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Conicality of Morse Limit Sets and Stability

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In loving memory of my grandfather, Sam.

ABSTRACT OF THE DISSERTATION

Conicality of Morse Limit Sets and Stability

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One of the most successful techniques for studying groups acting on metric spaces has been to study actions on spaces which admit hyperbolic properties. We study a group G acting by isometries on a proper, geodesic metric space X by studying interactions between the group action on the space and hyperbolic-like boundaries for X. We present results regarding two different hyperbolic-like boundaries on X: the Morse boundary and the sublinearly Morse boundary. Both of these boundaries are quasi-isometry invariants for proper geodesic metric spaces.

Subgroup stability is a strong notion of quasiconvexity that generalizes convex cocompactness in a variety of settings. A characterization of convex cocompact Kleinian groups is that the limit set of the group is composed entirely of conical limit points in the boundary of the three dimensional hyperbolic space. We show that stable subgroups admit an identical conical limit point characterization in the Morse boundary. We also, additionally, show that stable subgroups are characterized by having an entirely horospherical limit set.

A group G is non-elementary if G is not virtually cyclic and if its boundary is not empty. We show that every non-elementary group acts minimally on its sublinearly Morse boundary, i.e., for any element $\mathfrak{a}$ in the sublinearly Morse boundary, $G\mathfrak{a}$ is dense in the boundary. This result, which is joint with Yulan Qing and Elliott Vest, is an important step towards understanding the dynamics of groups acting on their own sublinearly Morse boundary.

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Chapter 1

Introduction

In the study of groups, a fruitful tool for discovery and understanding lies in the study of its group actions, and in geometric group theory, we study groups by studying actions on metric spaces. Note that every group acts trivially on the metric space which consists of a single point, for this reason it is common to only study group actions which are nice in some way, such as cobounded actions, proper actions, or isometric actions, see Definition 2.2. In geometric group theory, the spaces that have yielded the most interesting properties in the last few decades have been spaces that either have a notion of negative curvature, called δ-hyperbolicity, or spaces which exhibit hyperbolic properties, see Definition 2.6. Many recent results are generalizations of group actions on $\mathbb{H}^n$. Good examples of these group actions include Kleinian groups, which are discrete groups of orientation preserving isometries on $\mathbb{H}^3$, and surface groups acting on $\mathbb{H}^2$.

Recall that $\mathbb{H}^n$ has a natural visual boundary of S^{n-1}, and this boundary plays an important role in understanding group actions on $\mathbb{H}^n$, both by analyzing the fixed points on the boundary given by the action of a group element, and by studying the limit points of the group action's orbit onto the boundary, ΛG. (Definition 2.16)

This idea unifies the topics found in this dissertation, where we study group actions on spaces with hyperbolic-like properties, and use a boundary of the metric space to study this group action. First, we look at a generalization of convex cocompact groups called stable subgroups, and we characterize these groups by studying the accumulation points of this group on the Morse boundary. Second, we explore another related boundary, the sublinearly Morse boundary, and we show that every finitely generated group acts minimally on its sublinearly Morse boundary. The results on minimality are joint with Elliott Vest and Yulan Qing.

1.1 Convex Cocompactness and Stability

Convex cocompact groups are an important example of Kleinian groups. These are exactly the subgroups H whose orbit in $\mathbb{H}^3$ is convex cocompact. Additionally these groups admit compact Kleinian manifolds via the quotient of the group on $\mathbb{H}^3$, and every infinite order element of a convex cocompact group is loxodromic. Furthermore, convex cocompactness is an open condition: given any matrix representation for the group, small perturbations of the entries results in another convex cocompact group. We highlight some of the other interesting properties of convex cocompact groups in the following theorem.

Theorem 1.1. *([Mar74, Sul85]) A Kleinian group $H < Iso^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$ is called convex cocompact if one of the following equivalent conditions hold:*

1. *H acts cocompactly on the convex hull of its limit set ΛH.*

2. *Any H-orbit in $\mathbb{H}^3$ is quasiconvex.*

3. *Every limit point of H is conical.*

4. *H acts cocompactly on $\mathbb{H}^3 \cup \Omega$, where $\Omega = \partial\mathbb{H}^3 \setminus \Lambda H$.* □

However, other more recent versions of this relationship have been shown. Swenson showed a generalization of this theorem for Gromov hyperbolic groups equipped with their visual boundaries [Swe01], and there has been recent interest in generalizing these relationships beyond the setting of word-hyperbolic groups. For example, convex cocompact subgroups of mapping class groups acting on Teichmüller space, equipped with the Thurston compactification, have been characterized by Farb and Mosher [FM01] and Hamenstädt [Ham05] as exactly the subgroups which determine Gromov hyperbolic surface group extensions. Specifically, given a subgroup $\Gamma \leq Mod(S)$, and it's surface group extension

$$1 \to \pi_1(S) \to E_\Gamma \to \Gamma \to 1,$$

the results of Farb, Mosher and Hamenstädt show that Γ is convex cocompact exactly when E_Γ is hyperbolic.

Additional work done in this direction has been done for subgroups of $Out(F_n)$, relating convex cocompact subgroups to hyperbolic extensions of free groups [HH18, ADT17]. In particular, given a free group $\mathbb{F}$ of rank $r \geq 3$, one can construct the short exact sequence

$$1 \to \mathbb{F} \to \mathrm{Aut}(\mathbb{F}) \to \mathrm{Out}(\mathbb{F}) \to 1.$$

By taking the preimage of a subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$ under the map $\mathrm{Aut}(\mathbb{F}) \to \mathrm{Out}(\mathbb{F})$, one obtains the extension on $\mathbb{F}$, denoted E_Γ, via $1 \to \mathbb{F} \to E_\Gamma \to \Gamma \to 1$.

Using work on Bestvina and Feighn [BF14], Dowdall and Taylor show that E_Γ is hyperbolic if all inifinte order elements of Γ are atoroidal and the action of Γ on the free factor complex of $\mathbb{F}$ has a quasi-isometric orbit map [DT18], i.e., when Γ is convex cocompact.

There has also been interest in creating generalizations which are applicable for any finitely generated group. An important generalization comes from [DT15], where Durham and Taylor introduced *stability* (see Definition 2.35) to characterize convex cocompact subgroups of a mapping class group in a way which is intrinsic to the geometry of the mapping class group, and in fact, generalizes the notions of convex cocompactness to any finitely generated group. The concept of stability was later generalized to *strongly quasiconvex subgroup*, introduced in [Tra19]. We note that a subgroup is stable when it is undistorted and strongly quasiconvex.

In the Kleinian, hyperbolic, and mapping class group settings, convex cocompactness is characterized by properties of the limit set on an appropriate boundary [Swe01, KL08]. For an arbitrary finitely generated group, it is possible to construct a (quasi-isometric invariant) boundary called the Morse boundary, which was introduced by Cordes in [Cor17] and expanded by Cordes and Hume in [CH17]. A generalization of convex cocompactness developed by Cordes and Durham, called *boundary convex cocompactness* (see Definition 2.36), uses both the Morse boundary and stability to generalize item (1) of Theorem 1.1, see [CD17].

In this dissertation, we fully generalize item (3) of Theorem 1.1 to the setting of finitely generated groups, thereby answering [CD17, Question 1.15]. In fact, we additionally generalize some other characterizations from the hyperbolic setting found in [Swe01]. We summarize these results in the following theorem:

Theorem 1.2. *Let H be a finitely generated group acting by isometries on a proper geodesic metric space X. The following are equivalent:*

1. *Any H-orbit in X is a stable embedding of $H \to X$.*
2. *H acts boundary convex cocompactly on X.*
3. *Every point in ΛH is a conical limit point of H, $\Lambda H \neq \emptyset$, and ΛH is a compact subset of the Morse boundary of H.*
4. *Every point in ΛH is a horospherical limit point of H, $\Lambda H \neq \emptyset$, and ΛH is a compact subset of the Morse boundary of H.*

Remark 1.3. *The result* (1) $\Leftrightarrow$ (2) *is found in the main theorem of [CD17]. We show* (3) $\Rightarrow$ (4) *in a combination of Propostion 3.4 and Theorem 3.8, using methods similar to [Swe01]. We show* (4) $\Rightarrow$ (2) *in Theorem 4.3, by first showing that non-cobounded actions on the weak convex hull of ΛH admit a sequence of points p_n which diverge quickly from the orbit (see Lemma 4.1), but then showing that the p_n converge to an element of ΛH, which ultimately contradicts the horospherical limit point assumption. We also give an alternate proof to* (2) $\Rightarrow$ (3) *in Proposition 4.5 which does not use the main theorem from [CD17].*

A limit point in ΛH is **conical** if the limit point is accumulated by the orbit in a strong way: every geodesic ray representing the limit point gets close to the orbit, see Definitions 2.12 and 3.2. In general, a geodesic ray which is constructed from geodesic segments $[x, hx]$ need not stay close to the orbit of H, even when H is nicely embedded into a hyperbolic space. For an example, see [Swe01, Lemma 3]. A limit point in ΛH is **horospherical** if it is accumulated by the orbit in a similar way: every horoball around a geodesic ray representing the limit point intersects the orbit. See Definition 2.12 and Definition 3.2 for definitions of horoball and a horospherical limit point, respectively.

We take a moment to provide a broad overview of stability in the recent literature. In addition to results for the mapping class group from above in [FM01, KL08, Ham05, DT15], it is also known that infinite index Morse subgroups of the mapping class group exactly coincide with stable subgroups [Kim19], and stable subgroups of mapping class groups (and more generally, stable subgroups of Morse local-to-global groups) have interesting combination theorems [RST21]. Stability has also been studied in the context of Morse local-to-global groups [CRSZ22], relatively hyperbolic groups [ADT17], and hierarchically hyperbolic groups [ABD21, RST23]. It is also known that stable subgroups admit finite height [AMST19]. Also related, work of Karrer-Miraftab-Zbinden [KMZ24] losens the condition of stability by analyzing the core of a weak convex hull.

Comparing Theorem 1.2 to Theorem 1.1, we see a cocompact action involving a domain of discontinuity in Theorem 1.1 which does not appear in Theorem 1.2. This is because the standard methods used for showing this property rely on the fact that the (Gromov-)hyperbolic boundary for a word hyperbolic group is a compactification, and thus finding the requisite compact set needed for a cocompact action boils down to finding an appropriate closed subset. In contrast, the Morse boundary usually does not compactify the underlying group. In fact, the Morse boundary compactifies a finitely generated group

H if and only if H is word hyperbolic, see [Cor17, Theorem 3.10] and [CD17, Lemma 4.1]. This leads to an open question:

Open Question 1.4. *Does there exist a classification of boundary convex cocompactness via an appropriate action on a domain of discontinuity analog?*

For other properties in Theorem 1.2, we are able to address the need for some compactness in the Morse boundary by assuming that the limit set of the group, ΛH, is a compact subset of the Morse boundary, see Definition 2.28 and Corollary 2.30. It is not possible to remove the compactness condition in either point (3) or (4) of Theorem 1.2, as we illustrate in the following example.

Example 1.5. *Consider the group $G = \mathbb{Z}^2 * \mathbb{Z} * \mathbb{Z} = \langle a, b\rangle * \langle c\rangle * \langle d\rangle$ with subgroup $H = \langle a, b, c\rangle$. As discussed in [CD17, Remark 1.8], H is isometrically embedded and convex in G, and so every point of ΛH is conical with respect to H. In fact, all rays representing a point in ΛH travel through H infinitely often. However H is not hyperbolic, so H is not stable, see [CD17, Section 1.2] for a complete discussion.*

1.1.1 Applications to Mapping Class Groups

Convex cocompact subgroups of mapping class groups have been well studied, see [FM01, Ham05], but in particular conical limit point characterizations have been analyzed before. Let S be a finite type surface, $\mathrm{Mod}(S)$ its associated mapping class group, and let $\mathcal{T}(S)$ be its associated Teichmüller space. In [KL08, Theorem 1.2], Kent and Leininger show that a subgroup H of $\mathrm{Mod}(S)$ is convex cocompact if and only if all the limit points of H in the Thurston compactification of $\mathcal{T}(S)$ are conical. By combining Theorem 1.2 with a theorem of Cordes and Durham, [CD17, Thorem 1.18], we obtain the following direct comparison, which uses the intrinsic geometry of $\mathrm{Mod}(S)$ instead of the geometry of $\mathcal{T}(S)$.

Theorem 1.6. *Let S be a finite type surface, let $H < \mathrm{Mod}(S)$ be finitely generated, and consider ΛH in the Morse boundary of $\mathrm{Mod}(S)$. Then H is a convex-cocompact subgroup of $\mathrm{Mod}(S)$ if, and only if, every point in ΛH is a conical limit point of $H \curvearrowright \mathrm{Mod}(S)$, $\Lambda H \neq \emptyset$, and ΛH is compact in the Morse boundary of $\mathrm{Mod}(S)$.*

This theorem, combined with the above result of [KL08], gives the following immediate corollary, which shows that conicality is a strong condition in the setting of mapping class groups. We discuss these results in Chapter 5.

