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THE VALIDITY OF THE ISOTOPIC SPIN QUANTUM NUMBER FOR LIGHT NUCLEI

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ABSTRACT OF THESIS

The validity of the isotopic spin quantum number, which exists for charge independent nuclear forces, is affected only by the Coulomb potential. Unless charge independence is to be renounced, it must be possible to attribute violations of certain isotopic spin selection rules on nuclear reactions solely to the effect of the Coulomb potential in mixing states of different isotopic spin.

The isotopic spin impurity can be considered to arise in two ways: (1) through perturbation of the wave functions of nucleons in open shells by their Coulomb interaction with each other and with the field produced by protons in closed shells, and (2) through perturbation of the wave functions of nucleons in closed shells by their mutual interaction. The second effect, termed the "core impurity", has been neglected in previous work by L. A. Radicati. But a simple calculation using plane waves to represent the nuclear states (statistical model) gives a much larger figure, for the core impurity than that which Radicati found for the impurity of the state of nucleons in open shells.

Calculations of both types of impurity are then made using the jj coupling model of M. Mayer and harmonic oscillator wave functions. An understanding is obtained of the operation of the Coulomb potential in introducing isotopic spin impurity into the states of two and three nucleons in an open shell by studying the mixing of states of the representative configurations $(1p_{3/2})^2$, $(1p_{1/2})^2$, and $(1p_{3/2})^3$ to other low-lying states. The impurities of these states are found to be even

smaller than the results of Radicati. The principal reason for the difference lies in our use of the r^2 potential produced by a uniform sphere of charge rather than the singular $(1/r)$ core potential of Radicati. A general formula is derived for the equivalent potential produced by nucleons in closed jj shells when the field arises from scalar, spin independent, two-body potential. For the specific case of the Coulomb interaction one obtains an equivalent core potential which differs very little from that produced by a uniform sphere of charge.

The core impurity for the nucleons in closed jj shells can then be calculated in terms of the two-particle matrix elements which were just found. The figure obtained for the isotopic spin impurity of C^{12} is in excellent agreement with the simple statistical model calculation and confirms the predominance of this core impurity over the impurity of states of nucleons in open shells. One can conclude in fact that only the isotopic spin impurity of the core need be considered to be present in nuclear states until quite high excitations are reached.

An isotopic spin selection rule on electric dipole transitions provides a basis for the actual experimental determination of isotopic spin impurity provided that one can predict uninhibited El widths with sufficient reliability. A discussion is presented of the effects which produce variations in El matrix elements and the conclusion is reached that these effects are not important in light nuclei (A $\leqslant$ 20) at moderate excitation energies ($<$ 15 Mev). Consequently the limits on isotopic spin impurity determined experimentally by Wilkinson and collaborators from the El selection rule are approximately correct. These limits are easily explained by the presence of the Coulomb interaction.

Higher order correction terms to the El selection rule are also calculated and found to be $\sim$ 1000 times less effective than isotopic spin impurity in promoting forbidden transitions. In the course of the calculation formulas are derived for the exact quantum mechanical transition probabilities for radiation of any multipolarity.

Finally, the isotopic spin selection rule on β -decay is studied, but one can show that no violations of the selection rule are likely to be found in light nuclei. The effect of the isotopic spin impurity on the Fermi coupling constant derived from $0^+ \rightarrow 0^+$ transitions is shown to be just beyond the present limits of experimental accuracy.

CONTENTS

THE VALIDITY OF THE ISOTOPIC SPIN QUANTUM NUMBER FOR LIGHT NUCLEI

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1. Introduction

The introduction of the isotopic spin quantum number into nuclear physics occurred in a paper by Heisenberg (H32) on the systematics of nuclei. Heisenberg proposed to write the nuclear hamiltonian in a way which was symmetric with respect to the nucleons interacting by taking the wave functions for a neutron or a proton to be eigenfunctions corresponding to the eigenvalues $+1$ and -1 respectively of an isotopic spin operator τ_ζ represented in its diagonal form by the 2 x 2 matrix $\tau_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In order to treat correctly terms arising from the space exchange of a neutron and a proton it is also necessary to introduce the other three components of the isotopic spin vector

$$\vec{\tau} = \{\tau_\xi, \tau_\eta, \tau_\zeta\}$$

$$\tau_\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_\eta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_\zeta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

With these definitions (which differ from those of later authors; e.g., Wigner (W37), (W41),) one has a formalism which is the precise counterpart of the Pauli spin theory and in fact the Pauli spin matrices σ_x, σ_y, and σ_z have this matrix representation. We should remember, however, that the isotopic spin operates not in 3-space but in "isotopic spin space", and we shall, for this reason, use the ξ, η, ζ to denote the components of the isotopic spin vector $\vec{\tau}$. It is trivial to verify that these operators satisfy the commutation relations

$$[\tau_\xi, \tau_\eta] = 2i\tau_\zeta \qquad [\tau_\eta, \tau_\zeta] = 2i\tau_\xi \qquad [\tau_\zeta, \tau_\xi] = 2i\tau_\eta \tag{2}$$

The simplicity with which the isotopic spin formalism can represent the four nuclear interactions in common use was recognized by Cassen and Condon (C36) who gave the following formulas for the Wigner, Bartlet, Heisenberg, and Majorana potentials

$$V_M(r) = V(r) \qquad \text{Wigner (ordinary)}$$

$$V_B(r) = -\tfrac{1}{2}(1 + \sigma_1 \cdot \sigma_2)\, V(r) \qquad \text{Bartlett (spin exchange)}$$

$$V_H(r) = -\tfrac{1}{2}(1 + \tau_1 \cdot \tau_2)\, V(r) \qquad \text{Heisenberg (space and spin exchange)}$$

$$V_M(r) = -\tfrac{1}{4}(1 + \sigma_1 \cdot \sigma_2)(1 + \tau_1 \cdot \tau_2) V(r)$$
$$\text{Majorana (space exchange)}$$

$$(3)$$

Following a suggestion by Breit and Feenberg (B36) that the nuclear forces are independent of the character of the interacting nucleons, i.e. whether neutron or proton, Wigner applied the concept of isotopic spin to the formulation of a theory of the symmetric hamiltonian (W37a). Using as the two assumptions

(1) the nuclear hamiltonian is independent of the character of the interacting nucleons

(2) nuclear forces are spin independent,

Wigner developed the elegant supermultiplet theory of nuclear structure in which one can attach three quantum numbers to the nuclear levels which characterize certain multiplets. Using group theory the relative separations of the multiplets can be given in terms of a direct and an exchange integral, to be evaluated by use of some model or set of wave functions, for a potential which is the sum of a Wigner and a Majorana interaction (W37b, F37b). Perhaps the most striking success of Wigner's

theory is in the interpretation of the irregularities in binding energies of the nuclei beyond O^{16} (W37b, W40) e.g. energy differences between odd-odd and even-even nuclei, instability of odd-odd nuclei, etc.

The requirement of a description of the matrix element for β-decay, which should be symmetrical in the nucleons, led Nordheim and Yost (N37) to formulation of the theory in terms of the isotopic spin coordinate. Wigner (W39) also looked at β-decay from the standpoint of supermultiplet theory and was able to give selection rules for β-decay between different supermultiplets for both Fermi and Gamow-Teller matrix elements. The prediction of favored allowed β-transitions was and is a striking success, unique to Wigner's theory.

The validity of the "first approximation", in which both of Wigner's assumptions are made, and of the "second approximation" in which only the first holds, have been discussed in considerable detail by Wigner and Feenberg (W41). We shall not review the evidence for the validity of the "first approximation" here but will only point out that the whole theory of the rather successful jj coupling shell model rests on a contradictory assumption (F49), the existence of a strong spin-orbit potential in order to provide the closure of subshells at the "magic numbers" (E33, E34, M48, M49, H49). The suggestion of such a force has also been made by Case and Pais (C50) to explain the 340 Mev pp scattering data. In addition, work by A. M. Feingold (F52) and Keilson (51) indicates that an appreciable spin-orbit coupling is to be expected from a tensor force via second order perturbation theory. This origin of the spin-orbit force is not generally accepted, but other sources have not yet been found. The work of Breit (B37a,b, B38) shows that one cannot expect from relativistic corrections to two particle equations a spin orbit force of sufficient magnitude to give the jj coupling

model. This result has also been verified by Dresner (D53) using pseudoscalar meson theory.

Even when the first approximation is assumed to be invalidated by strong spin dependent forces, however, the isotopic spin of a nucleus remains a good quantum number in the second approximation and extremely useful selection rules for nuclear reactions emerge from its properties. In the next section we shall discuss the significance of the isotopic spin for nuclei and give the consequences of its validity.

2. The Isotopic Spin Quantum Number for Nuclei

(2.1) Validity of the Isotopic Spin and Selection Rules

Just as we have defined the isotopic spin vector $\vec{z}$ in 1^1 in strict analogy to the Pauli spin vectors from the requirement that the neutron and proton wave functions be eigenfunctions of τ_ζ with eigenvalues $+1$ and -1, we can proceed to define the isotopic spin $\vec{t}$ of a particle and of a nucleus $\vec{T}$ by the equations

$$\vec{t} = \tfrac{1}{2} \vec{z}$$
$$\vec{T} = \sum_{i=1}^{A} \vec{t}_i \qquad\qquad t_\zeta \psi_\pm = \pm \tfrac{1}{2} \psi_\pm \qquad (1)$$

The ζ-component of a nucleus will be given from these definitions by the formula

$$T_\zeta = \tfrac{1}{2}(N - Z) \qquad (2)$$

The result follows from the conservation of charge that nuclear states are eigenfunction of T. From the fact that a state of charge T_ζ cannot belong to a state of lower multiplicity than $(2T_\zeta + 1)$ the inequality follows that $T \geqslant T_\zeta$. The condition that T be a good quantum number is the same as that which holds; for example, for S,

the spin quantum number; viz.

$$\left[T^2, H \right] = 0 \tag{3}$$

or stated otherwise---T is a good quantum number if an only if H is invariant to rotations in isotopic spin space.

The common statements that T is a good quantum number only if nuclear forces are "charge independent" or "does not depend on the isotopic spin" are seen to be misleading and erroneous. The charge of the neutron and proton really have little to do with isotopic spin formalism except insofar as the charge of a proton distinguishes it from a neutron (W52). Further, the absence of the isotopic spin from the nuclear hamiltonian is certainly sufficient but not at all necessary for the validity of isotopic spin. The interaction of the charge symmetrical meson theory $(\vec{\tau}_1 \cdot \vec{\tau}_2)$ is obviously invariant under rotations in the isotopic spin space of the particles and therefore leaves T a good quantum number, just as does the neutral meson theory.

The present active interest in the isotopic spin began primarily with the experimentalists in nuclear physics and those otherwise interested in cataloguing and understanding the large amount of experimental data on light nuclei which is being accumulated at an increasing rate. For them the isotopic spin provides several important selection rules in each of the three types of nuclear reactions: (1) β -decay, (2) isomeric transitions, and (3) reactions resulting in heavy particle emission. Selection rules for processes of type (1) are usually simple and forbid such reactions such as (d, α) going from the ground state of an N = Z nucleus to the T = 1 states of the final N = Z nucleus. Selection rules for the first process were first given by Wigner (W39) and are different for the Fermi and Gamow-Teller matrix element.

Fermi matrix element $\qquad \Delta T = 0$

Gamow-Teller matrix element $\qquad \Delta T = 0, \quad 1 \text{ no } 0 \rightarrow 0$.

There are of course additional selection rules for spin and angular momentum. Finally, selection rules for electric dipole transitions were recently derived by Trainor (T52) in supermultiplet theory and by Christy (C52), Radicati (R52), and Gell-Mann and Telegdi (G53) for the second approximation.

The validity of these selection rules is affected only by a nuclear interaction which does not commute with T^2, i.e. by a "charge dependent" interaction, and by the ordinary Coulomb interaction

$$\frac{e^2}{8} \sum_{i \neq j} \frac{(1 - \tau_{\zeta i})(1 - \tau_{\zeta j})}{r_{ij}} \qquad \text{which also does not commute with } T^2.$$

Before any conclusions can be drawn about the equality of the neutron-neutron, protron-proton, and neutron-proton nuclear force, we must consequently know quantitatively the effect of the Coulomb force on the isotopic spin quantum number. Dismissal of any observed departures from the above selection rules by ascribing them to the Coulomb force would strongly suggest "charge independence" of the nuclear interaction. Conversely, the observation of large departures from the isotopic spin selection rules which could not be explained by Coulomb forces would certainly imply the existence of forces which were not independent of the character of the interacting nucleons. Of course, in case Coulomb forces should be shown to give rise to considerable mixing of the states of different isotopic spin the usefulness of the isotopic spin quantum number would be destroyed.

An investigation of the effectiveness of the Coulomb interaction in mixing states of different isotopic spin has been carried out by

Radicati (R53a) for the ground states of nuclei with two or four nucleons outside a closed core, for the 1.74 Mev and 5.11 Mev states of B^{10}, and for the 7.12 Mev state of O^{16}. These calculations are limited somewhat in their validity by (1) the use of the supermultiplet theory (especially for excited levels), (2) by the neglect of core excitation to higher isotopic spin states, and (3) by the neglect to a large extent of configuration interaction. The first of these should tend to give somewhat lower values for the percentage mixing of higher isotopic spin states to the ground state because of the fact that the Coulomb excitation leaves L^2 and S^2 good quantum numbers. In a classification of nuclear states which has L^2 and S^2 good quantum numbers, as in supermultiplet theory, the Coulomb perturbation should be much less effective in mixing states. The second assumption is completely unjustified without further investigation in view of the fact that the first isotopic spin state which can be mixed to the $T = 0$ ground state in even-even nuclei lies at only ~ 15 Mev, whereas the first such state in light odd-odd nuclei, as we shall show, lies at ~ 8 Mev. The fact that energy denominators for mixing are only twice as large in the even-even nuclei as in odd-odd nuclei suggests strongly that core excitation and configuration interaction may be very important even in light nuclei where Radicati's assumptions have the best chance of being correct. For heavy nuclei these effects of core excitation and configuration interaction are almost surely of considerable importance.

We shall restate the problem of isotopic spin impurity and our approach to the whole question.

We are interested in the extent to which one can assign an isotopic spin quantum number to the states of light nuclei, by which we mean $A \leq 20$. Since an obvious manifestation of an isotopic spin

quantum number would be the existence of certain selection rules for nuclear reactions, we shall pose the more concrete question--How much admixture of states of different isotopic spin is introduced into an eigenstate of the total isotopic spin by the Coulomb interaction? We shall call the amount of the admixture--"the isotopic spin impurity of the state".

In the shell model the simplest nuclear state is the ground state of a nucleus consisting of closed shells in neutrons and protons. Although the number of such states is an insignificant fraction of the states of interest, we can describe approximately the ground state and low-lying states of a nucleus with one or more particles in open shells as the state of a system consisting of a closed shell nucleus and one or more extra nucleons. Since the states of different isotopic spin are separated by a rather large energy, this neglect of exchange terms of the Coulomb interaction between closed shell and open shell states is legitimate (cf. 4.42). The effect of the Coulomb interaction in promoting the isotopic spin impurity of the ground state of the "core", or closed shells, and of the state of the extra-shell nucleons can be treated separately. In calculating the latter, the isotopic spin impurity of the nucleons in open shells (assuming no core excitation), the inter-action with the core is introduced by an "effective" potential which represents the non-exchange terms in the matrix element of the Coulomb interaction.

In this way we can approach the problem of isotopic spin impurity in nuclei having several closed shells and one or more nucleons in open shells by answering as completely as possible the following two questions: (1) In light nuclei ($A \leqslant 20$) to what extent does the Coulomb energy mix to the ground state of the core, taken as the closed shells of the

nucleus, excited states of the core with different isotopic spin?

(2) What is the effect of the Coulomb interaction between particles in open shells upon the impurity of the isotopic spin quantum number of the ground state?

We shall first answer question (1) by using methods based on the statistical model of the nucleus and thus obtaining results which are somewhat less dependent on the details of nuclear structure than later calculations which use the jj coupling model. Following this discussion a study is made of the mixing of states belonging to two or three particles in an open shell. The whole problem of the mixing of isotopic spin states for light nuclei then is completed and made consistent by using harmonic oscillator wave functions also to find the mixing of the core (closed shells) of light nuclei. Comparison of the latter results with the results of the statistical model are in agreement.

Following the development of the answers to these questions, we shall be concerned with the implications of the experimental results which shed light on the isotopic spin impurity of nuclear states. These results can be classified into β-decay, isomeric transitions, and reactions resulting in heavy particle emission. By far the greater emphasis will be placed on the data on isomeric transitions which comes from the work of D. H. Wilkinson and collaborators. These experiments on the electric dipole radiation were specifically designed to answer the question of isotopic spin impurity.

It has been suggested by M. Gell-Mann and V. L. Telegdi, however, that the <u>lower bounds</u> on isotopic spin impurity obtained by Wilkinson from electric dipole transition are invalidated by the effect of higher order terms in the electric dipole moments. These terms are ordinarily neglected in the approximation $kr \ll 1$ where "k" is the wave number of the emitted γ-ray and "R" is the nuclear radius. We shall show that for the energy of the electric dipole

transitions which Wilkinson considers these higher order terms are a factor of ~ 1000 less important than the isotopic spin impurity in promoting transitions which violate the selection rule on electric dipole transitions derived by Radicati. In so doing we offer a derivation of the multipole transition probabilities for the non-relativistic hamiltonian which is exact in (kR) and in a form suitable for estimating corrections to the Weisskopf formulae (W51). This has not been done before for the non-relativistic hamiltonian and has just recently been done for the Dirac equation by Stech (S54).

(2.2) Dynamic Validity of the Isotopic Spin

In addition to questions concerning the mixing of states belonging to different eigenvalues of the total isotopic spin T^2 Wigner has raised a point which he has termed the question of the "dynamic validity of the isotopic spin". To see what this means we observe that from the commutation relations of 1^a and the definitions (1) it follows that the operator $(T_\xi \pm i\,T_\eta)$ raises or lowers the T_ζ eigenvalue i.e. the charge, of a state characterized by eigenvalue T unless $T_\zeta = T$ or $T_\zeta = -T$ respectively, when the state is annihilated by the operator.

$$(T_\xi + i\,T_\eta)\,\Psi_{T,T_\zeta} = \sqrt{(T-T_\zeta)(T+T_\zeta+1)}\;\Psi_{T,T_\zeta+1}$$

$$(T_\xi - i\,T_\eta)\,\Psi_{T,T_\zeta} = \sqrt{(T+T_\zeta)(T-T_\zeta+1)}\;\Psi_{T,T_\zeta-1} \qquad (4)$$

In the theory of Wigner's second approximation, the corresponding states of an isobaric triad, such as $Be^{10}\; B^{10}\; C^{10}$, are simply the different T_ζ components of the same T-multiplet and should possess the same nuclear spin J, parity, and energy except for the perturbation by the Coulomb interaction. Wigner's question then is--To what extent

are the equations (4) valid in the presence of the Coulomb force? This question is of obvious significance for such reactions as the allowed-favored β-decays.

Assume first that we are given a complete set of commuting constants of the motion including T , which is not quite a constant of the motion because of the Coulomb interaction. We can then characterize every nuclear level by a set of quantum numbers, which are valid in the presence of the Coulomb interaction, and by the isotopic spin quantum number T . The question of the dynamic validity is now equivalent to the question of impurity of isotopic spin states, since the Coulomb force can only mix states of different T .

We can now see clearly that only when the quantum numbers, which we use to characterize a nuclear state, do not form a complete set of commuting constants of the motion does this question arise. For example in the calculations which are made on the jj coupling shell model the quantum numbers are the configurations e.g. $(1p_{3/2})^2$ and the spin and isotopic spin J, T . For this case a state of definite T is mixed by the Coulomb force with states of other configurations having the same J, T . For this reason the possibility arises of the result that the Coulomb interaction leaves T a good quantum number but invalidates the relation (4). We shall give also an answer in this thesis to the question of dynamic validity.

(2.3) Location of the Low-lying Isotopic Spin States

For a discussion of the isotopic spin selection rules and isotopic spin mixing produced by the Coulomb interaction a knowledge of the low-lying isotopic spin states is crucial. We are interested in the ground state isotopic spin, the first excited isotopic spin state, and the

first excited isotopic spin state of the same nuclear spin J and parity

as the ground state. The latter level is the most important one for

consideration of the mixing of excited isotopic spin states to the ground

state.

For identification of the isotopic spin states in nuclei with

$A \leq 20$ we have the convenient rule that the ground state of all nuclei

up to Cl^{34} (A53) have as the ground state $T = T_\zeta$. The origin of

this rule (W37b) lies in the simple fact that the Coulomb energy will

make the normal state of the nucleus with the largest value of T_ζ the

most stable one unless the Coulomb perturbation becomes so large that

the binding energy is lower for the next higher isotopic spin multiplet

and same T_ζ . This is illustrated by the (Wigner-type) following

diagram in which the ordinate represents the excitation energy above

some (negative) reference energy and the abscissa is T_ζ . In the

diagram Fig. 1c the effect of the Coulomb force has been to bring the

$T = 1$ multiplet below the $T = 0$ multiplet for nuclei with $T_\zeta \geq 4$.

A detailed discussion of the stability relations for heavier nuclei

which Wigner diagrams suggest has been given by Feenberg (F47).
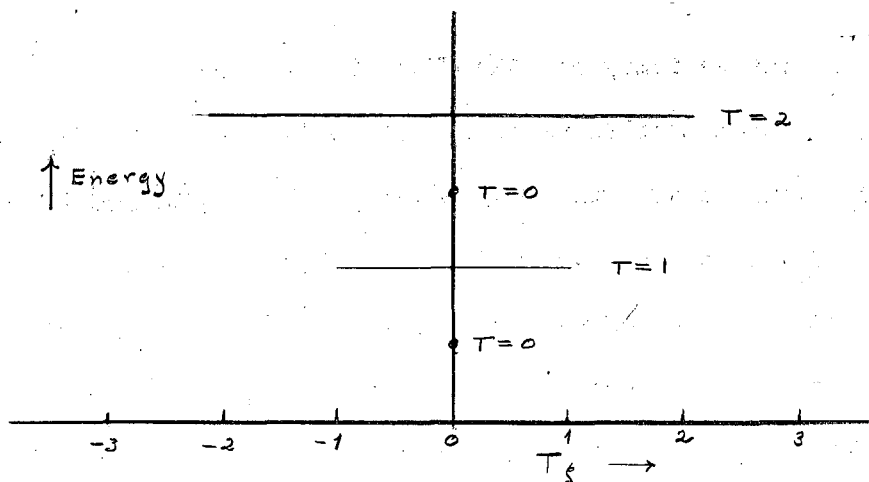
Representation of Multiplets
in Absence of Coulomb
Force
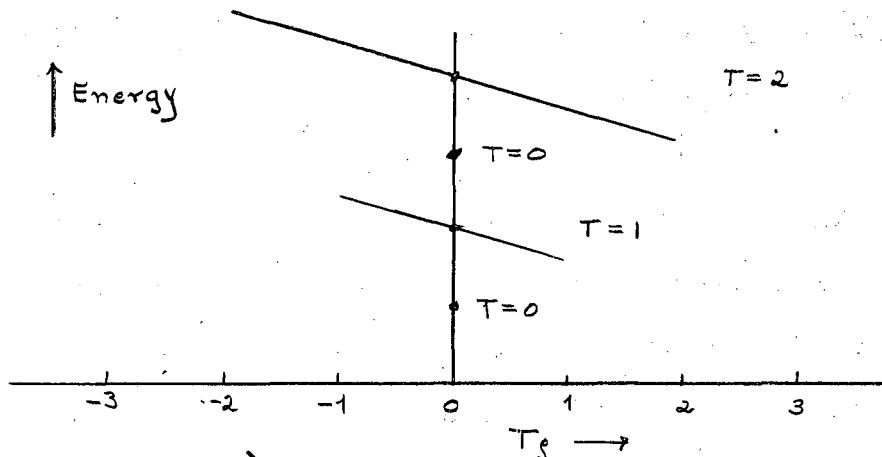
Effect of Coulomb Perturbation
on Multiplets
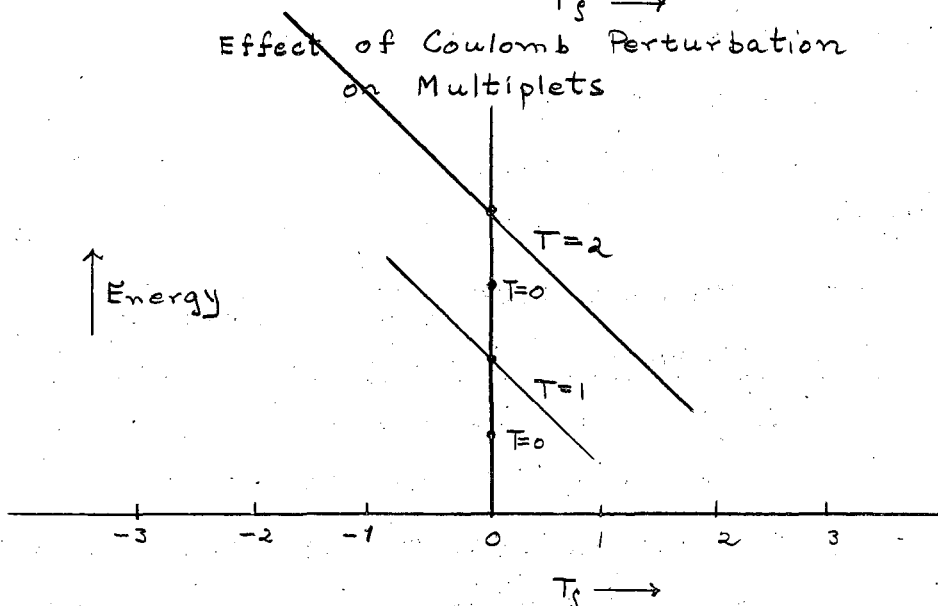
Elevation of Ground State Multiplet
by Coulomb Perturbation

The validity of this rule that the ground state has $T = T_\zeta$ for light nuclei enables us to determine the $T = 1$ level in odd-odd and even-even nuclei with $T_\zeta = 0$ as the analogue (always after corrections for the Coulomb energy difference and neutron-proton mass difference) of the ground state of the neighboring $|T_\zeta| = 1$ isobars. The relations say further that every level of the $|T_\zeta| = 1$ nucleus must have an analogue level of the same nuclear spin and parity in the $T_\zeta = 0$ isobar. The identification of the first excited isotopic spin state with the same spin and parity as the ground state is considerably more difficult because such states occur in a region of high level density in the stable $T_\zeta = 0$ nucleus while only a few spins and parities have usually been measured for the unstable $|T_\zeta| = 1$ isobars. There is difficulty in locating even the first excited isotopic spin state in even-odd nuclei, so the first excited state with the same spin and parity is unknown in all these cases. We shall make no attempt to locate the low-lying states in the odd-odd members of even A nuclei since no information, even on the ground states of their isobars with $|T_\zeta| = 2$, is available.

We shall now list the low-lying isotopic spin states for $T_\zeta = 0$ nuclei which are even-even or odd-odd and for odd-even nuclei. The level separations in these cases form three separate series.

$T_\zeta = 0$ Even-even Nuclei--(The first $T = 1$ $J = 0^+$ state is unknown in all)

Be^8	$Li^8(\beta^-)Be^8$ with Coulomb correction gives $T = 1$ state of Be^8 at ~ 16.8 Mev. The observation that over 95% of the reactions $C^{12}(\gamma,\alpha)Be^8$ leave Be^8 in an excited state at 17.0 ± 0.2 Mev (W51, T53) strongly suggests $T = 1$ for this state. Strong angular correlations indicate a $J \geqslant 2$ and Telegdi (T53) assigns $J = 2^+$. The spins and parities of the ground states

of Li^8 and B^8 are unknown but we expect $J = 2^+$ if Telegdi's assignment is correct.

C^{12} $B^{12}(\beta^-)C^{12}$ with Coulomb correction gives position of $T = 1$ as 15.09 Mev. A state is listed by Ajzenberg and Lauritsen at this energy but there are several near by levels, so positive identification is not possible.

O^{16} $N^{16}(\beta^-)O^{16}$ with Coulomb correction gives 12.2 to 13.3 Mev. A number of states in O^{16} lie in this range including two $J = 2^-$ states at 12.51 Mev and 12.95 Mev. Tentative identification of the ground state of N^{14} as $J = 2^-$ favors either of these two states as the $T = 1$ analogue in O^{16}. Positive identification has not been made.

Ne^{20} $F^{20}(\beta^-)Ne^{20}$ with Coulomb correction gives 10.1 Mev. A 10.1 Mev level is listed by Ajzenberg and Lauritsen as occurring in $F^{19}(d, n)Ne^{20}$ and $O^{16}(\alpha, \alpha)O^{16}$. The latter reaction, however, can show a 10.1 Mev level only by violation of the <u>strict</u> isotopic spin selection rule.

$T_\xi = 0$ Odd-odd Nuclei

Li^6 The level in Li^6 at 3.58 Mev has the correct position, nuclear spin and parity to correspond to the ground state of $He^6(J = 0^+)$. The $T = 1$ $J = 1^+$ state corresponding to ground state is unknown.

B^{10} The 1.74 Mev state has the correct position, spin, and parity $(J = 0^+)$ to be the first $T = 1$ level.

The 8.89 Mev level of B^{10} is suggested (A52) as $T = 1$ corresponding to the $J = 3^+$ state at 7.37 Mev in Be^{10}. This state has the same spin and parity as the ground state of B^{10}.

N^{14}. The 2.31 Mev level has the correct position, spin, and parity

$(J = 0^+)$ to be the first $T = 1$ state.

The 8.1 Mev level is quite possibly (S52, T52) the analogue

of the 6.1 Mev level of C^{14}, both being $J = 1$ states.

This level has the same spin and <u>perhaps</u> parity as the $J = 1$

ground state.

F^{18} The Coulomb energy produces the first stable $T = 1$ state.

$F^{18}(\beta^-)O^{18}$ with Coulomb correction gives the $T = 1$ state in

F^{18} at 1.1 Mev. No identifications of this $T = 1$ state or a

state of the same spin and parity as the ground state has been

made.

Odd-even Nuclei

Li^7 Peaslee and Telegdi (P53) conclude from data on $Li^7(\gamma,t)He^4$

and $Li^7(\gamma,n)Li^6$ that the first $T = 3/2$ level of Li^7 occurs

at 10.8 Mev.

Be^9 14.1 Mev separation from Li^9 when corrected for Coulomb energy

gives 15 Mev for the first $T = 3/2$ level.

N^{15} 8.8 Mev separation from C^{15} leads to 10.9 Mev for $T = 3/2$

level of N^{15}.

O^{17} 8.8 Mev separation from N^{17} leads to 11.6 Mev for $T = 3/2$

level of O^{17}.

F^{19} $O^{19}(\beta^-)F^{19}$ has a Q of 4.5 Mev giving 7 Mev for the $T = 3/2$

state while known states occur at 4.76 and 8.56 Mev.

We shall summarize these level locations in the following

tables.

TABLE I

Separations of $T = 0$ and $T = 1$ Multiplets for $T_\zeta = 0$ Nuclei.

A	6	8	10	12	14	16	18	20
Nucleus	Li^6	Be^8	B^{10}	C^{12}	N^{14}	O^{16}	F^{18}	Ne^{20}
$T = 1$ level Mev	3.58	16.8	1.74	15.09	2.31	12-13	1.05	10.1
$T = 1$ level of ground state spin and parity			8.89		8.1			

TABLE II

Separation of $T = \frac{1}{2}$ and $T = 3/2$ Multiplets for $T_\zeta = \frac{1}{2}$ Nuclei.

A	9	15	17	19
Nucleus	Be^9	O^{15}	O^{17}	F^{19}
$T = \frac{3}{2}$ level Mev	15	10.9	11.2	7

It is hardly necessary to emphasize that although these levels are incomplete and experimentally unverified in many cases, the locations are certainly correct to 10% or less and are therefore useful in providing an orientation and for computing the impurities of isotopic spin states.

We again mention that the first $T = 1$ state of the same spin and parity as the ground state $T = 0$ in odd-odd nuclei with $T_\zeta = 0$ lies much higher than the first $T = 1$ state. The first state which can be mixed to the ground state by Coulomb forces does not lie more than a factor of two lower in odd-odd nuclei $T_\zeta = 0$ than the first $T = 1$ level in even-even $T_\zeta = 0$ nuclei.

3. First Approximations to the Impurity of Isotopic Spin States of the Core

(3.1) Sum Rule Techniques

The so-called Coulomb perturbation which actually contains also the neutron-proton mass difference is given by

$$H_c = \frac{e^2}{8} \sum_{i \neq j} \frac{(1-\tau_{\zeta i})(1-\tau_{\zeta j})}{\lambda_{ij}} + (M_n - M_p)c^2 T_\zeta + \frac{A}{2}(M_n + M_p)c^2 \quad (1)$$

The last two terms commute with T^2 and can produce no mixing of nuclear states of either the same or different isotopic spin if these states are orthogonal, but only energy shifts. The first term can be decomposed into parts which are irreducible tensors in isotopic spin space with separate selection rules for each part.

$$\mathcal{H}_c = S + T_0^{(1)} + T_0^{(2)}$$

Scalar $\quad S = \frac{e^2}{8} \sum_{i \neq j} \frac{1}{r_{ij}} \left(1 - \frac{1}{3} \vec{\tau}_i \cdot \vec{\tau}_j \right) \qquad \Delta T = 0$

Vector $\quad T_0^{(1)} = -\frac{e^2}{8} \sum_{i \neq j} \frac{\tau_{si} + \tau_{sj}}{r_{ij}} \qquad \Delta T = 0, \pm 1$

Tensor $\quad T_0^{(2)} = \frac{e^2}{8} \sum_{i \neq j} \frac{1}{r_{ij}} \left(\tau_{si} \tau_{sj} - \frac{1}{3} \vec{\tau}_i \cdot \vec{\tau}_j \right) \quad \begin{array}{l} \Delta T = 0, \pm 1, \pm 2 \\ 0 \not\to 0 \quad 0 \not\to 1 \end{array}$

$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad$ (2)

The scalar part commutes with T^2 and therefore has matrix elements which are independent of T_ζ. Since the sole effect of S is a displacement of the T levels, it can always be included in the nuclear hamiltonian. The tensor $T_0^{(2)}$ can only mix the $T = 0$ with $T = 2$ state, and the energy separations of these states will enable us to neglect $T_0^{(2)}$ in computing the impurity of $T = 0$ states. We shall limit ourselves to this case in this section and can therefore confine ourselves to the vector component of the Coulomb perturbation.

When one applies a perturbation $\mathcal{H}_p$ to an eigenstate Ψ_0 of a hamiltonian, the perturbed state corresponding to Ψ_0 can be expanded in the eigenstates Ψ_ν of the hamiltonian

$$\Psi_0' = \sum_{\nu = 0}^{\infty} a_\nu \Psi_\nu \qquad \qquad a_\nu = \frac{(\Psi_\nu, \mathcal{H}_p \Psi_0)}{E_0 - E_\nu} \qquad (3)$$

The coefficients a_ν give the amplitude of higher states Ψ_ν in the perturbed state Ψ_0' with $1 - |a_0|^2 \ll 1$ for validity of the expressions. The percentage admixture of higher states to Ψ_0 is simply

$$p = \sum_{\nu = 1}^{\infty} \frac{|(\Psi_\nu, \mathcal{H}_p \Psi_0)|^2}{(E_0 - E_\nu)^2} \qquad (4)$$

By closure the inequality holds

$$p \leqq \frac{(\Psi_o, H_p^2 \Psi_o) - (\Psi_o, H_p \Psi_o)^2}{(E_o - E_1)^2} \tag{5}$$

It is therefore convenient to subtract from $T_o^{(1)}$ an expression which makes $(\Psi_o, H_p \Psi_o)$ zero and is of course a scalar in isotopic spin. We shall use for the Coulomb perturbation C.

$$C = \frac{e^2}{8} \sum_{i \neq j} \left(\frac{4T_\zeta}{A} - \tau_{\zeta i} - \tau_{\zeta j} \right) \hbar_{ij}^{-1} \tag{6}$$

whose expectation value is indeed zero in a $T = 0$ ground state as we shall see in the next section. The percentage mixing of $T = 1$ states to the $T = 0$ ground state is limited by

$$p \leqq \frac{(\Psi_o, C^2 \Psi_o)}{(E_o - E_1)^2} \tag{7}$$

We shall evaluate the matrix element of C^2 by a method commonly used in deriving sum rules. We first write

$$\langle C^2 \rangle = \frac{e^4}{64} \left\langle \sum_{i,j}' \left[\frac{32 T_\zeta^2}{A^2} - \frac{16 T_\zeta}{A} (\tau_{\zeta i} + \tau_{\zeta j}) + 2 (\tau_{\zeta i} + \tau_{\zeta j})^2 \right] \hbar_{ij}^{-2} \right.$$

$$+ 2 \sum_{i,j,\ell}' \left[\frac{32 T_\zeta^2}{A^2} - \frac{8 T_\zeta}{A} (2\tau_{\zeta i} + \tau_{\zeta j} + \tau_{\zeta \ell}) + (\tau_{\zeta i} + \tau_{\zeta j})(\tau_{\zeta i} + \tau_{\zeta \ell}) \right] \hbar_{ij}^{-1} \hbar_{i\ell}^{-1}$$

$$+ \left. \sum_{i,j,k,\ell}' \left[\frac{32 T_\zeta^2}{A^2} - \frac{4 T_\zeta}{A} (\tau_{\zeta i} + \tau_{\zeta j} + \tau_{\zeta k} + \tau_{\zeta \ell}) + (\tau_{\zeta i} + \tau_{\zeta j})(\tau_{\zeta k} + \tau_{\zeta \ell}) \right] \hbar_{ij}^{-1} \hbar_{k\ell}^{-1} \right\rangle$$

$$\tag{8}$$

where the primed summations indicate that different indices never assume the same value. We shall now assume that the expectation value of the reciprocal separation distances is the same for every term of each of the sums and shall designate these average values by $\overline{r_{12}^{-2}}$, $\overline{r_{12}^{-1} r_{13}^{-1}}$, and $\overline{r_{12}^{-1} r_{34}^{-1}}$. Each of the resulting expressions in the isotopic spin coordinates can be summed explicitly to give

$$\langle C^2 \rangle = \frac{e^4}{16} A(A-1) \left(1 - \frac{4 T_\zeta^2}{A^2}\right) \left\{ \overline{r_{12}^{-2}} + (A-4) \overline{r_{12}^{-1} r_{13}^{-1}} - (A-3) \overline{r_{12}^{-1} r_{34}^{-1}} \right\}$$

(9)

This expression satisfies the requirements of vanishing for $T_\zeta = A/2$ and for $\overline{r_{12}^{-2}} = \overline{r_{12}^{-1} r_{13}^{-1}} = \overline{r_{12}^{-1} r_{34}^{-1}}$. The first case corresponds to all nucleons being neutrons so that no Coulomb interaction exists. The second result follows from the fact that when these averages are equal the sum vanishes.

$$\langle C^2 \rangle = \frac{e^4}{64} \left(\sum_{i \neq j} \frac{4 T_\zeta}{A} - \tau_{\zeta i} - \tau_{\zeta j} \right)^2 \overline{r_{12}^{-2}} = 0$$

(10)

The averages which appear above can be evaluated on the uniform model which regards every nucleus as a sphere of nuclear matter with uniform density everywhere. The averages are then given by

$$\overline{r_{12}^{-2}} = \frac{1}{V^2} \oint \frac{d\tau_1 d\tau_2}{r_{12}^2}$$

$$\overline{r_{12}^{-1} r_{13}^{-1}} = \frac{1}{V^3} \oint \frac{d\tau_1 d\tau_2 d\tau_3}{r_{12} r_{13}}$$

$$\overline{r_{12}^{-1} r_{34}^{-1}} = \frac{1}{V^4} \oint \frac{d\tau_1 d\tau_2 d\tau_3 d\tau_4}{r_{12} r_{34}}$$

where V is the volume of the nucleus $\frac{4\pi}{3}R^3$ and $\oint$ represents a volume integral in which the points $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, and $\vec{r}_4$ move over the interior of a sphere of radius R.

