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Analysis of Length Distribution of Drainage Basin Perimeter

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To establish a theoretical base for the study of the length distribution of basin perimeters, the paper introduces a descriptive model of the topology of interlocking channel and ridge networks. Assuming topological randomness within and between both, the expected number of links of basin perimeters is derived; for large basin magnitudes n , it approximates a square root function in n . Observed link numbers of perimeters deviate significantly, showing a 0.69 regression exponent for their growth rate relative to the basin magnitude rather than the expected value of 0.5. The spatial constraint of possible perimeter/(area)^{1/2} proportions as defined by the circle is translated into a corresponding topological constraint but fails to provide a sufficient explanation. The paper then explores the possibility that the relatively large length of the perimeter reflects the basin elongation which, following Hack, might be linked to the length of the mainstream. Although basin perimeter, elongation, and mainstream length are highly correlated and the elongation axis is oriented to the outlet in two-thirds of the sample basins, the data indicate that the mainstream link number does not account for the basin elongation, nor does it account for the number of links of the basin perimeter.

1. INTRODUCTION

It has been known for many years that the length L of the mainstream of a natural channel network tends to grow relatively faster than the square root of the area A of the corresponding basin. A satisfactory explanation was finally provided by Shreve's model of random network topology [Werner and Smart, 1973, p. 291; Shreve, 1974, pp. 1174, 1176]. The observed relation $L \sim A^{0.6}$ is to be expected if channel networks conform with the assumptions of the random model [Smart, 1972, pp. 318, 329]. It seems plausible to assume that, in maturely eroded terrain free of environmental control, this model might also produce good predictions for another, related morphometric measure, the basin perimeter. It is the purpose of this paper to develop this hypothesis into an operational statement and test it against empirical data.

There are at least two reasons why the method of analysis of the mainstream length cannot readily be applied to the analysis of the basin perimeter:

1. Whereas the mainstream length is, by definition, the maximum value of a set of stream lengths, the perimeter is the length of a single line.
2. The mainstream and the perimeter are paths of different networks, the former belonging to the channel network under consideration, and the latter belonging to the network of ridges that delineate the basin and subbasins of the channel network and its subnetworks.

It is therefore necessary to introduce a number of concepts and postulates that will permit the construction of a theoretical framework encompassing both channel and ridge networks. On the basis of such a framework it will then be possible to establish a theoretical relationship between the perimeter of a basin and its mainstream and to test it against measurements from natural channel networks and their basins.

The paper is organized into eight sections. The second section develops a graph-theoretical model of interlocking ridge and channel networks, which includes an operational definition of the perimeter as a particular ridge path corresponding to a particular channel (sub)network. The third and

fourth sections give mathematical derivations for the expected number of links of network paths connecting exterior links in topologically random networks. The fifth section discusses the data and the methods of their measurement on topographic maps. The sixth section compares observed and predicted distributions of the number of links of basin perimeters and other ridge paths. The discrepancies are traced back to the postulate of independence between ridge and channel network topologies. The seventh section converts the spatial constraint of minimum perimeter length for a given area into a topological constraint to account for the observed nonrandomness of the lengths of perimeter paths. A second approach investigates a possible linkage between the length distributions of the mainstream and the perimeter via the basin elongation. The last section provides a summary of the theoretical and empirical results of the paper.

2. INTERLOCKING RIDGE AND CHANNEL NETWORKS

There is a third obvious reason why the methodological approach, which led to a formal explanation of the mainstream length distribution, is insufficient for the analysis of the basin's perimeter: whereas the mainstream can be identified as a particular path of the channel network without any knowledge about the corresponding network of the ridges, the equivalent statement does not hold for the basin perimeter. Whether a particular ridge path is also a basin perimeter depends on whether or not this path delineates the basin of a channel network or subnetwork. In fact, being a path of the ridge network and being simultaneously associated with a particular channel network by delineating its basin makes the perimeter the one morphological feature that is responsible for, and describes the intimate interdependence between, the patterns of channels and ridges of maturely eroded landscapes. Thus basin perimeters form a subset of the set of all ridge paths, and they account for the number, size, and distribution of the basins and subbasins, which in turn are a direct reflection of the spatial organization of channel networks and their subnetworks.

