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by

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Structures and Materials Research
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Division of Structural Engineering

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I. Introduction

The problem under study involves a body of infinite extent through which runs an infinitely long cylindrical hole of circular cross section. The material of the body is viscoelastic which is modeled in the study as a Standard Solid. The problem is that of finding the response in the body to a uniform pressure against the surface of the hole that has an arbitrary dependence on time.

We choose to solve the problem using the method of characteristics. The choice seems suitable for several reasons. The problem is one of plane strain with axisymmetry which means that the response is dependent only on the radial space variable and time. Further the governing equations are hyperbolic. The method of characteristics accommodates a variety of initial and boundary conditions so long as dependence on only two independent variables in the problem is maintained. From our experience we feel that numerical results are easier to obtain using the method of characteristics than by transform techniques.

Using the method of characteristics the governing differential equations are reduced to two different forms. The first is the decay equation which can be integrated directly to find the behaviour along the wave front. The behaviour behind the wave front is found by reducing the equations to their canonical form. In this form they can be integrated only along characteristic lines but the canonical form is simple and lends itself to integration by finite differences.

In our search of the literature we are unable to find a record of an attempt to solve the problem involving a viscoelastic body. The closest problem seems to be that solved by Kromm[1,2]. Kromm found the response in an infinite elastic plate to an input on the surface of a single circular hole in the plate. The input was either a uniform radial velocity or a uniform pressure each with a step distribution in time. Using a transform technique he obtained numerical results. Even though the problem is quite different from ours it is useful as a check. As the theory of viscoelasticity contains the theory of elasticity as a special case, our solution is easily adapted for the elastic case. As Kromm's problem is one of generalized plane stress and ours is plane strain it is necessary for comparison to take Kromm's elastic constants and from them find "equivalent" constants for plane strain. Using these elastic constants we find the response numerically and compare it to that found by Kromm. The two responses are so close together that in Figures (2-5) they have to be shown by a single line.

Chou and Koenig[3] studied the Kromm problem but instead of solving it using integral transforms they used the method of characteristics. When they compared their results with those of Kromm they stated that the two sets of responses were identical.

To get a feeling for the influence of the viscoelastic parameters on the response, we study two different viscoelastic materials. They are chosen so that both materials have an instantaneous Poisson's ratio of 0.2308 and so that the second viscoelastic material is more viscous than the first. We have therefore, in effect, three comparable viscoelastic materials starting with one that has no viscosity (elastic) and progressing with a second and a third that are characterized by increasing viscosity. The responses cannot be appraised in light of any other published results but they satisfy what one would expect by intuition. We see by examining Figures 2-5 that the more viscous the material is the less steep is the decay behind the wave front and for stations apart from the cylindrical surface the smaller is the amplitude of the discontinuity at the front of the wave.

II. Formulation of the Problem

Our study is of an infinite viscoelastic body, initially at rest, having an infinitely long circular cylindrical hole of radius "a". The surface of the hole is subjected to a uniform pressure applied with an arbitrary dependence on time.

The body is referred to a cylindrical coordinate system (r, θ, z) within which the z axis coincides with the axis of the hole. In the development, when it is appropriate, we use indicial notation and all of the rules that apply to its use. Because of the axisymmetry of the problem, we assume the displacement field in the form:

$$u_r = u_r(r, t)$$
$$u_\theta \equiv 0 \qquad (1)$$
$$u_z \equiv 0 \quad ;$$

then, using strain-displacement relations in cylindrical coordinate system, we obtain

$$\varepsilon_{rr} = u_{r,r}$$
$$\varepsilon_{\theta\theta} = \frac{u_r}{r} \qquad (2)$$
$$\varepsilon_{zz} = \varepsilon_{r\theta} = \varepsilon_{rz} = \varepsilon_{\theta z} = 0 \quad .$$

The constitutive equations for a linear isotropic viscoelastic material are given by

$$P_1(D)\tau'_{ij} = Q_1(D)\varepsilon'_{ij}$$
$$(3)$$
$$P_2(D)\tau_{kk} = Q_2(D)\varepsilon_{kk} \ ,$$

where

$$P_1(D) = \sum_{k=0}^{n_1} a_k D^k \; ; \quad Q_1(D) = \sum_{k=0}^{m_1} b_k D^k \; ;$$
$$P_2(D) = \sum_{k=0}^{n_2} c_k D^k \; ; \quad Q_2(D) = \sum_{k=0}^{m_2} d_k D^k \quad , \tag{4}$$

in which a_k, b_k, c_k, d_k are specified constants and $D^k = \frac{\partial^k}{\partial t^k}$. In Eqs. (3), τ'_{ij}, ε'_{ij} are the components of the stress and strain deviators:

$$\tau'_{ij} = \tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk}$$
$$\varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk} \quad , \tag{5}$$

where δ_{ij} is the Kroneker delta. If the initial values of τ'_{ij}, ε'_{ij}, τ_{kk}, ε_{kk} satisfy certain conditions[4] the constitutive equations, Eqs. (3), can be written in terms of integral equations as

$$\tau'_{ij}(\underline{x}, t) = G_1(t)\varepsilon'_{ij}(\underline{x}, 0) + \int_0^t G_1(t - \tau) \frac{\partial \varepsilon'_{ij}}{\partial \tau} (\underline{x}, \tau) d\tau$$
$$\tau_{kk}(\underline{x}, t) = G_2(t)\varepsilon_{kk}(\underline{x}, 0) + \int_0^t G_2(t - \tau) \frac{\partial \varepsilon_{kk}}{\partial \tau} (\underline{x}, \tau) d\tau \quad , \tag{6}$$

where $G_1(t)$, $G_2(t)$ are the shear and bulk relaxation functions respectively and $\underline{x}$ is the position vector of the particle considered.