Corollary 1.7. *Let S be a finite type surface, and let $H < \mathrm{Mod}(S)$ be finitely generated. The following are equivalent:*

1. *Every limit point of H in the Morse boundary of $\mathrm{Mod}(S)$ is a conical limit point of $H \curvearrowright \mathrm{Mod}(S)$ and ΛH is compact.*
2. *Every limit point of H in the Thurston compactification of $\mathcal{T}(S)$ is a conical limit point of $H \curvearrowright \mathcal{T}(S)$.*

We also show that there exists a natural $\mathrm{Mod}(S)$-equivariant map from $\mathrm{Mod}(S)$ to $\mathcal{T}(S)$ which sends conical limit points of $H < \mathrm{Mod}(S)$ in the Morse boundary of $\mathrm{Mod}(S)$ to conical limit point of H in the the Thurston compactification of $\mathcal{T}(S)$. This directly proves the implication (1) $\Rightarrow$ (2) in Corollary 1.7 without requiring results of [KL08], and in fact, does not require H to be a convex cocompact subgroup, see Theorem 5.2 for details.

1.2 Sub-linearly Morse Boundary and Minimality

The boundary of focus in the prior section, the Morse boundary, is one of many boundaries that can be constructed for a given metric space. A foundational result of Gromov [Gro87] constructs the visual boundary of a δ-hyperbolic space and shows that this boundary is a quasi-isometry invariant of these spaces, see Definition 2.6 and Definition 2.9. In the setting of CAT(0) spaces, the visual boundary is not an invariant, as shown by Croke and Kleiner in [CK00]. However, this was addressed by Charney and Sultan [CS14] who introduced the contracting boundary for CAT(0) spaces and showed that contracting boundaries are quasi-isometry invariants on these spaces. Every geodesic ray in a δ-hyperbolic space is a contracting geodesic, so we can think of the contracting boundary of a CAT(0) space as the part of the boundary which only sees the "hyperbolic directions" of the space.

The Morse boundary, as discussed above, was introduced by Cordes [Cor17]. Every Morse geodesic (see Definition 2.17) is a contracting geodesic in the CAT(0) setting, and every geodesic ray in a δ-hyperbolic space is uniformly Morse, see Lemma 2.7. In this way, the Morse boundary is a generalization of the contracting boundary for arbitrary geodesic metric spaces. However, this is not the only generalization of the contracting property found in the literature, see also [Min94, ACGH17, Mur19].

In [QR22], Qing and Rafi introduce the concept of a sublinearly contracting ray, and its associated sublinearly contracting boundary. As a set, this boundary consists of all sublinearly contracting rays and is a quasi-isometry invariant of CAT(0) spaces. A further generalization to the sublinearly Morse boundary was then introduced in [QRT] by Rafi, Qing, and Tiozzo, and this boundary, like the Morse boundary, is a quasi-isometry invariant of any proper, geodesic metric space, see Definition 6.3 and Definition 6.8.

Recall that a group is non-elementary if it is not virtually cyclic and if its boundary is non-empty. A boundary of a non-elementary group is said to be minimal if any boundary-orbit of the group is dense in the boundary, see Definition 6.11. Note that this is a stronger condition than requiring that there exists at least one dense orbit, see Remark 6.12. Every non-elementary hyperbolic group is minimal, see [KB02, Proposition 4.2] for a discussion. Murray showed that non-elementary CAT(0) groups have minimal contracting boundaries [Mur19], and Liu showed that Morse non-elementary groups have minimal Morse boundaries [Liu21]. The action of a CAT(0) group on its sublinearly contracting boundary is minimal by work of Qing and Zalloum, see [QZ19].

Of the boundaries discussed here, there is only one where it is was not known if the boundary is minimal: the sublinearly Morse boundary of a proper, geodesic metric space. We show that this boundary is indeed minimal. This result is joint with Yulan Qing and Elliott Vest.

Theorem 1.8. *Let G be a finitely generated group, and let $\partial_\kappa G$ be its sublinearly Morse boundary. If $|\partial_\kappa G| \geq 3$, then the group action $G \curvearrowright \partial_\kappa G$ is minimal.*

Notice that the requirement that $|\partial_\kappa G| \geq 3$ guarantees that G is non-elementary. The proof of Theorem 1.8 relies heavily on Lemma 6.15, which, given $\mathfrak{b} \in \partial_\kappa G$ and $\mathfrak{a} \in \partial_\kappa G$, constructs uniform quality quasi-geodesics that begin by projection to a geodesic representative of $g\mathfrak{a}$, then eventually fellow-travel a geodesic representative of $\mathfrak{b}$. The sublinearly Morse property shows that these quasi-geodesics sublinearly converge, see Theorem 6.23.

Chapter 2

Background

In the introduction, we established context for our main results by describing how our main results compare with the current literature. This chapter has the same goal of establishing context, but instead we provide context by exploring the mathematical constructions and proofs required for our results, and by exploring some key details and constructions in familiar settings. We begin by providing a brief overview of introductory topics in geometric group theory, which may be found in many introductory textbooks on geometric group theory, such as [CM17, Hd00, DK18]. We also prove some claims in the setting of hyperbolic spaces, before moving on to more recent foundational results required for Theorem 1.2.

Although the constructions required for Theorem 1.8 are similar to the constructions in this chapter, we delay our discussion of the sublinear boundary to Chapter 6.

2.1 Groups and Hyperbolicity

We first explore the topic of groups acting on metric spaces. We take a moment to recall some base definitions from the realm of metric geometry and to set some notation. An **isometric embedding** is a function between metric spaces $f : X \to Y$ such that the distance is preserved by f, i.e. so that for any $x_1, x_2 \in X$, we have $d(x_1, x_2) = d(f(x_1), f(x_2))$. Such a function is always injective. An **isometry** is a surjective isometric embedding, and a **geodesic** is an isometric embedding of a closed interval. In particular, an isometric embedding of $[a, b]$ is a **geodesic segment**, an isometric embedding of $[a, \infty)$ is a **geodesic ray**, and an isometric embedding of $(-\infty, \infty)$ is a **geodesic line**. Given two points x and y in a metric space, we denote $[x, y]$ to be a geodesic between them, and we call a metric space X a **geodesic metric space** if there exists at least one geodesic between every pair of points. Given $A \subset X$ and $M \geq 0$, we denote the M**-neighborhood of** A by $\mathcal{N}_M(A) = \{x \in X : d(x, A) \leq M\}$.

2.1.1 Finitely Generated Groups and Cayley Graphs

Unless otherwise stated, we'll use 1 for the identity element for a group, and we'll write the group operation as multiplication. If G is a finitely generated group, then it is straightforward to show that there exists a generating set S such that $G = \langle S \rangle$ where S is finite, $1 \notin S$, and $S^{-1} \subseteq S$. Indeed, given some other finite generating set S', one can simply remove the identity element creating $S'' = S' \setminus \{1\}$, then we can define $S = S'' \cup (S'')^{-1}$. While assuming these extra conditions on a generating set S are not required for the constructions to follow, they do make the constructions more convenient.

Given a group G with the above finite generating set S, we can construct a metric graph that encodes it called the **Cayley graph**. Although we do not use this construction in any of the novel results in this work, this construction does provide a plethora of examples where our results can be used and is the motivating setting for our results.

Definition 2.1. *Let G be a finitely generated group, and say $G = \langle S \rangle$. The* **Cayley graph of G with respect to S**, *denoted $Cay(G, S)$, is the metric graph such that:*

1. *The vertex set of $Cay(G, S)$ is G.*

2. *For $g, h \in G$, there exists an edge between g and h if and only if $g^{-1}h \in S$ or $h^{-1}g \in S$.*

3. *Every edge has length 1.*

The third point in the definition defines a metric on $Cay(G, S)$, denoted d_S: the distance between two points in $Cay(G, S)$ is the length of the shortest path between them. It is clear from this definition that $Cay(G, S)$ is a proper and geodesic: since $|S| < \infty$ all closed balls are compact, and between any two points in $Cay(G, S)$, there exists a path between them whose length realises the distance between them. We also note that restricting d_S to the vertices of $Cay(G, S)$ produces the word metric of G with respect to S.

A key property of Cayley graphs is that they admit very nice group actions. Recall the following properties of group actions on metric spaces.

Definition 2.2. *Let G be a group, let X be a metric space. The group action $G \curvearrowright X$ is...*

- **proper** *if, for all $K > 0$ and every $x, y \in X$, the set $\{g \in G : d(gx, y) \leq K\}$ is finite.*

- **cobounded** *if there exists $K > 0$ so that, for every $x, y \in X$, there exists $g \in G$ so* *that* $d(gx, y) \leq K$. *Equivalently, if there exists a ball of radius* K, $B_K(x_0)$, *so that* $X = \bigcup_{g \in G} gB_K(x_0)$.

- **by isometries** *if the map* $x \mapsto gx$ *is a isometry for every* $g \in G$.

If the group action satisfies all of these conditions, we call the group action **geometric**.

Notice that every Cayley graph of G admits a very natural G action via left multiplication, indeed, g sends the vertices h and k to gh and gk, respectively, and if there is an edge between h and k, there will be an edge between gh and gk. It is a straightforward exercise to see that this group action is geometric, and that any subgroup $H \leq G$ acts properly and by isometries on $Cay(G, S)$. Note that the action $H \curvearrowright Cay(G, S)$ is cobounded exactly when H has finite index in G.

This gives the following very useful observation: every finitely generated group acts geometrically on a proper geodesic metric space. However, most finitely generated groups G have many different finite generating sets, and these graphs can look very different. For example, consider $Cay(\mathbb{Z}, \{1\})$ and $Cay(\mathbb{Z}, \{2, 3\})$.

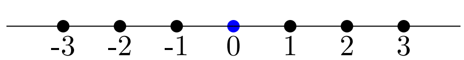

Figure 2.1: Cayley Graph with generating set $\{1\}$

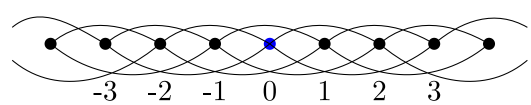

Figure 2.2: Cayley Graph with generating set $\{2, 3\}$

These both are metric graph representatives of $\mathbb{Z}$, however they induce different metrics on $\mathbb{Z}$: Notice that $d_{\{1\}}(-2, 3) = 5$, but $d_{\{2,3\}}(-2, 3) = 2$. In particular, this shows the identity map $G \to G$ is not an isometry. In addition, $Cay(\mathbb{Z}, \{1\})$ and $Cay(\mathbb{Z}, \{2, 3\})$ are not graph isomorphic, not homeomorphic, and not homotopic. However these two graphs do admit some similarities, in that there appear to be only two primary directions of travel in both graphs. In fact, if one squints very hard while looking at these two graphs, they appear to be the same! We now introduce a notion of equivalence that formalizes this idea.

Definition 2.3. *Let X and Y be metric spaces, and let $K \geq 1$ and $C \geq 0$. A function $f : X \to Y$ is called a (K, C)*-**quasi-isometric embedding** *if, for all $x_1, x_2 \in X$,*

$$\frac{1}{K} d(x_1, x_2) - C \leq d(f(x_1), f(x_2)) \leq K d(x_1, x_2) + C.$$

Additionally, f is called C-**coarsely surjective** *if for all $y \in Y$ there exists $x \in X$ so that $d(f(x), y) \leq C$. If f is a (K, C)-quasi-isometric embedding and is C-coarsely surjective, we say f is a (K, C)*-**quasi-isometry**.

It there exists constants K, C so that f is a (K, C)-quasi-isometric embedding or a (K, C)-quasi-isometry, then we call f a **quasi-isometric embedding** *or a* **quasi-isometry**, *respectively. We additionally define a* **quasi-geodesic** *to be a quasi-isometric embedding of a closed interval.*

Remark 2.4. *In general, quasi-geodesics do not need to be continuous, for example the floor function $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{R}$ is a $(1, 1)$-quasi-geodesic. However, if X is a geodesic metric space, every quasi-geodesic $\varphi : I \to X$ is bounded Hausdorff distance from a continuous quasi-geodesic: Let $S = \{\text{endpoints of } I\} \cup \{n \in \mathbb{Z} : n \in I\}$, then create a new quasi-geodesic by sequentially connecting the points in $\varphi(S)$ with geodesic segments.*

Note that there is a $(3, 2)$-quasi-isometry between $Cay(\mathbb{Z}, \{1\})$ and $Cay(\mathbb{Z}, \{2, 3\})$. In fact, any two Cayley graphs for a finitely generated group G admit quasi-isometric Cayley graphs. Although it is straightforward to show this directly, (see [CM17, Theorem 7.5]) it is also a consequence of the following foundational lemma.

Lemma 2.5 (Milnor-Swartz)**.** *Let G be a group and X be a geodesic metric space. If there exists a geometric group action $G \curvearrowright X$, then*

1. *$G = \langle S \rangle$ for a finite set S*

2. *There exists a quasi-isometry between $Cay(G, S)$ and X.*

Thus we show the equivalence of Cayley graphs for a finitely generated group G by recalling that G acts geometrically on each of its Cayley graphs. We also note that the existence of a quasi-isometry between two metric spaces induces an equivalence relation on the set of all metric spaces, see [CM17, Chapter 7, Exercise 10].