To make this largely impressionistic description operational, we define C and R to be 'interlocking channel and ridge networks' if R consists of all, and only, the ridges that delineate the basin and subbasins drained by the channel network C and its subnetworks. For the purpose of theory

Fig. 1. Example of two interlocking ridge and channel networks as defined for the purpose of graph-theoretical analysis.

construction, we will assume that both C and R are trivalent, and that every bifurcation of C creates one and only one ridge link, which functions as the drainage divide between the two upstream channels of the bifurcation (Figure 1). If C is of magnitude n , then there are a total of $n - 1$ channel bifurcations (nodes, junctions), each of which gives rise to an exterior ridge link. In addition, the ridge path of R delineating the drainage basin of C has two exterior links, which are located on either side of C 's outlet. The one on the left side will be defined as the root of R . Thus, just like C , R has a total of n exterior links and one root.

We now traverse C in the familiar fashion by starting at the outlet, then turning left at every bifurcation and reversing direction at the end of every exterior link. We denote the root and the exterior links of C by $c_0, c_1, \dots, c_n$ in the order they are encountered in traversing the network. For any pair of consecutive exterior links c_k, c_{k+1} , there exists exactly one network bifurcation Q_k , namely, the network node that the three paths interconnecting c_k, c_{k+1} , and c_0 share in common. This mapping holds in the reverse direction as well. For every node Q there exist exactly two consecutive exterior links in C such that the paths connecting them and c_0 intersect in Q . As postulated earlier, the bifurcation in Q_k produces one and only one exterior ridge link r_k , which forms the drainage divide between the two upstream links of Q_k . They in turn are located between c_k and c_{k+1} in the sense that they are links of the path connecting c_k and c_{k+1} and are being traversed after the former and before the latter. We will therefore say that r_k is located 'between' c_k and c_{k+1} . This definition of 'betweenness' of exterior ridge links relative to exterior channel links is also plausible as it corresponds to the customary meaning of the word. Thus the roots and the exterior links of C and R form a cyclical and alternating sequence $c_0, r_0, c_1, r_1, \dots, c_n, r_n, c_0$, which describes, in graph-theoretical terms, the characteristic interlocking pattern of channels and ridges and their respective networks in maturely eroded terrain.

Several properties follow immediately from the above construction.

Interlocking channel and ridge networks have the same magnitude and therefore the same number of links; for each channel link there exist exactly two exterior ridge links (or one exterior link and the root); they are located on opposite

sides of it and define the ridge path that delineates the drainage basin of the channel subnetwork defined by the channel link. In particular, if the root and the exterior links of R are labeled $0, 1, \dots, n$ in a clockwise sense, if x is a channel link and if i, j are the two exterior ridge links associated with x , then the magnitude of the channel subnetwork defined by x is $j - i$ and the ridge path defined by i and j is the perimeter of the corresponding basin. It is important to note that our definition of interlocking networks does not make any reference to their respective topologies beyond their magnitudes; indeed, interlocking networks can assume any two topologies, and knowing one of them does not provide any information about the other. This permits us to extend the notion of topologically random channel networks to interlocking pairs of networks: we postulate that the $[N(n)]^2$ different topologies of interlocking ridge and channel networks of magnitude n are equally likely.

Quite aside from the practical problems associated with its application, the model of interlocking ridge and channel networks as presented here has some serious shortcomings of a principal nature. Whereas it establishes a one-to-one correspondence between channel (sub)networks and particular ridge paths as their respective basin perimeters, the exterior links of these perimeter paths refer to different channel bifurcations, thereby violating the familiar concept of a drainage basin. When compared with natural ridge and channel networks, the model's definition of basin perimeter does not affect the correct value of the basin magnitude. However, by assuming one and only one exterior ridge link for each channel bifurcation, the number of links of the perimeter is frequently misrepresented to a degree that the model will generate sizable errors for very small basins, the relative error term approaching zero for increasing basin magnitude. These built-in error terms for small magnitudes were judged to be acceptable in view of the model's graph-theoretical simplicity and mathematical appeal.

Having established a theoretical base for the description of the peculiar interaction of ridge and channel networks, we will now introduce additional assumptions that will enable us to express this interaction in mathematical terms. Specifically, we will assume that, under environmentally homogeneous conditions, (1) the densities of channel and ridge links are approximately equal and fairly uniform throughout the area under consideration; (2) the geometrical lengths of channel and ridge paths are approximately proportional to their respective numbers of links; and (3) interlocking channel and ridge networks form trivalent planted plane trees of equal magnitude and alternating exterior links that are topologically random and independent of each other.

