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Ehrhart Theory of Combinatorially Defined Polytopes

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Ehrhart Theory of Combinatorially Defined Polytopes

by

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Abstract

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In geometric, algebraic, and topological combinatorics, properties such as symmetry, unimodality, and real-rootedness of combinatorial generating polynomials are frequently studied. In this thesis, we present three projects exploring various properties of polynomials arising from Ehrhart theory, the study of counting integer points in lattice polytopes.

Many of the open questions on real-rootedness and unimodality of polynomials pertain to the enumeration of faces of cell complexes. When proving that a polynomial is real-rooted, we often rely on the theory of interlacing polynomials and their recursive nature. We relate the theory of interlacing polynomials to the shellability of cell complexes. We first derive a sufficient condition for stability of the h-polynomial of a subdivision of a shellable complex. To apply it, we generalize the notion of reciprocal domains for convex embeddings of polytopes to abstract polytopes and use this generalization to define the family of stable shellings of a polytopal complex. We characterize the stable shellings of cubical and simplicial complexes, and apply this theory to answer a question of Brenti and Welker on barycentric subdivisions for the well-known cubical polytopes. We also give a positive solution to a problem of Mohammadi and Welker on edgewise subdivisions of cell complexes. We end by relating the family of stable line shellings to the combinatorics of hyperplane arrangements.

The Ehrhart polynomial $\mathrm{ehr}_P(t)$ of a lattice polytope P counts the number of integer points in the n-th dilate of P. The f^*-vector of P, introduced by Felix Breuer in 2012, is the vector of coefficients of $\mathrm{ehr}_P(n)$ with respect to the binomial coefficient basis $\left\{\binom{n-1}{0}, \binom{n-1}{1}, ..., \binom{n-1}{d}\right\}$, where $d = \mathrm{dim} P$. Similarly to h/h^*-vectors, the f^*-vector of P coincides with the f-vector of its unimodular triangulations (if they exist). We present several inequalities that hold among the coefficients of f^*-vectors of polytopes. These inequalities resemble striking similarities with existing inequalities for the coefficients of f-vectors of simplicial polytopes; e.g., the first half of the f^*-coefficients increases and the last quarter decreases. Even though f^*-vectors

of polytopes are not always unimodal, there are several families of polytopes that carry the unimodality property. We also show that for any polytope with a given Ehrhart h^*-vector, there is a polytope with the same h^*-vector whose f^*-vector is unimodal.

Posets can be viewed as subsets of the type-A root system that satisfy certain properties. Geometric objects arising from posets, such as order cones, order polytopes, and chain polytopes, have been widely studied, though many open questions concerning them remain open, such as the unimodality of their h^*-vectors. In 1993, Vic Reiner introduced signed posets, which are subsets of the type-B root system that satisfy the same properties. We introduce the analogue of order and chain polytopes in this setting, focusing on the Ehrhart theory of these objects. We are able to determine when these signed order polytopes have symmetric h^*-vectors, and end with a discussion of open questions regarding signed chain polytopes.

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Chapter 1

Introduction

This thesis consists of several projects, each concerning polynomials related to enumerative problems about discrete geometric objects, such as lattice polytopes and simplicial complexes. A lattice polytope is the convex hull of finitely many integer points in $\mathbb{R}^d$; enumerative data corresponding to these objects show up in many fields of mathematics, such as representation theory, algebraic geometry, and commutative algebra. In particular, in 1962, for a lattice polytope $P \subset \mathbb{Z}^d$, Eugene Ehrhart introduced and proved polynomiality of the Ehrhart enumerator $\mathrm{ehr}(n) = |nP \cap \mathbb{Z}|$ [38]. This *Ehrhart polynomial* can be seen as a discrete measure of volume, and many objects in combinatorics, such as order polynomials, chromatic polynomials of graphs, and magic squares have interpretations in Ehrhart theory (for examples, see [13]).

It can be helpful to view the information encoded in the Ehrhart polynomial in other forms. The generating series $\mathrm{E}hr(P; z)$ of a d-dimensional lattice polytope can be written as

$$\mathrm{E}hr(P; z) = 1 + \sum_{n \geq 1} \mathrm{ehr}_P(n) z^n = \frac{h_0^* + h_1^* z + \cdots + h_{d+1}^* z^d}{(1-z)^{d+1}}.$$

The numerator of this expression is called the *h^*-polynomial* of P. Unlike the Ehrhart polynomial, Stanley showed that that the coefficients of the h^*-polynomial of a lattice polytope are always non-negative integers [64].

When studying a combinatorial polynomial, an often studied question is one of classification: Exactly which polynomials can be realized as this specific type of combinatorial polynomial? This question is open for Ehrhart polynomials and h^*-polynomials, and one way that people have been chipping away at this is by studying specific properties that these polynomials may have. For example, there is a large collection of work asking if specific families of polytopes have h^*-polynomials that are unimodal or real-rooted, for e.g, [10], [31]. The three projects making up the bulk of this thesis are all related to this broad goal in some way, and are titled:

1. Subdivisions of shellable complexes.

2. Inequalities of f^*- polynomials.
3. Signed poset polytopes.

Project 1 is joint work with Liam Solus and published in [46], and is discussed in Chapter 3 of this thesis; it contributes to work exploring properties of h-polynomials of certain collections of simplicial complexes. A foundational result regarding simplicial complexes, the g-theorem ([15], [54], [66]), implies that the h-polynomial of the boundary complex of a simplicial polytope is unimodal. Since then, extensions via the relationship between h-polynomials of simplicial complexes and their subdivisions became of interest. For example, the *barycentric subdivision* of a polytope is a simplicial complex based on the combinatorial data of the face lattice of the polytope.

In [27], Brenti and Welker strengthened the unimodality result implied by the g-theorem for a specific family of simplicial complexes, the barycentric subdivision of the boundary complex of a simplicial polytope, by showing that the h-polynomial in question is real-rooted, a stronger property than unimodality. They then propose the following question: Does the barycentric subdivision of the boundary complex of any polytope yield a real-rooted h-polynomial? As a starting point, they suggested to look at the boundary complex of cubical polytopes (polytopes where all the faces are combinatorially equivalent to cubes).

In Chapter 3 of this thesis, we develop a tool based around a shelling argument and results from Ehrhart theory that can be used to prove real-rootedness results of h-polynomials of subdivisions of complexes; this tool is given by Theorem 3.3.1. We then apply this tool to various specific families of subdivisions of polytopal complexes, proving results such as, for example, the real-rootedness of the h-vector of barycentric subdivisions of specific families of cubical complexes, shown in Corollaries 3.4.9, 3.4.8, and 3.4.10. In the process of proving these results, we describe a specific type of shelling order, a *stable shelling* that we are interested in studying further.

Project 2 is joint work with Matthias Beck, Danai Deligeorgaki, and Jerónimo Valencia, published in [14], and is discussed in Chapter 4 of this thesis. It concerns expanding the current knowledge on a relatively new Ehrhart-related polynomial, the f^*-polynomial of a lattice polytope. There is a well-known basis change that transforms the f-polynomial of a simplicial complex, enumerating its faces, into what is called the h-polynomial of the simplicial complex and vice versa. The same basis change can be applied to the h^*-polynomial of a lattice polytope, giving rise to the polytope's *f^*-polynomial.* In 2012, Breuer proved that the f^*-polynomial has nonnegative coefficients not only for lattice polytopes but also for any complex of lattice polytopes [29].

In Chapter 4, we prove some additional inequalities regarding the coefficients of the f^*-vector mirroring inequalities for the f-vectors of simplicial polytopes [19]. These inequalities are listed in Theorem 4.0.1. We also give additional results strengthening these inequalities for specific families of polytopes, and also give a method for constructing polytopes with unimodal f^*-vectors, detailed in Corollary 4.0.6.

Project 3 is the subject of Chapter 5 of this thesis, and explores the Ehrhart theory of polytopes constructed from a generalization of posets, signed posets, introduced in 1994 by

Vic Reiner [58]. In 1986, Stanley introduced two polytopes coming from posets, the *order polytope* and the *chain polytope* [68]. For a given poset P, the order polytope is

$$\mathcal{O}(P) = \{\phi \in \mathbb{R}^P : 0 \leq \phi(p) \leq 1 \text{ for all } p \in P \text{ and } \phi(a) \leq \phi(b) \text{ when } a \leq b \text{ in } P\},$$

and the chain polytope is defined using inequalities constructed from chains in the poset.

Since then, the Ehrhart theory of the order polytope has been well studied, and is still the subject of large open problems. One well-known conjecture in algebraic combinatorics, the Neggers–Stanley conjecture [56], can be rephrased in Ehrhart theory: The h^*-polynomial of any order polytope is real-rooted. In 2007, Stembridge disproved this conjecture by finding an explicit counterexample [73]. However, a weaker question is still open: Is the h^*-polynomial of any order polytope unimodal?

In 1993, Reiner introduced *signed posets*, a type B generalization of posets [58]. In Chapter 5, we describe analogous type B generalizations of order and chain polytopes, focusing on the Ehrhart theory of these polytopes. Results such as Proposition 5.3.5 show analogous results appearing in [68], such as describing the lattice points and facets of signed order polytopes. We also classify when a given signed poset yields a *Gorenstein* signed order polytope in Proposition 5.5.4. We end by describing signed chain polytopes, giving some analagous results to those appearing in [68] and end with some open questions.

Chapter 2

Background

This chapter presents the main objects of study and establishes notation used throughout this thesis.

2.1 Polytopes

A ***polyhedron*** P is the solution set of finally many linear inequalities,

$$P := \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\},$$

for some real matrix $A \in \mathbb{R}^{x \times d}$ and some real vector $\mathbf{b} \in \mathbb{R}^m$. A bounded polyhedron is called a ***polytope***.

Equivalently, a polytope can be described as the ***convex hull*** of finitely many points in $\mathbb{R}^d$, where the convex hull of a set of points $\mathbf{v}_1, \mathbf{v_2}, \ldots, \mathbf{v_n}$ is given by

$$\operatorname{conv}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \left\{ \sum_{i=1}^{n} \lambda_i \mathbf{v}_i : \lambda_i \geq 0 \text{ for } 1 \leq i \leq n \text{ and } \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

For reasons we will see later, this is referred to as the ***vertex description*** of a polytope. For a proof of the equivalence between the vertex description and the hyperplane description of a polytope, see, for e.g., [23], and as we will see in the following discussion of some properties of polytopes, sometimes it is more helpful to use the vertex description, and sometimes the hyperplane description is more natural.

The ***dimension*** of a polytope is the dimension of the ***affine span*** of its vertices, defined as $\operatorname{aff}(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \{\sum_{i=1}^{n} \lambda_i \mathbf{v}_i : \lambda_i \geq 0 \text{ for } 1 \leq i \leq n\}$. Note that the dimension of a polytope is not necessarily equal to the dimension of the space it is embedded in. For example, the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ is a 2-dimensional polytope living in $\mathbb{R}^3$. We call a polytope ***full-dimensional*** if its dimension is equal to the dimension of its ambient space.

Given a polytope $P \subset \mathbb{R}^d$, consider a hyperplane $H = \{x : \langle \mathbf{w}, x \rangle = \mathbf{a}\}$ such that $P \subset \{x : \langle \mathbf{w}, x \rangle \leq \mathbf{a}\}$, or in other words P is entirely contained underneath H. Such a

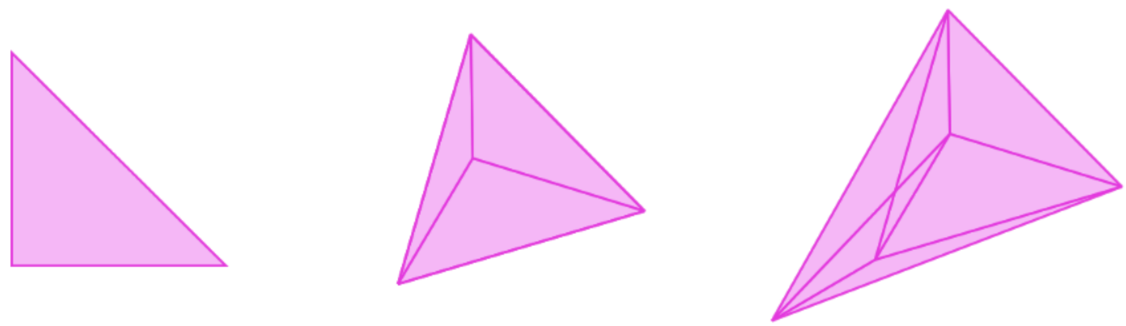


Figure 2.1: Simplices in $2, 3$, and 4 dimensions.

hyperplane is called an ***admissible hyperplane*** of P. A ***face*** of P is any intersection of P with an admissible hyperplane. We also consider the empty set and P to be faces of P. The dimension of a non-empty face is the dimension of its affine span. We call 0-dimensional faces ***vertices***, 1-dimensional faces ***edges***, and $(d-1)$-dimensional faces ***facets***. Note that any polytope can be expressed the convex hull of its vertices. In this thesis, we are mostly concerned with polytopes whose vertices are integer points; we will refer to these here as ***lattice polytopes***, though in the literature they are sometimes also refered to as integral polytopes. We call all faces of P except for P itself and the empty set to be ***proper faces*** of P. The ***relative interior*** of a polytope $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\}$ can be thought of as P with all of its proper faces removed, and is formally defined below as

$$P^\circ := \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} < \mathbf{b}.\}$$

Now that we have the fundamental definitions of polytopes, let's look at some examples.

Example 2.1.1. A d-***simplex*** a polytope with exactly $d+1$ vertices. For example, a triangle is a 2-dimensional simplex, and a tetrahedron is a 3-dimensional simplex. We often mention to the standard simplex, which by convention will refer to

$$\text{conv}((1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots (1, 1, \ldots, 1)) \subset \mathbb{R}^d.$$

The equivalent hyperplane description of the standard simplex is $\{x \in \mathbb{R}^d : 0 \leq x_1 \leq x_2 \leq \cdots \leq x_d \leq 1\}$.

A d-simplex has exactly $\binom{d+1}{i+1}$ faces of dimension i for $0 \leq i \leq d$, since the convex hull of any subset of vertices of a simplex forms a face. Each face is itself a simplex of some dimension.

Example 2.1.2. The standard d-cube is given by the hyperplane description $\{x \in \mathbb{R}^d : 0 \leq x_i \leq 1 \text{ for } 1 \leq i \leq d\}$., or alternatively, the convex hull of the points in $\{0, 1\}^d$. A standard d-cube has $2^{d-i}\binom{d}{i}$ faces of dimension i, each of which is a possibly shifted version of an i-dimensional cube.

Polytopal operations and maps

There are many ways to construct new polytopes from old; the one that we will need the most for what follows is the ***Minkowski sum*** of two polyhedra. For two polyhedra P and Q, their Minkowski sum is denoted

$$P + Q = \{\mathbf{v} + \mathbf{w} : v \in P, w \in Q\}$$

There are also various notions for what it means for two polytopes to be considered equivalent. First, there is a notion of ***combinatorial equivalence***, which generally means that the polytopes in question have the same facial structure. A more precise definition will be given in Section 2.2.

Since much of the material in this thesis concerns questions of counting integer points in lattice polytopes, we introduce a notion of equivalence that preserves such enumerative properties introduced in the Ehrhart Theory section of this chapter. We call a matrix $U \in \mathbb{Z}^{d\times d}$ ***unimodular*** if its determinant is ± 1. We call two lattice polytopes $P, Q \subseteq \mathbb{R}^d$ ***unimodularly equivalent*** if there exists a unimodular matrix U and a vector $\mathbf{b} \in \mathbb{Z}^d$ such that $P = f_U(Q) + \mathbf{b}$, where $f_U(Q) = \{U\mathbf{x} : \mathbf{x} \in Q\}$.

Triangulations

When considering a polytope P, it is often helpful to break it into bite-sized pieces. A ***dissection*** is a d-dimensional polytope P is a set of d-dimensional polytopes $P_1, \ldots, P_n$, such that $\bigcup P_i = P$ and $P_i^\circ \cap P_j^\circ = \emptyset$.

It is helpful to have a stricter condition on how the bite-sized pieces in the decomposition of a polytope intersect. A ***polytopal complex*** is a nonmpty finite set $\mathcal{S}$ of polytopes that satisfies the conditions:

- If $P \in \mathcal{S}$ and F is a face of P, then $F \in \mathcal{S}$.
- If $P, Q \in \mathcal{S}$, then $P \cap Q$ is a face of both P and Q.

The elements of $\mathcal{S}$ are called its ***faces***, and its maximal faces (with respect to inclusion) are called its ***facets***. If all the facets of $\mathcal{S}$ have the same dimension then $\mathcal{S}$ is called ***pure***.

Example 2.1.3. For any polytope P, the set of all proper faces of P is a polytopal complex, which we call the ***boundary complex*** of P and denote δP. For a proof of this, see, for e.g., [23].

A ***subdivision*** of a polytope P is a polytopal complex whose union is P. A subdivision of a P whose cells are all simplices is called a ***triangulation***. If P is a lattice polytope, a triangulation of P into lattice simplices is called a ***lattice triangulation***.

Theorem 2.1.1. *Every lattice polytope admits a lattice triangulation.*

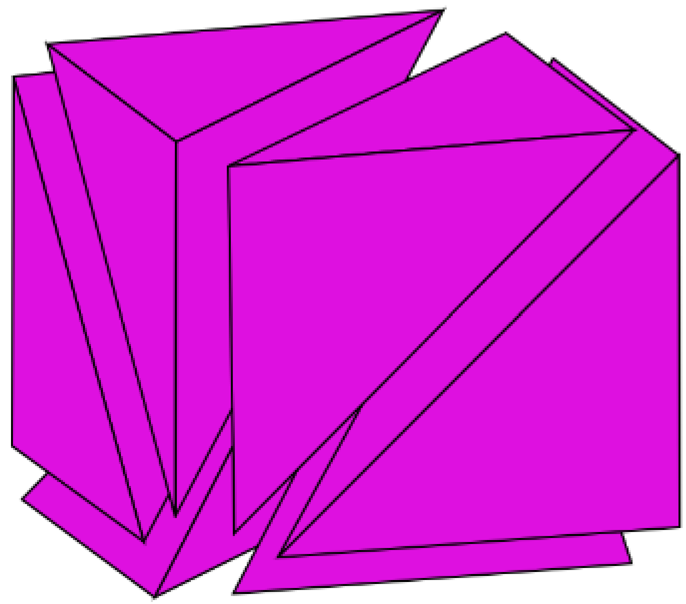


Figure 2.2: A triangulation of the 3-cube into 6 simplices.

For a full proof of this result, see, for e.g., [23]. The proof of this actually shows a stronger result: that every lattice polytope has a ***regular triangulation***. Since these triangulations appear later, we will introduce them and vaguely sketch their construction. For a more detailed definition and construction, see, for e.g., [33].

Let $V = \{\mathbf{v_1}, \dots, \mathbf{v_n}\} \subset \mathbb{R}^d$ be the set of lattice points in a polytope P, and consider a ***lifting*** of $\omega \in \mathbb{R}^n$ of these points as follows:

$$V^\omega = \{(\mathbf{v_i}, \omega_i) : \mathbf{v_i} \in V\}.$$

Now, let $\mathbb{R}^{\text{up}} = \{(0, \dots, 0, t) \in \mathbb{R}^{d+1} : t \geq 0\}$, and consider the unbounded polyhedra $\text{conv}(V^\omega) + \mathbb{R}^{\text{up}}$. The set of bounded faces of this polyhedra, when projected back down to $\mathbb{R}^d$, gives a subdivision of P. Any subdivision (or triangulation) that can be constructed in this way for some choice of ω is called a ***regular subdivision*** (or triangulation).

Another special type of triangulation is a ***unimodular triangulation***, which is a lattice triangulation whose simplices are all unimodularly equivalent to the standard simplex. Unimodular triangulations are often very helpful for computing volume and lattice point enumerations of polytopes.

2.2 Posets, with a view towards polytopes

A ***partially ordered set*** $(P, \preceq)$, called a ***poset*** for short, is a set P together with a relation $\preceq$ that satisfies:

- reflexivity: $a \preceq a$ for every $a \in P$.
- transitivity: for $a, b, c \in P$, if $apreceqb$ and $b \preceq c$, then $a \preceq c$.
- antisymmetry: for $a, b \in P$, if $a \preceq b$ and $bpreceqa$, then $a = b$.

A poset if ***finite*** if P is a finite set. We call a sequence of $n+1$ related elements, $x_0 \preceq x_1 \preceq \cdots \preceq x_n$ a ***chain*** of length n (note that we allow for repeated elements). We say that there is a ***cover relation*** between $x, y \in P$ and that y ***covers*** x if $x \preceq y$ and there is no such z such that $x \preceq z \preceq y$. Posets are often visualized by diagrams called ***Hasse diagrams***, in which the elements of P are represented with labeled dots, and for every pair (x, y) in which y covers x, the dot representing y is drawn above the dot representing y, and they are connected with an edge.

Throughout this thesis, we call a poset ***graded*** if all of its maximal chains are of the same length. (Note that there a few related but different definitions of graded posets in the literature.) A graded poset can always be given a ***rank function*** $\rho : P \to \mathbb{Z}$ that satisfies the following properties:

- if $x \preceq y \in P$, then $\rho(x) \leq \rho(y)$.
- if y covers x, then $\rho(y) = \rho(x) + 1$.

Below, we list a few examples of posets.

Example 2.2.1. Consider $([n], \leq)$, where $\leq$ is the standard less-than-or-equal relation on real numbers. One can verify that this satisfies all the properties needed to be a poset, and in fact the relation actually gives a total order, since each pair of elements is related. We call this the ***chain*** on n elements.

Example 2.2.2. Let P be the set of subsets of $[d]$, and let $\preceq$ be the inclusion relation on sets, for e.g., $\{1, 4\} \preceq \{1, 3, 4, 5\}$. One can verify that this is a poset; we call this the ***Boolean lattice*** on d and denote it B_d.

Example 2.2.3. Consider $([n], =)$. Note that no two distinct elements are related. We call this the ***antichain*** on n elements.

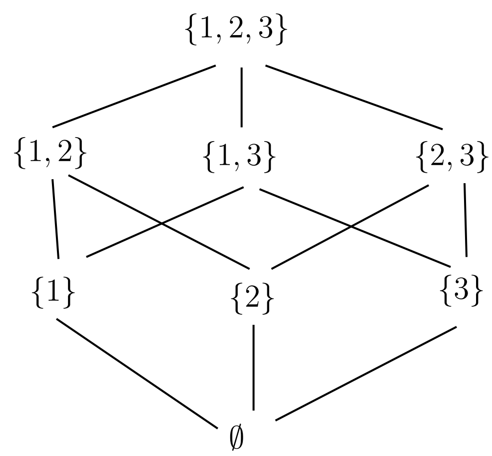


Figure 2.3: The Hasse diagram of the Boolean lattice on a set of size 3.

Face lattice of a polytope

Let P be a polytope and $\Phi(P)$ be the set of its faces, including P itself and $\emptyset$. Note that $(\Phi(P), \subseteq)$ is a poset (and also a lattice). We refer to this as the ***face lattice*** of P.

Example 2.2.4. Consider a d-simplex Δ^d, with vertices $v_1, \ldots, v_d, v_{d+1}$. As noted before, the faces of Δ^d are in bijection with the subsets of $[d+1]$, and the mapping respects inclusion. Thus, the face lattice of a d-simplex is isomorphic to the boolean lattice on $[d+1]$.

We are now ready to formally define another type of equivalence between polytopes. Two polytopes are ***combinatorially equivalent*** if their face lattices are isomorphic.

Order and chain polytopes

In the previous subsection, we discussed one example of a poset constructed from any given polytope. We note that there are at least two examples of polytopes constructed from any given poset, the ***order polytope*** and ***chain polytope***, both of which first appeared in [67] and have been well-studied since. In Chapter 5 of this thesis, we will give a detailed definition and overview of these, as well as describing a type B generalization of these polytopes.

2.3 Ehrhart Theory

Given a lattice polytope P, it makes sense to count the number of integer points (also called lattice points) in P. Perhaps one of the first such results involving counting lattice points is Pick's Theorem, concerning polygons, or 2-dimensional polytopes.

Theorem 2.3.1 (Pick's Theorem (1899))**.** *For a polygon $P \subset \mathbb{R}^2$ whose vertices are integer coordinates, let I be the number of integer points in the interior of P, let B be the number of integer points on the boundary of P, and let A be the area of P. Then,*

$$A = I + \frac{B}{2} - 1.$$

In 1962, for a lattice polytope $P \subset \mathbb{Z}^d$, while trying to generalize the ideas in Pick's Theorem to higher dimensions, Eugene Ehrhart introduced and proved polynomiality of the Ehrhart enumerator $\mathrm{ehr}_P(t) = |tP \cap \mathbb{Z}|$, [38]. Note that the t^{th} ***dilate of*** P is $tP := \{tp \in \mathbb{R}^n : p \in P\}$.

Theorem 2.3.2. *(Ehrhart's Theorem) For any d-dimensional lattice polytope $P \subseteq \mathbb{R}^n$, the quantity $ehr_P(t) = |tP \cap \mathbb{Z}^n|$ agrees with a polynomial of degree d.*

This ***Ehrhart polynomial*** can be seen as a discrete measure of volume. In general, the coefficients of the Ehrhart polynomial of a lattice polytope are rational numbers that can be negative. Certain families of polytopes have been shown to have Ehrhart polynomials with non-negative coefficients, a property known as ***Ehrhart positivity.*** One major open problem is to classify which polytopes are Ehrhart positive; a survey of these results are in[52].

The generating series $\mathrm{E}hr(P; z)$ of a d-dimensional lattice polytope can be written as

$$\mathrm{E}hr(P;z) = 1 + \sum_{t \geq 1} \mathrm{ehr}_P(t) z^t = \frac{h_0^* + h_1^* z + \cdots + h_{d+1}^* z^d}{(1-z)^{d+1}}.$$

The numerator of this expression is called the h^****-polynomial*** of P and denoted $h^*(P; z)$. Sometimes we view the coefficients of this polynomial as a vector called the h^*-vector. Note that $h^*(P; z)$ can have degree less than d, and the degree of $h^*(P; z)$ is called the ***degree*** of P. It can be helpful to view the information encoded by the Ehrhart polynomial in this form, since this h^*-polynomial satisfies nice properties. Unlike the Ehrhart polynomial, the coefficients of the h^*-polynomial of a lattice polytope are always non-negative integers [64]. One major open problem is classifying exactly when these polynomials are ***unimodal***. A polynomial p is called unimodal if there exists $t \in [d]$ such that $p_0 \leq \cdots \leq p_t \geq \cdots \geq p_d$. It is called ***log-concave*** if $p_k^2 \geq p_{k-1}p_{k+1}$ for all $k \in [d]$, and it is called ***real-rooted*** (or ***(real) stable***) if $p \equiv 0$ or p has only real zeros. A classic result states that p is both log-concave and unimodal whenever it is real-rooted [26, Theorem 1.2.1]. Since real-rootedness is the strongest of these three conditions, many conjectures in the literature ask when certain generating polynomials are not only unimodal or log-concave, but also real-rooted. There is currently a well-studied chain of conjectures by Stanley and others about which families of polytopes have h^*-vectors satisfying these properties, for an overview see [43].

One helpful property of Ehrhart polynomials, that will be used in Chapter 3 of this thesis, for example, is that they satisfy a reciprocity theorem.

n	$\mathrm{ehr}_P(n)$
1	3
2	6
3	10
4	15

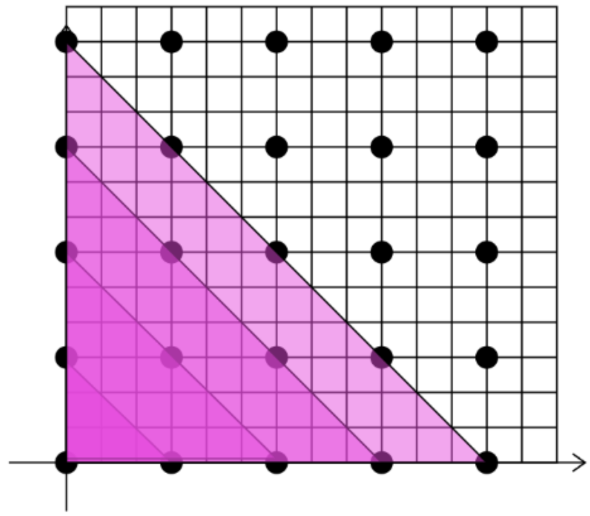


Figure 2.4: Several values of the ehrhart polynomial of the standard 2-simplex.

Theorem 2.3.3 (Ehrhart-Macdonald Reciprocity Theorem[38])**.** *Let P be a d-dimensional lattice polytope. Then:*

$$ehr_P(t) = (-1)^d ehr_{P^\circ}(-t)$$

For a full survey of the other various properties of Ehrhart and h^*-polynomials, see, for, example, [23].

We end this subsection with an example of Ehrhart and h^*-polynomials.

Example 2.3.1. The standard d-simplex Δ^d has Ehrhart polynomial $\binom{t}{d}$. We now compute the h^*-polynomial:

$$\mathrm{E}hr(\Delta^d; z) = 1 + \Sigma_{t\geq 1}\binom{t}{d} z^t = \frac{1}{(1-z)^{d+1}}.$$

Thus, $h^*_{\Delta^d}(z) = 1$.

2.4 Simplicial complexes

An (abstract) ***simplicial complex*** is any family of sets closed under taking subsets. Figure 2.5 illustrates why theses are called simplicial complexes, as any (abstract) simplicial complex can be visualized using a polytopal complex of simplices. Each face in the polytopal complex is identified by its set of vertices.

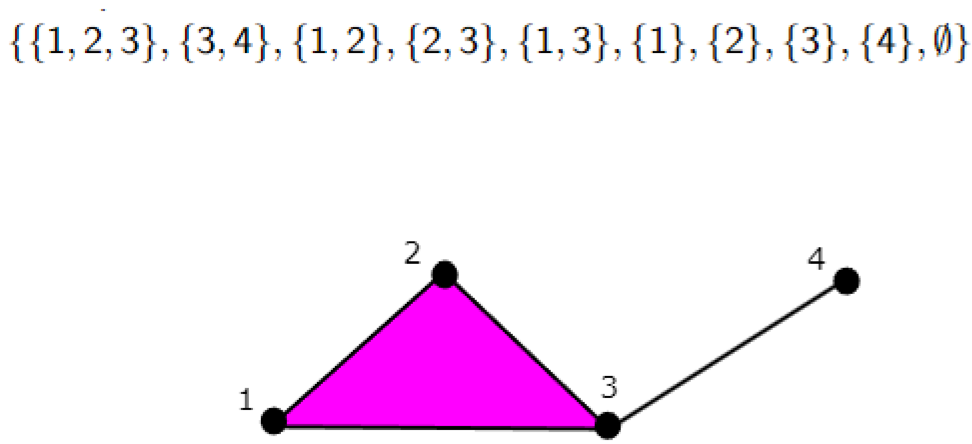


Figure 2.5: An example of an (abstract) simplicial complex and its realization as a polytopal complex of simplices.

Given a simplicial complex, the ***f-polynomial*** of a $(d-1)$-dimensional simplicial complex $\mathcal{C}$ is

$$f(\mathcal{C};x) = f_{-1}(\mathcal{C}) + f_0(\mathcal{C})x + f_1(\mathcal{C})x^2 + \cdots + f_{d-1}(\mathcal{C})x^d,$$

where $f_{-1}(\mathcal{C}) = 1$ and $f_k(\mathcal{C})$ is the number of k-dimensional faces of $\mathcal{C}$ for $0 \leq k \leq d-1$. The ***h-polynomial*** of $\mathcal{C}$ is given by

$$h(\mathcal{C};x) = (1-x)^d f(\mathcal{C};x).$$

The h-polynomial is closely related to the h^*-polynomial described in Section 1.1; if a lattice polytope P has a *unimodular triangulation*, then the h-polynomial of the triangulation is the h^*-polynomial of P. Like with the h^*-polynomial, there is interest in which simplicial complexes have h-polynomials that satisfy properties such as real-rootedness and unimodality, for e.g. see [28].

Chapter 3

Subdivisions of shellable complexes

3.1 Introduction

In algebraic, geometric, and topological combinatorics, we are often interested in properties such as unimodality and real-rootedness of the f- or h-polynomial associated to a cell complex. A foundational result in the field, known as the g-theorem, implies that the h-polynomial associated to the boundary complex of a simplicial polytope is unimodal [63]. In the years following the proof of the g-theorem, extensions via the relationship between h-polynomials of simplicial complexes and their subdivisions became of interest [3, 65]. In [27], Brenti and Welker strengthen the unimodality result implied by the g-theorem for one family of simplicial complexes when they showed that the h-polynomial of the barycentric subdivision of the boundary complex of a simplicial polytope is real-rooted. They then asked if their result generalizes to all polytopes [27, Question 1]. In [55], Mohammadi and Welker again raised this question and suggested cubical polytopes as a good starting point.

Most proofs of real-rootedness conjectures rely on ***interlacing polynomials***, [22], which are inherently tied to recursions associated to the polynomials of interest. In the same way that proofs of real-rootedness via interlacing polynomials often rely on polynomial recursions, proofs pertaining to the geometry of polytopal complexes often make use of the recursive structure of the complex (when it exists). This recursive property of polytopal complexes is termed ***shellability***. In this chapter, we relate the recursive structure of interlacing polynomials to the notion of shellability so as to derive a sufficient condition for the h-polynomial of a subdivision of a shellable complex to be real-rooted. It turns out that, in many cases, this sufficient condition can be applied to the same shelling order of a complex for different subdivisions. Shelling orders to which this phenomenon applies are termed ***stable shellings*** in this chapter, and they arise via a generalization of the notion of ***reciprocal domains*** for convex embeddings of polytopes, as studied by Ehrhart [38, 36] and Stanley [61]. By generalizing reciprocal domains to abstract polytopes, we introduce the family of stable shellings for an abstract polytopal complex. The stable shellings of both simplicial and cubical complexes are then characterized. As an application, we recover a positive answer to the question of Brenti and Welker for the well-known families of cubical polytopes; namely, the ***cuboids*** [41], the ***capped cubical polytopes*** [49], and the ***neighborly cubical polytopes*** [8].

The remainder of the chapter is structured as follows: In Section 3.2, we develop the necessary preliminaries pertaining to polytopal complexes, interlacing polynomials, and lattice point enumeration. In Section 3.3, we derive a sufficient condition for the real-rootedness of the h-polynomial of a subdivision of a shellable complex (Theorem 3.3.1). We then define the family of stable shellings, and we characterize the such shellings for simplicial and cubical complexes. In Section 3.4, we apply the results of Section 3.3 to deduce that the h-polynomial of the barycentric subdivision of the boundary complexes of the aforementioned cubical polytopes are real-rooted. We then apply these techniques to give an alternative proof of the original result of Brenti and Welker [27], establishing stable shelling methods as a common solution to [27, Question 1] for all known examples. We also apply stable shellings to solve a second problem proposed by Mohammadi and Welker [55] pertaining to edgewise subdivisions of cell complexes, thereby demonstrating how the same stable shelling can be used to recover real-rootedness results for multiple different subdivisions of a given complex. In Section 3.5, we end by relating the theory of stable line shellings to the combinatorics of hyperplane arrangements and the geometry of realization spaces of polytopes. Some questions are proposed; answers to which could potentially settle the question of Brenti and Welker in its fullest generality.

The results in this chapter can be found in [46], and are joint work with Liam Solus.