In our study we choose the standard solid as the viscoelastic model. For this model the constitutive equations take special forms. In differential equation form they are given by Eqs. (3) where Eqs. (4) have the specific form

$$P_1(D) = \sum_{k=0}^{1} a_k D^k \; ; \quad Q_1(D) = \sum_{k=0}^{1} b_k D^k \; ;$$

$$P_2(D) = \sum_{k=0}^{1} c_k D^k \; ; \quad Q_2(D) = \sum_{k=0}^{1} d_k D^k \quad . \tag{7}$$

In integral form the constitutive equations are given by Eqs. (6) in which the shear and bulk moduli for the standard solid are given by

$$G_1(t) = G_{1F} + (G_{10} - G_{1F})e^{-t/\tau_1}$$

$$G_2(t) = G_{2F} + (G_{20} - G_{2F})e^{-t/\tau_2} \quad \text{respectively} \quad . \tag{8}$$

In Eqs. (8) the constants τ_1, τ_2 are relaxation times of shear and bulk moduli respectively, and

$$G_{1F} = G_1(\infty) \; ; \quad G_{10} = G_1(0) \; ;$$

$$G_{2F} = G_2(\infty) \; ; \quad G_{20} = G_2(0) \quad . \tag{9}$$

The constants in Eqs. (7) and (8) are related according to

$$G_{10} = \frac{b_1}{a_1} ; \; G_{1F} = \frac{b_0}{a_0} ; \; \tau_1 = \frac{a_1}{a_0} ;$$

$$G_{20} = \frac{d_1}{c_1} ; \; G_{2F} = \frac{d_0}{c_0} ; \; \tau_2 = \frac{c_1}{c_0} \quad . \tag{10}$$

From Eqs. (2), and the constitutive relations we see that $\tau_{ij} \equiv 0$ for $i \neq j$, and that

$$P_1(D)\tau'_{rr} = \frac{1}{3} Q_1(D) \left(2u_{r,r} - \frac{u_r}{r}\right)$$

$$P_1(D)\tau'_{\theta\theta} = \frac{1}{3} Q_1(D) \left(2 \frac{u_r}{r} - u_{r,r}\right) \tag{11}$$

$$P_2(D)\tau_{kk} = Q_2(D) \left(u_{r,r} + \frac{u_r}{r}\right) \quad .$$

Noting that τ_{rr}, $\tau_{\theta\theta}$ and τ_{zz} are the functions of r and t only, the stress equation of motion becomes

$$\tau_{rr,r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = \rho\ddot{u}_r \quad . \tag{12}$$

In terms of stress deviators it can be written as

$$\left(\tau'_{rr} + \frac{1}{3} \tau_{kk}\right)_{,r} + \frac{\tau'_{rr} - \tau'_{\theta\theta}}{r} = \rho\ddot{u}_r \quad . \tag{13}$$

The other two equations are satisfied identically.

For the condition on the cylindrical boundary of the hole we specify that only normal pressure will exist and that it will be uniform. The method of characteristics will accommodate any boundary condition that does not violate axisymmetry so this particular condition was chosen chiefly so that our results can be compared with what published results there are. The body is taken to be initially at rest.

The boundary condition takes the form

$$\tau_{rr}(a, t) = \tau'_{rr}(a, t) + \frac{1}{3} \tau_{kk}(a, t) = - f(t)H(t) \quad , \tag{14}$$

where H(t) is the usual Heaviside step function and f(t) is a prescribed, continuous function of t.

The initial conditions are

$$u_r(r, 0) = \dot{u}_r(r, 0) = 0 \quad . \tag{15}$$

The problem is now completely described. It is one of finding the four variables τ'_{rr}, $\tau'_{\theta\theta}$, τ_{kk} and u_r. The variables are governed by Eqs. (11) and (13) and are subject to the boundary and initial conditions specified by Eqs. (14) and (15) respectively.

III. Solution of the Problem

The choice here is to solve the problem using the method of characteristics. There are many qualities of the problem that indicate this choice. There are two independent variables and the governing differential equations are hyperbolic; both conditions that satisfy the dictates of the method. Further, we will be dealing in the problem with wave fronts which are particularly well handled by this method.

There is no need to review the method in detail here as it is covered in books such as the one by R. Courant and D. Hilbert[5]. However, it makes the development more complete if we explain that the method of characteristics is one of reducing the governing differential equations to two much simpler forms, each of which is amenable to numerical analysis. The first of these forms is called the canonical form and the second, the decay equation. They are not applicable everywhere on the space-time plane. The canonical form of the differential equations is valid only along characteristic lines and is used in the domain of disturbed material. The decay equation is used along the boundary between disturbed and undisturbed material, namely the wave front.

Each of the two forms is developed separately.

(a) Canonical Form of the Governing Equations

We first transform the governing differential equations, Eqs. (11) and (13), into a set of first order differential equations. We do so by introducing new dependent variables

$$u_1 = \dot{u}_r \; ; \quad u_2 = u_{r,r} \quad , \tag{16}$$

and take into account the relation

$$u_{1,r} = \dot{u}_2 \quad . \tag{17}$$

The five governing differential equations (three Eqs. (11), Eq. (13) and Eq. (17)) take the form

$$v_{i,t} + B_{ij} v_{j,r} = c_i \quad (i,\ j = 1\text{-}5) \tag{18}$$

where

$$(v_i) = (u_1,\ \tau'_{rr},\ \tau'_{\theta\theta},\ \tau_{kk},\ u_2)$$

$$(B_{ij}) = \begin{bmatrix} 0 & -1/\rho & 0 & -1/3\rho & 0 \\ -\dfrac{2b_1}{3a_1} & 0 & 0 & 0 & 0 \\ \dfrac{b_1}{3a_1} & 0 & 0 & 0 & 0 \\ -\dfrac{d_1}{c_1} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

$$(c_i) = \begin{bmatrix} \dfrac{(\tau'_{rr} - \tau'_{\theta\theta})}{\rho r} \\ \left(-\dfrac{a_0}{a_1}\tau'_{rr} - \dfrac{b_1}{3a_1}\dfrac{u_1}{r} + \dfrac{2b_0}{3a_1}u_2 - \dfrac{b_0}{3a_1}\dfrac{u_r}{r}\right) \\ \left(-\dfrac{a_0}{a_1}\tau'_{\theta\theta} + \dfrac{2b_1}{3a_1}\dfrac{u_1}{r} + \dfrac{2b_0}{3a_1}\dfrac{u_r}{r} - \dfrac{b_0}{3a_1}u_2\right) \\ \left(-\dfrac{c_0}{c_1}\tau_{kk} + \dfrac{d_1}{c_1}\dfrac{u_1}{r} + \dfrac{d_0}{c_1}u_2 + \dfrac{d_0}{c_1}\dfrac{u_r}{r}\right) \\ 0 \end{bmatrix} \quad . \tag{19 (cont'd)}$$

Before we establish the canonical form of Eqs. (18), we first establish the characteristic lines along which they are valid. The equation, dependent on Eq. (18), governing the characteristic lines is given by (see reference 5)

$$\det(B_{ij} - \lambda\delta_{ij}) = 0 \quad , \tag{20}$$

where $\lambda = \frac{dr}{dt}$ defines the characteristic lines on the (r - t) plane. The conanical forms of the governing equations along the characteristic lines are (see reference 5)

$$\ell_m^{(i)} \frac{dv_m}{dt} = \ell_m^{(i)} c_m \quad \text{along} \quad \frac{dr}{dt} = \lambda^{(i)} \quad (i,\ m = 1 - 5), \tag{21}$$

where the $\ell_m^{(i)}$ is the left-hand eigenvector of the B_{mp} defined as

$$\ell_m^{(i)} B_{mp} = \lambda^{(i)} \ell_p^{(i)} \quad (\text{no sum on } i;\ i,\ m,\ p = 1 - 5) \quad . \tag{22}$$