The first two postulates translate the metric measures of area and length into the numbers of links of networks and their paths, and vice versa. Elsewhere in the literature they are further reduced to postulates about link length and associated link area distributions [Shreve, 1974, p. 1169; Smart, 1968, p. 1005]. But since our data base does not contain this type of information, the present, more restrictive formulations were adopted. The third postulate briefly restates the assumptions of the model of interlocking ridge and channel networks and adds their topological randomness and independence. It permits the deduction of the expected link number of the perimeter as a function of the magnitude of the corresponding channel network; with the help of postulates (1) and (2), the expected topological relationship

Fig. 2. Scatter diagram of the magnitudes n_r , n_c of 50 pairs of interlocking ridge and channel networks.

can then be expressed in terms of the metric measures of basin area and perimeter.

How do these postulates compare with reality? With regard to natural channel links, the applicability of the first postulate has repeatedly been confirmed by data published in the literature. For the study area of this paper, Figure 2 demonstrates the nearly equal densities of exterior channel and ridge links and, therefore, of channel and ridge links in general, provided that the junctions of the respective networks are indeed trivalent. The second postulate has been confirmed for channel network paths under a variety of different environmental conditions [e.g., Smart and Werner, 1976, p. 224]. The data of Figure 3 support the applicability of the postulate to ridge paths, at least inasmuch as our study area is concerned. Regarding the third postulate, so far it has only been shown for natural channel networks that topological randomness constitutes a good approximation [Smart, 1972, p. 320]. Empirical studies of ridge networks are exceedingly scarce and do not permit a similar confirmation. There are even less data available that address the relationship between the topological configurations of interlocking ridge and channel networks [Werner, 1972a, 1973]. The lack of adequate data might at least in part be due to practical difficulties in identifying ridge networks on topographic maps. Ridges frequently lack the well-defined, morphologically conspicuous expression of channels. This can create considerable difficulty (if not impossibility) in determining the location of ridge links and, as a consequence, the location and degree of ridge junctions. Thus the trivalency of ridge networks as required by the third postulate remains at this point an assumption with which we will have to deal later.

According to postulate (3), ridges, like channels, form topologically random networks. Since basin perimeters are ridge network paths connecting exterior ridge links, it will be the task of the next section to calculate the expected number of links of such paths.

3. EXPECTED LINK NUMBER OF PATHS CONNECTING EXTERIOR LINKS

Let G be a network of magnitude n ; we assign the number 0 to the network's root, and the numbers 1, 2, 3, ..., n , successively, to its exterior links in clockwise direction,

starting with the first exterior link left of the root. We call the j th exterior link the j th neighbor of the root, and, for any two exterior links i, j with $i < j \leq n$ and $j - i = k$, we call j the k th neighbor of i and k the neighborhood distance between i and j . We call the number of links of the path connecting i and j the topological length of the path [Werner and Smart, 1973, p. 276] or, alternatively, the topological distance between i and j . Let z be a link of this path: z decomposes the network G into two subnetworks G_1, G_2 with magnitudes u and $v = n + 1 - u$ serving as root link for both of them. Note that i is an exterior link of G_1 and j is an exterior link of G_2 . G_1 and G_2 may assume any of $N(u)$ and $N(n + 1 - u)$ topological configurations independently of each other, where

$$N(x) = \frac{\binom{2x-1}{x}}{2x-1} \quad (1)$$

is the familiar expression for the number of topologically different networks of magnitude x [Shreve, 1966, p. 29]. In combination a total of $N(u)N(n + 1 - u)$ different topological configurations is possible. Furthermore, for $u < k \leq n + 1 - k$ the exterior link i can be identical with any one of the u exterior links of G_1 . Similarly, the link j can be identical with any of the exterior links of G_2 if $v < k \leq n + 1 - k$. Thus, if the magnitude of either G_1 or G_2 is smaller than k , there are a total of

$$2 \sum_{u=1}^{k-1} N(u)N(n + 1 - u)u \quad (2)$$

topologically different configurations of G in which z is a link of the path connecting i and j . If, on the other hand, both u and v are equal to or larger than k , then the link i can assume exactly k different positions in G_1 (i.e., coincide with k of the u exterior links of G_1), and j will simultaneously assume k different positions in G_2 . For each case, there are again $N(u)N(n + 1 - u)$ different topologies so that the total number of different subnetworks G_1, G_2 with z as the

Fig. 3. Scatter diagram of the number of links p and the geometric length L (in units of 120 m) for 50 drainage basin perimeters.