3.2 Preliminaries

The results in this chapter are concerned with the f- and h-polynomials of polytopal complexes. When all facets of a polytopal complex $\mathcal{C}$ are simplices, we call $\mathcal{C}$ a ***simplicial complex***, and when all facets of $\mathcal{C}$ are cubes, we call $\mathcal{C}$ a ***cubical complex***. Note that, in this definition, we do not require our polytopal complex to be embedded in some Euclidean space, but instead treat it as an abstract cell complex. Given an abstract polytope P, or convex polytope $P \subset \mathbb{R}^n$, we can naturally produce two associated (abstract) polytopal complexes: the complex $\mathcal{C}(P)$ consisting of all faces in P and the complex $\mathcal{C}(\partial P)$ consisting of all faces in ∂P, the boundary of P. The ***facets*** of the polytope P are the facets of the complex $\mathcal{C}(\partial P)$. We call $\mathcal{C}(\partial P)$ the ***boundary complex*** of P. In a similar fashion, given a collection of polytopes $P_1, \dots, P_m$, we can define the polytopal complex $\mathcal{C}(P_1 \cup \cdots \cup P_m) = \cup_{i\in[m]}\mathcal{C}(P_i)$. Given a polytopal complex $\mathcal{C}$, a polytopal complex $\mathcal{D}$ is called a ***subcomplex*** of $\mathcal{C}$ if every face of $\mathcal{D}$ is also a face of $\mathcal{C}$. We refer to the difference $\mathcal{C} \setminus \mathcal{D} = \{P \in \mathcal{C} : P \notin \mathcal{D}\}$ as a ***relative (polytopal) complex***, and we define the ***dimension*** of $\mathcal{C} \setminus \mathcal{D}$ to be the largest dimension of a polytope in $\mathcal{C} \setminus \mathcal{D}$. When $\mathcal{D} = \emptyset$, note that $\mathcal{C} \setminus \mathcal{D} = \mathcal{C}$.

The ***f-polynomial*** of a $(d-1)$-dimensional polytopal complex $\mathcal{C}$ is the polynomial

$$f(\mathcal{C}; x) := f_{-1}(\mathcal{C}) + f_0(\mathcal{C})x + f_1(\mathcal{C})x^2 + \cdots + f_{d-1}(\mathcal{C})x^d,$$

where $f_{-1}(\mathcal{C}) := 1$ when $\mathcal{C} \neq \emptyset$ and $f_k(\mathcal{C})$ denotes the number of k-dimensional faces of $\mathcal{C}$ for $0 \leq k \leq d-1$. Given a subcomplex $\mathcal{D}$ of $\mathcal{C}$, the ***f*-polynomial** of the relative complex $\mathcal{C} \setminus \mathcal{D}$ is then

$$f(\mathcal{C} \setminus \mathcal{D}; x) := f(\mathcal{C}; x) - f(\mathcal{D}; x).$$

The ***h*-polynomial** of the $(m-1)$-dimensional relative complex $\mathcal{C} \setminus \mathcal{D}$ is the polynomial

$$h(\mathcal{C} \setminus \mathcal{D}; x) := (1-x)^m f\left(\mathcal{C} \setminus \mathcal{D}; \frac{x}{1-x}\right).$$

We write $h(\mathcal{C} \setminus \mathcal{D}; x) = h_0(\mathcal{C} \setminus \mathcal{D}) + h_1(\mathcal{C} \setminus \mathcal{D})x + \cdots + h_m(\mathcal{C} \setminus \mathcal{D})x^m$ when expressing $h(\mathcal{C} \setminus \mathcal{D}; x)$ in the standard basis, and we similarly write $f(\mathcal{C} \setminus \mathcal{D}; x) = f_0(\mathcal{C} \setminus \mathcal{D}) + f_1(\mathcal{C} \setminus \mathcal{D})x + \cdots + f_m(\mathcal{C} \setminus \mathcal{D})x^m$. The following lemma, whose proof is an exercise, relates the h-polynomial of a polytopal complex to those of its relative complexes.

Lemma 3.2.1. *Let $\mathcal{C}$ be a $(d-1)$-dimensional polytopal complex and suppose that $\mathcal{C}$ can be written as the disjoint union*

$$\mathcal{C} = \bigsqcup_{i=1}^{s} \mathcal{R}_i$$

where $\mathcal{R}_i$ are relative $(d-1)$-dimensional polytopal complexes. Then

$$h(\mathcal{C}; x) = \sum_{i=1}^{s} h(\mathcal{R}_i; x).$$

Proof. From the definition of the f-polynomial of a relative complex, we have $f(\mathcal{C}; x) = \sum_{i=1}^{s} f(\mathcal{R}_i; x)$. Since the transformation from f-polynomial to h-polynomial is linear, and all $\mathcal{R}_i$ are $(d-1)$-dimensional, it follows that $h(\mathcal{C}; x) = (1-x)^{d-1} \sum_{i=1}^{s} f(\mathcal{R}_i; x/(1-x)) = h(\mathcal{R}_i; x)$. $\square$

Subdivisions and local h-polynomials

In this subsection, we expand our discussion of subdivisions of polytopes given in Section 2.1 and define subdivisions of abstract polytopal complexes. Given a polytopal complex $\mathcal{C}$, a ***(topological) subdivision*** of $\mathcal{C}$ is a polytopal complex $\mathcal{C}'$ such that each face of $F \in \mathcal{C}$ is subdivided into a ball by faces of $\mathcal{C}'$ such that the boundary of this ball is a subdivision of the boundary of F. The subdivision is further called ***geometric*** if both $\mathcal{C}$ and $\mathcal{C}'$ admit ***geometric realizations***, G and G', respectively; that is to say, each face of $\mathcal{C}$ and $\mathcal{C}'$ is realized by a convex polytope in some real-Euclidean space such that G and G' both have the same underlying set of vertices and each face of G' is contained in a face of G. When referring to a subdivision $\mathcal{C}'$ of $\mathcal{C}$, we may instead refer to its associated inclusion map $\varphi : \mathcal{C}' \to \mathcal{C}$. While the main result of this chapter applies to general topological subdivisions, the applications of these results will pertain to some special families of subdivisions that are well-studied in the literature. These include the barycentric subdivision and the edgewise subdivision of a complex.

Interlacing polynomials

Two real-rooted polynomials $p, q \in \mathbb{R}[x]$ are said to ***interlace*** if there is a zero of p between each pair of zeros of q (counted with multiplicity) and vice versa. If p and q are interlacing, it follows that the ***Wronskian*** $W[p,q] = p'q - pq'$ is either nonpositive or nonnegative on all of $\mathbb{R}$. We will write $p \prec q$ if p and q are real-rooted, interlacing, and the Wronskian $W[p,q]$ is nonpositive on all of $\mathbb{R}$. We also assume that the zero polynomial 0 is real-rooted and that $0 \prec p$ and $p \prec 0$ for any real-rooted polynomial p.

Remark 3.2.1. Notice that if the signs of the leading coefficients of two real-rooted polynomials p and q are both positive, then $p \prec q$ if and only if

$$\cdots \leq \beta_2 \leq \alpha_2 \leq \beta_1 \leq \alpha_1,$$

where $\ldots, \beta_2, \beta_1$ and $\ldots, \alpha_2, \alpha_1$ are the zeros of p and q, respectively. In particular, when we work with combinatorial generating polynomials, $p \prec q$ is equivalent to p and q being real-rooted and interlacing.

A polynomial $p \in \mathbb{C}[x]$ is called ***stable*** if p is identically zero or if all of its zeros have nonpositive imaginary parts. The Hermite-Biehler Theorem relates the relation $p \prec q$ to stability in such a way that we can derive some useful tools for proving results about interlacing polynomials:

Theorem 3.2.2. *[57, Theorem 6.3.4] If $p, q \in \mathbb{R}[x]$ then $p \prec q$ if and only if $q + ip$ is stable.*

Remark 3.2.1 and Theorem 3.2.2 allow us to quickly derive some useful results.

Lemma 3.2.3. *If p and q are real-rooted polynomials in $\mathbb{R}[x]$ then*

1. *$p \prec \alpha p$ for all $\alpha \in \mathbb{R}$,*
2. *$p \prec q$ if and only if $\alpha p \prec \alpha q$ for any $\alpha \in \mathbb{R} \setminus \{0\}$,*
3. *$p \prec q$ if and only if $-q \prec p$, and*
4. *if p and q have positive leading coefficients then $p \prec q$ if and only if $q \prec xp$.*

We will also require the following proposition.

Proposition 3.2.4. *[21, Lemma 2.6] Let p be a real-rooted polynomial that is not identically zero. Then the following two sets are convex cones:*

$$\{q \in \mathbb{R}[x] : p \prec q\} \quad \textit{and} \quad \{q \in \mathbb{R}[x] : q \prec p\}.$$

It follows from Proposition 3.2.4 that we can sum a pair of interlacing polynomials to produce a new polynomial with only real-roots. More generally, we will work with recursions for which we need to sum several polynomials to produce a new real-rooted polynomial. Let $(p_i)_{i=0}^s = (p_0, \ldots, p_s)$ be a sequence of real-rooted polynomials. We say that the sequence of polynomials $(p_i)_{i=0}^s$ is an ***interlacing sequence*** if $p_i \prec p_j$ for all $1 \leq i \leq j \leq s$. Note that, by Proposition 3.2.4, any convex combination of polynomials in an interlacing sequence is real-rooted.

For a polynomial $p \in \mathbb{R}[x]$ of degree at most d, we let $\mathcal{I}_d(p) := x^d p(1/x)$. When d is the degree of p, then $\mathcal{I}_d(p)$ is the reciprocal of p. A polynomial $p = p_0 + p_1 x + \cdots + p_d x^d \in \mathbb{R}[x]$ is called ***symmetric*** with respect to degree d if $p_k = p_{d-k}$ for all $k = 0, \ldots, d$. If p is a degree d generating polynomial that is both real-rooted and symmetric with respect to d then $\mathcal{I}_d(p) \prec p$. However, non-symmetric polynomials also satisfy the latter condition, making it a natural generalization of symmetry for real-rooted polynomials. In [23], the authors characterized the condition $\mathcal{I}_d(p) \prec p$ in terms of the ***symmetric decomposition*** of p, which has been of recent interest [4, 5, 6, 10, 12, 23, 60]. In this chapter, the polynomials that we aim to show have only real zeros are known to be symmetric with respect to their degree. However, we will make use of the more general phenomenon $\mathcal{I}_d(p) \prec p$ in some of the proofs.

Ehrhart Theory

In Section 3.4, we will use some techniques from Ehrhart theory as described in Section 2.3. Here, we expand the notions described there to half-open polytopes. Let $\mathcal{H} = \{\langle a, y\rangle = b\}$ be a subset of the facet-defining hyperplanes of P and set

$$S_{\mathcal{H}} := \{z \in \mathbb{R}^n : \langle a, z\rangle = b \text{ for some } \langle a, y\rangle = b \in \mathcal{H}\}.$$

We then define the ***half-open polytope*** $P \setminus S_{\mathcal{H}}$, (which we note is not exactly a polytope, however, this terminology is standard [13]). We may also use the notation $P \setminus \mathcal{H}$ when we need to highlight the facet-defining hyperplanes that capture the points in $S_{\mathcal{H}}$. Analogously to the construction for lattice polytopes described in Section 2.3, the t^{th} dilate of $P \setminus S_{\mathcal{H}}$ is $t(P \setminus S_{\mathcal{H}}) := \{tp \in \mathbb{R}^n : p \in P \setminus S_{\mathcal{H}}\}$ and the Ehrhart function of $P \setminus S_{\mathcal{H}}$ is defined to be $\mathrm{ehr}_{P\setminus S_{\mathcal{H}}}(t) := |t(P \setminus S_{\mathcal{H}}) \cap \mathbb{Z}^n|$ for $t > 0$ (see [13, Section 5.3]). Notice that if $\mathcal{H} = \emptyset$ then $P = P \setminus S_{\mathcal{H}}$, and if $\mathcal{H}$ is the complete collection of facet-defining hyperplanes of P then $P \setminus S_{\mathcal{H}} =: P^\circ$, the relative interior of P. The ***Ehrhart series*** of the relative interior of P is defined as

$$\mathrm{E}hr(P^\circ; x) := \sum_{t>0} \mathrm{ehr}_{P^\circ}(t) x^t.$$

In the case that $\mathcal{H}$ is not the complete set of facet-defining hyperplanes of P, the ***Ehrhart series*** of $P \setminus S_{\mathcal{H}}$ is defined as

$$\mathrm{E}hr(P \setminus S_{\mathcal{H}}; x) := \sum_{t\geq 0} \mathrm{ehr}_{P\setminus S_{\mathcal{H}}}(t) x^t,$$

and the constant term is computed to be the Euler characteristic of $P \setminus S$ (see [13, Theorem 5.1.8]). When written in a closed rational form, the Ehrhart series of $P \setminus S_{\mathcal{H}}$, for any choice of $\mathcal{H}$, is

$$\mathrm{E}hr(P \setminus S_{\mathcal{H}}; x) = \frac{h_0^* + h_1^* x + \cdots + h_d^* x^d}{(1-x)^{d+1}},$$

and the polynomial $h^*(P \setminus S_{\mathcal{H}}; x) := h_0^* + h_1^* x + \cdots + h_d^* x^d$ is called the ***(Ehrhart)*** ***h^*-polynomial*** of $P \setminus S_{\mathcal{H}}$. It is well-known that $h^*(P \setminus S_{\mathcal{H}}; x)$ has only nonnegative integral coefficients (see for instance [48]). Since a lattice polytope P is a subset of $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$, it is natural to consider its subdivisions into polyhedral complexes whose 0-dimensional faces correspond to the lattice points in $P \cap \mathbb{Z}^n$. When such a (geometric) subdivision consists of only simplices, we call it a ***triangulation*** of P. When each simplex Δ in a triangulation of P has $h^*(\Delta; x) = 1$, we call it a ***unimodular triangulation*** of P. The following lemma is also well-known, and a proof appears in [12, Chapter 10].

Lemma 3.2.5. *Let $P \subset \mathbb{R}^n$ be a d-dimensional lattice polytope and let T be a unimodular triangulation of P. Then*

$$h^*(P; z) = h(T; x).$$

We will need a slight generalization of Lemma 3.2.5, whose proof is analogous to that of Lemma 3.2.5. However, we provide it below for the sake of completeness. In the following, given a facet-defining hyperplane H of P, we will let F_H denote the facet of P defined by H.

Lemma 3.2.6. *Let $P \subset \mathbb{R}^n$ be a d-dimensional lattice polytope with a unimodular triangulation T, and let $\mathcal{H}$ be a subset of its facet-defining hyperplanes. If the Euler characteristic of $P \setminus S_{\mathcal{H}}$ is 0 then*

$$h^*(P \setminus S_{\mathcal{H}}; x) = h(T \setminus (\cup_{H \in \mathcal{H}} T\big|_{F_H}); x).$$

Proof. To prove the claim, we first write $P \setminus S_{\mathcal{H}}$ as a disjoint union of the (nonempty) open simplices in the relative complex $T \setminus (\cup_{H \in S_{\mathcal{H}}} T\big|_{F_H})$:

$$P \setminus S_{\mathcal{H}} = \bigsqcup_{\Delta \in T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})} \Delta^\circ,$$

and we note that

$$\mathrm{ehr}_{P \setminus S_{\mathcal{H}}}(t) = \sum_{\Delta \in T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})} \mathrm{ehr}_{\Delta^\circ}(t).$$

It then follows that

$$
\begin{aligned}
\mathrm{E}hr(P \setminus S_{\mathcal{H}}, x) &= \sum_{n \geq 0} \mathrm{ehr}_{P \setminus S_{\mathcal{H}}}(t) x^t, \\
&= \mathrm{ehr}_{P \setminus S_{\mathcal{H}}}(0) + \sum_{n>0} \left(\sum_{\Delta \in T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})} \mathrm{ehr}_{\Delta^\circ}(t) \right) x^t, \\
&= \mathrm{ehr}_{P \setminus S_{\mathcal{H}}}(0) + \sum_{\Delta \in T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})} \left(\sum_{n>0} \mathrm{ehr}_{\Delta^\circ}(t) x^t \right), \\
&= \sum_{\Delta \in T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})} \mathrm{E}hr(\Delta^\circ; x),
\end{aligned}
$$

where the last equality follows from the definition of the Ehrhart series of the relative interior of a lattice polytope and the fact that $\mathrm{ehr}_{P \setminus S_{\mathcal{H}}}(0)$ is the Euler characteristic of $P \setminus S_{\mathcal{H}}$ (which we have assumed to be zero). Since each Δ° is the interior of a unimodular simplex, it follows by Ehrhart-Macdonald reciprocity [12, Theorem 4.1] that

$$
\mathrm{E}hr(\Delta^\circ; x) = \frac{x^{\dim(\Delta)+1}}{(1-x)^{\dim(\Delta)+1}}.
$$

Therefore, in analogous fashion to the proof of [12, Theorem 10.3], we have that

$$
\begin{aligned}
\mathrm{E}hr(P \setminus S_{\mathcal{H}}; x) &= \sum_{\Delta \in T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})} \mathrm{E}hr(\Delta^\circ; x), \\
&= \sum_{\Delta \in T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})} \frac{x^{\dim(\Delta)+1}}{(1-x)^{\dim(\Delta)+1}}, \\
&= \sum_{k=-1}^{d} f_k(T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})) \left(\frac{x}{1-x} \right)^{k+1}, \\
&= \frac{\sum_{k=-1}^{d} f_k(T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})) x^{k+1} (1-x)^{d-k}}{(1-x)^{d+1}}, \\
&= \frac{\sum_{k=0}^{d+1} f_{k-1}(T \setminus (\cup_{H \in S_{\mathcal{H}}} T|_{F_H})) x^{k} (1-x)^{d-k+1}}{(1-x)^{d+1}}, \\
&= \frac{h(T \setminus (\cup_{H \in S_{\mathcal{H}}} T\big|_{F_H}); x)}{(1-x)^{d+1}},
\end{aligned}
$$

which completes the proof. □

To prove the desired results in Section 3.4, we will use well-chosen sets $\mathcal{H}$ and $S_{\mathcal{H}}$. Let $q \in \mathbb{R}^n$, and let $P \subset \mathbb{R}^n$ a d-dimensional convex polytope. A point $p \in P$ is called

visible from q if the open line segment (q, p) in $\mathbb{R}^n$ does not meet the interior of P. Let $B \subset \partial P$ denote the collection of all points visible from q, and set $D := \overline{\partial P \setminus B}$, the closure of $\partial P \setminus B$. Given a facet F of P, the point q is said to be ***beyond*** F if $q \notin T_F(P)$, the tangent cone of F in P. It follows that q is beyond F if and only if the closed line segment $[q, p]$ satisfies $[q, p] \cap P = \{p\}$ for all $p \in F$ [13, Section 3.7]. Otherwise, the point q is said to be ***beneath*** F. Hence, B consists of all points in ∂P that lie in a facet which q is beyond; that is, $P \setminus B = P \setminus \mathcal{H}_B$, where $\mathcal{H}_B$ denotes the collection of facets which q is beyond. Similarly, D consists of all points in ∂P that lie in a facet which q is beneath; that is, $P \setminus D = P \setminus \mathcal{H}_D$, where $\mathcal{H}_D$ denotes the collection of facets which q is beneath. Stanley observed in [61, Proposition 8.2], that $\mathrm{ehr}_{P\setminus B}(t)$ is a polynomial and that classical Ehrhart-Macdonald reciprocity, Theorem 2.3.3, [53, Theorem 4.6], [37] can be extended to

$$(-1)^d \mathrm{ehr}_{P\setminus D}(t) = \mathrm{ehr}_{P\setminus B}(-t). \tag{3.1}$$

In [38, 36], Ehrhart referred to the half-open polytopes $P \setminus B$ and $P \setminus D$ as ***reciprocal domains***, since they satisfy the reciprocity law in equation (3.1). The notion of reciprocal domains, and a natural generalization thereof, will be important to us in the coming sections as we derive real-rootedness results via shellings of polytopal complexes. A first lemma that will help us along the way is the following, which translates equation (3.1) into a statement about h^*-polynomials. To prove the the lemma, one can use the equation (3.1), together with the result of [12, Exercise 4.7], to compare the Ehrhart series and thus the h^*-polynomials of the two complexes, recalling that for any polynomial p of degree at most $d+1$, $\mathcal{I}_{d+1}(p) = x^{d+1}p(1/x)$.

Lemma 3.2.7. *Let $P \subset \mathbb{R}^n$ be a d-dimensional lattice polytope, and let $q \in \mathbb{R}^n$. Let B denote the points in ∂P that are visible from q, and set $D := \overline{\partial P \setminus B}$. If B and D are both nonempty then*

$$h^*(P \setminus D; x) = \mathcal{I}_{d+1} h^*(P \setminus B; x).$$

3.3 Subdivisions of Shellable Complexes and Stable Shellings

In this section, we provide a sufficient condition for the h-polynomial of a subdivision of a polytopal complex to have only real zeros. As part of this condition, we will require the complex to be shellable.

Definition 3.3.1. Let $\mathcal{C}$ be a pure d-dimensional polytopal complex. A ***shelling*** of $\mathcal{C}$ is a linear ordering $(F_1, F_2, \ldots, F_s)$ of the facets of $\mathcal{C}$ such that either $\mathcal{C}$ is zero-dimensional (and thus the facets are points), or it satisfies the following two conditions:

1. The boundary complex $\mathcal{C}(\partial F_1)$ of the first facet in the linear ordering has a shelling, and

2. For $j \in [s]$, the intersection of the facet F_j with the union of the previous facets is nonempty and it is the beginning segment of a shelling of the $(d-1)$-dimensional boundary complex of F_j; that is,

$$F_j \cap \bigcup_{i=1}^{j-1} F_i = G_1 \cup G_2 \cup \cdots G_r$$

for some shelling $(G_1, \ldots, G_r, \ldots, G_t)$ of the complex $\mathcal{C}(\partial F_j)$ and $r \in [t]$.

A polytopal complex is ***shellable*** if it is pure and admits a shelling.

A shelling of a polytopal complex presents a natural way to decompose the complex into disjoint, relative polytopal complexes. Given a shelling order $(F_1, \ldots, F_s)$ of a polytopal complex $\mathcal{C}$, and a subdivision $\varphi : \mathcal{C}' \to \mathcal{C}$, we can let

$$\mathcal{R}_i := \mathcal{C}'\big|_{F_i} \setminus \left(\bigcup_{k=1}^{i-1} \mathcal{C}'\big|_{F_k} \right),$$

to produce the decomposition of $\mathcal{C}'$ into disjoint relative complexes

$$\mathcal{C}' = \bigsqcup_{i=1}^{s} \mathcal{R}_i, \tag{3.2}$$

with respect to the shelling $(F_1, \ldots, F_s)$ of $\mathcal{C}$. For a fixed shelling $(F_1, \ldots, F_s)$ of a polytopal complex $\mathcal{C}$ and subdivision $\varphi : \mathcal{C}' \to \mathcal{C}$, for $i \in [s]$, we will call the relative complex $\mathcal{R}_i$ the ***relative complex associated to*** F_i ***by*** $(F_1, \ldots, F_s)$ ***and*** φ.

The recursive structure of shelling orders of polytopal complexes, and the additive nature of the h-polynomials of their associated relative simplicial complexes (see Lemma 3.2.1) pairs nicely with the properties of interlacing polynomials discussed in Section 3.2. In particular, Lemma 3.2.1 allows us to combine the fact that a shellable complex $\mathcal{C}$ always admits a decomposition as in equation (3.2) with the facts about interlacing sequences collected in Subsection 3.2. We can then prove a theorem that directly relates the recursive nature of shelling orders to the recursive nature of interlacing sequences of polynomials.

Theorem 3.3.1. *Let $\mathcal{C}$ be a shellable polytopal complex with shelling $(F_1, \ldots, F_s)$ and subdivision $\varphi : \mathcal{C}' \to \mathcal{C}$. If $(h(\mathcal{R}_{\sigma(i)}; x))_{i=1}^{s}$ is an interlacing sequence for some $\sigma \in \mathfrak{S}_s$, the symmetric group on $[s]$, then $h(\mathcal{C}'; x)$ is real-rooted.*

Proof. Notice first that since $\mathcal{C}$ is a shellable polytopal complex of dimension d, and since $\mathcal{C}'$ is a (topological) subdivision of $\mathcal{C}$, then each subcomplex $\mathcal{C}'\big|_{F_i} := \varphi^{-1}(2^F)$ is also d-dimensional. Moreover, since $\mathcal{R}_i$ only removes faces of dimension strictly less than d, then each $\mathcal{R}_i$ is a d-dimensional relative simplicial complex. So, by Lemma 3.2.1, we have that

$$h(\mathcal{C}'; x) = \sum_{i=1}^{s} h(\mathcal{R}_i; x).$$

Supposing now that there exists $\sigma \in \mathfrak{S}_s$ such that $(h(\mathcal{R}_{\sigma(i)}; x))_{i=1}^s$ is an interlacing sequence, it then follows from Proposition 3.2.4 that $h(\Omega; x)$ is real-rooted. □

While the proof of Theorem 3.3.1 is straightforward to derive (once we have carefully defined and identified all of the necessary ingredients), its applications are much more interesting. The key to applying Theorem 3.3.1 lies in our ability to identify a shelling order $(F_1, \ldots, F_s)$ of the polytopal complex $\mathcal{C}$ such that the relative complexes $\mathcal{R}_i$ constructed for each facet F_i with respect to the subdivision $\varphi : \mathcal{C}' \to \mathcal{C}$ will all have real-rooted and interlacing h-polynomials $h(\mathcal{R}_i; x)$. While the real-rootedness and interlacing conditions are inherently tied to the choice of subdivision φ, we will see in Section 3.4 that for many complexes, the type of shelling to which Theorem 3.3.1 applies is independent of the choice of φ when φ is chosen from amongst the most commonly studied subdivisions. Here, the 'most commonly studied subdivisions' refers to ***uniform subdivisions*** [2], such as the barycentric and edgewise subdivisions, which will be the focus of our results in Section 3.4. Indeed, when the geometry of a given polytopal complex is such that the barycentric subdivision of the complex has a real-rooted h-polynomial, it is often the case that the edgewise subdivision admits the same property for the given complex. This phenomenon was noted, and formalized, in the case of simplicial complexes in the recent preprint [2], where the author studied the class of ***$\mathcal{F}$-uniform triangulations***. In this chapter, we make a similar observation for more general polytopal complexes, and apply this reasoning to shelling orders. Namely, we are interested in the class of shellings $(F_1, \ldots, F_s)$ of a polytopal complex $\mathcal{C}$ for which the associated relative complexes $\mathcal{R}_1, \ldots, \mathcal{R}_s$ with respect to $(F_1, \ldots, F_s)$ and uniform subdivisions, such as the barycentric subdivision and edgewise subdivision, fulfill the hypotheses of Theorem 3.3.1. As it turns out, such shellings arise when we insist that the relative complexes $\mathcal{R}_i$ are given by a generalization of Ehrhart's reciprocal domains for convex lattice polytopes (see the related discusion in Section 3.2).

Stable shellings

The family of shellings to which we will apply Theorem 3.3.1 will be called ***stable shellings***. To define them we first need to generalize the reciprocal domains for convex embeddings of rational polytopes to general (abstract) polytopes. Let $\mathcal{P} = ([n], <_{\mathcal{P}})$ be a partially ordered set on elements $[n]$ with partial order $<_{\mathcal{P}}$. If $\mathcal{P}$ has a unique minimal element, which we will denote by $\hat{0}$, then we can define its set of ***atoms*** to be all elements $i \in [n]$ such that $\hat{0} <_{\mathcal{P}} i$ and there is no $j \in [n]$ such that $\hat{0} <_{\mathcal{P}} j <_{\mathcal{P}} i$. Given a poset $\mathcal{P}$ with a unique minimal element, we will denote its set of atoms by $A(\mathcal{P})$. The ***dual poset*** of $\mathcal{P}$ is the poset $\mathcal{P}^*$ on elements $[n]$ with partial order $<_{\mathcal{P}^*}$ in which $i <_{\mathcal{P}^*} j$ if and only if $j <_{\mathcal{P}} i$. Given two elements $i, j \in [n]$, the ***closed interval*** between i and j in $\mathcal{P}$ is the set $[i, j] := \{k \in [n] : i \leq_{\mathcal{P}} k \leq_{\mathcal{P}} j\}$. Note that we can view the closed interval $[i, j]$ as a subposet of $\mathcal{P}$ by allowing it to inherit the partial order $<_{\mathcal{P}}$ from $\mathcal{P}$.

Let P be a d-dimensional polytope with face lattice $L(P)$; that is, $L(P)$ is the partially ordered set whose elements are the faces of P and for which the partial order is given by inclusion. Since $L(P)$ is a ***lattice*** (see [62, Chapter 3.3]), it follows that $L(P)$ has a unique minimal and maximal element, corresponding to the faces $\emptyset$ and P of P. Let $\mathcal{C}(P)$ denote the polytopal complex consisting of all faces of P. Given a face F of P we call the pair of relative complexes

$$\mathcal{C}(P) \setminus \mathcal{C}(A([F,P]^*)) \qquad \text{and} \qquad \mathcal{C}(P) \setminus \mathcal{C}(A(L(P)^*) \setminus A([F,P]^*))$$

the ***reciprocal domains*** associated to F in P. We call a relative complex $\mathcal{R}$ ***stable*** if it is isomorphic to one of the reciprocal domains associated to a face F in some polytope P. Using this terminology, we can now define the family of shellings, to which we will apply Theorem 3.3.1.

Definition 3.3.2. Let $\mathcal{C}$ be a polytopal complex. A shelling $(F_1, \dots, F_s)$ of $\mathcal{C}$ is ***stable*** if for all $i \in [s]$ the relative complex $\mathcal{R}_i$ associated to F_i by the shelling $(F_1, \dots, F_s)$ and the trivial subdivision $\varphi : \mathcal{C} \to \mathcal{C}$ is stable.

The set of stable shellings of a (shellable) simplicial complex is, in fact, the set of all shellings of the complex.

Proposition 3.3.2. *Let $\mathcal{C}$ be a shellable simplicial complex. Then any shelling $(F_1, \dots, F_s)$ of $\mathcal{C}$ is stable.*

Proof. Suppose that $\mathcal{C}$ is d-dimensional, let $\varphi : \mathcal{C} \to \mathcal{C}$ denote the trivial subdivision of $\mathcal{C}$, and fix a facet F_i of $\mathcal{C}$. If $i = 1$, then since $(F_1, \dots, F_s)$ is a shelling, it follows that the relative complex associated to F_i by the shelling $(F_1, \dots, F_s)$ and φ is $\mathcal{C}(F_i)$. Fix the face $F = F_i$ of the simplex F_i. Then the reciprocal domains associated to F in F_i are $\mathcal{C}(F_i)$ and $\mathcal{C}(F_i) \setminus \mathcal{C}(A(L(F_i)^*))$. Hence, $\mathcal{C}(F_i)$ is a stable relative complex.

Suppose now that $i > 1$. Since $(F_1, \dots, F_s)$ is a shelling order, it follows that $F_i \cap (F_1 \cup \cdots \cup F_{i-1}) = G_1 \cup \cdots \cup G_r$ is the initial segment of a shelling order $(G_1, \dots, G_r, \dots, G_d)$ of the boundary complex $\mathcal{C}(\partial F_i)$. Hence, $G_1, \dots, G_r$ are facets of F_i. Since F_i is a simplex, each face of F_i corresponds to an intersection of a subset of its facets $G_1, \dots, G_d$. Consider the face F of F_i given by $G_1 \cap \cdots \cap G_r$. It follows that the reciprocal domains associated to F in F_i are

$$\mathcal{C}(F_i) \setminus \mathcal{C}(A([F,F_i]^*)) = \mathcal{C}(F_i) \setminus \mathcal{C}(G_1 \cup \cdots \cup G_r),$$

and

$$\mathcal{C}(F_i) \setminus \mathcal{C}(A(L(F_i)^*) \setminus A([F,F_i]^*)) = \mathcal{C}(F_i) \setminus \mathcal{C}(G_{r+1} \cup \cdots \cup G_d),$$

Hence, it follows that the complex $\mathcal{R}_i = \mathcal{C}(F_i) \setminus \mathcal{C}(G_1 \cup \cdots \cup G_r)$ is stable, which completes the proof. ☐

Hence, any shelling of a simplicial complex is stable. As we will see in Section 3.4, this observation coincides with the recent results on $\mathcal{F}$-uniform triangulations [2], in the sense that we will be able to apply Theorem 3.3.1 to any shelling of a simplicial complex with respect to uniform subdivisions like the barycentric and edgewise subdivisions.

Stable shellings of cubical complexes

In general, it is not the case that all shellings of a polytopal complex are stable. For example, in the case of cubical complexes, stable shellings become a proper subclass of the class of all shellings. To see this, we can characterize which relative complexes of the combinatorial d-cube are stable.

In the following, we let $\square_d$ denote the (abstract) d-dimensional cube. When we consider a standard geometric realization of $\square_d$, such as the cube $[-1,1]^d \in \mathbb{R}^d$, we assign each facet F of $\square_d$ to a facet-defining hyperplane $x_i = \pm 1$ of $[-1,1]^d$. Given a facet F of $\square_d$ we will say that F is ***opposite (or opposing)*** the facet G of $\square_d$ whenever F is identified with $x_i = 1$ and G is identified with $x_i = -1$ (or vice versa), for some $i \in [d]$. In this case, we call the pair of facets F, G an ***opposing pair***. Given the embedding $[-1,1]^d$ we can also define the half-open polytopes

$$[-1,1]^d_\ell := [-1,1]^d \setminus \{x_d = 1, \ldots, x_{d+1-\ell} = 1\} \tag{3.3}$$

for $0 \leq \ell \leq d$, and

$$[-1,1]^d_\ell := [-1,1]^d \setminus \{x_d = 1, \ldots, x_1 = 1\} \cup \{x_d = -1, \ldots, x_{2d+1-\ell} = 1\} \tag{3.4}$$

for $d+1 \leq \ell \leq 2d$. The following lemma gives two characterizations of stable relative subcomplexes of the cube, one in terms of its possible geometric realizations and the other in terms of opposing pairs of facets.

Lemma 3.3.3. *Let $\square_d$ be the d-dimensional (combinatorial) cube, and let $\mathcal{D}$ be a subcomplex of $\mathcal{C}(\square_d)$. Then the following are equivalent:*

1. *The relative complex $\mathcal{C}(\square_d) \setminus \mathcal{D}$ is stable,*
2. *$\mathcal{C}(\square_d) \setminus \mathcal{D}$ has geometric realization $[-1,1]^d_\ell$ for some $0 \leq \ell \leq 2d$.*
3. *The set of codimension 1 faces of $\mathcal{C}(\square_d) \setminus \mathcal{D}$ or the set of facets of $\mathcal{D}$ does not contain an opposing pair.*

Proof. We first prove the equivalence of (1) and (2). Suppose that $\mathcal{C}(\square_d) \setminus \mathcal{D}$ is stable. Then $\mathcal{C}(\square_d) \setminus \mathcal{D}$ is isomorphic to a reciprocal domain $\mathcal{C}(\square_d) \setminus \mathcal{C}(A([F, \square_d]^*))$ or $\mathcal{C}(\square_d) \setminus \mathcal{C}(A(L(\square_d)^*) \setminus A([F, \square_d]^*))$ for some face F of the cube $\square_d$. Let $F_1, \ldots, F_{2d}$ denote the set of facets of $\square_d$. Given the geometric realization $[-1,1]^d$ of $\square_d$, without loss of generality, we can assume that the facet F_i is identified with the facet-defining hyperplane $x_i = 1$ and that the facet F_{d+i} is identified with the facet-defining hyperplane $x_i = -1$ of $[-1,1]^d$, for all $i \in [d]$. Given this identification, the face F of $\square_d$ corresponds to the intersection of the facet-defining hyperplanes identified with the atoms in the closed interval $[F, \square_d]^*$. Since the geometric realization of any nonempty face F cannot lie in both $x_i = 1$ and $x_i = -1$ for any $i \in [d]$, it follows that the hyperplanes corresponding to the atoms in $[F, \square_d]^*$ are of the form

$$\{x_{i_1} = 1, \ldots, x_{i_s} = 1\} \cup \{x_{j_1} = -1, \ldots, x_{j_t} = -1\},$$

where the sets $\{i_1,\dots,i_s\},\{j_1,\dots,j_t\}\subset[d]$ are disjoint. Hence, by reflecting over the hyperplanes $x_{j_1}=0,\dots,x_{j_t}=0$, we can instead identify the face F with the intersection of the hyperplanes

$$\{x_{i_1}=1,\dots,x_{i_s}=1\}\cup\{x_{j_1}=1,\dots,x_{j_t}=1\}.$$

Finally, by a simple permutation of coordinates, we can identify F with the intersection of the hyperplanes

$$\{x_d=1,\dots,x_{d+1-\ell}=1\},$$

where $\ell=|\{x_{i_1}=1,\dots,x_{i_s}=1\}\cup\{x_{j_1}=1,\dots,x_{j_t}=1\}|$. Hence, a geometric realization of $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ is given either by $[-1,1]^d_\ell$ or $[-1,1]^d_{2d-\ell}$.