In Eq. (22), $\lambda^{(i)}$ is the (i)th eigenvalue of B_{mp}, which can be determined from Eq. (20). For our problem they are

$$\lambda^{(1)} = c \ ; \quad \lambda^{(2)} = - c \ ; \quad \lambda^{(3)} = \lambda^{(4)} = \lambda^{(5)} = 0 \quad , \tag{23}$$

where

$$c = \left\{\frac{1}{3\rho}\left(\frac{2b_1}{a_1} + \frac{d_1}{c_1}\right)\right\}^{1/2} = \left(\frac{2G_{10} + G_{20}}{3\rho}\right)^{1/2} \quad . \tag{24}$$

Here, we note that $\lambda^{(1)} = c$ and $\lambda^{(2)} = - c$ describe two family of straight lines with slopes (c) and (-c) respectively on the (r - t) plane (see figure 1) and correspond to physical characteristics which are the ones across which the vector v_i or its derivatives may suffer a finite jump (physical considerations dictate that u_r will be continuous). On the other hand $\lambda^{(3)} = \lambda^{(4)} = \lambda^{(5)} = 0$ describe the family of straight lines (r = const.) parallel to the t axis on the (r - t) plane, which have physically no meaning, but along which the governing equations can be put into canonical form.

When we introduce the dimensionless quantities

$$\bar{r} = \frac{r}{a} \; ; \; \bar{t} = \frac{ct}{a} \; ;$$

$$\bar{u}_1 = \frac{u_1}{c} \; ; \; (\bar{\tau}'_{rr}, \bar{\tau}'_{\theta\theta}, \bar{\tau}_{kk}) = \frac{1}{\rho c^2} (\tau'_{rr}, \tau'_{\theta\theta}, \tau_{kk}) \; ; \quad (25)$$

$$\bar{u}_2 = u_2; \quad \bar{u}_r = \frac{u_r}{a} \; ;$$

$$\bar{\tau}_1 = \frac{c\tau_1}{a} \; ; \quad \bar{\tau}_2 = \frac{c\tau_2}{a} \; ,$$

the conanical form of the governing equations, Eq. (21), together with the continuity condition

$$du_r = \dot{u}_r dt = u_1 dt \quad \text{along } r = \text{const.} \quad (26)$$

can be written in the form:

$$\alpha_{ij} \frac{dw_j}{d\bar{t}} = \beta_{ij} w_j \; (i, j = 1 - 6) \quad (27)$$

where

$$i = 1, 2 \text{ along the characteristics } \frac{d\bar{r}}{d\bar{t}} = + 1, -1$$

$$i = 3 - 6 \text{ along the lines } \frac{d\bar{r}}{d\bar{t}} = 0 \; ,$$

and

$$(w_j) = (\bar{u}_1, \bar{\tau}'_{rr}, \bar{\tau}'_{\theta\theta}, \bar{\tau}_{kk}, \bar{u}_2, \bar{u}_r)$$

$$
(\alpha_{ij}) = \begin{bmatrix} 1 & -1 & 0 & -1/3 & 0 & 0 \\ 1 & 1 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{G_{10}}{3G_{20}} & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{G_{20}}{\rho c^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (28)
$$

$$(\beta_{ij}) = \begin{bmatrix} -\frac{(G_{20}-G_{10})}{3\rho c^2}\frac{1}{\bar{r}} & \left(\frac{1}{\bar{r}}+\frac{1}{\bar{\tau}_1}\right) & -\frac{1}{\bar{r}} & \frac{1}{3\bar{\tau}_2} & -\frac{1}{3\rho c^2}\left(\frac{2G_{1F}}{\bar{\tau}_1}+\frac{G_{2F}}{\bar{\tau}_2}\right) & -\frac{1}{3\rho c^2}\left(\frac{G_{2F}}{\bar{\tau}_2}-\frac{G_{1F}}{\bar{\tau}_1}\right)\frac{1}{\bar{r}} \\ \frac{(G_{20}-G_{10})}{3\rho c^2}\frac{1}{\bar{r}} & \left(\frac{1}{\bar{r}}-\frac{1}{\bar{\tau}_1}\right) & -\frac{1}{\bar{r}} & -\frac{1}{3\bar{\tau}_2} & \frac{1}{3\rho c^2}\left(\frac{2G_{1F}}{\bar{\tau}_1}+\frac{G_{2F}}{\bar{\tau}_2}\right) & \frac{1}{3\rho c^2}\left(\frac{G_{2F}}{\bar{\tau}_2}-\frac{G_{1F}}{\bar{\tau}_1}\right)\frac{1}{\bar{r}} \\ \frac{G_{10}}{\rho c^2}\frac{1}{\bar{r}} & -\frac{1}{\bar{\tau}_1} & -\frac{2}{\bar{\tau}_1} & 0 & 0 & \frac{1}{\rho c^2}\frac{G_{1F}}{\bar{\tau}_1}\frac{1}{\bar{r}} \\ \frac{G_{10}}{\rho c^2}\frac{1}{\bar{r}} & 0 & -\frac{1}{\bar{\tau}_1} & -\frac{G_{10}}{3G_{20}}\frac{1}{\bar{\tau}_2} & \frac{1}{3\rho c^2}\left(\frac{G_{10}}{G_{20}}\frac{G_{2F}}{\bar{\tau}_2}-\frac{G_{1F}}{\bar{\tau}_1}\right) & \frac{1}{3\rho c^2}\left(\frac{G_{10}}{G_{20}}\frac{G_{2F}}{\bar{\tau}_2}+\frac{2G_{1F}}{\bar{\tau}_1}\right)\frac{1}{\bar{r}} \\ \frac{G_{20}}{\rho c^2}\frac{1}{\bar{r}} & 0 & 0 & -\frac{1}{\bar{\tau}_2} & \frac{1}{\rho c^2}\frac{G_{2F}}{\bar{\tau}_2} & \frac{1}{\rho c^2}\frac{G_{2F}}{\bar{\tau}_2}\frac{1}{\bar{r}} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (28)$$

(cont'd)

The conanical form of the governing equations, Eq. (27), is valid in that region of the $(\bar{r} - \bar{t})$ plane within which the vector v_i is continuous.