The integrations can be effected most easily by use of a general formula for the integral $\oint f(r_{12})\,d\tau_1\,d\tau_2$. To obtain this formula choose the coordinate $\vec{r}$ of the center of mass of particle 1 and particle 2 and $\vec{\rho} = \vec{r}_2 - \vec{r}_1$ the relative separation. The transformation is

$$\vec{r}_1 = \vec{r} - \tfrac{1}{2}\vec{\rho}$$
$$\vec{r}_2 = \vec{r} + \tfrac{1}{2}\vec{\rho}$$

of which the Jacobian $J\left(\frac{\vec{r}_1,\vec{r}_2}{\vec{r},\vec{\rho}}\right)$ is unity. The integral can then be written

$$I = \oint f(r_{12})\,d\tau_1\,d\tau_2 = \oint f(\rho)\rho^2 \sin\alpha\,d\alpha\,d\beta \times$$
$$\times r^2 \sin\vartheta\,d\vartheta\,d\varphi\,d\rho\,dr$$

with (α, β) the angular coordinates of $\vec{\rho}/2$ and (θ, ϕ) those of $\vec{r}$. The ranges of integration for the coordinates are

$$0 \leq \varphi \leq 2\pi$$
$$0 \leq \vartheta \leq \pi \qquad 0 \leq r \leq \tfrac{1}{2}\left(-\rho\cos\alpha + \sqrt{\rho^2\cos^2\alpha + 4R^2 - \rho^2}\right)$$

and for the $\vec{\rho}$ coordinate

$$0 \leq \alpha \leq \pi$$
$$0 \leq \beta \leq 2\pi$$
$$0 \leq \rho \leq 2R$$

Then $I = 2\cdot\frac{8\pi}{3}\oint d\rho\,\rho^2 f(\rho)\int_0^{\pi/2} d\alpha\,\sin\alpha\cdot\tfrac{1}{8}\left(-\rho\cos\alpha + \sqrt{\rho^2\cos^2\alpha + 4R^2 - \rho^2}\right)^3$

$$I = \frac{\pi^2}{3}\int_0^{2R} d\rho\,f(\rho)\rho^2\left(\rho^3 - 12\rho R^2 + 16 R^3\right) \qquad (11)$$

This result now gives $\overline{r_{12}^{-1}}$ with ease as

$$\overline{r_{12}^{-1}} = \frac{1}{V^2} \iint r_{12}^{-1} \, d\tau_1 d\tau_2 = \frac{1}{V^2} \cdot \frac{\pi^2}{3} \int_0^{2R} d\rho\, \rho\, (\rho^3 - 12\rho R^2 + 16 R^3) = \frac{6}{5R}$$

Since $\overline{r_{12}^{-1} r_{34}^{-1}} = \overline{(r_{12}^{-1})}^2$ we obtain for this average

$$\overline{r_{12}^{-1} r_{34}^{-1}} = 1.44 \, R^{-2} \tag{12}$$

Trivial integrations yield the other averages

$$\overline{r_{12}^{-1} r_{13}^{-1}} = \frac{51}{35} R^{-2} = 1.457 \, R^{-2}$$

$$\overline{r_{12}^{-2}} = \frac{9}{4} R^2 = 2.25 \, R^{-2} \quad . \tag{13}$$

Inserting these quantities in (9) the matrix element emerges as

$$\left\langle c^2 \right\rangle = \frac{e^4}{16R^2} A(A - 1)(0.76 + 0.017\, A) \quad . \tag{14}$$

This result is very interesting in its dependence on A. From equation (9) one might expect $\left\langle c^2 \right\rangle$ to be proportional to A^3, but this result is proportional to $A(A - 1)$ for $A < {\sim}100$. An interpretation of this result can be made which is best expressed by the following equation

$$P_{state} = P_{particle} \frac{A(A - 1)}{2} \qquad P_{particle} = \left(\frac{e^2}{4R}\right)^2 1.52 \tag{15}$$

where P_{state} is the perturbation of a nuclear state, $P_{particle}$ the perturbation of the state of a single nucleon and $A(A-1)/2$ represents the number of interacting pairs in the nucleus. This equation will be justified later by another calculation of $P_{particle}$, and will be seen to be very useful in estimating the mixing of excited states.

(3.2) The Statistical Model

This simple calculation completely neglects the effects of correlation embobied in the Pauli principle as is obvious from the expressions for $\overline{r_{12}^{-2}}$, $\overline{r_{12}^{-1} r_{13}^{-1}}$ and $\overline{r_{12}^{-1} r_{34}^{-1}}$. For example, two particles in the same spin and isotopic spin state can never come into coincidence according to the Pauli principle, and for such pairs of particles the above averages should be smaller. The effect of these correlations should be greatest on $\overline{r_{12}^{-2}}$, since contributions from the singularity are sharply reduced. The average $\overline{r_{12}^{-1} r_{13}^{-1}}$ is affected somewhat less because the third particle may be in a different spin and isotopic state from the particles 1 and 2, and this fact allows contributions from the r_{13}^{-1} singularity. Since there is in $\overline{r_{12}^{-1} r_{34}^{-1}}$ a complete lack of correlation between the positions of 1, 2, and 3, however, the average $\overline{r_{12}^{-1} r_{34}^{-1}}$ is affected least of all by correlations. A much more physical way of seeing the effect of correlations on the averages proceeds from the physical interpretation of these quantities. The significance of $\overline{r_{12}^{-2}}$ is that it provides a measure of the magnitude of the density fluctuations in the nucleus. The quantity $\overline{r_{12}^{-1} r_{13}^{-1}}$ on the other hand associates the "density" in one direction from the point 1 with the "density" in another direction from that point. We may interpret $\overline{r_{12}^{-1} r_{13}^{-1}}$ as measuring the "angular" uniformity in density about a point. The last quantity $\overline{r_{12}^{-1} r_{34}^{-1}}$ correlates the density at one point with that at another point. For this reason $\overline{r_{12}^{-1} r_{34}^{-1}}$ measures the uniformity of the density of the nucleus.

For a reasonably large nucleus it is clear that uniformity of the nuclear matter from point to point should become a good approximation first. One sees in this way that $\overline{r_{12}^{-1} r_{34}^{-1}}$ can be well approximated by its value for a uniform nucleus. The effect of the nuclear surface

will cause $r_{12}^{-1} r_{13}^{-1}$ to deviate somewhat more from its value for a uniform nucleus than $r_{12}^{-1} r_{34}^{-1}$ does. The local density fluctuations should certainly disappear last, if at all, so that r_{12}^{-2} will show a rather large deviation from its value for a uniform nucleus. Comparison of the expressions (9) and (14) shows that correlation effects may be quite important and may materially affect the magnitude of $\langle c^2 \rangle$. We shall therefore proceed to evaluate $\langle c^2 \rangle$ in a framework in which such effects may be considered.

In attempting a more correct evaluation of $\langle c^2 \rangle$, we work in the framework of the Hartree or single particle picture of the nucleus and shall use a formalism which is similar to that of Rosenfeld (R48). There are a number of his expressions which omit terms that are important for our problem, however, and the result which we obtain for the matrix $(i / g(x_1, x_2) / f)$ will differ in several significant respects from that of Rosenfeld. The extension of these methods to the evaluation of three and four particle operators has not been done previously.

We begin by giving the state of the nucleus as described by the wave function $\psi (Q_1, Q_2, \ldots, Q_n)$ which is antisymmetric in the collective coordinates Q_i representing five coordinates; three space coordinates, a spin coordinate, and an isotopic spin coordinate. That the nuclear wave function should also be anti-symmetric when the isotopic spin is used as a coordinate follows from the application of the exclusion principle to the neutrons and protons individually as was elegantly shown by Klein (K38). In the single particle model the anti-symmetry of ψ permits representation of the state by a Slater determinant

$$\psi = \begin{vmatrix} \psi_1(Q_1) & \cdots & \psi_1(Q_A) \\ \vdots & & \vdots \\ \psi_A(Q_1) & \cdots & \psi_A(Q_A) \end{vmatrix}$$

(16)

where we shall take $\mathscr{U}_\nu(Q_\nu)$ to be a single particle wave function of the form

$$\mathscr{U}_\nu(Q_\nu) = \varphi_{k_\nu}(1)\, u_\nu(\sigma_\nu, \tau_\nu) \tag{17}$$

where $u_\nu(\sigma_\nu, \tau_\nu)$ is a spin-isotopic spin function and $\varphi_{k_\nu}(1)$ is a space state. Our treatment will be quite general up to the point where matrix elements are actually evaluated, but from this point we use plane waves normalized to a box of volume V. This choice constitutes the use of the "statistical model" which has been discussed in detail by Bethe (Be36). The use here is similar to that by H. Euler (E37) and Watanabe (Wa39) in their respective studies on the saturation properties of nuclear forces and the effect of nuclear forces on a Fermi distribution of nucleons. Our justification of the use of the statistical model is that we expect the results to be more dependent on the symmetry of Ψ than on the functional form.

(3.21) Matrix Elements of General Two, Three, and Four Nucleon Operators

Two Nucleon Operators:

Consider an operator $W_{12}(Q_1, Q_2)$ of the collective coordinates Q_1 and Q_2. The expectation value of W_{12} will be

$$\langle W_{12}(Q_1, Q_2)\rangle = \int_{Q_1}\cdots\int_{Q_A} \Psi^*(Q_1,\cdots,Q_A) W_{12}(Q_1,Q_2)\Psi(Q_1,\cdots Q_A) \tag{18}$$

The anti-symmetry of $\Psi(Q_1,\cdots,Q_A)$ is defined by

$$P_{ij}\,\Psi(Q_1,\cdots,Q_A) = -\,\Psi(Q_1,\cdots,Q_A)$$

where P_{ij} is an exchange operator on the collective coordinates Q_i. We can express P_{ij} when operating on properly anti-symmetrized wave

functions as

$$P_{ij} = - P_{\tau}^{ij} P_{\sigma}^{ij}$$

where P_{τ}^{ij} is an isotopic spin exchange operator and P_{σ}^{ij} is the spin exchange operator. The expectation value of $W(Q_1, Q_2)$ can be written differently by separating the space and "total" spin (isotopic spin and intrinsic spin) coordinates of Q_1 and Q_2. Let "i" denote the total spin state of both nucleons and use x_1, x_2 for their space coordinates. Then

$$\langle W(Q_1, Q_2)\rangle = \sum_i \int \cdots \int_{Q_3}^{Q_A} dx_1 dx_2 \, \Psi^*(x_1, x_2, i\, ; Q_3, \cdots Q_A)\, W_{12}(Q_1, Q_2) \times$$
$$\times \Psi(x_1, x_2, i\, ; Q_3, \cdots Q_A) \tag{19}$$

The summation over "i" is simply the operation of taking a trace over the 16 dimensional total spin state of the two nucleons. We can make a unitary transformation in this space to the representation in which the states are eigenstates of τ_ζ^1 and τ_ζ^2, the ζ- components of the isotopic spin of the two particles. The expectation value of $W(Q_1, Q_2)$ in this space will be

$$\langle W(Q_1, Q_2)\rangle = \sum_{i,f} \int (i\, |g(x_1, x_2)| f)(f |W(Q_1, Q_2)| i)\, dx_1 dx_2 \tag{20}$$

where i and f are now quantum numbers and $(i\, |g\, (x_1, x_2)| f)$ is a matrix defined by

$$(i\, |g(x_1, x_2)| f) = \int_{Q_3} \cdots \int_{Q_A} \Psi^*(x_1, x_2, i\, ; Q_3, \cdots Q_A)\, \Psi(x_1, x_2, f\, ; Q_3, \cdots Q_A) \tag{21}$$

In the state Ψ given by (16) we shall suppose most of the space states φ_k to be occupied by 4 nucleons, two neutrons and

two protons, with each pair of like nucleons having opposite spin. We shall restrict ourselves, as Rosenfeld does, to the case of nuclei for which $A = 4n$ and $T = 0$. One can give formulae for the more general case when every space state is not filled with four particles, but the expressions for the matrix elements of multiple-nucleon operators become extremely complicated.

We now introduce for the φ_{k_ν} plane waves normalized to volume V

$$\varphi_{k_\nu} = \frac{1}{\sqrt{V}} e^{i \vec{k}_\nu \cdot \vec{r}}$$

and make the assumption that possible free particle states are dense in k-space. We can therefore replace sums over the k_ν by integrals up to some maximum k_m determined by

$$k_m = \frac{(9\pi)^{1/3}}{2 r_0} = \frac{1.523}{r_0} \qquad R = r_0 A^{1/3}$$

The general expression for $(i \,|g(x_1, x_2)|\, f)$ is the following

$$
(i \,|g(x_1, x_2)|\, f) = \frac{1}{A(A-1)} \Big\{ \sum_{\mu,\nu}^{k_m} \varphi_{k_\mu}^*(1) \varphi_{k_\nu}^*(2) \big[(i\,|1|\,f) \varphi_{k_\mu}(1) \varphi_{k_\nu}(2)
$$
$$
- (i\,| P_{\sigma\tau}^{12} |\,f) \varphi_{k_\mu}(2) \varphi_{k_\nu}(1)
$$
$$
+ \delta_{ii'} \sum_{\mu} \varphi_{k_\mu}^*(1) \varphi_{k_\mu}^*(2) \big[(i'\,|1|\,f) \varphi_{k_\mu}(1) \varphi_{k_\mu}(2)
$$
$$
- (i\,| P_{\sigma\tau}^{12} |\,f) \varphi_{k_\mu}(1) \varphi_{k_\mu}(2) \big]
$$
$$
+ \text{etc.} \Big\}
$$

(22)

where $i' = $ total spin state for two non-congruent nucleons (different spin or isotopic spin).

$$P_{\sigma\tau}^{12} = P_\sigma^{12} P_\tau^{12}$$ product of space and spin exchange operators. The $\delta_{ii'}$ is just intended to restrict the sum over spin states to those states compatible with the requirement that the single-particle wave function can not be taken twice from Ψ to give terms like

$$\sum_\mu \psi_\mu^*(1) \psi_\mu^*(2) \psi_\mu(1) \psi_\mu(2) .$$

With the normalization which we have chosen for the $\mathscr{G}_{k_\nu}$, the sums over states give the following expressions

$$\sum_{\mu,\nu}^{m} |\varphi_{k_\mu}(1)|^2 |\varphi_{k_\nu}(2)|^2 = \frac{A(A-1)}{16V^2}$$

$$\sum_{\mu,\nu}^{m} \varphi_{k_\mu}^*(1)\varphi_{k_\nu}^*(2)\varphi_{k_\mu}(2)\varphi_{k_\nu}(1) = \frac{A(A-1)}{16V^2} G^2(r_{12})$$

where

$$G(r) = \frac{3}{k_m r} j_1(k_m r) = 3\sqrt{\frac{\pi}{2}} \frac{J_{3/2}(k_m r)}{(k_m r)^{3/2}} \tag{23}$$

The function $G(r)$ will be designated as the equivalent nucleon correlation function, since it gives the probability of a certain relative separation r of two nucleons with the same spin and isotopic spin. This correlation function was first introduced by Wigner and Seitz (W34) in the course of an investigation of the binding energy of metallic sodium. In this case $G(r)$ gives the probability of the relative separation r of two electrons with parallel spin.

The specific form of $G(r)$ given here is the result of our choice of free particle wave functions as the free particle states. More generally, $G(r)$ might be termed the "incomplete delta function" and can be defined for any orthonormal set from the equation

$$G_N(r; r_1, r_2) = \sum_{\mu=1}^{N} \varphi_\mu(x_1)\varphi_\mu(x_2)$$

There exist general relationships of this kind, for example, for the Hermite polynomials (A26)

$$\sum_{n=0}^{n=k} \frac{1}{n!} H_n(x) H_n(y) = \frac{1}{k!} \frac{H_{k+1}(x)H_k(y) - H_k(x)H_{k+1}(y)}{x-y}$$

so that the simple treatment of such correlation effects need not be restricted to the statistical model of the nucleus. In fact, the application of formulae of the type (22) makes it possible to apply correctly the formalism we have developed to several single particle models which are much more realistic, e.g. the harmonic oscillator model.

If we insert the expressions (24) into (23) we obtain finally

$$g(x_1, x_2) = \frac{1}{16V^2} \left[1 - P_{\sigma\tau}^{12} G^2(r_{12}) \right] \cdot \left[1 + \frac{4}{A-1} \delta_{ii'_{12}}^{p} \right] \tag{24}$$

Three Nucleon Operators:

Just as in the case of the two nucleon operator we can write the expectation value of a three nucleon operator as

$$\langle W_{123} \rangle = \sum_{i,f} \int (i | g(x_1, x_2, x_3) | f)(f | W_{123} | i) dx_1 dx_2 dx_3 \tag{25}$$

where of course i and f now denote the total spin state of three nucleons. Similar considerations as led to (24) now give

$$g(x_1, x_2, x_3) = \frac{1}{64V^3} \left[1 - P_{\sigma\tau}^{12} G^2(r_{12}) - P_{\sigma\tau}^{13} G^2(r_{13}) - P_{\sigma\tau}^{23} G^2(r_{23}) \right.$$
$$\left. + P_{\sigma\tau}^{123} G(r_{12}) G(r_{13}) G(r_{23}) + P_{\sigma\tau}^{132} G(r_{12}) G(r_{13}) G(r_{23}) \right] \cdot$$
$$\cdot \left[1 + \frac{4}{A-1} (\delta_{ii'_{12}}^{p} + \delta_{ii'_{13}}^{p} + \delta_{ii'_{23}}^{p}) + \frac{16}{(A-1)(A-2)} (\delta_{ii'_{123}}^{p} + \delta_{ii'_{132}}^{p}) \right] \tag{26}$$

where

$\delta_{ii'_{123}}^{p}$ = total spin state for three non-congruent nucleons

(no two having the same spin or isotopic spin)

The terms which are multiplied by Kronecker δ's came from terms which had two or more particles in the same space state.

Four Nucleon Operator:

The matrix element of a four nucleon operator is just

$$\langle W_{1234}\rangle = \sum_{i,f}\int (i|g(x_1,x_2,x_3,x_4)|f)\,(f|W_{1234}|i)\,dx_1\,dx_2\,dx_3\,dx_4 \tag{27}$$

with the proper density matrix being

$$g(x_1,x_2,x_3,x_4) = \frac{1}{256\,V^4}\Big[1 - P_{\sigma\tau}^{12}\,G^2(r_{12}) - P_{\sigma\tau}^{13}\,G^2(r_{13}) - P_{\sigma\tau}^{14}\,G^2(r_{14})$$

$$- P_{\sigma\tau}^{23}\,G^2(r_{23}) - P_{\sigma\tau}^{24}\,G^2(r_{24}) - P_{\sigma\tau}^{34}\,G^2(r_{34})$$

$$+ (P_{\sigma\tau}^{123} + P_{\sigma\tau}^{132})G(r_{12})G(r_{13})G(r_{23}) + (P_{\sigma\tau}^{124} + P_{\sigma\tau}^{142})G(r_{12})G(r_{14})G(r_{24})$$

$$+ (P_{\sigma\tau}^{134} + P_{\sigma\tau}^{143})G(r_{13})G(r_{14})G(r_{34}) + (P_{\sigma\tau}^{234} + P_{\sigma\tau}^{243})G(r_{23})G(r_{24})G(r_{34})$$

$$+ P_{\sigma\tau}^{(12)(34)}G^2(r_{12})G^2(r_{34}) + P_{\sigma\tau}^{(13)(24)}G^2(r_{13})G^2(r_{24}) + P_{\sigma\tau}^{(14)(23)}G^2(r_{14})G^2(r_{23})$$

$$- P_{\sigma\tau}^{1234}G(r_{12})G(r_{23})G(r_{34})G(r_{14}) - P_{\sigma\tau}^{1432}G(r_{14})G(r_{34})G(r_{23})G(r_{12})$$

$$- P_{\sigma\tau}^{1324}G(r_{13})G(r_{23})G(r_{24})G(r_{14}) - P_{\sigma\tau}^{1243}G(r_{12})G(r_{24})G(r_{34})G(r_{13})$$

$$- P_{\sigma\tau}^{1423}G(r_{14})G(r_{24})G(r_{23})G(r_{13}) - P_{\sigma\tau}^{1342}G(r_{13})G(r_{34})G(r_{24})G(r_{12})\Big]\cdot$$

$$\cdot\Big[1 + \frac{4}{A-1}(\delta_{ii'_{12}} + \delta_{ii'_{13}} + \delta_{ii'_{14}} + \delta_{ii'_{23}} + \delta_{ii'_{24}} + \delta_{ii'_{34}}) +$$

$$+ \frac{16}{(A-1)(A-2)}(\delta_{ii'_{124}} + \delta_{ii'_{134}} + \delta_{ii'_{234}} + \delta_{ii'_{123}}) + \frac{16}{(A-1)(A-2)}(\delta_{ii'_{12}}\delta_{ii'_{34}} + \delta_{ii'_{13}}\delta_{ii'_{24}} + \delta_{ii'_{14}}\delta_{ii'_{23}})$$

$$+ \frac{64}{(A-1)(A-2)(A-3)}\,\delta_{ii'_{1234}}\Big] \tag{28}$$

These expressions seem rather lengthy at first, but there are two ways in which great simplifications can be achieved. It will be noticed that the exchange terms arise from all the permutation operators on the symmetric group of order two, three, or four. In evaluating the matrix element of an operator, however, terms of say (28) which are equivalent under the normal subgroup which leaves the operator invariant contribute equal integrals to (27). Thus only the permutations of the corresponding factor group, multiplied by their multiplicity in the symmetric group, need be considered.

Secondly, a large number of terms will vanish under the trace operation when the matrix elements of the operators are calculated. We shall see this in the development which follows.

(3.22) Evaluation of $\langle C^2 \rangle$ for $T_\xi = 0$ $A = 4n$ Nuclei

From equation (8) the squared Coulomb operator can be written as

$$C^2 = \frac{e^4}{16}\left\{\frac{1}{2}\sum{}'W_{ij}' + \sum{}'W_{ijk} + \frac{1}{4}\sum{}'W_{ijkl}\right\} \tag{29}$$

where we have defined

$$W_{ij} = \left[\frac{16\,T_\xi^2}{A^2} - \frac{8\,T_\xi}{A}(\tau_{3i} + \tau_{3j}) + (\tau_{3i} + \tau_{3j})^2\right]h_{ij}^{-2}$$

$$W_{ijk} = \left[\frac{16\,T_\xi^2}{A^2} - \frac{4\,T_\xi}{A}(2\tau_{3i} + \tau_{3j} + \tau_{3k}) + (\tau_{3i} + \tau_{3j})(\tau_{3k} + \tau_{3k})\right]h_{ij}^{-1}h_{ik}^{-1}$$

$$W_{ijkl} = \left[\frac{16\,T_\xi^2}{A^2} - \frac{4\,T_\xi}{A}(\tau_{3i} + \tau_{3j} + \tau_{3k} + \tau_{3l}) + (\tau_{3i} + \tau_{3j})(\tau_{3k} + \tau_{3l})\right]h_{ij}^{-1}h_{kl}^{-1}$$

From the complete antisymmetry of the wave function it follows that

$$\langle c^2 \rangle = \frac{e^4}{16} \left\{ \frac{A(A-1)}{2} \langle W_{12} \rangle + A(A-1)(A-2) \langle W_{123} \rangle \right.$$
$$\left. + \frac{1}{4} A(A-1)(A-2)(A-3) \langle W_{1234} \rangle \right\}$$

(30)

In the case of $T_\zeta = 0$ $A = 4n$ nuclei the operators whose matrix elements we shall need are

$$W_{12} = \frac{(\tau_{\zeta 1} + \tau_{\zeta 2})^2}{\hbar_{12}^2} \equiv \frac{\omega_{12}}{\hbar_{12}^2}$$

$$W_{123} = \frac{(\tau_{\zeta 1} + \tau_{\zeta 2})(\tau_{\zeta 1} + \tau_{\zeta 3})}{\hbar_{12}\hbar_{13}} \equiv \frac{\omega_{123}}{\hbar_{12}\hbar_{13}}$$

$$W_{1234} = \frac{(\tau_{\zeta 1} + \tau_{\zeta 2})(\tau_{\zeta 3} + \tau_{\zeta 4})}{\hbar_{12}\hbar_{34}} \equiv \frac{\omega_{1234}}{\hbar_{12}\hbar_{34}}$$

(31)

Evaluation of W_{12} :

In calculating $\langle W_{12} \rangle$ we shall omit terms of order $\frac{1}{A}$ or smaller. The result will be

$$\langle W_{12} \rangle = \frac{1}{4 V^2} \sum_{i,f} \int d\tau_1 d\tau_2 \, (f | W_{12} | i) \left[(i | 1 | f) - \frac{1}{2} (i | R_e^{12} | f) \omega_{12} \right]$$

(32)

Using the following sums

$$\sum_{i,f} (f | W_{12} | i)(i | 1 | f) = \sum_i (i | W_{12} | i) = 4 \hbar_{12}^{-2}$$

$$\sum_{i,f} (f | W_{12} | i)(i | R_e^{12} | f) = 4 \hbar_{12}^{-2}$$

we have for $\langle W_{12} \rangle$

$$\langle W_{12} \rangle = 2 \left\{ \overline{z_{12}^{-2}} - \frac{1}{2V^2} \oint d\tau_1 d\tau_2 \, z_{12}^{-2} \, \mathcal{G}^2(z_{12}) \right\}$$

(33)

where $\overline{z_{12}^{-2}}$ is defined as in section (3.1).

Evaluation of $\langle W_{123} \rangle$:

To evaluate $\langle W_{123} \rangle$ we shall use (25) and shall include in $(i \mid g(x_1, x_2, x_3) \mid f)$ all non-exchange terms ($\mathcal{G}(z)$ absent) to order $1/A$ and the exchange integrals from terms of order unity.

$$(i \mid g(x_1, x_2, x_3) \mid f) = \frac{1}{64} \left(i \mid 1 - P_{\sigma\tau}^{12} \mathcal{G}^2(z_{12}) - P_{\sigma\tau}^{13} \mathcal{G}^2(z_{13}) - P_{\sigma\tau}^{23} \mathcal{G}^2(z_{23}) \right.$$

$$+ \left(P_{\sigma\tau}^{123} + P_{\sigma\tau}^{132} \right) \mathcal{G}^2(z_{12}) \mathcal{G}^2(z_{13}) \mathcal{G}^2(z_{23})$$

$$\left. + \frac{1}{16(A-2)} \left(i \mid \delta_{ii'_{12}}' + \delta_{ii'_{13}}' + \delta_{ii'_{23}}' \right) \mid f \right)$$

In evaluating (25) the sums over i and f give expressions of the form

$$\sum_i (i \mid W_{123} \mid i) = 64 z_{12}^{-1} z_{13}^{-1} \qquad\qquad \sum_{i_{12}} (i_{12} \mid W_{123} \mid i_{12}) = 32 z_{12}^{-1} z_{13}^{-1}$$

$$\sum_{i,f} (f \mid W_{123} \mid i)(i \mid P_{\sigma\tau}^{12} \mid f) = 32 z_{12}^{-1} z_{13}^{-1} \qquad\qquad \sum_{i_{23}} (i_{23} \mid W_{12} \mid i_{23}) = 0$$

$$\sum_{i,f} (f \mid W_{123} \mid i)(i \mid P_{\sigma\tau}^{123} \mid f) = 4 z_{12}^{-1} z_{13}^{-1} \qquad\qquad \sum_{i,f} (f \mid W_{123} \mid i)(i \mid P_{\sigma\tau}^{(12)(13)} \mid f) = 16 z_{12}^{-1} z_{13}^{-1}$$

Making use of the symmetries of integrals containing the $\mathcal{G}(z)$ we obtain the terms of lowest order in $\mathcal{R} = k_m R$.

$$\langle W_{123}\rangle = \overline{k_{12}^{-1}k_{13}^{-1}}\left[1+\frac{6}{A-2}\right]-\frac{1}{V^3}\left\{\oint\frac{G^2(h_{12})}{h_{12}h_{13}}+\frac{1}{2}\oint\frac{G^2(h_{23})}{h_{12}h_{13}}\right\}$$

(34)

These are the lowest order terms for the correlation terms containing integrals over $G(h)$, because the appearance of each additional $G(h)$ under the integral introduces another factor of $\frac{1}{k_0}$.

Evaluation of $\langle W_{1234}\rangle$:

In $\langle W_{1234}\rangle$ we must include not only the terms of order $1/A$ arising when the space states of two particles are the same, as in the second line of (23), but also terms of order $1/A^2$ in which the space states of three particles are the same. In this calculation some peculiar features arise from the fact that in some isotopic spin states for four nucleons $\langle W_{1234}\rangle$ is positive, and in others it is negative. In the case of $\langle W_{12}\rangle$ and $\langle W_{123}\rangle$ only positive values were possible. Let us use the symbols $(++++/W_{1234}/++++)$ to denote the matrix element of a state with $\tau_{f_1}=\tau_{f_2}=\tau_{f_3}=\tau_{f_4}=1$ while

$(--++/W_{1234}/--++)$ means that the state is $\tau_{f_1}=\tau_{f_2}=-1$ and $\tau_{f_3}=\tau_{f_4}=1$. It then follows that

$$\sum_i (i/W_{1234}/i)=(++++/W_{1234}/++++)+(----/W_{1234}/----)$$
$$+(++--/W_{1234}/++--)+(--++/W_{1234}/--++)=0$$

where only the terms which are individually non-zero are displayed. The significance of this result is that the contributions from $\langle W_{1234}\rangle$ of order unity will vanish, and the last two terms of (30) will both be of order A^2. Evaluation of the other sums arising in (27) gives

$$\sum_{i,f} (f|W_{1234}|i)(i|P_\sigma^{12}|f) = 0 = \sum_{i,f}(f|W_{1234}|i)(i|P_\sigma^{34}|f)$$

$$\sum_{i,f}(f|W_{1234}|i)(i|P^{13}|f) = 64\,\hbar_{12}^{-1}\hbar_{13}^{-1}$$

$$\sum_{i,f}(f|W_{1234}|i)(i|P^{(13)(24)}|f) = 32\,\hbar_{12}^{-1}\hbar_{34}^{-1}$$

The only part of $\langle W_{1234}\rangle$ arising from particles 1, 2, 3, and 4 in different space states is that containing the nucleon correlation function. We give only the terms which are of lowest order, those in which $G(\hbar)$ appears under the integral no more than twice (as $G^2(\hbar)$ or $G(\hbar)G(\hbar')$)

$$\langle W_{1234}\rangle_1 = -\frac{2}{V^4}\oint\!\!\oint \frac{G^2(\hbar_{13})}{\hbar_{12}\hbar_{34}}$$

$$(35)$$

The terms arising from two particles in the same space state are of order $1/(A-3)$, but the extra factor of $(A-3)$ for $\langle W_{1234}\rangle$ gives terms of the same order as $\langle W_{123}\rangle$.

The contributions will be

$$(i'|g(x_1,x_2,x_3,x_4)|f)_2 = \frac{1}{64(A-3)}\left(i'\,\big|\,\delta_{ii'_{12}} + \delta_{ii'_{13}} + \delta_{ii'_{23}} + \delta_{ii'_{34}}\,\big|\,f\right)$$

where i'_k is the total spin state of two equivalent nucleons (same space state). The following traces over the total spin states are needed

$$\sum_{i'_{12}}(i'_{12}|W_{1234}|i'_{12}) = 0 = \sum_{i'_{34}}(i'_{34}|W_{1234}|i'_{34})$$

$$\sum_{i'_{13}}(i'_{13}|W_{1234}|i'_{13}) = -64\,\hbar_{12}^{-1}\hbar_{34}^{-1} = \sum_{i'_{23}}(i'_{23}|W_{1234}|i'_{23}) = \sum_{i'_{14}}(i'_{14}|W_{1234}|i'_{14})$$

$$= \sum_{i'_{24}}(i'_{24}|W_{1234}|i'_{24})$$

$$\sum_{i'_{123}}(i'_{123}|W_{1234}|i'_{123}) = -32\,\hbar_{12}^{-1}\hbar_{34}^{-1}$$

Using these sums in (28) we obtain for this contribution to $\langle W_{1234} \rangle$

$$\langle W_{1234} \rangle_2 = -\ \frac{4\,k_{12}^{-1}\,k_{34}^{-1}}{(A-3)} \tag{36}$$

The terms arising from three particles in the same space state are multiplied by $\dfrac{1}{(A-2)(A-3)}$ but are actually of the same order of magnitude as (36). Their contribution to $\langle W_{1234} \rangle$ can be shown to be

$$\langle W_{1234} \rangle_3 = -\ \frac{16\,k_{12}^{-1}\,k_{34}^{-1}}{(A-2)(A-3)} \tag{37}$$

although the analysis is somewhat long.

Finally there are terms from pairs of doubly occupied states, and their contribution can also be calculated to be

$$\langle W_{1234} \rangle_{22} = -\ \frac{12\,k_{12}^{-1}\,k_{34}^{-1}}{(A-2)(A-3)} \tag{38}$$

These last two results show that the contribution of the correlation terms of $\langle W_{1234} \rangle$ surely does not reside wholly in (35) but is affected considerably by contributions of doubly and triply occupied states and of pairs of doubly occupied states. The analysis of these contributions is rather involved, but we can say something about the form of the correlation term which comes from this part of $\langle W_{1234} \rangle$ without further calculation.

The largest correlation effects will come from terms with $G^2(k)$ or $G(k)G(k')$. In $\langle W_{1234} \rangle$ only the first type of term appears, as we see from (35). The contribution which $\langle W_{1234} \rangle$ makes to correlation terms of this order are therefore

$$\frac{4\lambda}{V^4} \oint\!\!\!\oint \frac{G^2(r_{13})}{r_{12}r_{34}} \qquad (39)$$

where λ is in general a function of A.

Combining the expressions (36), (37), (38), and (39), we have for $\langle W_{1234} \rangle$

$$\langle W_{1234} \rangle = -4\overline{r_{12}^{-1}r_{34}^{-1}} \frac{(A+5)}{(A-2)(A-3)} + \frac{4\lambda}{V^4} \oint\!\!\!\oint \frac{G^2(r_{13})}{r_{12}r_{34}} \qquad (40)$$

With these results for $\langle W_{12} \rangle$, $\langle W_{123} \rangle$, and $\langle W_{1234} \rangle$, we can derive the matrix element of $\langle C^2 \rangle$. Using (30), (33), (34), and (40), we obtain

$$\langle C^2 \rangle = \frac{e^4}{16} A(A-1) \left\{ \overline{r_{12}^{-2}} + (A+4)\overline{r_{12}^{-1}r_{13}^{-1}} - (A+5)\overline{r_{12}^{-1}r_{34}^{-1}} + f(A) \right\} \qquad (41)$$

where $f(A)$ contains the correlation terms

$$f(A) = -\frac{1}{2V^2} \oint \frac{G^2(r_{12})}{r_{12}^2} - \frac{(A-2)}{V^3} \oint\!\!\!\oint \frac{1}{r_{12}r_{13}} \left[G^2(r_{12}) + \frac{1}{2}G^2(r_{23}) \right]$$

$$+ (A-2)(A-3)\frac{\lambda}{V^4} \oint\!\!\!\oint \frac{G^2(r_{13})}{r_{12}r_{34}} \qquad (42)$$

If we insert the averages $\overline{r_{12}^{-2}}$, $\overline{r_{12}^{-1}r_{13}^{-1}}$, and $\overline{r_{12}^{-1}r_{34}^{-1}}$ given in (12) and (13), the final result for $\langle C^2 \rangle$ is

$$\langle C^2 \rangle = \frac{e^4}{16R^2} A(A-1) \left\{ 0.878 + 0.017\,A + R^2 f(A) \right\} \qquad (43)$$

where R is the nuclear radius. If the correlation effects introduced through $f(A)$ are small, this result is seen to be in remarkably close agreement with the equation (14) obtained by the simple sum rule calculations. In view of the extremely crude nature of the latter, the agreement is surprising.

(3.23) Correlation Effects

We observed in (3.1) that if the r_{ij} are all equal to the same constant, then $\langle c^2 \rangle = 0$. Turning to (41) we can conclude that when this is so $f(A)$ also must vanish. From (42) this implies that

$$f(A) = -\frac{1}{2} - \frac{3}{2}(A-2) + \lambda (A-2)(A-3) = 0$$

We can solve this for λ and obtain

$$\lambda = \frac{1}{2} \frac{3A-5}{(A-2)(A-3)} \tag{44}$$

Inserting this result in $f(A)$ we have

$$f(A) = -\frac{1}{2V^2} \oint \frac{G^2(r_{12})}{r_{12}} - \frac{(A-2)}{V^3} \oint \frac{1}{r_{12} r_{13}} \left[G^2(r_{12}) + \frac{1}{2} G^2(r_{13}) \right]$$
$$+ \frac{3A-5}{2V^4} \oint \frac{G^2(r_{23})}{r_{12} r_{34}} \tag{45}$$

The evaluation of these integrals is quite difficult and no exact expression has been obtained. The asymptotic behavior for $\frac{R}{l} \gg 4$ of these integrals is given in Appendix I, and these results bear out the assertion that the corrections to $\overline{r_{12}^{-2}}$ decrease most slowly, the corrections to $\overline{r_{12}^{-1} r_{34}^{-1}}$ less slowly than this, while the corrections

to $\overline{r_{12}^{-1} \, r_{34}^{-1}}$ soon become negligible for large nuclei. From equation (41) we see that the effect of correlations will be to decrease the expectation value of $\langle C^2 \rangle$ although the effect should not be large, as one sees from the way in which (45) was obtained. That this is reasonable can be seen in the following way.