Conversely, suppose that $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ has geometric realization $[-1,1]^d_\ell$ for some $0\leq\ell\leq 2d$. If $0\leq\ell\leq d$, then let F be the face of $\Box_d$ whose geometric realization is the intersection of the hyperplanes $x_d=1,\dots,x_{d+1-\ell}=1$. Then $\mathcal{C}(\Box_d)\setminus\mathcal{D}=\mathcal{C}(\Box_d)\setminus\mathcal{C}(A([F,\Box_d]^*))$. If $d<\ell\leq 2d$ then let F be the face of $\Box_d$ whose geometric realization is the intersection of the hyperplanes $x_1=-1,\dots,x_{2d-\ell}=-1$. Then $\mathcal{C}(\Box_d)\setminus\mathcal{D}=\mathcal{C}(\Box_d)\setminus\mathcal{C}(A(L(\Box_d)^*)\setminus A([F,\Box_d]^*))$.

We now prove the equivalence of (2) and (3). Suppose first that $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ has geometric realization $[-1,1]^d_\ell$ for some $0\leq\ell\leq 2d$. Then $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ is isomorphic to one of two types of half-open cubes described in equations (3.3) and (3.4). Suppose that $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ has geometric realization corresponding to a half-open cube given in equation (3.3). Then the subset of facets in $\mathcal{D}$ are, without loss of generality, given by a subset of the facets $x_1=1,\dots,x_d=1$ of $[-1,1]^d$. Hence, the set of facets of $\mathcal{D}$ cannot contain an opposing pair, as this set contains no facet with defining hyperplane $x_i=-1$. In that case that $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ has geometric realization corresponding to a half-open cube given in equation(3.4), the codimension 1 faces of $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ correspond to a subset of the facet-defining hyperplanes $x_1=-1,\dots,x_d=-1$ of $[-1,1]^d$, and therefore cannot contain an opposing pair.

Conversely, suppose that $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ is such that the set of codimension 1 faces of $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ or the set of facets of $\mathcal{D}$ does not contain an opposing pair. In the former of these two cases, the set of codimension 1 faces of $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ is given by a subset of facets of $[-1,1]^d$ of the form

$$\{x_{i_1}=1,\dots,x_{i_s}=1\}\cup\{x_{j_1}=-1,\dots,x_{j_t}=-1\},$$

for two disjoint subsets, $\{i_1,\dots,i_s\}$ and $\{j_1,\dots,j_t\}$, of $[d]$. As in the proof of equivalence of (1) and (2), by reflecting through the hyperplanes $x_{j_1}=0,\dots,x_{j_t}=0$, we obtain that $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ is isomorphic to

$$\mathcal{H}:=[-1,1]^d\setminus\{x_{i_1}=1,\dots,x_{i_s}=1\}\cup\{x_{j_1}=1,\dots,x_{j_t}=1\}.$$

By applying the correct permutation matrix to $\mathbb{R}^n$, we recover that $\mathcal{H}$ is unimodularly equivalent to $[-1,1]^d_\ell$ where $\ell=|\{x_{i_1}=1,\dots,x_{i_s}=1\}\cup\{x_{j_1}=1,\dots,x_{j_t}=1\}|\leq d$. Hence, $\mathcal{C}(\Box_d)\setminus\mathcal{D}$ has geometric realization $[-1,1]^d_\ell$ for some $0\leq\ell\leq d$.

In the second case, we assume that the set of facets of $\mathcal{D}$ does not contain an opposing pair. By applying a similar argument, we see that the set of facets of $\mathcal{D}$ can be identified with a subset of the hyperplanes $x_1=-1,\dots,x_d=-1$, which we assume without loss of generality is $x_1 = -1, \dots, x_\ell = -1$ for some $\ell \le d$. Following the same steps as in the previous case, we find that $\mathcal{C}(\square_d) \setminus \mathcal{D}$ is isomorphic to $[-1,1]^d_{d+\ell}$, which completes the proof. $\square$

Lemma 3.3.3 characterizes the possible relative complexes that can be associated to the facets of a shellable cubical complex by a stable shelling, and hence, it characterizes the stable shellings of cubical complexes. Furthermore, Lemma 3.3.3 shows that, in the case of cubical complexes, any linear ordering of the facets of a complex for which all associated relative complexes are stable is a shelling order.

Lemma 3.3.4. *Let $\mathcal{C}$ be a pure d-dimensional cubical complex and let $(F_1, \dots, F_s)$ be a linear ordering of the facets of $\mathcal{C}$ such that the relative complex*

$$\mathcal{R}_i := F_i \setminus (F_1 \cup \cdots \cup F_{i-1})$$

associated to F_i by $(F_1, \dots, F_s)$ is stable for all $i \in [s]$. Then $(F_1, \dots, F_s)$ is a shelling order.

Proof. A well-known result states that a set of facets of a d-dimensional cube $\square_d$ forms a shellable subcomplex of the boundary complex of $\square_d$ if and only if either it contains no facets of $\square_d$, contains all facets of $\square_d$, or if it contains at least one facet such that its opposing facet is not in the complex (see, for instance, [76, Exercise 8.1(i)]). Moreover, it follows from this result that the boundary complex of the d-dimensional cube is ***extendably shellable***; meaning that any partial shelling of the complex can be continued to a complete shelling. Hence, it suffices to prove that $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ determines a shellable subcomplex of the boundary of the d-dimensional cube for all $i \in [s]$.

Notice first that F_1 is a d-dimensional cube, and hence the boundary complex $\mathcal{C}(\partial F_1)$ is shellable. So let $i > 1$ and consider the complex determined by $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$. This complex has facets given by the set of facets of F_i that are not codimension 1 faces of $\mathcal{R}_i$. Since $\mathcal{R}_i$ is a stable complex, by Lemma 3.3.3, it follows that either the set of codimension 1 faces of $\mathcal{R}_i$ or the set of facets of $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ does not contain an opposing pair. Moreover, all facets of the cube F_i are either codimension 1 faces of $\mathcal{R}_i$ or facets of $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$.

Suppose first that $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ does not contain an opposing pair. Then either the set of facets of $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ is empty, or $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ contains a facet of F_i but not its opposite. In either case, according to the result cited above [76, Exercise 8.1(i)], the set of facets of $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ form a shellable subcomplex of $\mathcal{C}(\partial F_i)$. Suppose, on the other hand, that $\mathcal{R}_i$ does not contain an opposing pair. Then either $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ is the entire complex $\mathcal{C}(\partial F_i)$ (i.e., the boundary complex of a d-dimensional cube), or $\mathcal{R}_i$ contains a facet of F_i but not its opposite. In the latter case, it follows that $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ contains a facet of F_i but not its opposite. Hence, in a similar fashion to the previous case, $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ forms a shellable subcomplex of $\mathcal{C}(\partial F_i)$, which completes the proof. $\square$

In two dimensions, any relative subcomplex of the cube is stable. However, already in three dimensions there exist relative complexes that are not stable, and hence are forbidden

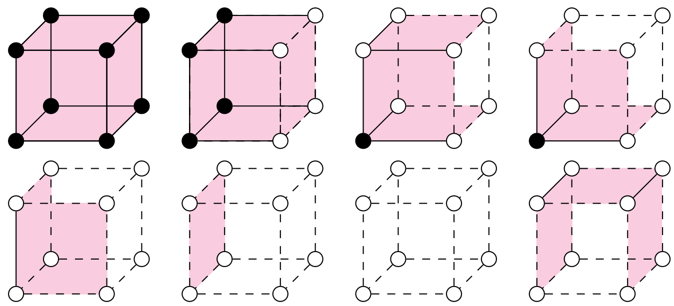


Figure 3.1: The eight possible relative complexes $\mathcal{R}_i$ for a facet F_i in a shelling order $(F_1, \ldots, F_s)$ if F_i is a 3-dimensional cube. All of the complexes are stable, excluding the bottom-right complex.

from being associated to a facet of a cubical complex in any of its stable shellings. For example, Figure 3.1 depicts the eight possible relative complexes of a three cube that may arise as the relative complex associated to a 3-dimensional facet of a cubical complex with respect to an arbitrary shelling. By Lemma 3.3.3, we see that only the first seven are stable. The eighth, with its table-top shape, is such that both the codimension 1 faces of $\mathcal{C}(\square_3) \setminus \mathcal{D}$ (depicted in blue) and the facets of $\mathcal{D}$ (depicted by their absence) contain an opposing pair. Hence, this relative complex cannot be included in any stable shelling of a 3-dimensional cubical complex.

Example 3.3.1 (The stable subcomplexes of the 3-cube)**.** Recall that our motivation for excluding certain relative complexes is that, upon subdivision, we need that all relative complexes associated to our shelling order have real-rooted and interlacing h-polynomials (see Theorem 3.3.1). Suppose we fix a 3-dimensional cubical complex $\mathcal{C}$ and consider its barycentric subdivision s$d(\mathcal{C})$ (as defined in subsection 3.4). We will see in Section 3.4 that any of the stable relative complexes depicted in Figure 3.1 have real-rooted and interlacing h-polynomials with respect to this subdivision. On the other hand, the barycentric subdivision of the table-top shaped relative complex in Figure 3.1 will have h-polynomial

$$6x + 36x^2 + 6x^3.$$

which is real-rooted, but it does not interlace all the h-polynomials of the other seven complexes. For instance, the complex just above it in Figure 3.1 has h-polynomial $8x+32x^2+8x^3$, and these two polynomials do not interlace. Hence, the stable relative subcomplexes of the 3-cube are the natural maximal subset of relative complexes to which we can apply Theorem 3.3.1 with respect to the barycentric subdivision. As we will see in Section 3.4, this will also be true for the other well-studied uniform subdivisions.

Given that not all relative subcomplexes of the cube are stable, it is then natural to ask which cubical complexes admit stable shellings. In the remainder of this section, we give some first examples of cubical complexes admitting stable shellings, and we provide an example of a shelling of a cubical complex that is not stable. These results will be used in Section 3.4, when we apply this theory to answer some open questions on the real-rootedness of h-polynomials of barycentric subdivisions of cubical complexes.

Example 3.3.2 (The boundary of the d-cube)**.** Let $\partial\Box_d$ denote the boundary of the d-dimensional cube $\Box_d$, and consider its geometric realization as the boundary of $[0,1]^d \subset \mathbb{R}^d$. Let F_i denote the facet of $\Box_d$ corresponding to the facet of $[0,1]^d$ defined by the hyperplane $x_i = 0$ for $i \in [d]$, and let F_{d+i} denote the facet corresponding to that of $[0,1]^d$ defined by $x_i = 1$ for $i \in [d]$. Hence F_i, F_{d+i} is an opposing pair for all $i \in [d]$. We claim that the linear ordering $(F_1, \ldots, F_d, \ldots, F_{2d})$ is a stable shelling of $\mathcal{C}(\partial\Box_d)$. By Lemma 3.3.4, it suffices to show that, for all $i \in \{1, \ldots, 2d\}$, the relative complex $\mathcal{R}_i$ associated to F_i by $(F_1, \ldots, F_d, \ldots, F_{2d})$ is stable.

Fix $i \in [d]$ and consider $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$. This intersection consists of the facets of $[0,1]^{d-1} \simeq F_i$ defined by $x_1 = 0, x_2 = 0, \ldots, x_{i-1} = 0$. Hence, $F_i \cap (F_1 \cup \cdots \cup F_{i-1})$ determines a subcomplex of $\mathcal{C}(\partial\Box_{d-1})$ whose set of facets does not contain an opposing pair. It follows from Lemma 3.3.3 that $\mathcal{R}_i = \mathcal{C}(F_i) \setminus \mathcal{C}(F_i \cap (F_1 \cup \cdots \cup F_{i-1}))$ is stable. Now consider a facet F_{d+i} for $i \in [d]$, and the subcomplex

$$G = F_{d+i} \cap (F_1 \cup \cdots \cup F_d \cup \cdots \cup F_{d+i-1})$$

of its boundary complex $\mathcal{C}(\partial F_{d+i})$. It follows that the codimension 1 faces of $\mathcal{R}_{d+i} = \mathcal{C}(F_{d+i}) \setminus \mathcal{C}(G)$ are determined by the hyperplanes $x_{i+1} = 1, \ldots, x_d = 1$. Hence, the set of codimension 1 faces of $\mathcal{R}_{d+i}$ does not contain an opposing pair of facets of F_{d+i}. By Lemma 3.3.3, $\mathcal{R}_{d+i}$ is stable, and we conclude that $(F_1, \ldots, F_d, \ldots, F_{2d})$ is a stable shelling of $\mathcal{C}(\partial\Box_d)$.

Example 3.3.3 (Piles of cubes)**.** For integers $a_1, \ldots, a_d \in \mathbb{Z}_{\geq 0}$, the ***pile of cubes*** $\mathcal{P}_d(a_1, \ldots, a_d)$ is the polytopal complex formed by all unit cubes with integer vertices in the d-dimensional box

$$B(a_1, \ldots, a_d) := \{x \in \mathbb{R}^d : 0 \leq x_i \leq a_i, i \in [d]\}.$$

Each facet of $B(a_1, \ldots, a_d)$ is uniquely associated to an integer point in the half-open box

$$B^\circ(a_1, \ldots, a_d) := \{x \in \mathbb{R}^d : 0 \leq x_i < a_i, i \in [d]\}.$$

In particular, the integer point $(z_1, \ldots, z_d) \in \mathbb{Z}^d \cap B^\circ(a_1, \ldots, a_d)$ indexes the unit cube whose lexicographically smallest vertex is $(z_1, \ldots, z_d)$. (For two points $a, b \in \mathbb{Z}^d_{\geq 0}$, $a >_{lex} b$ in the lexicographic ordering $>_{lex}$ whenever the left-most entry in $a - b$ is positive.) The lexicographic ordering induces a total (linear) ordering of the points in $B^\circ(a_1, \ldots, a_d)$. Consider the linear ordering of the facets of $\mathcal{P}_d(a_1, \ldots, a_d)$ induced by the lexicographic order on the integer points in $B^\circ(a_1, \ldots, a_d)$ indexing the facets (from smallest-to-largest). By [76, Example 8.2], this is a shelling order for $\mathcal{P}_d(a_1, \ldots, a_d)$. To see that this shelling order is stable, consider a facet $F_{(z_1,\ldots,z_d)}$ of $\mathcal{P}_d(a_1, \ldots, a_d)$, and suppose its associated relative

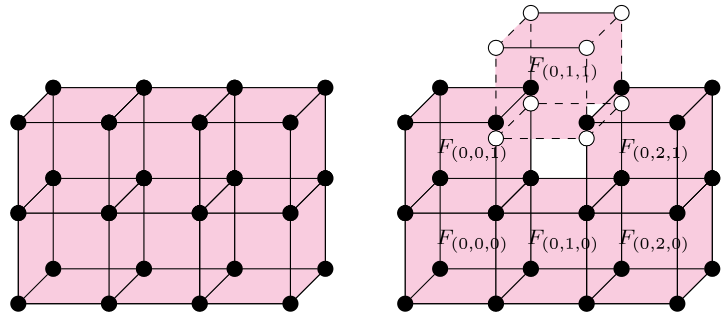


Figure 3.2: The pile of cubes $P_3(1,3,2)$ (on the left) and the final step in the shelling from Example 3.3.4(on the right). Since the last relative complex is not stable, then neither is this shelling.

complex $\mathcal{R}_{(z_1,\ldots,z_d)}$ does not contain its facet lying in the hyperplane $x_i = z_i + 1$ for some $i \in [d]$. It follows that $F_{(z_1,\ldots,z_i+1,\ldots,z_d)}$ was before $F_{(z_1,\ldots,z_d)}$ in the shelling order. However, $(z_1,\ldots,z_i+1,\ldots,z_d) >_{\mathrm{lex}} (z_1,\ldots,z_i,\ldots,z_d)$, so this cannot not be the case. Hence, the set of facets of the complex

$$\mathcal{C}\left(F_{(z_1,\ldots,z_d)} \cap \bigcup_{(z_1,\ldots,z_d)>_{lex}(y_1,\ldots,y_d)} F_{(y_1,\ldots,y_d)}\right)$$

does not contain an opposing pair because it does not contain any of the facets of $F_{(z_1,\ldots,z_d)}$ lying in a hyperplane $x_i = z_i + 1$ for any $i \in [d]$. It follows from Lemma 3.3.3 that $\mathcal{R}_{(z_1,\ldots,z_d)}$ is stable, and thus the lexicographic shelling of $\mathcal{P}_d(a_1,\ldots,a_d)$ is stable.

Example 3.3.4 (A non-stable shelling)**.** While the pile of cubes $\mathcal{P}_d(a_1,\ldots,a_d)$ always admits a stable shelling (as seen in Example 3.3.3), there exist piles of cubes with shellings that are not stable. For instance, the pile of cubes $P_3(1,3,2)$, depicted in Figure 3.2, has six facets $F_{(0,0,0)}, F_{(0,1,0)}, F_{(0,2,0)}, F_{(0,0,1)}, F_{(0,1,1)},$ and $F_{(0,2,1)}$. The linear ordering of these facets $(F_{(0,0,0)}, F_{(0,1,0)}, F_{(0,2,0)}, F_{(0,0,1)}, F_{(0,2,1)}, F_{(0,1,1)})$ is a shelling order of $P_3(1,3,2)$. However, the relative complex associated to $F_{(0,1,1)}$ by this order is the table-top complex depicted in the bottom-right of Figure 3.1. Since this complex is not stable, then neither is this shelling of $P_3(1,3,2)$.

The previous observations demonstrate that some of the classic examples of cubical complexes admit stable shellings but also that not all shellings of these complexes are stable. In the coming sections, we will show that more complicated examples of cubical complexes admit stable shellings, and we will use this fact, together with Theorem 3.3.1 to provide some answers to open questions in the literature. However, while we can prove the existence of stable shellings in the desired cases, it is not clear if they exist in all cases. So we end this section with the following question.

Question 3.3.1. Does there exist a shellable cubical complex $\mathcal{C}$ for which no shelling is stable?

3.4 Applications

In this section, we apply Theorem 3.3.1 and the notion of stable shellings to some classical subdivisions of the boundary complexes of polytopes that are of interest in algebraic, geometric, and topological combinatorics. In Subsection 3.4, we show that the barycentric subdivision of a cubical complex admitting a stable line shelling has a real-rooted h-polynomial. Applying this result, we positively answer a question of Brenti and Welker [27, Question 1] for the well-known constructions of ***cubical polytopes***; i.e., polytopes whose facets are all cubes. In its most general form, the question is as follows:

Problem 3.4.1. *[27, Question 1] Let $\mathcal{C}$ be the boundary complex of an arbitrary polytope. Is the h-polynomial of the barycentric subdivision of $\mathcal{C}$ real-rooted?*

In [55], cubical polytopes are proposed as the first case of interest, as the results of [27] already answered the question in the case of simplicial (and simple) polytopes. Within the literature on cubical polytopes there are surprisingly few explicitly constructed cubical polytopes. The most well-known constructions are the ***cuboids***, which were first introduced by Grünbaum in [41], the ***capped cubical polytopes*** [49], which are a cubical generalization of stacked simplicial polytopes [51], and the ***neighborly cubical polytopes*** [8]. Constructing cubical polytopes is, in general, a nontrivial task, as noted in [50] and the thesis [59]. However, by applying Theorem 3.3.1 and the notion of stable shellings, in Subsection 3.4 we will be able to positively answer Problem 3.4.1 for all three of these constructions.

At the same time, Theorem 3.3.1 and the associated stable shellings can also be applied to subdivisions other than the barycentric subdivision. In Subsections 3.4 and 3.4, we apply these techniques to the edgewise subdivision of simplicial and cubical complexes so as to solve a second problem of Mohammadi and Welker [55, Problem 27] for shellable complexes. We also observe that a (non-geometric) solution to this problem follows from a recent result of Jochemko [47].

In the following, we will make use of some well-studied real-rooted polynomials, which can be defined as follows: For $d, r \geq 1$ and $0 \leq \ell \leq d$, let $A_{d,\ell}^{(r)}$ be the polynomial defined by the relation

$$\sum_{t\geq 0}(rt)^{\ell}(rt+1)^{d-\ell}x^t = \frac{A_{d,\ell}^{(r)}}{(1-x)^{d+1}}. \tag{3.5}$$

We call $A_{d,\ell}^{(r)}$ the ***d^{th} r-colored ℓ-Eulerian polynomial***. When $r = 1$ and $\ell = 0$, $A_{d,\ell}^{(r)}$ is the classical Eulerian polynomial, which enumerates the elements of $\mathfrak{S}_d$ by the excedance statistic. When $r = 2$ and $\ell = 0$, $A_{d,\ell}^{(r)}$ is the ***type B Eulerian polynomial***, which enumerates signed permutations. For $d \geq 1$, the polynomials $A_{d,0}^{(1)}$ and $A_{d,0}^{(2)}$ are symmetric with respect to degree $d-1$ and d, respectively. When $\ell = 0$ and $r \geq 1$, $A^{(r)}_{d,\ell}$ is the d^{th} ***r-colored Eulerian polynomial***, which enumerates the the elements of the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_d$ with respect to their excedance statistic (see [23, Section 3], for example). It is an immediate consequence of [24, Theorem 4.6] that $A^{(r)}_{d,\ell}$ has only real, simple zeros for all $d, r \geq 1$ and $0 \leq \ell \leq d$.

Lemma 3.4.2. *For $d, r \geq 1$ and $0 \leq \ell \leq d$, the polynomial $A^{(r)}_{d,\ell}$ has only simple, real zeros. Moreover, for a fixed $d, r \geq 1$, $\left(A^{(r)}_{d,\ell}\right)_{\ell=0}^{d}$ is an interlacing sequence.*

Lemma 3.4.2 will play a key role in the coming subsections.

The barycentric subdivision of cubical complexes

Given a polytopal complex $\mathcal{C}$, let $L(\mathcal{C})$ denote its face lattice with partial order $<_{\mathcal{C}}$ given by inclusion. The ***barycentric subdivision*** of $\mathcal{C}$ is the simplicial complex $\mathrm{s}d(\mathcal{C})$ whose k-dimensional faces are the subsets $\{F_0, F_1, \ldots, F_k\}$ of faces of $\mathcal{C}$ for which

$$\emptyset <_{\mathcal{C}} F_0 <_{\mathcal{C}} F_1 <_{\mathcal{C}} \cdots <_{\mathcal{C}} F_k$$

is a strictly increasing chain in $L(\mathcal{C})$. Our goal in this subsection is to show that $h(\mathrm{s}d(\mathcal{C}); x)$ is real-rooted when $\mathcal{C}$ is a cubical complex admitting a stable shelling. To do so, we need to first consider the h-polynomials of relative complexes of barycentrically subdivided cubes. To determine these polynomials, we will make use of the lemmata developed in Subsection 3.2. In the following, we will let $\square_d$ denote the (abstract) d-dimensional cube. The following result is well-known, with the first equality appearing, for example, in [55].

Lemma 3.4.3. *For $d \geq 1$ we have that*

$$h(\mathrm{s}d(\partial\square_d); x) = A^{(2)}_{d,0} = h(\mathrm{s}d(\square_d); x).$$

We will also let $[-1,1]^d \subset \mathbb{R}^d$ denote the geometric realization of $\square_d$ in d-dimensional real-Euclidean space as the convex hull of all $(-1,1)$-vectors in $\mathbb{R}^d$. The following lemma is likely well-known to experts in the field and can be proved by induction on d.

Lemma 3.4.4. *Let T_d denote the triangulation of the d-cube $[-1,1]^d$ that is induced by the hyperplanes $x_i = \pm x_j$ for $0 \leq i < j \leq d$ and $x_i = 0$ for $i \in [d]$. Then T_d is abstractly isomorphic to the barycentric subdivision of the d-cube.*

The triangulation T_d from Lemma 3.4.4 has an h-polynomial with a well-known combinatorial interpretation: A ***signed permutation*** on $[d]$ is a pair $(\pi, \varepsilon) \in \mathfrak{S}_d \times \{-1,1\}^d$, which we sometimes denote as $\pi_1^{\varepsilon_1} \cdots \pi_d^{\varepsilon_d}$, where $\pi = \pi_1 \cdots \pi_d$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$. Set $\pi_0 := 0$ and $\varepsilon_0 := 1$ for all $(\pi, \varepsilon) \in \mathfrak{S}_d \times \{-1,1\}^d$ and all $d \geq 1$. Then $i \in [d-1]_0 := \{0, 1, \ldots, d-1\}$ is a ***descent*** of (π, ε) if $\varepsilon_i \pi_i > \varepsilon_{i+1}\pi_{i+1}$. We also let

$$\begin{aligned}
\mathrm{D}es(\pi, \varepsilon) &:= \{i \in [d-1]_0 : \varepsilon_i \pi_i > \varepsilon_{i+1}\pi_{i+1}\}, \text{ and} \\
\mathrm{d}es(\pi, \varepsilon) &:= |\,\mathrm{D}es(\pi, \varepsilon)|.
\end{aligned}$$

Let $0 \leq \ell \leq d$. Going one step further, we define the ***ℓ-descent set*** of (π, ε) to be

$$\mathrm{D}es_\ell(\pi,\varepsilon) := \begin{cases} \mathrm{D}es(\pi,\varepsilon) \cup \{d\} & \text{if } d+1-\ell \leq \varepsilon_d\pi_d \leq d, \\ \mathrm{D}es(\pi,\varepsilon) & \text{otherwise.} \end{cases}$$

We then let $\mathrm{d}es_\ell(\pi,\varepsilon) := |\,\mathrm{D}es_\ell(\pi,\varepsilon)|$. The ***type B ℓ-Eulerian polynomial*** is defined as

$$B_{d,\ell} := \sum_{(\pi,\varepsilon)\in\mathfrak{S}_d\times\{-1,1\}^d : \varepsilon_d\pi_d = d+1-\ell} x^{\mathrm{d}es_\ell(\pi,\varepsilon)}.$$

For $0 \leq \ell \leq d$, we can then make use of the following theorem from [10]:

Theorem 3.4.5. *[10, Theorem 5.1] For $d \geq 1$ and $0 \leq \ell \leq d$,*

$$\mathrm{E}hr([-1,1]^d_\ell; x) = \frac{B_{d+1,\ell+1}}{(1-x)^{d+1}}.$$

In [10], it is further noted that for $0 \leq \ell \leq d$,

$$\mathrm{ehr}_{[-1,1]^d_\ell}(t) = \sum_{[\ell]\subseteq S\subseteq[d]} (2t)^{|S|} = (2t)^\ell(2t+1)^{d-\ell}. \tag{3.6}$$

From this, it follows that $B_{d+1,\ell+1} = A^{(2)}_{d,\ell}$, for $0 \leq \ell \leq d$. The polynomials $B_{d+1,\ell+1}$ for $d+1 \leq \ell \leq 2d$ can also be computed using the polynomials $A^{(2)}_{d,\ell}$. However, it requires a small geometric trick.

Lemma 3.4.6. *For $d \geq 1$ and $0 \leq \ell < d$,*

$$h^*([-1,1]^d_{2d-\ell}; x) = \mathcal{I}_{d+1}h^*([-1,1]^d_\ell; x).$$

In particular, $h^*([-1,1]^d_{2d-\ell}; x) = x\mathcal{I}_d A^{(2)}_{d,\ell}$.

Proof. Notice first that for $0 \leq \ell \leq d$, the half-open polytope $[-1,1]^d_\ell$ corresponds to $[-1,1]^d \setminus \mathcal{H}$ where we have removed all facets visible from a point $q \in \mathbb{R}^n$ for a fixed choice of q. Let B_ℓ denote the set of all such points visible from q on $\partial[-1,1]^d$, and let $D_\ell := \overline{\partial[-1,1]^d \setminus B_\ell}$. Notice also, if $[-1,1]^d_\ell = [-1,1]^d \setminus B_\ell$ then $[-1,1]^d_{2d-\ell}$ is unimodularly equivalent to $[-1,1]^d \setminus D_\ell$ (namely, up to rotation).

Next, consider the case when $\ell = 0$. Then the desired statement follows directly from classic Ehrhart-Macdonald reciprocity [12, Theorem 4.1]. Thus, we need only to prove the statement when $0 < \ell < d$. Assuming this is the case, we then know that B_ℓ and D_ℓ are both nonempty, and $i(P \setminus B; t)$ is a polynomial in t (see equation (3.6)). Thus, by Lemma 3.2.7, we know that

$$h^*([-1,1]^d_{2d-\ell}; x) = h^*(P \setminus D_\ell; x) = \mathcal{I}_{d+1}h^*(P \setminus B_\ell; x) = \mathcal{I}_{d+1}h^*([-1,1]^d_\ell; x).$$

The fact that $h^*([-1,1]^d_{2d-\ell}; x) = x\mathcal{I}_d A^{(2)}_{d,\ell}$ then follows from equations (3.5) and (3.6). $\square$

Given our interpretation of the polynomials $h^*([-,1,1]^d_\ell;x)$ for $0\leq\ell\leq 2d$ in terms of the polynomials $A^{(2)}_{d,\ell}$, we can prove the following theorem for shellable cubical complexes.

Theorem 3.4.7. *Let $\mathcal{C}$ be a cubical complex with a stable shelling. Then $h(\mathrm{sd}(\mathcal{C});x)$ is real-rooted.*

Proof. Let $(F_1,\ldots,F_s)$ be a stable shelling of the d-dimensional cubical complex $\mathcal{C}$. Then for all $i\in[s]$, the relative complex $\mathcal{R}_i$ associated to F_i by $(F_1,\ldots,F_s)$ is stable. Since $\mathcal{C}$ is a d-dimensional cubical complex, each $\mathcal{R}_i$ is a relative subcomplex of a d-dimensional cube. By Lemma 3.3.3, it follows that each $\mathcal{R}_i$ is isomorphic to $\mathrm{sd}([-1,1]^d)_\ell$ for some $0\leq\ell\leq 2d$, where

$$\mathrm{sd}([-1,1]^d)_\ell := \mathrm{sd}([-1,1]^d)\setminus\{x_d=1,\ldots,x_{d+1-\ell}=1\}$$

for $0\leq\ell\leq d$, and

$$\mathrm{sd}([-1,1]^d)_\ell := \mathrm{sd}([-1,1]^d)\setminus\{x_d=1,\ldots,x_1=1\}\cup\{x_d=-1,\ldots,x_{2d+1-\ell}=-1\}$$

for $d+1\leq\ell\leq 2d$. So by Theorem 3.3.1, it suffices to show that the sequence of h-polynomials $(h(\mathrm{sd}([-1,1]^d)_\ell;x))_{\ell=0}^{2d}$ forms an interlacing sequence.

By Lemma 3.4.4, each $\mathrm{sd}([-1,1]^d)_\ell$ is a relative subcomplex of the unimodular triangulation T_d of the d-cube $[-1,1]^d$. In the case that $\ell=0$, this complex is a unimodular triangulation of the entire d-cube $[-1,1]^d$. So by Lemma 3.2.5, Theorem 3.4.5, and equation (3.6), we find that

$$h(\mathrm{sd}([-1,1]^d)_0;x)=h^*([-1,1]^d;x)=A^{(2)}_{d,0}.$$

When $\ell=2d$, $\mathrm{sd}([-1,1]^d)_{2d}$ is the relative complex produced by taking the barycentric subdivision of the d-cube and then removing its subdivided boundary. Hence, the f-polynomial satisfies $f(\mathrm{sd}([-1,1]^d)_{2d};x)=xf(\mathrm{sd}(\partial\Box_d);x)$. So by Lemma 3.4.3, we know that $h(\mathrm{sd}([-1,1]^d)_{2d};x)=xA^{(2)}_{d,0}$. Moreover, since $A^{(2)}_{d,0}$ is known to be symmetric with respect to degree d, it follows that

$$h(\mathrm{sd}([-1,1]^d)_{2d};x)=x\mathcal{I}_dA^{(2)}_{d,0}.$$

In the case that $0<\ell<2d$, the complex $\mathrm{sd}([-1,1]^d)_\ell$ has Euler characteristic 0. So it follows by Lemma 3.2.6, Theorem 3.4.5, and equation (3.6) that

$$h(\mathrm{sd}([-1,1]^d)_\ell;x)=h^*([-1,1]^d_\ell;x)=A^{(2)}_{d,\ell}$$

when $0<\ell\leq d$. Hence, by Lemma 3.4.6, for all $0\leq\ell\leq 2d$ it follows that $h(\mathrm{sd}([-1,1]^d)_\ell;x)$ is of the form $A^{(2)}_{d,\ell'}$ or $x\mathcal{I}_dA^{(2)}_{d,\ell'}$ for some $0\leq\ell'\leq d$. Thus, it suffices to show that the sequence

$$\left(A^{(2)}_{d,0},A^{(2)}_{d,1},\ldots,A^{(2)}_{d,d-1},A^{(2)}_{d,d},x\mathcal{I}_dA^{(2)}_{d,d},\ldots,x\mathcal{I}_dA^{(2)}_{d,1},x\mathcal{I}_dA^{(2)}_{d,0}\right)$$

is interlacing. By [24, Lemma 2.3], we need only check that each of the following relations are satisfied:

1. $A^{(2)}_{d,0} \prec x\mathcal{I}_d A^{(2)}_{d,0}$,
2. $A^{(2)}_{d,\ell} \prec A^{(2)}_{d,k}$ for all $0 \leq \ell < k \leq d$,
3. $A^{(2)}_{d,d} \prec x\mathcal{I}_d A^{(2)}_{d,d}$, and
4. $x\mathcal{I}_d A^{(2)}_{d,k} \prec x\mathcal{I}_d A^{(2)}_{d,\ell}$ for all $0 \leq \ell < k \leq d$.

Case (1) is immediate from the fact that $A^{(2)}_{d,0} = \mathcal{I}_d A^{(2)}_{d,0}$ and Lemma 3.2.3 (4). Case (2) follows from Lemma 3.4.2. Case (3) follows from [23, Theorem 3.1], which shows that $\mathcal{I}_d A^{(2)}_{d,d} \prec A^{(2)}_{d,d}$, and Lemma 3.2.3 (4), and case (4) follows directly from case (2). Thus, since this sequence is interlacing, it follows that $h(\mathrm{sd}(\mathcal{C}); x)$ is real-rooted, which completes the proof. $\square$

We now apply these results to give a positive answer to Problem 3.4.1 for the well-known constructions of cubical polytopes; namely, the cuboids, capped cubical polytopes, and the neighborly cubical polytopes.

Barycentric subdivisions of cuboids

Cuboids are a family of cubical polytopes described by Grünbaum in [41]. For each dimension $d \geq 1$, there are $d+1$ cuboids, denoted Q^d_ℓ for $0 \leq \ell \leq d$. The first cuboid in dimension d, denoted Q^d_0, is the d-cube and the rest are defined recursively as follows: To construct Q^d_ℓ for $\ell > 0$, glue two copies of $Q^d_{\ell-1}$ at a common $Q^{d-1}_{\ell-1}$. The boundary of the resulting complex is the boundary complex of the cuboid Q^d_ℓ. Equivalently, to construct the ℓ^{th} d-dimensional cuboid for $0 < \ell \leq d$, start by taking the geometric realization $[-1,1]^d$ of Q^d_0. Then consider the geometric subdivision of $[-1,1]^d$ given by intersecting $[-1,1]^d$ with the ℓ hyperplanes $x_1 = 0, x_2 = 0, \ldots, x_\ell = 0$. The boundary of the resulting cubical complex is the boundary complex of the cuboid Q^d_ℓ. Using this second construction of the cuboid Q^d_ℓ, we can deduce that all cuboids admit a stable shelling, yielding the following corollary to Theorem 3.4.7:

Corollary 3.4.8. *The barycentric subdivision of the boundary complex of a cuboid has a real-rooted h-polynomial.*

Proof. Recall that the cuboid Q^d_0 is the d-dimensional cube, whose barycentric subdivision is well-known to have a real-rooted h-polynomial See Lemma 3.4.3. So fix $0 < \ell \leq d$. By Theorem 3.4.7, it suffices to show that the boundary complex of Q^d_ℓ, denoted $\mathcal{C}(\partial Q^d_\ell)$, has a stable shelling. By construction, the boundary complex of Q^d_ℓ is isomorphic to the subdivision of the boundary complex of $[0,2]^d$ induced by the hyperplanes $x_1 = 1, \ldots, x_\ell = 1$. For $i \in [d]$, let F_i and F_{d+i} denote, respectively, the facets of $[0,2]^d$ lying in the hyperplanes $x_i = 0$ and $x_i = 2$. It follows that, to construct $\mathcal{C}(\partial Q^d_\ell)$, each facet F_i and F_{d+i} is subdivided into a complex isomorphic to the pile of cubes $\mathcal{P}_{d-1}(c^{(i,\ell)})$, where

$$c^{(i,\ell)} = (1,1,\ldots,1) + \sum_{j \in [\ell]\setminus\{i\}} e_j,$$

where $e_1, \ldots, e_{d-1} \in \mathbb{R}^{d-1}$ denote the standard basis vectors (see Example 3.3.3 for the definition of a pile of cubes). The facets of Q^d_ℓ are then the facets of the piles of cubes $\mathcal{P}_{d-1}(c^{(i,\ell)})$ for all $i \in [d]$. Here we have two copies of each pile of cubes, one for F_i and one for F_{d+i}.