common root becomes

$$k \cdot \sum_{u=k}^{n+1-k} N(u)N(n+1-u) \quad (3)$$

The upper limit of u , which is $n+1-k$, results from $u = n+1-v$ so that $u > n+1-k$ is equivalent to $v < k$, which case has already been incorporated in the first sum. Notice that the two sums (2) and (3) cover all possible magnitudes of G_1 and G_2 (u being the running index in both summations) as well as all different topological configurations of both G_1 and G_2 (expressed through the products $N(u)N(n+1-u)$) as well as all possible positions of G_1 and G_2 in G such that i and j are exterior links of G_1 and G_2 , respectively (accounted for by the multipliers u and k). Thus together the two sums represent the total number of links of the path between i and j in all $N(n)$ topologically different configurations of G . The expected number of links $E(l_{k,n})$ of the path connecting the exterior links i and j , where $j > i$ and $j-i = k$, is therefore

$$E(l_{k,n}) = \frac{1}{N(n)} \left[2 \sum_{u=1}^{k-1} N(u)N(n+1-u)u + k \sum_{n=k}^{n+1-k} N(u)N(n+1-u) \right] \quad (4)$$

where $N(n)$ is again given by equation (1).

Numerical values of the function show $E(l_{k,n})$ to approximate a square root function in n if n is large both absolute and relative to k . This result was later confirmed by J. W. Moon (private communication, 1978). For $n \rightarrow \infty$ the sequence defined by equation (4) starts with $E(l_{k,\infty}) = 4, 6, 7.5, 8.75, 9.84, \dots$, where $k = 1, 2, \dots$.

4. A PYTHAGOREAN THEOREM FOR BASIN PERIMETERS

Let C_k and R_k be interlocking channel and ridge networks of magnitude k . In nature, both are usually subnetworks of much larger interlocking networks C , R of magnitude n , where n for all practical purposes approaches infinity. Let p be the number of links of the particular ridge path of R that forms the perimeter of the basin drained by C_k , and let r_i, r_j be its exterior links. Since the exterior links $r_i, r_{i+1}, \dots, r_{j-1}, r_j$ alternate with the exterior links of C_k (postulate (3)), it follows that $j-i = k$ and that j is therefore the k th neighbor of i . The last section provided a derivation of the expected number of links of a path connecting an exterior link with its k th neighbor in a network of magnitude n , the expected number being a function of both k and n (equation (4)). For the present case of a ridge path that is the perimeter of the basin drained by the channel network C_k , this function describes the expected relationship between the channel network magnitude k and the topological length of its basin perimeter. For increasing values of n this relationship approximates a square root function in k : $E(p) \rightarrow b(k)^{1/2}$, where b is a constant, the numerical value of which we will now deduce from the assumption of topological randomness.

Let C_u and C_v be the two subnetworks of C_k , which together form C_k , i.e., which merge at the first node upstream from C_k 's outlet. If k_u, k_v and p_u, p_v are the magnitudes and the numbers of perimeter links of C_u and C_v , then $k = k_u + k_v$ and

$$E(p_x) \approx b(k_x)^{1/2} \quad x = u, v$$

so that

$$E(p)^2 \approx E(p_u)^2 + E(p_v)^2 \quad (5)$$

In words, if the assumptions of postulate (3) are applied to infinitely large interlocking networks, then the square of the expected topological length of a basin perimeter is approximately equal to the sum of the squares of the expected topological lengths of the two subbasin perimeters. The formal similarity to the famous Pythagorean theorem is obvious; applied to equation (5) it states that the expected link numbers of the perimeters of a basin and its two subbasins form the three sides of a (nearly) right triangle.

Since the two subnetworks C_u, C_v are themselves compositions of smaller subnetworks—provided, of course, that their respective magnitudes are larger than 1—the expected topological lengths of their perimeters can also be expressed in terms of those of the perimeters of their subbasins by repeated application of equation (5). This process of decomposition can be continued until the expected topological length of C_k 's perimeter is expressed by the expected lengths $E(p_i | i = 1, \dots, k)$ of the perimeters of all and only the k first-order basins of C_k :

$$E(p)^2 \approx \sum_{i=1}^k E(p_i)^2 \quad (6)$$

where p_i is the link number of the perimeter of the drainage basin of the i th exterior link of C_k . Again, this equation represents only an approximation inasmuch as it is based on equation (5), which itself is an approximation, especially for small network magnitudes. It has been shown elsewhere [Werner, 1972b, p. 944] that, for n approaching infinity, the probability $P(q, n)$ of a randomly selected first-order basin having a perimeter with k links is

$$P(q, n)_{n \rightarrow \infty} = \frac{q-1}{2^q} \quad (7)$$

Thus, combining equations (6) and (7), an estimate $\bar{E}(p)$ of the topological length p of the perimeter of the network C_k is