The quantity $\langle C^2 \rangle$ is a measure of the perturbation of the wave function which is assumed in the absence of the vector part of the Coulomb perturbation. Since the perturbation C for $T_\zeta = 0$ has the form

$$C = -\frac{e^2}{8} \sum_{i \neq j} \frac{\tau_{\zeta i} + \tau_{\zeta j}}{r_{ij}}$$

the effect of C on the wave function is to produce an apparent attraction of neutrons and an apparent repulsion of two protons. Since we actually evaluate $\langle C^2 \rangle$, the effect of C on any pair of neutrons or protons is a mutual repulsion. Now if the original wave function contains any correlation of this kind, the perturbation produced by C will be decreased. The statistical model does provide a certain amount of the required correlation when one brings in the exchange integrals containing the $G(r)$ viz., there exists a repulsion between any pair of neutrons or protons when both particles have the same spin. To the extent that this correlation coincides with the mutual repulsion required by C will the effect of C in bringing in higher isotopic spin states be reduced.

We can conclude from our discussion that setting $f(A) = 0$ makes $\langle C^2 \rangle$ a maximum and we shall use this expression for $\langle C^2 \rangle$ in computing the impurity of the $T_z = 0$ ground state of $T_\zeta = 0$ even-even nuclei as predicted by the statistical model. From (43)

$$\langle C^2 \rangle = \frac{e^4}{16} A(A - 1) \left\{ 0.878 + 0.017 A \right\} \, . \tag{46}$$

One more point should be mentioned concerning the relation of this value of $\langle c^2 \rangle$ to the value obtained from other models, in particular the single particle model with harmonic oscillator wave functions. When we say that the value of $\langle c^2 \rangle$ given by (44) neglects correlation effects, one must not conclude that (44) represents an upper bound on the predictions of more detailed single particle models. For although a nuclear wave function of the type (16) contains certain correlations, the individual particle wave functions are smooth $\left(|\psi_\nu|^2 = 1 \;\; ! \, ! \right)$ and do not provide a "positive inter-nucleon" correlation. A wave function or set of wave functions, which describes a state with pronounced maxima or minima of the nuclear density in certain regions of the nuclear volume could increase the values of $\overline{r_{12}^{-2}}$ and $\overline{r_{12}^{-1} r_{13}^{-1}}$ relative to $\overline{r_{12}^{-1} r_{34}^{-1}}$ and thus increase $\langle c^2 \rangle$.

(3.3) Impurities in the $T = 0$ Ground States of $T\zeta = 0$ ($N = Z$) Even-Even Nuclei.

Using the expression

$$ p \leq \frac{\langle c^2 \rangle}{(E - E_1)^2} $$

with $\langle c^2 \rangle$ given by (46) and $(E - E_1)^2$ from Table I in Chapter 2, we can give an upper limit on p. These are the following

A	8	12	16	20
P_{Max}	2.6×10^{-3}	7.5×10^{-3}	1.9×10^{-2}	3.9×10^{-2}

(3.4) Perturbation Theory

The perturbation upper bound on mixing

$$ p \leq \frac{\langle c^2 \rangle}{(E - E_1)^2} $$

where C is the perturbation, may be much larger than the value of p from the perturbation expression (4) if there is much contribution to $\langle C^2 \rangle$ from high-lying states. In this case, one should use

$$p \approx \frac{\langle C^2 \rangle}{(E - \bar{E})^2} \tag{47}$$

where $(E - \bar{E})^2$ is the reciprocal of $(E - E)^{-2}$ averaged with the matrix elements of C. It is, in a sense, the average energy separation of the ground state from the excited states which are mixed to the ground state by C. It is of interest to see whether (1) is a gross overestimate of the p one should obtain from the statistical model. We shall therefore do a perturbation calculation of p using for the ground and excited states wave functions of the type (16).

We first use the perturbation formalism for C^2 as a check of the expression for the matrix elements ($\Psi, C \Psi_0$) between the nuclear ground state and the excited states Ψ_γ .

The Coulomb perturbation (changing the sign for convenience) is

$$C = \frac{e^2}{4} \sum_{i > j} \frac{\tau_{\zeta i} + \tau_{\zeta j}}{n_{ij}}$$

for a $T_\zeta = 0$ state and we shall compute

$$(\gamma / C / 0) = e^2 \frac{A(A-1)}{2} \int \Psi_\gamma (1 \cdots, A) \frac{\tau_{\zeta 1} + \tau_{\zeta 2}}{4 n_{12}} \Psi_0 (1, \cdots, A) \tag{48}$$

where Ψ_γ is a wave function of the type (16), differing from Ψ_0 by either one or two single particle states. We shall not compute the matrix elements ($\gamma / C / 0$) for states Ψ_γ differing by two particles from Ψ_0 , because these can be shown to be of

order 1/A with respect to the matrix elements for which ψ_γ differs from ψ_0 in one single-particle state (TAS Chapter 6). The matrix elements for excitation of one particle are then

$$\left(\gamma \left| \frac{\tau_{\zeta 1} + \tau_{\zeta 2}}{4 r_{12}} \right| 0 \right) = \frac{1}{A(A-1)} \sum_j \int \psi_\gamma^*(1) \psi_j^*(2) \frac{\tau_{\zeta 1} + \tau_{\zeta 2}}{2 r_{12}} \times$$

$$\times \left[\psi_i(1) \psi_j(2) - \psi_i(2) \psi_j(1) \right] \quad (49)$$

where $\psi_q(1)$ is a single particle state consisting of a free particle wave $\varphi_{k_i'}(1) = \frac{1}{\sqrt{V}} e^{i \vec{k}_i' \cdot \vec{r}_1}$ multiplied by a spin and isotopic spin state. For the first term to be non-vanishing i and γ must be in the same isotopic spin state while in the second term i, j, and γ must all have the same total spin state. Summing $\psi_j(2)$ over spin and isotopic spin states, under the assumption that ψ_0 describes an A = 4n nucleus in its ground state, we have

$$\left(\gamma \left| \frac{\tau_{\zeta 1} + \tau_{\zeta 2}}{4 r_{12}} \right| 0 \right) = \pm \left(\frac{1}{2AV} \oint \varphi_i^*(1) \varphi_\gamma(1) \frac{1}{r_{12}} - \frac{1}{4AV} \oint \varphi_i^*(1) \varphi_\gamma(2) \frac{G(r_{12})}{r_{12}} \right)$$

$$(50)$$

We shall neglect the exchange integral, which is small compared to the direct integral. The first integral to be evaluated over a sphere is the following

$$\oint \varphi_i^*(1) \varphi_\gamma(1) \frac{1}{r_{12}} d\tau_1 d\tau_2 = \frac{1}{V} \int e^{-i(\vec{k}_i - \vec{k}_\gamma) \cdot \vec{r}_1} \frac{1}{r_{12}} d\tau_1 d\tau_2$$

Carrying out the necessary integration over the volume of a sphere we find

$$\oint \varphi_i^*(1) \varphi_\gamma(1) \frac{1}{r_{12}} d\tau_1 d\tau_2 = \frac{4\pi}{|\vec{k}_i - \vec{k}_\gamma|^2} \left\{ G(\rho) - \cos\rho \right\}$$

$$(51)$$

where

$$\rho = |\vec{k}_i - \vec{k}_\nu| R \qquad G(\rho) = 3\sqrt{\frac{\pi}{2}} \; \frac{J_{3/2}(\rho)}{\rho^{3/2}}$$

R being the radius of the sphere. The matrix element (48) becomes

$$(\nu/C/0) = \pm \frac{e^2}{4} A(A-1) \frac{1}{AV} \; \frac{4\pi \{G(\rho) - \cos\rho\}}{|\vec{k}_i - \vec{k}_\nu|^2} \qquad (52)$$

We note here that contrary to the result of evaluations of (48) which are not taken over a sphere the matrix element is finite even as

$$|\vec{k}_i - \vec{k}_\nu| \rightarrow 0 \quad .$$ Thus (52) will give a finite result for $\langle c^2 \rangle$.

The sum over these matrix elements is

$$\langle c^2 \rangle = \sum_\nu |(\nu/C/0)|^2 = \frac{e^4}{16} A(A-1) \frac{(4\pi)^2}{(2\pi)^6} \cdot 4 \int dk_i^3 \int dk_\nu^3 \frac{\{G(\rho) - \cos\rho\}^2}{|\vec{k}_i - \vec{k}_\nu|^4}$$

$$(53)$$

The factor of four in parenthesis arises from the fact that ψ_ν may differ from ψ_0 in any one of four "total" spin states associated with each space state. The integration on $\vec{k}_i$ is over the interior of the Fermi sphere with radius in k-space of

$$k_m = \frac{1}{R} \left(\frac{9\pi A}{8} \right)^{1/3} = 1.523 A^{1/3}$$

The range of $\vec{k}_\nu$ is from k_m to infinity, corresponding to the condition under which the relation $\sum_\nu |(\nu/C/0)|^2 = \langle c^2 \rangle$ is valid. Defining dimensionless variables

$$\vec{\sigma} = \vec{k}_i R \qquad \omega = k_m R = 1.523 A^{1/3}$$

$$\vec{z} = \vec{k}_\nu R$$

we write (53) as

$$\sum_{\gamma} |(\gamma|C|0)|^2 = \left(\frac{e^2}{R}\right)^2 \frac{(A-1)^2}{16} \frac{1}{\pi^4} \int d\vec{z} \int d\vec{z} \frac{\{G(\rho)-\cos\rho\}^2}{\rho^4} \tag{54}$$

We then transform $d\vec{z}$ to bipolar coordinates σ, ρ and φ the angle of rotation of $\vec{z}$ about $\vec{\sigma}$. Integrating over φ and the angle variables of $\vec{\sigma}$,

$$\sum_{\gamma} |(\gamma|C|0)|^2 = \left(\frac{e^2}{R}\right)^2 (A-1)^2 \frac{1}{2\pi^2} \int_{\omega}^{\infty} d\sigma\,\sigma \int_{0}^{\omega} d\tau\,\tau \int_{\sigma-\tau}^{\sigma+\tau} d\rho\,\rho \frac{\{G(\rho)-\cos\rho\}^2}{\rho^4} \tag{55}$$

A simple but crude approximation can be made

$$\left(\frac{G(\rho)-\cos\rho}{\rho^2}\right)^2 \approx \left(\frac{2}{5}\right)^2 e^{-a\rho} \qquad a \approx 1.5 \tag{56}$$

to check, at least approximately, the agreement of (56) with (46). As $\rho \to 0$ both sides of (~~11~~ 56) approach the same limit. The maximum of both sides also occurs at the same value of ρ. Inserting (56) in (55) we obtain

$$\sum_{\gamma} |(\gamma|C|0)|^2 = \left(\frac{e^2}{R}\right)^2 (A-1)^2 \frac{1}{\pi^2} \left(\frac{2}{5}\right)^2 \frac{f(a\omega)}{a^6} \tag{57}$$

where

$$f(\omega) = \{(\omega^3 + 5\omega^2 + 5\omega)\cosh\omega - (\omega^3 + 2\omega^2 + 5\omega + 5)\sinh\omega\}e^{-\omega}$$

For $\omega \gtrsim 3$ we can approximate $\cosh\omega \approx \sinh\omega \approx \frac{1}{2}e^{\omega}$ so that quite a good approximation to $f(\omega)$ is

$$f(\omega) = \frac{3\omega^2 - 5}{a} \tag{58}$$

Although "a" is determined by (11) to be approximately 1.5, from (57) and (58) it follows that the value of $\left\langle c^2 \right\rangle$ is rather sensitive to the exact choice. We shall therefore choose "a" by requiring that (55) give the same result as (46). This leads to the equation

$$\frac{A-1}{A} \frac{a(3a^2\omega^2 - 1)}{25\pi^2 a^6} = \frac{1}{16}\left\{0.878 + 0.017A\right\} \tag{59}$$

Using the approximation $3a^2\omega^2 >> 1$ we get

$$a = \left(\frac{\left(\frac{A-1}{A}\right)\frac{96\omega^2}{25\pi^2}}{0.878 + 0.017A}\right)^{1/4}$$

A	10	20	30	40
a	1.47	1.51	1.57	1.60

$$\tag{60}$$

The fact that a is not constant but $\propto A^{1/6}$ for large A is probably a result of the approximation (56). Considering the rough nature of the approximation (56), the result (60) shows that (54) and (46) do in fact agree. Having satisfied the requirement that the result of (54) converge and agree with (46), we look at the perturbation expression for p. Since the energy denominators are simply the difference in kinetic energy of the particle in the excited and the ground state, they are given simply by $E_o - E = \frac{\hbar^2}{2M}(k^2 - k_i^2)$. Placing this quantity in the denominator of (52) we can then square

and sum to get for

$$\wp = \left(\frac{e^2}{R}\right)^2 \left(\frac{2MR^2}{\hbar^2}\right)^2 \frac{(A-1)^2}{2\pi^2} \int_{-\Omega}^{\infty} d\sigma\,\sigma \int_{0}^{\omega} d\tau\,\tau \int_{\sigma-\tau}^{\sigma+\tau} d\rho\,\rho \; \frac{\left[(\varrho(\rho)-\cos\rho)/\rho^2\right]^2}{(\sigma^2-\tau^2)^2}$$

(61)

where we have again used dimensionless variables. We have made the further assumption that there exists a "gap" in the continuum of free particle states, so that the lowest excited state occurs for propagation vector K with $KR \equiv \Omega$, while the last filled state of the degenerate Fermi gas of nucleons is k_m, with $k_m R = \omega$. The introduction of the "gap" is necessary, since otherwise the expression for $\wp$ contains a logarithmic infinity. In so far as this assumption is an ad hoc addition to the statistical model, it is inconsistent with the model. The physical relevancy of this gap is quite clear, however, and in fact if one returns to the picture of free-particles in an infinite potential well of finite radius, one obtains a discrete set of levels. Regarding the statistical picture, one can argue that it is scarcely ad hoc to introduce a physical feature, ordinarily relatively inconsequential for obtaining the gross features of a model, when that feature becomes important.

The equation (61) then leads to the usual upper limit on $\wp$ when one uses the fact that the integrand possesses a strong maximum at $\sigma^2 - \tau^2 = \Omega^2 - \omega^2$

$$\wp = (\Omega^2-\omega^2)^{-2} \left(\frac{e^2}{R}\right)^2 \left(\frac{2MR^2}{\hbar^2}\right)^2 \frac{(A-1)^2}{2\pi^2} \int_{-\Omega}^{\infty} d\sigma\,\sigma \int_{0}^{\omega} d\tau\,\tau \int_{\sigma-\tau}^{\sigma+\tau} d\rho\,\rho \left\{\frac{\varrho(\rho)-\cos\rho}{\rho^2}\right\}^2$$

(62)

Graph of $\left(\dfrac{G(\rho)-\cos\rho}{\rho^2}\right)^2$

and $\rho\left(\dfrac{G(\rho)-\cos\rho}{\rho^2}\right)^2$
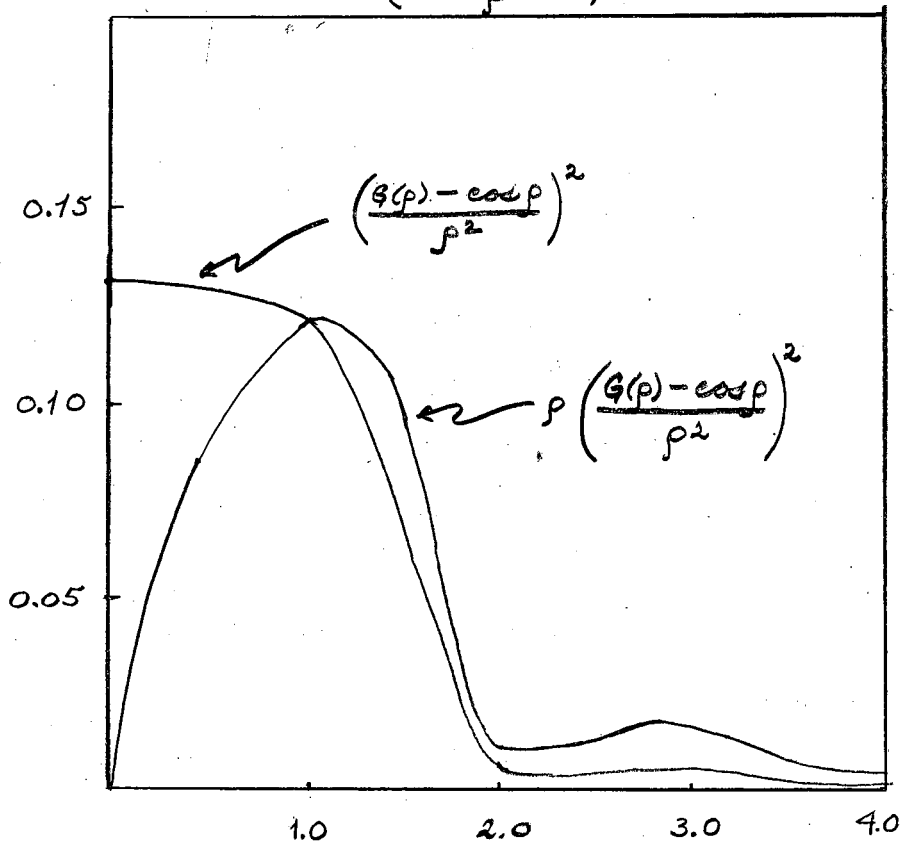
ρ	0	1.0	2.0	3.0	4.0	4.5
$\left(\dfrac{G(\rho)-\cos\rho}{\rho^2}\right)^2$	0.160···	0.132	0.00352	0.00513	0.00125	0.000225

We can draw several conclusions by comparing (61) and (62).

Since $\rho \left[\frac{G(\rho) - \cos\rho}{\rho^2} \right]^2$ rises to a sharp

maximum at $\rho \approx 1$, which corresponds to an energy of $\mathcal{E}_{max} \approx \frac{30}{A^{2/3}}$,

and decreases extremely rapidly beyond, the average energy separation

used in (47) will not exceed $\sim$ 10-15 Mev for $8 \leq A \leq 27$,

if the energy separation $E - E_1$ does not exceed this. For $E - E_1$

greater than E_m, the rapid drop of the integrand in (61) assures

that $E - \bar{E}$ will not be greater than $E - E_1$. For $E - E_1$ less than

E_m, a higher $E - \bar{E}$ is favored. As one sees from the graph of

$\rho \left[\frac{G(\rho) - \cos\rho}{\rho^2} \right]^2$ and from the integrand in (61), however, this

shift of $E - \bar{E}$ upwards from $E - E_1$ is very slight.

We conclude from this discussion of the perturbation expressions

for $\sum_\gamma \frac{|(\gamma|C|0)|^2}{(E_0 - E_\gamma)^2}$ and for $\frac{\langle C^2 \rangle}{(E_0 - E_1)^2}$ as

given by (61) and (62) that for the statistical model the upper limit

on ρ is very close to the value of ρ.

(3.5) Summary of Results from the Statistical Model on Core Mixing

The object of these calculations was to provide an answer to the

question of the extent to which the core of light nuclei may be expected

to be pure isotopic spin $T = 0$ states. The results of these calculations

on even-even $T_\zeta = 0$ nuclei indicate that the impurity of the $T = 0$

ground state is small but not negligible. For O^{16}, for example, the

statistical model gives 1.9 percent impurity in the ground state; this

is the same order of magnitude as Radicate (R53B) has provided for the

impurity of the 7.12 Mev state of O^{16} by neglecting the core impurity.

Our conclusion from the comparison of these results in that the isotopic

spin impurity of the core in light nuclei is of the same order of

magnitude as the isotopic spin impurity of the particles in open shells.
The question of whether the more detailed shell model calculation will
alter this result will be answered in a later chapter.

4. Shell Model Calculations

The calculations based on the uniform model were intended
primarily to provide an orientation with regard to the isotopic spin
impurity of the core of light nuclei. Using the more detailed shell
model, we wish to investigate the impurity of the states of the particles
in open shells. The model we shall use is the jj coupling shell model
whose origins and principal features have also been discussed in detail
by Wigner (W51) and by Redlich (R54). The order of levels and the
"magic numbers" of the jj model can be derived by using the levels
of either a square-well potential or an harmonic oscillator potential.
Consistent use of both of these wave functions will provide estimates
of the isotopic spin mixing which are in reasonable agreement with each
other. This being the case we shall use the harmonic oscillator
functions in the shell model calculations which follow since the
evaluation of matrix elements of two-particle operators is particularly
simple in this case (T52a).

The Scrodinger equation for a particle in an harmonic oscillator
potential is

$$- \frac{\hbar^2}{2M} \left(\Delta + \nu^2 r^2 \right) \psi = E \psi$$

(1)

The solutions of this equation can be expanded in eigenfunctions of
the angular momentum multiplied by a solution of the corresponding
radial equation. These are

$$\Psi(r, \vartheta, \varphi) = \frac{R_{n\ell}(r)}{r} Y_\ell^m(\vartheta, \varphi)$$

$$(2)$$

where Y_ℓ^m is a surface spherical harmonic and

$$R_{n\ell} = N_{n\ell} e^{-\frac{\nu}{2} r^2} r^{\ell+1} L_{n+\ell-\frac{1}{2}}^{\ell+\frac{1}{2}}(\nu r^2) \qquad \nu \equiv \frac{M\omega}{\hbar}$$

$$(3)$$

$L_{n+\ell-\frac{1}{2}}^{\ell+\frac{1}{2}}$ being an associated Laguerre polynomial. The form given here corresponds to the convention that $n = 1, 2, \ldots$. The radial wave functions for $n = 1, 2$ are then

$$R_{1\ell} = N_{1\ell} e^{-\frac{\nu}{2} r^2} r^{\ell+1} \qquad N_{1\ell}^2 = \frac{\sqrt{\nu} \, \nu^{\ell+1} \, 2^{\ell+2}}{\sqrt{\pi} \, 1 \cdot 3 \cdots (2\ell+1)}$$

$$R_{2\ell} = N_{2\ell} e^{-\frac{\nu}{2} r^2} r^{\ell+1} \left(1 - \frac{2\nu}{2\ell+3} r^2\right) \qquad N_{2\ell}^2 = \frac{\sqrt{\nu} \, \nu^{\ell+1} \, 2^{\ell+1} (2\ell+3)}{\sqrt{\pi} \, 1 \cdot 3 \cdots (1+2\ell)}$$

$$(4)$$

The parameter ν should be introduced from experimental data on features of the nuclear density distribution. There is considerable ambiguity in the way one should proceed, however, but one might make the assumption that ν should be related to $\langle r^2 \rangle$ by the formula

$$\langle r^2 \rangle = R^2$$

$$(5)$$

where R is given by $r_0 A^{\frac{1}{3}}$ where A is the atomic mass number, and r^2 is the radial coordinate of a nucleon in the center of mass system. The evaluation of $\langle r^2 \rangle$ would be found by using a nuclear wave function composed of individual particle wave functions of the form (4).

The equation (5) then gives the following values for γ .

A	4	12	16
γ	$\dfrac{11}{6R^2}$	$\dfrac{13}{6R^2}$	$\dfrac{9}{4R^2}$

$$(6)$$

In working in the 1p shell, however, we have found it convenient to determine γ from the equation $\overline{r^2} = R^2$ where $\overline{r^2}$ is the average of r^2 in the 1p state. In section (4.1) we shall also average r^2 in the state of the outside particle. Physically this means that we say that the particles in outer shells actually are moving in a region close to the nuclear boundary. In the 1p shell the value of $\gamma = \dfrac{5}{2R^2}$ and this is sufficiently close to the values listed in (16) to be reasonable as well as convenient. As one proceeds to larger A this method of choosing γ would tend to give larger values of γ than would (5). This happens because the relative weight of the contribution to γ of the outside particle in (5) grows smaller with increasing atomic number.

(4.1) The Coulomb Perturbation of Single Particle Levels

Using the wave functions (4) we can proceed to discuss the perturbation introduced by the Coulomb interaction on the wave function of a single proton outside a closed shell. We do not consider the impurity of the closed shell at this point. Although simple, the perturbation of a single nucleon outside a closed shell is interesting as the clearest and most obvious test of the dynamic validity of the isotopic spin quantum number for the nuclear states composed of jj wave functions.

Now a $T_\zeta = \pm \frac{1}{2}$ nucleus with a single particle outside closed shells is in the state $T = \frac{1}{2}$ and cannot be mixed with anything

but another $T = \frac{1}{2}$ state without excitation of the core to a $T = 1$ state. The perturbation of the outside nucleon therefore affects only the _dynamic_ validity of the isotopic spin for the nucleus. If we succeed in showing that the perturbation of the extra nucleon wave function is very small, we shall have demonstrated that the validity of the isotopic spin quantum number and the dynamic validity coincide in the case of closed shell nuclei with an extra particle (or hole). This last statement is true if states produced by excitation of two particles play only a small role in the core mixing. We shall see later that this is the case.

The perturbation of the wave function of the single proton is taken as that produced by a sphere of radius R and uniform charge density.

$$V(r) = \frac{3Ze^2}{2R^3}\left(R^2 - \frac{r^2}{3}\right) \quad 0 \leq r \leq R$$
$$= \frac{Ze^2}{r} \quad\quad\quad\quad r \geqslant R \quad\quad (7)$$

In describing the interaction of the extra nucleon with the core by an equivalent potential we are neglecting the exchange terms arising from the anti-symmetric part of the complete nuclear wave function (cf. 4.413). Although these terms are not describable by a central potential (TAS 10[6]), their contribution to isotopic spin impurity is negligible. We first show that the "outside" part of $V(r)$ for $r > R$ makes a very small contribution to the matrix elements of $V(r)$ and that one can actually take

$$V(r) = \frac{3Ze^2}{2R^3}\left(R^2 - \frac{r^2}{3}\right) \quad 0 \leq r \leq \infty \quad\quad (8)$$

with considerable accuracy. The single particle matrix elements of $V(r)$ are given in the following table and would all be equal to Ze^2/R if (8) were valid.

$$(1p \mid V \mid 1p) = \frac{8}{3}\sqrt{\frac{\nu}{\pi}} \, Ze^2 \left\{ \frac{15}{16} \cdot \xi_0^{-2} e^{-\xi_0^2} + \frac{9}{8\xi_0^2}\left(\xi_0^2 - \frac{5}{6}\right)\int_0^{\xi_0} d\xi \, e^{-\xi^2} \right\}$$

$$(1d \mid V \mid 1d) = \frac{16}{15}\sqrt{\frac{\nu}{\pi}} \, Ze^2 \left\{ \left(\frac{3}{8} + \frac{35}{32}\xi_0^{-2}\right)e^{-\xi_0^2} + \frac{45}{8\xi_0^3}\left(\xi_0^2 - \frac{7}{6}\right)\int_0^{\xi_0} d\xi \, e^{-\xi^2} \right\}$$

$$(2s \mid V \mid 2s) = \frac{2}{3}\sqrt{\frac{\nu}{\pi}} \, Ze^2 \left\{ \left(\frac{3}{2} + \frac{21}{4}\xi_0^{-2}\right)e^{-\xi_0^2} + \frac{9}{2\xi_0^3}\left(\xi_0^2 - \frac{7}{6}\right)\int_0^{\xi_0} d\xi \, e^{-\xi^2} \right\}$$

$$(3s \mid V \mid 3s) = \frac{1}{30}\sqrt{\frac{\nu}{\pi}} \, Ze^2 \left\{ \left(6\xi_0^4 - 37\xi_0^2 + \frac{129}{2} + 165\xi_0^{-2}\right)e^{-\xi_0^2} \right.$$
$$\left. + \frac{90}{\xi_0^3}\left(\xi_0^2 - \frac{11}{6}\right)\int_0^{\xi_0} d\xi \, e^{-\xi^2} \right\}$$

$$(3p \mid V \mid 3p) = \frac{1}{105}\sqrt{\frac{\nu}{\pi}} \, Ze^2 \left\{ \left(6\xi_0^2 + \frac{177}{2}\xi_0^2 + \frac{513}{2} + \frac{1365}{2}\xi_0^{-2}\right)e^{-\xi_0^2} \right.$$
$$\left. + \frac{315}{\xi_0^3}\left(\xi_0^2 - \frac{13}{6}\right)\int_0^{\xi_0} d\xi \, e^{-\xi^2} \right\}$$

where $\xi_0^2 = \nu R^2$ and $V(r)$ is given by (7).

If we insert reasonable values of ξ_0 into these expressions, we find that the deviation of the matrix elements from $\frac{Ze^2}{R}$ is only a few percent. As one expects, the deviation is slightly larger for states of higher angular momentum and larger principal quantum number. From the fact that even for non-diagonal matrix elements the factor $e^{-\xi_0^2}$ will multiply the contributions from $V \, (r > R)$, we can see that here too, (7), can be replaced by (8) with little error.

We have shown by these results that we may use (8) instead of (7) to evaluate the core perturbation of the single particle harmonic oscillator wave functions. Using this fact we can give an explicit formula for the effect of the core in mixing any harmonic oscillator wave function to higher states. To do this we note that the matrix elements $(n'\ell'| r^2 |n\ell)$ vanish unless $\ell'=\ell$ $n' = n, n+1$. The exact perturbation for the admixture of higher states therefore reduces to one term

$$p = \frac{(n+1\ \ell\ |\ V(r)\ |\ n\ \ell)^2}{(E_o - E_1)^2} \tag{9}$$

In evaluating the equation (11) the constant term of (8) contributes nothing, and we may in fact take $V(r)$ as

$$V(r) = -\frac{ze^2}{2R^3}\, r^2 \tag{10}$$

This is the form of the core perturbation which we shall use in the two and three particle calculations also. (The validity of this approximation of the core perturbation will be discussed in great detail in a later section.) The matrix element of (10) required in (9) follows from the formulas of Shaeffer (S44)

$$(n+1\ \ell\ |\ r^2\ |\ n\ \ell)^2 = \frac{n(n+\ell+\frac{1}{2})}{\nu^2} \tag{11}$$

Using $\overline{r^2} = R^2$ to determine ν approximately we find

$$\nu R^2 = \xi_o^2 = 2n+\ell-\frac{1}{2}$$

Combining this with (11) we find for the perturbation of the single particle outside a closed shell

$$p = \frac{1}{4} \frac{n(n+\ell+\frac{1}{2})}{(2n+\ell-\frac{1}{2})^2} \left(\frac{ze^2}{R}\right)^2 (E_o - E_1)^{-2} \tag{12}$$

where

$$E_o - E_1 = \frac{2\hbar^2 \nu}{M} = 4\xi_o^2 \left(\frac{\hbar^2}{2MR^2}\right) = \frac{41.2}{A^{2/3}} \xi_o^2 \ \mathrm{Mev.}$$

More explicitly we write p as

$$p \approx 1.3 \times 10^{-4} \frac{n(n+\ell+\frac{1}{2})}{(2n+\ell-\frac{1}{2})^4} z^2 A^{2/3} \tag{13}$$

When we apply this formula to the case of the $1p$ proton in N^{13} we find $p = 4.3 \times 10^{-3}$. Due to the fact that the r^2 potential brings a ν^{-4} into p, the ambiguity in determining ν is too serious to allow one to believe (13) for more than order of magnitude results.

These impurities of the single extra nucleon state are much smaller than the isotopic spin impurity of the core as estimated in (3.3). From this result we can conclude that--The dynamic validity of the isotopic spin in closed shell nuclei with one extra particle (or hole) holds to the same extent that isotopic spin is a good quantum number.

(4.11) Coulomb Expansion of Single Particle Functions

For harmonic oscillator wave functions the Coulomb perturbation (10) has an interesting significance: it produces only an expansion of the wave functions. Thus for the Schrodinger equation of a single nucleon which interacts with the core potential given by (10) we have

$$-\frac{\hbar^2}{2M}\left(\Delta + \gamma^2 r^2 - \frac{Ze^2}{2R^3}\frac{2M}{\hbar^2}r^2\right)\psi' = E'\psi'$$

(14)

But if we define

$$\gamma'^2 = \gamma^2 - \frac{Ze^2}{2R^3}\frac{2M}{\hbar^2}$$

(15)

equation (14) is just the oscillator equation (1) whose solutions are given by (2) and (3) with a new parameter γ'. If γ is determined by an equation of the form of (5), the change in γ can be related to an expansion of the nuclear wave function. From (15) one can find the change in γ to be in fact

$$\frac{\delta\gamma}{\gamma} \approx -\frac{1}{2}\frac{931}{137^2}\left(\frac{R}{e^2}\right)\frac{1}{S_0^4}$$

(16)

For the (1d) proton $\left(P^{17}\right)$ the expansion of the nuclear wave function is $\sim 0.13\%$. This is an extremely small effect. A similar result has been found by B. Jancovici (J54) using exact wave functions supplied by G. Breit (B54) for a neutron and proton in a nuclear potential described by a finite square well and Coulomb potential given by (7).

(4.2) Validity of the Isotopic Spin for Two-Particle States

When there is only one particle outside closed jj shells, there can be no mixing to higher isotopic spin states unless there is also excitation of the core. With the appearance of two particles outside closed shells, however, the possibility arises of mixing between $T = 0$ and $T = 1$ states as well as between $T = 0$ states of different configurations. The first type of mixing produces violations

of the isotopic spin selection rules which are derived for eigenstates of the isotopic spin. The second type of mixing destroys the dynamic validity of the isotopic spin since the eigenstates of T_ζ and T are not related by

$$ (T_\zeta + i\, T_\eta)\, \Psi_{T, T_\zeta} = \sqrt{(T \mp T_\zeta)(T \pm T_\zeta + 1)}\; \Psi_{T, T_\zeta \pm 1} $$

(17)

As we have noted in (2.2) this is merely a consequence of the fact that the jj configurations are not eigenstates of a hamiltonian which includes the Coulomb interaction.

Both of these two types of mixing are produced by both the interaction of the particles with the core and with each other. The core excitation, being a central potential proportional to r^2 (Eq. 10), commutes with the j of the individual particles and will produce mixing to excited configurations differing from the ground state only in the principal quantum number of one of the particles. On the other hand the particle interaction does not commute with the j of the individual particles and can mix the ground state to many configurations whose parities are the same.

For a specific investigation of two particle mixing we shall choose to study the $(1p_{3/2})^2$ and $(1p_{1/2})^2$ configurations of the Mayer jj coupling shell model. The results which we obtain will be specifically applicable to the isobaric triads at $A = 6$ and $A = 14$. We do not imply that jj coupling should be valid for so light a nucleus as Li^6, but only that the general results for $(1p_{3/2})^2$ will appear in other j^2 type configurations at larger A. We preferred, for the sake of being specific, to calculate the $(1p_{3/2})^2$ mixing for a definite nucleus, $A = 6$. From the results of these calculations,

there will emerge certain general conclusions which are applicable to any configuration γ^2.

(4.21) Calculation of the Matrix Elements

The total interaction of two particles in an open shell with the core through (12) and with each other through the Coulomb potential is

$$V = - \frac{Ze^2}{2R^3}\left[\left(\frac{1}{2}-t_{\zeta 1}\right)r_1^2 + \left(\frac{1}{2}-t_{\zeta 2}\right)r_2^2\right] + \frac{e^2}{r_{12}}\left(\frac{1}{2}-t_{\zeta 1}\right)\left(\frac{1}{2}-t_{\zeta 2}\right) \tag{18}$$

where t_ζ is the ζ-component of the vector $\vec{t}$ defined as

$$\vec{t} = \frac{1}{2}\vec{\tau} \tag{19}$$

and related to the total isotopic spin operator T and the ζ-component $T \rightarrow T_\zeta$ or (M_T) by

$$T = \sum_{i=1}^{A} t_{\zeta i} \qquad T_\zeta = \sum_{i=1}^{A} t_{\zeta i} = \frac{1}{2}\left(N-Z\right) \tag{20}$$

The choice of a core potential proportional to r^2 is justified by the results of (4.413). This dependence differs from the $1/r$ potential used by Radicati for which there is no justification and which leads to much larger matrix elements. The core potential represents only the direct terms of the interaction of the extra nucleon and the core (cf. 4.42).

In a manner similar to that adopted in (3.1) we shall decompose (18) into the irreducible components in isotopic spin space. Writing

$$V_C = - \frac{Ze^2}{2R^3}\left[\left(\frac{1}{2}-t_{\zeta 1}\right)r_1^2 + \left(\frac{1}{2}-t_{\zeta 2}\right)r_2^2\right] \tag{21}$$

for the core perturbation and

$$V_p = \frac{e^2}{\hbar_{12}} \left(\frac{1}{2} - t_{\zeta_1} \right) \left(\frac{1}{2} - t_{\zeta_2} \right)$$

$$(22)$$

for the particle interaction, we can begin by decomposing the core as

$$V_C = V_C^{(0)} + V_C^{(10)}$$

$$V_C^{(0)} = -\frac{Ze^2}{4R^3} \left(\lambda_1^2 + \lambda_2^2 \right)$$

$$V_C^{(10)} = \frac{Ze^2}{2R^3} \left(t_{\zeta_1} \lambda_1^2 + t_{\zeta_2} \lambda_2^2 \right)$$

$$(23)$$

where $V_C^{(0)}$ is a scalar and $V_C^{(10)}$ is the zero'th component (W31) of a vector in isotopic spin space. Similarly for the particle interaction, we obtain

$$V_p = V_p^{(0)} + V_p^{(10)} + V_p^{(20)}$$

$$V_p^{(0)} = \frac{e^2}{\hbar_{12}} \left(\frac{1}{4} + \frac{1}{3} \vec{t}_1 \cdot \vec{t}_2 \right)$$

$$V_p^{(10)} = - e^2 \frac{t_{\zeta_1} + t_{\zeta_2}}{2\lambda_{12}}$$

$$V_p^{(20)} = e^2 \frac{t_{\zeta_1} t_{\zeta_2} - \frac{1}{3} \vec{t}_1 \cdot \vec{t}_2}{\lambda_{12}}$$

$$(24)$$

with $V_p^{(0)}$ again being an isotopic spin scalar; $V_p^{(10)}$, the zero'th component of vector; and $V_p^{(20)}$, the zero'th component of a second rank tensor in isotopic spin space. For the two particle operators simple expressions in terms of T and T_ζ exist.

$$V_p^{(10)} = - \frac{e^2}{2\mathcal{l}_{12}} T_\zeta$$

$$\text{(25a)}$$

$$V_p^{(20)} = \frac{e^2}{2\mathcal{l}_{12}} \left(T_\zeta^2 - \frac{1}{3} \vec{T}^2 \right)$$

$$\text{(25b)}$$

These expressions clearly exhibit the fact that in the two particle

case no mixing of states of different T is caused by the interaction

of the particles. From (25a) and (25b) it further follows that $V_p^{(10)}$

and $V_p^{(20)}$ vanish in $T_\zeta = 0$ states and have in $T_\zeta = \pm 1$ states

the form

$$T_\zeta = 1 \qquad V_p^{(10)} = - \frac{e^2}{2\mathcal{l}_{12}} \qquad V_p^{(20)} = \frac{e^2}{6\mathcal{l}_{12}}$$

$$T_\zeta = -1 \qquad V_p^{(10)} = \frac{e^2}{2\mathcal{l}_{12}} \qquad V_p^{(20)} = \frac{e^2}{6\mathcal{l}_{12}}$$

$$\text{(26)}$$

In the calculation of the effect of the Coulomb perturbation

we shall use only the vector and tensor parts of V because the

scalar part of the interaction can be included in the nuclear hamiltonian

and T will still be rigorously a good quantum number for the nucleus.