Fix the stable shelling order $(F_1, \ldots, F_d, \ldots, F_{2d})$ of $\mathcal{C}(\partial\Box_d) \simeq \mathcal{C}(\partial[0,2]^d)$ from Example 3.3.2, and suppose that F_i and F_{d+i} have been subdivided into the pile of cubes $\mathcal{P}_{d-1}(c^{(i,\ell)})$ consisting of M_i cubes. Also set $M_{d+i} := M_i$. Just as in Example 3.3.3, we index the cubes in the facet F_i (for $i \in [2d]$) by their lexicographically smallest vertex. As proven in Example 3.3.3, this is a stable shelling order for $\mathcal{P}_{d-1}(c^{(i,\ell)})$. Suppose this shelling order for the cubes in F_i is $(C_{1_i}, \ldots, C_{M_i})$. We claim that the linear ordering

$$(C_{1_1}, \ldots, C_{M_1}, C_{1_2}, \ldots, C_{M_2}, \ldots, C_{1_{2d}}, \ldots, C_{M_{2d}}) \tag{3.7}$$

is a stable shelling of the boundary of Q^d_ℓ. To see this, fix C_{j_k} for $k \in [d]$ and suppose that C_{j_k} is indexed by the integer point $(a_1, \ldots, a_d) \in F_k$, the facet of $[0,2]^d$. Note here that F_k is assumed to be a facet of $[0,2]^d$ whose corresponding facet-defining hyperplane is $x_k = 0$. Since C_{j_k} is a cube in the subdivided facet F_k, it follows that $a_k = 0$ and that C_{j_k} has facet-defining hyperplanes

$$x_1 = a_1, \ldots, x_{k-1} = a_{k-1}, x_{k+1} = a_{k+1}, \ldots, x_d = a_d,$$

and $x_i = a_i + b_i$ for $i \in [d] \setminus \{k\}$ and some $b_i \in \{1, 2\}$. In this context, $x_i = a_i$ and $x_i = a_i + b_i$ contain the opposing pair of facets G_i, G_{d+i} of C_{j_k} for $i \in [d] \setminus \{k\}$.

We then have two cases: either $b_i = 1$ or $b_i = 2$. In the former case, the facet G_{d+i} defined by $x_i = a_i + b_i$ would only not be a codimension 1 face of the relative complex associated to C_{j_k} if the cube indexed by $(a_1, \ldots, a_k + 1, \ldots, a_d)$ in F_k had preceded C_{j_k} in the order (3.7). However, this cannot happen since $(a_1, \ldots, a_k + 1, \ldots, a_d) >_{\text{lex}} (a_1, \ldots, a_k, \ldots, a_d)$ in the lexicographic order. In the latter case, the facet-defining hyperplane $x_i = a_i + b_i$ is the hyperplane $x_i = 2$, and hence the facet G_{d+i} defined by this hyperplane would not be in the associated relative complex if and only if a cube lying in a facet F_{d+i} of $[0,2]^d$ for some $i \in [d]$ had preceded C_{j_k} in the order (3.7). Since this is impossible, we conclude that the facet of C_{j_k} defined by $x_i = a_i + b_i$ is in the relative complex associated to C_{j_k}. Since this argument holds for all $i \in [d] \setminus \{k\}$, it follows that the set of facets of the complex determined by

$$C_{j_k} \cap (C_{1_1} \cup \cdots \cup C_{M_1} \cup \cdots \cup C_{1_k} \cup \cdots \cup C_{j-1_k})$$

does not contain an opposing pair. Hence by Lemma 3.3.3, the relative complex associated to C_{j_k} by the order (3.7) is stable.

Now suppose that $k = d + k'$ for some $k' \in [d]$. Hence, C_{j_k} is a facet of $\mathcal{C}(\partial Q^d_\ell)$ lying in the facet $F_{d+k'}$ of $[0,2]^d$. Assume once more that C_{j_k} is indexed by the integer point $(a_1, \ldots, a_d)$ in the facet F_k of $[0,2]^d$. We claim that none of the facets of C_{j_k} defined by the hyperplanes

$$x_1 = a_1, \ldots, x_{k'-1} = a_{k'-1}, x_{k'+1} = a_{k'+1}, \ldots, x_d = a_d,$$

are facets of the relative complex $\mathcal{R}_{j_k}$ associated to C_{j_k} by the order (3.7). This follows via induction. Suppose first that $j_k = 1_k$. In this case, C_{j_k} is the first cube in the facet F_k that appears in the order (3.7). Hence, the integer point in F_k indexing C_{j_k} must be $e_{k'}$, the standard basis vector in $\mathbb{R}^d$. Thus, $a_i = 0$ for all $i \in [d] \setminus \{k'\}$ (and $a_{k'} = 1$). Since all cubes in the facets $F_1, \dots, F_d$ preceded C_{j_k} in the order (3.7), then none of the facets defined by the hyperplanes $x_i = a_i (= 0)$ for $i \in [d] \setminus \{k'\}$ can be a facet of the relative complex $\mathcal{R}_{j_k}$ associated to C_{j_k}. Hence, by Lemma 3.3.3, the relative complex $\mathcal{R}_{j_k}$ is stable.

Similarly, the next cube in the ordering C_{j_k+1} will be indexed by $(a_1, \dots, a_{k^*}+1, \dots, a_d)$, where k^* is the right-most coordinate in $(a_1, \dots, a_d)$ for which adding 1 produces a new point in $[0,2]^d \cap \mathbb{Z}^d$ that indexes a cube. It then follows that $a_{k^*} = 0$ and $b_{k^*} = 1$, by construction of Q_ℓ^d. Hence, C_{j_k+1} has the set of facet-defining hyperplanes

$$x_1 = a_1, \dots, x_{k^*} = a_{k^*} + 1, \dots, x_{k'-1} = a_{k'-1}, x_{k'+1} = a_{k'+1}, \dots, x_d = a_d, \tag{3.8}$$

and their opposites $x_i = a_i + b_i$ for $i \in [d] \setminus \{k'\}$ and some $b_i \in \{1, 2\}$. Since each of the hyperplanes listed in (3.8) is also a facet-defining hyperplane of C_{j_k}, it follows that none of them define a facet in the relative complex $\mathcal{R}_{j_k+1}$ associated to C_{j_k+1}. Hence the set of facets of $\mathcal{R}_{j_k+1}$ does not contain an opposing pair. Therefore, it is a stable complex by Lemma 3.3.3. By iterating this argument, we see that the relative complex of each facet following C_{j_k} in the order (3.7) is stable. The fact that the order (3.7) is a shelling order now follows from Lemma 3.3.4. Applying Theorem 3.4.7 completes the proof. □

Corollary 3.4.8 gives a positive answer to Problem 3.4.1 in the case of cuboids, one of the three well-known constructions of cubical polytopes. In the next subsection, we deduce a positive answer to Problem 3.4.1 for the remaining two.

Barycentric subdivisions of capped cubical polytopes

Capped cubical polytopes, or ***stacked cubical polytopes***, are the cubical analogue to ***stacked simplicial polytopes*** [51]. A polytope P is called ***capped*** over a given cubical polytope Q if there is a combinatorial cube C such that $P = Q \cup C$ and $F := Q \cap C$ is a facet of Q. In this case, we then think of P as produced by ***capping*** Q ***over*** F, and we write

$$P = capped(Q, F).$$

We say that a polytope is ***ℓ-fold capped cubical*** for some $\ell \in \mathbb{Z}_{\geq 0}$ if it can be obtained from a combinatorial cube by ℓ capping operations. In the following, let $\mathcal{C}$ denote a $(d-1)$-dimensional cubical complex that is the boundary complex of an ℓ-fold capped polytope. Let $\square_d$ denote the (abstract) d-cube. Our goal in this subsection is to show that $h(\mathrm{sd}(\mathcal{C}); x)$ is real-rooted whenever $\mathcal{C}$ is the boundary complex of an ℓ-capped cubical polytope P for some $\ell \in \mathbb{Z}_{\geq 0}$. To do so, we again use Theorem 3.4.7 and the machinery of stable shellings developed in Subsection 3.3.

Corollary 3.4.9. *Let $\mathcal{C}$ denote the boundary complex of a d-dimensional ℓ-capped cubical polytope. Then $h(\mathrm{sd}(\mathcal{C}); x)$ is real-rooted.*

Proof. By Theorem 3.4.7, it suffices to show that $\mathcal{C}$ admits a stable shelling. To prove this, we proceed by induction on $\ell \geq 0$. When $\ell = 0$, $\mathcal{C}$ is the boundary complex of a d-dimensional cube. Hence, as we saw in Example 3.3.2, $\mathcal{C}$ admits a stable shelling.

Suppose now that $\mathcal{C}$ is the boundary complex of a d-dimensional ℓ-capped cubical polytope $P = capped(Q, F)$, for $\ell > 0$. Then Q is a d-dimensional $(\ell - 1)$-capped cubical polytope, and hence, by our inductive hypothesis, the boundary complex $\mathcal{D}$ of Q admits a stable shelling

$$(F_1, \dots, F_M). \tag{3.9}$$

Since $P = capped(Q, F)$, then it follows that $\mathcal{C}$ is produced from $\mathcal{D}$ by subdividing the facet F of $\mathcal{D}$ into the Schlegel diagram [76, Definition 5.5] of the d-dimensional cube $\square_d$ based at a facet G of $\square_d$ that is identified with F in $\mathcal{D}$. Suppose that the facets of $\square_d$ are $G_1, \dots, G_{2d}$, and suppose that $(G_1, \dots, G_d, G_{d+1}, \dots, G_{2d})$ is the stable shelling order of $\mathcal{C}(\partial\square_d)$ given in Example 3.3.2, where we assume $G = G_1$. We claim that the linear ordering

$$(F_1, \dots, F_{k-1}, G_2, \dots, G_d, G_{d+1}, \dots, G_{2d}, F_{k+1}, \dots, F_M) \tag{3.10}$$

is a stable shelling of $\mathcal{C}$. Since the relative complex of each F_i for $i \neq k$ in the linear ordering (3.10) is the same is its relative complex in the ordering (3.9), it suffices to show that the relative complex associated to each G_i for $i \in [2d]$ is stable.

To see this, consider first a facet G_i for $1 < i \leq d$. Since $G = G_1$ is identified with the facet $F_k = F$ of $\mathcal{D}$, then

$$G_i \cap (F_1 \cup \ldots \cup F_{k-1} \cup G_2 \cup \cdots \cup G_{i-1}) = G_i \cap (G_1 \cup G_2 \cup \cdots \cup G_{i-1}) \tag{3.11}$$

if $G_i \cap (F_1 \cup \ldots \cup F_{k-1}) \neq \emptyset$, or

$$G_i \cap (F_1 \cup \ldots \cup F_{k-1} \cup G_2 \cup \cdots \cup G_{i-1}) = G_i \cap (G_2 \cup \cdots \cup G_{i-1}) \tag{3.12}$$

otherwise. In the former case, as in Example 3.3.2, the subcomplex (3.11) consists of the facets of G_i defined by the hyperplanes $x_1 = 0, \dots, x_{i-1} = 0$. Therefore, $G_i \cap (F_1 \cup \ldots \cup F_{k-1} \cup G_2 \cup \cdots \cup G_{i-1})$ determines a subcomplex of $\mathcal{C}(\partial\square_{d-1})$ whose set of facets does not contain an opposing pair. In the latter case, the subcomplex (3.12) consists of the facets of G_i defined by the hyperplanes $x_2 = 0, \dots, x_{i-1} = 0$, and it again determines a subcomplex of $\mathcal{C}(\partial\square_{d-1})$ whose set of facets does not contain an opposing pair. Hence, by Lemma 3.3.3, the relative complex $\mathcal{R}_i$ associated to G_i by the ordering (3.10) is stable in both cases.

Now consider the facet G_{d+i} for some $i \in [d]$ and the subcomplex

$$H := G_i \cap (F_1 \cup \ldots \cup F_{k-1} \cup G_2 \cup \cdots \cup G_d \cup \cdots \cup G_{d+i-1}).$$

of the boundary complex $\mathcal{C}(\partial G_{d+i})$. Just as in the case of G_i with $1 < i \leq d$, this subcomplex is equal to $G_i \cap (G_1 \cup G_2 \cup \cdots \cup G_d \cup \cdots \cup G_{d+i-1})$ if $G_i \cap (F_1 \cup \ldots \cup F_{k-1}) \neq \emptyset$. Otherwise, it is equal to $G_i \cap (G_2 \cup \cdots \cup G_d \cup \cdots \cup G_{d+i-1})$. In either case, the relative complex $\mathcal{R}_{d+i}$ associated to G_{d+i} by the ordering (3.10) is

$$\mathcal{R}_{d+i} = \mathcal{C}(G_{d+i}) \setminus \mathcal{C}(H).$$

In the former case, its codimension 1 faces are determined by the hyperplanes $x_{i+1} = 1, \ldots, x_d = 1$. Therefore, the set of codimension 1 faces of $\mathcal{R}_{d+i}$ does not contain an opposing pair. In the latter case, $G_i \cap (F_1 \cup \ldots \cup F_{k-1}) = \emptyset$, which can happen in one of two ways: either $i = 1$ or $i \neq 1$. In the case that $i = 1$, it follows that the set of codimension 1 faces of $\mathcal{R}_{d+i}$ is determined by the hyperplanes $x_{i+1} = 1, \ldots, x_d = 1$, and thus does not contain an opposing pair. In the case that $i \neq 1$, the set of codimension one 1 is determined by the hyperplanes $x_{i+1} = 1, \ldots, x_d = 1$ and the hyperplane $x_1 = 0$. However, since $i + 1 > 1$, this set still does not contain an opposing pair. Thus, by Lemma 3.3.3, the relative complexes $\mathcal{R}_i$ and $\mathcal{R}_{d+i}$ are stable for all $i \in [d]$. It follows that the ordering (3.10) is a stable shelling order of $\mathcal{C}$, which completes the proof. □

The third class of well-known cubical polytopes with an explicit construction are the neighborly cubical polytopes. These polytopes were introduced in [8], but no explicit constructions was given. However, in [50], Joswig and Zeigler proved that neighborly cubical polytopes exist by giving such explicit constructions, which they denoted by C_n^d. In [50, Comment 1], they note that the constructions C_n^d are also capped cubical polytopes. Hence, as a corollary to Corollary 3.4.9, we also obtain a positive answer to Problem 3.4.1 for the family of neighborly cubical polytopes.

Corollary 3.4.10. *The barycentric subdivision of the boundary complex of a neighborly cubical polytope has a real-rooted h-polynomial.*

By combining Corollary 3.4.8, Corollary 3.4.9, and Corollary 3.4.10, we obtain a positive answer to Problem 3.4.1 for the well-known families of cubical polytopes; namely, the cuboids, capped cubical polytopes, and the neighborly cubical polytopes. To deduce these results we used the theory of stable shellings, developed in Subsection 3.3. As we will see in Subsection 3.4, we can also use stable shelling techniques to give an alternate proof of Brenti and Welker's original solution to Problem 3.4.1 in the case of simplicial polytopes. This suggests that the framework of stable shellings is perhaps appropriate for all polytopes, as it yields a proof for all known cases. In fact, by a theorem of Bruggesser and Mani [30], it is not unreasonable that these same techniques may be useful in addressing Problem 3.4.1 in its fullest generality. In Section 3.5, we will explain some of the open questions pertaining to this approach in more detail. Before doing so, in Subsections 3.4 and 3.4, we will demonstrate how stable shellings can also be applied to subdivisions other than the barycentric subdivision. As one application, we will derive an answer to a second problem of Mohammadi and Welker [55] for shellable simplicial complexes.

Barycentric subdivisions of simplicial polytopes

In this section, we give an alternative proof of the result of [27] that motivated Problem 3.4.1. Namely, we apply Theorem 3.3.1 to show that the h-polynomial of the boundary complex of a simplicial polytope has only real zeros. We will prove this using a similar narrative as in Subsections 3.4 and 3.4, in that we will use a shelling argument to decompose the complex into relative simplicial complexes, and show that the h-polynomials of each complex form an interlacing family. Recall that, by Proposition 3.3.2, any shelling of a simplicial complex is stable, and that stable shellings were defined in Subsection 3.3 to capture those shellings to which we can apply Theorem 3.3.1 for multiple different uniform subdivisions. In this subsection and the next, we will indeed see that we can apply Theorem 3.3.1 to any shelling of a simplicial complex with respect to the two most common uniform subdivisions: the barycentric subdivision and the edgewise subdivision.

In this case, the relative complexes associated to facets of our shelling will be subdivided, half-open simplices. In the following, let Δ_d denote the d-dimensional simplex, and let $\Delta_{d,\ell}$ denote the relative simplicial complex given by removing ℓ of the facets of Δ_d for $0 \leq \ell \leq d{+}1$. Note that $\Delta_{d,0} = \Delta_d$. We will require the following well-known result.

Lemma 3.4.11. *Let $d \geq 1$ and $0 \leq \ell \leq d+1$. Then $h(\Delta_{d,\ell}; x) = x^\ell$.*

Proof. By the Principle of Inclusion-Exclusion, (as described in terms of linear transformations in [61, Theorem 2.1]), we deduce that

$$f(\Delta_{d,\ell}; x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f(\Delta_{d-j}; x).$$

Since $h(\Delta_d; x) = 1$ for all $d \geq 1$, it then follows that

$$h(\Delta_{d,\ell}; x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (1-x)^j = x^\ell.$$

□

To make use of this observation, we need to generalize the main result of [27] (i.e. [27, Theorem 1]) to relative complexes. We refer to the reader to [27] for the definition of a Boolean cell complex.

Lemma 3.4.12. *Let $\mathcal{C}$ be a $(d-1)$-dimensional Boolean cell complex and $\mathcal{D}$ a subcomplex of $\mathcal{C}$. If the relative complex $\mathcal{C} \setminus \mathcal{D}$ is also $(d-1)$-dimensional then*

$$h(\mathrm{sd}(\mathcal{C}) \setminus \mathrm{sd}(\mathcal{D}); x) = \sum_{\ell=0}^{d} h_\ell(\mathcal{C} \setminus \mathcal{D}) A^{(1)}_{d,\ell}.$$

Proof. The definition of the h-polynomial of a relative complex $\mathcal{C} \setminus \mathcal{D}$ is known to be equivalent to the condition that

$$h_r(\mathcal{C} \setminus \mathcal{D}) = \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} f_{i-1}(\mathcal{C} \setminus \mathcal{D})$$

for all $r = 0, \ldots, d$. From this formula it follows that for all $r = 0, \ldots, d$

$$\begin{aligned} h_r(\mathrm{sd}(\mathcal{C}) \setminus \mathrm{sd}(\mathcal{D})) &= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} f_{i-1}(\mathrm{sd}(\mathcal{C}) \setminus \mathrm{sd}(\mathcal{D})), \\ &= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} (f_{i-1}(\mathrm{sd}(\mathcal{C})) - f_{i-1}(\mathrm{sd}(\mathcal{D})), \\ &= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} f_{i-1}(\mathrm{sd}(\mathcal{C})) - \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} f_{i-1}(\mathrm{sd}(\mathcal{D})). \end{aligned}$$

Since $\mathrm{sd}(\mathcal{C})$ and $\mathrm{sd}(\mathcal{D})$ are both Boolean cell complexes then we can apply [27, Lemma 1]. This yields

$$\begin{aligned} h_r(\mathrm{sd}(\mathcal{C}) \setminus \mathrm{sd}(\mathcal{D})) = &\sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} f_{m-1}(\mathcal{C}) S(m,k) m! \\ &- \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} f_{m-1}(\mathcal{D}) S(m,k) m!. \end{aligned}$$

Notice here that the formula for $f_{m-1}(\mathrm{sd}(\mathcal{D}))$ given in [27, Lemma 1] still holds with respect to degree d even if $\mathcal{D}$ has dimension less than d. This is immediate from the proof of [27, Lemma 1] since $f_{m-1}(\mathcal{D}) = 0$ for all m greater than the dimension of $\mathcal{D}$. Hence, it follows that

$$\begin{aligned} h_r(\mathrm{sd}(\mathcal{C}) \setminus \mathrm{sd}(\mathcal{D})) &= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} S(m,k) m! (f_{m-1}(\mathcal{C}) - f_{m-1}(\mathcal{D})), \\ &= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} S(m,k) m! f_{m-1}(\mathcal{C} \setminus \mathcal{D}). \end{aligned}$$

Since we have assumed that $\mathcal{C} \setminus \mathcal{D}$ is $(d-1)$-dimensional, it follows that

$$f_{m-1}(\mathcal{C} \setminus \mathcal{D}) = \sum_{\ell=0}^{m} \binom{d-\ell}{d-m} h_\ell(\mathcal{C} \setminus \mathcal{D}),$$

and so

$$\begin{aligned} h_r(\mathrm{sd}(\mathcal{C}) \setminus \mathrm{sd}(\mathcal{D})) &= \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \sum_{m=0}^{d} S(m,k) m! \sum_{\ell=0}^{m} \binom{d-\ell}{d-m} h_\ell(\mathcal{C} \setminus \mathcal{D}), \\ &= \sum_{\ell=0}^{d} \left(\sum_{m=0}^{d} \sum_{i=0}^{r} (-1)^{r-i} \binom{d-i}{r-i} \binom{d-\ell}{d-m} S(m,k) m! \right) h_\ell(\mathcal{C} \setminus \mathcal{D}). \end{aligned}$$

In the proof of [27, Theorem 1], it is shown that the coefficient of $h_\ell(\mathcal{C} \setminus \mathcal{D})$ in the above expression is equal to the k^{th} coefficient of $A^{(1)}_{d,\ell}$. Hence,

$$h(\mathrm{sd}(\mathcal{C}) \setminus \mathrm{sd}(\mathcal{D}); x) = \sum_{\ell=0}^{d} h_\ell(\mathcal{C} \setminus \mathcal{D}) A^{(1)}_{d,\ell}.$$

□

From Lemma 3.4.11 and Lemma 3.4.12, we recover the following proposition:

Proposition 3.4.13. *Let Δ_{d-1} be a $(d-1)$-dimensional simplex, and let $0 \leq \ell \leq d$ be the number of facets of Δ_{d-1} missing in $\Delta_{d-1,\ell}$. Then,*

$$h(sd(\Delta_{d-1,\ell}), x) = A^{(1)}_{d,\ell}.$$

Applying Lemma 3.4.2 to Proposition 3.4.13, we see that the h-polynomials of the barycentric subdivision of a simplex restricted to half-open simplices form an interlacing family. Since a shelling of the boundary of a simplicial polytope will decompose this complex into such half-open simplices, the h-polynomial of the barycentric subdivision of the complex will be real-rooted. We summarize this observation in the following theorem, which is originally due to Brenti and Welker [27].

Theorem 3.4.14. *Let $\mathcal{C}$ be a $(d-1)$-dimensional shellable simplicial complex. Then $h(\mathrm{sd}(\mathcal{C}); x)$ is real-rooted. In particular, the h-polynomial of the barycentric subdivision of the boundary complex of a d-dimensional simplicial polytope is real-rooted.*

Proof. The result is an immediate consequence of Theorem 3.3.1, Lemma 3.4.2, and Proposition 3.4.13. The special case of boundary complexes of simplicial polytopes follows from the fact that the boundary complex of any polytope admits a shelling [30]. □

The proofs of Theorems 3.4.7 and 3.4.14 given here suggest that stable shellability of all boundary complexes of polytopes could be key to answering Problem 3.4.1 in its fullest generality. In Section 3.5, we offer some first results in this direction and pose some related open questions. However, we first examine some applications of Theorem 3.3.1 and stable shellings to subdivisions other than the barycentric subdivision.

Edgewise subdivisions of simplicial complexes

The edgewise subdivision of a simplicial complex is another well-studied subdivision that arises frequently in algebraic and topological contexts (see for instance [28, 31, 35, 40]). Within algebra, it is intimately tied to the Veronese construction, and it is considered to be the algebraic analogue of barycentric subdivision [28, Acknowledgements]. For $r \geq 1$, the r^{th} edgewise subdivision of a simplex is defined as follows: Suppose that $\Delta :=$

$conv(e^{(1)}, \ldots, e^{(d)}) \subset \mathbb{R}^d$ is a $(d-1)$-dimensional simplex with 0-dimensional faces $e^{(1)}, \ldots, e^{(d)}$, the standard basis vectors in $\mathbb{R}^d$. For $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$, we let

$$supp(x) := \{i \in [d] : x_i \neq 0\},$$

and we define the linear transformation $\iota : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\iota : x \longmapsto (x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_d).$$

The r^{th} ***edgewise subdivision*** of Δ is the simplicial complex $\Delta^{\langle r \rangle}$ whose set of 0-dimensional faces are the lattice points in $r\Delta \cap \mathbb{Z}^d$ and for which $F \subset r\Delta \cap \mathbb{Z}^d$ is a face of $\Delta^{\langle r \rangle}$ if and only if

$$\bigcup_{x \in F} \{supp(x)\} \in \Delta,$$

and for all $x, y \in F$ either $\iota(x) - \iota(y) \in \{0,1\}^d$ or $\iota(y) - \iota(x) \in \{0,1\}^d$. Given a simplicial complex $\mathcal{C}$, the r^{th} ***edgewise subdivision*** of $\mathcal{C}$, denoted $\mathcal{C}^{\langle r \rangle}$, is given by gluing together the r^{th} edgewise subdivisions of each of its facets. In [55], the authors proposed the following problem.

Problem 3.4.15. *[55, Problem 27] If $\mathcal{C}$ is a d-dimensional simplicial complex with $h_k(\mathcal{C}) \geq 0$ for all $0 \leq k \leq d+1$, is $h(\mathcal{C}^{\langle r \rangle}; x)$ real-rooted whenever $r > d$?*

Applying Theorem 3.3.1 and Proposition 3.3.2, we will give a positive answer to Problem 3.4.15 for shellable simplicial complexes via geometric methods. We then also observe that a positive answer to Problem 3.4.15 for both shellable and non-shellable complexes follows from some recent enumerative results of Jochemko [47]. To do this, we first note that for every $r \geq 1$ and polynomial $p \in \mathbb{R}[x]$ there are uniquely determined polynomials $p^{(0)}, p^{(1)}, \ldots, p^{(r-1)} \in \mathbb{R}[x]$ satisfying

$$p = p^{(0)}(x^r) + xp^{(1)}(x^r) + x^2 p^{(2)}(x^r) + \cdots + x^{r-1} p^{(r-1)}(x).$$

We define the linear operator

$$^{\langle r,\ell \rangle} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x] \quad \text{where} \quad ^{\langle r,\ell \rangle} : p \longrightarrow p^{(\ell)},$$

and the polynomial $p_{(r,d)} := (1 + x + \cdots + x^{r-1})^d$. It is well-known that the sequence

$$\left(p_{(r,d)}^{\langle r,r-\ell \rangle}\right)_{\ell=1}^{r} = \left(p_{(r,d)}^{\langle r,r-1 \rangle}, p_{(r,d)}^{\langle r,r-2 \rangle}, \ldots, p_{(r,d)}^{\langle r,0 \rangle}\right), \tag{3.13}$$

is an interlacing sequence (see [60, Remark 4.2] or [47], for instance). On the other hand, [4, Equation 21] shows that for any d-dimensional simplicial complex $\mathcal{C}$

$$h(\mathcal{C}^{\langle r \rangle}; x) = ((1 + x + x^2 + \cdots + x^{r-1})^{d+1} h(\Delta; x))^{\langle r,0 \rangle} \tag{3.14}$$

for all $r \geq 1$. Let Δ_d denote the d-dimensional simplex, and let $\Delta_{d,\ell}$ denote the relative simplicial complex given by removing ℓ of the facets of Δ_d for $0 \leq \ell \leq d+1$.

Lemma 3.4.16. *Let $d \geq 1$, $r > d$, and $0 < \ell \leq d+1$. Then*

$$h(\Delta_{d,\ell}^{\langle r \rangle}; x) = x p_{(r,d+1)}^{\langle r, r-\ell \rangle}.$$

Proof. Notice first that the relative complex $\Delta_{d,\ell}^{\langle r \rangle}$ can be constructed in two equivalent ways: Either we first remove the ℓ facets of Δ_d and then apply the subdivision procedure outlined in the definition of the edgewise subdivision to the corresponding geometric realization of the half-open simplex $\Delta_{d,\ell}$, or we first compute $\Delta_d^{\langle r \rangle}$ and then remove the faces of $\Delta_d^{\langle r \rangle}$ lying in the ℓ facets of Δ_d scheduled for removal. For the purposes of this proof, we work with the latter construction. Our first goal, then, is to prove the following fact in analogy to equation (3.14):

$$h(\Delta_{d,\ell}^{\langle r \rangle}; x) = ((1 + x + \cdots + x^{r-1})^{d+1} h(\Delta_{d,\ell}; x))^{\langle r, 0 \rangle}.$$

Given the chosen construction of $\Delta_{d,\ell}^{\langle r \rangle}$, we know that

$$f(\Delta_{d,\ell}^{\langle r \rangle}; x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f(\Delta_{d-j}^{\langle r \rangle}; x),$$

and so

$$\begin{aligned} h(\Delta_{d,\ell}^{\langle r \rangle}; x) &= (1-x)^{d+1} \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} f\left(\Delta_{d-j}^{\langle r \rangle}; \frac{x}{1-x}\right), \\ &= \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (1-x)^j h(\Delta_{d-j}^{\langle r \rangle}; x). \end{aligned}$$

Since Δ_{d-j} is a simplicial complex, it follows from equation (3.14) that

$$h(\Delta_{d-j}^{\langle r \rangle}; x) = ((1 + x + \cdots + x^{r-1})^{d+1-j} h(\Delta_{d-j}; x))^{\langle r, 0 \rangle}.$$

Since

$$\begin{aligned} &(1-x)^j ((1 + \cdots + x^{r-1})^{d+1-j} h(\Delta_{d-j}; x))^{\langle r, 0 \rangle} \\ &\qquad = ((1-x^r)^j (1 + \cdots + x^{r-1})^{d+1-j} h(\Delta_{d-j}; x))^{\langle r, 0 \rangle}, \end{aligned}$$

it follows that

$$\begin{aligned} h(\Delta_{d,\ell}^{\langle r \rangle}; x) &= \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} ((1-x^r)^j (1 + \cdots + x^{r-1})^{d+1-j} h(\Delta_{d-j}; x))^{\langle r, 0 \rangle}, \\ &= \left((1 + \cdots + x^{r-1})^{d+1} \left(\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (1-x)^j h(\Delta_{d-j}; x) \right) \right)^{\langle r, 0 \rangle}, \\ &= \left((1 + \cdots + x^{r-1})^{d+1} h(\Delta_{d,\ell}; x) \right)^{\langle r, 0 \rangle}, \end{aligned}$$

as desired. We then note that $h(\Delta_{d,\ell};x) = x^\ell$, by Lemma 3.4.11. Since $r > d$ and $0 < \ell \leq d+1$, it follows that

$$h(\Delta_{d,\ell}^{\langle r\rangle};x) = xp_{(r,d+1)}^{\langle r,r-\ell\rangle},$$

which completes the proof. □

Lemma 3.2.3, Theorem 3.3.1, and Lemma 3.4.16 give the necessary tools to positively answer Problem 3.4.15 for shellable simplicial complexes.

Theorem 3.4.17. *Let $\mathcal{C}$ be a d-dimensional shellable simplicial complex, and let $r > d$. Then $h(\mathcal{C}^{\langle r\rangle};x)$ is real-rooted.*

As an immediate corollary to Theorem 3.4.17 we get that, for $r \geq d$, the r^{th} edgewise subdivision of the boundary complex of any d-dimensional simplicial polytope has a real-rooted h-polynomial.

Corollary 3.4.18. *Let $\mathcal{C}$ be the boundary complex of a d-dimensional simplicial polytope. Then for $r \geq d$, the edgewise subdivision $\mathcal{C}^{\langle r\rangle}$ of $\mathcal{C}$ has a real-rooted h-polynomial.*

On the other hand, Theorem 3.4.17 holds more generally. This, in fact, follows directly from some recent results of Jochemko [47], in particular, from combining [47, Theorem 1.1] with [47, Lemma 3.1].

Theorem 3.4.19 (Essentially due to [47])**.** *If $\mathcal{C}$ is a d-dimensional simplicial complex with $h_k(\mathcal{C}) \geq 0$ for all $0 \leq k \leq d+1$ then $h(\mathcal{C}^{\langle r\rangle};x)$ is real-rooted whenever $r > d$.*

Theorem 3.4.19 shows that the geometric approach used in Theorem 3.4.17 was not necessary, as it was for the solution to Problem 3.4.1 for cuboids, capped cubical polytopes, and neighborly cubical polytopes given in Subsection 3.4. On the other hand, the geometric proof of Theorem 3.4.17 highlights that the applications of Theorem 3.3.1 are not limited to barycentric subdivisions. In the next subsection, a similar result for the edgewise subdivision of a cube is derived. In this case, there is currently no other proof aside from the geometric methods developed in this chapter.

Edgewise subdivisions of cubical complexes

In Subsection 3.4 we defined the edgewise subdivision of a simplicial complex. We now extend this definition to cubical complexes. To do so, we perform the same operations on a unit cube that were performed on the standard simplex Δ in the construction of the r^{th} edgewise subdivision of a simplex. To reiterate, let $\square_d$ denote the (abstract) d-dimensional cube, and consider its geometric realization $[0,1]^d$ and $r[0,1]^d = [0,r]^d$, the r^{th} dilation of $[0,1]^d$. Recall the map ι defined in Subsection 3.4 that sends $(x_1,\ldots,x_d) \in C_d \cap \mathbb{Z}^d$ to $(x_1, x_1+x_2,\ldots,x_1+\cdots+x_d)$. We define the r^{th} ***edgewise subdivision*** $\square_d^{\langle r\rangle}$ of the d-dimensional cube in terms of a subdivision of its geometric realization $[0,1]^d$ as follows: Let $A \subset [0,1]^d \cap \mathbb{Z}^d$. Then $conv(A)$ is a face of the subdivision if and only if $\iota(v-v')$ or

$-\iota(v-v')$ is in $\{0,1\}^d$ for all $v, v' \in A$. We first note that this a unimodular triangulation of $[0,1]^d$, as it splits the dilated cube $[0,r]^d$ into unit cubes which are each triangulated according to (a rotated version of) the standard unimodular triangulation of $[0,1]^d$; that is, the triangulation induced by the hyperplanes $x_i = x_j$ for all $1 \leq i < j \leq d$. Given a cubical complex $\mathcal{C}$, its r^{th} ***edgewise subdivision***, denoted $\mathcal{C}^{\langle r\rangle}$ is given by gluing together the r^{th} edgewise subdivisions of each of its facets.