$$\bar{E}(p)^2 = k \cdot \sum_{q=1}^{\infty} \frac{q-1}{2^q} q^2 \quad (8)$$

We can rewrite the sum in equation (8) as

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{q-1}{2^q} q^2 &= 2 \cdot \sum_{q=1}^{\infty} \frac{q-1}{2^q} q^2 - \sum_{q=1}^{\infty} \frac{q-1}{2^q} q^2 \\ &= \sum_{q=0}^{\infty} \frac{q}{2^q} (q+1)^2 - \sum_{q=1}^{\infty} \frac{q-1}{2^q} q^2 = \sum_{k=1}^{\infty} \frac{3q^2 + q}{2^q} \end{aligned} \quad (9)$$

and, treating the sum on the right side of equation (9) in a similar fashion,

$$\begin{aligned} \sum_{q=1}^{\infty} \frac{3q^2 + q}{2^q} &= 2 \sum_{k=1}^{\infty} \frac{3q^2 + q}{2^q} - \sum_{q=1}^{\infty} \frac{3q^2 + q}{2^q} \\ &= \sum_{q=0}^{\infty} \frac{3(q+1)^2 + (q+1)}{2^q} - \sum_{q=1}^{\infty} \frac{3q^2 + q}{2^q} \\ &= 4 + \sum_{q=1}^{\infty} \frac{6q + 4}{2^q} \end{aligned} \quad (10)$$

By repeating the same operation of lowering q 's exponent in the sum once more, we finally get

$$\sum_{q=1}^{\infty} \frac{q-1}{2^q} q^2 = 4 + \left(10 + \sum_{q=1}^{\infty} \frac{6}{2^q} \right) = 14 + 6 \cdot \sum_{q=1}^{\infty} \frac{1}{2^q} \quad (11)$$

The sum on the right hand of (11) is a geometric series, which converges to 1 for q approaching infinity, and the substitution of equation (11) into (8) results in

$$\bar{E}(p) = 2(5)^{1/2} \cdot (k)^{1/2} \quad (12)$$

As indicated before, this estimate is based on large values of k , with the error term becoming increasingly significant for the smallest network magnitudes.

To summarize, the constant $b = 2(5)^{1/2} = 4.47$ is a coefficient expressing the relationship between the estimated topological length of a basin perimeter and the square root of the basin magnitude. The total number of channel links in the basin is, of course, $2k - 1$. Thus we can rephrase the above result as follows. Under the assumptions of postulate (3), the number of enclosing links and the number of those being enclosed are, approximately and on the average, related by a square root function with an estimated constant coefficient $t \approx 10^{1/2} = 3.16$. In analogy to its metric counterpart, we will call $p/(2k - 1)^{1/2}$ the 'topological shape measure' of C_k , the expected value of which is approximately t .

5. THE DATA

The data base of the paper is quite modest, and its analysis cannot claim to be more than a pilot study. Fortunately, the main empirical findings are quite systematic and conspicuous, reducing the drawbacks normally associated with small data samples. There is one major limitation, however: almost all of the data refer to basins with magnitudes of up to a few hundred. As *Shreve* [1974] has shown in a careful study, at least some network parameters show appreciable differences in their behavior when network magnitudes become rather large. Thus an extrapolation of our empirical results to very large basins is not readily justified.

Data for both channel and ridge networks were sampled from the 1:24,000 USGS map, Varney, Kentucky, quadrangle. This area is part of a region which has been studied by *Krumbein and Shreve* [1970, pp. 3, 4] and was judged 'to be a good example of a mature landscape developed in the absence of geologic controls.' Their study, as well as subsequent studies, has shown that under these conditions channel networks tend to be topologically random [*Smart*, 1972, p. 320], which is one of the assumptions of our third postulate.

Whereas the methodology of sampling channel network data has been developed and applied by a number of researchers, the methodological issues posed by the measurement of ridges are largely unexplored. Therefore, before reporting particular numerical results, we will first review the key problems encountered in the sampling of ridge data.

There is no evidence in the literature that the identification of drainage divides of the surface runoff in maturely eroded terrain causes any difficulty. Nor is there any indication that ridges, which are always part of the drainage divide system, pose any problem to the investigator. To quote *Krumbein and Shreve* [1970, p. 10]:

In this study, drainage basins were outlined by starting at the outlet and drawing lines perpendicular to the contours upward to the tops of adjacent ridges, then following the ridge lines around the drainage basin.