The perturbation we shall use is therefore

$$\mathscr{P} = V_C^{(10)} + V_p^{(10)} + V_p^{(20)}$$

$$\text{(27)}$$

The evaluation of matrix elements of $\mathscr{U}$, a potential containing

isotopic spin dependence, is most easily performed by transforming to

the representation in which the states are characterized by the eigen-

values of the individual t_ζ . In this representation the problem

of finding matrix elements of an isotopic spin dependent two-particle

interaction between states completely anti-symmetric in space, spin, and isotopic spin coordinates has become the problem of evaluating matrix elements of several ordinary potentials between wave functions either symmetric or anti-symmetric in the space and spin coordinates of the two particles but containing no isotopic spin coordinates. The procedure will be clear in a moment.

We first transform matrix elements of the general form $(\gamma T M_T | \mathcal{V} | \gamma' T M_T)$, where γ and γ' represent all the auxiliary quantum numbers, into the representation in which states are characterized by the individual $t_i^2 = t_{\xi i}^2 + t_{\eta i}^2 + t_{\rho i}^2 (= \frac{1}{2})$ and $t_{\xi i}$.

$$(\gamma T M_T | \mathcal{V} | \gamma' T' M_T) = \sum_{t_{\xi 1}, t_{\xi 2}} (T M_T | t_{\rho 1} t_{\rho 2})(t_{\rho 1} t_{\rho 2} | T' M_T) \cdot$$
$$\cdot (\gamma t_{\xi 1} t_{\xi 2} | \mathcal{V} | \gamma' t_{\xi 1} t_{\xi 2})$$

Defining $\mathcal{V}_{TT'}^{(M_T)}$ by the equation

$$(\gamma T M_T | \mathcal{V} | \gamma T' M_T) \equiv (\gamma | V_{TT'}^{(M_T)} | \gamma') \qquad (28)$$

a table of Clebsch-Gordon coefficients for spin $\frac{1}{2}$ particles taken from TAS yields for $\mathcal{V}_{TT'}^{(M_T)}$ the following expressions

I $M_T = 0$

$$V_{00}^{(0)} = 0$$
$$V_{01}^{(0)} = \frac{Ze^2}{4R^3}(\Lambda_1^2 - \Lambda_2^2)$$
$$V_{11}^{(0)} = 0$$

II $M_T = -1$

$$V_{11}^{(1)} = -\frac{Ze^2}{2R^3}(\Lambda_1^2 + \Lambda_2^2) + \frac{2}{3}\frac{e^2}{\Lambda_{12}}$$

III $M_T = 1$

$$V_{11}^{(1)} = \frac{Ze^2}{2R^3}(\Lambda_1^2 + \Lambda_2^2) - \frac{1}{3}\frac{e^2}{\Lambda_{12}} \qquad (29)$$

The anti-symmetric character of $V_{01}^{(0)}$ arises from the difference in symmetry of the spin-space part of wave functions with $T = 0$ and $T = 1$. The vanishing of the $V_{00}^{(0)}$ and $V_{11}^{(0)}$ follows from the general selection rule on the zero'th component of isotopic spin vectors in $M_T = 0$ nuclei, $\Delta T = \pm 1$. This rule is of very great importance for nuclear electric dipole transitions. The sign of $V_{01}^{(0)}$ is determined by the phases of Clebsch-Gordon coefficients and does not have a uniquely determined sign. This apparent arbitrariness disappears, however, when one comes to the matrix element of $V_{01}^{(0)}$ since the sign of the anti-symmetric space-spin function associated with the $T = 1$ state is also determined by the phase of the Clebsch-Gordon coefficients.

The matrix elements of the potentials can now be evaluated in the following way. Since the $V(r)$ of (25) are diagonal in the L, S representation, one can compute matrix elements between the $(1p)^2$ and the low-lying configurations by using formula (38) of Racah (R42) for the particle interaction.

$$(n_1 \ell_1 n_2 \ell_2\, LM \,|\, \tfrac{1}{r_{12}} \,|\, n_3 \ell_3 n_4 \ell_4\, LM) = \sum_k (\ell_1 \ell_2 LM | C_1^{(k)} \cdot C_2^{(k)} | \ell_3 \ell_4 LM) \cdot$$

$$\cdot\ \rho^{(k)}(n_1 \ell_1 n_2 \ell_2 ;\ n_3 \ell_3 n_4 \ell_4)$$

$$= (-1)^{\ell_1 + \ell_4 - L} \sum_k (\ell_1 \| C^{(k)} \| \ell_3)(\ell_2 \| C^{(k)} \| \ell_4) W(\ell_1 \ell_2 \ell_3 \ell_4 ; L k)$$

$$\cdot\ \rho^{(k)}(n_1 \ell_1 n_2 \ell_2 ;\ n_3 \ell_3 n_4 \ell_4)$$

$$(30)$$

$W(\ell_1 \ell_2 \ell_3 \ell_4 ; L k)$ is the Racah coefficient tabulated by Biedenharn (Bi52) while $(\ell_1 \| C^{(k)} \| \ell_2)$ is given in (51) of (R42). The integral $\rho^{(k)}$ is the matrix element of the expansion of $\dfrac{1}{r_{12}}$ in Legendre polynomials i.e.

$$F^{(k)}(n_1 \ell_1 n_2 \ell_2 ; n_3 \ell_3 n_4 \ell_4) = \int_0^\infty d(1) \int_0^\infty d(2) \; R_{n_1 \ell_1}(1) R_{n_2 \ell_2}(2) f_k(1,2) R_{n_3 \ell_3}(1) R_{n_4 \ell_4}(2)$$

$$(31)$$

where $f_k(1,2)$ is defined as

$$f_k(1,2) = \frac{2k+1}{2} \int_{-1}^{1} V(r_{12}) P_k(\mu) d\mu$$

$$\mu \equiv \cos(\vec{r}_1, \vec{r}_2)$$

The evaluation of the $F^{(k)}$ can be accomplished very simply for harmonic oscillator wave functions as Talmi (T52) shows explicitly. For completeness we give the procedure. Using the definition of $f_k(1,2)$, Eq. (31) can be written

$$F^{(k)} = \frac{2k+1}{2} \int_{-1}^{1} d\mu \int_0^\infty dr_1 \int_0^\infty dr_2 \; V(r_{12}) \Phi_k(r_1, r_2, \mu) r_1^2 r_2^2 \, \mu$$

where

$$\Phi_k(r_1, r_2, \mu) = R_{n_1 \ell_1}(1) R_{n_2 \ell_2}(2) R_{n_3 \ell_3}(1) R_{n_4 \ell_4}(2) r_1^{-2} r_2^{-2} P_k(\mu)$$

Introducing the center of mass coordinate and the relative coordinate by

$$\vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

$$\vec{r} = \tfrac{1}{2}(\vec{r}_2 - \vec{r}_1)$$

When we let $\eta = \dfrac{\vec{r} \cdot \vec{R}}{r R}$ and use R, r, and η as new variables, we can express $V(r_{12}) \Phi_k(r_1, r_2, \mu)$ as a rational polynomial in R, r, and μ, for the $F^{(k)}$ which appear in (30). To do this we use the obvious relations

$$r_1 r_2 \mu = R^2 - \frac{r^2}{4} \qquad r_1^2 r_2^2 = \left[R^2 + \frac{r^2}{4} \right]^2 - R^2 r^2 \eta$$

$$r_1^2 + r_2^2 = 2R^2 + \frac{r^2}{2} \qquad r_2^2 = R^2 + \frac{r^2}{4} + r R \eta$$

$$r_1^2 = R^2 + \frac{r^2}{4} - r R \eta$$

$$(32)$$

A table of $F^{(k)}$ for various diagonal and non-diagonal matrix elements of $\frac{e^2}{\lambda_{12}}$ is given in Appendix II.

Having evaluated the LS matrix elements of the expressions in (29), we transform to jj coupling matrix elements by the use of the $(LSJM|j_1j_2JM)$ transformation coefficients given by Racah (R50), or less explicitly by a formula due to Hope and quoted by Edmonds and Flowers (E52a). The final expressions for the non-zero matrix elements between $(1p_{3/2})^2$ and a number of jj configurations is given in Table I. The corresponding matrix elements for $(1p_{1/2})^2$ are given in Table IV.

Using the tabulated $F^{(k)}(n_1\ell_1n_2\ell_2;n_3\ell_3n_4\ell_4)$ we can evaluate these jj matrix elements for the nuclei Li^6, Be^6, N^{14}, and O^{14}. Since the shell model predicts level positions also, although these will be modified by various forces which we believe to exist in the nucleus (T52) and (R54), we can also evaluate the percent admixture of higher states to the ground state of $(1p_{3/2})^2$ or $(1p_{1/2})^2$. These are given in Tables II, III and Tables V, VI respectively.

(4.22) Mixing of Different Isotopic Spin States

The first observation is that in the $(1p_{3/2})^2$ configuration no mixing to the $(1p_{1/2})^2$ configuration can occur with change of isotopic spin T. This means that all mixing of different T states must occur to configurations which are separated by a rather large energy $\sim$ 20-30 Mev. The particle interaction vanishes for $M_T = 0$ components and can therefore mix only $T = 1$ states. The core inter-action on the other hand can only mix states of different isotopic spin. In doing this there is an additional selection rule, however, which arises from the central force character of the core potential. The rule is that $\Delta\ell_i = 0$ where ℓ_i are the angular momenta of the

individual particles. The result of these two selection rules on $\mathcal{U}$ is that the only possible mixing of T = 0 and T = 1 states is that between

$$(1p^{3/2})^2 \text{ and } 1p^{3/2}2p^{3/2} \text{ or between } (1p^{1/2})^2 \text{ and } 1p^{1/2}2p^{1/2}.$$

(4.23) Mixing of the Isotopic Spin Triplet States

Although no mixing of different isotopic spin states of the two outside particles can occur when $M_T = \pm 1$ i.e. when the particles are both neutrons or both protons, the matrix elements between the T = 1 states of different configurations have several interesting features. We note first of all that mixing can occur to configurations in which the individual angular momentum j of one or both particles is different. The core interaction does not play any role in these matrix elements, so we should expect them to be somewhat smaller than those to the $1p^{3/2}2p^{3/2}$ state. What we actually find is that the latter matrix elements are larger by a factor of $\sim 10 - 15$ than the matrix elements to states in which one or both j's are different and that this difference is not due to the core alone. This result holds in fact for any spin-independent "long range force" for which the $F^{(0)}$ defined in (31) are appreciably larger than the $F^{(k)}$.

That the $\mathcal{l}$ -value of the individual particle should be preserved nearly follows when we show that the coefficient of $F^{(0)}$ in the Slater expansion of the matrix element vanishes unless the excitation of the individual particle is to a state of the same $\mathcal{l}$. Any central potential commutes with $\vec{\mathcal{l}_i}$ and therefore satisfies the selection rule on its matrix elements, $\Delta \mathcal{l}_i = 0$. Now the coefficient of $F^{(0)}$ for any two particle force is the matrix element of $f_0(1,2) P_0(\cos \vartheta) = f_0(1,2)$ which is an average of the two particle interaction over all angles. The $f_0(1,2) P_0(\cos \vartheta)$ must commute

therefore with the individual ℓ ; and therefore also satisfies the selection rule $\Delta\ell_i = 0$. Since the particle interaction is assumed to be spin independent we immediately have the additional result that the coefficient of $F^{(0)}$ vanishes between jj states differing in the individual j_i.

This result follows trivially from the expression for

$(\ell_1 \ell_2 | e^2/r_{12} | \ell_3 \ell_4)$ given in (30) when one uses the relation

$$W(\ell_1 \ell_2 \ell_3 \ell_4 ; L0) = W(\ell_1 \ell_3 \ell_2 \ell_4)$$

$$= (-1)^{\ell_3 + \ell_4 - L} \frac{\delta_{\ell_1 \ell_3} \delta_{\ell_2 \ell_4}}{\sqrt{(2\ell_3+1)(2\ell_2+1)}}$$

We have shown therefore that $F^{(0)}_0$ appears only in the expansion of the matrix element of the particle interaction between states which do not differ in the individual ℓ or j quantum numbers. From $F^{(0)} \gg F^{(k)}$ follows then the approximate diagonality of long range scalar particle interactions in the $(\ell_i \, j_i)$ representation.

Besides the fact that matrix elements from $(1p^{3/2})^2$ to the $(p^{3/2} \ell p^{3/2})$ states are much larger than any others, we also note that the largest matrix elements to jj' states arising from some $(\ell \ell')$ configuration are those to states where both particles have parallel spin and orbit. Almost equally large are matrix elements from $(1p^{3/2})^2$ to states in which both particles have anti-parallel spin and orbit. Matrix elements from $(1p^{3/2})^2$ to states in which one particle has parallel spin and orbit and the other anti-parallel spin and orbit are generally smaller than the other two matrix elements.

We have calculated the $(1p^{3/2})^2$ matrix elements for A = 6 where the core consists of only two protons and two neutrons. As we have said, we do not wish to imply that jj coupling should be valid

for such a light nucleus, but merely wish to study the case of two

particles outside closed shells in order to see the general results

which one may expect. Even in such a light nucleus, however, the core

interaction, which is proportional to the charge of the core, has matrix

elements (where they exist) of nearly twice the magnitude of the

particle interaction $\frac{e^2}{\hbar_{12}}$ matrix elements. The core interaction is

even more important in heavier nuclei as one sees clearly in the mixing

of the $\left(1p_{1/2}\right)^2$ state which is calculated for $A = 14$. There the core

interaction is approximately $6 \sim 10$ times as effective as the particle

interaction in producing mixing. While some of this large difference

between the relative effectiveness of the core interaction in Be^6 and

N^{14} is due to the relatively smaller matrix elements of $\left(1p_{1/2}\right)^2$ to

$1p_{1/2}2p_{1/2}$, we expect the core interaction to increase in

importance proportional to A.

In this discussion we have made two points, (1) that the Coulomb

interaction between two particles in an open shell has an effect like a

central potential in that it mixes states of the same j and ℓ and

(2) that the core potential predominates over the extra-nucleon inter-

action in producing mixing. From these two conclusions we can make

the statement--Unless the energy separation of the ground state of two

outside nucleons in a $(nj)^2$ configuration from the same JT state

of $\left(nj, n+1\, j\right)$ is much larger (5-10 times) than that from

another $\left(nj'\, nj''\right)$ of the same parity, the mixing may be computed

as though due to an "equivalent" central potential. We shall show in

detail later that a not unreasonable equivalent central potential is

that due to a uniform sphere of charge.

This discussion of the two-particle matrix elements may seem

somewhat irrelevant since these are matrix elements between states of

the same isotopic spin. As we shall see in the discussion of the three particle mixing, however, these effects will play a part in determining the matrix elements between states of different isotopic spin for more than two particles outside closed shells.

It seems clear that any detailed study of the relative contributions to mixing made by configurations other than the $1p_{1/2} 2p_{1/2}$ or $1p_{3/2} 2p_{3/2}$ is unjustified in view of the considerable splitting and displacement of levels which is necessary even to give the jj shell model. In order to obtain some idea of the extent to which higher configurations appear in the ground state, we have given the matrix elements to the jj states arising from $(1f)^2$. Although the $(1f)^2$ states lie twice as far from the ground state as the states of $(1d)^2$, the matrix elements to the jj states of these configurations are of the same order of magnitude. In fact the largest difference between matrix elements of the same type (to parallel spin and orbit, etc.) is only $2 \sim 3$, while several matrix elements are nearly equal. In this case then the energy denominators are decisive in determining the relative amounts of mixing.

TABLE I

The jj Matrix Elements between $(1p_{3/2})^2$ and Other Low-lying Configurations

$$\underline{M_T = 0}$$

Only the matrix elements to $1p_{3/2}\,2p_{3/2}$ are non-zero

$$\left((1p_{3/2})^2\,JT\,|\mathcal{V}|\,1p_{3/2}\,2p_{3/2}\,JT'\right) = \frac{ze^2}{4R^3}\frac{1}{\nu}\sqrt{\frac{5}{2}}$$

The non-zero matrix elements are for these JTT' values

J	T	T'
0	1	0
1	0	1
2	1	0
3	0	1

$$\underline{M_T = 1}$$

Matrix element $\left((1p_{3/2})^2\,JT\,|\mathcal{V}|\,jj'\,JT'\right)$ $T = T' = 1$.

For $M_T = 1$ multiply every $F^{(k)}$ by $(-1/3)$; for $M_T = -1$ multiply every F^k by $2/3$.

jj'	T	J	$\left((1p_{3/2})^2\,JT\,|\mathcal{V}|\,jj'\,JT\right)$
$(1p_{1/2})^2$	1	0	$\frac{\sqrt{2}}{15}\,F^{(2)}((1p)^2;(1p)^2)$
$1p_{3/2}1p_{1/2}$	1	2	$-\frac{2\sqrt{2}}{25}\,F^{(2)}((1p)^2;(1p)^2)$
$1p_{1/2}1f_{5/2}$	1	2	$\frac{1}{25}\sqrt{\frac{2}{3}}\,F^{(2)}((1p)^2;1p1f)$
$1p_{3/2}1f_{5/2}$	1	2	$-\frac{2}{75}\sqrt{\frac{3}{7}}\,F^{(2)}((1p)^2;1p1f)$
$1p_{3/2}1f_{7/2}$	1	2	$\frac{2}{25}\sqrt{\frac{2}{7}}\,F^{(2)}((1p)^2;1p1f)$
$1p_{1/2}1f_{7/2}$	1	2	0
$(1f_{5/2})^2$	1	0	$\frac{6}{35}\sqrt{\frac{2}{3}}\,F^{(2)}((1p)^2;(1f)^2)+\frac{4}{7}\sqrt{\frac{2}{3}}\,F^{(4)}((1p)^2;(1f)^2)$
	1	2	$-\frac{4}{175}\sqrt{\frac{3}{7}}\,F^{(2)}((1p)^2;(1f)^2)+\frac{4}{21}\sqrt{\frac{3}{7}}\,F^{(4)}((1p)^2;(1f)^2)$
$1f_{7/2}1f_{5/2}$	1	2	$\frac{144}{175}\sqrt{\frac{2}{7}}\,F^{(2)}((1p)^2;(1f)^2)-\frac{20}{63}\sqrt{\frac{2}{7}}\,F^{(4)}((1p)^2;(1f)^2)$

TABLE I (Cont.)

$(1f_{7/2})^2$ 1   0 $\frac{18}{35}\sqrt{2}\,F^{(2)}\big((1p)^2;(1f)^2\big) + \frac{10}{63}\sqrt{2}\,F^{(4)}\big((1p)^2;(1f)^2\big)$

        1   2 $\frac{18}{35}\sqrt{\frac{6}{7}}\,F^{(2)}\big((1p)^2;(1f)^2\big) - \frac{26}{189}\sqrt{\frac{6}{7}}\,F^{(4)}\big((1p)^2;(1f)^2\big)$

$(1d_{3/2})^2$ 1   0 $-\frac{2}{15}\,F^{(1)}\big((1p)^2;(1d)^2\big) - \frac{18}{35}\,F^{(3)}\big((1p)^2;(1d)^2\big)$

        1   2 $-\frac{8}{75}\,F^{(1)}\big((1p)^2;(1d)^2\big) - \frac{12}{175}\,F^{(3)}\big((1p)^2;(1d)^2\big)$

$1d_{3/2}\,1d_{5/2}$ 1   2 $-\frac{4}{25}\sqrt{\frac{14}{3}}\,F^{(1)}\big((1p)^2;(1d)^2\big) + \frac{144}{1225}\sqrt{\frac{14}{3}}\,F^{(3)}\big((1p)^2;(1d)^2\big)$

$(1d_{5/2})^2$ 1   0 $-\frac{2}{5}\sqrt{\frac{2}{3}}\,F^{(1)}\big((1p)^2;(1d)^2\big) - \frac{24}{35}\sqrt{\frac{2}{3}}\,F^{(3)}\big((1p)^2;(1d)^2\big)$

        1   2 $-\frac{28}{25}\sqrt{\frac{3}{7}}\,F^{(1)}\big((1p)^2;(1d)^2\big) + \frac{44}{175}\sqrt{\frac{3}{7}}\,F^{(3)}\big((1p)^2;(1d)^2\big)$

$1p_{1/2}\,2p_{1/2}$ 1   0 $\frac{2\sqrt{2}}{5}\,F^{(2)}\big((1p)^2;1p2p\big)$

$1p_{3/2}\,2p_{1/2}$ 1   2 $-\frac{4}{25}\,F^{(2)}\big((1p)^2;1p2p\big)$

$1p_{1/2}\,2p_{3/2}$ 1   2 $\frac{4}{25}\,F^{(2)}\big((1p)^2;1p2p\big)$

$1p_{3/2}\,2p_{3/2}$ 1   0 $\varepsilon\,\frac{Ze^2}{2R^3}\frac{1}{\nu}\sqrt{\frac{5}{2}} + 2F^{(0)}\big((1p)^2;1p2p\big) + \frac{2}{5}F^{(2)}\big((1p)^2;1p2p\big)$

        1   2 $\varepsilon\,\frac{Ze^2}{2R^3}\frac{1}{\nu}\sqrt{\frac{5}{2}} + 2F^{(0)}\big((1p)^2;1p2p\big) - \frac{6}{25}F^{(2)}\big((1p)^2;1p2p\big)$

$$\varepsilon \equiv -M_T$$

TABLE II

Interaction Matrix Elements of $(1p_{3/2})^2$ to Low-lying Configurations and the Mixing for Each.

$$\underline{M_T = 0 \ (\text{Li}^6)}$$

$$m = ((1p_{3/2})^2 \, JT \, |U| \, 1p_{3/2} 2p_{3/2} \, JT') = 0.107 \text{ Mev}$$

$$p = \frac{m^2}{(E_0 - E_1)^2} = 3.31 \times 10^{-5}$$

$$J = 0,2 \quad T, T' = 1, 0$$
$$J = 1,3 \quad T, T' = 0, 1$$

$$\underline{M_T = 1 \ (\text{He}^6)}$$

jj'	J	$((1p_{3/2})^2 J \|U\| jj' J 1) \equiv m$	$p = \dfrac{m^2}{(E_0-E_1)^2}$
$(1p_{1/2})^2$	0	−0.0177 Mev	
$1p_{3/2} 1p_{1/2}$	2	0.00696	
$1p_{1/2} 1f_{5/2}$	2	−0.0145	2.20×10^{-7}
$1p_{3/2} 1f_{5/2}$	2	0.00774	6.28×10^{-8}
$1p_{3/2} 1f_{7/2}$	2	−0.0190	3.78×10^{-7}
$1p_{1/2} 1f_{7/2}$	2	0	0
$(1f_{5/2})^2$	0	−0.0346	3.14×10^{-7}
$(1f_{5/2})^2$	2	−0.00589	9.08×10^{-9}
$1f_{7/2} 1f_{5/2}$	2	−0.00954	2.39×10^{-8}
$(1f_{7/2})^2$	0	−0.0352	3.24×10^{-7}
$(1f_{7/2})^2$	2	−0.0235	1.45×10^{-7}
$(1d_{3/2})^2$	0	0.0499	2.61×10^{-6}
$(1d_{3/2})^2$	2	0.0163	2.78×10^{-7}
$1d_{3/2} 1d_{5/2}$	2	0.0200	4.19×10^{-7}
$(1d_{5/2})^2$	0	0.0740	5.93×10^{-6}
$(1d_{5/2})^2$	2	0.0681	4.86×10^{-6}
$1p_{1/2} 2p_{1/2}$	0	−0.0139	2.02×10^{-7}
$1p_{3/2} 2p_{1/2}$	2	0.00394	1.62×10^{-8}
$1p_{1/2} 2p_{3/2}$	2	−0.00394	1.62×10^{-8}
$1p_{3/2} 2p_{3/2}$	0	−0.453	2.15×10^{-4}
$1p_{3/2} 2p_{3/2}$	2	−0.438	2.01×10^{-4}

TABLE III

Interaction of $(1p_{3/2})^2$ with Low–lying Configurations in

$\underline{M_T = -1}$ Nucleus (Be[6])

jj'	J	$((1p_{3/2})^2 J1|V|jj'J1) = m_b$	$p = m_b^2/(E_0-E_1)^2$
$(1p_{1/2})^2$	0	0.0354	1.32×10^{-6}
$1p_{3/2}1p_{1/2}$	2	−0.0139	2.03×10^{-7}
$1p_{1/2}1f_{5/2}$	2	0.0290	8.80×10^{-7}
$1p_{3/2}1f_{5/2}$	2	−0.0155	2.51×10^{-7}
$1p_{3/2}1f_{7/2}$	2	0.0380	1.51×10^{-6}
$1p_{1/2}1f_{7/2}$	2	0	0
$(1f_{5/2})^2$	0	0.0692	1.26×10^{-6}
$(1f_{5/2})^2$	2	0.0118	3.63×10^{-8}
$1f_{5/2}1f_{7/2}$	2	0.0191	9.53×10^{-8}
$(1f_{7/2})^2$	0	0.0704	1.30×10^{-6}
$(1f_{7/2})^2$	2	0.0470	5.78×10^{-7}
$(1d_{3/2})^2$	0	−0.0998	1.04×10^{-5}
$(1d_{3/2})^2$	2	−0.0326	1.11×10^{-6}
$1d_{3/2}1d_{5/2}$	2	−0.0400	1.68×10^{-6}
$(1d_{5/2})^2$	0	−0.1480	2.37×10^{-5}
$(1d_{5/2})^2$	2	−0.1362	1.94×10^{-5}
$1p_{1/2}2p_{1/2}$	0	0.0278	8.08×10^{-7}
$1p_{3/2}2p_{1/2}$	2	−0.00788	6.48×10^{-8}
$1p_{1/2}2p_{3/2}$	2	0.00788	6.48×10^{-8}
$1p_{3/2}2p_{3/2}$	0	0.552	1.79×10^{-4}
$1p_{3/2}2p_{3/2}$	2	0.520	1.68×10^{-4}

TABLE IV

The jj Matrix Elements between $\left(1p_{1/2}\right)^2$ and Other Low-lying Configurations

$\underline{M_T = 0 \text{ Nucleus}}$

Only the matrix elements to $1p_{1/2}2p_{1/2}$ are non-zero

$$\left(\left(1p_{1/2}\right)^2 JT \mid \mathcal{U} \mid 1p_{1/2}2p_{1/2} JT'\right) = \frac{Ze^2}{4R^3} \frac{1}{\gamma}\sqrt{\frac{5}{2}}$$

Non-zero elements are for $J = 0$, $T = 1$, $T' = 0$ and $J = 1$, $T = 0$, $T' = 1$.

$\underline{M_T = \pm 1}$

For $M_T = +1$ multiply every $F^{(k)}$ by $(-1/3)$; for $M_T = -1$, by $(+2/3)$.

jj'	T	J	$\left(\left(1p_{1/2}\right)^2 JT \mid V \mid jj' JT'\right)$ $(T=T'=1)$
$(1d_{3/2})^2$	1	0	$-\frac{2\sqrt{2}}{3}F^{(1)}\left((1p)^2;(1d)^2\right)$
$(1d_{5/2})^2$	1	0	$-\frac{2\sqrt{3}}{7}F^{(3)}\left((1p)^2;(1d)^2\right)$
$(1f_{5/2})^2$	1	0	$\frac{2\sqrt{3}}{5}F^{(2)}\left((1p)^2;(1d)^2\right)$
$(1f_{7/2})^2$	1	0	$\frac{4}{9}F^{(4)}\left((1p)^2;(1d)^2\right)$
$1p_{1/2}2p_{1/2}$	1	0	$\varepsilon\frac{Ze^2}{2R^3}\frac{1}{\gamma}\sqrt{\frac{5}{2}} + 2F^{(0)}\left((1p)^2;1p2p\right)$
$1p_{3/2}2p_{3/2}$	1	0	$\frac{2\sqrt{2}}{15}F^{(2)}\left((1p)^2;1p2p\right)$

$$\varepsilon \equiv -M_T$$

TABLE V

Interaction Matrix Elements of $(1p_{1/2})^2$ to Low-lying Configurations and the Mixing for Each.

$$M_T = 0 \quad \text{Nucleus } (N^{14})$$

Only the matrix elements to $1p_{1/2}2p_{1/2}$ are non-zero.

jj'	T	J	T	$((1p_{1/2})^2 JT \mid \mathcal{V} \mid jj' JT) \equiv m_b$	$p = \dfrac{m_b^2}{(E_o - E_1)^2}$
$1p_{1/2}2p_{1/2}$	0	1	0		
			1		
	1	0	0	0.403 Mev	5.18×10^{-4}
			1		

$$M_T = +1 \quad \text{Nucleus } (C^{14})$$

jj'	$((1p_{1/2})^2 01 \mid \mathcal{V} \mid jj' 01) \equiv m_b$	$p = \dfrac{m_b^2}{(E_o - E_1)^2}$
$(1d_{3/2})^2$	0.0772 Mev	1.90×10^{-5}
$(1d_{5/2})^2$	0.0258	2.13×10^{-6}
$(1f_{5/2})^2$	-0.0336	9.49×10^{-7}
$(1f_{7/2})^2$	-0.0184	5.20×10^{-7}
$1p_{1/2}2p_{1/2}$	-0.873	2.44×10^{-3}
$1p_{3/2}2p_{3/2}$	-0.00351	3.94×10^{-8}

The energy separation of $(1p)^2$ and $(1d)^2$ was taken as $E_o - E_1 = 17.7$ Mev. The separation of $(1p)^2$ and $(1f)^2$ was taken as 35.4 Mev.

TABLE VI

Interaction Matrix Elements of $(1p_{1/2})^2$ to Low-lying Configurations for

$$M_T = -1 \qquad O^{14}$$

jj'	$((1p_{1/2})^2 01 \| \mathcal{V} \| jj' 01) \equiv m_0$	$p = \dfrac{m_0^2}{(E_0 - E_1)^2}$
$(1d_{3/2})^2$	-0.154 Mev.	7.58×10^{-5}
$(1d_{5/2})^2$	-0.0517	8.53×10^{-6}
$(1f_{5/2})^2$	0.0672	3.80×10^{-6}
$1f_{5/2}1f_{7/2}$	0.0368	2.08×10^{-6}
$1p_{1/2}2p_{1/2}$	0.940	9.76×10^{-3}
$1p_{3/2}2p_{3/2}$	0.00702	1.58×10^{-7}

$$E_0 - E_1 = 2\hbar^2 \nu = 17.7 \text{ Mev.}$$

Energy separation of $(1f)^2$ from $(1p)^2$ is 35.4 Mev.

(4.3) Isotopic Spin Impurities in a Three Particle Configuration

The results of the calculation of the mixing of the states of

two particles outside a closed shell disclosed certain interesting

general features concerning the relative magnitude of the matrix elements

to excited states. The two particle configuration, however, was seen

to have the special characteristic that no mixing of different isotopic

spin states could be produced by the particle interaction. Consequently

the efficiency of the particle interaction in mixing states of different

isotopic spin is first to be observed in the case of three particles

outside a closed shell. In addition there was in the two particle case

no mixing of states of different isotopic spin belonging to configurations

which would be degenerate on a strict harmonic oscillator model with no

particle interaction other than the Coulomb potential, e.g. $\left(1p^{3/2}\right)^{2}$ and

$1p^{3/2} \; 1p^{1/2}$.

In order to investigate these features of the Coulomb mixing we

shall consider the isotopic spin impurity of the $\left(1p^{3/2}\right)^{3}$ state,

such a configuration being the one to which the low-lying states of Li^{7}

should belong.

(4.31) Fractional Parentage Coefficients and Tensor Operators

For the calculation of matrix elements for states belonging

to configurations of many particles we shall use the fractional parentage

coefficients (c.f. p.) defined by Racah (R43) and their extension to

the jj states of non-equivalent particles as given by Redlich (R54).

Redlich's derivation is not generally accessible and we shall give it

here in an abbreviated form for reference. We shall then develop

expressions for the matrix elements of general tensor operators in

isotopic spin space in preparation for calculating matrix elements of

the Coulomb interaction.

Consider a properly anti-symmetrized state for n particles in the configuration $j_1, \ldots, j_n$ with angular momentum J, Z-component of angular momentum M_J, isotopic spin T, and ζ-component of isotopic spin M_T. The total J can be obtained from the successive addition of the j_i

$$\vec{J} = \left[\left((\vec{j}_1 + \vec{j}_2) + \vec{j}_3 \right) + \vec{j}_4 \right] + \cdots$$

so that the set "a" of the angular momenta $|\vec{j}_1 + \vec{j}_2|, |\vec{j}_1 + \vec{j}_2 + \vec{j}_3|$ may be added to a similar set "b" for the $\vec{t}_i$ to complete the specification of the state ψ. Such a state $\psi(n, \alpha\, J\, M_J\, T\, M_T)$ where $\underline{n}$ denotes the set $\{j_i\}$ and α the collective set of a and b, can be found by first adding a particle with angular momentum $\vec{j}_\lambda$ to the correctly symmetrized state $\psi(\underline{n} - \lambda, \alpha'\, J'\, M_J'\, T'\, M_T')$ and then anti-symmetrizing. Thus,

$$\psi(\underline{n}, \alpha\, J\, M_J\, T\, M_T) = \sideset{}{'}\sum_\lambda \sum_{\alpha' J' T'} \varphi \left(\left[\underline{n} - \lambda, (\alpha'\, J'\, M_J'\, T'\, M_T') j_\lambda ; \alpha\, J\, M_J\, T\, M_T \right) \right.$$
$$\left. \cdot \left(\underline{n} - \lambda\, (\alpha'\, J'\, T') j_\lambda \,|\} \alpha\, J\, T \right) \right.$$

$$(33)$$

where $\varphi \left(\left[\underline{n} - \lambda (\alpha'\, J'\, M_J'\, T'\, M_T') j_\lambda ; \alpha\, J\, M_J\, T\, M_T \right) \right.$ is the unsymmetrized state formed by vector addition of the particle with spin j_r to the correctly anti-symmetrized $\psi(n, -\lambda, \alpha'\, J'\, M_J'\, T'\, M_T')$. The coefficients $\left(\underline{n} - \lambda, (\alpha'\, J'\, T') j_\lambda \,|\} \alpha\, J\, M_J\, T\, M_T \right)$ are the coefficients of fractional parentage.

The c.f.p. can now be used to reduce the matrix elements for configurations for n particles to those for configurations of $n - 1$ particles. We can consider two classes of operators

1.) $\quad F^{(n)} = \sum_{i=1}^{n} t_i$ $\hfill (34a)$

2.) $\quad G^{(n)} = \sum_{i,j}^{n} t_{ij}$ $\hfill (34b)$

where t_i depends only on the coordinates of one particle and t_{ij} depends on the coordinates of two particles.

In discussing matrix elements of (34a) we shall take $t_i = t_q^{(k)}(i)$ to be the q'th component of a general tensor operator of degree k in isotopic spin space (R42), a scalar in space and spin coordinates. The more general case when t_i is a tensor operator in all coordinates is obtained as a trivial generalization of the discussion which follows.