2.1.2 Hyperbolic Spaces

One of the benefits of introducing Lemma 2.5 is that it produces a large number of immediate interesting examples. Given any closed Riemmanian surface S, $\pi_1(S)$ acts geometrically on the universal cover of S. With the exception of the sphere and the torus, the universal cover of all other closed Riemannian surfaces is $\mathbb{H}^2$. This shows that most surface groups have a Cayley Graph which is quasi-isomorphic to $\mathbb{H}^2$. Around 1985, Gromov showed that many of the properties of surface groups which arise from the connection to $\mathbb{H}^2$ come from a coarse version of hyperbolicity, see [Gro87, KB02].

Definition 2.6. *Let X be a geodesic metric space. We call X a* **δ-hyperbolic** *metric space if every geodesic triangle is δ-slim, i.e., if for every $x, y, z \in X$, $[x, z] \subseteq \mathcal{N}_\delta([x, y] \cup [y, z])$. We call X a* **hyperbolic space** *if X is δ-hyperbolic for some $\delta \geq 0$.*

As a first example of a hyperbolic space, one can consider metric trees, for example, $Cay(\mathbb{Z}, \{1\})$ and $Cay(\langle a, b \rangle, \{a, b\})$. The second example, the Cayley graph of the free group on two generators, is familiar as the universal cover of the wedge of two circles. It is clear to see that these examples, and in fact any metric tree, are all 0-hyperbolic.

Another easy example of a hyperbolic space is, as the name suggests, $\mathbb{H}^n$. To see why this space is hyperbolic in the sense of Definition 2.6, we can use the fact that every triangle in $\mathbb{H}^n$ has an area bounded by π. (See [BH09, Proposition III.H.1.4]) Suppose there exists a point p on the side of a geodesic triangle such that the distance from p to both of the other sides is greater than 2. Then there exists a semicircle of radius 2 centered at p contained in the triangle, but this semicircle has an area greater than π.

A key fact about δ-hyperbolic spaces is that quasi-geodesics must travel close to geodesics. This idea motivates the definition of the Morse boundary that we use in Theorem 1.2, and it leads to a useful tool in the setting of hyperbolic spaces.

Lemma 2.7 (Morse Lemma)**.** *Let X be a proper, geodesic δ-hyperbolic space. There exists a (non-decreasing) function $N : [1, \infty) \times [0, \infty) \to [0, \infty)$ such that, for any geodesic α and any (K, C)-quasi-geodesic $\varphi : [a, b] \to X$ such that $\varphi(a), \varphi(b) \in \alpha$, we have that $\varphi \subseteq \mathcal{N}_{N(K,C)}(\alpha)$.*

A detailed proof of this lemma can be found in [BH09, Theorem III.H.1.7]. Note that the Morse lemma is not true for Euclidean spaces: Let α be the x-axis in $\mathbb{R}^2$ and let φ_n be the concatenation of the geodesic segments $[(0, 0), (n, n)]$ and $[(n, n), (2n, 0)]$. Then for each $n \in \mathbb{N}$, φ_n is a $(\sqrt{2}, 0)$-quasi-geodesic with endpoints on α, but $\varphi_{n+1} \not\subseteq \mathcal{N}_n(\alpha)$.

Besides being a property of hyperbolic spaces, the Morse lemma exactly classifies a geodesic metric space as a δ-hyperbolic space, where δ depends only on the Morse function N. This key insight shows that the geodesic rays which exhibit fellow-travelling behavior are essentially the "hyperbolic directions" of the space, see Definition 2.17.

Proposition 2.8. *Let X be a proper geodesic space. Suppose there exists a (non-decreasing) function $N : [1, \infty) \times [0, \infty) \to [0, \infty)$ such that, for any geodesic α and any (K, C)-quasi-geodesic $\varphi : [a, b] \to X$ with $\varphi(a), \varphi(b) \in \alpha$, we have that $\varphi \subseteq \mathcal{N}_{N(K,C)}(\alpha)$. Then X is a δ-hyperbolic space where $\delta = N(3, 0)$.*

Proof. Let $x, y, z \in X$ be arbitrary. Recalling that $[x, z]$ is closed, let $w \in [x, z]$ be such that $d(y, w) = d(y, [x, z])$. Notice that $d(y, w) \leq d(y, s)$ for any $s \in [x, z]$. Let φ be the concatenation of $[x, w]$ and $[w, y]$, and let ψ be the concatenation of $[z, w]$ and $[w, y]$. We show that φ is a $(3, 0)$-quasi-geodesic.

Let $u, v \in \varphi$. By Definition 2.3, it suffices to show that $d(u, v) \leq d_\varphi(u, v) \leq 3d(u, v)$, where d_φ measures the distance between points along the path φ. The first inequality is always satisfied since the φ-subpath is a path from u to v. The second inequality is trivial if u and v are on the same segment of φ, so we assume that $u \in [x, w]$ and $v \in [w, y]$, see Figure 2.3.

Since $v \in [y, w]$, notice that $d(v, w) \leq d(v, s)$ for any $s \in [x, z]$. So in particular, $d(w, v) \leq d(u, v)$. Using this fact together with the triangle inequality, we get

$$d_\varphi(u, v) = d(u, w) + d(w, v) \leq d(u, v) + d(v, w) + d(w, v) \leq d(u, v) + d(u, v) + d(u, v).$$

Thus φ is a $(3, 0)$-quasi-geodesic. By a similar argument, ψ is also a $(3, 0)$-quasi-geodesic. Therefore, $\varphi \subseteq \mathcal{N}_{N(3,0)}([x, y])$ and $\psi \subseteq \mathcal{N}_{N(3,0)}([z, y])$. So in particular, $[x, z] \subseteq \varphi \cup \psi \subseteq \mathcal{N}_{N(3,0)}([x, y] \cup [z, y])$. So by Definition 2.6, X is $N(3, 0)$-hyperbolic. $\square$

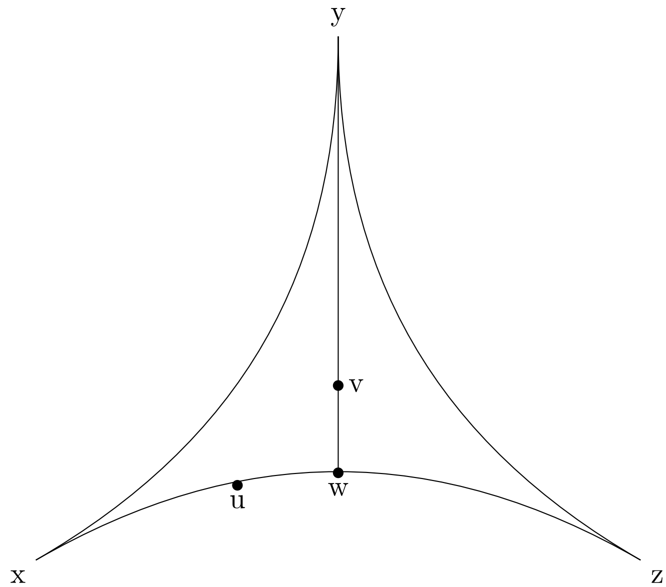

Figure 2.3: Diagram for Proposition 2.8. Since the path $\varphi = [x, w] * [w, y]$ is a $(3, 0)$-quasi-geodesic, the Morse lemma guarantees that $[x, w]$ is at least within $N(3, 0)$ of $[x, y]$.

2.1.3 Visual Boundaries of Hyperbolic Spaces

Motivated by the fact that $\mathbb{H}^n$ has a natural boundary of S^n, which compactifies the space, we now provide a definition for an analogous boundary called the visual boundary, and in fact reproduces the S^n boundary for $\mathbb{H}^n$.

Definition 2.9. *Let X be a proper geodesic space, and let $\mathfrak{o} \in X$. Let $R_{\mathfrak{o}}(X)$ be the collection of all geodesic rays $\alpha : [a, \infty) \to X$ such that $\alpha(a) = \mathfrak{o}$. Then we can define an equivalence relation on $R_{\mathfrak{o}}(X)$ by setting $\alpha \sim \beta$ whenever the Hausdorff distance between α and β is bounded. The* **visual boundary** *of X based at $\mathfrak{o}$ is defined to be $\partial_\infty X_{\mathfrak{o}} = R_{\mathfrak{o}}(X)/\sim$. We equip $\partial_\infty X_{\mathfrak{o}}$ with the topology generated by the neighborhood basis for α, $U(\alpha, r, n) = \{\beta \in \partial_\infty X_{\mathfrak{o}} : d(\alpha(t), \beta(t) \leq r$ for all $t \leq n\}$.*

To give a colloquial interpretation of this definition, the visual boundary encodes points "at infinity" by identifying the geodesic rays that travel in the same direction. The topology described by the neighborhood basis states that two points at infinity are close together if the sight lines corresponding to those points fellow travel for a long time. Note that if X is a hyperbolic space, then $X \cup \partial_\infty X_{\mathfrak{o}}$ is a compact space, see [BH09, Proposition III.H.3.7]. Whenever $G \curvearrowright X$ acts by isometries, then $G \curvearrowright R_{\mathfrak{o}}(X)$ is well defined, and this passes down to an action $G \curvearrowright \partial_\infty X_{\mathfrak{o}}$.

Remark 2.10. *In the case where X is δ-hyperbolic, we may use the Morse lemma (Lemma 2.7) to equivalently define $\partial_\infty X_{\mathfrak{o}}$ by instead considering the collection of all quasi-geodesic rays $\varphi : [0, \infty) \to X$ with $\varphi(0) = \mathfrak{o}$.*

One of the key properties of the visual boundary is that it is a quasi-isometry invariant of δ-hyperbolic spaces. We formalize this statement below:

Proposition 2.11. *Let X and Y be proper geodesic spaces and assume Y a δ-hyperbolic space. If $f : X \to Y$ is a (K, C)-quasi-isometry, then*

1. *X is δ'-hyperbolic where δ' depends only on K, C, and δ.*

2. *f induces a homeomorphism $\partial f : \partial_\infty X_{\mathfrak{o}} \to \partial_\infty Y_{f(\mathfrak{o})}$.*

2.2 Morse Boundaries and Stability

In this section, we begin by exploring the definitions and statements found in [Swe01], whose main theorem generalizes Theorem 1.1 into the setting of δ-hyperbolic spaces. We then introduce analogs appropriate for the setting of Morse boundaries and explore their properties. We begin by setting some notation.

Definition 2.12. *Let (X, d) be a proper, geodesic metric space, and let $\alpha : [a, \infty) \to X$ and $\beta : [b, \infty) \to X$ be two geodesic rays.*

- *Given a closed set $S \subseteq X$, we define the* ***closest point projection to*** *S as $\pi_S(x) = \{s \in S : d(s, x) = d(S, x)\}$.*
- *We say α and β* ***N-asymptotically fellow-travel****, denoted by $\alpha \sim_N \beta$, if there exists $T \in \mathbb{R}$ so that whenever $t \geq T$, we have $d(\alpha(t), \beta(t)) \leq N$.*

In addition, if X is δ-hyperbolic we have the following definitions from [Swe01]:

- *We denote the* ***horoball about*** *α by $H(\alpha)$ and define it as $H(\alpha) = \bigcup\{\beta([b, \infty)) : \beta \sim_{6\delta} \alpha,\ b \geq a\}$.*
- *We denote the* ***funnel about*** *α by $F(\alpha)$ and define it as $F(\alpha) = \{x \in X : d(x, \alpha) \leq d(\pi_\alpha(x), \alpha(a))\}$.*
- *Given a point x in the visual boundary of X and a subset $A \subseteq X$, we say x is a* ***horopherical limit point of*** *A if, for every geodesic ray α with $\alpha(\infty) = x$, we have $H(\alpha) \cap A \neq \emptyset$.*
- *Given a point x in the visual boundary of X and a subset $A \subseteq X$, we say x is a* ***funneled limit point of*** *A if, for every geodesic ray α with $\alpha(\infty) = x$, we have $F(\alpha) \cap A \neq \emptyset$.*

- *Given a point x in the visual boundary of X and a subset $A \subseteq X$, we say x is a* ***conical limit point of*** *A if there exists $K > 0$ such that, for every geodesic ray α with $\alpha(\infty) = x$, we have $\mathcal{N}_K(\alpha) \cap A \neq \emptyset$.*

Remark 2.13. *Notice that, given $S \subseteq X$ and a point $x \in X$, the nearest point projection $\pi_S(x)$ need not have a uniform bound, even in the case where S is a geodesic. For example, one can take x to be the origin in $\mathbb{R}^2$ equipped with the L_1 metric, and take S to be the geodesic between $(n, 0)$ and $(0, n)$ which passes through (n, n). Then $\pi_S(x) = \{(n, 0), (0, n)\}$.*

We present here for completeness a relaxed version of a claim in [Swe01, pg 125] which shows that every horoball of a geodesic ray contains a funnel of an equivalent geodesic ray in a δ-hyperbolic space.

