P. H., Sztajn, P., Edgington, C., & Confrey, J. (2014). Teachers' use of their mathematical knowledge for teaching in learning a mathematics learning trajectory. *Journal of Mathematics Teacher Education*, *17*(2), 149–175.
- Yeager, D. S., & Dweck, C. S. (2012). Mindsets that promote resilience: When students believe that personal characteristics can be developed. *Educational Psychologist*, *47*(4), 302–314.
- Zazkis, R., & Hazzan, O. (1999). Interviewing in mathematics education research: Choosing the questions. *The Journal of Mathematical Behavior*, *17*(4), 429–439.
- Zhang, M., Lundeberg, M., Koehler, M. J., & Eberhardt, J. (2011). Understanding affordances and challenges of three types of video for teacher professional development. *Teaching and Teacher Education*, *27*(2), 454–462. <https://doi.org/10.1016/j.tate.2010.09.015>