Ridges are defined here as drainage divides with convex shapes, i.e., divides that appear on topographic maps as sequences of cusps in consecutive contour lines, and were identified accordingly, following the procedure for the identification of channel networks on topographic maps described by *Krumbein and Shreve* [1970, p. 12]. The same procedure was used for the identification and measurement of channels. Since under humid conditions closed drainage systems cannot develop, the lines of drainage divides have, by necessity, the graph-theoretical structure of trees, and the same statement holds for the system of ridges inasmuch as they are subgraphs of the drainage divide trees. Ridge trees are, of course, plane and are planted if one of their exterior links is defined as the root. As to their magnitudes and nodal degrees, considerable difficulties were encountered in the actual map work, despite the above quotation (which, incidentally, refers to the same general area from which the data for this study were sampled). The identification of the precise location of ridges becomes, at times, impossible, especially when erosional processes have either not sufficiently advanced so as to reduce remnants of plateau-like forms to a pattern of well-defined slopes and crests or, for reasons of climate and geology, will never produce a sharply defined dissection. In either case, even ridge lines that are well developed elsewhere, will frequently lose their morphological prominence and flatten out into a broad, undifferentiated surface before interconnecting with adjacent ridges. Thus there is often considerable uncertainty—if not outright impossibility—in the determination of the precise location and degree of ridge nodes and the location of the ridges in their immediate vicinity. The morphometric part of this dilemma was resolved in this paper by projecting the location of an unidentifiable ridge node on the basis of the spatial trends of the ridges approaching it. In this way, a particular node with ridges emanating from it is imposed on what is in reality a small, undifferentiated 'nodal area.' It would not be reasonable to think of an error term in the measurement, as there is no 'true' or 'correct' location of the node from which our measurements would deviate. Instead, reality has been transformed by a particular transformation function in order to facilitate a particular modeling approach. With regard to the topological description of ridge nodal areas, this transformation consisted of two steps: (1) it was assumed that ridge nodes are always of degree three (which, for example, calls for two nodes if four ridges emanate from the same nodal area) and (2) the connectivity of the ridge links was determined by random choice—unless, of course, the evidence from the map permitted determination of the actual link connections.

The end points of the exterior ridge links turned out to be another source of uncertainty. Fortunately, the large-scale configuration of ridge patterns is usually quite pronounced, so that these difficulties at the microlevel do not prohibit their graph-theoretical analysis as long as the uncertainties are resolved through a consistent application of explicit rules. Collectively, these rules constitute the transformation discussed earlier, the merits of which will ultimately be decided by the explanatory/predictive power of the modeling approach.

Fig. 4. Scatter diagram of the link numbers p of 50 basin perimeters as a function of basin magnitude n . The dashed line represents the estimated link number distribution as derived from the assumption of topological randomness.

6. EMPIRICAL FINDINGS

Figure 4 shows the scatter diagram for 50 drainage basins, each dot representing the number of perimeter links and the magnitude of a basin. The broken line represents the expected relationship (equation (12)) based on postulate (3). The discrepancy is considerable and systematic, and calls for the examination of each of our postulates. With regard to the first, the approximate proportionality of channel link number and area under homogeneous conditions has repeatedly been tested and verified (see, for example, *Onesti and Miller* [1978], page 146). Figure 2 shows a near perfect fit between the magnitudes of 50 channel networks and their interlocking ridge networks, which indicates that, if the postulate holds for channel links, it will also hold for the ridge links—at least in our study area. The second postulate calls for approximate proportionality between the geometric length of channel and ridge paths and their respective number of links. Figures 3 and 5 compare the geometric length of basin perimeter and mainstream with their link numbers. The linear trend and the small variance of the data provide sufficient support for the acceptance of this postulate, at least for these particular types of network paths in our study area. The third postulate consists of seven individual assumptions. Some of these do not pose particular problems for their acceptance: many studies have demonstrated that, by and large, channel networks in maturely eroded terrain free from environmental control form trivalent planted plane trees; the same statement holds for ridge networks—although in this case it is an artifact of their graph-theoretic representation in this paper rather than their actual appearance. The assumption that channel networks are topologically random has survived many tests but not without a number of surprises (e.g., *James and Krumbein's* [1969] discovery of the nonrandom distribution of Cis and Trans links). On balance, it is still considered to be a good approximation. The near equality of the magnitudes of interlocking ridge and channel networks is demonstrated by Figure 2. The assumption that exterior channel and ridge links in drainage basins alternate is also approximately correct. The assumption was not met for roughly one third of the data, which seems to be a rather large proportion. However, in comparison to the $(2n/n)$ possibilities of arranging $2n$ exterior channel and ridge