Stable shellings were defined so as to capture those shellings to which Theorem 3.3.1 can be applied for multiple different subdivisions. In Subsections 3.4 and 3.4, we saw this to be the case for simplicial complexes. To further substantiate this claim, we now show that the analogous result to Theorem 3.4.7 holds for the edgewise subdivision of a cubical complex. To do so, we will first use the fact that the r^{th} edgewise subdivision of a cube has a geometric realization that is a unimodular triangulation of the r^{th} dilation of $[0,1]^d$ so as to give a formula for the h-polynomials of the stable relative complexes $\mathcal{R}_i$ associated to the edgewise subdivision of a cubical complex. This formula will be in terms of the colored Eulerian polynomials $A^{(r)}_{d,\ell}$, which were introduced at the beginning of Section 3.4.

We first give a combinatorial interpretation of the coefficients of $A^{(r)}_{d,\ell}$ in terms of a descent statistic for the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_d$. Denote the elements of the wreath product $\mathbb{Z}_r \wr \mathfrak{S}_d$ as pairs (π,ε), where $\pi = \pi_1 \cdots \pi_d \in \mathfrak{S}_d$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{0, \dots, r-1\}^d$. We will typically denote the pair (π,ε) as $\pi_1^{\varepsilon_1} \cdots \pi_d^{\varepsilon_d}$. Like the classical symmetric group, $\mathbb{Z}_r \wr \mathfrak{S}_d$ admits combinatorial statistics such as descents and excedances, and there exist several different well-studied versions of each. A brief survey of these different definitions, as well as their uses, can be found in [11]. For our purposes, we use the following definition:

Definition 3.4.1. [11, Definition 2.5] Order the elements in the set $\{i^{\varepsilon_i} : i \in [d], \varepsilon_i \in \{0,1,\dots,r-1\}\}$ such that $i^{\varepsilon_i} < j^{\varepsilon_j}$ if $\varepsilon_i > \varepsilon_j$ or $\varepsilon_i = \varepsilon_j$ and $i < j$. For $(\pi,\varepsilon) \in \mathbb{Z}_r \wr S_d$, the ***descent set*** of (π,ε) is

$$\mathrm{D}es(\pi,\varepsilon) := \{j \in \{0,1,\dots,d-1\} : \pi_j^{\varepsilon_j} > \pi_{j+1}^{\varepsilon_{j+1}}\},$$

where we use the convention that $\pi_0 = 0$ and $c_0 = 0$. The ***descent statistic*** is $\mathrm{d}es(\pi,\varepsilon) := |Des(\pi,\varepsilon)|$.

As in the case of the signed permutations used in Subsection 3.4, we can extend this Definition 3.4.1 to ℓ-descents: Let $(\pi,\varepsilon) \in \mathbb{Z}_r \wr \mathfrak{S}_d$ and let $0 \leq \ell \leq d$. Then the ***ℓ-descent set*** of (π,ε) is

$$\mathrm{D}es_\ell(\pi,\varepsilon) := \begin{cases} \mathrm{D}es(\pi,\varepsilon) \cup \{0\} & \text{if } \pi_1 \in [\ell] \\ \mathrm{D}es(\pi,\varepsilon) & \text{otherwise.} \end{cases}$$

The ***ℓ-descent statistic*** is then $\mathrm{d}es_\ell(\pi,\varepsilon) := |\,\mathrm{D}es_\ell(\pi,\varepsilon)|$. Using this new statistic, we can now give a combinatorial interpretation of $A^{(r)}_{d,\ell}$.

Proposition 3.4.20. *For $d, r \geq 1$ and $0 \leq \ell \leq d$,*

$$A^{(r)}_{d,\ell} = \sum_{(\pi,\varepsilon)\in\mathbb{Z}_r\wr\mathfrak{S}_d} x^{\mathrm{d}es_\ell(\pi,\varepsilon)}.$$

Moreover, $A^{(r)}_{d,\ell} = h^*([0,r]^d_\ell; x)$.

Proof. We prove this by using the following unimodular triangulation of the cube $[0,r]^d$: Subdivide $[0,r]^d$ into the pile of (unit) cubes $\mathcal{P}_d(r,\ldots,r)$, and triangulate each cube in the resulting cubical complex according to the standard triangulation of $[0,1]^d$. That is, the triangulation of the cube $C_{\mathbf{z}}$, for $\mathbf{z} = (z_1,\ldots,z_d) \in B^\circ(r,r,\ldots,r)$ is given by triangulating $[0,1]^d$ via the hyperplanes $x_i = x_j$ for $1 \leq i < j \leq d$ and then translating this triangulated version of $[0,1]^d$ as $[0,1]^d + \mathbf{z} = C_{\mathbf{z}}$. Each simplex in this triangulation of $C_{\mathbf{z}}$ is then of the form

$$\Delta^d_{(\pi,\varepsilon)} := \{x \in \mathbb{R}^d : 0 \leq x_{\pi_1} - z_{\pi_1} \leq \cdots \leq x_{\pi_d} - z_{\pi_d} \leq 1\},$$

for $(\pi,\varepsilon) \in \mathbb{Z}_r \wr S_d$ given by $\pi_1^{z_{\pi(1)}} \ldots \pi_d^{z_{\pi(d)}}$. This correspondence between elements of $\mathbb{Z}_r \wr S_d$ and facets of this triangulation is similar in spirit to the triangulation defined in [72, Section 3]. However, we use slightly different conventions.

We now define the following half-open simplices, with π, $\mathbf{z}$, and ε defined as above:

$$\Delta^{d,\ell}_{(\pi,\varepsilon)} := \left\{ \mathbf{x} \in \mathbb{R}^d : \begin{array}{l} 0 \leq x_{\pi_1} - z_{\pi_1} \leq \cdots \leq x_{\pi_d} - z_{\pi_d} \leq 1, \\ x_{\pi_i} - z_{\pi_i} < x_{\pi_{i+1}} - z_{\pi_{i+1}} \text{ for } i \in \mathrm{D}es(\pi,\varepsilon) \text{ , and} \\ 0 < x_{\pi_1} - z_{\pi_1} \text{ if } 0 \in \mathrm{D}es_\ell(\pi,\varepsilon) \end{array} \right\}.$$

Note that $\Delta^{d,\ell}_{(\pi,\varepsilon)}$ is a unimodular half-open simplex with the number of missing facets equal to $\mathrm{d}es_\ell(\pi,\varepsilon)$. Hence, $h^*(\Delta^{d,\ell}_{(\pi,\varepsilon)}; x) = x^{\mathrm{des}_\ell(\pi,\varepsilon)}$. (A proof of this fact is an exercise analogous to the proof of Lemma 3.4.11.) We now show that for fixed $d, r \geq 1$ and $0 \leq \ell \leq d$, the disjoint union

$$\bigsqcup_{(\pi,\varepsilon)\in\mathbb{Z}_r\wr\mathfrak{S}_d} \Delta^{d,\ell}_{(\pi,\varepsilon)}$$

is $[0,r]^d_\ell$. To do so, we first decompose $[0,r]^d_\ell$ into a disjoint union of half-open unit cubes. As before, let $\mathbf{z} \in B^\circ(r,\ldots,r)$, and set

$$C^\ell_{\mathbf{z}} = C_{\mathbf{z}} \setminus \{x_i = z_i : z_i \neq 0 \text{ or } z_i = 0 \text{ and } i \in [\ell]\}.$$

It follows, as in Example 3.3.3, that $[0,r]^d_\ell$ is the disjoint union of the half-open cubes $C^\ell_{\mathbf{z}}$ for $\mathbf{z} \in B^\circ(r,\ldots,r)$. We now show that for a fixed $\mathbf{z}$, the half-open cube $C^\ell_{\mathbf{z}}$ is the disjoint union of $\Delta^{d,\ell}_{(\pi,\varepsilon)}$ where π ranges over all elements of $\mathfrak{S}_d$ and $(\pi,\varepsilon) = \pi_1^{z_{\pi_1}} \cdots \pi_d^{z_{\pi_d}}$. First, we observe that the closed cube $C_{\mathbf{z}}$ is the disjoint union of half-open simplices of the form

$$\left\{ \mathbf{x} \in \mathbb{R}^d : \begin{array}{l} 0 \leq x_{\pi_1} - z_{\pi_1} \leq \cdots \leq x_{\pi_d} - z_{\pi_d} \leq 1, \\ x_{\pi_i} - z_{\pi_i} < x_{\pi_{i+1}} - z_{\pi_{i+1}} \text{ for } i \in \mathrm{D}es(\pi,\varepsilon) \cap \{1,\ldots,d-1\} \end{array} \right\},$$

since this is simply a translated version of the standard half-open decomposition of the unit cube described, for example, in [13]. From this, we can construct a half-open decomposition of $[0,r]^d_\ell$ by removing all of the points in the simplices described above that satisfy $x_{\pi_1} = z_{\pi_1}$ for $0 \in \mathrm{D}es_\ell(\pi,\varepsilon)$. Removing these points gives us the half-open simplex $\Delta^{d,\ell}_{(\pi,\varepsilon)}$. Hence,

$C^\ell_{\mathbf{z}}$ is a disjoint union of $\Delta^{d,\ell}_{(\pi,\varepsilon)}$ where $(\pi,\varepsilon) = \pi_1^{z_{\pi_1}} \cdots \pi_d^{z_{\pi_d}}$. Since $[0,r]^d_\ell$ is the disjoint union of all $C^\ell_{\mathbf{z}}$ for $z \in B^\circ(r,\ldots,r)$, it follows that $[0,r]^d_\ell$ is the disjoint union of $\Delta^{d,\ell}_{(\pi,\varepsilon)}$ over all $(\pi,\varepsilon) \in \mathbb{Z}_r \wr \mathfrak{S}_d$.

We now compute the h^*-polynomial of $[0,r]^d_\ell$ in two different ways. First, since $[0,r]^d_\ell$ is the disjoint union of $\Delta^{d,\ell}_{(\pi,\varepsilon)}$ over all $(\pi,\varepsilon) \in \mathbb{Z}_r \wr \mathfrak{S}_d$, it follows that

$$\mathrm{E}hr([0,r]^d_\ell; x) = \sum_{(\pi,\varepsilon)\in\mathbb{Z}_r \wr S_d} h^*(\Delta^{d,\ell}_{(\pi,\varepsilon)}; x) = \sum_{(\pi,\varepsilon)\in\mathbb{Z}_r \wr S_d} x^{\mathrm{des}_\ell(\pi,\varepsilon)},$$

where the last equality follows from the fact that the h^*-polynomial of a unimodular simplex with m facets removed is x^m. On the other hand, since $\ell \leq d$, we know that $[0,r]^d_\ell$ is the product of ℓ copies of the half-open 1-dimensional cube $[0,r)$ and $d-\ell$ copies of the 1-dimensional cube $[0,r]$. Since Ehrhart polynomials are multiplicative,

it follows that

$$\mathrm{ehr}_{[0,r]^d_\ell}(t) = \mathrm{ehr}_{[0,r)}(t)^\ell \mathrm{ehr}_{[0,r]}(t)^{d-\ell}) = (rt)^\ell (rt+1)^{d-\ell}.$$

Thus,

$$\mathrm{E}hr([0,r]^d_\ell; x) = \sum_{t\geq 0} (rt)^\ell (rt+1)^{d-\ell} x^t = \frac{h^*([0,r]^d_\ell; x)}{(1-x)^{d+1}}.$$

From the definition of r-colored ℓ-Eulerian polynomials given in Equation (3.5), we see that $h^*([0,r]^d_\ell; x) = A^{(r)}_{d,\ell}$. Thus, $A^{(r)}_{d,\ell} = \sum_{(\pi,\varepsilon)\in\mathbb{Z}_r\wr S_d} x^{\mathrm{des}(\pi,\varepsilon)}$, as desired. □

Corollary 3.4.21. *For all $d \geq 1$, if $r = 1$ and $0 \leq \ell < d$, then the degree of $A^{(r)}_{d,\ell}$ is $d-1$. Otherwise, the degree of $A^{(r)}_{d,\ell}$ is d.*

Proof. By Proposition 3.4.20, the degree of $A^{(r)}_{d,\ell}$ is the maximum number of ℓ-descents of $(\pi,\varepsilon) \in \mathbb{Z}_r \wr S_d$. Since $\mathrm{D}es_\ell(\pi,\varepsilon)$ is a subset of $\{0,\ldots,d-1\}$, the degree is at most d. When $r = 1$, $0 \in \mathrm{D}es_\ell(\pi,\varepsilon)$ if and only if $\ell > 0$ and $\pi_1 \leq \ell$. If $0 \in \mathrm{D}es_\ell(\pi,\varepsilon)$ and $r = 1$, then $i \in \mathrm{D}es_\ell(\pi,\varepsilon)$ for all $i \in [d-1]$ if and only if $\pi_i > \pi_{i+1}$ for all $i \in [d-1]$. However, this is only possible if $\pi = d(d-1)\cdots 1$. Hence, $A^{(1)}_{d,\ell}$ has degree d if and only if $\ell = d$. Otherwise, $A^{(1)}_{d,\ell}$ has degree $d-1$, since $(\pi,\varepsilon) = \pi_1^0\pi_2^0\cdots\pi_d^0$ has exactly $d-1$ descents. If $r > 1$ then $A^{(r)}_{d,\ell}$ has degree d since the element $(\pi,\varepsilon) = d^1(d-1)^1\cdots 1^1$ has d descents. □

Using these facts, we can apply the results on stable shellings developed in Section 3.3 to prove the following:

Theorem 3.4.22. *Let $\mathcal{C}$ be a cubical complex with a stable shelling. Then $h(\mathcal{C}^{\langle r\rangle}; x)$ is real rooted for $r \geq 2$.*

Proof. Using the shelling argument detailed in the proof of Theorem 3.4.7, we see that we can decompose $\mathcal{C}^{\langle r\rangle}$ into relative complexes $\mathcal{R}_j$ in which each $\mathcal{R}_j$ has geometric realization the unimodular triangulation $T_{d,r}$ of $[0,r]^d_\ell$, induced by the r^{th} edgewise subdivision as described above, for some $0 \leq \ell \leq 2d$. We first consider the cases in which $0 \leq \ell \leq d$. In this case, $[0,r]^d_\ell$ is either a closed, convex polytope and hence has Euler characteristic 1, or it is missing a subset of facets that forms a contractible subcomplex, and hence it has Euler characteristc 0. So by Lemmas 3.2.5 and 3.2.6,

$$h(\mathcal{R}_j;x) = h^*([0,r]^d_l;x) = A^{(r)}_{d,\ell}.$$

where the last equality follows from Proposition 3.4.20.

We now must consider the case in which more than d facets are removed. From Lemma 3.2.7, we know that for $0 \leq \ell \leq d$:

$$h^*([0,r]^d_{2d-\ell};x) = \mathcal{I}_{d+1}h^*([0,r]^d_\ell;x) = x\mathcal{I}_d A^{(r)}_{d,\ell},$$

where the last equality follows from Corollary 3.4.21 stating that the degree of $A^{(r)}_{d,\ell}$ is d. Thus, we see that $h(\mathcal{R}_j;x)$ is either $A^{(r)}_{d,\ell}$ or $x\mathcal{I}_d A^{(r)}_{d,\ell}$ for some $0 \leq \ell \leq d$. Thus it suffices to show that for a fixed $r \geq 2$ and $d \geq 1$ the concatenated sequence

$$\left((A^{(r)}_{d,\ell})^d_{\ell=0}, (x\mathcal{I}_d A^{(r)}_{d,\ell})^0_{\ell=d}\right)$$

is interlacing. The justification is identical to the one found in the proof of Theorem 3.4.7. □

In the proofs of Corollary 3.4.8, Corollary 3.4.9, and Corollary 3.4.10, it was argued that the boundary complexes of cuboids, capped cubical polytopes, and neighborly cubical polytopes all admit stable shellings. Hence, as a corollary to Theorem 3.4.22, we recover that the r^{th} edgewise subdivision of the boundary complex of all well-known examples of cubical polytopes have real-rooted h-polynomials.

Corollary 3.4.23. *If $\mathcal{C}$ is the boundary complex of a cuboid, a capped cubical polytope, or a neighborly cubical polytope, then $h(\mathcal{C}^{\langle r\rangle};x)$ is real-rooted for all $r \geq 2$.*

The results of Corollary 3.4.23, Corollary 3.4.8, Corollary 3.4.9, and Corollary 3.4.10 collectively show how the same stable shelling allows one to deduce the real-rootedness of h-polynomials for a variety of different uniform subdivisions.

3.5 Stable Line Shellings

One of our applications of stable shellings in Section 3.4 was that any cubical complex admitting such a shelling has a barycentric subdivision with a real-rooted h-polynomial (Theorem 3.4.7). This gave a positive answer to Problem 3.4.1 for all well-known constructions of cubical polytopes. A classic result of Bruggesser and Mani [30] states that, in fact, the boundary complex of any polytope admits a shelling. Hence, the result of Theorem 3.4.7 suggests the following general question:

Question 3.5.1. Does every polytope (cubical or otherwise) admit a stable shelling?

The result of Bruggesser and Mani [30] is, in fact, stronger than stated since they further demonstrate that the boundary complex of any polytope admits a special type of shelling known as a line shelling. Suppose that $\mathcal{C}$ is the boundary complex of a d-dimensional polytope P with s facets and m vertices. The ***realization space*** of $\mathcal{C}$, denoted $R_{\mathcal{C}}$, is the space of all geometric realizations $\Sigma \subset \mathbb{R}^d$ of $\mathcal{C}$ as the boundary complex of a convex polytope in $\mathbb{R}^d$. Hence, each realization $\Sigma \in R_{\mathcal{C}}$ is the boundary complex of a convex polytope in $\mathbb{R}^d$, which we will denote by Q_Σ. Note that $R_{\mathcal{C}}$ can be thought of as a semialgebraic set living in $\mathbb{R}^{d\times m}$ where each realization Σ corresponds to a $d \times m$ matrix whose columns are the realizations of the vertices of $\mathcal{C}$. Since $\mathcal{C}$ is $(d-1)$-dimensional, then for all $\Sigma \in R_{\mathcal{C}}$, the geometric realization $\sigma_i \in \Sigma$ of a given facet F_i of $\mathcal{C}$ spans an affine hyperplane $H_i \subset \mathbb{R}^d$. Let $\mathcal{A}_\Sigma := \{H_i : i \in [s]\}$ denote the corresponding hyperplane arrangement. Fix $\Sigma \in R_{\mathcal{C}}$, and let $\ell \subset \mathbb{R}^d$ be a line that intersects each hyperplane $H_i \in \mathcal{A}_\Sigma$ at a point $q_i := \ell \cap H_i$. Assume that $q_1, \dots, q_s$ are all distinct and that ℓ intersects the interior of the convex polytope Q_Σ. Without loss of generality, the point q_1 then lies in the interior of some facet σ_1 of Σ. Consider $\mathcal{A}_\Sigma$ and ℓ in the one-point compactification of $\mathbb{R}^d$, denoted $\mathbb{R}^d \cup \{\infty\}$. By fixing an orientation of ℓ and following this orientation outwards from the initial point q_1, we obtain a linear ordering of the points of intersection of ℓ with $\mathcal{A}_\Sigma$ and the point ∞:

$$(q_1, \dots, q_t, \infty, q_{t+1}, \dots, q_s).$$

If the corresponding linear order $(F_1, \dots, F_t, F_{t+1}, \dots, F_s)$ of the facets of $\mathcal{C}$ is a shelling, we call it a ***line shelling*** of $\mathcal{C}$ (induced by $\mathcal{A}_\Sigma$ and ℓ). A well-known fact about line shellings is the following (see, for instance, [76] or [30]): If $i \leq t$, then the set of facets of F_i that are not included in the relative complex $\mathcal{R}_i$ associated to F_i by $(F_1, \dots, F_s)$ is the collection of facets F_i realized by facets of σ_i that are ***visible*** from the point q_i; that is, the set of all facets containing a point q that is visible from q_i in H_i (as defined in Subsection 3.2). On the other hand, if $i > t$, then the set of facets of F_i that are not included in $\mathcal{R}_i$ are those facets realized by facets of σ_i that are not visible from q_i in H_i. These are callled the ***covisible facets*** of F_i.

Line shellings are a well-studied tool in geometric and algebraic combinatorics (see for instance [76]). The result of Bruggesser and Mani [30] states that the boundary complex of any convex polytope admits a line shelling. Given this result, a positive answer to Question 3.5.1 that gives a stable line shelling for the boundary complex of every polytope would yield an answer to Problem 3.4.1 in the case of all cubical polytopes (via Theorem 3.4.7). Hence, it would be of general interest to better understand the geometry of stable line shellings. Namely, given the boundary complex $\mathcal{C}$ of a d-dimensional polytope P, we let $\mathcal{L}_{\mathcal{C}}$ denote the collection of all pairs (Σ, ℓ) for which $\Sigma \in R_{\mathcal{C}}$ and $\ell \in \mathbb{R}^d$ induce a line shelling of $\mathcal{C}$. We further let $\mathcal{L}_{\mathcal{C}}^s \subseteq \mathcal{L}_{\mathcal{C}}$ denote the space of all pairs that induce stable line shellings of $\mathcal{C}$.

Problem 3.5.1. *Let $\mathcal{C}$ be the boundary complex of a d-dimensional polytope P. Describe the space of stable line shellings $\mathcal{L}_{\mathcal{C}}^s$.*

The geometry of $\mathcal{L}^s_{\mathcal{C}}$ is tied to the geometry of realization spaces and hyperplane arrangements, both of which have a long history in algebraic and geometric combinatorics. A description of the space $\mathcal{L}^s_{\mathcal{C}}$ would likely bring the study of such ideas closer to the theory of interlacing polynomials, and in doing so, could lead to a full answer to Problem 3.4.1 in the case of cubical polytopes. To do so, we need to see that the space $\mathcal{L}^s_{\mathcal{C}}$ is nonempty whenever $\mathcal{C}$ is the boundary complex of a cubical polytope. This is true in the simplest case; i.e., when $\mathcal{C}$ is the boundary of the d-cube.

Example 3.5.1 (The boundary of the d-cube)**.** Consider the geometric realization of the boundary of the d-dimensional cube as the boundary of $[0,1]^d \subset \mathbb{R}^d$. Let $\ell \subset \mathbb{R}^d$ be a general line passing through the interior of $[0,1]^d$. It follows that ℓ intersects each facet-defining hyperplane of $[0,1]^d$ at a distinct point, and ordering these points with respect to an orientation of ℓ yields a line shelling $(F_1,\ldots,F_{2d})$ of the facets of $[0,1]^d$. Let H_i denote the facet-defining hyperplane of $[0,1]^d$ containing the facet F_i for all $i \in [2d]$, and consider the point q_i lying in the hyperplane H_i defining the facet F_i for some $i \in [2d]$. Without loss of generality, this hyperplane is of the form $x_j = 0$ for some $j \in [d]$. By identifying the affine subspace of $\mathbb{R}^d$ defined by the hyperplane $x_j = 0$ with $\mathbb{R}^{d-1}$, we see that the induced arrangement $\mathcal{A}_i := \{H_i \cap H_k : k \in [2d] \setminus \{i\}\}$ lying in the hyperplane H_i is, up to reindexing, the hyperplane arrangement $\mathcal{A}_{[0,1]^{d-1}}$. Since q_i does not lie in any facet-defining hyperplane for a facet F_k of $[0,1]^d$ for $k \neq i$, it follows that q_i lies in an open region of $\mathbb{R}^{d-1} \setminus \{x \in \mathbb{R}^{d-1} : x \in H \text{ for some } H \in \mathcal{A}_{[0,1]^{d-1}}\}$. Since the arrangement $\mathcal{A}_{[0,1]^{d-1}}$ consists of the hyperplanes $x_j = 0$ and $x_j = 1$ for all $j \in [d-1]$, it follows that this region is either the interior of $[0,1]^{d-1}$ or it is all $x \in \mathbb{R}^{d-1}$ satisfying $x_k < 0$ and $x_{k'} > 1$ for the facet-defining hyperplanes $x_k = 0$ and $x_{k'} = 1$ of the tangent cone $T_F([0,1]^{d-1})$ of some face F of $[0,1]^{d-1}$. In the former case the set of visible facets of F_i from q_i is empty and the set of covisible facets is the complete set of facets of F_i. In the latter case, the set of visible facets of F_i from q_i is the set of all facets containing the face F, and the set of covisible facets is their complement. In either case, the relative complex $\mathcal{R}_i$ is stable, and hence, so is the line shelling $(F_1,\ldots,F_s)$.

When attempting to generalize Example 3.5.1, we see that a stable line shelling of the boundary complex $\mathcal{C}$ of a d-dimensional polytope P should be a line shelling induced by a hyperplane arrangement $\mathcal{A}_\Sigma$, for $\Sigma \in R_{\mathcal{C}}$, and a line $\ell \in \mathbb{R}^d$ for which the points $q_1,\ldots,q_s$ are all 'sufficiently close' to the complex Σ in the following sense: Note that the hyperplane arrangement $\mathcal{A}_\Sigma$ naturally subdivides $\mathbb{R}^d$ into a collection of disjoint connected components. A ***region*** of the arrangement $\mathcal{A}_\Sigma$ is a connected component of the complement of the hyperplanes in $\mathcal{A}_\Sigma$:
$$\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{A}_\Sigma} H.$$
We let $\mathfrak{R}(H_\Sigma)$ denote the collection of all regions of the $\mathcal{H}_\Sigma$. Similarly, given any subset $Y \subset [s]$, we can consider the hyperplane arrangement $\mathcal{A}^Y_\Sigma$ living in the real-Euclidean space

$\bigcap_{i\in Y} H_i \subset \mathbb{R}^d$ given by

$$\mathcal{A}_\Sigma^Y := \{H_j \cap H_i : j \in [s] \setminus Y\}.$$

It follows that $\mathcal{A}_\Sigma$ subdivides $\mathbb{R}^d$ into the disjoint, connected components

$$comp(\mathcal{A}_\Sigma) := \bigcup_{Y\subset[s]} \mathfrak{R}(\mathcal{A}_\Sigma^Y).$$

Since $\mathcal{C}$ is the boundary complex of a convex polytope, then for each facet F_i of $\mathcal{C}$, there is a subset of hyperplanes $Y_i \subset \mathcal{A}_\Sigma$ such that for all $H \in Y_i$, the intersection $H \cap H_i$ is a facet-defining hyperplane in the affine subspace H_i of the realization σ_i of the facet F_i of $\mathcal{C}$. Let $\mathcal{A}_i$ denote the hyperplane arrangement $\{H \cap H_i : H \in Y_i\} \subset H$. Since $q_i \in H_i$, it follows that q_i is contained in a unique region $C^{(i)} \in \mathfrak{R}(\mathcal{A}_i)$. The following proposition notes that a line shelling of $\mathcal{C}$ in which the points q_i are sufficiently close to Σ with respect to the (combinatorial) geometry of the hyperplane arrangement $\mathcal{A}_\Sigma$ will be stable.

Proposition 3.5.2. *Let $\mathcal{C}$ be the boundary complex of a d-dimensional polytope P and let $(F_1, \ldots, F_s)$ be a line shelling of $\mathcal{C}$ induced by $\mathcal{A}_\Sigma$ and a line $\ell \in \mathbb{R}^d$. Then $(F_1, \ldots, F_s)$ is stable if, for all $i \in [s]$, the closure of the region $C^{(i)} \in \mathfrak{R}(\mathcal{A}_i)$ containing the point q_i also contains a point of the geometric realization σ_i of F_i.*

Proof. We work directly with the shelling $(\sigma_1, \ldots, \sigma_s)$ of the geometric realization Σ of $\mathcal{C}$. Suppose that $(\sigma_1, \ldots, \sigma_s)$ is a line shelling induced by ℓ such that, for all $i \in [s]$, the closure of the region $C^{(i)} \in \mathfrak{R}(\mathcal{A}_i)$ containing the point q_i also contains a point of σ_i. Notice first that, since Σ is the boundary complex of a convex polytope Q_Σ, the facet-defining hyperplanes of σ_i are given by a subset $\{H_1, \ldots, H_M\}$ of the hyperplanes in $\mathcal{A}_\Sigma \setminus \{H_i\}$ in the sense that

$$\mathcal{A}_i = \{G_1 := H_i \cap H_1, \ldots, G_M := H_i \cap H_M\}.$$

Moreover, since $H_i \simeq \mathbb{R}^{d-1}$, each hyperplane $G \subset \mathcal{A}_i$ consists of the set of solutions $x \in \mathbb{R}^{d-1}$ to a linear equation $\langle a_G, x\rangle = b_G$ for some $a_G \in \mathbb{R}^{d-1}$ and $b_G \in \mathbb{R}$.

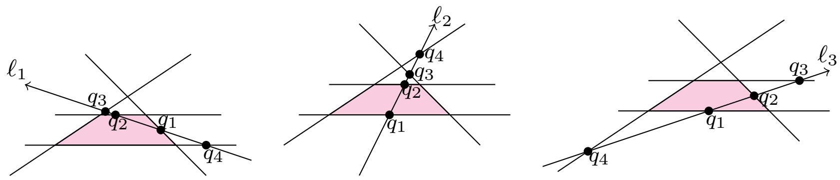
Since $(\sigma_1, \ldots, \sigma_s)$ is a line shelling it is induced by the ordering

$$(q_1, \ldots, q_t, \infty, q_{t+1}, \ldots, q_s)$$

of points in $\mathbb{R}^d \cup \{\infty\}$.

Let $vis(q_i)$ denote the set of facet-defining hyperplanes of σ_i that define facets visible from q_i, and let $covis(q_i)$ denote the set of facet-defining hyperplanes of all covisible facets from q_i.

As noted in Subsection 3.2, a point q lying in a facet σ of σ_i defined by the hyerplane $G \in \mathcal{A}_i$ is visible from q_i if and only if $q_i \notin T_\sigma(\sigma_i)$, the tangent cone of σ_i at σ. Hence, q is visible from q_i if and only if $\langle a_G, q_i\rangle > b_G$. It follows that the point q_i lies in the region $C^{(i)}$ of $\mathcal{A}_i$ consisting of all points $x \in H_i$ satisfying $\langle a_G, x\rangle > b_G$ for all $G \in vis(q_i)$ and $\langle a_G, x\rangle < b_G$ for all $G \in covis(q_i)$. Therefore, the closure of this region is all $x \in H_i$ satisfying $\langle a_G, x\rangle \geq b_G$



(a) (b) (c)

Figure 3.3: Some examples of (stable) line shellings.

for all $G \in vis(q_i)$ and $\langle a_G, x\rangle \leq b_G$ for all $G \in covis(q_i)$. From the inequality description of $C^{(i)}$ we know that the closure of $C^{(i)}$ contains the face σ of σ_i defined by $\langle a_G, x\rangle \leq b_G$ for all $G \in vis(q_i)$. We claim that for all $1 < i < s$, the face σ is nonempty.

To see this, recall that we are assuming that the closure of $C^{(i)}$ contains a point q of σ_i. Since q is in the closure of $C^{(i)}$ then $\langle a_G, q\rangle \geq b_G$ for all $G \in vis(q_i)$. On the other hand, since q is a point in the convex polytope σ_i, which is defined as the set of solutions $x \in \mathbb{R}^d$ to the system of inequalities $\langle a_G, x\rangle \leq b_G$ for $G \in \mathcal{A}_i$, then $\langle a_G, q\rangle \leq b_G$ for all $G \in vis(q_i)$. Hence, $\langle a_G, q\rangle = b_G$ for all $G \in vis(q_i)$, and so the face σ is nonempty.

Now, if $1 < i \leq t$, since the facets of σ_i not included in the relative complex $\mathcal{R}_i$ is precisely the set of visible facets from q_i, it follows that

$$\mathcal{R}_i = \mathcal{C}(\sigma_i) \setminus \mathcal{C}(L([\sigma, \sigma_i]^*)).$$

On the other hand, if $t < i < s$, then the set of facets of σ_i not included in the relative complex $\mathcal{R}_i$ is precisely the set of covisible facets $covis(q_i)$, and so it follows that

$$\mathcal{R}_i = \mathcal{C}(\sigma_i) \setminus \mathcal{C}(A(L(\sigma_i)^*) \setminus A([\sigma, \sigma_i]^*)).$$

Hence, $\mathcal{R}_i$ is stable for all $1 < i < s$. Finally, when $i = 1$ or $i = s$, the point q_i, by definition of a line shelling, lies in the interior of σ_i. Hence, the set of visible facets from q_i is empty. Therefore, the relative complex $\mathcal{R}_1$ is simply the entire complex $\mathcal{C}(\sigma_1)$, and the relative complex $\mathcal{R}_s$ is $\mathcal{C}(\sigma_s) \setminus \mathcal{C}(\partial\sigma_s)$. Each of these arises as a reciprocal domain for the face $\emptyset$ of σ_1 and σ_s, respectively. Therefore, for all $i \in [s]$, the relative complex $\mathcal{R}_i$ is stable, and we conclude that $(F_1, \dots, F_s)$ is a stable line shelling of $\mathcal{C}$. □

Example 3.5.2. In Figure 3.3, we see three different lines ℓ_1, ℓ_2, and ℓ_3 in $\mathbb{R}^2$ that induce line shellings of the quadrilateral defined by the given arrangement of facet-defining hyperplanes. The blue quadrilateral is the polytope Q_Σ with respect to the embedding Σ of the boundary complex of the quadrilateral. The four unlabeled lines in each figure constitute the hyperplane arrangement $\mathcal{A}_\Sigma$. Each of these line shellings is stable because it satisfies the conditions of Proposition 3.5.2. This can also be seen by Proposition 3.3.2 and the fact that all 2-dimensional polytopes are simplicial.

These small examples help us to see some of the intricacies of stable line shellings. The line shellings given by Figure 3.3 (a) and (c) are seen to be stable since each point q_i for $i \in \{1, 2, 3, 4\}$ lies in the closure of a region of the hyperplane arrangement $\mathcal{A}_\Sigma$ containing a face of σ_i (the facet of Q_Σ on the line in $\mathcal{A}_\Sigma$ containing the point q_i). Hence, each point q_i must also lie in a region of $\mathcal{A}_i$ whose closure contains a point of σ_i, and so stability follows by Proposition 3.5.2. On the other hand, this same reasoning does not apply when deducing that the line shelling given by Figure 3.3 (b) is stable. Here, the point q_4 lies in the closure of a region of $\mathcal{A}_\Sigma$ that does not contain a point of σ_4. This is because the point q_4 is separated from σ_4 by the hyperplane that defines the facet σ_3. However, this line shelling is still stable by Proposition 3.5.2, since we do not include the hyperplane $H_3 \cap H_4$ in the arrangement $\mathcal{A}_4$.

Figure 3.3 (a) and (c) show that there are line shellings that can be deduced to be stable by using the fact that each point q_i is contained in the closure of a region of $\mathcal{A}_\Sigma$ that also contains a point of F_i. We can call such line shellings ***strongly stable***, since the shelling can be deduced to be stable via the global geometry of the arrangement $\mathcal{A}_\Sigma$, without restricting to the induced arrangements $\mathcal{A}_i$. In fact, reflecting on Example 3.5.1, we see that all line shellings of the boundary complex of the d-dimensional cube are strongly stable.

In general, the authors hope that the boundary complex $\mathcal{C}$ of every cubical polytope admits a stable line shelling induced by some realization $\Sigma \in R_\mathcal{C}$ and line $\ell \in \mathbb{R}^d$ that is sufficiently close to Σ, in the sense of Proposition 3.5.2.

Conjecture 3.5.3. *The boundary complex of any cubical polytope admits a stable line shelling.*

It would also be of interest to know whether or not the boundary complex of every cubical or simplicial (or otherwise) polytope admits a strongly stable line shelling.

Chapter 4

Inequalities for f^*-vectors of Lattice Polytopes

For a d-dimensional lattice polytope $P \subset \mathbb{R}^d$ and a positive integer n, consider its Ehrhart polynomial $\mathrm{ehr}_P(t)$, denoting the number of integer lattice points in tP. Similarly to other combinatorial polynomials, it is useful to express $\mathrm{ehr}_P(n)$ in different bases; here we consider two such bases consisting of binomial coefficients:

$$\mathrm{ehr}_P(t) \;=\; \sum_{k=0}^{d} h_k^* \binom{t+d-k}{d} \;=\; \sum_{k=0}^{d} f_k^* \binom{t-1}{k}. \tag{4.1}$$

Note that $(h_0^*, h_1^*, \ldots, h_d^*)$ is the h^*-vector of P described in Section 2.3, and we call $(f_0^*, f_1^*, \ldots, f_d^*)$ the ***f^*-vector*** of P. Recall that Stanley [69] proved that the h^*-vector of any lattice polytope is nonnegative (whereas the coefficients of $\mathrm{ehr}_P(n)$ written in the standard monomial basis can be negative). Breuer [29] proved that the f^*-vector of any lattice polytopal complex is nonnegative (whereas the h^*-vector of a complex can have negative coefficients); his motivation was that various combinatorially-defined polynomials can be realized as Ehrhart polynomials of complexes and so the nonnegativity of the f^*-vector yields a strong constraint for these polynomials.