Lemma 2.14. *Let (X, d) be a proper, geodesic, δ-hyperbolic space, and let $\alpha : [0, \infty) \to X$ be a geodesic ray. Define $\alpha' : [0, \infty) \to X$ by $\alpha'(t) = \alpha(t + 6\delta)$. Then $F(\alpha') \subseteq H(\alpha)$.*

Proof. See Figure 2.4. Let $p \in F(\alpha')$. Construct a geodesic ray $\beta : [b, \infty) \to X$ such that $\beta \sim_{6\delta} \alpha$ and $\beta(b) = p$ (for details on the existence of such a geodesic ray, we refer to [BH09, pg 427-428]). Let $q \in \pi_{\alpha'}(p)$ such that $d(\alpha(0), q) = \min\{d(\alpha(0), x) : x \in \pi_{\alpha'}(p)\}$, i.e., so that q is the point in $\pi_{\alpha'}(p)$ closest to $\alpha(0)$. Notice that since $p \in F(\alpha')$ we have that $d(p, q) \leq d(q, \alpha'(0))$. Choose $T \geq 6\delta$ so that $q \in [\alpha'(0), \alpha'(T)]$ and so that for all $t \geq T$, we have $d(\alpha(t), \beta(t)) < 6\delta$. Then

$$T - b = d(\beta(T), p) \leq d(\beta(T), \alpha(T)) + d(\alpha(T), q) + d(q, p)$$

$$\leq 6\delta + d(\alpha'(T - 6\delta), q) + d(q, \alpha'(0)) = 6\delta + (T - 6\delta).$$

This shows that $b \geq 0$, and so $p = \beta(b) \in H(\alpha)$. ∎

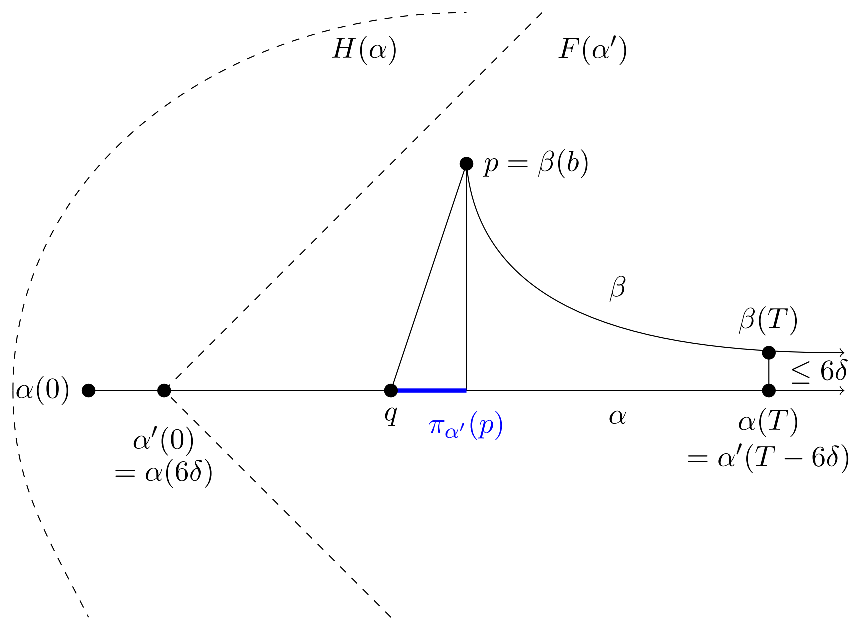

Figure 2.4: Diagram for Lemma 2.14

We also include the complementary statement that every funnel of a geodesic ray contains a horoball of an equivalent geodesic ray.

Lemma 2.15. ([Swe01, Lemma 5]) *Let (X, d) be a proper, geodesic, δ-hyperbolic metric space, and let $\alpha : [0, \infty) \to X$ be a geodesic ray. Define $\alpha' : [0, \infty) \to X$ by $\alpha'(t) = \alpha(t+12\delta)$. Then $H(\alpha') \subseteq F(\alpha)$.* □

The combination of Lemma 2.14 and Lemma 2.15 give the following relationship, which was originally stated as a corollary in [Swe01].

Corollary 2.16. *In a proper, geodesic, δ-hyperbolic metric space, the funneled limit points are exactly the horospherical limit points.* □

In a proper, geodesic, δ-hyperbolic metric space, any two geodesic rays α and β with $d_{Haus}(\alpha, \beta) < \infty$ can be re-parameterized so that they asymptotically fellow-travel (in the sense of Definition 2.12, for a fellow-travelling constant depending only on δ, see [Swe01, Lemma 4]. This is an important part of the definition of a horoball in Definition 2.12. In order to define a horoball in a non-hyperbolic space, it will be necessary to develop an analogue of this fact for Morse rays. We begin by recalling the definition.

Definition 2.17. ([Cor17, Definition 1.3]) *A (quasi)-geodesic γ in a metric space is called N-**Morse**, where N is a function $[1,\infty)\times[0,\infty)\to[0,\infty)$, if for any (K,C)-quasi-geodesic φ with endpoints on γ, we have $\varphi\subset\mathcal{N}_{N(K,C)}(\gamma)$. We call the function N a **Morse gauge**. We say γ is **Morse** if there exists a Morse gauge N so that γ is N-Morse.*

Definition 2.18. ([Cor17, CH17]) *Given a Morse gauge N and a basepoint $\mathfrak{o}\in X$, the N-**Morse stratum**, denoted $X_{\mathfrak{o}}^N$, is defined as the set of all points x such that $[\mathfrak{o},x]$ is an N-Morse geodesic. Each such stratum is δ-hyperbolic for δ depending only on N [CH17, Proposition 3.2], and thus has a well defined visual boundary, which we denote as $\partial X_{\mathfrak{o}}^N$. If $\mathcal{M}$ is the set of all Morse gauges, then there is a natural partial order on $\mathcal{M}$: $N\leq N'$ if $N(K,C)\leq N'(K,C)$ for all K and C. Note the natural inclusion $\partial X_{\mathfrak{o}}^N\hookrightarrow\partial X_{\mathfrak{o}}^{N'}$ is continuous whenever $N\leq N'$ by [Cor17, Corollary 3.2]. We define the **Morse boundary based at $\mathfrak{o}$** as*

$$\partial X_{\mathfrak{o}}=\varinjlim_{\mathcal{M}}\partial X_{\mathfrak{o}}^N$$

with the induced direct limit topology. Given a Morse geodesic ray α, we denote the associated point in $\partial X_{\mathfrak{o}}$ as $\alpha(\infty)$.

Remark 2.19. *Often when studying the Morse boundary, the basepoint is suppressed from the notation, as the Morse boundary is basepoint independent [Cor17, Proposition 2.5].*

However, we will often make use of the basepoint explicitly in the arguments to come, thus we keep it in the notation.

The following fact states that subrays of Morse rays are also Morse. This will be especially useful in Chapter 3, as many of the arguments which describe the relationships between horoballs, funnels, and cones require restriction to a subray, as illustrated in the proof of Lemma 2.14.

Lemma 2.20. *([Liu21, Lemma 3.1]) Let X be a geodesic metric space. Let $\alpha : I \to X$ be an N-Morse (λ, ϵ)-quasi-geodesic where I is an interval of $\mathbb{R}$. Then for any interval $I' \subseteq I$, the (λ, ϵ)-quasi-geodesic $\alpha' = \alpha|_{I'}$ is N'-Morse where N' depends only on λ, ϵ, and N.* □

We now present a combination of statements which will show that, given one Morse ray and another ray which fellow-travels with the first, then eventually the fellow-travelling constant is determined only by the Morse gauge of the first ray.

Proposition 2.21. ([Cor17, Proposition 2.4]) *Let X be a geodesic metric space. Let $\alpha : [0, \infty) \to X$ be an N-Morse geodesic ray. Let $\beta : [0, \infty) \to$ be a geodesic ray such that $d(\alpha(t), \beta(t)) < K$ for $t \in [A, A + D]$ for some $A \in [0, \infty)$ and $D \geq 6K$. Then for all $t \in [A + 2K, A + D - 2K]$, $d(\alpha(t), \beta(t)) < 4N(1, 2N(5, 0)) + 2N(5, 0) + d(\alpha(0), \beta(0))$.* □

The proof of Proposition 2.21, as presented in [Cor17], shows the following additional facts:

Corollary 2.22. *Let X be a geodesic metric space. Let $\alpha : [0, \infty) \to X$ be an N-Morse geodesic ray. Let $\beta : [0, \infty) \to$ be a geodesic ray such that $d(\alpha(t), \beta(t)) < K$ for $t \in [A, A + D]$ for some $A \in [0, \infty)$ and $D \geq 6K$. Then there exists $x, y \in [0, A + 2K]$ such that $d(\alpha(x), \beta(y)) < N(5, 0)$.* □

Corollary 2.23. ([Cor17, Corollary 2.6]) *Let X be a geodesic metric space. Let $\alpha : [0,\infty) \to X$ be an N-Morse geodesic ray. Let $\beta : [0,\infty) \to$ be a geodesic ray such that $d(\alpha(t), \beta(t)) < K$ for all $t \in [0,\infty)$ (i.e. $\beta(\infty) = \alpha(\infty)$). Then for all $t \in [2K,\infty)$, $d(\alpha(t), \beta(t)) < \max\{4N(1, 2N(5,0)) + 2N(5,0), 8N(3,0)\} + d(\alpha(0), \beta(0))$.* □

Combining Corollaries 2.22 and 2.23, we get the following generalization of [BH09, Chapter 3, Lemma 3.3].

Proposition 2.24. *Let X be a geodesic metric space. Let $\alpha : [0,\infty) \to X$ be an N-Morse geodesic ray. Let $\beta : [0,\infty) \to X$ be a geodesic ray such that $d(\alpha(t), \beta(t)) < K$ for all $t \in [0,\infty)$ (i.e. $\beta(\infty) = \alpha(\infty)$). Then there exists $T_1, T_2 > 0$ such that for all $t \in [0,\infty)$, $d(\alpha(T_1 + t), \beta(T_2 + t)) < \max\{4N(1, 2N(5,0)) + 2N(5,0), 8N(3,0)\} + N(5,0)$.*

Proof. By Corollary 2.22, there exists $x, y \geq 0$ so that $d(\alpha(x), \beta(y)) < N(5,0)$. Define $\alpha'(t) = \alpha(x+t)$ and $\beta'(t) = \beta(y+t)$, and note in particular that $\alpha'(0) = \alpha(x)$ and $\beta'(0) = \beta(y)$. Applying Corollary 2.23 to α' and β' produces the desired result. □

For convenience, we will denote $\delta_N = \max\{4N(1, 2N(5,0)) + 2N(5,0), 8N(3,0)\} + N(5,0)$. Using this notation, Proposition 2.24 leads to the following generalization of [Swe01, Lemma 4].

Corollary 2.25. *Let X be a geodesic metric space. Let $\alpha : [0,\infty) \to X$ be an N-Morse geodesic ray. Let $\beta : [0,\infty) \to X$ be a geodesic ray such that $\beta(\infty) = \alpha(\infty)$. Then there exists $a \in \mathbb{R}$ and an isometry $\rho : [a,\infty) \to [0,\infty)$ so that $\alpha \sim_{\delta_N} \beta \circ \rho$.*

Proof. Apply Proposition 2.24 to find $T_1, T_2 > 0$ so that for all $t \in [0,\infty)$, $d(\alpha(T_1+t), \beta(T_2+t)) < \delta_N$. Then let $\rho : [a,\infty) \to [0,\infty)$ be the unique isometry such that $\rho(T_1) = T_2$. □

Proposition 2.26. *Suppose $\alpha : [a, \infty) \to X$ is an N-Morse geodesic ray and $\beta : [b, \infty) \to X$ is a geodesic ray such that $\beta \sim_{\delta_N} \alpha$ and $\alpha(a) = \beta(b)$. Then β is M-Morse where M depends only on N.*

Proof. It suffices to show that $d_{Haus}(\alpha, \beta) \leq K$ where $K \geq 0$ depends only on N. Choose $T > 0$ so that $d(\alpha(t), \beta(t)) \leq \delta_N$ for all $t \geq T$. Note that $[\beta(b), \beta(t)] * [\beta(t), \alpha(t)]$ is a $(1, 2\delta_N)$ quasi-geodesic, so by [Cor17, Lemma 2.1], $d_{Haus}([\alpha(a), \alpha(t)], [\beta(b), \beta(t)] * [\beta(t), \alpha(t)]) \leq L$ for some L depending only on N. But since $l([\alpha(t), \beta(t)]) \leq \delta_N$, we have $d_{Haus}([\alpha(a), \alpha(t)], [\beta(b), \beta(t)]) \leq L + \delta_N$. □

The above statement leads to the following generalization, which is very similar to [Cor17, Lemma 2.8]. This statement will be useful for showing a generalization of Corollary 2.16, since our horoballs and funnels will be restricted to a single Morse stratum, see Theorem 3.8.