In writing matrix elements we shall suppress the M_J since all matrix elements are independent of this quantum number. From the symmetry of $\psi(n, \alpha\, J T\, M_T)$ there follows

$$\left\langle n, \alpha\, J\, T\, M_T \middle| F^{(n)} \middle| n\, \alpha\, J\, T'\, M_T \right\rangle = \frac{n}{n-1} \left\langle n, \alpha J T M_T \middle| F^{(n-1)} \middle| n, \alpha J T' M_T \right\rangle \tag{35}$$

where $F^{(n-1)}$ is defined precisely as in (34a) and no longer contains the coordinates ~~of one~~ of the r'th particle. We are also assuming that $F^{(n)}$ commutes with T_ζ so that M_T is a good quantum number at all times. This is an expression of the law of conservation of charge. Using (33) we can write for the matrix element on the right hand side

$$\left\langle n, \alpha\, J\, T\, M_T \middle| F^{(n-1)} \middle| n, \alpha J T M_T \right\rangle = \sum_{\alpha_1 J_1 T_1} \left(\alpha J T \{ \middle| n-1\, (\alpha_1 J_1 T_1) j n \right) \cdot$$
$$\cdot \left(n-1\, (\alpha_2 J_2 T_2) j n \middle| n, \alpha\, J T' \right) \left\langle n-1\, (\alpha_1 J_1 T_1) j n \alpha J T M_T \middle| F^{(n-1)} \middle|$$
$$n-1\, (\alpha_2 J_2 T_2) j n \alpha' J T' M_T \right\rangle \tag{36}$$

We now use the fact that $F^{(n-1)}$ is assumed to be an irreducible tensor operator and the fact that it commutes with t_r, the isotopic spin vector of the r'th particle, to derive from (44a) of Racah (R42) the expression

$$\langle \underline{n-1} \, (\alpha_1 J_1 T_1) \, j_\lambda \, \alpha \, J T M_T | \, F^{(n-1)} | \, \underline{n-1} \, (\alpha_2 J_2 T_2) \, j_\lambda \, \alpha \, J T' M_T \rangle$$
$$= (-1)^{\frac{1}{2}+\lambda-T_2+M_T} \sqrt{(2T+1)(2T'+1)} \, W(T_1 T \, T_2 T'; \tfrac{1}{2} \lambda) \cdot$$
$$\cdot V(T T' \lambda; -M_T \, M_T \, q) \langle \underline{n-1} \, (\alpha_1 J_1 T_1) \| \, F^{(n-1)} \| \, \underline{n-1} \, (\alpha_2 J_2 T_2) \rangle$$

$$(37)$$

The double-barred matrix element of $F^{(n-1)}$ appearing on the right is

independent of M_{T_1} and can be related to the more usual matrix element

of $F^{(n-1)}$ by formula (29) of Racah (R42)

$$\langle \underline{n-1} \, (\alpha_1 J_1 T_1 M_{T_1}) | \, F^{(n-1)} | \, \underline{n-1} \, (\alpha_2 J_2 T_2 M_{T_2}) \rangle = (-1)^{T_1+M_{T_1}} V(T_1 T_2 \lambda; -M_{T_1} M_{T_2} q) \cdot$$
$$\cdot \langle \underline{n-1} \, (\alpha_1 J_1 T_1) \| \, F^{(n-1)} \| \, \underline{n-1} \, (\alpha_2 J_2 T_2) \rangle$$

$$(38)$$

The quantities $W(a\,b\,c; \gamma\,\beta\,\gamma)$ are the Racah functions and

the $V(a\,b\,c; \alpha\,\beta\,\gamma)$ are symmetrical quantities closely related to

the Clebsch-Gordon coefficients. Now by combining (35), (36), (37),

and (38) we have the desired reduction of the matrix elements of $F^{(n)}$

to the n-1 particle configurations. The treatment of the two particle

operators is precisely the same except for the equation (35) which

becomes

$$\langle \underline{n} \, \alpha \, J T M_T | \, G^{(n)} | \, \underline{n} \, \alpha \, J T' M_T \rangle = \frac{n}{n-2} \langle \underline{n} \, \alpha \, J T M_T | \, G^{(n-1)} | \, \underline{n} \, \alpha \, J T' M_T \rangle$$

$$(39)$$

Assuming that the matrix elements of F and G are known for

the (n-1) particle case, the problem is now merely that of calculating

the c.f.p. For many configurations of equivalent particles j^n the

c.f.p. have been given by Edmonds and Flowers (E52a). For configurations

of inequivalent particles of the kind $j_1^2 j_2$, it is easy to show (R54)

that the c.f.p are given by the expressions

$$\left(j_1^2(J'T')j_2\, JT \,|\}\, j_1^2 j_2 (J'T')JT\right) = \frac{1}{\sqrt{3}} \tag{40}$$

$$\left(j_1 j_2(J''T'')j_1\, JT \,|\}\, j_1^2 j_2 (J'T')JT\right) = \sqrt{\tfrac{2}{3}}\; h(J'J''J)\, g(T'T''T)$$

where

$$h(J'J''J) = \sqrt{(2J'+1)(2J''+1)}\; W(j_1 j_1 J j_2; J'J'')(-1)^{\frac{3}{2}-J-J''} \tag{41}$$

$$g(T'T''T) = \sqrt{(2T'+1)(2T''+1)}\; W(\tfrac{1}{2}\tfrac{1}{2}T\tfrac{1}{2}; T'T'')(-1)^{\frac{1}{2}-T-T''}$$

With formulas (40) and (41) we can construct tables of c.f.p for the three particle configurations in which we are interested, viz. $(1p^{3/2})^3$, $(1p^{3/2})^2\, 1p^{1/2}$ and $(1p^{3/2})^2\, 2p^{3/2}$. Tables of these c.f.p. follow with those for $(1p^{3/2})^3$ being constructed from the tables of Edmonds and Flowers (E52) and John and van Wieringen (J51)

$$\left((\tfrac{3}{2})^2(J'T')\tfrac{3}{2}\,|\}(\tfrac{3}{2})^3(J'T')\,JT\right)$$

T	J	T'=0	0	1	1
		J'=1	3	0	2
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$			$\frac{1}{\sqrt{2}}$
	$\frac{3}{2}$	$\sqrt{\frac{3}{20}}$	$\sqrt{\frac{7}{20}}$	$-\sqrt{\frac{5}{12}}$	$-\sqrt{\frac{1}{12}}$
	$\frac{5}{2}$	$\sqrt{\frac{7}{30}}$	$-\sqrt{\frac{4}{15}}$		$-\frac{1}{\sqrt{12}}$
	$\frac{7}{2}$		$\frac{1}{\sqrt{2}}$		$-\frac{1}{\sqrt{2}}$
	$\frac{3}{2}$			$\frac{1}{\sqrt{6}}$	$-\sqrt{\frac{5}{6}}$

In this table and those which follow, vacant squares are to be understood as zero elements.

Left table: $((\tfrac{3}{2})^2(J'T')\tfrac{1}{2}|\}(\tfrac{3}{2})^2\tfrac{1}{2}(J'T')JT)$

T	J	T'=0	0	1	1
		J'=1	3	0	2
$\frac{1}{2}$	$\frac{1}{2}$ a				
	$\frac{1}{2}$ b			$\frac{1}{\sqrt3}$	
	$\frac{3}{2}$ a	$\frac{1}{\sqrt3}$			
	$\frac{3}{2}$ b				$\frac{1}{\sqrt3}$
	$\frac{5}{2}$ a		$\frac{1}{\sqrt3}$		
	$\frac{5}{2}$ b				$\frac{1}{\sqrt3}$
	$\frac{7}{2}$ a		$\frac{1}{\sqrt3}$		
$\frac{3}{2}$	$\frac{1}{2}$			$\frac{1}{\sqrt3}$	
	$\frac{3}{2}$				$\frac{1}{\sqrt3}$
	$\frac{5}{2}$				$\frac{1}{\sqrt3}$

Right table: $(\tfrac{3}{2}\tfrac{1}{2}(J''T'')\tfrac{3}{2}|\}(\tfrac{3}{2})^2\tfrac{1}{2}(J'T')JT)$

T	J	T''=0	0	1	1
		J''=1	2	1	2
$\frac{1}{2}$	$\frac{1}{2}$ a	$-\frac{1}{4}\sqrt{\frac{5}{3}}$	$\frac{1}{4}$	$-\frac{\sqrt5}{4}$	$\frac{\sqrt3}{4}$
	$\frac{1}{2}$ b	$-\frac{\sqrt3}{4}$	$-\frac{\sqrt5}{4}$	$\frac{1}{4}$	$\frac{1}{4}\sqrt{\frac{5}{3}}$
	$\frac{3}{2}$ a	$\frac{1}{2\sqrt6}$	$\frac{1}{2\sqrt2}$	$\frac{1}{2\sqrt2}$	$-\frac{1}{2}\sqrt{\frac{3}{2}}$
	$\frac{3}{2}$ b	$-\frac{1}{2}\sqrt{\frac{3}{2}}$	$-\frac{1}{2\sqrt2}$	$-\frac{1}{2\sqrt2}$	$-\frac{1}{2\sqrt6}$
	$\frac{5}{2}$ a	$-\frac{1}{4}\sqrt{\frac{7}{3}}$	$\frac{1}{4\sqrt3}$	$-\frac{\sqrt7}{4}$	$\frac{1}{4}$
	$\frac{5}{2}$ b	$-\frac{1}{4}$	$-\frac{\sqrt7}{4}$	$\frac{1}{4\sqrt3}$	$\frac{1}{4\sqrt3}\sqrt{7}$
$\frac{3}{2}$	$\frac{1}{2}$			$-\frac{1}{\sqrt2}$	$-\frac{1}{\sqrt6}$
	$\frac{3}{2}$			$\frac{1}{\sqrt2}$	$\frac{1}{\sqrt6}$
	$\frac{5}{2}$			$-\frac{1}{2\sqrt3}$	$-\frac{1}{2\sqrt3}\sqrt{7}$

In both of these tables "a" designates three particle states arising from the $T=0$ multiplet of the parent $\left(1p\tfrac{3}{2}\right)^2$ particle states; "b" designates those arising from the $T=1$ multiplet.

With these preliminaries we are ready to analyze the isotopic spin impurity in the $\left(1p\tfrac{3}{2}\right)^3$ ground state.

(4.32) Impurity of the $\left(1p_{3/2}\right)^3$ Ground State

The greatest impurity in the $T = 1/2$ $\left(1p_{3/2}\right)^3$ state we may expect to arise from low-lying states, perhaps solely because of smaller energy denominators, as we have suggested in the two particle case. The nearest states of different isotopic spin, viz. $T = 3/2$, are those belonging to the same configuration, $\left(1p_{3/2}\right)^3$. Although of course these states are all degenerate on the strict single particle model which does not consider particle interactions, we know from the data that the separation of the first $T = 3/2$ and $T = 1/2$ states in Li^7 is probably $\sim$ 10.8 Mev (P53) or a little higher. The first isotopic spin state of the same spin and parity may well lie several Mev higher.

Another configuration which should lie very near to $\left(1p_{3/2}\right)^3$ is $\left(1p_{3/2}\right)^2 1p_{1/2}$. This state also introduces $T = 3/2$ impurity into the $T = 1/2$ state of $\left(1p_{3/2}\right)^3$. The splitting of the $1p_{3/2} \, 1p_{1/2}$ states of the same isotopic spin has been suggested by Inglis (I54) as 0.48 Mev in Li^7 and by Adair (A53) as $\geqslant$ 1.5 Mev. Although these estimates are not in agreement with the dependence on $\mathcal{l}$ and A usually assumed $\sim \left(\mathcal{l} + \frac{1}{2}\right) A^{-2/3}$ (I53, M50b), both values are much smaller than the energy separation of the T multiplets. This is in agreement with Wigner's isotopic multiplet picture of nuclear levels.

The significance of this rather obvious point is that the energy separations of different isotopic states are always large in light nuclei even when the states belong to the same or an adjacent configuration. In fact, if we take the separation of different orbits to be given by the oscillator or square well models, the energy denominator for the admixture of $T = 3/2$ of $\left(1p_{3/2}\right)^2 2p_{3/2}$ is only three times (30.9 Mev) the minimum separation which could exist between

the $T = \frac{1}{2}$ state of $(1p^{3/2})^3$ and another $T = 3/2$ state, i.e. $\sim$ 10.8 Mev. Actually, the calculations of Kurath (Ku52) on the $(1p^{3/2})^3$ lead to a separation of the $J = 3/2$ $T = 1/2$ and $J = 3/2$ $T = 3/2$ states of $\sim$ 43 Mev. We shall therefore compute the amount of $T = 3/2$ impurity which comes from states of $(1p^{3/2})^2 2p^{3/2}$ as well as states of

$$(1p^{3/2})^3 \quad \text{and} \quad (1p^{3/2})^2 1p^{1/2} \quad .$$

Impurities from configurations like $(1p^{3/2})^2 1f^{7/2}$ may also appear and we should like to know what are the relative magnitudes of the mixing to this configuration and to $(1p^{3/2})^2 2p^{3/2}$. From equations (35), (36), (37), and (38) we see that three-particle matrix elements are linear combinations of two-particle matrix elements. The c.f.p. concerned satisfied the sum rule (R54)

$$\sum_{\alpha_1 J_1 T_1} \left| \left(n_1 - 2 (\alpha_1 J_1 T_1) j_2 \left| \right\} n_1 J T \right) \right|^2 = \frac{\gamma_\lambda}{n}$$

where the configuration $(\underline{n}) = (j_1)^{\gamma_1} \cdots (j_k)^{\gamma_k}$ and $n = \sum_{i=1}^{k} \gamma_i$. The c.f.p. are therefore always less than unity. Since for three particles the sum is distributed over not more than four parent states and the c.f.p vary among themselves in magnitude by no more than a factor of $\sim$ 2 (as one verifies from the above tables or from the more extensive tables of Edmonds and Flowers (E52)), the relative magnitude of corresponding matrix elements (same J, M_T) to different configurations is determined mostly by the relative magnitudes of the two particle matrix elements of the parent states. Observing that matrix elements from $(1p^{3/2})^2$ to $1p^{3/2} 2p^{3/2}$ are 10 $\sim$ 15 times larger than those to all other configurations, we conclude that the three particle matrix elements from $(1p^{3/2})^3$ to $(1p^{3/2})^2 2p^{3/2}$ will also predominate by such a factor.

We may therefore expect that in the three particle case also the mixing to $\left(1p^{3/2}\right)^2 2p^{3/2}$ should predominate over the mixing to any other configuration except possibly $\left(1p^{3/2}\right)^3$ or $\left(1p^{3/2}\right)^2 1p^{1/2}$. Even here $\left(1p^{3/2}\right)^2 2p^{3/2}$ may be as important as the others, in spite of the larger energy separation involved, just because the corresponding two-particle matrix elements are so much larger.

For a system of n nucleons the Coulomb interaction is simply

$$V = -\frac{Ze^2}{2R^3} \sum_{i=1}^{n} \left(\tfrac{1}{2}-t_{\zeta i}\right)r_i^2 + e^2 \sum_{i>j} \left(\tfrac{1}{2}-t_{\zeta i}\right)\left(\tfrac{1}{2}-t_{\zeta j}\right)r_{ij}^{-1} \tag{41}$$

Analogous to equation (27) follows the result

$$V = V_c^{(10)} + V_p^{(10)} + V_p^{(20)}$$

where

$$V_c^{(10)} = \frac{Ze^2}{2R^3} \sum_{i} t_{\zeta i} r_i^2$$

$$V_p^{(10)} = -e^2 \sum_{i>j} \frac{t_{\zeta i}+t_{\zeta j}}{2r_{ij}}$$

$$V_p^{(20)} = e^2 \sum_{i>j} \frac{t_{\zeta i}t_{\zeta j}-\tfrac{1}{3}\vec{t}_1\cdot\vec{t}_2}{r_{ij}} \tag{42}$$

The $V_c^{(10)}$ and $V_p^{(10)}$ are the zero components of vectors in isotopic spin space arising from the core and particle interactions respectively. We can no longer use the simple relations of (25a) and (25b) where T and T_ζ are the total isotopic spin operators for the system and its ζ-component, but instead must use the methods given in (4.31). In the calculation of Radicati (R53a,b) the different transformation properties of the irreducible components of the Coulomb

energy are ignored with the results that the isotopic spin scalar and tensor parts are either treated incorrectly or neglected.

From the table of c.f.p. for $(1p^{3/2})^3$ we observe that the only component of the $T = 1/2$ multiplet which can be mixed to a $T = 3/2$ state is the $J = 3/2$ component. We shall calculate the matrix elements of the interaction $\mathcal{U}$ between these states by evaluating the elements for each of the three operators in (42). One might a priori expect these matrix elements to be quite large since they are between two states of the same configuration. That this is not so can easily be seen in the following calculation of $\langle V_C^{(10)} \rangle$ which we give in some detail for this reason, as well as to illustrate the techniques we use.

The matrix element of any of the three operators in (42) follows from (36) and the c.f.p. table for $(1p^{3/2})^3$ as

$$\langle (\tfrac{3}{2})^3 \tfrac{3}{2} \tfrac{1}{2} M_T | \,_3 V^{(11)} | (\tfrac{3}{2})^3 \tfrac{3}{2} \tfrac{3}{2} M_T \rangle$$

$$= \left((\tfrac{3}{2})^3 \tfrac{3}{2} \tfrac{1}{2} \left\{ \left| (\tfrac{3}{2})^2 (01) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{1}{2} \right) \left((\tfrac{3}{2})^2 (01) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{3}{2} \right\} (\tfrac{3}{2})^3 \tfrac{3}{2} \tfrac{3}{2} \right) \cdot$$

$$\cdot \langle (\tfrac{3}{2})^2 (01) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{1}{2} M_T | \,_3 V | (\tfrac{3}{2})^2 (01) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{3}{2} M_T \rangle +$$

$$+ \left((\tfrac{3}{2})^3 \tfrac{3}{2} \tfrac{1}{2} \left\{ \left| (\tfrac{3}{2})^2 (21) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{1}{2} \right) \left((\tfrac{3}{2})^3 \tfrac{3}{2} \tfrac{3}{2} \left\{ \left| (\tfrac{3}{2})^2 (21) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{3}{2} \right) \right. \right.$$

$$\langle (\tfrac{3}{2})^3 (21) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{1}{2} M_T | \,_3 V | (\tfrac{3}{2})^2 (21) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{3}{2} M_T \rangle$$

$$(43)$$

where $\,_3 V^{(11)}$ denotes a general three-particle operator and the matrix elements on the right are between unsymmetrized states obtained by vector coupling a single nucleon to the properly symmetrized two particle states of $(1p^{3/2})^2$. These matrix elements can be expressed for $\,_2 V_C^{(10)}$ as

$$\left\langle \left(\tfrac{3}{2}\right)^2 (JT) \tfrac{3}{2} \tfrac{1}{2} M_T \left| {}_2 V_c^{-(10)} \right| \left(\tfrac{3}{2}\right)^2 (JT) \tfrac{3}{2} \tfrac{3}{2} M_T \right\rangle = (-1)^{\frac{1}{2} + M_T} 2\sqrt{2}$$

$$W(1\tfrac{1}{2} 1 \tfrac{3}{2} ; \tfrac{1}{2} 1) V(\tfrac{1}{2} \tfrac{3}{2} 1 ; -M_T M_T 0)\left\langle \left(\tfrac{3}{2}\right)^2 (JT) \| {}_2 V_c^{(10)} \| \left(\tfrac{3}{2}\right)^2 JT \right\rangle$$

$$(44)$$

The matrix elements in (43) of $\quad {}_3 V_c^{-(10)} \quad$ can be expressed as matrix

elements of $\quad {}_2 V_c^{-(10)} \quad$, which does not contain the coordinates of one

particle, and the proper c.f.p. inserted from the table to give

$$\left\langle \left(\tfrac{3}{2}\right)^3 \tfrac{3}{2} \tfrac{1}{2} M_T \left| {}_3 V_c^{-(10)} \right| \left(\tfrac{3}{2}\right)^3 \tfrac{3}{2} \tfrac{3}{2} M_T \right\rangle$$

$$= -\tfrac{1}{4}\sqrt{\tfrac{5}{2}} \left[\left\langle \left(\tfrac{3}{2}\right)^2 (01) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{1}{2} M_T \left| {}_2 V_c^{(10)} \right| \left(\tfrac{3}{2}\right)^2 (01) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{3}{2} M_T \right\rangle \right.$$

$$\left. - \left\langle \left(\tfrac{3}{2}\right)^3 (21) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{1}{2} M_T \left| {}_2 V_c^{-(10)} \right| \left(\tfrac{3}{2}\right)^3 (21) \tfrac{3}{2} ; \tfrac{3}{2} \tfrac{3}{2} M_T \right\rangle \right]$$

$$(43')$$

This result is quite interesting since from (44) we see that only

$$\left\langle \left(\tfrac{3}{2}\right)^2 JT \| {}_2 V_c^{-(10)} \| \left(\tfrac{3}{2}\right)^2 JT \right\rangle$$ can be different for the two terms

of (43'). This, however, means that the matrix element of $\quad {}_3 V_c^{-(10)}$ will

vanish since the double-barred matrix elements of $\quad {}_2 V_c^{-(10)} \quad$ for $J = 0, 2$

are equal. Further, only the $F^{(2)}$ terms of the Slater expansion for

the matrix elements of $\quad \dfrac{e^2}{r_{12}} \quad$ will not cancel in (43'), as we have

explained in our discussion of the two particle terms. The matrix

elements between $T = 1/2$ and $T = 3/2$ of the $\left(1 p \tfrac{3}{2}\right)^3$ configuration

will be much smaller than one might expect.

Combining two equations similar to (43') and (44), we can find

the matrix elements for $\quad {}_3 V_p^{-(10)} \quad$ and $\quad {}_3 V_p^{-(30)} \quad$ and therefore of $\mathcal{U}$.

These are given in the table which summarizes the intermultiplet matrix elements having the initial state $(1p^{3/2})^3$.

The mixing of $(1p^{3/2})^3$ and $(1p^{3/2})^2 1p^{1/2}$ can actually occur for the $J = 1/2$, $3/2$, and $5/2$ components of the $T = 1/2$ and $T = 3/2$ multiplets. As a matter of interest we have found the intermultiplet matrix elements for all these J states. It is very interesting to note that the smallest isotopic spin mixing between these configurations occurs for the $J = 3/2$ state, which should also be the ground state for $(1p^{3/2})^3$ according to Nordheim's empirically derived rules. This prediction of the ground state spin has been verified by direct calculation (K52) on the jj coupling shell model.

The calculation of the mixing between $(1p^{3/2})^3$ and $(1p^{3/2})^2 2p^{3/2}$ is somewhat more lengthy due to the existence of two $\frac{3}{2}\frac{3}{2}$ states having the parents (01) and (21) in the $(1p^{3/2})^2$ configuration (cf. the derivation of the c.f.p.). As we expected these matrix elements are quite large and are the only ones into which the core interaction enters in producing isotopic spin impurities in the $\frac{3}{2}\frac{1}{2}$ state of $(1p^{3/2})^3$. We observe that the matrix elements to the $(1p^{3/2})^2 2p^{3/2}$ state having the parent (21) are much smaller than those to the state with (01) parent. This is a reflection of the fact that all the two particle matrix elements between $J = 2$ states are much smaller than those between $J = 0$ states.

The matrix elements for isotopic spin mixing between all these configurations are given in table VII. Their magnitudes are what one should expect from considering the corresponding two particle matrix elements. The determination of the percentage impurity which these matrix elements imply is somewhat ambiguous considering the lack of

knowledge on separations of states of various configurations. We have

therefore given what one might consider to be an upper limit on the

impurity from various $T = 3/2$ states by taking the separation of the

$T = 1/2$ and $T = 3/2$ states to be $E_o - E_1 \sim 10$ Mev and calculating

$$p = \frac{m^2}{(E_o - E_1)^2}$$ in each case. From the discussion which preceded

this calculation we must conclude that this p is really an extreme

upper limit for the mixing of the ground state <u>if one neglects excitation</u>

<u>of the core.</u>

Perhaps the most significant feature of Table VII is the large

factor by which the mixing produced by the core exceeds the mixing

produced by the particle interaction. Again this conclusion reflects

the corresponding situation in the mixing of two particle states. The

domination of the perturbation by the core is quite significant when

one considers that Z is only 2 for the "core" in the $A = 6$ triad.

TABLE VII

$$\langle (1p_{3/2})^3 \ \tfrac{3}{2} \ \tfrac{1}{2} \ M_T \ | \ \mathcal{V} \ | \ (1p_{3/2})^3 \ \tfrac{3}{2} \ \tfrac{3}{2} \ M_T \rangle \equiv m_\ell$$

M_T	m_ℓ	$A=7$ Nucleus	$p=10^{-2} m_\ell^2$
$\tfrac{1}{2}$	0		
$-\tfrac{1}{2}$	$\dfrac{5}{48}\sqrt{\dfrac{1}{\pi}}\dfrac{e^2}{R}$	0.0660 Mev	4.35×10^{-5}

$$\langle (1p_{3/2})^3 \ J \ \tfrac{1}{2} \ M_T \ | \ \mathcal{V} \ | \ (1p_{3/2})^2 \ 1p_{1/2} \ J \ \tfrac{3}{2} \ M_T \rangle \equiv m_6$$

J	M_T	m_6	$A=7$ Nucleus	$p=10^{-2} m_6^2$
$\tfrac{1}{2}$	$\tfrac{1}{2}$	0		
	$-\tfrac{1}{2}$	$-\dfrac{1}{24}\sqrt{\dfrac{5}{3\pi}}\dfrac{e^2}{R}$	-0.0170 Mev	2.88×10^{-6}
$\tfrac{3}{2}$	$\tfrac{1}{2}$	0		
	$-\tfrac{1}{2}$	$-\dfrac{1}{72}\sqrt{\dfrac{5}{2\pi}}\dfrac{e^2}{R}$	-0.00696	4.83×10^{-5}
$\tfrac{5}{2}$	$\tfrac{1}{2}$	0		
	$-\tfrac{1}{2}$	$\dfrac{1}{24}\sqrt{\dfrac{35}{6\pi}}\dfrac{e^2}{R}$	$+0.0320$	1.02×10^{-5}

$$\langle (1p_{3/2})^3 \ \tfrac{3}{2} \ \tfrac{3}{2} \ M_T \ | \ \mathcal{V} \ | \ (1p_{3/2})^2 \ 2p_{3/2} \ \tfrac{3}{2} \ \tfrac{3}{2} \ (J'T') \ M_T \rangle \equiv m_6$$

J'	T'	M_T	m_6	$A=7$ Nucleus	$p=10^{-2} m_6^2$
0	1	$\tfrac{1}{2}$	$-\dfrac{Ze^2}{R}\dfrac{1}{6\sqrt{2}}$	-0.132	1.74×10^{-4}
		$-\tfrac{1}{2}$	$-\dfrac{Ze^2}{R}\dfrac{1}{6\sqrt{2}}-\dfrac{23}{72}\sqrt{\dfrac{1}{\pi}}\dfrac{e^2}{R}$	-0.233	5.42×10^{-4}
2	1	$\tfrac{1}{2}$	$\dfrac{Ze^2}{R}\dfrac{1}{6\sqrt{2}}$	0.0593	3.52×10^{-5}
		$-\tfrac{1}{2}$	$\dfrac{Ze^2}{R}\dfrac{1}{6\sqrt{2}}-\dfrac{31}{360}\sqrt{\dfrac{5}{\pi}}\dfrac{e^2}{R}$	-0.0017	2.9×10^{-7}

(4.4) The Excitation of the Isotopic Spin State of Closed Shells

In the calculations which we have done on the mixing of

the isotopic spin states of two or three nucleons we have always neglected

the excitation of the isotopic spin state of the core. Since all our

calculations have been based on the jj coupling shell model of the

nucleus, this assumption is a particularly convenient one. All nuclei

with mass numbers between $A = 12$ and $A = 20$ will then require nothing

more than a calculation involving the states of either two particles or

two holes. We now purpose to consider the excitation of the core and

the isotopic spin impurity in a nucleus with closed shells. The problem

can be treated by two different methods which illuminate two different

aspects of the Coulomb effect on the isotopic spin state of closed shells.

These two methods are (1) the reduction to the two nucleon matrix

elements, and (2) the method of an "equivalent" potential of a closed

shell. The first approach will show how one may regard the excitation

of the isotopic spin state of the core as an extension of the problem

of Coulomb mixing of the states of two nucleons. The second approach

is a rigorous derivation of the method by which we have included the

effect of the interaction of the core on two particle states, but will

show in fact how we can also use the method to calculate the impurity

of a closed shell.

Finally we shall give some consideration to the question of how

one may obtain approximately the isotopic spin impurity in the ground

and excited states of nuclei. The result of this investigation will be

that one may relatively easily estimate isotopic spin impurities in

ground and excited states of nuclei.

(4.41) Reduction of Closed Shell Mixing to Two Particle

Mixing

The wave function for a closed shell nucleus corresponds to a $J = 0$ $T = 0$ state and is an antisymmetric linear combination of products of the single particle wave functions $\psi_\gamma = \varphi_{j,m_\gamma}(\vec{x},\vec{\sigma})u(\vec{t})$. The function of space and spin coordinate $\varphi_{j,m_\gamma}(\vec{x},\vec{\sigma})$ is characterized by a definite value of angular momentum j and the Z-component m_γ. The isotopic spin function $u(\vec{t})$ is an eigenfunction of t_ζ with eigenvalues $\pm 1/2$. The nuclear wave function for the ground state of a shell will be designated by ψ_0 and can be written as a Slater determinant of these wave functions with all the $(2j_\gamma + 1)$ values of m_γ appearing as well as the eigenfunctions $u(+\frac{1}{2})$ and $u(-\frac{1}{2})$. Excited states of the closed shell nucleus ψ_γ can also be written as Slater determinants, or as linear combinations of Slater determinants, in which one or more single particle wave functions have been changed to states of a single particle lying outside the shell. Since the Coulomb perturbation is a two-particle operator, matrix elements will vanish to states ψ_γ differing from ψ_0 by more than two single particle states. We shall first assume that ψ_γ differs from ψ_0 by only one state since we expect the contributions to be small from states which correspond to the excitation of two nucleons. This approximation will be checked by deriving a general expression for the contributions from states of two-particle excitation and evaluating it for C^{12} (closed $1S_{1/2}$ and $1P_{3/2}$ on jj coupling model).

The Coulomb interaction of system of nucleons is given by

$$V = \sum_{i>j} \frac{e^2}{\mathcal{A}_{ij}}\left(\frac{1}{2}-t_{\zeta i}\right)\left(\frac{1}{2}-t_{\zeta j}\right) = V^{(0)} + V^{(10)} + V^{(20)}$$

and has the irreducible components

$$V^{(0)} = \sum_{i>j} \frac{e^2}{\lambda_{ij}} \left(\frac{1}{4} + \frac{1}{3} \vec{t}_i \cdot \vec{t}_j \right)$$

$$V^{(10)} = -\sum_{i>j} \frac{e^2}{2\lambda_{ij}} \left(t_{3i} + t_{3j} \right)$$

$$V^{(20)} = \sum_{i>j} \frac{e^2}{\lambda_{ij}} \left(t_{3i} t_{3j} - \frac{1}{3} \vec{t}_i \cdot \vec{t}_j \right)$$

where $V^{(0)}$ is a scalar in isotopic spin space; $V^{(10)}$, the zero'th component of a vector; and $V^{(20)}$, the zero'th component of a tensor of rank two. The selection rule on the matrix elements of $V^{(20)}$ which reads, $\Delta T = 2$, insures that the only mixing produced by $V^{(20)}$ will be to states which lie so high that their contributions to isotopic spin impurity can be neglected. Thus only matrix elements $V^{(10)}$ need be calculated, and we shall put

$$C = -\sum_{i>j} \frac{e^2}{2\lambda_{ij}} \left(t_{3i} + t_{3j} \right) \tag{46}$$

for convenience. The selection rule $\Delta T = 1$ on C insures that all its contributions to mixing are from $T = 1$ states and none are from $T = 0$ states. The impurities produced in ψ_e by C are therefore isotopic spin impurities.

Designate the excited single particle state in ψ_ν by ψ_σ with momentum j_3, suppressing the principal quantum number n for the present. The Z-component of the state must be the same as that of the corresponding state in ψ_0, $j_2 m_2$ say, so that $M_J = 0$. The matrix element of C is (TAS 5[6])

$$(\Psi_v, C\Psi_o) = -\frac{e^2}{2} \sum_\nu \left[\int \psi_\sigma^*(1)\psi_j^*(2) \frac{t_{\zeta 1} + t_{\zeta 2}}{\hbar_{12}} \psi_\mu(1)\psi_\nu(2) \right.$$

$$\left. - \int \psi_\sigma^*(1)\psi_j^*(2) \frac{t_{\zeta 1} + t_{\zeta 2}}{\hbar_{12}} \psi_\nu(1)\psi_\mu(2) \right]$$

(47)

Summing over the isotopic spin states of particle 2 this becomes

$$(\Psi_v, C\Psi_o) = -\frac{e^2}{2} \sum_\nu \left[\int \varphi_\sigma^*(1)\varphi_j^*(2) \frac{1}{\hbar_{12}} \varphi_\mu(1)\varphi_\nu(2) - \right.$$

$$\left. - \int \varphi_\sigma^*(1)\varphi_j^*(2) \frac{1}{\hbar_{12}} \varphi_\nu(1)\varphi_\mu(2) \right] \qquad (48)$$

Using the Dirac notation (48) can be written more explicitly as

$$(\Psi_v, C\Psi_o) = -\frac{e^2}{2} \sum_{j_1} \sum_{m_1} \left[(j_3 m_2 j_1 m_1 | \frac{1}{\hbar_{12}} | j_2 m_2 j_1 m_1) - \right.$$

$$\left. - (j_3 m_2 j_1 m_1 | \frac{1}{\hbar_{12}} | j_1 m_1 j_2 m_2) \right]$$

(49)

We shall transform the two particle matrix elements into the JM_J representation, obtaining

$$(\Psi_v, C\Psi_o) = -\frac{e^2}{2} \sum_{j_1} \sum_{m_1} \sum_{J,M_J} \left[(j_3 m_2 j_1 m_1 | j_3 j_1 JM_J)(j_2 j_1 JM_J | j_2 m_2 j_1 m_1) \cdot \right.$$

$$\cdot (j_3 j_1 J | \frac{1}{\hbar_{12}} | j_2 j_1 J)$$

$$- (j_3 m_2 j_1 m_1 | j_3 j_1 JM_J)(j_1 j_2 JM_J | j_1 m_1 j_2 m_2) \cdot$$

$$\cdot (j_3 j_1 J | \frac{1}{\hbar_{12}} | j_1 j_2 J) \right]$$

(50)

We have omitted the index M_J on the matrix elements of $\frac{1}{\hbar_{12}}$ since these are independent of M_J. For simplifying this sum we introduce the $V(abc; \alpha\beta\gamma)$ discussed by Racah (R42) and defined by him as

$$(j_1 j_2 jm | j_1 j_2 m_1 m_2) = (-1)^{j+m} \sqrt{2j+1} \; V(j_1 j_2 j; m_1 m_2 -m)$$

(51)

The first term in (50) when summed only over m_1, M_J is

$$\sum_{m_1, M_J} (j_3 m_2 j_1 m_1 | j_3 j_1 JM_J)(j_2 j_1 JM_J | j_2 m_2 j_1 m_1)$$

$$= (-1)^{2(J+M_J)} \sum_{m_1, M_J} (2J+1) V(j_3 j_1 J; m_2 -m_1 M_J) \cdot$$
$$\cdot V(j_2 j_1 J; m_2 m_1 -M_J)$$

$$= \sum_{m_1, M_J} V(j_1 J j_2; m_1 -M_J m_2) V(j_1 J j_3; m_1 -M_J m_2)(2J+1) \quad (52)$$

where we have used the symmetries of the $V(abc, \alpha\beta\gamma)$. Using the

orthogonality relation

$$\sum_{\alpha, \beta} V(abc; \alpha\beta\gamma) V(abc'; \alpha\beta\gamma') = \frac{\delta(c,c')\delta(\gamma,\gamma')}{2c+1}$$

the equation (52) becomes

$$\sum_{m_1, M_J} (j_3 m_2 j_1 m_1 | j_3 j_1 JM_J)(j_2 j_1 JM_J | j_2 m_2 j_1 m_1)$$

$$= \frac{2J+1}{2j_2+1} \delta(j_2, j_3) \quad (53)$$

By similar processes one gets for the second term in (50)

$$\sum_{m_1, M_J} (j_3 m_2 j_1 m_1 | j_3 j_1 JM_J)(j_1 j_2 JM_J | j_1 m_1 j_2 m_2)$$

$$= - \frac{2J+1}{2j_2+1} \delta(j_2, j_3)(-1)^{j_2 -j_1 +J} \quad (54)$$

Insertion of equations (53) and (54) into equation (50) gives finally

$$(\Psi_1, C\Psi_0) = - \frac{e^2}{2} \sum_{j_1, J} \left(\frac{2J+1}{2j_2+1}\right)\left[(\gamma j_2 j_1 J | \frac{1}{\hbar_{12}} | \gamma' j_2 j_1 J) + \right.$$
$$\left. + (-1)^{J+j_2 -j_1} (\gamma j_1 j_2 J | \frac{1}{\hbar_{12}} | \gamma' j_2 j_1 J) \right]$$

(55)

The γ and γ' represent the principal quantum numbers of the single particle states j_1 and j_2. The significance of the equations (53) and (54) which require that a particle of the state ψ_o must be excited without change of the individual angular momentum j_2 is very easily seen by using the fact that the spin of ψ_o is $J = 0$. The excited state ψ_ν can be regarded as a shell with a single hole of spin j_2 coupled to an outside particle, i.e. the excited nucleon, of spin j_3. But unless $j_2 = j_3$ ψ_ν will not have spin $J = 0$ and the matrix element $(\psi_\nu, C \psi_o)$ will vanish.

The form of equation (50) displays the relation of the problem of core excitation to the existence of a two particle interaction. The individual terms are in fact the "matrix elements" for the excitation of a single particle which is interacting with another particle of the core through the Coulomb potential. These are not matrix elements between properly symmetrized two particle states as one sees from the derivation of (55).

From (49) we observe that the matrix element (55) corresponds to the excitation of just the single particle state $j_2 m_2$, while m_2 may assume $(2j_2 + 1)$ values and j_2 should assume a different value for each closed shell. The value of t_ζ for the excited nucleon may also assume two values corresponding to whether a neutron or a proton was excited. Let $\alpha (j_2)$ designate

$$\alpha(j_2) = \frac{(\psi_\nu, C \psi_o)}{E_o - E_{ji}} \tag{56}$$

where $(\psi_\nu, C \psi_o)$ is the matrix element (55) between ψ_o and the state ψ_ν corresponding to excitation of particle $j_2 m_2$ and $E_o - E_{ji}$ is the appropriate energy denominator. The total impurity p

in Ψ_o is then

$$p = \sum_{j_i} 2\left(2j_i + 1\right) \alpha^2\left(j_i\right) \qquad \text{(sum over closed shells)}$$

(57)

(4.411) Calculation of the Impurity in C^{12} and He^4

We shall apply these formula to the specific case of C^{12} and shall then compare our results with the amount of isotopic spin impurity obtained from the calculations on the statistical model.