links in different sequences, the observed deviation from the assumption is relatively small, as is evident from the narrow scatter of the data in Figure 2 even for small magnitudes. The postulated topological randomness of ridge networks is in part assured by the particular method we have adopted here for their representation as trivalent plane trees: whenever more than three ridge links converged in an undifferentiated 'nodal area' they were interconnected in trivalent nodes, the particular interconnections being determined by random choice. It is this last step that has a 'randomization' effect upon the graph representing the natural ridge network. Therefore, if our ridge network data indicate randomness, they might at least in part do so as a result of this method. Figure 6 compares the expected topological path length distribution (equation (12)) with the observed number of links of 180 ridge paths from our study area. The topological path length is paired with the neighborhood distance of the exterior links of each path. In contrast to Figure 4 the correspondence between expected path length and observed data is quite good and demonstrates that the random model, when applied to the path length distribution of ridge networks, is at least adequate to predict the trend of the observed distribution in our study area. Thus it seems to be the last assumption of the third postulate—the topological independence of the interlocking ridge and channel networks—which is responsible for the discrepancy between predicted and observed number of links of basin perimeters (Figure 4). This interpretation of our model will now be examined in more detail.

In Figure 6 the solid dots refer to ridge paths that form the perimeters of drainage basins, and the circles refer to other ridge paths. If channel and ridge networks had independent topologies, the property of a ridge path being a basin perimeter should be independent from its link number. This is clearly not the case. Instead, basin perimeters turn out to be a nonrandom subset of all ridge paths that have in common a large positive deviation from the expected path length. Moreover, their variance is quite small in comparison to that of ridge paths in general. The least squares regression line for the perimeter data is

$$p = 4.17 \cdot n^{0.69}$$

Fig. 5. Scatter diagram of the number of links d and the geometric length of the mainstream L (measured in units of 120 m) for 50 drainage basins.

The exponent is of particular interest, indicating that the topological length of the basin perimeter—and therefore its geometric length—grows significantly faster than the square root of the basin magnitude (and therefore of the basin area). A similar observation has been consistently made with regard to the mainstream length of channel networks and its topological counterpart, the network diameter. Thus we might be able to shed some light on the observed distribution of the topological length of the perimeter by studying its relationship to the diameter.

7. COMPARING MAINSTREAM LENGTH AND PERIMETER OF NATURAL DRAINAGE BASINS

Figures 7 and 8 show that p and d , the link numbers of the perimeter and mainstream of drainage basins, are highly correlated, the regression line for the diameter as a function of the basin magnitude being

$$d = 1.5 \cdot n^{0.67}$$

As was pointed out in the introduction, the diameter, like the mainstream length, is a maximum value from a set of path length values. The perimeter, on the other hand, refers to the length of a single path. It was this particular definition of the diameter which permitted *Werner and Smart* [1973, p. 291] and *Shreve* [1974, p. 1176] to show that the accelerated growth of the mainstream length is essentially a consequence of the random topology of natural channel networks. Individual paths like basin perimeters, however, should have an expected growth rate proportional to the square root of the network magnitude (equation (12)). Thus we reach the somewhat puzzling conclusion that the mainstream and the perimeter of drainage basins grow at similar rates and significantly faster than the square root of the basin area, the first, however, as a result of the topological randomness of channel networks, and the second, in contrast, as a result of the topological nonrandomness of perimeter paths in ridge networks. Despite their opposite nature, both conditions are far from implausible: the randomness of channel network topology is a plausible result of the absence of environmental control, and the nonrandomness of perimeter paths in ridge networks is a plausible result of the constraint created by the uniformity of link density. If exterior and interior ridge links,

Fig. 6. Scatter diagram of the number of links l of 180 ridge paths as a function of the neighborhood distance k of their exterior links. The solid circles refer to ridge paths that are perimeters of drainage basins. Superimposed is the theoretically estimated link number distribution.

Fig. 7. Scatter diagram of the number of links of the basin perimeter p (solid circles) and of the mainstream d (open circles) as a function of the basin magnitude n for 40 drainage basins.

like those in channel networks, have particular link length distributions independent of their location within the study area, it follows that there exists a lower limit to the number of links of a perimeter path enclosing a given number of links. This lower limit corresponds to a basin of circular shape and follows directly from the application of the first and second postulates to the familiar condition that the perimeter of an area A is larger than or equal to $2(\pi A)^{1/2}$. But, whereas the topological counterpart to this geometric perimeter/area constraint explains why the link number of most ridge paths is incompatible with their being basin perimeters, it does not explain the deviation of the p versus n regression exponent from 0.5.