Proposition 2.27. *Suppose $x \in \partial X^N_{\mathfrak{o}}$ for a Morse gauge N. Then any geodesic ray $\alpha : [a, \infty) \to X$ with $\alpha(\infty) = x$ is M-Morse, where M depends only on N and the Morse gauge of $[\alpha(a), \mathfrak{o}]$.*

Proof. See Figure 2.5. Let $\beta : [b, \infty) \to X$ be N-Morse with $\beta(b) = \mathfrak{o}, \beta(\infty) = \alpha(\infty)$, and let N' be the Morse gauge of $[\alpha(a), \beta(b)]$. For each $n \in \mathbb{N}$, let $\gamma_n = [\alpha(a), \beta(b + n)]$. Note that $\beta|_{[b,b+n]}$ is Morse for some Morse gauge depending only on N by Lemma 2.20, and so by [Cor17, Lemma 2.3], γ_n in N''-Morse for N'' depending only on $\max\{N, N'\}$. Then via a straightforward generalization of [Cor17, Lemma 2.10], there exists an N''-Morse geodesic ray γ with $\gamma_n \to \gamma$ (uniformly on compact sets) and $\gamma(\infty) = \beta(\infty)$. Then Proposition 2.26 shows that α is Morse for an appropriate Morse gauge. □

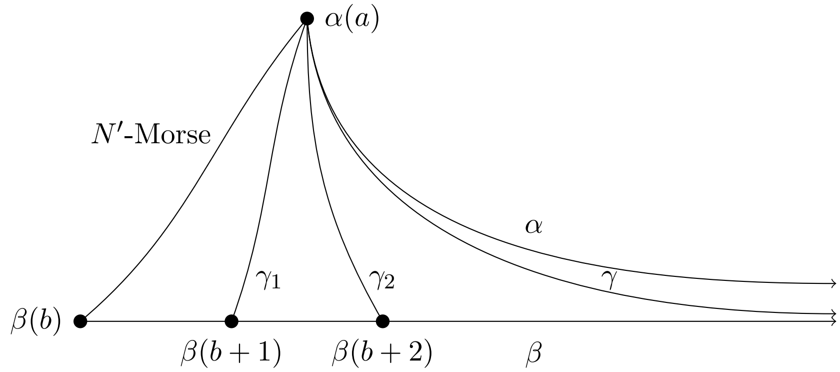

Figure 2.5: Diagram for Proposition 2.27

2.2.1 Limit Sets and Weak Convex Hulls

We now introduce limit sets and weak convex hulls, and give some useful properties that these sets have. We use these constructions to turn subsets of X into subsets of the Morse boundary, and vice versa.

Definition 2.28. *([CD17, Definition 3.2]) Let X be a proper, geodesic metric space and let $A \subseteq X$. The* ***limit set of*** *A, denoted as ΛA, is the set of points in $\partial X_{\mathfrak{o}}$ such that, for some Morse gauge N, there exists a sequence of points $(a_k) \subset A \cap X_{\mathfrak{o}}^N$ such that $[\mathfrak{o}, a_k]$ converges (uniformly on compact sets) to a geodesic ray α with $\alpha(\infty) = x$. (Note α is N-Morse by [Cor17, Lemma 2.10].) In the case where H acts properly by isometries on X, we use ΛH to denote the limit set of $H\mathfrak{o}$.*

Remark 2.29. *By [CD17, Lemma 3.3], ΛH is well-defined as the limit set of* ***any*** *orbit of H, we merely choose the orbit $H\mathfrak{o}$ for convenience and simplicity in future arguments.*

We also prove a fact about limit sets, see also [CD17, Lemma 4.1, Proposition 4.2].

Corollary 2.30. *Let X be a proper, geodesic metric space and suppose $A \subseteq X$. If $\Lambda A \subseteq \partial X_{\mathfrak{o}}^{N}$ for some Morse gauge N, then ΛA is compact.*

Proof. By [Cor17, Proposition 3.12], this follows from the fact that ΛH is closed. □

Remark 2.31. *By Corollary 2.30 and by [CD17, Lemma 4.1], the requirement that ΛH is compact is equivalent to the requirement that ΛH is contained in the boundary of a single Morse stratum.*

Definition 2.32. *([Swe01, CD17]) Let X be a proper, geodesic metric space, and let $A \subseteq X \cup \partial X_{\mathfrak{o}}$. Then the **weak convex hull** of A, denoted $WCH(A)$, is the union of all geodesic (segments, rays, or lines) of X which have both endpoints in A.*

We take a moment to highlight some nice interactions between the weak convex hull of a compact limit set with the Morse boundary.

Lemma 2.33. *([CD17, Proposition 4.2]) Let X be a proper geodesic metric space and let $A \subseteq X$ such that $\Lambda A \subseteq \partial X_{\mathfrak{o}}^{N}$ for some Morse gauge N. Then there exists a Morse gauge N', depending only on N, such that $WCH(\Lambda A) \subset X_{\mathfrak{o}}^{N'}$.* □

Lemma 2.34. *Let X be a proper geodesic metric space and let $A \subseteq X$ such that $\Lambda A \subseteq \partial X_{\mathfrak{o}}^{N}$ for some Morse gauge N. Then $\Lambda(WCH(\Lambda A)) \subseteq \Lambda A$.*

Proof. We may assume $|\Lambda A| > 1$. Let $x \in \Lambda(WCH(\Lambda A))$. By Definition 2.28, there exists $x_n \in WCH(\Lambda A)$ such that $[\mathfrak{o}, x_n]$ converges to a geodesic ray γ with $\gamma(\infty) = x$. We show that there exists $K > 0$ so that for all n there exists a_n with $[\mathfrak{o}, x_n] \subseteq \mathcal{N}_K([\mathfrak{o}, a_n])$. Thus, (a subsequence of) the geodesics $[\mathfrak{o}, a_n]$ converge to a geodesic ray $\alpha : [0, \infty) \to X$

with $\alpha(0) = \mathfrak{o}$, and $\alpha(\infty) = \gamma(\infty) = x$ and so $x \in \Lambda A$. It remains to find K so that $[\mathfrak{o}, x_n] \subseteq \mathcal{N}_K([\mathfrak{o}, a_n])$.

Fix n. Since $x_n \in WCH(\Lambda A)$, $x \in \eta$ where $\eta : (-\infty, \infty) :\to X$ is a geodesic with $\eta(\pm\infty) \in \Lambda A$. So, by Definition 2.28, there exists $a_k^+, a_k^- \in A \cap X_{\mathfrak{o}}^N$ so that $[\mathfrak{o}, a_k^+]$ and $[\mathfrak{o}, a_k^-]$ converge to geodesics β^+ and β^-, respectively, with $\beta^+(\infty) = \eta(\infty)$ and $\beta^-(-\infty) = \eta(-\infty)$. Since $\Lambda A \subseteq \partial X_{\mathfrak{o}}^N$, the triangle $\eta \cup \beta^+ \cup \beta^-$ is L-slim for L depending only on N by [CD17, Proposition 3.6], and as $x \in \eta$, there exists $y \in \beta^+ \cup \beta^-$ so that $d(x_n, y) \leq L$. Without loss of generality, assume $y \in \beta^+$. Since $[\mathfrak{o}, a_k^+]$ converges to β^+ uniformly on compact sets, choose m large enough so that $d(y, [\mathfrak{o}, a_m^+]) \leq 1$. Let $z \in [\mathfrak{o}, a_m^+]$ so that $d(y, z) \leq 1$, see Figure 2.6.

Note that the concatenation $[\mathfrak{o}, x_n] * [x_n, z]$ is a $(1, L+1)$-quasi-geodesic with endpoints on $[\mathfrak{o}, a_m^+]$. Since $[\mathfrak{o}, a_m^+]$ is N-Morse, we have that $[\mathfrak{o}, x_n] \subseteq [\mathfrak{o}, x_n] * [x_n, z] \subseteq \mathcal{N}_{N(1,L+1)}([\mathfrak{o}, a_m^+])$. Since $K := N(1, L+1)$ did not depend on the choice of n, this completes the proof. □

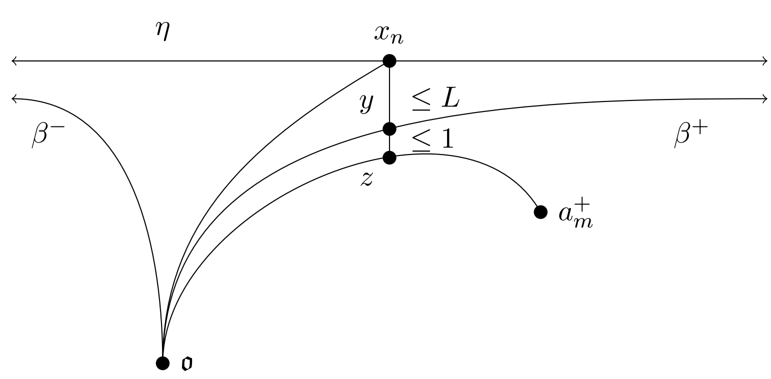

Figure 2.6: Diagram for Lemma 2.34

Finally, we finish this section by stating the definitions of stability and boundary convex cocompactness here for reference.

Definition 2.35. ([DT15] [CD17, Definition 1.3]) *If $f : X \to Y$ is a quasi-isometric embedding between geodesic metric spaces, we say X is a* ***stable*** *subspace of Y if there exists a Morse Gauge N such that every pair of points in X can be connected by an N-Morse quasi-geodesic in Y; we call f a* ***stable embedding****.*

If $H < G$ are finitely generated groups, we say H is ***stable in*** *G if the inclusion map $i : H \hookrightarrow G$ is a stable embedding.*

Definition 2.36. ([CD17, Definition 1.4]) *We say that H acts* ***boundary convex cocompactly*** *on X if the following conditions hold:*

1. *H acts properly on X,*
2. *ΛH is nonempty and compact,*
3. *The action of H on $WCH(\Lambda H)$ is cobounded.*

Chapter 3

Limit point characterizations in the Morse Boundary

3.1 Limit points in the Morse boundary

The goal of this section is show that, given a set $A \subseteq X$, if $x \in \partial X_{\mathfrak{o}}$ is a Morse conical limit point of A, then x is a Morse horospherical limit point of A. This was first shown in the hyperbolic case in [Swe01], we generalize this fact into the setting of proper geodesic spaces. We begin by introducing horospheres and funnels for Morse rays.

Definition 3.1 (Horoballs, Funnels)**.** *Let X be a proper, geodesic metric space and let $\mathfrak{o} \in X$ be some designated point. Let $\alpha : [a, \infty) \to X$ be an N'-Morse geodesic ray, and let N be some, potentially different, Morse gauge. We define the N-**Morse horoball around** α **based at** $\mathfrak{o}$ as*

$$H^N_{\mathfrak{o}}(\alpha) = \{x \in X^N_{\mathfrak{o}} \mid \exists \beta : [b, \infty) \to X \text{ with } \beta \sim_{\delta_{N'}} \alpha \text{ and } b \geq a \text{ and } \beta(b) = x\}.$$

*We define the N-**Morse funnel around** α **based at** $\mathfrak{o}$ as*

$$F_{\mathfrak{o}}^N(\alpha) = \{x \in X_{\mathfrak{o}}^N \mid d(x, \pi_\alpha(x)) \leq d(\alpha(a), \pi_\alpha(x))\}.$$

Comparing these definitions to Definition 2.12 shows that a Morse horoball is a horoball about a Morse geodesic intersected with an appropriate Morse stratum, and similarly, a Morse funnel is a funnel about a Morse geodesic intersected with an appropriate Morse stratum. The following three definitions classify points on the Morse boundary by asking if every horoball, funnel, or cone intersects a given subset of X.

Definition 3.2. *Let X be a proper, geodesic metric space and let $\mathfrak{o} \in X$ be some designated point. Let $A \subset X$.*

- *We say that $x \in \partial X_{\mathfrak{o}}$ is a **Morse horospherical limit point of** A if for every Morse geodesic α with $\alpha(\infty) = x$, there exists a Morse gauge N such that $H_{\mathfrak{o}}^N(\alpha) \cap A \neq \emptyset$.*
- *We say that $x \in \partial X_{\mathfrak{o}}$ is a **Morse funneled limit point of** A if for every Morse geodesic α with $\alpha(\infty) = x$, there exists a Morse gauge N such that $F_{\mathfrak{o}}^N(\alpha) \cap A \neq \emptyset$.*
- *We say that $x \in \partial X_{\mathfrak{o}}$ is a **Morse conical limit point of** A if there exists $K > 0$ such that, for every Morse geodesic α with $\alpha(\infty) = x$, we have that $\mathcal{N}_K(\alpha) \cap A \neq \emptyset$.*

Remark 3.3. *Notice that, in the case where X is a δ-hyperbolic space, these definitions agree with the definitions given in Definition 2.12, as every geodesic in a δ-hyperbolic space is N-Morse for N depending only on δ. In light of this, we will use "conical limit point" instead of "Morse conical limit point" for the rest of this paper, except in cases where the difference between these definitions causes confusion. We similarly reduce "Morse horospherical limit* *point" and "Morse funneled limit point" to "horospherical limit point" and "funneled limit point," respectively.*

We now begin proving the new implications found in Theorem 1.2. We will first show that every conical limit point of A is a funneled limit point of A, and then we will show that the funneled limit points of A exactly coincide with the horospherical limit points of A. These arguments generalize the arguments found in [Swe01].