The boundary and initial conditions, Eqs. (14) and (15), in dimensionless form become

$$\bar{\tau}'_{rr}(1, \bar{t}) + 1/3\, \bar{\tau}_{kk}(1, \bar{t}) = -\, g(\bar{t})H(\bar{t})$$
$$\bar{u}_r(\bar{r}, 0) = \bar{u}_1(\bar{r}, 0) = 0 \quad , \tag{29}$$

where

$$g(\bar{t}) = \frac{f(t)}{\rho c^2} \quad .$$

(b) Decay Equation along the Wave Front

We are seeking the response of an infinite viscoelastic body having infinitely long cylindrical hole whose lateral surface is subjected to a uniform pressure. As the resulting disturbance will move into the medium, the behaviour is best understood if it is described using the notion of a wave front. The wave front is defined as the boundary between disturbed and undisturbed regions of the medium. When the material at a point becomes suddenly disturbed from an undisturbed state it can only do so if some derivative of the displacement u_r suffers a finite jump at the point, that is v_i or its derivatives suffer a finite jump. On the $(\bar{r} - \bar{t})$ plane, a wavefront can be represented by a line and, by definition, that line will be a characteristic. This particular line corresponds to a cylindrical wave front propagating into the medium with the velocity "c". In our problem the initial conditions are homogeneous which means that of a family of characteristic lines, it is the one emanating from the point with the coordinates (1, 0) on the $(\bar{r} - \bar{t})$ plane that will represent the wave front. The order of the discontinuity of the characteristic line describing the wave front will depend on the boundary condition at $\bar{r} = 1$, specifically the dependence on time in the neighborhood of $\bar{t} = 0$.

The line $(\bar{r} - 1) = \bar{t}$ describing the wave front "S" is shown in Fig. 1. In the boundary condition, the first of Eqs. (29), $g(\bar{t})$ is an arbitrary function of $\bar{t}$. If $g(0) \neq 0$, we will show shortly that along S the vector v_i will suffer a finite jump. Since discontinuities of v_i are not permitted with the use of the conanical form of the governing equations, it will be necessary to develop decay equations that are valid along the line "S".

These are developed from Eq. (27) by recognizing that even though the conanical equations are not valid along the characteristic across which the v_i suffer finite jumps, they are valid on each side of it.

We begin by using Eqs. (27) with the choice of $i = 1$, which is the conanical form of the governing equations along $\frac{d\bar{r}}{d\bar{t}} = 1$. We write this equation on both sides of the characteristic line and take the difference. We obtain

$$\alpha_{1j} \frac{d[w_j]}{d\bar{t}} = \beta_{1j}[w_j] \quad . \tag{30}$$

We enclose a function by square brackets to denote the finite jump of the function across a characteristic line, i.e., [f] designates the finite jump of the function (f) across a characteristic line $\bar{r} = \bar{r}(\bar{t})$.

On the other hand, from the continuity conditions we have

$$[\bar{u}_r] = 0 \quad . \tag{31}$$

Here we use Hadamard's lemma which states that $[f] = 0$ along $\bar{r} = \bar{r}(\bar{t})$

implies that $[f,_{\bar{t}}] + \frac{d\bar{r}}{d\bar{t}} [f,_{\bar{r}}] = 0$. If we apply the lemma to Eq. (31) along $\frac{d\bar{r}}{d\bar{t}} = 1$ we obtain

$$[\bar{u}_1] = - [\bar{u}_2] \quad \text{along } \frac{d\bar{r}}{d\bar{t}} = 1 \quad . \tag{32}$$

From the constitutive relations, Eqs. (6), we also have

$$\begin{aligned} [\bar{\tau}'_{rr}] &= \frac{G_{10}}{\rho c^2} [\varepsilon'_{rr}] = \frac{2G_{10}}{3\rho c^2} [\bar{u}_2] \\ [\bar{\tau}'_{\theta\theta}] &= \frac{G_{10}}{\rho c^2} [\varepsilon'_{\theta\theta}] = \frac{-G_{10}}{3\rho c^2} [\bar{u}_2] \\ [\bar{\tau}_{kk}] &= \frac{G_{20}}{\rho c^2} [\varepsilon_{kk}] = \frac{G_{20}}{\rho c^2} [\bar{u}_2] \quad . \end{aligned} \tag{33}$$

When, using Eqs. (28), we expand Eq. (30) and employ Eqs. (31, 32, 33) we obtain decay equations in the form

$$\frac{d[\bar{u}_2]}{d\bar{r}} + \frac{1}{2\bar{r}} [\bar{u}_2] + m[\bar{u}_2] = 0 \quad \text{along } \frac{d\bar{r}}{d\bar{t}} = 1 \quad , \tag{34}$$

where

$$m = \frac{1}{6\rho c^2} \left\{ 2(G_{10} - G_{1F}) \frac{1}{\bar{\tau}_1} + (G_{20} - G_{2F}) \frac{1}{\bar{\tau}_2} \right\} \tag{35}$$

The solution of Eq. (34) is

$$[\bar{u}_2] = A \left(\frac{1}{\bar{r}}\right)^{1/2} e^{-m\bar{r}} \quad \text{along } \frac{d\bar{r}}{d\bar{t}} = 1 \ , \tag{36}$$

where A is a constant to be determined from boundary and initial conditions.

We return to our problem. Since $g(\bar{t})$ is a continuous function of $\bar{t}$ for $\bar{t} > 0$, using Eq. (36) it can be shown that in the disturbed region behind the wave front S, the v_i are continuous so that the conanical form of the equations, Eq. (27), are appropriate. On the other hand along the wave front S, the constant A in the decay equation, Eq. (36), will not be zero (implying the v_i suffer finite jumps across S) if $g(0) \neq 0$. The constant A can be obtained from the behaviour of the boundary condition and initial conditions in the neighborhood of the point with coordinates (1, 0) on the $(\bar{r} - \bar{t})$ plane. Using boundary and initial conditions Eqs. (29), and noting that $[\bar{u}_r] = 0$ everywhere on the $(\bar{r} - \bar{t})$ plane one obtains

$$[\bar{u}_2(1, 0)] = - g(0) \quad . \tag{37}$$

Using Eq. (37), the constant in Eq. (36) can be determined. It is

$$A = - e^m g(0) \tag{38}$$

Accordingly, the decay equation becomes

$$[\bar{u}_2] = - g(0) \left(\frac{1}{\bar{r}}\right)^{1/2} e^{-m(\bar{r}-1)} \quad \text{along S} \quad . \tag{39}$$

Knowing the jump in $\bar{u}_2$, the jumps $\bar{u}_1$, $\bar{\tau}'_{rr}$, $\bar{\tau}'_{\theta\theta}$ and $\bar{\tau}_{kk}$ along S can be determined from Eqs. (32) and (33).