As an alternative interpretation of the observed behavior of the perimeter, one might follow *Hack* [1957] and postulate a causal relationship between mainstream length and basin elongation. We will explore the hypothesis that the accelerated growth rate of the mainstream length is responsible for the elongation of the basin, which in turn is reflected in a similar growth rate of the basin perimeter. As *Smart and Surkan* [1967] and *Shreve* [1974] have argued and empirically demonstrated, this hypothesis is constrained by the fact that part of the mainstream length can be accounted for by the sinuosity of its channel. Nevertheless, we will pursue the

Fig. 8. Scatter diagram of the number of links of the basin perimeter p and of the mainstream d for 50 drainage basins.

Fig. 9. Scatter diagram of the length of the perimeter P and the basin elongation $e_{\max}$ (both measured in units of 120 m) for 45 drainage basins.

formal interrelationship between mainstream length, perimeter, and elongation of drainage basins one step further. Figure 9 shows the plot of basin perimeter and basin elongation for 45 basins, where elongation is defined as the maximum straight line distance between any two points in the basin. The data demonstrate the linear relationship and the high correlation of the two measures. Figure 10 compares the number of links of the basin diameter, i.e., the path with the largest number of links originating in the outlet, with the link numbers of the n paths that start in each of the n exterior links of the network and have a maximum number of links. Both mean and ± 2 standard deviations of the $n + 1$ topological path length values are given for each of 18 networks. Clearly, there is no particular topological orientation of the networks toward their outlets, i.e., our data are in line with the postulate of topological randomness of channel networks. In contrast, examining the location of the end points of the elongation axis in each basin revealed that in two thirds of all basins one of these end points coincided with the network outlet. This last observation, when compared with Figure 10, indicates that the accelerated growth of the network diameter does not account for the elongation of the basins; instead, it appears that the basin elongation and its tendency to be oriented to the outlet may be related to the smaller sinuosity of the mainstream relative to that of the other paths with maximum links numbers originating elsewhere in the network. In any case, judging by our data the growth rate of both the basin elongation and the perimeter cannot be explained by any topological argument. Rather, metric studies concentrating on the distribution of link lengths and the angles between the links of network junctions might be able to give answers where the topological approach does not succeed [Smart, 1981; Abrahams, 1980; Jarvis, 1976].

8. SUMMARY OF RESULTS

The answer to the main question of this paper—whether the length distribution of the basin perimeter can be derived

from the random model as it was done for the main stream length—is negative both on theoretical and empirical grounds. In the course of the investigation, several intermediate results were generated. The paper provides a descriptive topological model for interlocking ridge and channel networks. The expected number of links of the path connecting exterior links i and j in a network of magnitude n is derived as a function of $|j - i|$ and n ; it approximates a square root function in n for n approaching infinity and $j - i$ finite. Whereas the link numbers of randomly sampled ridge paths show good agreement with the expected values, those ridge paths p that are perimeters of drainage basins deviate consistently and significantly, the regression exponent of p versus n being 0.69 rather than the expected value of 0.5. A partial explanation is provided by the translation of the spatial constraint, which limits the minimum possible length of perimeters for a given basin area, into a corresponding topological constraint. The constraint expresses the minimum possible number of links of a perimeter path as a function of the number of links enclosed, i.e., as a function of the basin magnitude. However, since this constraint is again a square root function, it cannot account for the observed growth rate of the perimeter. In pursuing a possible connection between perimeter and mainstream length for which the random model gives a good prediction, data on basin perimeter, mainstream length, and basin elongation were collected. Although the three variables are highly correlated, the data indicate that the observed growth rate of the mainstream link number with basin magnitude is unrelated to the elongation of the basin, despite the fact that one end point of the elongation axis coincides with the outlet in 68% of the basins surveyed. Consequently, if the accelerated growth rate of the perimeter is the result of basin elongation, it cannot be explained by the similar growth rate (0.67) of the mainstream. Thus, whereas the random model gives good estimates for the average length of ridge paths in general, it can apparently not account for the average length of perimeter paths and, therefore, the peculiar interaction that exists between interlocking ridge and channel networks.

Fig. 10. Mean $\bar{d}$ and ± 2 standard deviations σ of the link number distribution of the paths with the largest number of links originating in the exterior links and the outlet for each of 18 channel networks. For comparison, the value d (open circle) of the link number of the path starting in the outlet (i.e., the network diameter) has been superimposed in each case.

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