Proposition 3.4. *Let X be a proper, geodesic metric space and let $\mathfrak{o} \in X$. Let $A \subseteq X$. If $x \in \partial X_{\mathfrak{o}}$ is a conical limit point of A, then x is a funneled limit point of A.*

Proof. See Figure 3.1. Let $x \in \partial X_{\mathfrak{o}}$ be a conical limit point of $A \subseteq X$. Let $\alpha : [0, \infty) \to X$ be an N-Morse geodesic with $\alpha(\infty) = x$. By Lemma 2.20, there exists a Morse gauge M so that every geodesic sub-ray of α is M-Morse. Thus by Definition 3.2, there exists $K \geq 0$ so that every subray of α gets at least K close to A.

Now define $\alpha' = \alpha|_{[3K,\infty)}$, and let $a \in A$ such that $a \in \mathcal{N}_K(\alpha')$. Then note that $d(a, \pi_\alpha(a)) \leq d(a, \pi_{\alpha'}(a)) \leq K$, and so $d(\pi_\alpha(a), \pi_{\alpha'}(a)) \leq 2K$. By the triangle inequality, $\pi_\alpha(a) \subseteq \alpha|_{[K,\infty)}$. Therefore, $d(\pi_\alpha(a), a) \leq K = d(\alpha(0), \alpha(K)) \leq d(\alpha(0), \pi_{\alpha'}(a))$. It remains to show that $a \in X_{\mathfrak{o}}^{N'}$ for a Morse gauge N' which is independent of the choice of $a \in A$.

Let $L = d(\mathfrak{o}, \alpha(0))$, and let $p \in \pi_{\alpha'}(a)$, and note that $d(p, a) \leq K$. Thus, $[\mathfrak{o}, \alpha(0)]$ and $[p, a]$ are both N''-Morse depending only on $\max\{K, L\}$, and $[\mathfrak{o}, p]$ is N'''-Morse depending only on N by Lemma 2.20. Since $[\mathfrak{o}, a]$ is one side of a quadrilateral whose other three sides are $\max\{N'', N'''\}$-Morse, $[\mathfrak{o}, a]$ is N'-Morse where N' does not depend on choice of $a \in A$ by [Cor17, Lemma 2.3]. □

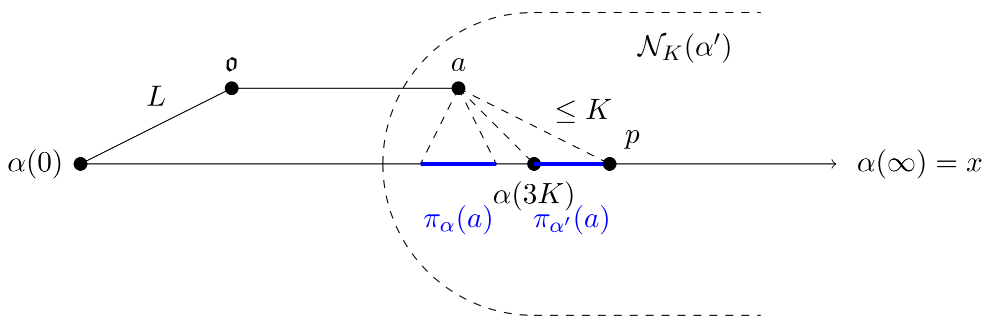

Figure 3.1: Diagram for Lemma 3.4

3.2 Equivalence of Horospherical and Funneled limit points

Our next goal is to show that the funneled limit points of A coincide with the horospherical limit points of A. Towards this end, we show that, given a point x in a horoball of a subray, the projection of x to the subray is coarsely the same as the projection to the base ray.

Lemma 3.5. *Suppose α is an N-Morse geodesic ray and let α' be a subray. Suppose $x \in H_{\mathfrak{o}}^{N'}(\alpha')$. If $\alpha(\infty) \in \partial X_{\mathfrak{o}}^{N''}$, then $d_{Haus}(\pi_\alpha(x), \pi_{\alpha'}(x)) \leq K$, where $K \geq 0$ depends only on N, N', and N''.*

Proof. See Figure 3.2. Let $\alpha : [0, \infty) \to X$ be an N-Morse geodesic ray and let $\alpha' = \alpha|_{[a,\infty)}$ for some $a \geq 0$. By Lemma 2.20, α' is M-Morse for M depending only on N. Let $x \in H_{\mathfrak{o}}^{N''}(\alpha')$, thus there exists $\beta : [b, \infty) \to X$ a geodesic ray with $b \geq a$, $\beta(b) = x$, and $\beta \sim_{\delta_M} \alpha'$. By Proposition 2.27, β is M'-Morse for M' depending only on N, N', and N''. We note that if $\pi_\alpha(x) \subseteq \alpha'$, then $\pi_\alpha(x) = \pi_{\alpha'}(x)$. So, we assume that $\pi_\alpha(x) \nsubseteq \alpha'$. We shall show that in this case, $d(x, \pi_\alpha(x))$ and $d(x, \pi_{\alpha'}(x))$ are both bounded above by an appropriate constant, and this gives the desired result.

Let $p \in \pi_\alpha(x) \setminus \alpha'$, and let $q \in \pi_{\alpha'}(x)$. Without loss of generality, let T be large enough so that $q \in [\alpha'(a), \alpha'(T)]$ and $d(\alpha'(T), \beta(T)) \leq \delta_N$.

Put $\gamma = [\beta(b), \alpha'(a)] * [\alpha'(a), \alpha'(T)] * [\alpha'(T), \beta(T)]$, and note that γ is a $(3, 4\delta_N)$ quasi-geodesic. Thus there exists $w \in [\beta(b), \beta(T)]$ and $L \geq 0$ such that $d(\alpha'(a), w) \leq L$, where L depends only on M' by [Cor17, Lemma 2.1]. Notice now that $|d(\alpha'(a), \alpha'(T)) - d(w, \beta(T))| \leq \delta_N + L$. However, since $b \geq a$ and $w \in [\beta(b), \beta(T)]$, we know $|d(\alpha'(a), \alpha'(T)) - d(w, \beta(T))| = d(\alpha'(a), \alpha'(T)) - d(w, \beta(T))$. But then by the definition of the nearest point projection and the triangle inequality, we have

$$\begin{aligned} d(x,p) \leq d(x,q) \leq d(x,\alpha'(a)) &\leq d(x,w) + d(w,\alpha'(a)) \\ &= d(x,\beta(T)) - d(w,\beta(T)) + d(w,\alpha'(a)) \\ &= d(\alpha'(b),\alpha'(T)) - d(w,\beta(T)) + d(w,\alpha'(a)) \\ &\leq d(\alpha'(a),\alpha'(T)) - d(w,\beta(T)) + L \\ &\leq \delta_N + L + L. \end{aligned}$$

Therefore, $d(\pi_\alpha(x), x)$ and $d(\pi_{\alpha'}(x), x)$ are both bounded above by L, which is a constant depending only on N, N', and N'', as desired. □

We're now ready to show that Morse funneled limit points are exactly Morse horospherical limit points. We proceed using the same strategy as the one found in [Swe01], by showing direct generalizations of Lemma 2.14 and Lemma 2.15 for the Morse case.

Proposition 3.6. *Let $x \in \partial X_{\mathfrak{o}}^N$. Let $\alpha : [0, \infty) \to X$ be an N'-Morse geodesic with $\alpha(\infty) = x$. Then for every Morse gauge N'', there exists $T \geq 0$ such that, for any subray α' of α with $d(\alpha(0), \alpha') \geq T$, we have $H_{\mathfrak{o}}^{N''}(\alpha') \subseteq F_{\mathfrak{o}}^{N''}(\alpha)$.*

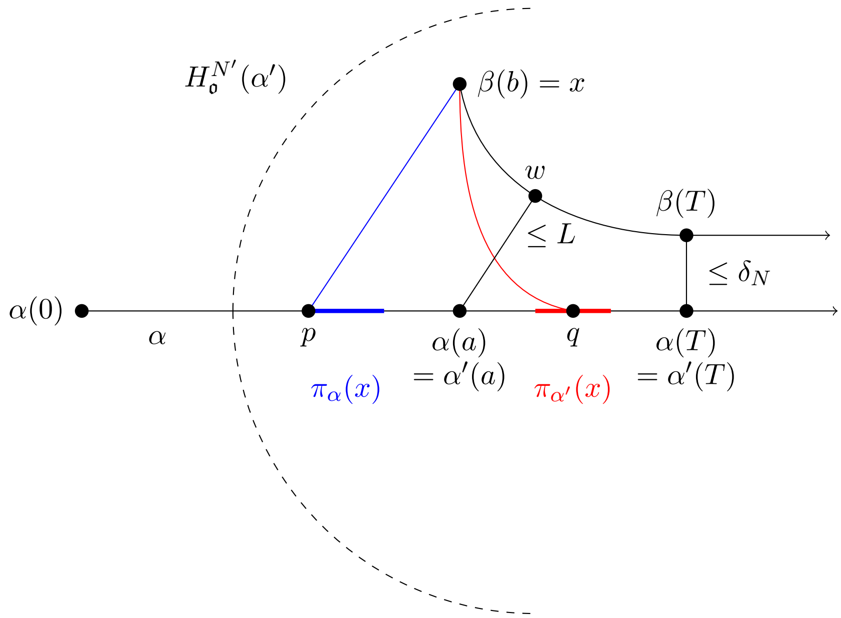

Figure 3.2: Diagram for Lemma 3.5

Proof. See Figure 3.3. Let $\alpha' = \alpha|_{[a,\infty)}$ be a subray of α. By Lemma 2.20, α' is M-Morse where M depends only on N'. Let $y \in H_{\mathfrak{o}}^{N''}(\alpha')$. Thus there exists $\beta : [b, \infty) \to X$ be a geodesic ray such that $b \geq a$, $\beta(b) = y$, and $\beta \sim_{\delta_M} \alpha'$. Note that β is M'-Morse where M' depends only on N, N', and N'' by Proposition 2.27. Choose $z \in \pi_\alpha(y)$ such that $d(\alpha(0), z) = d(\alpha(0), \pi_\alpha(y))$, i.e., so that z is closest to $\alpha(0)$. By Lemma 3.5, there exists $p \in \pi_{\alpha'}(x)$ so that $d(z, p) \leq L$ for some L depending only on N, N', and N''. Choose t large enough so that $d(\alpha'(t), \beta(t)) = d(\alpha(t), \beta(t)) \leq \delta_M$ and $p, \alpha(b) \in [\alpha(a), \alpha(t)]$. Note that $[y, p] * [p, \alpha(t)] * [\alpha(t), \beta(t)]$ is a $(3, 4\delta_M)$-quasi-geodesic, thus there exists $q \in [\beta(b), \beta(t)]$ and $\lambda \geq 0$ such that $d(p, q) \leq \lambda$ where λ depends only on M' by [Cor17, Lemma 2.1]. It suffices to show that $d(y, z) \leq d(\alpha(0), z)$.

Using the triangle inequality and the definition of π_α, we find

$$d(y,z) \leq d(y,p) \leq d(y,q) + d(q,p) \leq d(y,q) + \lambda$$

$$= d(y,\beta(t)) - d(q,\beta(t)) + \lambda = d(\alpha(b),\alpha(t)) - d(q,\beta(t)) + \lambda$$

$$\leq d(\alpha(a),\alpha(t)) - d(p,\alpha(t)) + \lambda + \delta_M + \lambda$$

$$= d(\alpha(a),p) + 2\lambda + \delta_M \leq d(\alpha(a),z) + L + 2\lambda + \delta_M.$$

So, if $a \geq L + 2\lambda + \delta_M$, we have $d(y,z) \leq d(\alpha(a),z) + L + 2\lambda + \delta_M \leq d(\alpha(a),z) + d(\alpha(0),\alpha(a)) = d(\alpha(0),z)$. □

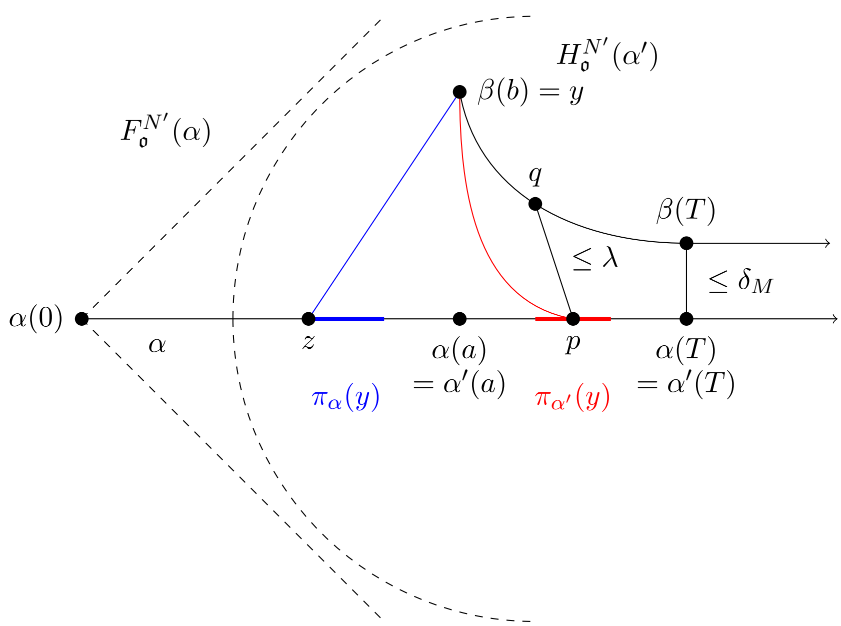

Figure 3.3: Diagram for Lemma 3.6

Proposition 3.7. *Let $x \in \partial X_{\mathfrak{o}}^N$. Let $\alpha : [0,\infty) \to X$ be an N'-Morse geodesic with $\alpha(\infty) = x$. Suppose $S = \delta_{N'}$. Define $\alpha' = \alpha|_{[S,\infty)}$. Then $F_{\mathfrak{o}}^{N''}(\alpha') \subseteq H_{\mathfrak{o}}^{N''}(\alpha)$ for any Morse gauge N''.*