IV. Numerical Analysis

We seek $(w_i) = (\bar{u}_1, \bar{\tau}'_{rr}, \bar{\tau}'_{\theta\theta}, \bar{\tau}_{kk}, \bar{u}_2, \bar{u}_r)$ at a station $\bar{r}$ and a time $\bar{t}$, and having these, we can calculate the strains and stresses. We refer to Figure 1, which shows the $(\bar{r} - \bar{t})$ plane. On this plane, the line S: $\bar{r} - 1 = \bar{t}$ divides the space-time domain into two parts, the domain D_1 representing undisturbed particles and D_2 representing particles of the body which are in motion. The part D_2, which is the part that interests us, is subdivided by means of a grid. The grid shown by fine solid lines is formed by two sets of parallel lines. The first set ($\bar{r} - \bar{t} = $ const.) is parallel to the line S, and the second set ($\bar{r} + \bar{t} = $ const.) has equal but opposite slopes. Each diamond shaped element has diagonals measuring $2\Delta\bar{r}$ and $2\Delta\bar{t}$.

To establish w_i in the region D_2, we start at the origin and along S where it is known from the decay equations, and fan out into region element by element. To be more explicit, we know w_i at the points 0 and 1 in Figure 1, and using a technique to be explained shortly, we find w_i at the point 2. Having w_i at the points 1, 2, and 3, we use the same technique to find w_i at the point 4, and so forth.

In explaining the technique we refer to element M shown in Figure 1. w_i is known at points A_1, A_2 and A_3 and is sought at the point A. As there are six unknowns, we need six equations to establish them.

The boundary lines AA_1 and AA_2 are the characteristic lines $\bar{r} - \bar{t} =$ const., and $\bar{r} + \bar{t} =$ const. respectively. Two of the six equations come from using the conanical form of the governing equations along the characteristic lines AA_1 and AA_2 in the element converging on A (Eq. (27) with i = 1, 2).

The four remaining equations are the conanical forms of the governing equations along the line AA_3, $\bar{r} =$ const. (Eq. (27) with i = 3 - 6). The six elements of w_i are found at A by solving the six equations by the method of finite differences.

For the element L adjacent to the line $\bar{r} = 1$, the procedure is the same except that the equation along the line $\bar{r} - \bar{t} =$ const. must be replaced by the boundary condition at $\bar{r} = 1$, namely the first of Eqs. (29):

$$\bar{\tau}'_{rr}(A) + \frac{1}{3}\,\bar{\tau}_{kk}(A) = -\,g(A) \quad . \tag{40}$$

V. Numerical Results

Our choice is to calculate and exhibit three quantities; the radial stress τ_{rr}, the tangential stress $\tau_{\theta\theta}$ and the radial velocity $\dot{u}_r$. Each quantity is found at two stations; the first at r = a, the edge of the cylindrical hole, and the second at r = 2.5a. We use $f(t) = P_0$ for the time dependency of the input though with the method of characteristics any function could have been chosen. This particular choice enables us to compare our results, for the elastic case, with those due to Kromm[1,2]. This comparison is possible because the theory of viscoelasticity contains the theory of elasticity as a special case.

Kromm studied the response in an infinite elastic sheet to a uniform input applied to the surface of a circular hole. One of his inputs was a pressure having a step distribution in time and it is the response due to this input with which we compare ours. The comparison is additionally significant because Kromm established his response using integral transforms.

In establishing a response to compare to that of Kromm we first recognized that his problem was one of generalized plane stress and ours is plane strain. However it is well known that the solution of one problem can be taken as the solution of the other provided the elastic constants are adjusted. Kromm used a Poisson's ratio of 0.30 from which we found the "equivalent" Poisson's ratio for plane strain to be 0.2308. Using this value we found the responses at the same stations as Kromm and compared the two sets of results. The results are almost identical. They are so close that they are indistinguishable from one another in Figures 2-5.

Our main interest however, is in the response in viscoelastic bodies. We choose two separate viscoelastic materials to show the influence of the viscosity on the response. The second of the two materials is the more viscous. As each material is modeled by the Standard Solid each is identified by the quantities listed below. The quantities are chosen so that both materials have the instantaneous Poisson's ratio of 0.2308 which will mean that the elastic response we have found can be considered the limiting case for the three responses.

(a) Material one

$$\frac{G_{1F}}{G_{10}} = 0.40 \ ; \quad \bar{\tau}_1 = 3.0 \ ;$$

$$\frac{G_{20}}{G_{10}} = 2.28571; \quad \frac{G_{2F}}{G_{10}} = 1.142855; \quad \bar{\tau}_2 = 5.0 \ , \tag{41}$$

(b) Material two

$$\frac{G_{1F}}{G_{10}} = 0.20 \ ; \quad \bar{\tau}_1 = 1.50 \ ;$$

$$\frac{G_{20}}{G_{10}} = 2.28571; \quad \frac{G_{2F}}{G_{10}} = 0.5714275; \quad \bar{\tau}_2 = 2.50. \tag{42}$$

We have then, in effect, three comparable viscoelastic materials starting with one having no viscosity (elastic) and progressing with a second and a third characterized by increasing viscosity. The influence of viscosity is revealed in Figures 2-5. The Figures show that the more viscous the material is the less steep is the decay behind the wave front and for stations apart from the cylindrical surface of the hole the smaller is the amplitude of the discontinuity at the front of the wave.

ACKNOWLEDGMENT

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CAPTIONS FOR FIGURES

Fig. 1 Description of characteristic lines and wave front on the $(\bar{r} - \bar{t})$ plane.

Fig. 2 Radial stress for the stations r = a and r = 2.5a.

Fig. 3 Tangential stress for the stations r = a and r = 2.5a.

Fig. 4 Radial velocity for the station r = a.

Fig. 5 Radial velocity for the station r = 2.5a.