The f^*- and h^*-vector can also be defined through the Ehrhart series of P:

$$\mathrm{E}hr(P;z) \;:=\; 1 + \sum_{t \ge 1} \mathrm{ehr}_P(t)\, z^t \;=\; \frac{\sum_{k=0}^{d} h_k^*\, z^k}{(1-z)^{d+1}} \;=\; 1 + \sum_{k=0}^{d} f_k^* \left(\frac{z}{1-z}\right)^{k+1}.$$

It is thus sometimes useful to add the definition $f_{-1}^* := 1$.

The f^*- and h^*-vectors share the same relation as f- and h-vectors of polytopes/polyhedral complexes, namely

$$\sum_{k=0}^{d} h_k^*\, z^k \;=\; \sum_{k=0}^{d+1} f_{k-1}^*\, z^k (1-z)^{d-k+1} \tag{4.2}$$

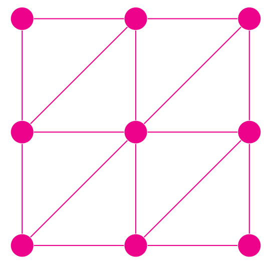


Figure 4.1: A (regular) unimodular triangulation of the cube $[-1,1]^2$.

$$h_k^* = \sum_{j=-1}^{k-1} (-1)^{k-j-1} \binom{d-j}{k-j-1} f_j^* \tag{4.3}$$

$$f_k^* = \sum_{j=0}^{k+1} \binom{d-j+1}{k-j+1} h_j^* . \tag{4.4}$$

The (very special) case that P admits a unimodular triangulation yields the strongest connection between f^*/h^*-vectors and f/h-vectors: in this case the f^*/h^*-vector of P equals the f/h-vector of the triangulation, respectively.

Example 4.0.1. Let P be the 2-dimensional cube $[-1,1]^2$. The unimodular triangulation of P shown in Figure 4.1, has f-vector $(f_0, f_1, f_2) = (9, 16, 8)$, as f_i counts its i-dimensional faces. Equivalently,

$$f^*(P) = (9, 16, 8) ,$$

and one easily checks that (4.1) yields the familiar Ehrhart polynomial $\mathrm{ehr}_P(t) = (2t+1)^2$.

Example 4.0.2. The f^*-vector of a d-dimensional unimodular simplex Δ equals

$$\left[\binom{d+1}{1}, \binom{d+1}{2}, \ldots, \binom{d+1}{d+1} \right],$$

coinciding with the f-vector of Δ considered as a simplicial complex. If we append this vector by $f_{-1}^* = 1$, it gives the only instance of a ***symmetric*** f^*-vector of a lattice polytope P, since the equality $f_{-1}^* = f_d^*$ implies that $h_i^* = 0$ for all $1 \le i \le d$.

There has been much research on (typically linear) constraints for the h^*-vector of a given lattice polytope (see, e.g., [71, 70]). On the other hand, f^*-vectors seem to be much less studied, and the goal of this chapter is to rectify that situation. The motivating question is how close the f^*-vector of a given lattice polytope is to being unimodal. The main results of this chapter are joint work with Matthias Beck, Danai Deligeorgaki, and Jerónimo Valencia, and are also published in [14]. The results are as follows.

Theorem 4.0.1. *Let $d \geq 2$ and let P be a d-dimensional lattice polytope. Then*

(a) $f_0^* < f_1^* < \cdots < f^*_{\lfloor \frac{d}{2} \rfloor - 1} \leq f^*_{\lfloor \frac{d}{2} \rfloor}$;

(b) $f^*_{\lfloor \frac{3d}{4} \rfloor} > f^*_{\lfloor \frac{3d}{4} \rfloor + 1} > \cdots > f_d^*$;

(c) $f_k^* \leq f^*_{d-1-k}$ *for* $0 \leq k \leq \frac{(d-3)}{2}$.

Examples 4.0.1 and 4.0.2 yield cases of polytopes for which the inequalities $f^*_{\lfloor \frac{3d}{4} \rfloor - 1} < f^*_{\lfloor \frac{3d}{4} \rfloor}$ and $f^*_{\lfloor \frac{d}{2} \rfloor} > f^*_{\lfloor \frac{d}{2} \rfloor + 1}$ hold, respectively.

We record the following immediate consequence of Theorem 4.0.1.

Corollary 4.0.2. *Let P be a d-dimensional lattice polytope. Then for $0 \leq k \leq d$,*

$$f_k^* \geq \min\{f_0^*, f_d^*\} .$$

Theorem 4.0.3. *The f^*-vector of a d-dimensional lattice polytope, where $1 \leq d \leq 13$, is unimodal. On the other hand, there exists a 15-dimensional lattice simplex with nonunimodal f^*-vector.*

Even though f^*-vectors are quite different from f-vectors of polytopes, the above results resemble striking similarities with existing theorems on f-vectors. Namely, Björner [20, 17, 18] proved that the f-vector of a simplicial d-polytope satisfies all inequalities in Theorem 4.0.1 (with the *s removed, and the last coordinate dropped). In fact, Björner also showed that in the f-analogue of Theorem 4.0.1(b) the decrease starts from $\lfloor \frac{3(d-1)}{4} \rfloor - 1$ instead of $\lfloor \frac{3d}{4} \rfloor$, and that the inequalities in Theorem 4.0.1(a) and (b) cannot be further extended, by constructing a simplicial polytope with f-vector that peaks at f_j, for any $\lfloor \frac{d}{2} \rfloor \leq j \leq \lfloor \frac{3(d-1)}{4} \rfloor - 1$.

Corollary 4.0.2 compares the entries of the f^*-vector with the minimum between the first and the last entry. Note that a similar relation for f-vectors of polytopes was recently proven by Hinman [45], answering a question of Bárány from the 1990s. (Hinman also proved a stronger result, namely certain lower bounds for the ratios $\frac{f_k}{f_0}$ and $\frac{f_k}{f_{d-1}}$.)

The f-analogue of Theorem 4.0.3 is again older: Björner [20] showed that the f-vector of any simplicial d-polytope is unimodal for $d \leq 15$ (later improved to $d \leq 19$ by Eckhoff [34]), and he and Lee [16] produced examples of 20-dimensional simplicial polytopes with nonunimodal f-vectors.

For a special class of polytopes we can increase the range in Theorem 4.0.1(b). A lattice polytope P is ***Gorenstein of index*** g if

- nP contains no interior lattice points for $1 \leq n < g$,
- gP contains a unique interior lattice point, and

- $\mathrm{ehr}_P(n-g)$ equals the number of interior lattice points in nP, for $n > g$.

This is equivalent to P having degree $d+1-g$ and a symmetric h^*-vector (with respect to its degree).

Theorem 4.0.4. *Let P be a d-dimensional Gorenstein polytope of index g. Then*

$$f^*_{k-1} > f^*_k \qquad \text{for } \ \tfrac{1}{2}\left(d+1+\left\lfloor \tfrac{d+1-g}{2} \right\rfloor\right) \le k \le d\,.$$

Going even further, for a certain class of polytopes we can prove unimodality of the f^*-vector, a consequence of the following refinement of Theorem 4.0.1(b) for polytopes with degree $< \frac{d}{2}$.

Theorem 4.0.5. *Let P be a d-dimensional lattice polytope with degree $\le s$. Then*

$$f^*_{k-1} > f^*_k \qquad \text{for } \ \left\lceil \tfrac{d+s}{2} \right\rceil \le k \le d\,,$$

unless the degree of P is 0, i.e., P is a unimodular simplex with f^*-vector as in Example 4.0.2.

This theorem implies that lattice d-polytopes of degree s satisfying $s^2 - s - 1 \le \frac{d}{2}$ have a unimodal f^*-vector (see Proposition 4.1.1 below for details). One family with asymptotically small degree, compared to the dimension, is given by taking iterated pyramids. Given a polytope $P \subset \mathbb{R}^d$, we denote by $\mathrm{Pyr}(P) \subset \mathbb{R}^{d+1}$ the convex hull of P and the $(d+1)$st unit vector. It is well known that P and $\mathrm{Pyr}(P)$ have the same h^*-vector (ignoring an extra 0), and so we conclude:

Corollary 4.0.6. *If P is any lattice polytope then $\mathrm{Pyr}^n(P)$ has unimodal f^*-vector for sufficiently large n.*

4.1 Proofs

We start with a few warm-up proofs which only use the fact that h^*-vectors are nonnegative.

Proof of Theorem 4.0.1(a). It follows by (4.4) and the nonnegativity of $h^*(P)$ that, for $1 \le k \le \left\lfloor \frac{d}{2} \right\rfloor$,

$$f^*_k - f^*_{k-1} \;=\; \sum_{j=0}^{k+1} \left(\binom{d+1-j}{k+1-j} - \binom{d+1-j}{k-j} \right) h^*_j \;\ge 0.$$

In fact, $f^*_k - f^*_{k-1}$ is bounded below by $\left(\binom{d+1}{k+1} - \binom{d+1}{k}\right) h^*_0 > 0$ for $1 \le k < \left\lfloor \frac{d}{2} \right\rfloor$, since $h^*_0 = 1$. □

Proof of Theorem 4.0.1(c). For $0 \le k \le \frac{(d-3)}{2}$, equation (4.4) gives

$$\begin{aligned} f^*_{d-1-k} - f^*_k &= \sum_{j=0}^{d-k} \left(\binom{d+1-j}{d-k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \\ &= \sum_{j=0}^{d-1-2k} \left(\binom{d+1-j}{k+1} - \binom{d+1-j}{k+1-j} \right) h^*_j + \sum_{j=d-2k}^{d-k} \left(\binom{d+1-j}{d-k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j. \end{aligned}$$

We have $\binom{d+1-j}{k+1} - \binom{d+1-j}{k+1-j} \ge 0$ since $k+1-j \le k+1 \le \frac{d+1-j}{2}$ holds for $0 \le j \le d-1-2k$. Similarly, $\binom{d+1-j}{d-k-j} - \binom{d+1-j}{k+1-j} \ge 0$ holds because $k+1-j \le d-k-j \le \frac{d+1-j}{2}$ for all $d-2k \le j$. Therefore, it follows by the nonnegativity of h^*-vectors that $f^*_{d-1-k} - f^*_k \ge 0$. □

Proof of Theorem 4.0.5. Since $h^*_j = 0$ for $j \ge s+1$, (4.4) gives

$$f^*_{k-1} - f^*_k = \sum_{j=0}^{s} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j = \sum_{j=0}^{s} \frac{2k-d-j}{k+1-j} \binom{d+1-j}{k-j} h^*_j.$$

For $\frac{d+s}{2} \le k \le d$, we have $k+1-j > 0$ and $2k-d-j > 0$ for all $j = 0, ..., s-1$, and $k+1-j > 0$, $2k-d-j \ge 0$ for $j = s$. Therefore, the claim follows by the nonnegativity of h^*-vectors and the positivity of h^*_0. □

Proposition 4.1.1. *Let P be a d-dimensional lattice polytope that has degree at most s for some positive s. If $d \ge 2s^2-2s-2$ then the f^*-vector of P is unimodal with a (not necessarily "sharp") peak at f^*_p, where $\lfloor \frac{d}{2} \rfloor \le p \le \lceil \frac{d+s}{2} \rceil - 1$.*

Proof. By Theorems 4.0.1(a) and 4.0.5, it suffices to show that $f^*_{\lfloor \frac{d}{2} \rfloor + i} \ge f^*_{\lfloor \frac{d}{2} \rfloor + i+1}$ implies $f^*_{\lfloor \frac{d}{2} \rfloor + i+1} \ge f^*_{\lfloor \frac{d}{2} \rfloor + i+2}$, i.e., that $2f^*_{\lfloor \frac{d}{2} \rfloor + 1+i} - f^*_{\lfloor \frac{d}{2} \rfloor + 2+i} - f^*_{\lfloor \frac{d}{2} \rfloor + i} \ge 0$ for $0 \le i \le \frac{s}{2} - 2$.

As $h^*_j = 0$ for $j \ge s+1$, by (4.4) we can express $2f^*_{\lfloor \frac{d}{2} \rfloor + 1+i} - f^*_{\lfloor \frac{d}{2} \rfloor + 2+i} - f^*_{\lfloor \frac{d}{2} \rfloor + i}$ as the sum

$$\begin{aligned} &\sum_{j=0}^{s} \left(2\binom{d+1-j}{\lfloor \frac{d}{2} \rfloor + 2 - j + i} - \binom{d+1-j}{\lfloor \frac{d}{2} \rfloor + 3 - j + i} - \binom{d+1-j}{\lfloor \frac{d}{2} \rfloor + 1 - j + i} \right) h^*_j \\ &= \sum_{j=0}^{s} \left(\frac{2\left(\lceil \frac{d}{2} \rceil - i\right)}{\lfloor \frac{d}{2} \rfloor + 2 - j + i} - \frac{(\lceil \frac{d}{2} \rceil - i)(\lceil \frac{d}{2} \rceil - 1 - i)}{(\lfloor \frac{d}{2} \rfloor + 2 - j + i)(\lfloor \frac{d}{2} \rfloor + 3 - j + i)} - 1 \right) \binom{d+1-j}{\lfloor \frac{d}{2} \rfloor + 1 - j + i} h^*_j. \end{aligned}$$

Since $d \ge \max\{2s^2 - 2s - 2, 0\}$ we have that $(\lfloor \frac{d}{2} \rfloor + 3 - j + i)(\lfloor \frac{d}{2} \rfloor + 2 - j + i)$ is positive for $j = 0, ..., s$ and since h^*_j is nonnegative, it remains to show that

$$\begin{aligned} &2(\lceil \tfrac{d}{2} \rceil - i)(\lfloor \tfrac{d}{2} \rfloor + 3 - j + i) - (\lceil \tfrac{d}{2} \rceil - i)(\lceil \tfrac{d}{2} \rceil - 1 - i) - (\lfloor \tfrac{d}{2} \rfloor + 2 - j + i)(\lfloor \tfrac{d}{2} \rfloor + 3 - j + i) \\ &\quad = d - (2j-1)(\lceil \tfrac{d}{2} \rceil - \lfloor \tfrac{d}{2} \rfloor) + 4i(\lceil \tfrac{d}{2} \rceil - \lfloor \tfrac{d}{2} \rfloor) - 12i + 4ij - 4i^2 - 6 + 5j - j^2 \qquad (4.5) \\ &\quad = \begin{cases} d - 4i^2 + 4ij - 12i - j^2 + 5j - 6 & \text{if } d \text{ is even,} \\ d - 4i^2 + 4ij - 8i - j^2 + 3j - 1 & \text{if } d \text{ is odd,} \end{cases} \end{aligned}$$

is nonnegative for $0 \leq j \leq s$. Indeed, the conditions $j \leq s$ and $i \leq \frac{s}{2} - 2$ imply that (4.5) is bounded below by

$$d - 4i^2 - 12i - j^2 - 6 \geq d - 4(\tfrac{s}{2} - 2)^2 - 12(\tfrac{s}{2} - 2) - s^2 - 6 = d - 2s^2 + 2s + 2,$$

which is nonnegative by assumption. $\square$

The next proofs use more than just the nonnegativity of h^*-vectors. The first result needs the following elementary lemma on binomial coefficients.

Lemma 4.1.2. *Let j, k, n be positive integers such that $k \leq n + 1 - j$. Then*

$$\left|\binom{n}{k} - \binom{n}{k-1}\right| \geq \left|\binom{n-j}{k} - \binom{n-j}{k-1}\right|$$

whenever $n \neq 2k - 1$.

Proof. It suffices to prove the statement for the cases i) $j = 1$ and the quantities $\binom{n}{k} - \binom{n}{k-1}$ and $\binom{n-1}{k} - \binom{n-1}{k-1}$ having the same sign, and ii) the point when the signs change, i.e., $n = 2k$ and $j = 2$.

To show case i), we simplify

$$\left|\binom{n}{k} - \binom{n}{k-1}\right| = \frac{(n-1)!}{k!(n-k)!} \frac{n}{n-k+1} |n - 2k + 1|$$

and

$$\left|\binom{n-1}{k} - \binom{n-1}{k-1}\right| = \frac{(n-1)!}{k!(n-k)!} |n - 2k| .$$

If $n \geq 2k$ then the inequalities

$$\frac{n}{n-(k-1)}(n - 2k + 1) \geq n - 2k + 1 > n - 2k$$

imply that

$$\left|\binom{n}{k} - \binom{n}{k-1}\right| > \left|\binom{n-1}{k} - \binom{n-1}{k-1}\right|. \tag{4.6}$$

If $n \leq 2k - 2$, we have $k(-2k + 2 + n) \leq 0$ which is equivalent to

$$\frac{n}{n-(k-1)}(2k - n - 1) \geq 2k - n$$

and so again (4.6) holds as a weak inequality.

To show case ii), we compute

$$\left|\binom{2k}{k} - \binom{2k}{k-1}\right| = \frac{(2k)!}{k!(k+1)!} = \frac{(2k-2)!}{k!(k-1)!} \frac{2k(2k-1)}{k(k+1)}$$

and

$$\left|\binom{2k-2}{k} - \binom{2k-2}{k-1}\right| = \frac{(2k-2)!}{k!(k-1)!}.$$

Since $2(2k-1) \geq (k+1)$ for any positive k, we conclude that

$$\left|\binom{2k}{k} - \binom{2k}{k-1}\right| \geq \left|\binom{2k-2}{k} - \binom{2k-2}{k-1}\right|. \qquad \square$$

Proof of Theorem 4.0.1(b). The inequality $f^*_{d-1} > f^*_d$ holds by Theorem 4.0.5. Now, let $\lfloor \frac{3d}{4} \rfloor + 1 \leq k < d$. By (4.4),

$$f^*_{k-1} - f^*_k = \sum_{j=0}^{k+1} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j . \tag{4.7}$$

The difference $\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j}$ is nonnegative whenever $k - j \geq \lfloor \frac{d+1-j}{2} \rfloor$ and negative otherwise, i.e., the difference is nonnegative whenever $j \leq 2k - d$ and negative whenever $j > 2k - d$. Since $2d - 2k < 2k + 1 - d$ for $\lfloor \frac{3d}{4} \rfloor + 1 \leq k$, from (4.7) we obtain

$$f^*_{k-1} - f^*_k \geq \sum_{j=0}^{2d-2k} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \tag{4.8}$$

$$+ \sum_{j=2k+1-d}^{k+1} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \tag{4.9}$$

where the differences appearing in (4.8) are nonnegative and the ones in (4.9) are negative. Our aim is to compare the sums in (4.8) and (4.9) to conclude that $f^*_{k-1} - f^*_k$ is positive.

Using standard identities for binomial coefficients, the right hand-side of (4.8) equals

$$\begin{aligned}
&\sum_{j=0}^{2d-2k} \left(\sum_{l=j}^{2d-2k-1} \left(\binom{d-l}{k-l} - \binom{d-l}{k+1-l} \right) + \left(\binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \right) h^*_j \\
&= \sum_{l=0}^{2d-2k-1} \left(\left(\binom{d-l}{k-l} - \binom{d-l}{k+1-l} \right) \sum_{j=0}^{2d-2k-1-l} h^*_j \right) \\
&\quad + \left(\binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=0}^{2d-2k} h^*_j,
\end{aligned}$$

hence we conclude that right hand-side of (4.8) is bounded below by

$$\begin{aligned}
&\left(\binom{d}{k} - \binom{d}{k+1} \right) h^*_0 + \left(\binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=0}^{2d-2k} h^*_j \\
&> \left(\binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=0}^{2d-2k} h^*_j
\end{aligned} \tag{4.10}$$

since $\binom{d}{k} - \binom{d}{k+1} > 0$ for $\lfloor \frac{3d}{4} \rfloor + 1 \leq k < d$, and $h_0^* = 1, h_j^* \geq 0$ for $j = 1, ..., 2d - 2k - 1$.

On the other hand, for the differences appearing in (4.9), using that $2d - 2k < j$ and $j \leq k + 1$, it follows by Lemma 4.1.2 that

$$\left| \binom{d+1-(2d-2k)}{d+1-k} - \binom{d+1-(2d-2k)}{d-k} \right| \geq \left| \binom{d+1-j}{d+1-k} - \binom{d+1-j}{d-k} \right|,$$

i.e.,

$$\left| \binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right| \geq \left| \binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right|.$$

Hence for $j \geq 2k + 1 - d$,

$$-\left(\binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \leq \binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j}.$$

Since both $-\binom{d+1-j}{k-j} + \binom{d+1-j}{k+1-j}$ and h_j^* are nonnegative for $j \geq 2k + 1 - d$, the sum in (4.9) is bounded below by

$$-\left(\binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \sum_{j=2k+1-d}^{d} h_j^* \,. \tag{4.11}$$

Now (4.10) and (4.11) yield

$$f_{k-1}^* - f_k^* > \left(\binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \left(\sum_{j=0}^{2d-2k} h_j^* - \sum_{j=2k+1-d}^{d} h_j^* \right).$$

Hibi [42] showed that the inequality

$$\sum_{j=0}^{m+1} h_j^* \;\geq\; \sum_{j=d-m}^{d} h_j^* \tag{4.12}$$

holds for $m = 0, ..., \lfloor \frac{d}{2} \rfloor - 1$. Since $2d - 2k - 1 \leq \lfloor \frac{d}{2} \rfloor - 1$ for $\lfloor \frac{3d}{4} \rfloor + 1 \leq k$, we can use (4.12) to finally obtain

$$f_{k-1}^* - f_k^* > 0 \,.$$

□

Proof of Theorem 4.0.3. If $d = 1$ or 2, there is nothing to prove.

If $3 \leq d \leq 6$, then by Theorem 4.0.1, either

$$f_0^* \leq \cdots \leq f_{\lfloor \frac{d}{2} \rfloor}^* \geq f_{\lfloor \frac{3d}{4} \rfloor}^* \geq \cdots \geq f_d^*$$

or

$$f_0^* \leq \cdots \leq f_{\lfloor \frac{d}{2} \rfloor}^* \leq f_{\lfloor \frac{3d}{4} \rfloor}^* \geq \cdots \geq f_d^*.$$

For $7 \leq d \leq 13$, we will show that if $f_i^* \geq f_{i+1}^*$, then $f_{i+1}^* \geq f_{i+2}^*$, for all $\left\lfloor \frac{d}{2} \right\rfloor \leq i \leq \left\lfloor \frac{3d}{4} \right\rfloor - 2$. By Theorem 4.0.1, this will imply the unimodality of $(f_0^*, f_1^*, ..., f_d^*)$.

We will examine each value of d separately.

Suppose that $d = 7$ and $f_3^* \geq f_4^*$. Then, by (4.4), we compute

$$2f_4^* - f_3^* - f_5^* = 14h_0^* + 14h_1^* + 10h_2^* + 5h_3^* + h_4^* - h_5^* - h_6^* > h_0^* + h_1^* + h_2^* + h_3^* - h_5^* - h_6^* - h_7^*,$$

which is always nonnegative by (4.12). Hence $f_4^* - f_5^* \geq f_3^* - f_4^*$.

Likewise, for $d = 8$, (4.12) implies that

$$2f_5^* - f_4^* - f_6^* = 6h_0^* + 14h_1^* + 14h_2^* + 10h_3^* + 5h_4^* + h_5^* - h_6^* - h_7^* > h_0^* + h_1^* + h_2^* + h_3^* - h_6^* - h_7^* - h_8^* \geq 0.$$

For $d = 9$, we similarly get

$$\begin{aligned} &f_5^* - f_6^* - 2(f_4^* - f_5^*) = 6h_0^* + 42h_1^* + 42h_2^* + 28h_3^* + 13h_4^* + 3h_5^* - h_6^* - h_7^* > \\ &h_0^* + h_1^* + h_2^* + h_3^* + h_4^* - h_6^* - h_7^* - h_8^* - h_9^* \geq 0 \end{aligned}$$

by (4.12).

A similar argument works for $d = 10$. By (4.12),

$$\begin{aligned} &2f_6^* - f_5^* - f_7^* = 33h_0^* + 48h_1^* + 42h_2^* + 28h_3^* + 14h_4^* + 4h_5^* - h_6^* - 2h_7^* - h_8^* > \\ &2(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* - h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^*) \geq 0. \end{aligned}$$

For $d = 11$, we need to consider two values: $i = 5$ and $i = 6$. The claim follows again by (4.12), since

$$\begin{aligned} &f_6^* - f_7^* - 2(f_5^* - f_6^*) = 33h_0^* + 132h_1^* + 126h_2^* + 84h_3^* + 42h_4^* + 14h_5^* + h_6^* - 2h_7^* - h_8^* > \\ &2(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^*) \geq 0, \end{aligned}$$

and

$$f_7^* - f_8^* - \frac{4}{5}(f_6^* - f_7^*) > 3(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^*) \geq 0.$$

For $d = 12$, there are also two cases: $i = 6$ and $i = 7$. Using (4.12), it follows that

$$\begin{aligned} &f_7^* - f_8^* - \frac{5}{4}(f_6^* - f_7^*) > \\ &3(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* + h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^*) \geq 0, \end{aligned}$$

and

$$\begin{aligned} &f_8^* - f_9^* - \frac{1}{2}(f_7^* - f_8^*) > \\ &3(h_0^* + h_1^* + h_2^* + h_3^* + h_4^* + h_5^* + h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^*) \geq 0. \end{aligned}$$

For $d = 13$, we employ a stronger form of (4.12). The expression

$$\begin{aligned} &f_7^* - f_8^* - \frac{7}{3}(f_6^* - f_7^*) \geq \\ &3(h_1^* + h_2^* + h_3^* + h_4^* + h_5^* + h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^* - h_{13}^*) \end{aligned}$$

is nonnegative by Theorem (6) in [71].

Similarly, using Theorem (6) in [71] we have

$$\begin{aligned} &2f_8^* - f_7^* - f_9^* \geq \\ &4(h_1^* + h_2^* + h_3^* + h_4^* + h_5^* + h_6^* - h_7^* - h_8^* - h_9^* - h_{10}^* - h_{11}^* - h_{12}^* - h_{13}^*) \geq 0. \end{aligned}$$

To construct a polytope with nonunimodal f^*-vector, we employ a family of simplices introduced by Higashitani [44]. Concretely, denote the jth unit vector by e_j and let

$$\Delta_w \;:=\; \mathrm{conv}\big\{0, e_1, e_2, ..., e_{14}, w\big\}$$

where

$$w \;:=\; (\underbrace{1, 1, \ldots, 1}_{7}, \underbrace{131, 131, \ldots, 131}_{7}, 132)\,.$$

It has h^*-vector

$$(1, \underbrace{0, 0, \ldots, 0}_{7}, 131, \underbrace{0, 0, \ldots, 0}_{7})$$

and, via (4.4), f^*-vector

$$\begin{aligned} (&16, 120, 560, 1820, 4368, 8008, 11440, 13001, \\ &12488, 11676, 11704, 10990, 7896, 3788, 1064, 132)\,. \end{aligned}$$

□

Corollary 4.1.3. *Every lattice polytope of degree at most 5 has unimodal f^*-vector.*

Proof. Let P be a d-dimensional lattice polytope of degree at most 5. We know from Theorem 4.0.3 that f^* is unimodal when $d \leq 13$.

Suppose that $d \geq 14$. The proof is similar to the proof of Proposition 4.1.1, but we need to be a bit more precise with bounds. By Theorems 4.0.1(a) and 4.0.5, it suffices to show that $f^*_{\lfloor \frac{d}{2} \rfloor + i} \geq f^*_{\lfloor \frac{d}{2} \rfloor + i+1}$ implies $f^*_{\lfloor \frac{d}{2} \rfloor + i+1} \geq f^*_{\lfloor \frac{d}{2} \rfloor + i+2}$, for $i = 0, ..., \lceil \frac{d+5}{2} \rceil - \lfloor \frac{d}{2} \rfloor - 3$. Notice that $\lceil \frac{d+5}{2} \rceil = \lfloor \frac{d}{2} \rfloor + \lceil \frac{5}{2} \rceil$, hence $i = 0$. Arguing as in the proof of Proposition 4.1.1, we can reduce the proof to showing that the expression in (4.5) in Proposition 4.1.1 is nonnegative for $0 \leq j \leq 5$ and $i = 0$, i.e., that

$$d - (2j-1)\left(\left\lceil \frac{d}{2} \right\rceil - \left\lfloor \frac{d}{2} \right\rfloor\right) - 6 + j(5-j) \;\geq\; 0\,. \tag{4.13}$$

For $0 \le j \le s$, we have

$$d - (2j-1)\left(\left\lceil \frac{d}{2} \right\rceil - \left\lfloor \frac{d}{2} \right\rfloor\right) - 6 + j(5-j) \ \ge\ d - 15\,,$$

hence (4.13) holds if $d \ge 15$. Finally, if $d = 14$ then (4.13) holds because

$$d - (2j-1)\left(\left\lceil \frac{d}{2} \right\rceil - \left\lfloor \frac{d}{2} \right\rfloor\right) - 6 + j(5-j) \ \ge\ d - 6\,. \qquad \square$$

Proof of Theorem 4.0.4. Let $s := d+1-g$. We first consider the case that s is odd; the case s even will be similar. Since $h_j^* = 0$ for $j > s$ and $h_j^* = h_{s-j}^*$,

$$\begin{aligned} f_{k-1}^* - f_k^* \ &= \ \sum_{j=0}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h_j^* \\ &= \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h_j^* + \sum_{j=\lfloor \frac{s}{2} \rfloor + 1}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h_j^* \\ &= \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \right) h_j^*\,. \end{aligned}$$

Because we assume $k \ge \frac{1}{2}(d+1+\lfloor \frac{s}{2} \rfloor)$,

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \ > \ 0$$

for $0 \le j \le \lfloor \frac{s}{2} \rfloor$. The inequality

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} > 0$$

follows directly if $\binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \ge 0$ or $k-s+j+1 < 0$. Otherwise, Lemma 4.1.2 implies that, for the same range of j,

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \ \ge \ 0\,.$$

In fact, the last inequality is strict for $k \ge \frac{1}{2}(d+1+\lfloor \frac{s}{2} \rfloor)$, as seen in the proof of Lemma 4.1.2. Finally we use that $h_j^* \ge 0$ and $h_0^* = 1$ to deduce that $f_{k-1}^* - f_k^* > 0$.

The computations in the case s even is very similar. Now we write

$$
\begin{aligned}
f^*_{k-1} - f^*_k &= \sum_{j=0}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j \\
&= \sum_{j=0}^{\frac{s}{2}-1} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \right) h^*_j \\
&\quad + \left(\binom{d-\frac{s}{2}+1}{k-\frac{s}{2}} - \binom{d-\frac{s}{2}+1}{k-\frac{s}{2}+1} \right) h^*_{\frac{s}{2}}
\end{aligned}
$$

and use the same argumentation as in the case s odd. □

4.2 Concluding Remarks

There are many open questions surrounding f^*-vectors, for example, those inspired by analogous studies of h^*-vectors. We conclude with a few open questions which are natural followups to the results presented in this chapter.

The techniques in the proof of Theorem 4.0.3 do not offer much insight in the case of 14-dimensional lattice polytopes as there are candidates for f^*-vectors with corresponding h^*-vectors that satisfy all inequalities discussed in [71]. It is unknown though if such polytopes exist.

Higashitani [44, Theorem 1.1] provided examples of d-dimensional polytopes with nonunimodal h^*-vector for all $d \geq 3$. Therefore, by Theorem 4.0.3 we have examples of polytopes that have such a h^*-vector but their f^*-vector is unimodal. It would be interesting to know if the opposite can be true, that is, if there exist polytopes with unimodal h^*-vector and nonunimodal f^*-vector. By Corollary 4.1.3, such polytopes would need to have degree at least 6.

Whenever we are able to show that a combinatorial polynomial is unimodal, it is natural to ask whether the polynomial satisfies stronger properties, such as log concavity or real-rootedness. It would be interesting if one could extend, e.g., Proposition 4.1.1 along these lines.

Finally, starting with Stapledon's work [71], there has been much recent attention to symmetric decompositions of h- and h^*-polynomials; see, e.g., [7, 9] and, in particular, [25] where analogous decompositions for f-vectors are discussed. We believe this line of research is worthy of attention with regards to understanding f^*-vectors and the inequalities that hold among their coefficients.

Chapter 5

Signed Poset Polytopes

5.1 Introduction

In the seminal paper [67], Stanley introduced two geometric incarnations of a given finite partially ordered set (*poset*) Π:

Definition 5.1.1. The ***order polytope*** of Π is given by

$$\mathcal{O}(\Pi) \ := \ \left\{x \in \mathbb{R}^{\Pi} : \, 0 \le x_p \le 1 \text{ for all } p \in \Pi \text{ and } x_a \le x_b \text{ when } a \le_{\Pi} b\right\}.$$

Definition 5.1.2. The ***chain polytope*** of Π is given by

$$\mathcal{C}(\Pi) \ := \ \{x \in \mathbb{R}^{\Pi} : \, x_p \ge 0 \text{ for all } p \in \Pi \text{ and } x_{c_1} + \cdots + x_{c_k} \le 1 \\ \text{for every chain } c_1 <_{\Pi} \cdots <_{\Pi} c_k\}.$$

There is also a natural unbounded conical analogue of an order polytope:

Definition 5.1.3. The ***order cone*** of Π is given by

$$\mathcal{K}(\Pi) \ := \ \left\{x \in \mathbb{R}^{\Pi} : \, 0 \le x_p \text{ for all } p \in \Pi \text{ and } x_a \le x_b \text{ when } a \le_{\Pi} b\right\}.$$

These polyhedra contain much information of the given poset, e.g., about its filters, chains, and linear extensions. Conversely, order polytopes (and, to a lesser extent, chain polytopes) have given a fertile ground in polyhedral and discrete geometry as a class of 0/1-polytopes with many, sometimes extreme, at other times only conjectured, properties; see, e.g., [13, Chapter 6].

It is a short step (which we will detail in Section 5.2 below) to think about a given partial order on $[n] := \{1, 2, \ldots, n\}$ as a certain subset of the type-A root system $A_n := \{e_i - e_j : 1 \le i < j \le n\}$, where e_j is the jth unit vector in $\mathbb{R}^n$. Here we need the notion of the positive linear closure of a subset of a root system:

Definition 5.1.4. Let Φ be a root system. For a subset $S \subset \Phi$, let $\overline{S}^{\text{PLC}}$ be the set of positive linear combinations of elements in S.

In [58], Reiner generalised the Coxeter description of a classical poset to type B and named these objects ***signed posets***.

Definition 5.1.5. Let B_n be the root system $\{\pm e_i : 1 \le i \le n\} \cup \{\pm e_i \pm e_j : 1 \le i < j \le n\}$. A signed poset P (on n elements) is a subset $P \subset B_n$ satisfying

1. $\alpha \in P$ implies $-\alpha \notin P$;
2. $\overline{P}^{\mathrm{PLC}} = P$.

Reiner proved several type-B analogues of classical posets results, e.g., on order ideals, permutation statistics, and P-partitions. While [58] has given rise to further work in algebraic combinatorics, the analogous polyhedral constructions seem to not have been introduced/studied. (An exception of sorts is [74], and we indicate its relation to the present article below.) Our goal is to remedy this situation and explore the geometry of signed posets. We define order polytopes and order cones for signed posets, modeled after the Coxeter-group descriptions of these objects for classical posets.