On the jj model C^{12} consists of closed $|S_{1/2}$ and $|p_{3/2}$ shells and the isotopic spin impurity follows from (55) and (57)

$$p = 8\alpha^2\left(|p_{3/2}\right) + 4\alpha^2\left(|S_{1/2}\right)$$

(58)

$$\left(E_o - E_{|p_{3/2}}\right)\alpha\left(|p_{3/2}\right) = -\frac{e^2}{2}\left\{ \sum_J \frac{(2J+1)}{4}\left(2p_{3/2}|p_{3/2}J\left|\frac{1}{\lambda_{12}}\right|\left(|p_{3/2}\right)^2 J\right)\left(1+(-1)^J\right)\right.$$

$$\left. + \sum_J \frac{2J+1}{4}\left[\left(2p_{3/2}|S_{1/2}J\left|\frac{1}{\lambda_{12}}\right||p_{3/2}|S_{1/2}J\right)-(-1)^J\left(2p_{3/2}|S_{1/2}J\left|\frac{1}{\lambda_{12}}\right||S_{1/2}|p_{3/2}J\right)\right.\right.$$

(59)

$$\left(E_o - E_{|S_{1/2}}\right)\alpha\left(|S_{1/2}\right) = -\frac{e^2}{2}\left\{ \sum_J \frac{2J+1}{2}\left[\left(2S_{1/2}|p_{3/2}J\left|\frac{1}{\lambda_{12}}\right||S_{1/2}|p_{3/2}J\right)-\right.\right.$$

$$\left. - (-1)^J\left(2S_{1/2}|p_{3/2}J\left|\frac{1}{\lambda_{12}}\right||p_{3/2}|S_{1/2}J\right)\right]$$

$$\left. + \left(2S_{1/2}|S_{1/2}0\left|\frac{1}{\lambda_{12}}\right|\left(|S_{1/2}\right)^2 0\right)\right\}$$

(60)

We now apply the same procedure with which we calculated two particle

matrix elements in (4.21). We find first the "matrix elements" in LS

coupling, calculate all the required Slater integrals, and transform

to jj coupling. The necessary jj "matrix elements" are listed in

the following table while the appropriate F^k are to be found in

Appendix II

Direct Terms

$$(2p_{3/2}\,1p_{3/2}\,0\,|\tfrac{1}{r_{12}}|\,1p_{3/2}\,1p_{1/2}\,0) = F^0((1p)^2;\,1p2p) + \tfrac{1}{5}\,F^2((1p)^2;\,1p2p)$$

$$(2p_{3/2}\,1p_{3/2}\,2\,|\tfrac{1}{r_{12}}|\,1p_{3/2}\,1p_{1/2}\,2) = F^0((1p)^2;\,1p2p) - \tfrac{3}{25}\,F^2((1p)^2;\,1p2p)$$

$$(2p_{3/2}\,1S_{1/2}\,0\,|\tfrac{1}{r_{12}}|\,1p_{3/2}\,1S_{1/2}\,0) = 0$$

$$(2p_{3/2}\,1S_{1/2}\,1\,|\tfrac{1}{r_{12}}|\,1p_{3/2}\,1S_{1/2}\,1) = F^0(2p1s;\,1p1s)$$

$$(2S_{1/2}\,1p_{3/2}\,2\,|\tfrac{1}{r_{12}}|\,1S_{1/2}\,1p_{3/2}\,2) = F^0(2s1p;\,1p1s)$$

$$(2S_{1/2}\,1S_{1/2}\,0\,|\tfrac{1}{r_{12}}|\,2S_{1/2}\,1S_{1/2}\,0) = F^0(2s1s;\,(1s)^2)$$

Exchange Terms

$$(2p_{3/2}\,1S_{1/2}\,0\,|\tfrac{1}{r_{12}}|\,1S_{1/2}\,1p_{3/2}\,0) = 0$$

$$(2p_{3/2}\,1S_{1/2}\,1\,|\tfrac{1}{r_{12}}|\,1S_{1/2}\,1p_{3/2}\,1) = -\tfrac{1}{9}\,F^1(2p1s;\,1s1p)$$

$$(2p_{3/2}\,1S_{1/2}\,2\,|\tfrac{1}{r_{12}}|\,1S_{1/2}\,1p_{3/2}\,2) = -\tfrac{1}{3}\,F^1(2p1s;\,1s1p)$$

$$(2S_{1/2}\,1p_{3/2}\,0\,|\tfrac{1}{r_{12}}|\,1p_{3/2}\,1S_{1/2}\,0) = 0$$

$$(2S_{1/2}\,1p_{3/2}\,1\,|\tfrac{1}{r_{12}}|\,1p_{3/2}\,1S_{1/2}\,1) = -\tfrac{1}{9}\,F^1(2s1p;\,1p1s)$$

$$(2S_{1/2}\,1p_{3/2}\,2\,|\tfrac{1}{r_{12}}|\,1p_{3/2}\,1S_{1/2}\,2) = -\tfrac{1}{3}\,F^1(2p1s;\,1s1p)$$

We emphasize that these are not matrix elements between properly symmetrized two particle states, but only between unsymmetrized states of two particles coupled to form different angular momenta. With these values of the "matrix elements" the mixing coefficients given by (59) and (60) become

$$(E_0 - E_{1p^{3/2}})\alpha(1p^{3/2}) = -\frac{e^2}{2}\left\{3F^0(2p1p;(1p)^2) - \frac{1}{5}F^2(2p1p(1p)^2)\right.$$
$$\left. + 2F^0(2p1s;1p1s) + \frac{1}{3}F^1(2p1s,1s1p)\right\}$$

$$(61)$$

$$(E_0 - E_{1s^{1/2}})\alpha(1s^{1/2}) = -\frac{e^2}{2}\left\{4F^0(2s1p,1s1p) + \frac{2}{3}F^1(2s1p,1p1s)\right.$$
$$\left. + F^0(2s1s,(1s)^2)\right\}$$

$$(62)$$

Using the value for $\gamma = \frac{5}{2R^2}$ these equations yield for the mixing coefficients for C^{12}

$$(E_0 - E_{1p^{3/2}})\alpha(1p^{3/2}) = -0.624\frac{e^2}{R}$$

$$(E_0 - E_{1s^{1/2}})\alpha(1s^{1/2}) = -0.858\frac{e^2}{R}$$

According to equation (57) the impurity in the ground state of C^{12} is

$$p = 6.06\left(\frac{e^2/R}{E_0 - E_1}\right)^2$$

$$(63)$$

where we have taken $E_0 - E_{1p^{3/2}} = E_0 - E_{1s^{1/2}} = E_0 - E_1$. This value of p for C^{12} is to be compared with the value obtained from the expression for p found from the statistical model

$$p_s = \frac{A(A-1)}{16}\left\{0.878 + 0.017A\right\}\left(\frac{e^2/R}{E_0 - E_1}\right)^2$$

The value of p_s for C^{12} is found from this expression to be

$$p_s = 9.0\left(\frac{e^2/R}{E_0 - E_1}\right)^2$$

$$(64)$$

The agreement between (63) and (64) is really quite remarkable, but the value of p found from (63) will be somewhat lower than the upper limits given in our work on the statistical model. We now know that $E_0 - E_1$ is equal to the separation of levels for a single particle having the same j value but differing by unity in the principal quantum number n. Where we took $E_0 - E_1 \sim 15$ Mev for an upper limit on p (and should again for an unquestionable upper limit), we will assume now that $E_0 - E_1$ is at least as large as the separtion of the $1p$ and $2p$ levels, or ~ 20 Mev. The impurity in C^{12} is therefore 3.02×10^{-3} as compared to the previously estimated 7.5×10^{-3}. We must remember, however, that most of this factor of ~ 2.5 difference is a result of a different choice of $E_0 - E_1$ and the agreement of statistical model and these shell model calculations is really surprisingly, perhaps fortuitously, close.

The impurity of the α-particle is very easily found by using just the $\alpha(1s_{1/2})$ found above. If we use the $2S, 1S$ separation for $He^4 \sim 25$ Mev, the isotopic spin impurity in the ground state of He^4 is

$$p = \frac{4\alpha^2(1s_{1/2})}{(E_0 - E_1)^2} \approx 2 \times 10^{-3}$$

The fact that the impurity for He^4 is nearly as large as that for C^{12} is a result of the much smaller radius of He^4.

(4.412) Contributions to Impurities from Doubly

Excited Shells

The contribution to isotopic spin impurity from excited states Ψ_ν formed by excitation of two particle states in Ψ_0 has been assumed to be much smaller than contributions from states Ψ_ν of single particle excitation. We shall now justify

this assumption by first deriving a general expression for the isotopic spin impurity from "doubly excited" states and then evaluating the expression for C^{12}.

Just as we wrote equation (47), we first write the matrix element of C between ψ_0 and a doubly excited state ψ_ν (TAS 5^6) as

$$(\psi_\gamma, C\psi_0) = -\frac{e^2}{2}\left[\int \psi_\delta^*(1)\psi_\omega^*(2)\frac{t_{s1}+t_{s2}}{r_{12}}\psi_\mu(1)\psi_\nu(2)\right.$$
$$\left. - \int \psi_\omega^*(1)\psi_\delta^*(2)\frac{t_{s1}+t_{s2}}{r_{12}}\psi_\mu(1)\psi_\nu(2)\right]$$

(65)

This matrix element vanishes unless $t_{\rho1} = t_{\rho2} = \pm\frac{1}{2}$, i.e. two protons or two neutrons are excited. In this case we find

$$(\psi_\gamma, C\psi_0) = \pm\frac{e^2}{2}\left[\int \varphi_\delta^*(1)\varphi_\omega^*(2)\frac{1}{r_{12}}\varphi_\mu(1)\varphi_\nu(2) -\right.$$
$$\left. - \int \varphi_\omega^*(1)\varphi_\delta^*(2)\frac{1}{r_{12}}\varphi_\mu(1)\varphi_\nu(2)\right]$$

(66)

or using Dirac notation

$$(\psi_\gamma, C\psi_0) = \pm\frac{e^2}{2}\left[(j_3 m_3 j_4 m_4|\frac{1}{r_{12}}|j_1 m_1 j_2 m_2) -\right.$$
$$\left. - (j_4 m_4 j_3 m_3|\frac{1}{r_{12}}|j_1 m_1 j_2 m_2)\right]$$

(67)

Again we transform to the JM_J representation

$$(\psi_\gamma, C\psi_0) = \pm\frac{e^2}{2}\sum_{J,M_J}\left\{(j_3 m_3 j_4 m_4|j_3 j_4 JM_J)(j_1 j_2 JM_J|j_1 m_1 j_2 m_2)\cdot\right.$$
$$\cdot(j_3 j_4 J|\frac{1}{r_{12}}|j_1 j_2 J)$$
$$- (j_4 m_4 j_3 m_3|j_4 j_3 JM_J)(j_1 j_2 JM_J|j_1 m_1 j_2 m_2)\cdot$$
$$\left. \cdot(j_4 j_3 J|\frac{1}{r_{12}}|j_1 j_2 J)\right\}$$

(68)

This matrix element can be formed for any of the values of m_1, m_2, m_3, and m_4 so that the isotopic spin impurity due to excitation of particles of spin j_1 and j_2 will be the square of (68) summed over all the possible values of m_1, m_2, m_3, and m_4. A factor of two comes in from the possibility of exciting two neutrons or two protons. This sum is then

$$\sum |(\Psi_{-1}, C\Psi_o)|^2 = 2\left(\frac{e^2}{2}\right)^2 \left\{ A - 2B + C \right\} \equiv S(j_1 j_2 ; j_3 j_4)$$

$$A = \sum_{m_1, m_2} \sum_{J', M_J'} \sum_{J, M_J} (j_3 m_3 j_4 m_4 | j_3 j_4 J M_J)(j_3 m_3 j_4 m_4 | j_3 j_4 J' M_J') \cdot$$
$$(j_1 j_2 J M_J | j_1 m_1 j_2 m_2)(j_1 j_2 J' M_J' | j_1 m_1 j_2 m_2)$$
$$(j_3 j_4 J | \tfrac{1}{r_{12}} | j_1 j_2 J)(j_3 j_4 J' | \tfrac{1}{r_{12}} | j_1 j_2 J')$$

$$B = \sum_{J, M_J} \sum_{J', M_J'} \sum_{\substack{m_1, m_2 \\ m_3, m_4}} (j_3 m_3 j_4 m_4 | j_3 j_4 J M_J)(j_4 m_4 j_3 m_3 J' M_J')$$
$$(j_1 j_2 J M_J | j_1 m_1 j_2 m_2)(j_1 j_2 J' M_J' | j_1 m_1 j_2 m_2)$$
$$(j_3 j_4 J | \tfrac{1}{r_{12}} | j_1 j_2 J) \cdot (j_4 j_3 J' | \tfrac{1}{r_{12}} | j_1 j_2 J')$$

$$C = \sum_{m_1, m_2} \sum_{J', M_J'} \sum_{J, M_J} (j_4 m_4 j_3 m_3 | j_4 j_3 J M_J)(j_4 m_4 j_3 m_3 | j_4 j_3 J' M_J')$$
$$(j_1 j_2 J M_J | j_1 m_1 j_2 m_2)(j_1 j_2 J' M_J' | j_1 m_1 j_2 m_2)$$
$$(j_4 j_3 J | \tfrac{1}{r_{12}} | j_1 j_2 J)(j_1 j_2 J' | \tfrac{1}{r_{12}} | j_1 j_2 J')$$

$$(69)$$

These three expressions can all be reduced by the methods which gave (53) and (54). The resulting expression for (69) is

$$S(j_1 j_2 ; j_3 j_4) = 2\left(\frac{e^2}{2}\right)^2 \sum_J (2J+1)\left\{\left(j_3 j_4 J \left|\frac{1}{r_{12}}\right| j_1 j_2 J\right)^2\right.$$

$$- 2\left(j_3 j_4 J \left|\frac{1}{r_{12}}\right| j_1 j_2 J\right)\left(j_4 j_3 J \left|\frac{1}{r_{12}}\right| j_1 j_2 J\right)$$

$$\left. + \left(j_4 j_3 J \left|\frac{1}{r_{12}}\right| j_1 j_2 J\right)^2\right\}$$

(70)

where the sum on J is from the larger of $\left|j_1 - j_2\right|$, $\left|j_3 - j_4\right|$ to the smaller of $\left|j_1 + j_2\right|$, $\left|j_3 + j_4\right|$. To obtain the total isotopic spin impurity due to excitation of a pair of particles from any two states of the nucleus consisting only of closed shells, we must sum (70) over all possible values of j_1, j_2, j_3, and j_4 after dividing each term like (70) by the correct energy difference. This gives for the isotopic spin impurity

$$\rho = \sum_{\substack{j_1 < j_2 \\ j_3 < j_4}} \frac{S(j_1 j_2 ; j_3 j_4)}{\left(E_0 - E(j_1 j_2 ; j_3 j_4)\right)^2}$$

(71)

The largest contribution to the sum for C^{12} might be expected to come from the term which has the smallest energy denominator viz. excitation of two $|p^{3/2}$ particles to $|p^{1/2}$ states. For either j_1, j_2 or j_3, j_4 in the same state, however, the term given in (70) vanishes. Since parity is a good quantum number the excitation of particles from $|p^{3/2}$ and $|s_{1/2}$ must be to an odd-parity

"two-particle state". The lowest lying is $1p_{1/2} \, 2S_{1/2}$ and this has

only, $J = 1$ in common with $1p_{3/2} , 1S_{1/2}$. For this excitation (70)

leads to the isotopic spin impurity

$$P_2 = \left(\frac{e^2}{2}\right)^2 2 \cdot 3 \left\{ \left(1p_{1/2} \, 2S_{1/2} | \frac{1}{\lambda_{12}} | 1p_{3/2} \, 1S_{1/2}\right)^2 - \right.$$

$$- 2 \left(1p_{1/2} \, 2S_{1/2} | \frac{1}{\lambda_{12}} | 1p_{3/2} \, 1S_{1/2} 1\right)\left(2S_{1/2} \, 1p_{1/2} | \frac{1}{\lambda_{12}} | 1p_{3/2} \, 1S_{1/2} 1\right)$$

$$\left. + \left(2S_{1/2} \, 1p_{1/2} | \frac{1}{\lambda_{12}} | 1p_{3/2} \, 1S_{1/2} 1\right)^2 \right\}$$

Since $\left(1p_{1/2} \, 2S_{1/2} 1 | \frac{1}{\lambda_{12}} | 1p_{3/2} \, 1S_{1/2} 1\right) = 0$ we are left with

$$P_2 = \frac{\left(\frac{e^2}{2}\right)^2}{(E_0 - E_1)^2} 2 \cdot 3 \left(2S_{1/2} \, 1p_{1/2} 1 | \frac{1}{\lambda_{12}} | 1p_{3/2} \, 1S_{1/2} 1\right)^2$$

or evaluating the matrix element

$$P_2 = 7.72 \times 10^{-3} \left(\frac{e^2/R}{E_0 - E_1}\right)^2 \tag{72}$$

This is to be compared with the value of p for C^{12} given in (63).

$$\frac{P_2}{P} = 1.27 \times 10^{-3} \tag{73}$$

The energy denominators for excitation of $1p_{3/2} \, 1S_{1/2}$ to

all other states are twice as large, making the contributions negligible

from all states except possibly a $2p_{3/2} \, 2S_{1/2}$ combination. From

the results for the two particle mixing and the ensuing discussion one

sees that these matrix elements might exceed those to $1p_{1/2} \, 1S_{1/2}$ by

a factor of 10. Considering the additional factor of ~ 2 in the

energy denominator of $2p_{3/2} 2s_{1/2}$ these contributions to (71) cannot exceed (72) by more than 50% Therefore even in this case the contributions from doubly excited states is negligible.

We conclude--States formed from the ground state of closed shell nuclei by excitation of two nucleons contribute negligibly to isotopic spin impurity of the ground state.

(4.413) An Equivalent Potential for Closed jj Shells

If we return to any of the equations (47), (48), or (49) we can see that if the sum over the states $\varphi_\gamma(2)$ were performed and then the integration over just the coordinate of the second particle performed, we should have for the first term of (48) a matrix element for the excitation of a single nucleon moving in the equivalent potential of the other nucleons in the closed shells. The second term, which we call the exchange term, is actually equal to the first term when $j_1 = j_2$ and $m_1 = m_2$ (remember $j_3 = j_2$ according to (54) and (55)) and reduces the sum over m_1 when $j_1 = j_2$ by a factor of

$$\left(1 - \frac{2j_1+1}{(2j_1+1)^2} \right) = \frac{2j_1}{(2j_1+1)}$$

This means that if one wished to calculate (47) by neglecting the second term, one should multiply the sum over the $j_1 = j_2$ terms by

$$\left(\frac{2j_1}{2j_1+1} \right)$$ i.e. by 3/4 when $j_1 = j_2 = 1p_{3/2}$ or by $\frac{1}{2}$ when $j_1 = j_2 = 1s_{1/2}$. This is merely subtracting the self-perturbation effect. The terms of the second sum have no simple interpretation in terms of an equivalent potential (TAS 10^6) when $m_1 j_1 \neq m_2 j_2$. These "exchange" integrals will be much smaller, however, than the "direct" integrals.

The equivalent potential is given by the sum

$$V_E(r_2) = \sum_{j m_j} \int \varphi_{j m_j}(1) \frac{1}{r_{12}} \varphi_{j m_j}(1) \qquad (74)$$

carried out over all closed shells of the nucleus. The integral in (74) can be expressed in terms of "spinless" wave functions

$$\chi_{n_\ell}^{m_\ell} = \frac{R_{n\ell}}{\hbar} Y_\ell^m(\vartheta,\varphi)$$

by the transformation

$$V_E = \sum_{m_\ell, ms} (\ell\tfrac{1}{2} j m_j | \ell m_\ell \tfrac{1}{2} ms)(\ell m_\ell \tfrac{1}{2} ms | \ell \tfrac{1}{2} j m_j) \cdot$$
$$\cdot \int \chi_{n_\ell}^{m_\ell}(1) \frac{1}{\hbar_{12}} \chi_{n_\ell}^{m_\ell}(1) \tag{74}$$

Now using a Slater-type expansion of the integral (TAS 8[6]), we can write

$$\int \chi_{n_\ell}^{m_\ell}(1) \frac{1}{\hbar_{12}} \chi_{n_\ell}^{m_\ell}(1) = \sum_k m_6^{(k)}(\ell;\ell)(\ell m_\ell | P_k(\cos\omega_{12}) | \ell m_\ell) \tag{76}$$

where

$$m_6^{(k)}(\ell,\ell) = \frac{2k+1}{2} \int R_{r\ell}(1) R_{n\ell}(1) \frac{1}{\hbar_{12}} P_k(\cos\omega_{12}) d(\cos\omega_{12}) d\hbar_1$$
$$\cos\omega_{12} \equiv \frac{\vec{\hbar}_1 \cdot \vec{\hbar}_2}{\hbar_1 \hbar_2} \tag{77}$$

Now choose the coordinate system so that $\vartheta_2 = 0$ and $\cos\omega_{12} = \cos\vartheta_2$. From equations (47) and (52) of Racah (R42)

$$(\ell m_\ell | P_k(\cos\omega_{12}) | \ell m_\ell) = (-1)^{m_\ell}(2\ell+1) V(\ell\ell k; 000) \cdot$$
$$\cdot V(\ell\ell k; -m_\ell m_\ell 0) \tag{78}$$

where the $V(abc; \alpha\beta\gamma)$ are the functions previously introduced as related to the Clebsch-Gordon coefficients. With this notation V_E becomes

$$V_E(\hbar_2) = \sum_{j,k} m_6^{(k)}(\ell,\ell)(2\ell+1) V(\ell\ell k; 000)(2j_1+1)$$
$$W(\ell j \ell j; \tfrac{1}{2} k) \sum_{m_j} (-1)^{m_j + k} V(j j k; -m_j m_j 0) \tag{79}$$

with $W(\ell j \; \ell j; \tfrac{1}{2}k)$ being a Racah coefficient. In summing over j the ℓ values also change. Only even k appear in the sum since $V(\ell\ell k; 000)$ vanishes unless $\ell + \ell + k \equiv 2g$ is even.

The second sum in (49) in the same way gives rise to a sum over "non-diagonal" integrals of the form

$$\int \varphi_{j_1 m_{j_1}}(1) \frac{1}{r_{12}} \varphi_{j_2 m_{j_2}}(1) = \sum_{\substack{m\ell_1, m_{s_1}, \\ m\ell_2, m_{s_2}}} (\ell_1 \tfrac{1}{2} j_1, m_{j_1}, | \ell_1, m\ell_1, \pm m_{s_1})(\ell_2 m\ell_2 \; \tfrac{1}{2} m_{s_2} | \ell_2 \tfrac{1}{2} j_2 m_{j_2})$$
$$\cdot \int \chi^{m\ell_1}_{n_1 \ell_1}(1) \frac{1}{r_{12}} \chi^{m\ell_2}_{n\ell_2}(1) \tag{80}$$

Just as we obtained (79) we obtain for the sum over the "incomplete" matrix elements (80).

$$V_E' = \sum_{j, m_{j_1}} \int \varphi_{j_1 m_{j_1}}(1) \frac{1}{r_{12}} \varphi_{j_2 m_{j_2}}(1) = \sum_{j_1, k} m_C^{(k)}(\ell_1, \ell_2)$$
$$\sqrt{(2\ell_1+1)(2\ell_2+1)(2j_1+1)(2j_2+1)} \; V(\ell_1 \ell_2 k; 000)(-1)^{k - \ell_1 - \ell_2 + \frac{1}{2} + m_{j_2}}$$
$$W(\ell_1 j_1 \ell_2 j_2 ; \tfrac{1}{2} k) \; V(j_1 j_2 k; -m_{j_2} m_{j_2} 0) \tag{81}$$

with

$$m_C^{(k)}(\ell_1, \ell_2) = \frac{2k+1}{2} \int R_{n_1 \ell_1}(1) R_{n_2 \ell_2}(1) \frac{1}{r_{12}} P_k(\cos\omega_{12}) d(\cos\omega_{12}) dr_1 \tag{82}$$

The expressions (81) and (82) are quite general and suffice to determine the effective potential produced by any scalar two-body force. In the next section, however, we limit ourself to the Coulomb potential.

(4.414) Electrostatic Potential Produced by Protons in He^4 and C^{12}

The sum in (74) has been carried out over all the states

which protons (or neutrons) occupy in a closed shell nucleus and therefore $V_E(r_\chi)$ is the electrostatic potential produced by protons in the nucleus. We shall now proceed to calculate $V_E(r_\chi)$ for simple closed shell nuclei He^4 and C^{12} to compare this $V_E(r)$ with the electrostatic potential which is produced by a uniform sphere of charge. In this way we shall justify the use of this latter potential in the calculation of the perturbation of the states of one, two, and three particles outside a core of closed shells.

We shall derive the $V_E(r)$ for C^{12} first and shall immediately be able to give V_E for He^4 also. Since for C^{12}, k can only be 0 or 2 the sum in (79) reduces to

$$V_E(h_\chi) = 12\, m_6^{(0)}(1p;1p)\, V(110;000)\, W(1\tfrac{3}{2}1\tfrac{3}{2};\tfrac{1}{2}0)$$
$$\cdot \sum_{m_j=-3/2}^{3/2} (-1)^{m_j+1/2}\, V(\tfrac{3}{2}\tfrac{3}{2}0;-m_j\,m_j\,0)$$
$$+12\, m_6^{(2)}(1p;1p)\, V(112;000)\, W(1\tfrac{3}{2}1\tfrac{3}{2};\tfrac{1}{2}0)$$
$$\cdot \sum_{m_j=-3/2}^{3/2} (-1)^{m_j+1/2}\, V(\tfrac{3}{2}\tfrac{3}{2}2;-m_j\,m_j\,0)$$
$$+2\, m_6^{(0)}(1s;1s)\, V(000;000)\, W(0\tfrac{1}{2}0\tfrac{1}{2};\tfrac{1}{2}0)$$
$$\sum_{m_j=-1/2}^{1/2} (-1)^{m_j+1/2}\, V(\tfrac{1}{2}\tfrac{1}{2}0;-m_j\,m_j\,0) \tag{83}$$

The sum over the $V(3/2\ 3/2\ 2;\ -m_j\ m_j\ 0)$ in the second term can be shown to vanish, so one has only terms in which $k = 0$. By considering the series expression for the $V(abc;\ \alpha\beta\gamma)$ one can show that

$$\sum_\alpha (-1)^{\alpha+1/2}\, V(jj'0;-\alpha\,\alpha\,0) = (-1)^{j-1/2}\sqrt{2j+1}$$

Combining this with

$$W(\ell j\ell j';\tfrac{1}{2}0) = \frac{(-1)^{\ell+j-1/2}}{\sqrt{(2\ell+1)(2j+1)}} \qquad V(\ell\ell0;000) = \frac{(-1)^\ell}{\sqrt{2\ell+1}}$$

we derive the result

$$(2\ell+1)\,V(\ell\ell 0;000)\,W(\ell j \ell j;\tfrac{1}{2}0)\sum_{m_j}(-1)^{m_j+\frac{1}{2}}\,V(jj0;-m_j m_j 0)=1 \tag{84}$$

From this it follows that

$$V_E(\lambda_2) = 4\,m_6^{(0)}(1p;1p) + 2\,m_6^{(0)}(1s;1s) \tag{85}$$

for C^{12}. The potential for He^4 is obviously

$$V_E(\lambda_2) = 2\,m_6^{(0)}(1s;1s) \tag{86}$$

Performing the integration indicated in (77) we find

$$m_6^{(0)}(1s;1s) = \frac{e^2}{R}\,\frac{1}{\xi}\,Erf(\sqrt{\nu}\,R\xi) \tag{87}$$

$$m_6^{(0)}(1p;1p) = e^2\,2\sqrt{\frac{\nu}{\pi}}\left\{-\frac{1}{3}e^{-\nu R^2\xi} + \frac{1}{2}\sqrt{\frac{\pi}{\nu}}\,\frac{1}{\xi}\,Erf(\sqrt{\nu}\,R\xi)\right\} \tag{88}$$

where

$$\xi \equiv \frac{\lambda_2}{R}$$

$$Erf\,x = \frac{2}{\sqrt{\pi}}\int_0^x dy\,e^{-y^2}$$

We have used $\nu = \frac{5}{2R^2}$ in our calculations with harmonic oscillator wave functions in the 1p shell and will therefore proceed to discuss the effective potentials for this value of ν. Later we shall see how the discussion applies to (87) and (88) for any value of ν.

The asymptotic forms of (87) and (88) are

$$\mathcal{M}_6^{(0)}(1s; 1s) \sim \sqrt{\frac{10}{\pi}}\left(1 - \frac{5}{6}\xi^2\right)\frac{e^2}{R} \qquad \xi \ll 1$$

$$\sim \frac{e^2}{\lambda_2} \qquad \xi \gg 1$$

$$\tag{89}$$

$$\mathcal{M}_6^{(0)}(1p; 1p) \sim \frac{4}{3}\sqrt{\frac{5}{2\pi}} + 0\left(\xi^4\right) \qquad \xi \ll 1$$

$$\sim \frac{e^2}{\lambda_2} \qquad \xi \gg 1$$

$$\tag{90}$$

These expressions are very interesting as a representation of the
physical picture. The potential produced by $1s$ particles is an
oscillator potential for small ξ, similar to the potential of a
uniform sphere of charge. This is a reflection of the fact that a
$1s$ wave function represents a distribution of charge which is non-
zero at the origin and is constant to first order in ξ. The
constants are not precisely those which one finds for a uniform sphere
of charge

$$V_u = \frac{3Ze^2}{2R}\left(1 - \frac{1}{3}\xi^2\right)$$

$$\tag{91}$$

but these are determined by the overall charge distribution and depend
on γ. At large distances $\xi \gg 1$ the potential must become
$\frac{e^2}{\lambda_2}$. For the $1p$ wave function the potential at the
origin is constant. This reflects the fact that the $1p$ wave function
represents a particle in an orbit, so that one is "inside" the charge
at the origin. By Gauss' Theorem the potential must be constant here.

The potential produced by the protons in He^4 is given by (86) and is

$$V_E(\xi) = \frac{e^2}{R} \frac{2}{\xi} \, Erf\left(\sqrt{\frac{5}{2}} \, \xi\right)$$

$$(92)$$

The potential produced by the protons in C^{12} is given by (85) and is

$$V_E(\xi) = \left(\frac{e^2}{R}\right)^2 \left\{ -\frac{8}{3}\sqrt{\frac{5}{2\pi}} \, e^{-\frac{5\xi^2}{2}} + \frac{6}{\xi} \, Erf\left(\sqrt{\frac{5}{2}} \, \xi\right) \right\}$$

$$(93)$$

The asymptotic forms of (93) are

$$V_E(\xi) \sim \frac{e^2}{R} \frac{28}{3}\sqrt{\frac{5}{2\pi}} \left(1 - \frac{5}{14}\xi^2\right) \qquad \xi \ll 1$$

$$\sim \frac{e^2}{R} \frac{6}{\xi} \qquad\qquad \xi \gg 1$$

$$(94)$$

The numerical values of these potentials as a function of ξ are given in Tables VIII and IX. Also given are the potentials of a uniform sphere of charge for Z = 2 and Z = 6

$$Z=6 \qquad V_u = \frac{e^2}{R} 3\left(1 - \frac{1}{3}\xi^2\right) \qquad \xi \leq 1$$

$$= \frac{e^2}{R} \frac{2}{\xi} \qquad\qquad \xi \gg 1$$

$$Z=12 \qquad V_u = \frac{e^2}{R} 9\left(1 - \frac{1}{3}\xi^2\right) \qquad \xi \leq 1$$

$$= \frac{e^2}{R} \frac{6}{\xi} \qquad\qquad \xi \gg 1$$

$$(95)$$

The potential for He^6 is also given for the value of $\nu = \dfrac{3}{2R^2}$ which is perhaps more reasonable for this nucleus. The data in Tables VIII and IX are plotted on the graph which gives $\dfrac{R}{e^2} V_u(\xi)$ and $\dfrac{R}{e^2} V_E(\xi)$ for He^4 and C^{12} as functions of ξ.

These graphs show very clearly that the electrostatic potential for a uniform sphere is quite a good approximation to that produced by protons in the proper harmonic oscillator states for single particles. By choosing $\nu = \dfrac{2.925}{R^2}$ instead of $\dfrac{2.5}{R^2}$ the two potentials can be made to coincide at the origin and differ by less than 5% everywhere. The effect of changing ν is most easily seen by writing the expressions for $\mathcal{M}_6^{(0)}(1s\,;\,1s)$ and $\mathcal{M}_6^{(0)}(1p\,;\,1p)$ in terms of a new variable $\eta = \sqrt{\nu}\, r_2$. Equations (87) and (88) become simply

$$\mathcal{M}_6^{(0)}(1s\,;\,1s) = e^2 \frac{\sqrt{\nu}}{\eta} \operatorname{erf}\eta$$

$$\mathcal{M}_6^{(0)}(1p\,;\,1p) = e^2 2\sqrt{\frac{\nu}{\pi}} \left\{ -\frac{1}{3} e^{-\eta^2} + \frac{\sqrt{\pi}}{2} \frac{1}{\eta} \operatorname{erf}\eta \right\} \tag{96}$$

$$\tag{97}$$

The effect of ν is to determine first, the value of the potential at the origin, and secondly, to accelerate the approach of the potential to the asymptotic form $1/r$. Using (96) and (97) we obtained the last two columns in Table VIII.

TABLE VIII

Coulomb Potential of Protons in He^4

ξ	$\frac{R}{e^2} V_u(\xi)$	$\frac{R}{e^2} V_E(\xi)$ $\nu = \frac{3}{2R^2}$	$\frac{R}{e^2} V_E(\xi)$ $\nu = \frac{5}{2R^2}$
0	3.00	2.77	3.57
0.2	2.96	2.76	3.49
0.4	2.84	2.56	3.14
0.6	2.64	2.35	2.74
0.8	2.36	2.09	2.31
1.0	2.00	1.84	1.95
1.4	1.43	1.41	1.43
1.8	1.11	1.11	1.11
2.2	0.909	0.909	0.909
2.6	0.769	0.769	0.769
3.0	0.667	0.667	0.667

TABLE IX

Coulomb Potential of Protons in C^{12}

ξ	$\frac{R}{e^2}V_u(\xi)$	$\frac{R}{e^2}V_E(\xi); \nu=\frac{5}{2R^2}$	ξ'	$\frac{R}{e^2}V_E(\xi'); \nu=\frac{2.925}{R^2}$
0	9.00	8.33	0	9.00
0.2	8.88	8.20	0.185	8.86
0.4	8.52	7.84	0.371	8.47
0.6	7.92	7.23	0.556	7.81
0.8	7.08	6.47	0.741	6.98
1.0	6.00	5.65	0.926	6.10
1.4	4.29	4.28	1.30	4.63
1.8	3.33	3.33	1.67	3.59
2.2	2.73	2.73	2.13	2.95
2.6	2.31	2.31	2.41	2.49
3.0	2.00	2.00	2.78	2.18

Coulomb Potential for Harmonic Oscillator Wave Functions and for Uniform Model

$\frac{e^2}{R} \, V \, (\cdot 5)$
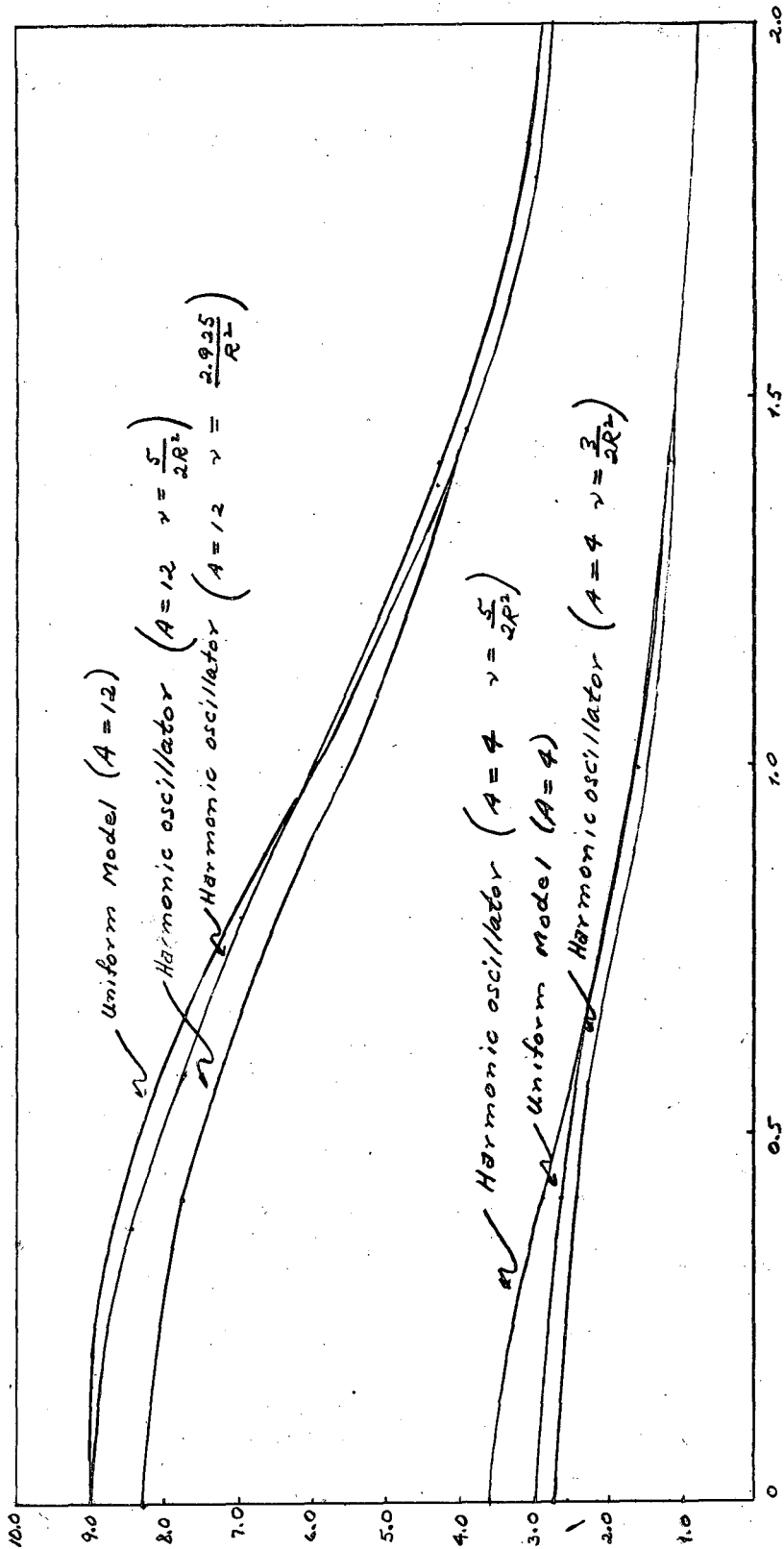
In summarizing the results of this section we can say

(1) the use of the Coulomb potential of a uniform

sphere is justified for calculating the mixing

of two-particle states when the core is not excited,

(2) the electrostatic potential for the C^{12} nucleus

produced by protons in harmonic oscillator states

differs from that predicted by the uniform model by

less than 10%, perhaps much less.

The last remark has great significance for the interpretation of high

energy electron scattering experiments on C^{12}.

(4.414) Simplified Calculation of Impurity in a Closed
Shell Nucleus

The close agreement between the electrostatic potential

produced by a closed shell of protons and the potential produced by a

uniform sphere of charge suggests a very simple method of calculating

very approximately the impurity in the ground states of closed shell

nuclei. For the equivalent potential $V_E(r)$ we use

$$V_E(r) = \int \frac{1}{r_{12}} \left(\sum_{jm} |\varphi_{jm}(2)|^2 \right) d\tau_2$$
$$= \frac{3(A-1)}{4 R^3} \left(R^2 - \frac{r^2}{3} \right)$$

(98)

In using $V_E(r)$ we must be careful to subtract the self-perturbation

effect, i.e. the perturbation of an orbit by the particle itself. This

is subtracted in the exact calculation by the appearance of the proper

matrix element in the second term of (47). The equivalent potential

(98) gives the quantity $\alpha(1p^{3/2})$ defined in (56) as

$$\left(E_0 - E_{1p^{3/2}} \right) \alpha(1p^{3/2}) = \frac{e^2}{2} \left(\frac{3}{4} \right) \frac{A}{4R^3} (2p|r^2|1p) = \frac{\sqrt{10}}{4} \frac{e^2}{R}$$

The factor of $3/4$ approximately corrects for the self-perturbation; it is just $\frac{2j}{2j+1}$. Similarly .

$$\left(E_0 - E_{1s_{1/2}}\right)\Big/ \propto (1s_{1/2}) = \frac{1}{3}\sqrt{\frac{2}{3}}\frac{e^2}{R} = 0.408\frac{e^2}{R}$$

These quantities now give for the perturbation of C^{12} .

$$\rho = 5.64\left(\frac{e^2/R}{E_0 - E_1}\right)^2$$

This differs from the exact result only by 7%, but this accuracy is at least partly coincidence.