Proof. Let $y \in F_{\mathfrak{o}}^{N''}(\alpha')$. By definition, $d(y, \pi_{\alpha'}(y)) \leq d(\alpha(S), \pi_{\alpha'}(y))$. Let $p \in \pi_{\alpha'}(y)$ such that $d(\alpha(S), p) = d(\alpha(S), \pi_{\alpha'}(y))$, i.e., let p be the element of $\pi_{\alpha'}(y)$ which is closest to $\alpha(S)$. Then $d(y, p) \leq d(\alpha(S), p)$. Construct $\beta : [b, \infty) \to X$ such that $\beta(b) = y$ and $\beta \sim_{\delta_{N'}} \alpha$. We want to show that $b \geq 0$. Choose $T \geq 0$ so that $d(\beta(T), \alpha(T)) \leq \delta_{N'}$. Then

$$\begin{aligned} T - b = d(y, \beta(T)) &\leq d(y, p) + d(p, \alpha(T)) + d(\alpha(T), \beta(T)) \\ &\leq d(\alpha(S), p) + d(p, \alpha(T)) + \delta_{N'} = d(\alpha(S), \alpha(T)) + \delta_{N'} \\ &= d(\alpha(0), \alpha(T)) - d(\alpha(0), \alpha(S)) + \delta_{N'} = T - S + \delta_{N'} = T \end{aligned}$$

In summary, $T - b \leq T$, but this immediately shows that $0 \leq b$, as desired. □

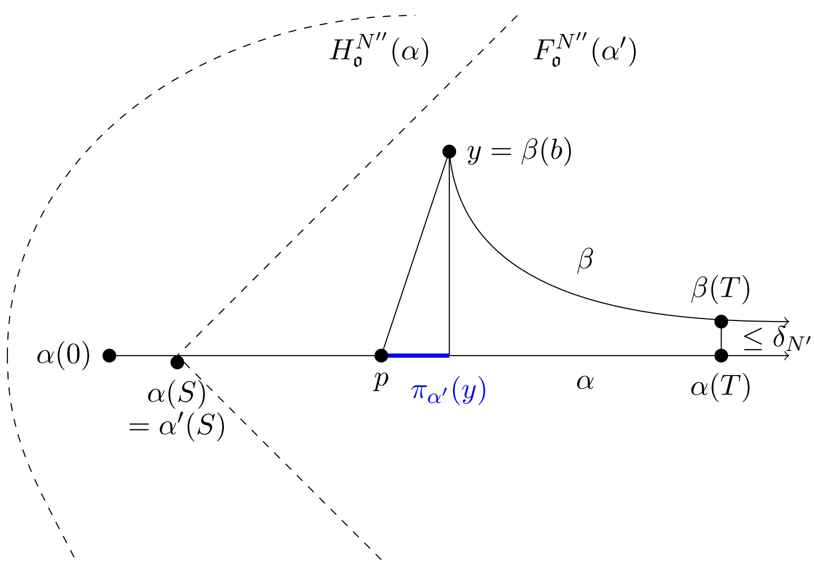

Figure 3.4: Diagram for Lemma 3.7

Theorem 3.8. *Let $x \in \partial X_{\mathfrak{o}}$. Then x is a Morse horospherical limit point of $A \subseteq X$ if and only if x is a Morse funneled limit point of A.*

Proof. This is a direct consequence of Propositions 3.6 and 3.7. □

Chapter 4

Limit Set Conditions For Stability

In this section, we show that the horospherical limit point condition, combined with the limit set being compact, is enough for to show that the group action on the weak convex hull is cobounded. The main idea behind this argument is to show the contrapositive: when the group action is not cobounded, then geodesic rays in the space eventually end up very far from the orbit of the group. We begin by showing the following helpful fact, which states that if a group acts non-coboundedly on the weak convex hull of its limit set, there exists a sequence of points p_n in the weak convex hull that "maximally avoids" the orbit.

Lemma 4.1. *Suppose X is a proper geodesic metric space and suppose that H acts properly on X by isometries. Assume that $\Lambda H \neq \emptyset$. If the action $H \curvearrowright WCH(\Lambda H)$ is not cobounded, then there exists an increasing sequence of positive integers, $(n_i)_i$, such that for each $i \in \mathbb{Z}_{\geq 1}$ there exists $p_i \in WCH(\Lambda H)$ satisfying*

1. *$B_{n_i}(p_i) \cap H\mathfrak{o} = \emptyset$,*

2. *$d(p_i, \mathfrak{o}) \leq n_i + 1$.*

This fact is a straightforward consequence of the definition of a cobounded group action, however we include a proof for the sake of completeness.

Proof. Set $n_0 = 1$. We define q_i and n_i for $i \geq 1$ via an inductive process. Since the action of $H \curvearrowright WCH(\Lambda H)$ is not cobounded, there exists a point $q_i \in WCH(\Lambda H)$ such that $n_{i-1} + 1 < d(H\mathfrak{o}, q_i)$. By the definition of $WCH(\Lambda)$, there exists a bi-infinite Morse geodesic γ with $\gamma(\pm\infty) \in \Lambda H$ such that $q_i \in \gamma$. Set n_i to be the unique positive integer such that $n_i < d(H\mathfrak{o}, q_i) \leq n_i + 1$. Note that the sequence $(n_i)_i$ is increasing because $n_{i-1} + 1 \leq n_i$.

Since $d(H\mathfrak{o}, q_i) \leq n_i + 1$ there exists $h_i \in H$ so that $d(q_i, h_i\mathfrak{o}) \leq n_i + 1$. Recalling that the action of H on X is by isometries, we define $p_i = h_i^{-1} q_i$, and so $B_{n_i}(p_i) \cap H\mathfrak{o} = \emptyset$, and $d(\mathfrak{o}, p_i) \leq n_i + 1$. Finally, by [CD17, Lemma 3.3], $h_i^{-1}\gamma$ is a bi-infinite Morse geodesic with endpoints in ΛH, and so $p_i \in WCH(\Lambda H)$. ☐

We note that, under the additional assumption that ΛH is compact and that every point in ΛH is conical, we get a stronger conclusion to this lemma, namely, we can take $n_i = i$ for large i. We formally state and prove this observation.

Lemma 4.2 (Sliding Spheres)**.** *Suppose X is a proper geodesic metric space and suppose that H acts properly on X by isometries. Assume that $\Lambda H \neq \emptyset$, every point of ΛH is a conical limit point of $H\mathfrak{o}$, and that $\Lambda H \subseteq \partial X_{\mathfrak{o}}^N$ for some Morse gauge N. If the action $H \curvearrowright WCH(\Lambda H)$ is not cobounded, there exists a sequence of points $p_n \in WCH(\Lambda H)$ such that, for sufficiently large n, $B_n(p_n) \cap H\mathfrak{o} = \emptyset$ and $\mathfrak{o} \in B_{n+1}(p_n)$.*

Proof. Let $K > 0$ be the conical limit point constant. Let $n \in \mathbb{N}$ with $n > K + 1$. By [Liu21, Corollary 5.8], we may assume that ΛH has at least two distinct points. Since

$H \curvearrowright WCH(\Lambda H)$ is not cobounded, there exists $p \in WCH(\Lambda H)$ with $d(p, H\mathfrak{o}) > n$. By definition, $p \in \gamma$ for some bi-infinite geodesic γ with $\gamma(\pm\infty) \in \Lambda H$. Since $\Lambda H \subseteq \partial X_{\mathfrak{o}}^N$, we have by [CD17, Proposition 4.2] that γ is is Morse for some Morse gauge depending only on N. Since every point in ΛH is a conical limit point of $H\mathfrak{o}$, there exists $h' \in H$ such that $d(h'\mathfrak{o}, \gamma) < K$. Put $q \in \pi_\gamma(h'\mathfrak{o})$.

We may assume that $\gamma(s) = q$ and $\gamma(s') = p$ with $s < s'$. Let $A = \{r \in [s, s'] \ : \ n < d(\gamma(r), H\mathfrak{o})\}$. (Equivalently, one may define $A = \{r \in [s, s'] \ : \ B_n(\gamma(r)) \cap H\mathfrak{o} = \emptyset\}$.) Note that $s' \in A$. Put $t = \inf A$. By the definition of t, we have $n \leq d(\gamma(t), H\mathfrak{o})$, see Figure 4.1. We now claim that $d(\gamma(t), H\mathfrak{o}) < n + 1$.

Suppose for contradiction that $n + 1 \leq d(\gamma(t), H\mathfrak{o})$. By the triangle inequality, $n \leq d(\gamma(t-1), H\mathfrak{o})$. So if $t-1 \in [s, s']$, then $t-1 \in A$, however $t = \inf A$. Thus $t-1 \notin [s, s']$. Therefore, $t \in [s, s+1]$, and so

$$n + 1 \leq d(\gamma(t), h\mathfrak{o}) \leq d(\gamma(t), h'\mathfrak{o}) \leq d(\gamma(t), q) + d(q, h'\mathfrak{o})$$

$$= d(\gamma(t), \gamma(s)) + d(q, h'\mathfrak{o}) \leq 1 + K \leq n$$

But then $n + 1 \leq n$, a contradiction.

Thus, there exists $h \in H$ such that $h\mathfrak{o} \in B_{n+1}(\gamma(t))$, but $B_n(\gamma(t)) \cap H\mathfrak{o} = \emptyset$. Put $p_n = h^{-1}(\gamma(t))$. By [CD17, Lemma 3.3], $h\gamma$ is a bi-infinite Morse geodesic with endpoints in ΛH. As $H \curvearrowright X$ by isometries, $B_n(p_n) \cap H\mathfrak{o} = \emptyset$, and $\mathfrak{o} \in B_{n+1}(p_n)$, as desired. □

We now prove that (4) implies (2) in the language of Theorem 1.2. We show that, if the action is not cobounded on the weak convex hull, then using Lemma 4.1 we can find a sequence of points which maximally avoid the orbit of H, but by Arzelà-Ascoli,

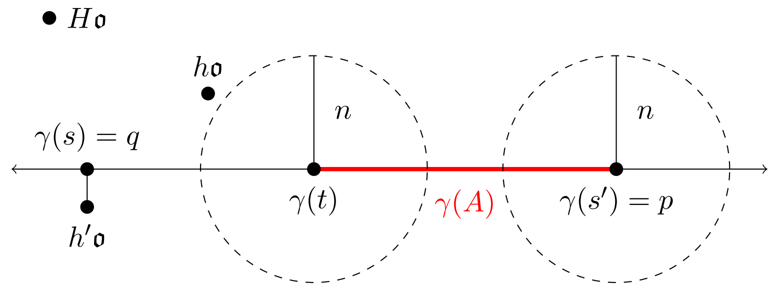

Figure 4.1: Diagram for Lemma 4.2. We can think of this proof as sliding the ball on the right towards the left until it is "up against" the orbit $H\mathfrak{o}$, such as the ball centered at $\gamma(t)$.

geodesics connecting these points to the basepoint eventually fellow-travel a ray which has orbit points that at most linearly diverge from the ray.

Theorem 4.3. *Suppose X is a proper geodesic metric space and suppose H acts properly on X by isometries. Assume that $\Lambda H \neq \emptyset$, every point of ΛH is a horospherical limit point of $H\mathfrak{o}$, and that there exists a Morse gauge N such that $\Lambda H \subset \partial X_{\mathfrak{o}}^N$. Then the action of $H \curvearrowright WCH(\Lambda H)$ is cobounded.*

Proof. For contradiction, assume that $H \curvearrowright WCH(\Lambda H)$ is not a cobounded action. By Lemma 4.1, there exists a sequence of points $p_i \in WCH(\Lambda H)$ and an increasing sequence of positive integers $(n_i)_i$ such that $B_{n_i}(p_i) \cap H\mathfrak{o} = \emptyset$, and $\mathfrak{o} \in B_{n_i+1}(p_i)$. Let $\gamma_i : [0, d(0, p_i)] \to X$ be a geodesic connecting $\mathfrak{o}$ and p_i with $\gamma_i(0) = \mathfrak{o}$. Notice that since $\Lambda H \subset \partial X_{\mathfrak{o}}^N$, we have that γ_i is N'-Morse for some N' depending only on N. By restricting to a subsequence, we may assume that γ_i converges, uniformly on compact subsets, to an N'-Morse geodesic ray γ with $\gamma(0) = \mathfrak{o}$.