Definition 5.1.6. Let P be a signed poset on $[n]$. Then

$$\mathcal{K}_P \ := \ \{x \in \mathbb{R}^n : \langle \alpha, x \rangle \ge 0 \text{ for all } \alpha \in P\}$$

and

$$\mathcal{O}_P \ := \ \{x \in \mathbb{R}^n : \langle \alpha, x \rangle \ge 0 \text{ for all } \alpha \in P\} \cap [-1,1]^n.$$

In Section 5.2 we give detailed connections between (signed) posets, Coxeter groups, bidirected graphs, permutation statistics, and the resulting polyhedral objects. Section 5.3 contains convex-hull and halfspace description of signed order polytopes, as well as canonical triangulations. In Section 5.4 we compute the Ehrhart h^*-polynomial of a signed polytope, encoding the integer-point structure in dilates of the polytope. Analogous to classical posets, this is related to permutation statistics (now of type B). Section 5.5 gives a characterization of signed order polytopes that are Gorenstein; as a consequence these polytopes have a symmetric and unimodal h^*-polynomial. Finally, in Section 5.6 we propose one definition of a signed chain polytope and study its properties, giving yet another new class of Gorenstein polytopes.

5.2 Signed Posets, Their Cones, and Their Polytopes

We start by detailing the interpretation of a poset in terms of type-A root systems, as introduced by Reiner [58].

Proposition 5.2.1. *Let f be the following map from the set of posets on $[n]$ to subsets of A_n:*

$$f(\Pi) \ := \ \{e_j - e_i : i <_\Pi j\}$$

Then f gives a bijection between partial orders of $[n]$ and subsets $P \subseteq A_n$ satisfying

(1) $\alpha \in P$ *implies* $-\alpha \notin P$*;*

(2) $\overline{P}^{\mathrm{PLC}} = P$*.*

The first property comes from the antisymmetry property of a poset, and the second from the transitivity property.

Remark 5.2.1. In several works, including [58], the map from posets to subsets of A_n is defined as $f(\Pi) = \{e_i - e_j : i <_\Pi j\}$. We switched the direction of the inequality to make our upcoming description of order cones consistent with the description of P-partitions found in [58].

Classically-defined order cones and order polytopes can be reformulated via the Coxeter group description of the poset. Let Π be a poset and recall that $f(\Pi) = \{e_j - e_i : i <_\Pi j\}$. Then

$$\mathcal{K}(\Pi) \;=\; \left\{x \in \mathbb{R}^{\Pi}_{\geq 0} : \langle \alpha, x\rangle \geq 0 \;\text{ for all }\; \alpha \in f(\Pi)\right\}$$

and

$$\mathcal{O}(\Pi) \;=\; \left\{x \in \mathbb{R}^{\Pi}_{\geq 0} : \langle \alpha, x\rangle \geq 0 \;\text{ for all }\; \alpha \in f(\Pi)\right\} \cap [0,1]^{\Pi}.$$

We often visually view signed posets as bi-directed graphs, using the definitions found in [75].

Definition 5.2.1. Let $\Gamma = (V, E)$ be a graph. The ***incidence set*** $I(\Gamma)$ of Γ consists of all pairs (e, v) for $v \in e$.

Definition 5.2.2. A ***bidirected graph*** is a graph Γ together with a ***bidirection*** σ, which is defined as any map $I(\Gamma) \to \{+, -\}$.

We view a signed poset P on $[n]$ as a bidirected graph with vertex set $[n]$: the elements of P are the edges of the bidirected graph, and the bidirection σ is defined so that

- e_j corresponds to a loop e on j with $\sigma(e, j) = +$,
- $-e_j$ corresponds to a loop e on j with $\sigma(e, j) = -$,
- $e_i + e_j$ corresponds to an edge $e = ij$ with $\sigma(e, i) = \sigma(e, j) = +$,
- $-e_i - e_j$ corresponds to an edge $e = ij$ with $\sigma(e, i) = \sigma(e, j) = -$,
- $e_i - e_j$ corresponds to an edge $e = ij$ with $\sigma(e, i) = +$ and $\sigma(e, j) = -$, and
- $-e_i + e_j$ corresponds to an edge $e = ij$ with $\sigma(e, i) = -$ and $\sigma(e, j) = +$.

Example 5.2.1. Below, we list a collection of signed posets on 2 elements:

- $P = \{e_1, e_2, e_1 + e_2\}$

- $P = \{\}$
- $P = \{e_2\}$
- $P = \{e_1 + e_2, -e_1 + e_2, e_1, e_2\}$
- $P = \{-e_1 + e_2\}$
- $P = \{-e_1 + e_2, e_1 + e_2, e_2\}$
- $P = \{e_2, e_1 + e_2\}$

The bidirected graph representations of these signed posets is given in Figure 5.1.

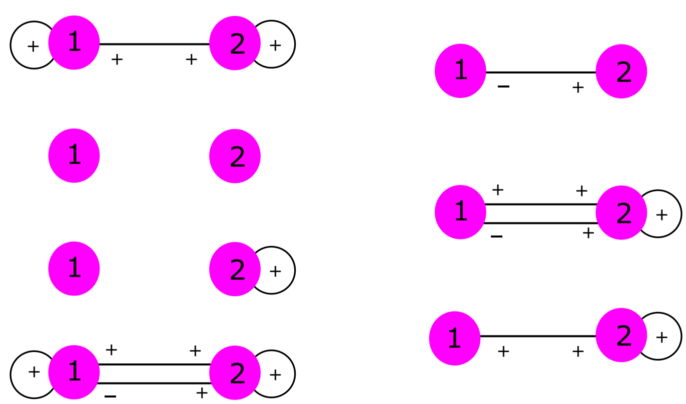


Figure 5.1: A collection of bidirected graph representations of signed posets on 2 elements.

Example 5.2.2. The left image in Figure 5.2 shows a bidirected graph representing the signed poset $P = \{e_1 + e_2, -e_1 + e_2, e_2\}$. A discussion in [58, p. 329] states that every signed poset has a unique minimal representation, i.e., a minimal subset whose positive linear closure is the whole signed poset. However, finding this minimal representation is not as straightforward as finding the cover relations of a classical poset. The minimal representation of P is shown in the right image in Figure 5.2.

Reiner also gave a notion of homomorphism of signed posets in [58]. Before we discuss this definition, we establish some facts and definitions pertaining to S_n^B, mostly following the notation set in [11].

Definition 5.2.3. A ***signed permutation*** on $[n]$ is a bijection ω on $\{\pm 1, \pm 2, \ldots, \pm n\}$ such that $\omega(-i) = -\omega(i)$ for all $i \in [n]$. We refer to the group of all such bijections as S_n^B, the signed permutation group on $[n]$, where the group operation is composition.

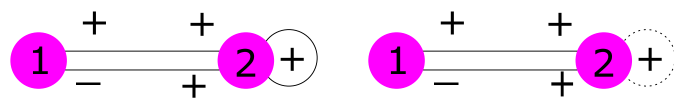


Figure 5.2: The left hand side shows the bidiricted graph representation of $P = \{-e_1 + e_2, e_1 + e_2, e_2\}$. The right hand side indicates that the unique minimal representation of P is $\{-e_1 + e_2, e_1 + e_2\}$. Since $e_2 = \frac{1}{2}(-e_1 + e_2) + \frac{1}{2}(e_1 + e_2)$, we see that e_2 is in the positive linear closure of the other two elements, and thus is not in the minimal representation of P. This is indicated on the right hand side by using a dotted loop instead of a solid loop to represent e_2.

We will later use the notation $\omega = (\pi, \epsilon)$ where $\pi \in S_n$ is defined by $\pi_i := |\omega(i)|$ and $\epsilon \in \{1, -1\}^n$ is defined via $\epsilon_i := \text{sign}(\omega(i))$. We will also use the following fact.

Proposition 5.2.2. *The set* $\{s_1, \ldots, s_{n-1}, s_0\}$ *generates* S_n^B*, where* $s_i := [1, \ldots, i-1, i+1, i, \ldots, n]$ *for* $i \in [n-1]$ *and* $s_0 := [-1, 2, \ldots, n]$

The elements of S_n^B have a natural action on the type-B root system.

Definition 5.2.4. The elements of S_n^B have a linear action on B_n generated as follows, where $i, j \in [n]$. If $\omega(i) = j$, then $\omega e_i = e_j$. If $\omega(i) = -j$, then $\omega e_i = -e_j$.

Definition 5.2.5. Let P_1 and P_2 be signed posets on $[n]$. Then P_1 and P_2 are ***isomorphic*** if there exists $\omega \in S_n^B$ such that $\omega P_1 = P_2$.

Example 5.2.3. We see from Figure 5.3 that there are many more combinatorial types of order polytopes in the signed poset setting than in the classical poset setting, even in two dimensions. The order polytopes shown come from the following signed posets:

(A) $P = \{e_1, e_2, e_1 + e_2\}$

(B) $P = \{\}$

(C) $P = \{e_2\}$

(D) $P = \{e_1 + e_2, -e_1 + e_2, e_1, e_2\}$

(E) $P = \{-e_1 + e_2\}$

(F) $P = \{-e_1 + e_2, e_1 + e_2, e_2\}$

(G) $P = \{e_2, e_1 + e_2\}$

The polytope labeled (F) is the order polytope of the signed poset illustrated in Figure 5.2. Note that the supporting hyperplanes of this polytope are given by $x_2 = 1$ and $\langle \alpha, x\rangle = 0$, where α is an element of the unique minimal representation described in Example 1.6. We will generalize this observation below‘.

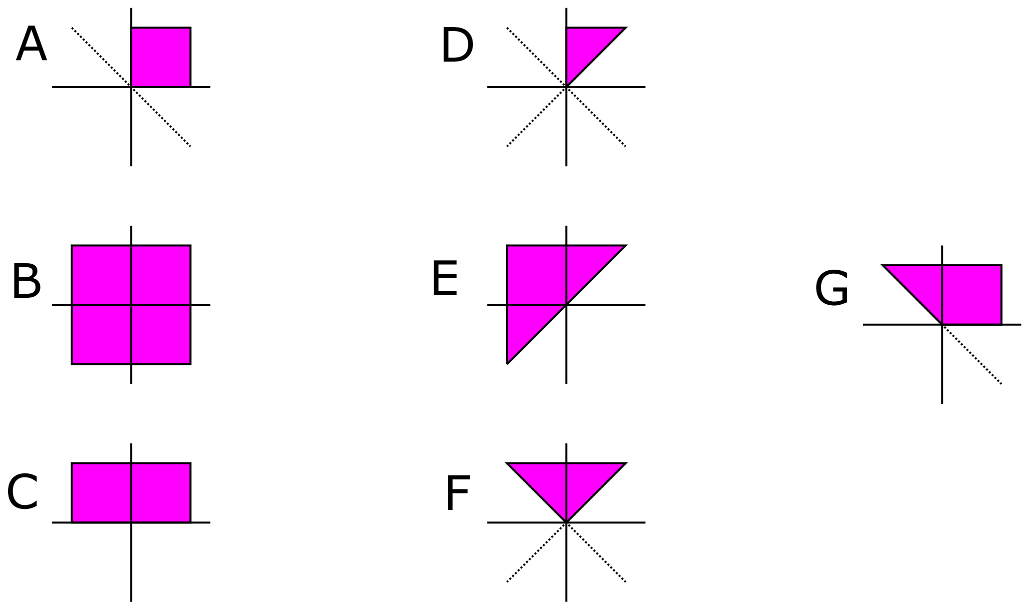


Figure 5.3: Some order polytopes for signed posets on two elements.

We note that our geometric constructions play nicely with Reiner's definition of signed poset isomorphism.

Definition 5.2.6. Two lattice polytopes Q and Q' are ***unimodularly equivalent*** if there is an affine lattice isomorphism of the ambient lattices mapping Q' onto Q.

Proposition 5.2.3. *Suppose P and P' are isomorphic signed posets on $[n]$. Then $\mathcal{O}_P$ and $\mathcal{O}'_P$ are unimodularly equivalent.*

Proof. Since P' is isomorphic to P, then $P' = \omega P$ for some $\omega \in S_n^B$. By Proposition 5.2.2, it suffices to verify Proposition 5.2.3 for the cases in which ω is equal to the generators $s_i = [1, \ldots, i-1, i+1, i, \ldots, n]$, and $s_0 = [-1, 2, \ldots, n]$.

Suppose $\omega = s_i$ for $i \in [n-1]$. Then

$$\mathcal{O}_P \;=\; \{x \in \mathbb{R}^n : \langle \alpha, x\rangle \geq 0 \text{ for all } \alpha \in P\} \cap [-1,1]^n$$

and

$$\mathcal{O}'_P \ = \ \{x \in \mathbb{R}^n : \langle s_i\alpha, x\rangle \geq 0 \text{ for all } \alpha \in P\} \cap [-1,1]^n.$$

From this description, we see that the hyperplanes defining $\mathcal{O}'_P$ are reflections of the hyperplanes defining $\mathcal{O}_P$ about the hyperplane $x_i = x_{i+1}$ (since the cube $[-1,1]^n$ is symmetric about this hyperplane). This is an affine lattice isomorphism, so $\mathcal{O}_P$ and $\mathcal{O}'_P$ are unimodularly equivalent.

Similarly, for $\omega = s_0$, the hyperplanes defining $\mathcal{O}'_P$ are reflections of the hyperplanes defining $\mathcal{O}_P$ about the hyperplane $x_1 = 0$, which is again an affine lattice isomorphism. □

We note that the order cone of a classical poset is always pointed. This is not the case for order cones of signed posets. This makes it so that one cannot always write down a rational generating function for the integer point transform of such an order cone. However, we can construct a pointed cone encoding the same information by homogenizing the order polytope, defined below following, e.g., [13]. As is the case for classical posets, the order polytope is related to the order cone of a signed poset as follows:

Definition 5.2.7. Let Q be a polytope in $\mathbb{R}^n$. The ***homogenization*** of Q is given by

$$\operatorname{Hom}(Q) \ := \ \left\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : t \in \mathbb{R}_{\geq 0},\ \mathbf{x} \in tQ\right\}.$$

Proposition 5.2.4. *For a signed poset P with n elements,* $\hom(O_P) = K_{\hat{P}}$*, where $\hat{P}$ is given by $P \cup \{e_{n+1} \pm e_i : 1 \leq i \leq n\}$.*

Proof. Both $\hom(O_P)$ and $K_{\hat{P}}$ are given by

$$\left\{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : -t \leq x_i \leq t \text{ for all } i \in [n] \text{ and } \langle \alpha, \mathbf{x}\rangle \leq 0 \text{ for all } \alpha \in P\right\}. \qquad \square$$

We now show that $\mathcal{K}_P$ (and thus $\mathcal{O}_P$) is full dimensional:

Proposition 5.2.5. *Let P be a signed poset on $[n]$. Then* $\dim(\mathcal{K}_P) = n$.

Proof. We prove that we can construct a point $\phi : [n] \to \mathbb{R}$ in the interior of $\mathcal{K}_P$. We argue inductively on n. The base case $n = 1$ holds, since the order cone is either the non-negative ray, the non-positive ray, or $\mathbb{R}$.

Now suppose we have a signed poset P on $[n]$ with $n \geq 2$. Let P' be the signed poset on $[n-1]$ obtained from restricting P to the set $\{\pm e_i : 1 \leq i \leq n-1\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq n-1\}$. By our inductive hypothesis, there exists a point $\phi' : [n-1] \to \mathbb{R}$ in the interior of $\mathcal{K}'_P$. Our strategy is to show that we can extend ϕ' to a point ϕ in the interior of $\mathcal{K}_P$. We first let $\phi(i) = \phi'(i)$ for $1 \leq i \leq n-1$. We now need to show that there exists a choice of $\phi(n)$ such that ϕ is in the interior of $\mathcal{K}_P$. To make the following equations easier to look at, let $\phi(n) = x$.

The only way in which there is no viable choice for x is if the hyperplanes defining $\mathcal{K}_P$ together with the choice of ϕ' give rise to inequalities of the form $a < x < b$ for some $a \geq b$. We first list the ways that we could get a restriction of the form $x > a$:

(i) Suppose $e_n - e_j \in P$ where $j \neq n$. Then $\phi \in \mathcal{K}_P^\circ$ implies $x > \phi(j)$.

(ii) Suppose $e_n + e_j \in P$ where $j \neq n$. Then $\phi \in \mathcal{K}_P^\circ$ implies $x > -\phi(j)$.

(iii) Suppose $e_n \in P$. Then $\phi \in \mathcal{K}_P^\circ$ implies $x > 0$.

We next list the ways that we could get a restriction of the form $x < b$.

(iv) Suppose $-e_n + e_k \in P$ where $k \neq n$. Then $\phi \in \mathcal{K}_P^\circ$ implies $x < \phi(k)$.

(v) Suppose $-e_n - e_k \in P$ where $k \neq n$. Then $\phi \in \mathcal{K}_P^\circ$ implies $x < -\phi(k)$.

(vi) Suppose $-e_n \in P$. Then $\phi \in \mathcal{K}_P^\circ$ implies $x < 0$.

We now show that in all cases in which we have a restriction of the form $x < b$ and a restriction of the form $x > a$, we get that $a < b$. (And thus, there is a solution for x).

Case 1: We have situations (i) and (iv) above and thus the restriction $\phi(j) < x < \phi(k)$. Since $e_n - e_j, -e_n + e_k \in P$, we know that $e_k - e_j$ is in P and P'. Now $\phi' \in \mathcal{K}_P'^\circ$ implies $\phi(j) < \phi(k)$.

Case 2: We have situations (i) and (v), and thus $\phi(j) < x < -\phi(k)$. Since $e_n - e_j, -e_n - e_k \in P$, we know that $-e_j - e_k$ is in both P and P'. Since ϕ' was chosen to be in $\mathcal{K}_P'^\circ$, we have $\phi(j) < -\phi(k)$.

Case 3: We have the situations (i) and (vi) and thus $\phi(j) < x < 0$. Since $e_n - e_j, -e_n \in P$, we know that $-e_j \in P, P'$ and so $\phi(j) < 0$.

Case 4: The arguments for the situations (ii) and (iv), (ii) and (v), (ii) and (vi), (iii) and (iv), (iii) and (v) are similar as the previous three cases.

Case 5: We have the situations (iii) and (vi). This would imply that $e_n, -e_n \in P$, which violates the asymmetry property of a signed poset.

Thus, in all viable cases it is possible to choose a value of $\phi(n)$ so that $\phi \in \mathcal{K}_P a^\circ$. $\square$

The map $\phi(x)$ constructed above can be modified to give a signed permutation in the Jordan–Hölder set of P, which will be necessary below when we describe triangulations of $\mathcal{K}_P$.

Definition 5.2.8. Let P be a signed poset on $[n]$. The ***Jordan–Hölder set*** JH(P) of P is the set of signed permutations $\omega \in S_n^B$ such that ω is order-preserving, that is, $\langle \omega, \alpha \rangle \geq 0$ for all $\alpha \in P$, where we think of ω as the point $(\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$.

Remark 5.2.2. Reiner gives a definition of Jordan–Hölder set in [58] that is in the same spirit, but slightly different to the one here. He defines the Jordan–Hölder set of a signed poset P on $[n]$ as the set of all $\sigma \in B_n$ such that $\alpha \in \sigma B_n^+$ for all $\alpha \in P$. Here, B_n^+ refers to the set $\{e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : 1 \leq i \leq j \leq n\}$, the ***positive roots*** of B_n. Our definition $\mathrm{JH}(P)$ can be rewritten similarly, as the set of all $\sigma \in B_n$ such that $\alpha \in \sigma S$, where $S = \{e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : 1 \leq j \leq i \leq n\}$. The difference between our definition and Reiner's is analogous to the slight differences in our early definitions discussed in Remark 5.2.1.

To get from the point ϕ described in the proof of Proposition 5.2.4 to the corresponding signed permutation $\omega \in \mathrm{JH}(P)$, do the following: Consider the coordinates of ϕ and determine which one has the highest absolute value. Replace this one with n or $-n$, corresponding to the sign of this coordinate. Repeat with the remaining coordinates, this time replacing with $n-1$ or $-(n-1)$. Repeat until you have replaced every coordinate. Call this point p. Then the corresponding signed permutation is the unique element of $\omega \in S_n^B$ such that $(\omega(1), \ldots, \omega(n)) = p$.

We now introduce the idea of a naturally labeled signed poset, mimicking a similar notion for classical posets.

Definition 5.2.9. Let P be a signed poset on $[n]$, and let $\mathrm{id} \in S_n^B$ be the identity element. We say P is ***naturally labeled*** if and only if $\mathrm{id} \in \mathrm{JH}(P)$.

Proposition 5.2.6. *Every signed poset is isomorphic to a naturally labeled signed poset.*

Proof. Let P be a signed poset. As a consequence of the proof of Proposition 5.2.5, P has some linear extension $\mathbf{l}$. Let ω be the unique element of S_n^B such that ω restricted to $[n]$ gives $\mathbf{l}$. Consider $\omega^{-1}P$ which, by definition, is isomorphic to P. To show that id is a linear extension of $\omega^{-1}P$, it suffices to show that $\langle \mathrm{id}, \omega^{-1}\alpha \rangle \geq 0$ for all $\alpha \in P$.

For any α in the type-B root system and any $\omega \in S_n^B$, we have $\langle \mathrm{id}, \omega^{-1}\alpha \rangle = \langle \omega, \alpha \rangle$. This can be shown by noting that $\langle \mathrm{id}, \omega^{-1}e_i \rangle = \langle \omega, e_i \rangle = \omega(i)$ and then using linearity.

Thus, for any $\alpha \in P$, $\langle \mathrm{id}, \omega^{-1}\alpha \rangle = \langle \omega, \alpha \rangle \geq 0$, since ω is a linear extension of P. $\square$

5.3 Alternative Descriptions of $\mathcal{K}_P$ and $\mathcal{O}_P$

We now discuss a few other descriptions of the order cones and polytopes of signed posets. Definition 5.1.6 gives a hyperplane description of $\mathcal{O}_P$ and $\mathcal{K}_P$, but this hyperplane description may be redundant. This is also true for the definition of classical order polytopes and cones. The following proposition from [67] gives an irredundant representation for the order polytopes of classical posets.

Proposition 5.3.1 (Stanley [67])**.** *Let Π be a classical poset. Then, an irredundant representation of $\mathcal{O}(\Pi)$ is given by*

$$\mathcal{O}(\Pi) \;=\; \left\{ x \in \mathbb{R}^{\Pi} : \begin{array}{ll} x_a \le x_b & \textit{if } a \lessdot b \\ x_a \ge 0 & \textit{if } a \textit{ is a minimal element} \\ x_a \le 1 & \textit{if } a \textit{ is a maximal element} \end{array} \right\},$$

where $a \lessdot_{\Pi} b$ means that $a \le_{\Pi} b$ is a cover relation in Π.

As hinted in Example 5.2.3, we can give an similar irredundant hyperplane representation for order polytopes of signed posets. First, we need a few propositions and definitions. The next proposition from [58] was briefly described earlier in Example 5.2.2.

Proposition 5.3.2 (Reiner [58])**.** *Let P be a signed poset. Then there exists a unique minimal subset of P, called the* ***minimal representation*** *of P and denoted* $\mathrm{minrep}(P)$*, such that*

$$\overline{\mathrm{minrep}(P)}^{\mathrm{PLC}} \;=\; P\,.$$

Note that the relations in $\mathrm{minrep}(P)$ are analogous to the cover relations of a classical poset, since both give an irredundant description of the (signed) poset up to transitivity. We now give a definition for two types of maximal elements of a signed poset.

Definition 5.3.1. Let P be a signed poset on $[n]$. We say $i \in [n]$ is a ***positive maximal element*** of P if and only if every relation adjacent to i is of the form e_i or $e_i \pm e_j$, i.e., the e_i portion of the adjacent relations is positive. We denote the set of positive maximal elements of P as $\mathrm{pmax}(P)$. We say $i \in [n]$ is a ***negative maximal element*** of P if and only if every relation adjacent to i is of the form $-e_i$ or $-e_i \pm e_j$, i.e., the e_i portion of the adjacent relations is negative. We denote the set of negative maximal elements of P as $\mathrm{nmax}(P)$.

We can now give an irredundant representation of $\mathcal{O}_P$.

Proposition 5.3.3. *Let P be a signed poset on $[n]$. Then an irredundant representation of $\mathcal{O}_P$ is given by*

$$\mathcal{O}_P \;=\; \left\{ x \in \mathbb{R}^{n} : \begin{array}{ll} \langle \alpha, x\rangle \ge 0 & \textit{if } \alpha \in \mathrm{minrep}(P) \\ x_i \le 1 & \textit{if } i \in \mathrm{pmax}(P) \\ x_i \ge -1 & \textit{if } i \in \mathrm{nmax}(P) \end{array} \right\}. \tag{5.1}$$

Proof. We first show that (5.1) does indeed describe $\mathcal{O}_P$. Since (5.1) consists of a subset of the inequalities listed in Definition 5.1.6, it suffices to show that each inequality in Definition 5.1.6 is implied by the inequalities listed in (5.1). First of all, each inequality of the form $\langle \alpha, x\rangle \ge 0$ for $\alpha \in P$ is implied by the inequalities of the form $\langle \alpha, x\rangle \ge 0$ for $\alpha \in \mathrm{minrep}(P)$ from the definition of $\mathrm{minrep}(P)$.

We now show that the inequalities $-1 \le x_i \le 1$ for all $i \in [n]$ are implied by the inequalities listed in the proposition. It suffices to show that for each $i \in [n]$, one of the following holds (to ensure that $x_i \le 1$):

(i) $i \in \mathrm{pmax}(P)$,

(ii) $-e_i \in P$, since $x_i \leq 0$ implies $x_i \leq 1$,

(iii) $-e_i + e_p \in P$, where $p \in \text{pmax}(P)$,

(iv) $-e_i - e_n \in P$, where $n \in \text{nmax}(P)$,

and one of the following holds (to ensure that $-1 \leq x_i$):

(v) $i \in \text{nmax}(P)$,

(vi) $e_i \in P$, since $x_i \geq 0$ implies $x_i \geq -1$,

(vii) $e_i + e_p \in P$, where $p \in \text{pmax}(P)$,

(viii) $e_i - e_n \in P$, where $n \in \text{nmax}(P)$.

We show by induction on n that this is always true. It is true for $n = 1$ since in that case one of (i) or (ii) must be true, and one of (v) or (vi) must be true.

Now, consider some $i \in P$, and let P' be the signed poset on $[n] - \{i\}$ with all relations adjacent to i removed (similarly as in the proof of Proposition 5.2.4). We first show that i satisfies (i), (ii), (iii), or (iv). Assume that i does not satisfy (i) or (ii), with the aim to show that i satisfies (iii) or (iv). By our assumptions, i must be adjacent to a relation of the form $-e_i + e_j$ or $-e_i - e_j$.

Case 1: Assume i is adjacent to a relation of the form $-e_i + e_j$. We know by our inductive hypothesis that within P', j must satisfy (i), (ii), (iii), or (iv). We go through each of these subcases.

- Suppose j satisfies (i) and is thus an element of $\text{pmax}(P')$. We show that j is also an element of $\text{pmax}(P)$. The only obstruction to this is if $-e_i - e_j$ or $e_i - e_j$ are in P. However, $e_i - e_j$ cannot be in P since its opposite $-e_i + e_j$ was already assumed to be in P. Suppose $-e_i - e_j$ is in P. Then, by transitivity with $-e_i + e_j$, we have that $-e_i$ is in P which is not the case since we assumed i does not satisfy (ii). Thus, j is an element of $\text{pmax}(P)$, so i satisfies (iii).

- Now suppose j satisfies (ii), so $-e_j$ is an element of P' and thus P. However, by transitivity with $-e_i + e_j$, $-e_i$ is in P, which again is not possible by our assumptions.

- Now suppose j satisfies (iii). Then $-e_j + e_p$ is in P' and thus P, where p is in $\text{pmax}(P')$. We first note that by transitivity, $-e_i + e_p$ is in P. We now show that p is not only in $\text{pmax}(P')$, but also in $\text{pmax}(P)$. The only obstructions to this are if $-e_i - e_p$ or $e_i - e_p$ are in P. Note that $e_i - e_p$ cannot be in P since we already established that its opposite, $-e_i + e_p$, is in P. We also see that $-e_i - e_p$ cannot be in P since by transitivity then $-e_i$ would be in P, which we assumed was not the case. Thus, $p \in \text{pmax}(P)$ and so i satisfies (iii).

- Now suppose j satisfies (iv). Then $-e_j-e_n$ is in P' and thus P, for some n in $\text{nmax}(P')$. By transitivity, we see that $-e_i-e_n$ is in P. It suffices to show that n is not only in $\text{nmax}(P')$ but also in $\text{nmax}(P)$. The only obstructions to this are if $-e_i+e_n$ or e_i+e_n are in P. We see that e_i+e_n cannot be in P because we already established that that its opposite $-e_i-e_n$ is. If $-e_i+e_n$ is in P, then by transitivity with $-e_i-e_n$, $-e_i$ is in P, which we assumed was not the case. So n is in $\text{nmax}(P)$ and thus i satisfies (iv).

Case 2: Assume i is a adjacent to a relation of the form $-e_i-e_j$. We know by our inductive hypothesis that within P', j must satisfy (v),(vi),(vii), and (viii). We go through each of these subcases and show that each case implies i satisfies (iii) of (iv).

- Suppose j satisfies (v) and thus is an element of $\text{nmax}(P')$. We show that j is also an element of $\text{nmax}(P)$. The only possible obstructions to this are $-e_i+e_j$ or e_i+e_j being in P. We know its impossible for e_i+e_j to be in P since its opposite is. If $-e_i+e_j$ were to be in P, then by transitivity $-e_i$ would be in P which we assumed was not the case much earlier. So j is an element of $\text{nmax}(P)$ and thus i satisfies (iv).

- Suppose j satisfies (vi) and thus e_j is in P' and thus P. However, by transitivity with $-e_i-e_j$ this implies that $-e_i$ is in P, which we earlier assumed was not the case.

- Suppose j satisfies (vii) and thus e_j+e_p is in P, where $p \in \text{pmax}(P')$. By transitivity, we see that $-e_i+e_p \in P$. It suffices to show that p is in $\text{pmax}(P)$. Suppose not. Then either e_i-e_p or $-e_i-e_p$ is in P. We see that e_i-e_p cannot be in $\text{pmax}(P)$ since its opposite $-e_i+e_p$ was already shown to be in P. If $-e_i-e_p$ is in P, then by transitivity with $-e_i+e_p$ we get that $-e_i$ is in P, which we assumed much earlier not to be the case. Thus, p is in $\text{pmax}(P)$ and thus i satisfies (iii).

- Suppose j satisfies (viii) and e_j-e_n is in P, where n is in $\text{nmax}(P')$. By transitivity, we know that $-e_i-e_n$ is in P. To show i satisfies (iv), it suffices to show that n is in $\text{nmax}(P)$. The only obstructions to this are $-e_i+e_n$ or e_i+e_n being in P. We see that e_i+e_n cannot be in P because we already established that its opposite is. Suppose that $-e_i+e_n$ is in P. Then by transitivity with $-e_i-e_n$, we know that $-e_i$ is in P, which we already assumed was not the case. Thus, n must be in $\text{nmax}P$ and so i satisfies (iv).

Thus, in all cases, (i) must satisfy at least one of (i), (ii), (iii), or (iv). The proof that the i must satisfy at least one of (v), (vi), (vii), or (viii) is similar.

We have thus shown that the given inequalities in this proposition do describe $\mathcal{O}_P$. We now argue that this set of inequalities is a minimal set describing $\mathcal{O}_P$. From the definition of $\text{minrep}(P)$, we see that we cannot remove any of the hyperplanes of the form $\langle \alpha, x\rangle \geq 0$ without changing $\mathcal{O}_P$. Now let $i \in \text{pmax}(P)$. All of the adjacent relations to i must be of the form e_i-e_j, e_i+e_j, or e_i. So all of restrictions on x_i coming from rest of the poset are of the form $x_i \geq x_j$, $x_i \geq -x_j$ and $x_i \geq 0$. No combination of these can imply $x_i \leq 1$, so this inequality is necessary in this description of $\mathcal{O}_P$. $\square$

In order to describe a convex hull description of $\mathcal{O}_P$, we first need the idea of filters. For classical posets, they are defined as follows, as appears in, e.g., [13].

Definition 5.3.2. Let Π be a classical poset. Then, a filter is a subset F of Π such that for $a \leq_\Pi b$, $a \in F$ implies $b \in F$.

The filters give a vertex description of K_Π through the following proposition, which again appeared in [67].

Proposition 5.3.4 (Stanley [67])**.** *The vertex set of $\mathcal{O}(\Pi)$ is the set of all e_F, where F is a filter of Π and $e_F : \Pi \to \{0,1\}$ is the point in $\mathbb{R}^\Pi$ given by*

$$e_F(a) \;=\; \begin{cases} 1 & \textit{if } a \in F, \\ 0 & \textit{otherwise.} \end{cases}$$

Note that the filters are exactly the points $x \in \{0,1\}^\Pi$ such that $\langle \alpha, x \rangle \geq 0$ for all $\alpha \in f(\Pi)$. (Recall that f is the map from a classical poset to its corresponding subset of the type-A root system). This observation allows for a similar definition for signed posets. The following definition appears in [58].

Definition 5.3.3. Let P be a signed poset on $[n]$. Then $x \in \{-1,0,1\}^n$ is a ***signed filter*** of P if $\langle \alpha, x \rangle \geq 0$ for all $\alpha \in P$.

Note that in [58], due to the earlier mentioned difference in convention (up to a sign) in defining the map from classical posets to subsets of the type-A root system, Reiner calls these objects signed order ideals instead of signed filters.

We can now give a convex hull description of $\mathcal{O}_P$.

Proposition 5.3.5. *Let P be a signed poset on $[n]$ and let F be the set of signed filters on P. Then $\mathcal{O}_P = \mathrm{conv}(F)$.*

Remark 5.3.1. A signed filter may not necessarily be a vertex of $\mathcal{O}_P$ (in contrast to classical order polytopes). For example $x = (0, \ldots, 0)$ is always a signed filter of $\mathcal{O}_P$, but need not be a vertex (as seen in Figure 5.3).

Before we give the proof of this proposition, we need to discuss a specific unimodular triangulation of $\mathcal{O}_P$ for any signed poset. Along the way, we discuss a way of dividing $\mathbb{R}^n$ into cones indexed by elements of the signed permutation group S_n^B.

Definition 5.3.4. Consider a signed permutation $\sigma = (\pi, \epsilon) \in S_n^B$. The simplicial cone associated with σ is

$$K_\sigma \;:=\; \left\{x \in \mathbb{R}^n : 0 \leq \epsilon_1 x_{\pi_1} \leq \cdots \leq \epsilon_n x_{\pi_n}\right\}.$$

These simplicial cones induce a triangulation on the $[-1,1]^n$ cube, as described in [11, Proof of Theorem 6.9]. Each maximal face of this triangulation is of the form $\Delta_\sigma := \{x \in \mathbb{R}_n : 0 \leq \epsilon_1 x_{\pi_1} \leq \cdots \leq \epsilon_n x_{\pi_n} \leq 1\}$. Note that the defining hyperplanes of $\mathcal{O}_P$ are a subset of the union of the set of hyperplanes defining the described triangulation of $[-1,1]^n$. Thus, restricting this triangulation to $\mathcal{O}_P$ gives a triangulation of $\mathcal{O}_P$. We will refer to this triangulation as $\mathcal{T}$. Later, we will use the fact the maximal faces $\mathcal{T}$ are exactly the set Δ_σ, where $\sigma \in \mathrm{JH}(P)$.

Now, we will use this triangulation to prove Proposition 5.3.5.

Proof. We first observe that $conv(F) \subseteq \mathcal{O}_P$. By definition, all the points in F are in $\mathcal{O}_P$, and thus the convex hull of F is in $\mathcal{O}_P$.

We now show that $\mathcal{O}_P \subseteq conv(F)$. Suppose we have some point $x \in \mathcal{O}_P$. Then, x must be in one of the full-dimensional simplices of $\mathcal{T}$, i.e., $x \in \Delta_\omega$ for some $\omega \in S_n^B$, and thus can be written as a convex combination of the vertices of Δ_ω. These vertices have entries of $0, 1, -1$, and thus are filters of P. □

5.4 Computing the h^*-polynomial of $\mathcal{O}_P$

In 1962, for a lattice polytope $P \subset \mathbb{Z}^d$, Eugene Ehrhart introduced and proved polynomiality of the Ehrhart enumerator $\mathrm{ehr}_P(t) = |tP \cap \mathbb{Z}|$, [38]. Note that the t^{th} ***dilate of*** P is $tP := \{tp \in \mathbb{R}^n : p \in P\}$.