(4.42) Isotopic Spin Impurity for Closed Shell Plus Two Nucleons

Having discussed separately the isotopic spin impurity for the ground states of two nucleons outside a closed shell and for the ground state of a closed shell nucleus, we are now in a position to give the isotopic spin impurity for a closed shell nucleus with two holes or two outside particles. Just as we did in (47) we can give the matrix elements between the ground state ψ_0 and various kinds of excited states. Let χ_ν designate particle states found in the core;

χ_γ , particle states occupied by the outside nucleons. The various excited states ψ_s can be grouped into classes distinguished by the possible ways of exciting nucleons

(A) Single nucleon excited

    (1) core nucleon

    (2) "outside" nucleon

(B) Two nucleons excited

    (1) both core nucleons

    (2) both "outside" nucleons

    (3) one core and one outside nucleon.

The corresponding matrix elements of the Coulomb perturbation

$$\mathcal{V} = \sum_{i > j} \mathcal{V}_{ij}$$

given in (42) are as follows:

(A1) $(\Psi_v, C\Psi_0) = \sum_{\nu} \left[\int \psi_\sigma^*(1)\psi_\tau^*(2) \mathcal{V}_{12} \psi_\mu(1)\psi_\nu(2) - \int \psi_\sigma^*(1)\psi_\tau^*(1)\mathcal{V}_{12}\psi_\nu(1)\psi_\mu(2) \right]$

$\qquad + \sum_{\nu} \left[\int \psi_\sigma^*(1)\chi_\tau^*(2) \mathcal{V}_{12} \psi_\mu(1)\chi_\nu(2) - \int \psi_\sigma^*(1)\chi_\tau^*(2)\mathcal{V}_{12}\chi_\nu(1)\psi_\mu(2) \right]$

(A2) $(\Psi_v, C\Psi_0) = \sum_{\nu} \left[\int \chi_\sigma^*(1)\psi_\tau^*(2) \mathcal{V}_{12} \chi_\mu(1)\psi_\nu(2) - \int \chi_\sigma^*(1)\psi_\tau^*(2)\mathcal{V}_{12}\psi_\nu(1)\chi_\mu(2) \right]$

$\qquad + \sum_{\nu} \left[\int \chi_\sigma^*(1)\chi_\tau^*(2) \mathcal{V}_{12} \chi_\mu(1)\chi_\nu(2) - \int \chi_\sigma^*(1)\chi_\tau^*(2)\mathcal{V}_{12}\chi_\nu(1)\chi_\mu(2) \right]$

(B1) $(\Psi_v, C\Psi_0) = \int \psi_\sigma^*(1)\psi_\tau^*(2) \mathcal{V}_{12} \psi_\mu(1)\psi_\nu(2) - \int \psi_\sigma^*(1)\psi_\tau^*(2)\mathcal{V}_{12}\psi_\nu(1)\psi_\mu(2)$

(B2) $(\Psi_v, C\Psi_0) = \int \chi_\sigma^*(1)\chi_\omega^*(2) \mathcal{V}_{12} \chi_\mu(1)\chi_\nu(2) - \int \chi_\sigma^*(1)\chi_\omega^*(2) \mathcal{V}_{12}\chi_\nu(1)\chi_\mu(2)$

(B3) $(\Psi_v, C\Psi_0) = \int \psi_\sigma^*(1)\chi_\omega^*(2)\mathcal{V}_{12} \psi_\mu(1)\chi_\nu(2) - \int \psi_\sigma^*(1)\chi_\omega^*(2) \mathcal{V}_{12} \chi_\nu(1)\psi_\mu(2)$

Inspection of (A1) shows it to be the sum of the similar matrix with just a closed shell and a term which can be regarded as the contribution of the outside nucleons to the equivalent potential. This contribution should be small for large nuclei but in any case is easily taken into account approximately. (A2) has been calculated, the second line exactly, the first line by an equivalent potential. This procedure neglects the exchange term, but this can be shown to be $\sim$ 5% of the direct term. The matrix elements (B1) and (B2) have both been treated exactly and shown to be negligible compared to type A matrix elements. For similar reasons (necessity for recoupling of excited states, very

different space functions, large energy denominators, lack of a sum for matrix elements between states differing by two particles (TAS 5^6)) we also expect (B3) to be small.

The conclusion we must draw for nuclei consisting of two holes or two particles outside closed shells is that the isotopic spin impurity of T = 1 nuclei (He^6, Be^6, Be^{10}, C^{10}, etc.) is just the isotopic spin impurity of the core. The isotopic spin impurity of the T = 0 nuclei (Li^6, B^{10}, N^{14}, etc.) is the sum of the impurities for the core and for the two particle state.

The isotopic spin impurity in the state of the outside neutron and proton in Li^6 is 3.3. x 10^{-5}. This is negligible compared to the impurity of the core as one sees from the value of p = 2.0 x 10^{-3} for He^4. The core impurity is increased in Li^6 by the outside particle and decreased by the slightly larger spring constant γ and slightly larger radius. From the results on the reduction of the core matrix elements by an equivalent potential and (A1), we see that the effect of the outside nucleons can be included by multiplying $\alpha(1s_{1/2})$ by $\sim \frac{6}{4}$. Making the other corrections we find for the isotopic spin impurity of the ground state of Li^6 p = 2.1 x 10^{-3}.

In N^{14} the isotopic spin impurity is again due mostly to the core. In making the correction for the effect of the two outside nucleons on the impurity of the core we must multiply $\alpha(1p_{3/2})$ and $\alpha(1s_{1/2})$ by a factor of 14/12. The result after all corrections is p = 3.9 x 10^{-3} for N^{14}.

We can also treat the ground state of B^{10} by regarding it as a closed shell nucleus with two holes in the $1p_{3/2}$ shell. In this case the isotopic spin impurity does not come from excited states of

type (A2), but only from (A1) states. Making the corrections for the lack of two nucleons to interact with the core and the different radii, we find for the ground state of B^{10} $p = 2.2 \times 10^{-3}$.

This value of the isotopic spin impurity in N^{14} may be compared with that of Radicati who obtains 2.5×10^{-3} from a less rigorous calculation. The principal ways in which Radicati's work differs from this thesis have been pointed out in the appropriate sections of this thesis and are (1) failure to remove the isotopic scalar parts of the Coulomb interaction, (2) use of $1/r_i$ for the core potential instead of the more correct r_i^2 approximation, (3) neglect of excitation of the isotopic spin state of the core, and (4) use of supermultiplet theory for excited states. Although supermultiplet theory may very well contain some element of the true description of ground states (intermediate coupling seems necessary I54, Z53, L53, T53, T54, Sh54) its validity for excited states seems questionable (see however C53). In LS coupling with the correct core potential and Radicati's assumptions for the state contributing the greatest impurity and the energy separation, the impurity of N^{14} would be $p \sim 1.0 \times 10^{-4}$. The only difference between this corrected value of p and our value for the mixing of isotopic spin state of the outside neutron and proton is that produced by the different 1p, 2p level separations, 30 Mev in our work and 40 Mev in that of Radicati.

The impurity of a N^{14} may here be somewhat underestimated since the effect of two outside particles is seen in a more careful analysis not to be treated in a completely correct way by increasing the effective potential of the core. The presence of two outside particles does in fact introduce the possibility of excitation of a core particle to

single particle state of different j with recoupling of the spins of the two outside particles, the excited particle, and the core to the same spin as the initial state of the two outside particles (which was also the spin of the nucleus in its ground state). The matrix element to such an excited state however should not be particularly large and the energy denominator is not decreased appreciably since the energy difference which enters is still that between nuclear states of the same J in different T multiplets. The additional contribution of these states should therefore not be large. The effect of perturbations by excited states formed in such a way but having the same isotopic spin as the ground state could, on the other hand, be more important. The extreme single particle model would predict an energy denominator about half of that which we used when the particle excited from the core did not change its j value. Even here the contribution of these matrix elements to the impurity already correctly found should be negligible.

5. Isomeric Transitions

Of the several methods by which one may test the validity of the isotopic spin quantum number, the existence of certain selection rules on the electric dipole transitions has been the only phenomenon which has been experimentally investigated. These selection rules were first derived on the basis of supermultiplet theory by Trainor (T52b) but it was pointed out by Radicati (R52) and Christy (C52) that these restrictions on electric dipole radiation actually follow from very general properties of the electromagnetic interaction with a system of nucleons. A more complete statement of the selection rules on isotopic spin change in El transitions was then given by Gell-Mann and Telegdi (G53) who also discussed in considerable detail the effect of these selection rules on

the absorption cross section for γ -rays.

The experimental investigation of the validity of the isotopic spin selection rules was undertaken by Wilkinson (W53I) who first determined a reliable method of predicting uninhibited El radiation widths and then checked for the uninhibited γ -rays expected in nuclei belonging to the same isobaric triad as the nucleus in which selection rules were predicted to operate. Wilkinson (W53a, J53, W53I, II, III, IV) was then able to isolate a number of strongly inhibited γ -rays which were violations of the selection rules, and thus to place upper and lower limits on the isotopic spin impurity in several nuclei. The results of Wilkinson were thrown into doubt by the remark of Gell-Mann and Telegdi (G53) that higher order terms in the El moments are sufficient to promote "forbidden" El transitions in an even greater amount than the isotopic spin impurity would provide. A more careful investigation of these higher order terms, however, shows them to produce effects $\sim$ 1000 times smaller than the effects of isotopic spin impurity in the transitions studied by Wilkinson. Wilkinson's estimates of isotopic spin still stand, subject to the somewhat questionable predictability of the El radiation widths.

(5.1) Inhibition of El Transitions

The non-relativistic Schrodinger hamiltonian for the interaction of particle of charge e and mass M with a radiation field represented by the vector potential A is (Kr38)

$$\mathscr{H} = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 - \mu\,\vec{\sigma}\cdot\vec{\nabla}\times\vec{A}$$

$$(1)$$

The quantity $\mu \vec{\sigma}$ is the magnetic moment in magnetons $\beta = \dfrac{e\hbar}{2mc}$.

Expanding the quadratic term in $\mathcal{H}$ and discarding those terms in A^2, we obtain the approximate hamiltonian

$$H = \frac{p^2}{2m} - \frac{e}{2mc}\left(\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A}\right) - \mu\vec{\sigma}\cdot\vec{\nabla}\times\vec{A}$$

(2)

where we have taken the static Coulomb field to be zero, $\varphi = 0$.

According to the usual perturbation approach to radiation theory one

assumes the particles to be in eigenstates of a more complete hamiltonian

for nucleons would contain also the terms involving the central nuclear

potential, the electrostatic interaction of the particles, etc.

Correspondingly the radiation field will also be in some eigenstate

describable by a quantum analogue of Maxwell's equations. The process

of radiation is then describable as a transition of the system of

particles and of the radiation field between two of their respective

eigenstates, these transitions being induced by the parts of the complete

hamiltonian which couple the particle field with the radiation field.

From (2) this interaction hamiltonian is just

$$H_I = -\frac{e}{mc}\left(\vec{A}\cdot\vec{p}\right) - \mu\vec{\sigma}\cdot\left(\vec{\nabla}\times\vec{A}\right)$$

(3)

where we have written (3) in the unsymmetrized form which the gauge

condition $\vec{\nabla}\cdot A = 0$. For the interaction of protons with an

electromagnetic field one has an interaction of the form of (3) with

$\mu = \mu_p$, the anomalous magnetic moment of the proton and $m = M$

the proton mass

$$H_I = -\frac{e}{Mc}\left(\vec{A}\cdot\vec{p}\right) - \mu_p\,\vec{\sigma}\cdot\vec{\nabla}\times\vec{A}$$

(4)

For neutrons, which have no charge, but only an anomalous magnetic moment μ_N the interaction (3) reduces to

$$H_{\mathcal{I}}^N = -\mu_N \, \vec{\sigma} \cdot (\vec{\nabla} \times \vec{A})$$

(5)

Using the isotopic spin formalism the interaction of a system of A nucleons with the radiation field is just

$$H_{\mathcal{I}} = -\sum_{i=1}^{A} \left\{ \frac{e}{mc} (\vec{p}_i \cdot \vec{A})(\tfrac{1}{2} - t_{\zeta i}) + \left[\mu_P \left(\tfrac{1}{2} - t_{\zeta i} \right) + \mu_n \left(\tfrac{1}{2} + t_{\zeta i} \right) \right] \cdot \, \vec{\sigma}_i \cdot \nabla \times A(\vec{\gamma}_i) \right\}$$

(6)

The neutron proton mass difference has been neglected in writing this formula as is certainly justified in first approximation. This result differs from that of Radicati (R52) and Gell-Mann and Telegdi (G53) by a sign because our e is positive for protons and negative for electrons.

The $H_{\mathcal{I}}$ in (6) is the sum of a part which is a scalar in isotopic spin space H_0 and of a part which is the zero'th component of an isotopic spin vector H_1.

$$H_0 = -\sum_{i=1}^{A} \left[\frac{e}{2mc} \vec{p}_i \cdot \vec{A} + \tfrac{1}{2} (\mu_P + \mu_N) \vec{\sigma}_i \cdot \vec{\nabla} \times \vec{A} \right]$$

(7)

$$H_1 = \sum_{i=1}^{A} \left[\frac{e}{mc} (\vec{p}_i \cdot \vec{A}) + (\mu_P - \mu_N) \vec{\sigma}_i \cdot \vec{\nabla} \times A \right] t_{\zeta i}$$

(8)

If one makes the assumption $(kR) \ll 1$, one can demonstrate that the electric dipole transitions induced by H_0 are proportional to the square of the electric dipole moment between the initial and final nuclear states (B52, Chapter XII).

$$Q_{lm}^{(0)} = e \sum_{k=1}^{A} \int z_k \, Y_{lm}^{*}(\Theta_k, \varphi_k) \, \varphi_f^{*}(1, \cdots, A) \, \varphi_i(1, \cdots, A)$$

$$(9a)$$

Correspondingly El transitions induced by H_1 have the transition probabilities proportional to the square of

$$Q_{lm}^{(1)} = e \sum_{k=1}^{A} \tau_{\zeta k} \int z_k \, Y_{lm}^{*}(\Theta_k, \varphi_k) \, \varphi_f^{*}(1, \cdots A) \varphi_i(1, \cdots, A)$$

$$(9b)$$

If one now neglect nuclear recoil upon emission of a γ -ray, the center of mass of the nucleus remains fixed and $\sum_{k=1}^{A} z_k = 0$. Since (9a) vanishes for electric dipole transitions only H_1 can produce El transitions.

From the fact that H_1 is the zero'th component of a vector in isotopic spin space, the following selection rules follow immediately (W31)

$$H_1 : \qquad \Delta T = 0, \pm 1 \qquad T_\zeta \neq 0$$

$$\Delta T = \pm 1 \qquad T_\zeta = 0$$

The selection rule for $N = Z$ nuclei is a generalization of Radicati's requirement of no $T = 0 \rightarrow T = 0$ transitions. The vanishing of the matrix elements of H_0 makes the selection rules for H_1 those which El transitions satisfy in all nuclei. Their absolute validity for $T_\zeta = 0$ nuclei can be impaired either by the impurity of isotopic spin states or by certain higher order terms in the multipole matrix elements.

In addition to the inhibition of the El transitions by isotopic spin selection rules there exists the possibility of inhibition of El transitions due to the correlation of the neutrons and protons in the nucleus. This effect upon H_1 follows immediately from (9b) where we note that the pairing of neutrons and protons produces terms of equal magnitude but opposite in sign $\sum_{k=1}^{A} \tau_{\zeta k} \lambda_k$. This effect was first pointed out by Delbruck and Gamow (D31), and was later invoked by Bethe (Be37) to explain the anomalously small El transitions thought to occur in heavy nuclei. The evidence adduced by Bethe to support this widely held belief was of several kinds:

(1) anaomalously long-lifetimes in heavy nuclei (10^{-12} sec. versus theoretical estimated 10^{-16} sec. for El)

(2) γ-ray widths for (n, γ) reactions (1/10 ev to a few ev versus 100 ev expected for 4-6 Mev γ -rays)

(3) small Γ_γ for Li^7(p,γ)Be^8 by 17.63 Mev γ -ray

(4) Γ_γ widths observed in (p,γ) capture in F^{19}.

As has been pointed out by Kinsey and Bartholemew (K54) the number of El transitions in heavy nuclei was suddenly reduced when the first accurate tables of internal conversion coefficients appeared (R51). Although several El transitions in heavy nuclei have been announced recently (Be52, S53) and these have rather long lives, even here it has been formed by a complex coupling of several nucleons with consequent reduction of the extreme single particle matrix element for El. That complex coupling can produce the necessary reduction by factors $\sim$ 100 of the single particle matrix elements (W51) has been demonstrated by the work of Lane and Radicati (L54). Their results show that a variation by a factor of 50 of the El matrix element from the 7.48 Mev

state of B^{10} to the ground state is possible. The reason for the small number of true E1 transitions may be simply the absence of low-lying states of the proper spin and parity in heavy nuclei (K54). The evidence given in (1) and (2), which was always circumstantial, is now even doubtful. Turning to (3) we find that the 17.63 Mev state in Be^8 is now suggested to be $1+$ (A52) in spin and parity. If the ground state is $0+$ as one expects for such an even-even nucleus, the parity change required for E1 transitions excludes the evidence in (3). Even if the 17.63 Mev should be 1-, the small Γ_γ may very well be due to the isotopic spin selection rule operating in Be^8. The evidence in (4) is also spurious, since all of the levels in Ne^{20} formed by proton capture in F^{19} are now known to have the wrong spin and parity for E1 transitions to the ground state (A52).

The belief that E1 radiation has the same transition probability as E2, indicating inhibition of E1 by an amount $(kR)^2$ also has been investigated by Kinsey and Bartholemew (K52, K54). Their conclusion, based on one direct comparison of an E1 and an E2 gamma ray in Mg^{25} and on indirect evidence in a series of other nuclei, is that even from states as low as 7 Mev E2 matrix elements are at least an order of magnitude smaller than E1 matrix elements. Kinsey and Bartholemew conclude further that while the Weisskopf formulas (W51) overestimate the E1 rates by about a factor of ten, the ratio of E1 to M1 is correctly predicted to be 200 for $A \sim 30$.

Evidence for the existence of an effect on E1 transitions arising from the collective motion of neutrons and protons in the nucleus can also be drawn from the giant resonances in photo-emission cross sections (B39, B48, L48, P48, M49, S51, M53) which occur at 15-25 Mev excitation

of the product nucleus. These resonances have been interpreted by Goldhaber and Teller as due to relative vibration of two interpenetrating spheres containing separately the neutrons and protons. In this way they have predicted for the dependence on A of the energy at which the maximum of the photo-emission occurs

$$E_m \propto A^{-\frac{1}{6}}$$

(11)

This agrees reasonably well with the experimental result Mc53)

$$E_m \approx 37A^{-0.186} \text{ MeV.}$$

(12)

Levinger and Bethe (L50) have also shown, however, that many of the features of the (n,γ) resonance can be deduced merely by the use of sum rules for electric dipole absorption. The position of the maximum is predicted to occur slightly above the average nucleon kinetic energy, however, and this is constant at $\sim$ 15 Mev on the simple Fermi model of the nucleus. Obviously any features which follow from simple sum rule arguments must follow from any model so the principal success which can be claimed specifically for Goldhaber and Teller's model is that of predicting the A dependence of E_m nearly correctly. Whether a model not embodying the ideas of collective motion of the neutrons and protons can also predict this dependence remains to be seen.

Finally we shall merely repeat the earlier remark that complex coupling of the nucleons can also inhibit the E1 transitions (L54).

Our conclusion from this discussion of the E1 transitions is that at moderate energies ($<$ 15 Mev) there is slight evidence for some inhibition of E1 transitions, but that this inhibition is not

nearly so pronounced as one thought previously. It is not even certain that this remaining apparent diminution of E1 matrix elements is not due to overestimates inherent in the crude nature of the Weisskopf formula.

(5.2) Experimental Results on Isotopic Spin Selection Rules

The establishment of reliable estimates for the uninhibited (by isotopic spin selection rules) widths for E1 transitions is clearly the first step in verifying the isotopic spin selection rules. Wilkinson began his investigation of the selection rules at precisely this point (W53b) by giving a list of E1 transitions and calculating the quantity $(2J + 1)$ $|M|^2$. This quantity, first defined by Goldhaber and Sunyar (G51) is the product of the spin of the initial state times $|M|^2$, the ratio of experimental width Γ_λ to the Weisskopf width (W51). The quantity is used by Wilkinson because his data are taken from (p,γ) and (α,γ) reaction cross sections from which one automatically obtains $(2J + 1)$ Γ_λ by use of the Breit-Wigner formula. The factor $2J + 1$ should be removed for comparison of experiment and theory but is usually not known for high energy resonances (H54). In the E1 transitions chosen by Wilkinson $(2J + 1)$ $|M|^2$ fluctuated by a factor of less than ~ 2.5 from the value of 1/5. Wilkinson's estimates of the uninhibited widths Γ_λ of E1 transitions were therefore made from

$$(2J+1)\,|M|^2 \sim 0.2 \tag{13}$$

or using Weisskopf's formula

$$(2J+1)\,\Gamma_{\gamma x} \sim 0.022\,A^{2/3}\,E_{\gamma x}^3 \quad \text{e.v.} \tag{14}$$

$\Gamma_{\gamma x}$ indicates a partial width.

Several remarks should be made about (14). The previous discussion (5.1) has shown clearly that one should place very little reliability on formulae for E1 widths extrapolated to different nuclei and especially to higher energies. Since Wilkinson's study actually contained one E1 transition in B^{10}, two in B^{11}, one in N^{13}, four in N^{14}, one in O^{15}, one in O^{16}, and one in F^{17} for a total of eleven transitions with only one above 10 Mev, we can probably trust (14) up to $A \sim 20$ and $E_{\gamma x} \lesssim 12$ Mev; within a factor of 2 or 3 either way. The mere presence of the statistical factor $(2J + 1)$ guarantees this much fluctuation.

The next step taken by Wilkinson (W53I) was to verify the existence of uninhibited E1 transitions between two T = 0 states of a T $\neq$ 0 nucleus. This is done by comparing several cross sections for radiative capture of neutrons as calculated from (14) by use of the Breit-Wigner one-level formula with the experimentally measured (n, γ) cross section. The nuclei used were Li^8, Be^{10}, B^{12}, and C^{12} and the levels were respectively at 2.04 Mev, 6.81 Mev, 3.36 Mev, and 8.17 Mev. Although the analysis is far from being convincing as to the agreement between theory and experiment, within an order of magnitude one can conclude that there is no inhibition of the E1 transitions.

Following these preliminaries the E1 transitions in several nuclei were studied for evidence of the operation of the isotopic spin selection rule (W53a, W53II, W53III, W53IV). The results of the analysis of Wilkinson and collaborators is tabulated below.

Nucleus	Level	Energy (Mev)	Limits on Isotopic Spin Impurity
B^{10}	(2-)?	5.11	$< 3 \times 10^{-3}$
N^{14}	$1^- \to 0^+$	$8.06 \to 2.31$	$< 2 \times 10^{-2}$
O^{16}	$1^- \to 0^+$	$7.116 \to$ ground	$> 4 \times 10^{-6}$
	$2^+ \to 3^-$	$6.913 \to 6.137$	$< 10^{-3}$
	$1^- \to 0^+$	$13.09 \to$ ground	$> 3 \times 10^{-2}$
Ne^{20}	$2^- \to 2^+$	$13.70 \to 1.63$	$< 3 \times 10^{-2}$

Although these are the figures adopted by Wilkinson, as is evident from his analysis considerable doubt should be attached to the exact value and in some cases to the order of magnitude. Perhaps the most reliable value is that for N^{14} were the forbidden 8.06 - 2.31 transition is compared with the allowed transition to the ground state. The impurities obtained for the levels of O^{16} are obtained by a comparison of E1 and E2 branching ratios from the two levels (W53a, W53II) using the estimates of Weisskopf. Among the isomeric transitions, however, the E2 have been known for some time (G51) to be unusual because of their large matrix elements, which equal or exceed the Weisskopf estimates. In Mg^{25} Kinsey and Bartholemew find the known E2 to be correctly predicted by the Weisskopf formula. It therefore is quite possible that the impurity estimates are low by an order of magnitude. The limit obtained for B^{10} is based on rather tentative level assignments (J53) but agrees with the theoretical predictions.

All of these experimental results are in agreement with the estimates of isotopic spin impurity given in (4.42). As we expect if the isotopic spin impurity is predominately due to the core, these limits on mixing are very nearly the same for a wide range of energies and in several different nuclei.

(5.3) Higher Order Effects on Isotopic Spin Selection Rules

The selection rules on isotopic spin are based on the vanishing of the H_0 matrix element in the lowest order approximation. There are three corrections to be considered:

(1) neutron-proton mass difference,

(2) dipole matrix element of the spin dependent part of H_0, and

(3) higher order terms in the expressions for the electric dipole moment.

The first of these is easily dismissed by noting that the dipole moment of the nucleus involves $\sum\limits_{k=1}^{A} r_k$. Taking the center of mass as the origin

$$m_p \sum_{k=1}^{Z} r_k + m_N \sum_{k=Z+1}^{A} r_k = 0 = (m_p - m_N)\sum_{k=1}^{Z} r_k + m_N \sum_{k=1}^{A} r_k$$

and therefore

$$\sum_{k=1}^{A} r_k = \left(\frac{m_N - m_p}{m_N}\right)\sum_{k=1}^{Z} r_k \tag{15}$$

where

$$m_p = \text{proton mass} = 1.00758 \text{ amu}$$

$$m_N = \text{neutron mass} = 1.00893 \text{ amu}.$$

We conclude from (15) that the effect of the center of mass and the center of charge nearly coinciding is to reduce single particle transition probabilities by

$$\left(\frac{m_N - m_p}{m_N}\right)^2 \approx 2 \times 10^{-6}$$

Since the isotopic spin impurity produces E1 widths which are $\sim 10^{-3}$ times the single particle widths, the neutron–proton mass difference can be neglected in considering the violation of the selection rules on isotopic spin.

(5.31) Exact Quantum Mechanical Multipole Transition Probabilities

In order to treat the corrections (2) and (3) we shall have to derive expressions for the electromagnetic transition probabilities in terms of exact multipole moments. The treatment of Blatt and Weisskopf (B52) is valid only when (kR) $\ll$ 1 where k is the propagation number for the emitted photon and R is the nuclear radius. Higher order terms

in (kR) will therefore enter the exact electric dipole matrix elements of H_0 and produce transitions which violate the selection rules.

The derivation of the multipole fields for electromagnetic radiation has been given by a number of authors (H35, H36, D39, C40, G46, F50, Wa51, W51, S52, M54) for both the classical Maxwell field and for non-relativistic quantum mechanics using a variety of techniques. Only in the paper of Wallace which treats the classical Maxwell field, however, are the multipole moments given correctly to all orders of kR. Quite recently French and Shimoto (F53) have studied the connection between the longitudinal and transverse electric fields for a bounded source in the long wavelength approximation. In doing this they have also derived exact expressions for the multipole moments, but again only for the Maxwell field.

We shall now present a derivation of the quantum mechanical transition probabilities for emission of electric and magnetic multipole radiation of all orders correct to all orders of (kR). We begin directly with the non-relativistic Schrodinger equation for charged particles in an electromagnetic field.

In terms of the interaction hamiltonian H_I between charged particles and a field, the transition probabilities for the emission of radiation are given by (S44)

$$T_{a \to b} = \frac{2\pi}{\hbar} |H_{ab}|^2 \rho(E) \tag{16}$$

where a and b are indices denoting states of the nucleus. In the use of this formula we note that H_{ab} is not the matrix element of H_I but of the part of H_I arising from the positive frequency component of A, i.e. from A(r) defined in

$$\vec{A} = \vec{A}(\vec{\lambda})e^{-i\omega t} + \vec{A}^{*}(\vec{\lambda})e^{i\omega t}$$

(17)

(cf. B51 (18.16) or Sc49, chapters 8 and 14). Of the two $H_{\underline{x}} - H_{o}$ and H_{1} we are interested only in H_{0}, which can produce violations of the isotopic spin selection rules (10). The expressions for the corresponding transition probabilities for H_{1} would follow trivially from those for H_{0}. In what follows, however, we shall actually use the interaction of equation (3) so that direct comparison of our multipole moments with the usual formulas can be made.

We begin by expanding A into a series characterized by the fact that each term of the series is an eigenfunction of the single particle angular momentum operators J^{2} and J_{z} with Z-component M and a definite parity. This expansion is of the form

$$\vec{A} = \sum_{J=0}^{\infty} \sum_{M=-J}^{J} \vec{A}(J,M;\vec{\lambda},t)$$

$$\vec{A}(J,M,\vec{\lambda},t) = \lambda^{-1}\left[f(J,M;\vec{\lambda},t)\vec{Y}_{J,J,1}^{M} + g(J,M;\vec{\lambda},t)\vec{Y}_{J,J+1,1}^{M}\right.$$

$$\left. + h(J,M;\vec{\lambda},t)\,\vec{Y}_{J,J-1,1}^{M}\right]$$

(18)

where $\vec{A}$ can be a general vector field. The $\vec{Y}_{J,L,1}$ are the vector spherical harmonics discussed by Corben (C40), Goertzel (G46), and Franz (F50) whic are eigenfunctions of J^{2}, J_{z}, L^{2}, and L_{z}. The parity of the spherical harmonics $\vec{Y}_{J,J+1,1}^{M}$ and $\vec{Y}_{J,J-1,1}^{M}$ is $(-1)^{J+1}$ while the parity of $\vec{Y}_{J,J,1}^{M}$ is $(-1)^{J}$. Consequently when one introduces the concept of electric or magnetic multipole fields distinguished by the parity of the magnetic field $\vec{H}$ associated with each, either the

electric field $\vec{\mathcal{E}}$ or the magnetic field $\vec{H}$ can be expanded in a series which contains only $\vec{Y}_{J,J'}^{M}$ (F50). If we now perform the decomposition of $\vec{\mathcal{E}}$ and $\vec{H}$, the electric and magnetic fields, into the positive and negative frequency components

$$\vec{\mathcal{E}}(\vec{r},t) = \vec{\mathcal{E}}(r)\,e^{-i\omega t} + \vec{\mathcal{E}}^{*}(r)\,e^{i\omega t}$$

$$\vec{H}(\vec{r},t) = \vec{H}(r)\,e^{-i\omega t} + \vec{H}(r)\,e^{i\omega t}$$

$$(19)$$

Maxwell's equations for $\vec{\mathcal{E}}(\vec{r})$ and $\vec{H}(\vec{r})$ are

$$c\,\vec{\nabla}\times\vec{H}(\vec{r}) = -i\omega\,\vec{\mathcal{E}}(\vec{r}) \qquad c\,\vec{\nabla}\times\vec{\mathcal{E}}(\vec{r}) = i\omega\,\vec{H}(\vec{r})$$

$$(20)$$

Consequently an expansion of $\vec{\mathcal{E}}(r)$ in the $\vec{Y}_{J,J'}^{M}$ implies the possibility of an expansion of $\vec{H}(r)$ in the vector functions and conversely. The only vector harmonics which need enter our calculations are therefore the $\vec{Y}_{J,J'}^{M}$ which can also be simply expressed as

$$\vec{Y}_{J,J'}^{M} \equiv \vec{X}_{JM}(\vartheta,\varphi) = \frac{\vec{L}\,Y_{JM}(\vartheta,\varphi)}{\sqrt{J(J+1)\hbar^{2}}}$$

$$(21)$$

where $\vec{L}$ is the angular momentum operator $\vec{r}\times\vec{p}$ with $\vec{p} = \frac{\hbar}{i}\vec{\nabla}$ and $Y_{JM}(\vartheta,\varphi)$ is a surface spherical harmonic of order J,M. The multipole fields of order ℓ,m will be characterized by the parity of $\vec{H}$ with the electric multipole field being assigned parity $(-1)^{\ell}$ and the magnetic multipole field parity $-(-1)^{\ell}$. The respective fields and vector potentials are given as follows:

Electric multipole field:

$$\vec{A}_E(\ell,m) = \frac{1}{k^2}\,\vec{\nabla}\times\frac{f_\ell}{\mathcal{k}}\,\vec{X}_{\ell m}$$

$$\vec{E}_E(\ell,m) = \frac{i}{k}\,\vec{\nabla}\times\frac{f_\ell(\mathcal{k})}{\mathcal{k}}\,\vec{X}_{\ell m}$$

$$\vec{H}_E(\ell,m) = \frac{f_\ell(\mathcal{k})}{\mathcal{k}}\,\vec{X}_{\ell m} \tag{22}$$

Magnetic multipole field:

$$\vec{A}_M(\ell,m) = -\frac{i}{k}\,\frac{f_\ell(\mathcal{k})}{\mathcal{k}}\,\vec{X}_{\ell m}$$

$$\vec{E}_M(\ell,m) = \frac{f_\ell(\mathcal{k})}{\mathcal{k}}\,\vec{X}_{\ell m}$$

$$\vec{H}_M(\ell,m) = -\frac{i}{k}\,\vec{\nabla}\times\frac{f_\ell(\mathcal{k})}{\mathcal{k}}\,\vec{X}_{\ell m} \tag{23}$$

The relation between $\vec{A}_E$ and $\vec{E}_E$ follows from Maxwell's equations and the relation

$$\vec{E}(\ell,m) = ik\,\vec{A}(\ell,m)$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = k^2\vec{A} \tag{24}$$

When (22) and (23) are inserted in Maxwell's equations, we find that satisfies

$$\left(\frac{d^2}{d\mathcal{k}^2} - \frac{\ell(\ell+1)}{\mathcal{k}^2} + k^2\right)f_\ell(\mathcal{k}) = 0$$

Since we shall normalize to a sphere of volume V and radius a, we have the solution for $f_\ell(\mathcal{k})$

$$f_\ell (r) = j_\ell (kr)$$

$$f_\ell (a) = j_\ell (ka) = 0 \tag{25}$$

We first derive the transition rates for emission of magnetic multipole radiation of order (ℓ, m). The vector potential will be then

$$\vec{A}_M (\ell, m) = b_\ell \frac{\vec{L}}{\sqrt{\ell(\ell+1)\hbar^2}} j_\ell (kr) Y_{\ell m} (\vartheta, \varphi) \tag{26}$$

where we determine b_ℓ by normalizing to one quantum of energy $\hbar c k$ in the volume V. The energy in the volume V will be

$$E = \frac{1}{4\pi} \int \vec{E}^2 dv = \frac{k^2}{2\pi} \int |\vec{A}(r)|^2 d\tau$$

$$E = \frac{k^2}{2\pi} |b_\ell|^2 \int_0^a 4\pi j_\ell^2 (kr) r^2 dr$$

which is to equal $\hbar c k$. The integral is a monotonic increasing function of a so that the value of the integral obtained by using the asymptotic form of $j_\ell (kr)$

$$r j_\ell (kr) \sim \sin (kr - \tfrac{1}{2} \ell \pi) \tag{27}$$

will become asymptotically correct in the limit of large a. From the value of the integral

$$\int_{cycle} \sin^2 (kr - \tfrac{1}{2} \ell \pi) d(kr) = \pi$$

we deduce

$$\int_0^a 4\pi j_\ell^2(kr)\, r^2 dr \sim 4\pi\left(\frac{\pi}{k^3}\right)\frac{ka}{2\pi} = \frac{2\pi a}{k^2}$$

The value of $|b_\ell|^2$ emerges as

$$|b_\ell|^2 = (\hbar c k)\frac{2\pi}{k^2}\frac{k^2}{2\pi a} = \frac{\hbar c k}{a}$$

$$(28)$$

The vector potential for the magnetic multipole field can therefore be taken as

$$\vec{A}_M(\ell, m) = i\sqrt{\frac{\hbar c k}{a}}\frac{\vec{L}}{\sqrt{\ell(\ell+1)}\hbar^2}\, j_\ell(kr)\, Y_{\ell m}(\vartheta, \varphi)$$

$$(29)$$

From the relation $\vec{A}_E(\ell, m) = -\frac{i}{k}\vec{A}_M(\ell, m)$ we find immediately

$$A_E(\ell, m) = \frac{1}{k}\sqrt{\frac{\hbar c k}{a}}\frac{\nabla \times \vec{L}}{\sqrt{\ell(\ell+1)}\hbar^2}\, j_\ell(kr)\, Y_{\ell m}(\vartheta, \varphi) \qquad (30)$$

the vector potential for the electric multipole field.

Before inserting these expressions (28) and (29) into equation (16) using (3) (for reason given above), we shall transform the matrix element

$$H_{ab} = -\frac{e}{mc}\int \varphi_a^*\vec{p}\varphi_b \cdot \vec{A} - \mu\int \varphi_a^*\vec{\sigma}\cdot(\vec{\nabla}\times\vec{A})\varphi_b$$

into

$$H_{ab} = -\frac{1}{c}\int \vec{J}_{ab}\cdot\vec{A} - \mu\int \varphi_a^*\vec{\sigma}\cdot\nabla\times\vec{A}\,\varphi_b \qquad (31)$$

where

$$\vec{J}_{ab} = \frac{e}{2mc} \left[\varphi_a^* (\vec{p}\, \varphi_b) + (\vec{p}\, \varphi_a)^* \varphi_b \right]$$

the quantum mechanical "transition current". The density of final

states per unit energy follows from the asymptotic form of

(27) and the boundary condition (25) as

$$kR - \frac{1}{2}\ell\pi = m\pi \quad (m \text{ integral})$$

Thus

$$\rho(E) = \frac{1}{\Delta E} = \frac{a}{\hbar c \pi} \tag{32}$$

The magnetic multipole transition probability per unit time and per

unit solid angle for the direction of the emitted photon follows

immediately by combining (16), (29), (31), and (32)

$$T_{ab}^M = \frac{2}{\hbar} \frac{k}{\ell(\ell+1)\hbar^2} \left| -\frac{i}{c} \int \vec{J}_{ab} \cdot \vec{L}\, j_\ell\, Y_{\ell m} - i\mu \int \varphi_a^* \vec{\sigma} \cdot \vec{\nabla} \times \vec{L}\, j_\ell\, Y_{\ell m}\, \varphi_b \right|^2 \tag{33}$$

Similarly,

$$T_{ab}^E = \frac{2}{\hbar} \frac{k}{\ell(\ell+1)\hbar^2} \left| -\frac{1}{\hbar c} \int \vec{J}_{ab} \cdot \vec{\nabla} \times \vec{L}\, j_\ell\, Y_{\ell m} - \mu \hbar \int \varphi_a^* \vec{\sigma} \cdot \vec{L}\, j_\ell\, Y_{\ell m}\, \varphi_b \right|^2 \tag{34}$$

We have transformed the second term of T_{ab}^E by the use of

$$\nabla \times \vec{A}_E (\ell, m) = -ik \vec{A}_M (\ell, m)$$

to obtain a similar form.