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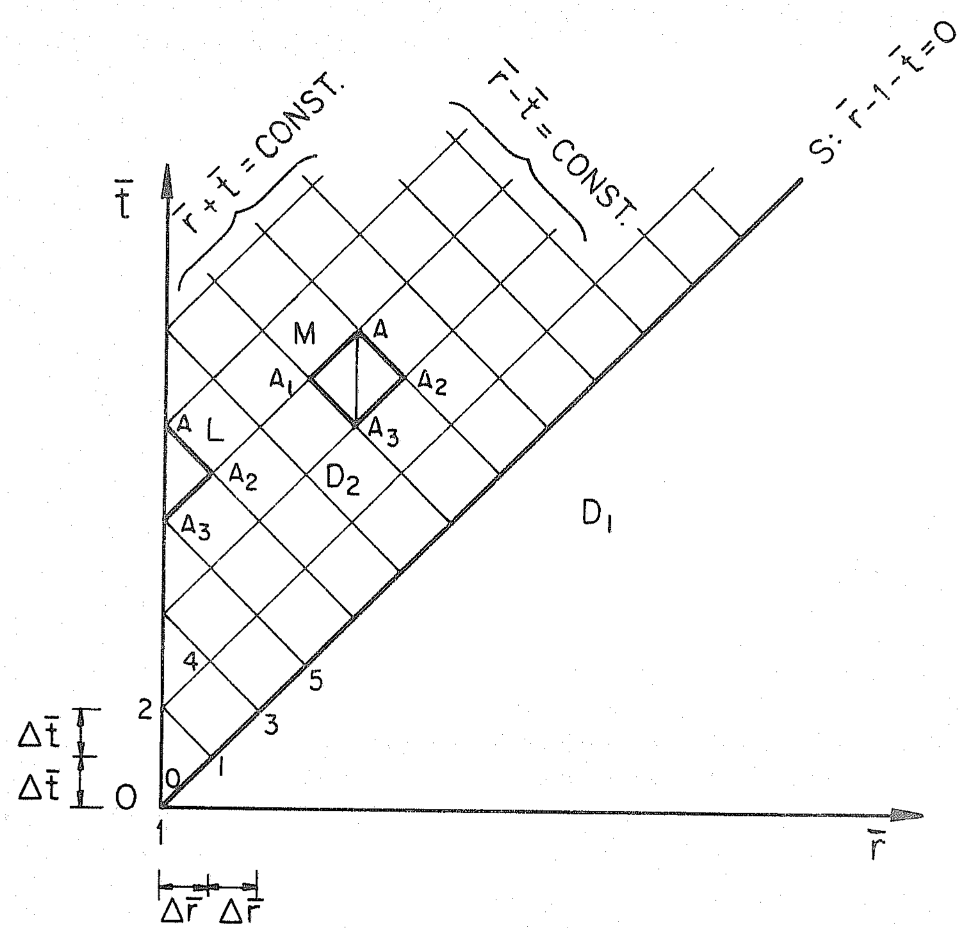
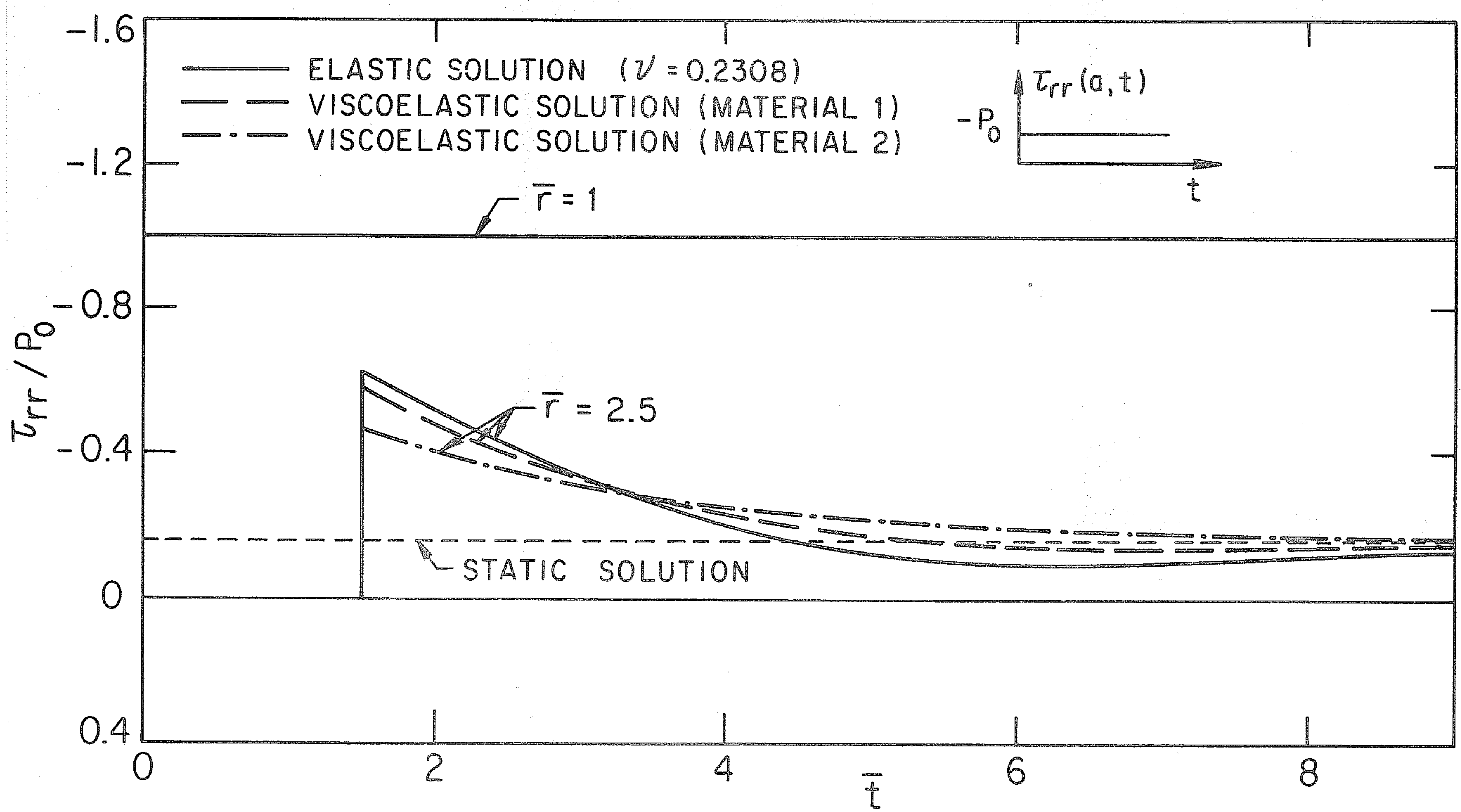


$\bar{r}+\bar{t}$ = CONST.
$\bar{r}-\bar{t}$ = CONST.
S: $\bar{r}$-1-$\bar{t}$=0
$\bar{t}$
$\bar{r}$
M
A
A_1
A_2
A_3
L
D_2
D_1
$\Delta\bar{t}$
$\Delta\bar{r}$

FIG. 1



ELASTIC SOLUTION (ν = 0.2308)
VISCOELASTIC SOLUTION (MATERIAL 1)
VISCOELASTIC SOLUTION (MATERIAL 2)
τrr (a, t)
-P0
t
r̄ = 1
r̄ = 2.5
STATIC SOLUTION
τrr / P0
-1.6
-1.2
-0.8
-0.4
0
0.4
0
2
4
6
8
t̄