Theorem 5.4.1. *(Ehrhart's Theorem) For any d-dimensional lattice polytope $P \subseteq \mathbb{R}^n$, the quantity $ehr_P(t) = |tP \cap \mathbb{Z}^n|$ agrees with a polynomial of degree d.*

This ***Ehrhart polynomial*** can be seen as a discrete measure of volume. In general, the coefficients of the Ehrhart polynomial of a lattice polytope are rational numbers that can be negative. The generating series $\mathrm{Ehr}(P; z)$ of a d-dimensional lattice polytope can be written as

$$\mathrm{Ehr}(P; z) = 1 + \sum_{t \geq 1} \mathrm{ehr}_P(t) z^t = \frac{h_0^* + h_1^* z + \cdots + h_{d+1}^* z^d}{(1-z)^{d+1}}.$$

The numerator of this expression is called the h^****-polynomial*** of P and denoted $h^*(P; z)$. It can be helpful to view the information encoded by the Ehrhart polynomial in this form. For example, unlike the Ehrhart polynomial, the coefficients of the h^*-polynomial of a lattice polytope are always non-negative integers [64]. Thus, one area of research in Ehrhart theory is to give combinatorial interpretations of these coefficients for specific families of polytopes. For classical order polytopes, we can describe the h^*-polynomial in terms of permutation statistics in the following way, as outlined, e.g, in [13, Chapter 5]. The Jordan–Hölder set of a classical poset is defined similarly as for a signed poset.

Definition 5.4.1. Let Π be a classical poset on $[n]$. The Jordan-Hölder Set Π the set of permutations $\omega \in S_n$ that are order-preserving, that is, $\langle \alpha, \omega \rangle \geq 0$ for all $\alpha \in f(\Pi)$, where f is the map described in Proposition 5.2.1 and where we think of ω as the point $(\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$.

Definition 5.4.2. Let $\omega = \omega_1\omega_2 \ldots \omega_n \in S_n$. A ***descent*** is a position i such that $\omega_i > \omega_{i+1}$. The ***descent set*** of ω is $\mathrm{Des}(\omega) := \{i : i \text{ is a descent of } \omega\}$, and the ***descent statistic*** of ω is $\mathrm{des}(\omega) := |\mathrm{Des}(\omega)|$.

See, for example, [62, Section 3.12] for a more complete discussion of permutation statistics and other equivalent definitions of the classical Jordan–Hölder set, relating for linear extensions of posets.

Proposition 5.4.1. *Let Π be a naturally labeled poset with Jordan–Hölder set $\mathrm{JH}(\Pi) \subset S_n$. Then*

$$h^*_{\mathcal{O}_\Pi}(z) \;=\; \sum_{\tau \in JH(\Pi)} z^{\mathrm{des}(\tau)}.$$

In this section, we use the triangulation $\mathcal{T}$ described in Section 5.3 to give an analogous description of the h^*-polynomial of $\mathcal{O}_P$ in terms of statistics on the type-B permutation group S_n^B. We first introduce some background on half-open polytopes and half-open decompositions of polytopes. A more complete treatment of this material can be found in [13, Chapter 5]. We first define what it means for a point to be beyond a face F of a polytope P in the special case in which F is a facet of P. For more generality, see [13, Chapter 3].

Definition 5.4.3. Let $P \subset \mathbb{R}^n$ be a full-dimensional polytope, and let F be a facet of P with defining hyperplane $\langle a, x\rangle = b$ such that P lies in the half space $\langle a, x\rangle \le b$. Then $p \in \mathbb{R}^n$ is ***beyond*** F if $\langle a, p\rangle > b$.

This concept of a point being beyond a facet is used to construct half-open polytopes.

Definition 5.4.4. Let $P \subset \mathbb{R}^n$ be a full-dimensional polytope with facets $F_1, \ldots F_m$. Let $q \in \mathbb{R}^n$ be generic relative to P, i.e., q does not lie on any facet-defining hyperplane of P. Then we define

$$\mathcal{H}_q P \;:=\; P \setminus \bigcup_{i \in I} F_i\,,$$

where $I := \{i \in [m] : q \text{ beyond } F_i\}$. We call $\mathcal{H}_q P$ a ***half-open polytope***.

Applying this construction to a triangulation of P allows us to write P as a disjoint union of half-open simplices.

Lemma 5.4.2 ([13])**.** *Let $P \subset \mathbb{R}^n$ be a full-dimensional polytope with dissection $P = P_1 \cup P_2 \cup \cdots \cup P_m$. If $q \in P^\circ$ is generic relative to each P_j, then*

$$P \;=\; \mathcal{H}_q P_1 \uplus \mathcal{H}_q P_2 \uplus \cdots \uplus \mathcal{H}_q P_m$$

We can apply these results to the unimodular triangulation $\mathcal{T}$ of $\mathcal{O}_P$ described in the previous section in order to write $\mathcal{O}_P$ as a disjoint union of half-open unimodular simplices. These simplices can be described in terms of the naturally ordered descent statistic of S_n^B. For more information about various statistics (including several definitions of the descent set) of B_n, see, e.g., [11].

Definition 5.4.5. For $\sigma \in S_n^B$, the ***naturally ordered descent set*** of σ is

$$\mathrm{NatDes}(\sigma) \;:=\; \{i \in \{0,\ldots,n-1\} : \sigma(i) > \sigma(i+1)\},$$

where we use the convention that $\sigma(0) = 0$. The ***natural descent statistic*** of σ is $\mathrm{natdes}(\sigma) := |\mathrm{NatDes}(\sigma)|$.

Proposition 5.4.3. *Let $\sigma = (\pi, \epsilon) \in S_n^B$ and $p := \left(\frac{1}{n+1}, \frac{2}{n+1}, \ldots, \frac{n}{n+1}\right)$. Then*

$$\mathbb{H}_p\Delta_\sigma \;=\; \left\{ x \in \mathbb{R}^\Pi : \begin{array}{ll} 0 \le \epsilon_1 x_{\pi_1} \le \cdots \le \epsilon_n x_{\pi_n} \le 1 & \\ \epsilon_i x_{\pi_1} < \epsilon_{i+1} x_{\pi_{i+1}} & \text{if } i \in \ \mathit{NatDes}(\sigma) \\ 0 < \epsilon_1 x_{\pi_1} & \text{if } 0 \in \mathit{NatDes}(\sigma) \end{array} \right\}.$$

Proof. We identify the facets of Δ_σ that are removed in the half-open polytope $\mathbb{H}_p\Delta_\sigma$, starting with the facets of the form $\epsilon_i x_{\pi_i} = \epsilon_{i+1} x_{\pi_{x+1}}$ for $i \in [n-1]$. These facets are removed when p is beyond the facet, which occurs exactly when $\epsilon_i p_{\pi_i} > \epsilon_{i+1} p_{\pi_{i+1}}$. Substituting our expressions for the coordinates of p yields

$$\epsilon_i \frac{\pi_i}{n+1} > \epsilon_{i+1} \frac{\pi_{i+1}}{n+1}$$

which simplifies to $\sigma(i) > \sigma(i+1)$. Thus, we see that a facet of Δ_σ of the form $\epsilon_i x_{\pi_i} = \epsilon_{i+1} x_{\pi_{i+1}}$ is removed exactly when $i \in \mathrm{NatDes}(\sigma)$.

We now consider the facet given by $\epsilon_1 x_{\pi_1} = 0$. We know that p is beyond this facet exactly when $\epsilon_1 p_{\pi_1} < 0$. Since all the coordinates of p are positive, this holds exactly when $\epsilon_1 < 0$, which coincides with the cases in which $0 \in \mathrm{NatDes}(\sigma)$.

We finally consider the facet given by $\epsilon_n x_{\pi_n} = 1$. Since $-1 \le p_i \le 1$ for any $i \in [n]$, we know that p is never beyond this facet. Thus, this facet is never removed in $\mathbb{H}_p\Delta_\sigma$. □

A classical result (see, e.g., [13]) gives the h^*-polynomials of half-open unimodular simplices.

Lemma 5.4.4. *Let $\mathbb{H}_p\Delta$ be a unimodular half open simplex with k missing facets. Then, $h^*_{\mathbb{H}_p\Delta}(z) = z^k$.*

We can now describe the h^*-polynomial of $\mathcal{O}_P$ for any naturally ordered signed poset P in terms of descent statistics in a statement analogous to Proposition 5.4.1.

Proposition 5.4.5. *Let P be a naturally labeled signed poset on $[n]$ with Jordan–Hölder set $\mathrm{JH}(P) \subset B_n$. Then*

$$h^*_{\mathcal{O}_P}(z) \;=\; \sum_{\tau \in \mathrm{JH}(P)} z^{\mathrm{natdes}(\tau)}.$$

Proof. Since P is naturally labeled, we know that $p = \left(\frac{1}{n+1}, \ldots, \frac{n}{n+1}\right)$ is an interior point of $\mathcal{O}_P$. Thus, we can use the triangulation $\mathcal{T}$ restricted to $\mathcal{O}_P$ to decompose $\mathcal{O}_P$ into a disjoint union of half-open simplices. Using Lemma 5.4.2 with respect to the point p and this triangulation, we obtain

$$\mathcal{O}_P \;=\; \biguplus_{\sigma \in \mathrm{JH}(P)} \mathbb{H}_p \Delta_\sigma .$$

From Proposition 5.4.3, we know that $\mathbb{H}_p\Delta_\sigma$ is a unimodular half-open simplex with $\mathrm{natdes}(\sigma)$ missing facets. From Lemma 5.4.4, we know that the h^*- polynomial of such an object is $z^{\mathrm{natdes}(\sigma)}$. Since the h^*-polynomials of disjoint half-open polytopes are additive,

$$h^*_{\mathcal{O}_P}(z) \;=\; \sum_{\tau \in \mathrm{JH}(P)} z^{\mathrm{natdes}(\tau)}. \qquad \square$$

Remark 5.4.1. This result only gives a description of the h^*-polynomial for naturally labeled signed posets. However, since unimodularly equivalent polytopes have identical h^*-polynomials, this encompasses all the unique h^*-polynomials corresponding to signed posets by Propsitions 5.2.3 and 5.2.6.

5.5 Which Signed Order Polytopes Are Gorenstein?

We now review a classification of Gorenstein order polytopes in the classical case and discuss how it extends to signed order polytopes.

Definition 5.5.1. A lattice polytope is ***Gorenstein*** if there exists a positive integer k such that $(k-1)P^\circ \cap \mathbb{Z}^d = \emptyset$, $|kP^\circ \cap \mathbb{Z}^d| = 1$, and $|tP^\circ \cap \mathbb{Z}^d| = |(t-k)P^\circ \cap \mathbb{Z}^d|$ for all integers $t > k$.

This is equivalent to the polytope having a symmetric h^*-vector. For classical posets, the following result is well known (see, e.g., [13]):

Proposition 5.5.1. *The order polytope of a poset P is Gorenstein if and only if P is graded.*

In this section, we will develop an analogue of this result for signed posets; we begin with a representation of a signed poset on $[n]$ as a classical poset on $[2n+1]$ that satisfies certain properties, first introduced in [39].

Definition 5.5.2. Let P be a signed poset on $[n]$. The ***Fischer represention*** $\hat{G}(P)$ is a poset on $[-n, n] = \{-n, -(n-1), \ldots, -1, 0, 1, \ldots, n-1, n\}$ whose relations are the transitive closure of the following:

$$\begin{array}{ll} i < j \text{ and } -j < -i & \text{for } -e_i + e_j \in P \\ i < -j \text{ and } j < -i & \text{for } -e_i - e_j \in P \\ -i < j \text{ and } -j < i & \text{for } e_i + e_j \in P \\ i < 0 \text{ and } 0 < -i & \text{for } -e_i \in P \\ -i < 0 \text{ and } 0 < i & \text{for } e_i \in P\,. \end{array}$$

Figure 5.4 shows an example of the bidirected graph representation and the Fischer representation of the same signed poset.

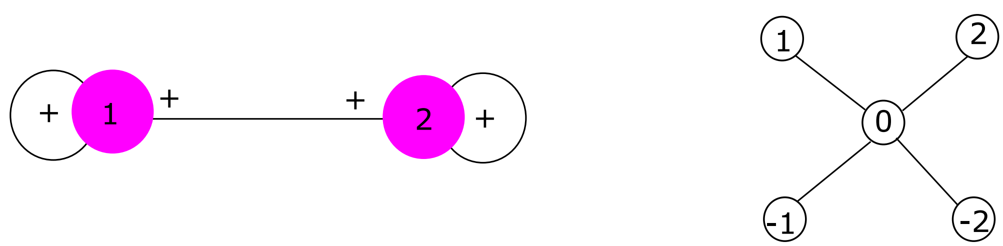


Figure 5.4: The left side shows the bidirected graph representation of $P := \{e_1, e_2, e_1 + e_2\}$ and the right side shows the Fischer representation of P.

Remark 5.5.1. This definition has all the inequalities reversed from Fischer's original definition to make this poset more consistent with our definition of signed order polytopes.

The following proposition from [39] classifies exactly when a poset on $[-n, n]$ equals $\hat{G}(P)$ for some signed poset P on $[n]$.

Proposition 5.5.2. *A poset $[-n, n]$ is $\hat{G}(P)$ for some signed poset P if and only if*

- *$i < j$ if and only if $-j < -i$ for all $i, j \in [-n, n]$;*
- *if $-i < i$ then $-i < 0 < i$ for all $i \in [-n, n]$.*

Next, we establish how we can view the previously defined signed order polytopes through this lens.

Proposition 5.5.3. *Let P be a signed poset on $[n]$. Define a polytope $\mathcal{O}_{\hat{G}(P)} \subset \mathbb{R}^n$ via the following inequalities:*

- $-1 \leq x_i \leq 1$ *for all* $i \in [n]$;
- $-x_i \leq x_j$ *for all* $-i \leq j$*, where* $i, j \in [n]$;
- $x_i \leq -x_j$ *for all* $i \leq -j$*, where* $i, j \in [n]$;
- $x_i \leq x_j$ *for all* $i \leq j$ *where* $i, j \in [n]$;
- $x_i \geq 0$ *for all* $i \geq 0$ *where* $i \in [n]$;
- $x_i \leq 0$ *for all* $i \leq 0$ *where* $i \in [n]$.

(Note that some of these inequalities will be equivalent to each other.) Then $\mathcal{O}_P = \mathcal{O}_{\hat{G}(P)}$.

Proof. Starting with a signed poset P, constructing $\mathcal{O}_P$ and $\mathcal{O}_{\hat{G}(P)}$ yields polytopes defined by exactly the same set of inequalities. □

We can now give the following analogue to Proposition 5.5.1.

Proposition 5.5.4. *Let* P *be a signed poset on* $[n]$*. The* h^* *polynomial of* P *is Gorenstein if and only if* $\hat{G}(P)$ *is graded.*

Proof. Suppose $\hat{G}(P)$ is graded. We first briefly establish some facts about its maximal chains. The element 0 must be in one of the maximal chains; consider the part of the chain $0 < c_1 \cdots < c_k$. Because of Proposition 5.5.2, the chain $-c_k < \cdots < -c_1 < 0$ must also exist in $\hat{G}(P)$, and when extended into a maximal chain must not contain any other elements below 0, otherwise $0 < c_1 \cdots < c_k$ could be similarly extended. Thus, $-c_k < -c_{k-1} < \cdots - c_1 < 0 < c_1 < \cdots < c_{k-1} < c_k$ is a maximal chain. Thus, all maximal chains in $\hat{G}$ are of the same even length.

Suppose $\hat{G}(P)$ is graded with maximal chains of length $2k-2$ and rank function $\rho : \hat{G}(P) \to \mathbb{N}$. We show that $\mathcal{O}_P$ is Gorenstein of degree k. We first verify that $(k-1)\mathcal{O}_P$ has no interior points. Consider a maximal chain $c_{-(k-1)} \leq c_{-(k-2)} \leq \cdots \leq c_0 \leq \cdots \leq c_{k-2} \leq c_{k-1}$ of $\hat{G}(P)$. In order for a point q to be in the interior of $(k-1)\mathcal{O}_p$, by Proposition 5.5.3 , q must satisfy $-(k-1) < q_{c_{-(k-1)}} < q_{c_{-(k-2)}} < \cdots < q_{c_0} < \cdots < q_{c_{k-2}} < q_{c_{k-1}} < k-1$. This is not possible, since there are not $2k-1$ distinct integers between $-(k-1)$ and $(k-1)$.

We now construct a point $p \in k\mathcal{O}_P$ and show that it is the unique interior point of $k\mathcal{O}_P$. If in $\hat{G}(P)$, $\rho(i) = \rho(0)$ for some $i \in [n]$, we note that because of the symmetries outlined in Proposition 5.5.2, it must also be true that $\rho(-i) = \rho(0)$. In this case, we set $p_i = 0$. Suppose $\rho(i) - \rho(0) = \ell$. Then, we assign $p_i = \ell$. Note that $-(k-1) \leq \ell \leq k-1$, so p satisfies the strict inequality $-(k-1) < x_i < k-1)$ for all $i \in [n]$. By construction, the coordinates of p satisfy the other strict inequalities in $k\mathcal{O}_{\hat{G}(P)}$. Thus p is an interior point of $k\mathcal{O}_P$.

We now show that such an interior point must be unique. As described above, since p is an interior point of $k\mathcal{O}_P$, for every maximal chain in $\hat{G}(P)$, $c_{-(k-1)} \leq c_{-(k-2)} \leq \cdots \leq c_0 \leq \cdots \leq c_{k-2} \leq c_{k-1}$, p must satisfy $-k < q_{c_{-(k-1)}} < q_{c_{-(k-2)}} < \cdots < q_{c_0} < \cdots < q_{c_{k-2}} < q_{c_{k-1}} < k$, so each coordinate corresponding to this maximal chain is uniquely determined. Since every element in a poset is part of a maximal chain, every coordinate of p is uniquely determined.

Finally, for all integers $t \geq k$, we establish a bijection between the sets $t\mathcal{O}_P^\circ \cap \mathbb{Z}^d$ and $(t-k)P \cap \mathbb{Z}^d$. Let $p \in (t-k)P \cap \mathbb{Z}^d$ and consider $\phi(p) = p + (\rho(1)-\rho(0), \ldots, \rho(n)-\rho(0))$. First, we know that $t < \phi(p)_i < t$, since $t-k \leq p_i \leq t-k$ and $-k < \rho(i)-\rho(0) < k$. Furthermore, since p satisfies the inequalities of $k\hat{G}(P)$ and $(\rho(1)-\rho(0), \ldots, \rho(n)-\rho(0))$ satisfy the strict inequalities, their sum $\phi(p)$ also satisfies the strict inequalities, so $\phi(p)$ is indeed an interior point of $t\mathcal{O}_P$. We now need to show that $\phi : (t-k)\mathcal{O}_P \to t\mathcal{O}_P^\circ$ is bijective. It suffices to show that the map is surjective. Suppose we have a point $q \in t\mathcal{O}_P^\circ$, and consider $q' = q - (\rho(1)-\rho(0), \ldots, \rho(n)-\rho(0))$, so that $\phi(q') = q$. Note that since $-(t-1) \leq q_i \leq t-1$ and $-(k-1) \leq \rho(i)-\rho(0) \leq k-1$, $-(t-k) \leq q'_i \leq t-k$. Now, suppose $i \leq j \in \hat{G}(P)$. Then, since q is in the relative interior of $t\mathcal{O}_P$, $q_i - \rho(i) \leq q_j - \rho(j)$, which implies $q'_i \leq q'_j$. So q' is indeed in $(t-k)\mathcal{O}_P$.

We now suppose that $\hat{G}(P)$ is not graded, and show that $\mathcal{O}_P$ cannot be Gorenstein. For $\mathcal{O}_P$ to be Gorenstein of degree k, the interior points of the cone $\hom \mathcal{O}_P$ must be the integer points of the shifted cone $\hom \mathcal{O}_P + (p, k)$, where p is the unique interior point of $k\mathcal{O}_P$. Let $2a-2$ be the length of the longest maximal chain in $\mathcal{O}_{\hat{G}(P)}$. Based on an argument above, we know that there are no interior points in $b\mathcal{O}_P$ for any nonnegative integer $b < a$. We now consider $a\mathcal{O}_P$, the first dilate with at least one interior point $\mathbf{p}$. However, since $\hat{G}(P)$ is not graded, there must be a maximal chain of a length k where $k < 2a-2$, $c_1 < \ldots c_k$. Without loss of generality, let $p_{c_1} = -(a-1)$ and $p_{c_k} = a-1$, which is always possible since there are no elements above c_k nor below c_1. Since the maximal chain we are considering has length less than $2a-a$, we know that there is some $1 \leq i \leq k-1$ such that $p_{c_{i+1}} - p_{c_i} > 1$. We now construct an interior point p' of $(a+1)\mathcal{O}_P$, where we add 1 to all the coordinates p_j where $j \geq c_i$ in $\hat{G}(P)$ except for $p_{c_{i+1}}$. Note that $(\mathbf{p}', a+1) \in \hom \mathcal{O}_P$ is formed from adding a point that is not compatible with the ordering to $(\mathbf{p}, a)$, so $(\mathbf{p}', a+1)$ is an interior point of $\hom \mathcal{O}_P$ that does not lie in the shifted version. Thus, $\mathcal{O}_P$ is not Gorenstein.

□

The following result applies to our situation.

Theorem 5.5.1 (Bruns–Römer [32]). *A Gorenstein lattice polytope P with a regular unimodular triangulation has a unimodal h^*-vector.*

Corollary 5.5.5. *Let P be a signed poset on $[n]$. If $\hat{G}(P)$ is graded, then the h^* polynomial of $\mathcal{O}_P$ is unimodal.*

Stembridge [74] extended Reiner's work with signed posets to any root system. He defines a generalization of order cones and signed order cones for other root systems, calling these Coxeter cones.

Definition 5.5.3 ([74]). Let Φ be any root system in $\mathcal{R}^n$, and let Ψ be a subset of Φ. Then the coxeter cone of Ψ is

$$\Delta(\Psi) := \{x \in \mathcal{R}^n : \langle x, \beta \rangle \geq 0 \text{ for any } \beta \in \Psi\}$$

Viewing these cones as simplicial complexes and defining a general notion of when these complexes are graded, he used algebraic methods to give a condition on when the h-vectors of these cones are symmetric and unimodal.

Theorem 5.5.2 ([74])**.** *If a Coxeter cone is **graded**, then its h-polynomial is symmetric and unimodal.*

The definition of graded is quite technical; for a full definition see [74].
Proposition 5.5.4 can be seen as an Ehrhart theoretic interpretation of the type-B case of Theorem 5.5.2, using a geometric proof method as opposed to the algebraic proof method in [74]. Below, we summarize the connection between Proposition 5.5.4 and Theorem 5.5.2.

We first interpret Theorem 5.5.2 in the Type B case. In Examples 5.2(b) and 6.4(b), Stembridge notes that his definition of a graded Coxeter cone, when restricted to the type B case, results in exactly the Coxeter cones of type B corresponding to signed posets with a graded Fischer representation. Thus, these Coxeter cones result in simplicial complexes with a symmetric and unimodal h-polynomial.

We then make the transition from the h-polynomial of a type B Coxeter cone to the h^*-polynomial of $\mathcal{O}_P$ of the corresponding signed poset P. In Section 4 of [74], Stembridge notes that the h-polynomial of his type A and B Coxeter cones are identical to the h^*-polynomial of a certain lattice polytope. His construction in the type B case, described in algebraic terms, gives the same polytope as $\mathcal{O}_P$. Thus we see that Proposition 5.5.4 can be seen as a special case of the broad algebraic result in [74].

5.6 Chain Polytopes

We note that, in the definition of the chain polytope $\mathcal{C}(\Pi)$, it suffices to have an inequality for each maximal chain, since the other inequalities are implied by these.

In [67], Stanley establishes some properties of chain polytopes.

Definition 5.6.1. Let Π be a poset. An ***antichain*** of Π is a subset I of the elements of Π such that for any $i, j \in I$, neither $i < j$ nor $j < i$ in Π.

Proposition 5.6.1 (Stanley [67])**.** *The vertices of $\mathcal{C}(\Pi)$ are given by the $\{0,1\}$-indicator vectors of the antichains of Π.*

Theorem 5.6.1 (Stanley [67])**.** *Let Π be a poset on $[n]$.*

- *$\mathcal{C}(\Pi)$ and $\mathcal{O}(\Pi)$ have the same h^*-polynomial.*
- *$\mathcal{C}(\Pi)$ and $\mathcal{O}(\Pi)$ are combinatorially equivalent if and only if Π does not contain the poset shown in Figure 5.5 as a subposet.*

In this section, we suggest a definition for signed chain polytopes and examine the properties of said polytopes. First, we define a chain of a signed poset.

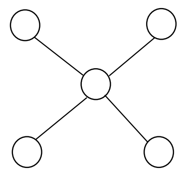


Figure 5.5: The forbidden poset in Theorem 5.6.1.

Definition 5.6.2. A chain on a signed poset P on $[n]$ is an ordered pair (C, S), where $C = (c_1, \ldots, c_m) \in [n]^m$ and $S = (s_1, \ldots, s_{m-1}) \in \{-1, 1\}^{m-1}$ such that for each $i \in [m-1]$ there exists $\alpha_i \in P$ that satisfy:

- if $s_i = 1$, then $\alpha_i = \pm(e_{c_i} - e_{c_{i+1}})$, and if $s_i = -1$, then $\alpha_i = \pm(e_{c_i} + e_{c_{i+1}})$;
- $\alpha_i + \alpha_{i+1} \in P$.

We now give one definition of a chain polytope.

Definition 5.6.3. The ***signed chain polytope*** $\mathcal{C}_P$ of a signed poset P on $[n]$ is the intersection of inequalities of the form

$$-1 \leq x_{c_1} + s_1 x_{c_2} + s_1 s_2 x_{c_3} + \cdots + s_1 s_2 \ldots s_{m-1} x_{c_m} \leq 1$$

for each chain (C, S) of P.

We also introduce another useful class of polytopes, directly related to Gorenstein polytopes.

Definition 5.6.4. A lattice polytope is ***reflexive*** if its hyperplane description can be written as $A\mathbf{x} \leq \mathbf{1}$ for an integral matrix A.

There are many equivalent definitions of reflexive polytopes, for example relating to the ***duals*** of polytopes. A reflexive polytope can also be described as a Gorenstein polytope with Gorenstein index 1. For a full description, see, for example, [76].

Directly from the definitions above, we make the following observation:

Proposition 5.6.2. *For any signed poset P, the signed chain polytope $\mathcal{C}_P$ is reflexive, and thus Gorenstein.*

Proof. From the definition, we can see that $\mathcal{C}_P$ is defined by a linear system $A\mathbf{x} \leq \mathbf{1}$ for an integral matrix A. □

One consequence of this is that it allows us to associate a Gorenstein polytope with every classical poset, since every classical poset can be viewed as a signed poset.

Similarly to Proposition 5.6.1, we can give a convex hull description of $\mathcal{C}_P$ in terms of antichains of P, defined below.

Definition 5.6.5. Let P be a signed poset on $[n]$. An element of $\mathbf{a} = (a_1 \dots a_n) \in \{-1, 0, 1\}^n$ is an antichain of P if for each element $\alpha \in P$ of the form $\pm e_i \pm e_j$ or $\pm e_i \mp e_j$, $\langle \alpha, \mathbf{a} \rangle \neq 0$ unless $a_i = a_j = 0$.

Example 5.6.1. Figure 5.6 shows a signed poset, with a chain indicated in blue. This chain $(C, S) = ((1, 2, 3), (1, -1))$ is the longest chain in this signed poset.

There are many antichains of this signed poset, one of which is $\mathbf{a} = (1, 0, 1, -1)$. It might seem like since 1 and 3 are related, they shouldn't both have a nonzero entry in the antichain, but the way the signs are arranged makes $\mathbf{a}$ fit the definition.

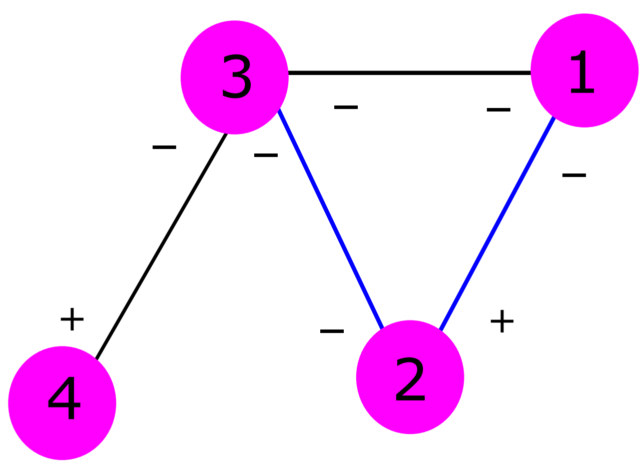


Figure 5.6: A bidirected graph representation of a signed poset on 4 elements, with a chain highlighted in blue.

Proposition 5.6.3. *The set of antichains of a signed poset P are exactly the integer points in $\mathcal{C}_P$.*

Proof. Suppose a point $\mathbf{p} \in \mathbb{R}^n$ is an integer point of $\mathcal{C}(P)$. Then $\mathbf{p}$ must satisfy all the inequalities specified in Definition 5.6.3; in particular:

- Suppose $\alpha \in P$ is of the form $\pm e_i \pm e_j$. Then, $(C, S) = (\{i, j\}, \{-1\})$ is a signed chain of P, and $\mathbf{p}$ must satisfy the inequalities $-1 \leq p_i - p_j \leq 1$. Thus, either $p_i = p_j = 0$, $p_i = \pm 1$ and $p_j = 0$, $p_i = 0$ and $p_j = \pm 1$, or $p_i = p_j = \pm 1$. In the latter three cases, it is true that $\langle \mathbf{p}, \alpha \rangle \neq 0$.

- Suppose $\alpha \in P$ is of the form $\pm e_i \mp e_j$. Then, $(C, S) = (\{i, j\}, \{1\})$ is a signed chain of P, and $\mathbf{p}$ must satisfy the inequalities $-1 \leq p_i + p_j \leq 1$. Thus, either $p_i = p_j = 0$, $p_i = \pm 1$ and $p_j = 0$, $p_i = 0$ and $p_j = \pm 1$, or $p_i = -p_j = \pm 1$. In the latter three cases, it is true that $\langle \mathbf{p}, \alpha \rangle \neq 0$.

Thus $\mathbf{p}$ satisfies all the properties of being an antichain of P.

Suppose $\mathbf{p}$ is not an integer point of $\mathcal{C}(P)$. Then there must be some chain $(\{c_1, \ldots, c_m\}, \{s_1, \ldots, s_{m-1}\})$ such that

$$p_{c_1} + s_1 p_{c_2} + s_1 s_2 p_{c_3} + \cdots + s_1 s_2 \ldots s_{m-1} p_{c_m} \leq -2$$

or

$$2 \leq p_{c_1} + s_1 p_{c_2} + s_1 s_2 p_{c_3} + \cdots + s_1 s_2 \ldots s_{m-1} p_{c_m}.$$

This implies that for some $i, j \in [m]$, $2 \leq s_1 \ldots s_i p_i + s_1 \ldots s_j p_j$ or $s_1 \ldots s_i p_i + s_1 \ldots s_j p_j \leq -2$. From this, we can determine that $s_1 \ldots s_i p_i = s_1 \ldots s_j p_j = \pm 1$. We have the following two cases:

- If $s_1 \ldots s_j = s_1 \ldots s_j$, then $p_i = p_j$. We also know that either $e_i - e_j \in P$ or $-e_i + e_j \in P$ from the transitivity of signed posets and the definition of signed chains. Since $\langle \mathbf{p}, \pm e_i \mp e_j \rangle = 0$, we deduce that $\mathbf{p}$ cannot be an antichain of P.

- If $s_1 \ldots s_j = -s_1 \ldots s_j$, then $p_i = -p_j$. We also know that either $e_i + e_j \in P$ or $-e_i - e_j \in P$ from the transitivity of signed posets and the definition of signed chains. Since $\langle \mathbf{p}, \pm e_i \pm e_j \rangle = 0$, we deduce that $\mathbf{p}$ cannot be an antichain of P. □

We note that there is no nice analogue for Theorem 5.6.1, since generally $\mathcal{C}_P$ and $\mathcal{O}_P$ are neither combinatorially equivalent nor Ehrhart equivalent. One example of the latter is the example in which P contains an element of the form $\pm e_i$. Observe that $\mathcal{O}_P$ has no interior lattice points, since the defining inequality $\pm x_i \geq 0$ prevents the origin from being an interior point and there are no other posibilities for an interior point of a polytope that is a subset of $[-1, 1]^n$. From the definition, we can see that $\mathcal{C}_P$ always has the origin as in interior point. Thus in this case, these two polytopes cannot have the same Ehrhart polynomial.

Chapter 6

Future Work

In this chapter, we give some ideas for future research related to each of the projects described in this thesis.

6.1 Subdivisions of Shellable Complexes

Since the paper describing this project [46] was released, Athanasiadis [1] answered the question posed by Brenti, Mohammadi, and Welker, showing that the boundary complex of any cubical polytope has a real-rooted h-polynomial. His paper used different techniques than the ones that we used to prove special cases of this result. We are interested in the following question: Does the boundary complex of any cubical polytope admit a stable shelling? If so, we can give an alternative proof of Athanasiadis's result using the framework developed in Chapter 3.

Furthering this line of thinking to non-cubical polytopes, we can ask: Are there other families of polytopes that admit stable shellings, and do real-rootedness results follow? Since it is known that all polytopes admit a stable shelling, we can also ask: Do all polytopes admit a ***stable line shelling***?

6.2 Inequalities for f^*-vectors of Lattice Polytopes

There are many open questions surrounding f^*-vectors, for example, those inspired by analogous studies of h^*-vectors. We conclude with a few open questions which are natural followups to the results presented in this Chapter 4.

The techniques in the proof of Theorem 4.0.3 do not offer much insight in the case of 14-dimensional lattice polytopes as there are candidates for f^*-vectors with corresponding h^*-vectors that satisfy all inequalities discussed in [71]. It is unknown though if such polytopes exist.

Higashitani [44, Theorem 1.1] provided examples of d-dimensional polytopes with nonunimodal h^*-vector for all $d \geq 3$. Therefore, by Theorem 4.0.3 we have examples of polytopes that have such a h^*-vector but their f^*-vector is unimodal. It would be interesting to know if the opposite can be true, that is, if there exist polytopes with unimodal h^*-vector and nonunimodal f^*-vector. By Corollary 4.1.3, such polytopes would need to have degree at least 6.

Whenever we are able to show that a combinatorial polynomial is unimodal, it is natural to ask whether the polynomial satisfies stronger properties, such as log concavity or real-rootedness. It would be interesting if one could extend, e.g., Proposition 4.1.1 along these lines.

Finally, starting with Stapledon's work [71], there has been much recent attention to symmetric decompositions of h- and h^*-polynomials; see, e.g., [7, 9] and, in particular, [25] where analogous decompositions for f-vectors are discussed. We believe this line of research is worthy of attention with regards to understanding f^*-vectors and the inequalities that hold among their coefficients.

6.3 Signed Poset Polytopes

At the end of Chapter 5, we noted that for the signed chain polytope definition given, there is no analogue for Theorem 5.6.1 [67], stating that for a classical poset Π, $\mathcal{C}(\Pi)$ and $\mathcal{O}(\Pi)$ have the same h^*-polynomial. The definition we give in Chapter 5 has the advantages of always giving reflexive polytopes and working well with a definition of signed antichains, but we wonder if there is an alternate definition of signed chain polytopes that gives rise to an analogue for Thereom 5.6.1. Our current definition does not take roots of the form $\pm e_i$ (denoted as loops in the bidirected graph representation) into acount, so perhaps a definition that does take these into account would be of interest.

It is currently unknown if classical order polytopes always have unimodal h^*-polynomials; it would be of interest to continue to make headway towards this difficult problem in the signed order polytope case.

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