We shall now proceed to transform these integrals into forms from which the usual multipole moments will emerge after one further approximation.

A.

$$-\frac{i}{c}\int \vec{J}_{ab}\cdot \vec{L}\, j_\ell(kr)\, Y_{\ell m} = -\frac{ie\hbar^2}{2mc}\int\left(\varphi_a^*\frac{\partial \varphi_b}{\partial x_i} - \varphi_b\frac{\partial \varphi_a^*}{\partial x_i}\right)\cdot$$
$$\cdot\left(x_\ell\frac{\partial}{\partial x_m}\varepsilon_{\ell m i}\right) j_\ell\, Y_{\ell m}$$

The $\varepsilon_{\ell m i}$ is the Levi-Civita tensor density

$$\varepsilon_{\ell m i} = \begin{cases} 1 & \text{even permutation of 1, 2, 3} \\ -1 & \text{odd permutation of 1, 2, 3} \\ 0 & \text{otherwise} \end{cases}$$

Integrating by parts

$$-\frac{i}{c}\int \vec{J}_{ab}\cdot \vec{L}\, j_\ell(kr)\, Y_{\ell m} = -\frac{e\hbar}{mc}\int j_\ell(kr)\, Y_{\ell m}\, \mathrm{div}\,(\varphi_a^*\vec{L}\varphi_b) \tag{35}$$

B.

$$-i\mu\int \varphi_a^*\,\vec{\sigma}\cdot\vec{\nabla}\times\vec{L}\, j_\ell\, Y_{\ell m}\,\varphi_b = \mu\hbar\int \varphi_a^*\,\vec{\sigma}\cdot\left[\vec{\nabla}\left(\frac{\partial}{\partial r}r\right) - r\nabla^2\right] j_\ell Y_{\ell m}$$

where we have used the operator identity

$$\vec{\nabla}\times\vec{L} = i\left[\vec{\nabla}\left(\frac{\partial}{\partial r}r\right) - r\nabla^2\right]$$

From

$$\nabla^2 j_\ell\, Y_{\ell m} = -k^2 j_\ell\, Y_{\ell m}$$

and integration by parts

$$-i\mu\int \varphi_a^*\,\vec{\sigma}\cdot(\vec{\nabla}\times\vec{L})\, j_\ell\, Y_{\ell m}\,\varphi_b = -\mu\hbar\int\left(\frac{\partial}{\partial r}r\right) j_\ell\, Y_{\ell m}\, \mathrm{div}\,(\varphi_a^*\vec{\sigma}\varphi_b)$$
$$+ \mu\hbar k^2\int j_\ell\, Y_{\ell m}\,(\varphi_a^*\,\vec{\sigma}\cdot r\varphi_b) \tag{36}$$

<u>C.</u> In a similar way, by using the continuity equation

$$\vec{\nabla} \cdot \vec{J}_{ab} = - \frac{\partial \rho_{ab}}{\partial t} = i k c \rho_{ab}$$

$$\rho_{ab} \equiv \varphi_a^* \varphi_b$$

we obtain

$$-\frac{1}{\hbar c} \int \vec{J}_{ab} \cdot \vec{\nabla} \times \vec{L} \, j_\ell Y_{\ell m} = -\hbar \int \left[\left(\frac{\partial}{\partial r} j_\ell\right) Y_{\ell m} \varphi_a^* \varphi_b\right.$$

$$\left. - \frac{i k \hbar}{c} \int j_\ell Y_{\ell m} (\vec{r} \cdot \vec{J}_{ab})\right) \tag{37}$$

<u>D.</u> As in A, we can also obtain

$$i \mu \int \varphi_a^* \vec{r} \cdot \vec{L} \, j_\ell Y_{\ell m} \varphi_b = i \mu \hbar \int j_\ell Y_{\ell m} \, div \left(\varphi_a \vec{r} \times \vec{\sigma} \varphi_b\right) \tag{38}$$

Define now the generalized multipole moments

Magnetic:

$$\mathfrak{m}_{\ell m} = - \frac{(2\ell+1)!!}{\hbar^\ell} \frac{e}{mc} \int j_\ell(kr) Y_{\ell m}(\vartheta, \varphi) \, div \left(\varphi_a^* \vec{L} \varphi_b\right)$$

$$\mathfrak{m}'_{\ell m} = - \frac{(2\ell+1)!!}{\hbar^\ell} \mu \left\{ \int j_\ell(kr) Y_{\ell m}(\vartheta, \varphi) div \left(\varphi_a^* \vec{\sigma} \varphi_b\right) \right.$$

$$\left. + k^2 \int j_\ell(kr) Y_{\ell m}(\vartheta, \varphi) \left(\varphi_a^* \vec{\sigma} \cdot \vec{r} \varphi_b\right)\right\} \tag{39}$$

Electric:

$$\mathfrak{Q}_{\ell m} = \frac{(2\ell+1)!!}{(\ell+1)\hbar^\ell} \left\{ e \int \left[\left(\frac{\partial}{\partial r} r\right) j_\ell Y_{\ell m}(\vartheta, \varphi)\right] \varphi_a^* \varphi_b \right.$$

$$\left. + \frac{i k}{c} \int j_\ell(kr) Y_{\ell m}(\vartheta, \varphi) \vec{r} \cdot \vec{J}_{ab}\right\}$$

$$\tag{40}$$

where $(2\ell+1)!! = (2\ell+1)(2\ell-1)(2\ell-3)\ldots 2$ or 1.

Using the relations established in (35), (36), (37), and (38) and the definitions of (39) and (40), the transition probabilities per unit time (integrated over solid angle) are

$$T_{ab}^{M} = \frac{8\pi}{\ell[(2\ell+1)!!]^2} \frac{k^{2\ell+1}}{\hbar} |m_{\ell m} + m'_{\ell m}|^2$$

(41)

$$T_{ab}^{E} = \frac{8\pi(\ell+1)}{\ell[(2\ell+1)!!]^2} \frac{k^{2\ell+1}}{\hbar} |Q_{\ell m} + Q'_{\ell m}|^2$$

(42)

These formulas are exact so far in that no assumption about the magnitude of (kR) has been made. If we make the "long wave length approximation" kR $\ll$ 1, i.e., only low energy gamma rays are considered, we put

$$j_\ell(kr) \sim \frac{(kr)^\ell}{(2\ell+1)!!}$$

Since

$$\nabla^2(r^\ell Y_{\ell m}) = 0$$

the multipole moments defined in (39) and (40) become

$$M_{\ell m}^{(0)} = -\frac{e}{mc} \int r^\ell Y_{\ell m} \, \text{div}(\varphi_a^* \vec{L} \varphi_b)$$

$$M_{\ell m}^{(0)\,'} = -\mu \int r^\ell Y_{\ell m} \, \text{div}(\varphi_a^* \vec{\sigma} \varphi_b)$$

(43)

$$Q_{\ell m}^{(0)} = \int r^\ell Y_{\ell m} \varphi_a^* \varphi_b$$

$$Q_{\ell m}^{(0)\,'} = -\frac{ik}{\ell+1} \mu \int r^\ell Y_{\ell m} \, \text{div}(\varphi_a^* \vec{r} \times \vec{\sigma} \varphi_b)$$

(44)

These are precisely the single particle moments of Blatt and Weisskopf's formulas XII (3.32) p. 599 (B52).

Although no derivation of the exact non-relativistic quantum mechanical multipole transition probabilities have been published, a very recent derivation of the exact multipole moments for the Dirac equation by B. Stech (S54) has appeared. The treatment is similar to the preceding development.

(5.32) Higher Order Contributions of H_0

We have pointed out in (5.3) two types of higher order terms in H_0 which produce contributions to El radiation and therefore violations of the selection rules in $T_5 = 0$ nuclei. We estimate the first of these, the contribution of the spin dependent part of H_0 to El radiation. This contribution appears in $Q_{1m}^{(0)}{}'$ in (44) and hence in (42). We shall show that the transition probability produced by Q_{1m}' is negligible by evaluating $\left| \dfrac{Q_{1m}'}{Q_{1m}} \right|$. Since we are interested in H_0 comparison of (3) with (7) shows that we should use

$$\mu = \tfrac{1}{2}\left(\mu_p + \mu_N\right) \qquad \left(\text{in nuclear magnetons } \frac{e\hbar}{2Mc}\right)$$

$$\frac{e}{mc} = \frac{e}{2Mc}$$

In using $Q_{1m}^{(0)}$ and $Q_{1m}^{(0)}{}'$ we are of course using the extreme single particle picture, but this approximation gives the proper orders of magnitude. To evaluate $Q_{\ell m}$ and $Q_{\ell m}'$ assume that φ_a and φ_b are constant over a sphere of radius R, the radius of the nucleus, and that $Y_{\ell m}$ is of order unity. In $Q_{\ell m}'$ we also take $\vec{\sigma}$ of order unity and $\tfrac{1}{2}\left(\mu_p + \mu_n\right) \sim 0.5 \dfrac{e\hbar}{2Mc}$.

The divergence in $Q'_{\ell m}$ is approximated by $(1/R)$. We then obtain

$$\left| \frac{Q_{1m}^{(0)'}}{Q_{1m}^{(0)}} \right|^2 \sim \frac{1}{25} \left(\frac{\hbar \omega}{Mc^2} \right)^2$$

(45)

for gamma rays of $\sim$ 10 Mev $\left| \frac{Q'_{1m}}{Q_{1m}} \right|^2 \sim 4 \times 10^{-6}$. The contribution of the magnetic moment to El transitions therefore provides a rate 4×10^{-6} times an uninhibited matrix element of H_o. This is negligible compared to the effect of isotopic spin impurities.

The higher order term in Q_{1m} arising from the second term of $j_\ell(kr)$ can be found by using

$$j_\ell(kr) = \frac{(kr)^\ell}{(2\ell+1)!!} - \frac{(kr)^{\ell+2}}{(2\ell+3)!!\,2}$$

The second term in an expansion of

$$Q_{\ell m} = Q_{\ell m}^{(0)} + Q_{\ell m}^{(1)} + \cdots$$

found from successive terms in $j_\ell(kr)$ is

$$Q_{\ell m}^{(1)} = - \frac{(2\ell+1)!!}{(2\ell+3)!!} \frac{k^{\ell+2}}{2\hbar\ell} \left\{ \int (\ell+3) r^{\ell+2} Y_{\ell m} \mathcal{G}_a^* \mathcal{G}_b + \frac{i\hbar}{c} \int r^{\ell+2} Y_{\ell m} r \cdot \vec{J}_{ab} \right\}$$

or

$$Q_{1m}^{(1)} = - \frac{k^2}{5} \left\{ 4 \int r^3 Y_{1m} \mathcal{G}_a^* \mathcal{G}_b + \frac{i\hbar}{c} \int r^3 Y_{1m} r \cdot \vec{J}_{ab} \right\}$$

The ratio of $\left| \frac{Q_{1m}^{(0)}}{Q_{1m}^{(0)}} \right|$ is found by the same approximations as before. The result is

$$\left| \frac{Q'_{1m}}{Q_{1m}} \right|^2 \sim \left(\frac{2}{15} \right)^2 (kR)^4$$

From $(\hbar R) \approx \dfrac{\hbar \omega (Mev)}{137}$ we find that for 10 Mev photons

$$\left| \frac{Q'_{1m}}{Q_{1m}} \right|^2 \sim 10^{-6}$$

We therefore conclude that all the higher order terms in H_0 give rise to Γ'_λ widths for E1 emission which are of order 10^{-6} times the uninhibited widths. The Γ'_λ widths produced by isotopic spin impurities are proportional to the impurity p and are at least 10^{-3} times a "normal" E1 width.

6. β -Decay

The Fermi theory of β -decay (F34) based on field theoretic concepts introduces an interaction hamiltonian

$$\mathcal{H} = G \left\{ \left(O^L \varphi \psi^* \right) O^H Q + \left(O^L \varphi \psi \right)^* O^{H^*} Q^* \right\} \tag{1}$$

between the field describing the nucleon, the electron, and the neutrino (or anti-neutrino). The operators O^H and Q operate on the nucleon wave function with the neutron and proton being regarded as two states of the same entity. O^L is an operator on the wave functions of the light particles, the electron and neutrino. The interaction H is now restricted to a linear combination of a few fundamental forms by the requirement that H be relativistically covariant scalar. If one uses Dirac spinors to describe all the fields present in (1), it

is known that there are only five fundamental forms which satisfy this requirement on H. Any linear combination of these five forms would be a suitable H.

For allowed transitions in the non-relativistic approximation the half-life t for β -decay can be found for any linear combination of these forms from

$$\frac{1}{f(z,W_0)t} = f_F^2 \, F^2 + f_G^2 \, G^2 \tag{2}$$

where $f(z,W_0)$ is a function of the charge of the nucleus and the maximum energy W_0 of the emitted β particle. The F is the Fermi matrix element.

$$F = \int \Psi_f^* \sum_{i=1}^A (t_{\xi i} \pm i t_{\eta i}) \Psi_i = \int \Psi_f^* (T_\xi \pm i T_\eta) \Psi_i \tag{3}$$

G^2 is called the Gamow-Teller matrix element squared and is given by

$$G^2 = |G_x|^2 + |G_y|^2 + |G_z|^2$$
$$G_x = \int \Psi_i^* \sum_{i=1}^A (t_{\xi i} + i t_{\eta i}) \sigma_{x i} \Psi_i \quad etc. \tag{4}$$

The f_F^2 and f_G^2 are the Fermi and Gamow-Teller coupling constants respectively. The suggestion of using the isotopic spin formalism for writing the Fermi and Gamow-Teller matrix elements in a form which is completely symmetrical in the coordinates of all nucleons is credited to Nordheim and Yost (N37), but the above form was first given by Wigner (W39) in his discussion of the theory of β -decay from the supermultiplet theory (W37a). The form of the operators appearing in (3) and (4) permits the formulation of selection rules on

the isotopic spin quantum number T and the total angular momentum J.

Fermi matrix element: $\Delta T = 0 \quad \Delta J = 0$

Gamow-Teller matrix element: $\Delta T = 0, \pm 1 \quad \Delta J = 0, \pm 1$

$$T = 0 \not\to T = 0 \quad J = 0 \not\to 0$$

$$(5)$$

The non-zero value of $\int G$ has been well established for a number of years (K43), but the existence of a non-zero $\int_F$ was not established until the discovery of $J = 0 \xrightarrow{+} J = 0^+$ β -decays. For such transitions the Gamow-Teller matrix element is zero so that the decay can only proceed by the Fermi interaction. The first known $0^+ \to 0^+$ β -decays were the $O^{14}(\beta^+)N^{14}$ (S49) to the 2.31 Mev level and the $C^{10}(\beta^+)B^{10}$ to the 1.74 Mev level. As a result of a number of recent experiments, however, a number of such decays have been given. These are summarized in the Table X due to Moskowski (M53).

(6.1) The Validity of Isotopic Spin Selection Rules for β -Decay

By the use of the selection rules (5) one might hope to test the validity of the isotopic spin quantum number by looking for a $J = 0 \to J = 0$ β -decay which disobeys the selection rule on isotopic spin, $\Delta T = 0$, for the Fermi interaction (R53a, K53). A look at the possible $0 \to 0$ transitions reveals that both of the first two transitions are certainly $T = 1 \to T = 1$, while this is also asserted by Stahelin (S53a, S53b) to be true of the next three. If one looks at the level schemes of B^{10} and N^{14}, one sees in fact that there is no $J = 0 \quad T = 0$ state to which the β -decay could go. This situation prevails generally among the light nuclei as one shows by the following simple argument.

TABLE X

$0^+ \rightarrow 0^+$ Transitions

	Positron Maximum	Half-life (sec.)	Log ft	
C^{10}	1.08 ± 0.1	$19.1(1.65 \pm 20\%)$	3.77 ± 0.2	a,b
O^{14}				
Al^{26}	(3.27 ± 0.05)	6.3	3.52 ± 0.03	c,d
Cl^{34}	4.45 ± 0.1	1.58 ± 0.05	3.47 ± 0.06	d,e,h
K^{38}	> 4.75	0.95 ± 0.03	> 3.37	d,f
V^{46}	> 6.0	0.40	> 3.45	g
$Mn^{49 or 50}$	> 6.3	0.28	> 3.38	g
Co^{54}	> 7.4	0.18	> 3.51	g

(a) R. Sherr and J. Gerhart, Phys. Rev. <u>91</u>, 909 (1953).

(b) R. Sherr, H. Muether, M. G. White, Phys. Rev. <u>75</u>, 282 (1949).

(c) L. Katz, private communication (Energy $Al^{27}(\gamma, n)Al^{26}$ Q value).

(d) P. Stahelin, Phys. Rev. <u>92</u>, 1077L (1953) (Half-lives).

(e) W. Arber and P. Stahelin, Helvetica, Physica Acta <u>26</u>, 433 (1953).

(f) D. Green, private communication (energy from indirect evidence).

(g) W. M. Martin and S. W. Breckon, Can. J. Phys. <u>30</u>, 643 (1952).

(h) P. Stahelin and P. Preiswerk, Nuovo Cimento <u>10</u>, 1214 (1953).

The $0^+ \to 0^+$ decays must occur among the even nuclei, which is to say, in the isobaric triads. By a triad we mean the three nuclei having the same atomic number and $T_\zeta = 0, \pm 1$. The $T_\zeta = 0$ member of these triads is always the stable one in its ground state and decay is from the neighboring isobars to it. The $T_\zeta = 0$ member of $A = 4n + 2$ triads is odd-odd and has non-zero spin for its ground state in all known cases. The neighboring even-even isobars do have $J = 0^+$ for the ground state, however, and hence $0^+ \to 0^+$ decay to an excited state of the odd-odd $T_\zeta = 0$ nucleus might be possible. The level in the $T_\zeta = 0$ nucleus which corresponds to the ground state of the even-even nucleus must, however, be $J = 0^+$ also and is a $T = 1$ level. Since this first $T = 1$ level comes at $\sim 1 - 2$ Mev, there is practically no probability of another $J = 0^+$ level below it, i.e., one having $T = 0$. One cannot expect, therefore, to find allowed $0^+ \to 0^+$ transitions with $|\Delta T| = 1$ among the $A = 4n + 2$ triads.

Among the $A = 4n$ triads the nucleus with the stable ground state for $A < 20$ is again the $T_\zeta = 0$ nucleus, now an even-even nucleus with a $J = 0^+$ ground state. Since the neighboring isobars are odd-odd, however, their ground states are not expected to be $J = 0^+$; the β-transition from their ground states could not be $0^+ \to 0^+$ then.

The only hope of observing $J = 0^+ \to J = 0^+$ transitions which violate the selection rule on isotopic spin would therefore seem to be β-decays from excited states of the unstable members of triads to the stable $T_\zeta = 0$ nucleus. In view of the long half-life for β-decay as compared to that for isomeric transitions, the possibility of such experimental tests is negligible unless the spin difference between the $J = 0$ state in the parent and its own ground state (and all lower states) is large. Such a situation does, indeed,

seem to occur in heavier nuclei $(_{13}Al^{26}_{13}, {}_{17}Cl^{34}_{17}, {}_{19}K^{38}_{19})$ (S53a,b) but the

β-decay has been determined as $T = 1 \rightarrow T = 1$ by the Coulomb energy

difference of the corresponding states.

We conclude— β-decay is unlikely to provide direct evidence on

the validity of the isotopic spin quantum number by violations of isotopic

spin selection rules.

(6.2) Effect of Isotopic Spin Impurity on the Fermi Matrix Element

From the commutation rules for the T_ξ , T_η , and T_ζ components

of isotopic spin the value of the Fermi matrix element (3) follows

immediately

$$\int \Psi^*_f \, (T_\xi \pm iT_\eta) \, \Psi_i = \sqrt{(T \mp T_\zeta)(T \pm T_\zeta + 1)} \tag{6}$$

independent of any assumptions on the nuclear hamiltonian other than that

of invariance under rotations in isotopic spin space. The Gamow-Teller

matrix element cannot be evaluated under such a general assumption, but

can be evaluated exactly in the LS coupling scheme of supermultiplet theory

(G39, W39). Since an evaluation of the Fermi element is possible without

any restriction on the wave functions, we may hope to obtain a test of the

validity of the isotopic spin by checking the constancy of the ft value

for $J = 0 \rightarrow J = 0$ transitions.

The validity of equation (6) in a certain sense involves more

than the validity of the isotopic spin as a good quantum number, as we

have already pointed out. The validity of (6) actually requires that

the wave functions for the numbers of a mirror pair or corresponding

states of an isobaric triad should be generated from each other by the

operator $(T_\xi \pm iT_\zeta)$. If one has actually found a correct set of

eigenfunctions for the description of nuclear levels, this requirement

must be satisfied. We have found that if one uses jj wave functions

there is a certain amount of mixing of the same isotopic spin states belonging to different configurations. Such mixing means that even though corresponding levels of numbers of mirror pairs or triads might have the same isotopic spin, the wave functions for these levels could not be generated from each other by ($T_\xi \pm i T_\eta$). We have actually only calculated the mixing of the same isotopic spin states belonging to one or two particles outside a closed shell. For three particles outside a closed shell such mixing might well be very large since the energy separations of two levels of the same isotopic spin is much less than if the levels have different isotopic spin. If one considers the possible excitation of closed shells, the mixing is undoubtedly even larger, although the actual difference between the wave functions of different numbers of a triad will not be correspondingly larger, of course, since the same closed shells belong to all of them.

In spite of this fault of the jj wave functions from the general arguments based on "charge independence" the value of the Fermi matrix element (3) is certain and this is true therefore no matter what the specific choice of wave functions for corresponding states between which the Fermi interaction causes β-decay. The Gamow-Teller matrix elements, which must be evaluated on a specific model or at least with a definite choice of coupling (T52c), are much more uncertain. If we restrict ourselves to $J = 0^+ \rightarrow J = 0^+$ transitions, however, we have seen that we have only the Fermi matrix element. This fact means that we can study a feature of β-decay which is connected only with the validity of T. This feature is the variation in the experimentally measured ft value for $0^+ \rightarrow 0^+$ transitions.

Let us see what the effect of the Coulomb interaction should be on the ft values for β-decay. Since the Coulomb interaction can

only mix states of the same $J = 0^+$ although different isotopic spin to
the two $J = 0^+$ states participating in the β-decay, the Gamow-Teller
interaction does not enter whether there are isotopic spin impurities or
not. Let the effect of the Coulomb interaction on corresponding states
$\Psi(T,T_\beta)$, $\Psi(T,T_\beta \pm 1)$ be to produce the impure states $\Psi'(T,T_\beta)$ and
$\Psi'(T,T_\beta,\pm 1)$

$$\Psi'(T,T_\beta) = \sqrt{1-a_1^2}\ \Psi(T,T_\beta) + a_1\Psi(T',T_\beta)$$

$$\Psi'(T,T_\beta \pm 1) = \sqrt{1-a_2^2}\ \Psi(T,T_\beta \pm 1) + a_2\Psi(T',T_\beta \pm 1) \tag{7}$$

where of course $T \neq T'$. Then the Fermi matrix element (3) will be

$$\int \Psi^*(T,T_\beta \pm 1)(T_\xi \pm i\,T_\eta)\Psi'(T,T_\beta) = \sqrt{(1-a_1^2)(1-a_2^2)}\ \sqrt{(T \mp T_\beta)(T \pm T_\beta + 1)}$$
$$+ a_1 a_2 \sqrt{(T' \mp T_\beta)(T' \pm T_\beta + 1)} \tag{8}$$

since a_1, a_2 and $\Psi(T',T_\beta \pm 1)$ are also corresponding states.
Since a_1, a_2 will be at most a few percent, the Fermi matrix element is
just

$$F'^2 \approx (1-2p)\,F^2 \tag{9}$$

if we assume $a_1^2 \approx a_2^2 = p$ as is appropriate when the impurity of
the core predominates. The ft value for $J = 0^+ \to J = 0^+$ transitions
being given by

$$ft = (g_F\,F)^{-2} \tag{10}$$

the effect of isotopic spin impurity is to increase ft by

$$\mathcal{S}(A) = \frac{(ft)'}{(ft)} = \frac{1}{1-2p}$$

(11)

From the result for C^{12} where $p \sim 3 \times 10^{-3}$ and the variation of $p \propto A^{8/3}$ predicted by the statistical model, we find for $A \sim 36$ that $p \sim 5.6 \times 10^{-2}$. This gives $\mathcal{S}(30) \sim 1.06$ or a $\sim 6\%$ variation in ft value. Although the previously measured ft values were not sufficiently accurate to detect such a variation, a recent measurement of the ft value for $O^{14}(\beta^+)N^{14}$ to 3% accuracy (G54) indicates that the required accuracy is attainable. The variation of p with atomic number taken here is maximal and assumes $E_0 - E_1 \propto A^{-2/3}$ as indicated by the $E(T = 1) - E(T = 0)$ energy separation of the ground state $T = 0$ and first $T = 1$ state in even-even nuclei with $T_\zeta = 0$ (M54a). The energy separation of $(n+1)\ell$ and $n\ell$ levels on the harmonic oscillator model is also proportional to $A^{-2/3}$.

Appendix I: Evaluation of Certain Correlation Integrals.

The integrals we require are

$$I_1 = \frac{1}{V^2} \oint \frac{G^2(\lambda_{12})}{\lambda_{12}} \qquad I_3 = \frac{1}{V^3} \oint \frac{G^2(\lambda_{23})}{\lambda_{23}}$$

$$I_2 = \frac{1}{V^3} \oint \frac{G^2(\lambda_{12})}{\lambda_{12}\lambda_{13}} \qquad I_4 = \frac{1}{V^4} \oint \frac{G^2(\lambda_{13})}{\lambda_{12}\lambda_{34}}$$

where $\oint$ designates the integration of all coordinates over the volume of a sphere of radius R. In these calculations a parameter whose magnitude determines the accuracy of the evaluations is $R = k_m$ $R = 1.523 A^{1/3}$ where k_m is the wave length of a particle at the top of the degenerate Fermi distribution of nucleons. The evaluations are expected to be asymptotically correct for $R_e \gtrsim 3$, whose R_e is related to A in the following table.

A	10	20	30	40
R_e	3.28	4.12	4.73	5.20.

Evaluation of I_1

Using formula (3.1 - 11) we write I_1 exactly as

$$I_1 = \frac{1}{V^2} \left(\frac{\pi^2}{3}\right) \int_0^{2R} G^2(\rho)(\rho^3 - 12\rho R^2 + 16 R^3) \tag{1}$$

We make the approximation of letting the upper limit go to infinity because $G^2(\rho)$ decreases so rapidly as ρ increases. Inserting the definition of $G^2(\rho) = \frac{9\pi}{2} \frac{J^2_{3/2}(\rho)}{\rho^3}$

$$I_1 = \frac{R^3}{V^2}\left(\frac{\pi^2}{3}\right)\int_0^\infty dx \frac{J^2_{3/2}(x)}{x^3}\left(16 - \frac{12x}{R_e} + \frac{x^3}{R_e^3}\right)$$

From Watson (W44)(13.33) we find Struve's integral

$$\int_0^\infty \frac{J_\mu(t) J_\nu(t)}{t^{\mu+\nu}} = \frac{\Gamma(\mu+\nu)\,\Gamma\left(\frac{1}{2}\right)}{2^{\mu+\nu}\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)\,\Gamma\left(\mu+\frac{1}{2}\right)\,\Gamma\left(\nu+\frac{1}{2}\right)}$$

which gives

$$\int_0^\infty dx\, \frac{J_{3/2}^2(x)}{x^3} = \frac{2}{15}$$

Lommel's method gives (W44)(5.12 - 3)

$$\int_0^z z^{-2\mu-1}\, \mathscr{C}_{\mu+1}^2(z)\, dz = -\frac{z^{-2\mu}}{4\mu+2}\left\{\mathscr{C}_\mu^2(z) + \mathscr{C}_{\mu+1}^2(z)\right\}$$

where $\mathscr{C}_\mu(z)$ is a general cylinder function of order μ.
Specialization to $\mu = 3/2$ yields

$$\int_0^\infty dx\, \frac{J_{3/2}^2(x)}{x^2} = \frac{1}{2\pi}$$

Finally Gubler's investigation of the Weber-Schafheitlin integral (W44)
supplies

$$\int_0^\infty J_\mu(at)\, J_\nu(bt)\, dt = \frac{b^\nu\,\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}\right)}{a^{\nu-1}\,\Gamma(\nu+1)\,\Gamma\left(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)}\,{}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu+1}{2}; \nu+1; \frac{b^2}{a^2}\right)$$

where ${}_2F_1$ is a generalized hypergeometric function defined by

$${}_pF_q\left(\alpha_1,\alpha_2\cdots;\alpha_p;\beta_1,\cdots,\beta_q;z\right) = \sum_{n=0}^\infty \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{n!\,(\beta_1)_n\cdots(\beta_q)_n}\,z^n$$

$$(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$$

For our case

$$\int_0^\infty J_{3/2}^2(x)\, dx = \frac{5}{2\sqrt{\pi}}\,{}_2F_1\left(2,\frac{1}{2};\frac{5}{2};1\right) = \frac{5}{2\sqrt{\pi}}\,1.828$$

When we combine these results we get

$$I_1 = \frac{27}{16 R_e^2} \frac{1}{R_e} \left(3.352 - \frac{3}{R_e} + \frac{2.67}{R_e^3} \right)$$

(2)

This result shows that for $R_e > 3$ we can expect $\sim 30\%$ error if we use only the first term in parenthesis. We shall do this because I_2, I_3, and I_4 cannot be evaluated explicitly to an accuracy better than this and we want only the asymptotic value. If we use only the R^3 term under the integral in (1), our result for I_1 is just

$$I_1 \sim \left(\frac{27}{5 R_e} \right) \frac{1}{R^2}$$

(3)

Evaluation of I_2

The integral will be evaluated by using bipolar coordinates for the point 3, viz. $\sigma = r_{13}$ $\omega = r_{23}$ and φ the angle of rotation of the plane containing r_{13} and r_{23} about the axis r_{12}. The volume element becomes

$$d\tau_3 = J \left(\frac{x_3, y_3, z_3}{\sigma \omega \varphi} \right) d\sigma d\omega d\varphi$$

or using the properties of Jacobians

$$J \left(\frac{x_3, y_3, z_3}{\sigma \omega \varphi} \right) = \left[J \left(\frac{\sigma \omega}{R z_3} \right) J \left(\frac{R z_3 \varphi}{x_3 y_3 z_3} \right) \right]^{-1}$$

The coordinate transformation is given by

$$x_3 = R \cos \varphi \qquad (z_3 - z_1)^2 + R^2 = \sigma^2$$
$$y_3 = R \sin \varphi \qquad (z_3 - z_2)^2 + R^2 = \omega^2$$
$$z_3 = z_3$$

where we have taken the Z-axis parallel to r_{12}.

$$J\left(\frac{\sigma\omega}{Rz_3}\right) = \frac{R\,|z_1 - z_2|}{\sigma\omega} \qquad\qquad J\left(\frac{x_3 y_3 z_3}{Rz_3\varphi}\right) = R$$

and

$$J\left(\frac{x_3 y_3 z_3}{\sigma\omega q}\right) = \frac{\sigma\omega}{\rho} \qquad\qquad \rho \equiv r_{12}$$

Consequently we have

$$d\tau_3 = \frac{\sigma\omega}{\rho}\,d\sigma\,d\omega\,d\varphi$$

Now applying this to I_2 there results

$$I_2 = \frac{1}{V^3}\oint \frac{G^2(\sigma)}{\rho\sigma}\frac{\sigma\omega\,d\sigma\,d\omega}{\rho}\,d\tau_1 d\tau_2$$

where the limits are rather complicated if the points 1, 2, 3 are restricted to a sphere. Instead, by using the fact that $G(\sigma)$ decreases rapidly, we can restrict only the points 1 and 3 to a sphere while allowing 3 to range to infinity. The contribution from points outside the sphere will be small. The limits on σ, ω, φ are now simple and we have for I_2

$$I_2 \approx \frac{2\pi}{V^3}\int\frac{d\tau_1 d\tau_2}{\rho^2}\left(\oint_0^\rho G^2(\sigma)\int_{\rho-\sigma}^{\rho+\sigma}d\omega\,\omega + \int_\rho^\infty d\sigma\,G^2(\sigma)\int_{\sigma-\rho}^{\sigma+\rho}d\omega\,\omega\right)$$

$$I_2 \approx \frac{4\pi}{V^3}\int\frac{d\tau_1 d\tau_2}{r_{12}}\int_0^\infty d\sigma\,G^2(\sigma) \qquad\qquad (3)$$

We evaluate this in the same way as I_1 was integrated to get

$$I_2 \sim \left(\frac{81}{10 R_e^2}\right)\frac{1}{R^2} \qquad\qquad (4)$$

A	10	20	30	40
$R^2 I_1$	0.74	0.47	0.36	0.30

Evaluation of I_3

Use bipolar coordinates for the point 3 and get

$$I_3 = \frac{1}{V^3} \oint \frac{G^2(\sigma)}{\sigma \omega} \frac{\sigma \omega}{\rho} d\sigma d\omega d\varphi \, d\tau_1 d\tau_2$$

$$I_3 \approx \frac{4\pi}{V^3} \int \frac{d\tau_1 d\tau_2}{\hbar_{12}} \left(\int\int_0^\rho d\sigma G^2(\sigma) \sigma + \rho \int_\rho^{2R} d\sigma \, G^2(\sigma) \right)$$

Since $G(\sigma)$ decreases so rapidly, and the contribution to $\int \frac{d\tau_1 d\tau_2}{\hbar_{12}}$ where $\int \rho = r_{12}$ is greatest for small ρ, the integral is asymptotically equal to

$$I_3 \approx \frac{4\pi}{V^3} \left(\int \frac{d\tau_1 d\tau_2}{\hbar_{12}} \right) \int_0^{\rho \phi} d\sigma \, G^2(\sigma) \sigma \tag{5}$$

Comparing (3) and (5) we see

$$I_3 \sim I_1.$$

Evaluation of I_4

Carrying out the integral on the coordinates (2) and (4) I_4 becomes

$$I_4 = \frac{9}{4R^6} \frac{1}{V^2} \int G^2(\hbar_{13}) \left(R^2 - \frac{\hbar_1^2}{3} \right)\left(R^2 - \frac{\hbar_3^2}{3} \right) d\tau_1 d\tau_3 \tag{6}$$

If we look at the volume as composed of concentric shells of radii r_1 and r_3, the rapid decrease of $G(\hbar_{13})$ insures that the greatest contribution will come from the shells with nearly equal radii, i.e. $r_1 = r_3$. We approximate

$$I_4 = \frac{9}{4R^6} \frac{1}{V^2} \int G^2(r_{13}) \left(R^2 - \frac{r_1^2}{3}\right)^2 d\tau_1 d\tau_2$$

We now use coordinates centered on the point 1 and let r_{13} go to infinity obtaining thus

$$I_4 = \frac{9}{4R^6} \frac{1}{V^2} \left(4\pi \int_0^\infty d\sigma \; G^2(\sigma) \sigma^2\right) \left(4\pi \int_0^R dr_1 \, r_1^2 \left(R^2 - \frac{r_1^2}{3}\right)^2\right)$$

$$\approx \frac{81}{20 R_e^3} \cdot \frac{1}{R^2} \cdot \frac{9\pi}{2} \int_0^\infty dx \; \frac{J_{3/2}^2(x)}{x}$$

From Watson (W)(13.42 - 1)

$$\int_0^\infty dt \; \frac{J_\mu(at) J_\nu(bt)}{t} = \begin{cases} \frac{1}{2} (b/a)^\mu / \mu & b \le a \\ \frac{1}{2} (a/b)^\mu / \mu & b \ge a \end{cases}$$

Inserting this in I_9

$$I_4 \approx \left(\frac{6\pi}{R_e^3}\right) \frac{1}{R^2} \tag{7}$$

Conclusion

The general expectations expressed in (3.1) concerning the behavior of corrections to $\overline{r_{12}^{-2}}$, $\overline{r_{12}^{-1} r_{13}^{-1}}$, and $\overline{r_{12}^{-1} r_{34}^{-1}}$ are verified. I_1 gives the correction to r_{12}^{-2} and decreases least slowly, being proportional to $1/R_e$. I_2 and I_3 give corrections to $\overline{r_{12}^{-1} r_{13}^{-1}}$ and decrease as $1/R_e^2$. Finally, the term which corrects $\overline{r_{12}^{-1} r_{34}^{-1}}$ is proportional to I_4 and decreases as $1/R_e^3$. The asymptotic values of the integrals are

$$I_1 \sim \left(\frac{27}{5 R_e}\right) \frac{1}{R^2} \quad I_2 \sim I_3 \sim \left(\frac{81}{10 R_e^2}\right) \frac{1}{R^2} \quad I_4 \sim \left(\frac{19}{R_e^3}\right) \frac{1}{R^2} \tag{8}$$

Appendix II

Table of $F^k(\ell_1 \ell_2; \ell_3 \ell_4)$.

$$F^{(0)}((1p)^2; (1p)^2) = e^2 \ \frac{75}{8} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^{(2)}((1p)^2; (1p)^2) = e^2 \ \frac{25}{24} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^{(1)}((1p)^2; (1d)^2) = e^2 \ \frac{11}{12} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^{(3)}((1p)^2; (1d)^2) = e^2 \ \frac{7}{12} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^{(2)}((1p)^2; 1p1f) = e^2 \ \frac{15}{4} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^{(2)}((1p)^2; (1f)^2) = e^2 \ \frac{13}{24} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^{(4)}((1p)^2; (1f)^2) = e^2 \ \frac{13}{28} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^{(0)}((1p)^2; 1p2p) = e^2 \ \frac{3}{8} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^{(2)}((1p)^2; 1p2p) = e^2 \ \frac{5}{24} \sqrt{\frac{\gamma}{2\pi}}$$

$$F^0 (2p1s; 1p1s) = \frac{7}{60} \sqrt{\frac{5\gamma}{\pi}}$$

$$F^1 (2p1s; 1s1p) = \frac{3}{20} \sqrt{\frac{5\gamma}{\pi}}$$

$$F^0 (2s1p; 1s1p) = \frac{1}{4} \sqrt{\frac{3\gamma}{\pi}}$$

$$F^1 (2s1p; 1p1s) = - \frac{1}{12} \sqrt{\frac{3\gamma}{\pi}}$$

$$F^0 (2s1s; (1s)^2) = \frac{1}{6} \sqrt{\frac{3\gamma}{\pi}}$$

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