FIG. 2

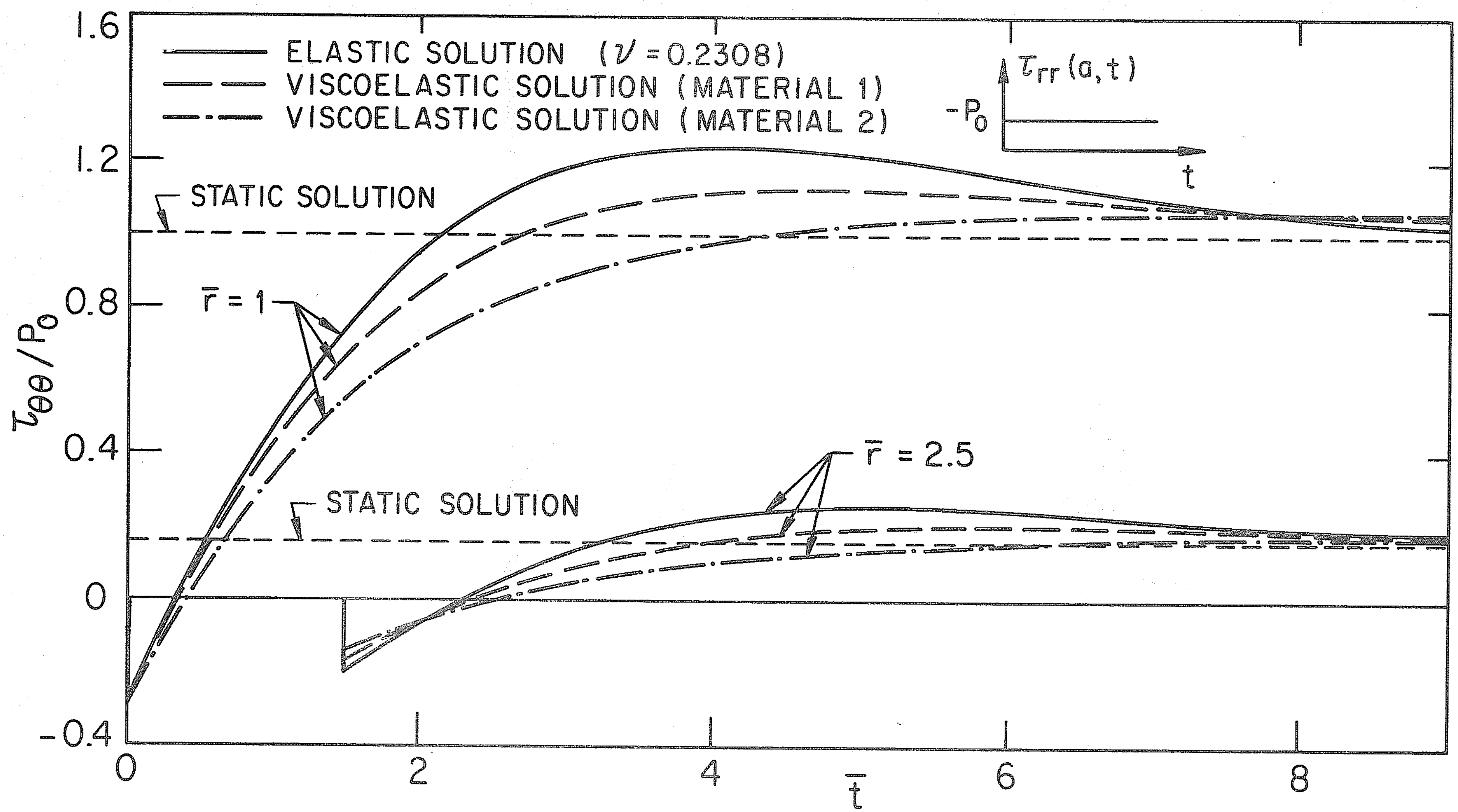
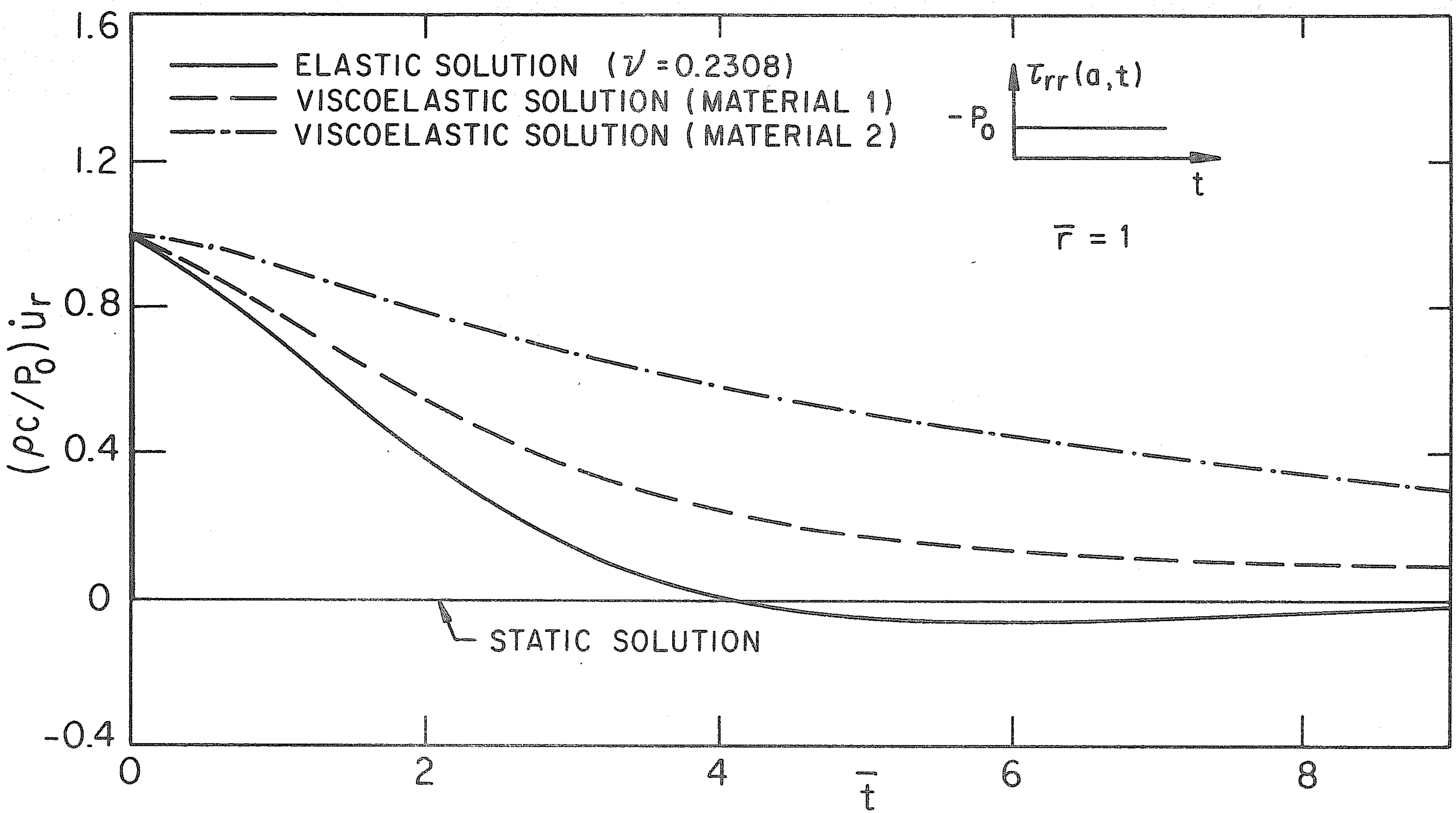
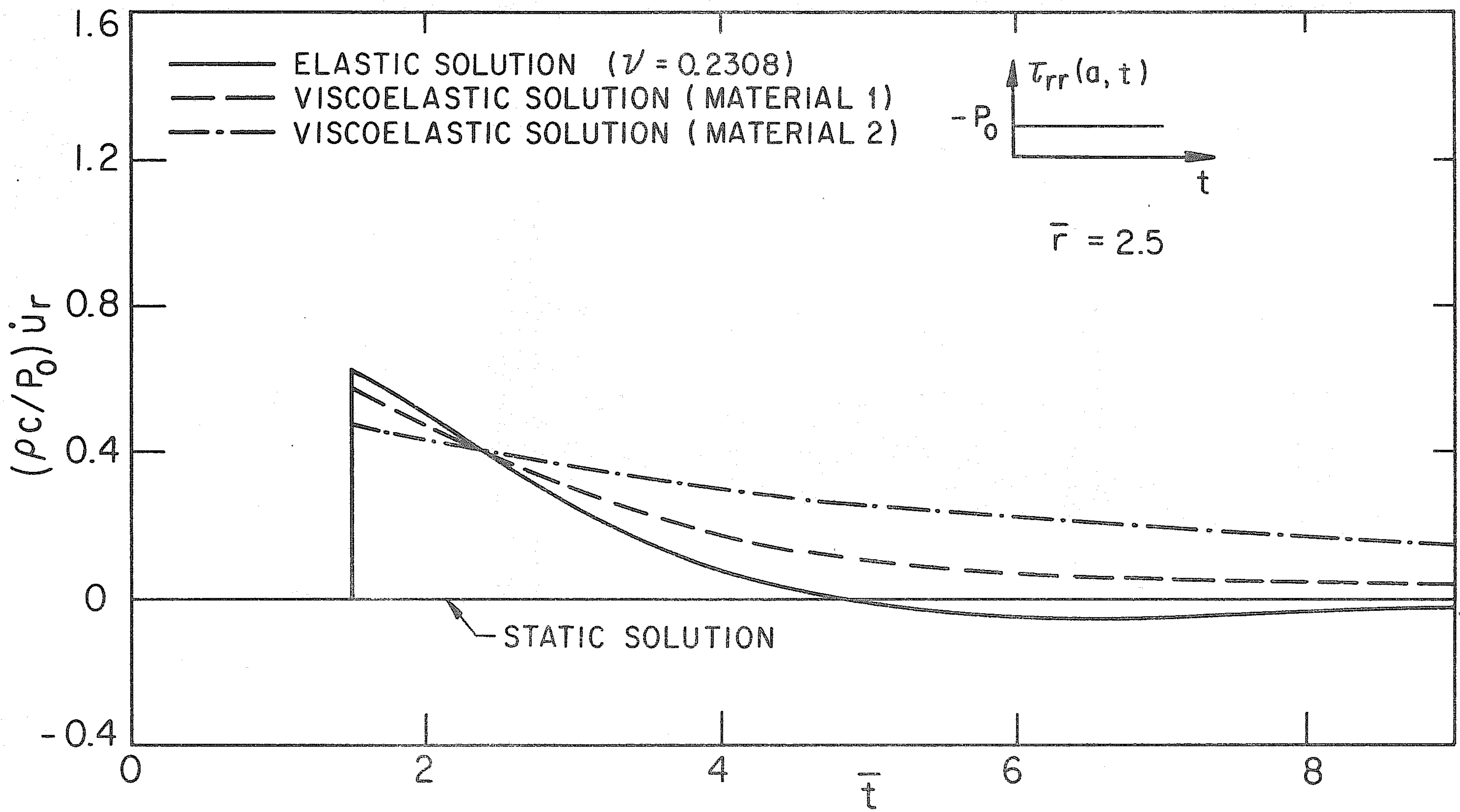

ELASTIC SOLUTION (ν = 0.2308)
VISCOELASTIC SOLUTION (MATERIAL 1)
VISCOELASTIC SOLUTION (MATERIAL 2)
$\tau_{rr}(a,t)$
$-P_0$
t
STATIC SOLUTION
$\bar{r} = 1$
$\bar{r} = 2.5$
STATIC SOLUTION
$\tau_{\theta\theta}/P_0$
1.6
1.2
0.8
0.4
0
-0.4
0
2
4
6
8
$\bar{t}$

FIG. 3



ELASTIC SOLUTION ($\bar{\nu}$ = 0.2308)
VISCOELASTIC SOLUTION (MATERIAL 1)
VISCOELASTIC SOLUTION (MATERIAL 2)
$\tau_{rr}(a,t)$
$-P_0$
t
$\bar{r} = 1$
STATIC SOLUTION
$(\rho c/P_0)\dot{u}_r$
1.6
1.2
0.8
0.4
0
-0.4
0
2
4
6
8
$\bar{t}$

FIG. 4



ELASTIC SOLUTION (ν = 0.2308)
VISCOELASTIC SOLUTION (MATERIAL 1)
VISCOELASTIC SOLUTION (MATERIAL 2)
$\tau_{rr}(a, t)$
$-P_0$
t
$\bar{r}$ = 2.5
STATIC SOLUTION
$(\rho c/P_0)\dot{u}_r$
1.6
1.2
0.8
0.4
0
-0.4
0
2
4
6
8
$\bar{t}$

FIG. 5