By construction and by Lemma 2.34, $\gamma(\infty) \in \Lambda(WCH(\Lambda H)) \subseteq \Lambda H$. So, by [Cor17, Corollary 2.6], there is an N-Morse geodesic ray α with $\alpha(0) = \mathfrak{o}$ and $d(\alpha(t), \gamma(t)) < D$ for all $t \geq 0$, where $D \geq 0$ is a constant that depends only on N. Let $T = 2D+4$, and put

$\alpha' = \alpha_{[T,\infty)}$. Since $\alpha'(\infty) \in \Lambda H$, and so by Theorem 3.8, $\alpha'(\infty)$ is a funneled limit point of H. Thus there exists $h \in H$ so that $h\mathfrak{o} \in F_{\mathfrak{o}}^N(\alpha')$. Let $t_0 = \min\{s : \alpha'(s) \in \pi_{\alpha'}(h\mathfrak{o})\}$. Since the sequence γ_i converges uniformly on compact sets to γ, we may choose i large enough so that $d(\gamma_i(t_0), \gamma(t_0)) \leq 1$, see Figure 4.2.

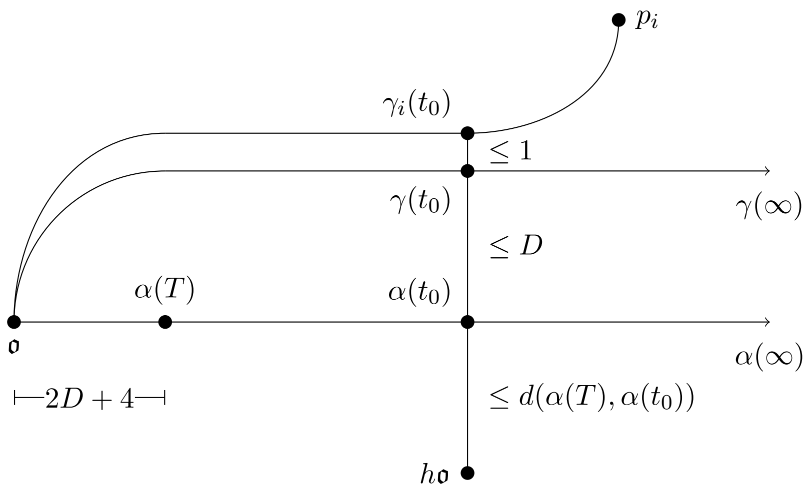

Figure 4.2: Diagram for Theorem 4.3

By the triangle inequality we have that $d(\gamma_i(t_0), \alpha'(t_0)) \leq D + 1$, and therefore $|d(0, \gamma_i(t_0)) - d(0, \alpha(t_0))| \leq D + 1$. Also, by construction we have that $d(h\mathfrak{o}, \alpha'(t_0)) \leq d(\alpha(T), \alpha(t_0))$. Therefore we have

$$\begin{aligned}
d(p_i, h\mathfrak{o}) &\leq d(p_i, \gamma_i(t_0)) + d(\gamma_i(t_0), \alpha'(t_0)) + d(\alpha('t_0), h_0) \\
&\leq d(0, \gamma_i(t_0)) + (D + 1) + d(\alpha(T), \alpha(t_0)) \\
&= d(\mathfrak{o}, p_i) - d(\mathfrak{o}, \gamma_i(t_0)) + (D + 1) + d(0, \alpha(t_0)) - d(0, \alpha(T)) \\
&\leq (n_i + 1) + (D + 1) + (D + 1) - (2D + 4) \leq n_i - 1.
\end{aligned}$$

However, this contradicts the assumption that $B_{n_i}(p_i) \cap H\mathfrak{o} = \emptyset$. $\square$

We now present an alternate definition of a conical limit point which agrees with Definition 3.2 in the case where ΛA is compact, and requires us to only consider of the geodesic rays which emanate from the given basepoint. By Corollary 2.30 and by [CD17, Lemma 4.1], the requirement that ΛH is compact is equivalent to the requirement that ΛH is contained in the boundary of a single Morse stratum.

Proposition 4.4. *Let X be a proper, geodesic metric space. Let $Y \subseteq X$. Suppose $\Lambda Y \neq \emptyset$. Then the following are equivalent:*

1. *$x \in \partial X_{\mathfrak{o}}$ is a conical limit point of Y*
2. *There exists $K > 0$ such that, for every N-Morse geodesic ray $\alpha : [0, \infty) \to X$ with $\alpha(0) = \mathfrak{o}$ and $\alpha(\infty) = x$, and for every $T > 0$, there exists $y \in Y$ such that $y \in \mathcal{N}_K(\alpha')$, where $\alpha' : [0, \infty) \to X$ is defined by $\alpha'(t) = \alpha(t + T)$.*

Proof. Showing (1) implies (2) is a direct consequence of Lemma 2.20 and Definition 3.2.

Instead assume (2). Let $\beta : [b, \infty) \to X$ be an N'-Morse ray with $\beta(\infty) = x$. Let $\alpha : [0, \infty) \to X$ an N-Morse geodesic ray with $\alpha(0) = \mathfrak{o}$ and $\alpha(\infty) = x$. Without loss of generality, by Cor 2.25 there exists $T > 0$ such that $d(\alpha(t), \beta(t)) < \delta_N$ for all $t > T$. Put $\alpha' : [0, \infty) \to X$ via $\alpha'(t) = \alpha(t + T)$. By hypothesis, there exists $y \in Y$ such that $y \in \mathcal{N}_K(\alpha')$. Say $s \in [0, \infty)$ such that $d(\alpha'(s), y) < K$, so via the triangle inequality we have $d(\beta(s + T), y) \leq d(\beta(s + t), \alpha(s + t)) + d(\alpha'(s), y) \leq \delta_N + K$. Thus, $y \in \mathcal{N}_{K+\delta_N}(\beta)$, which shows (1). $\square$

We conclude the chapter by showing that (3) $\Rightarrow$ (2) for Theorem 1.2. Although shown in [CD17, Corollary 1.14], we give a direct proof not relying on [CD17, Theorem 1.1].

Proposition 4.5. *Let X be a proper geodesic space and let H be a finitely generated group of isometries of X where the orbit map $H \to X$ via $h \mapsto h\mathfrak{o}$ is a stable mapping. If there is a Morse gauge N so that $\Lambda H \subseteq \partial X_{\mathfrak{o}}^N$, then every $x \in \Lambda H$ is a conical limit point of $H\mathfrak{o}$.*

Proof. Let $x \in \Lambda H$, and let $\alpha : [0, \infty) \to X$ be an N-Morse geodesic ray with $\alpha(\infty) = x$, $\alpha(0) = \mathfrak{o}$. Let $\alpha' = \alpha|_{[a,\infty)}$ be a subray of α. Notice that α' is N'-Morse where N' depends only on N by Lemma 2.20. By Proposition 4.4, it suffices to show that there exists some $K \geq 0$, depending only on N' and H, so that $H\mathfrak{o} \cap \mathcal{N}_K(\alpha') \neq \emptyset$.

Since H is a stable subgroup of isometries on X, we have that for any $h \in H$, there exists a (λ, λ)-quasi-geodesic γ from $\mathfrak{o}$ to $h\mathfrak{o}$ such that, for any $p \in \gamma$, $B_{2\lambda}(p) \cap H\mathfrak{o} \neq \emptyset$. (To find such a path γ, take a geodesic in a Cayley graph for H and embed it into X by extending the orbit map along appropriate geodesic segments.)

Now, since $x \in \Lambda H$, there exists a sequence $h_n \in H$ such that the sequence of geodesic segments, $\beta_n = [\mathfrak{o}, h_n\mathfrak{o}]$, converges (uniformly on compact subsets) to a geodesic ray $\beta : [b, \infty) \to X$ with $\beta(\infty) = x$ and $\beta(b) = \mathfrak{o}$. Since H is a stable group of isometries, β_n is N''-Morse by Definition 2.35. Up to potentially re-parameterizing β, there exists $T > a$ so that $d(\beta(T), \alpha(T)) < \delta_N$ by Corollary 2.25.

Since β_n converges to β uniformly on $\overline{B_{T+1}(\mathfrak{o})}$, the ball of radius $T+1$ centered at $\mathfrak{o}$, there exists $n \in \mathbb{N}$ and $p \in \beta_n$ so that $d(\beta(T), p) < 1$. Since γ_n is an (λ, λ)-quasi-geodesic with endpoints on β_n, there exists $q \in \gamma_n$ so that $d(p, q) \leq N''(\lambda, \lambda)$. Finally, there exists $h \in H$ so that $d(h\mathfrak{o}, q) \leq \lambda$.

Therefore by the triangle inequality, $d(\alpha(T), h\mathfrak{o}) \leq d(\alpha(T), \beta(T)) + d(\beta(T), p) + d(p, q) + d(q, h\mathfrak{o}) \leq \delta_N + 1 + N''(\lambda, \lambda) + 2\lambda$. As $\alpha(T) \in \alpha'$, this completes the proof. □

Chapter 5

Applications to Teichmüller Space

We conclude by illustrating applications to the above work in the setting of Teichmüller space for a finite type surface S. We begin by setting some notation. Let $\mathrm{Mod}(S)$ denote the mapping class group of S and let $\mathcal{T}(S)$ denote the associated Teichmüller space. We will denote the set of projective measured foliations on S by $\mathrm{PMF}(S)$. The Thurston compactification of Teichmüller space is $\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \mathrm{PMF}(S)$.

We take a moment to restate Corollary 1.7 using the above notation:

Corollary 5.1. *(Restatement of Corollary 1.7.) Let H be a finitely generated subgroup of* $\mathrm{Mod}(S)$*. The following are equivalent:*

1. *Every element of $\Lambda H \subset \partial\mathrm{Mod}(S)$ is a conical limit point of $H \curvearrowright \mathrm{Mod}(S)$ and ΛH is compact (in the Morse boundary of $\mathrm{Mod}(S)$).*

2. *Every element of $\Lambda H \subset \mathrm{PMF}(S)$ is a conical limit point of $H \curvearrowright \mathcal{T}(S)$.*

By work of Cordes, $\partial\mathrm{Mod}(S)$ is homeomorphic to $\partial\mathcal{T}(S)$ (where ∂ refers to the Morse boundary) [Cor17, Theorem 4.12], and there exists a natural continuous injective map $h_\infty : \partial\mathcal{T}(S) \hookrightarrow \mathrm{PMF}(S)$ [Cor17, Proposition 4.14]. Keeping this in mind, we denote the continuous inclusion $f_\infty : \partial\mathrm{Mod}(S) \hookrightarrow \mathrm{PMF}(S)$. The purpose of this section is to prove the following theorem.

Theorem 5.2. *Let H be a subgroup of $\mathrm{Mod}(S)$, and let $x_\infty \in \Lambda H \subseteq \partial\mathrm{Mod}(S)$ be a conical limit point of $H \curvearrowright \mathrm{Mod}(S)$. Then $f_\infty(x_\infty) \in \mathrm{PMF}(S)$ is a conical limit point of $H \curvearrowright \mathcal{T}(S)$.*

Remark 5.3. *This theorem directly proves (1) $\Rightarrow$ (2) of Corollary 1.7.*

Our proof of Theorem 5.2 uses several of the tools developed in [Cor17], so we take a moment to recall the construction and definitions presented therein and from [MM00]. The *curve graph*, denoted $\mathcal{C}(S)$, is a locally infinite simplicial graph whose vertices are isotopy classes of simple closed curves on S. We join two vertices with an edge it there exists representative from each class that are disjoint.

A set of (pairs of) curves $\mu = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_m, \beta_m)\}$ is called a complete clean *marking* of S if the $\{\alpha_1, \ldots, \alpha_m\}$ forms a pants decomposition of S, if each α_i is disjoint from β_j whenever $i \neq j$, and if each α_i intersects β_i once if the surface filled by α_i and β_i is a one-punctured torus. (Otherwise, α_i and β_i will intersect twice, and the filling surface is a four-punctured sphere.) We call $\{\alpha_1, \ldots, \alpha_m\}$ the *base* of μ and we call β_i the *transverse curve to α_i in μ*. For the sake of completeness, we also define the marking graph, $\mathcal{M}(S)$, although the definition is not needed in this paper. $\mathcal{M}(S)$ is the simplicial graph whose vertices are markings as defined above, and two markings are joined by an edge if they differ by an elementary move. The marking graph $\mathcal{M}(S)$ is quasi-isometric to the mapping class group $\mathrm{Mod}(S)$, see [MM00].

For each $\sigma \in \mathcal{T}(S)$ there is a *short marking*, which is constructed inductively by picking the shortest curves in σ for the base and repeating for the transverse curves. Now define a map $\Upsilon : \mathcal{M}(S) \to \mathcal{T}(S)$ by taking a marking μ to the region in the ϵ-thick part of $\mathcal{T}(S)$, denoted $\mathcal{T}_\epsilon(S)$, where μ is a short marking in that region. As stated in [Cor17], it is a well known fact that Υ is a coarsely well defined map which is coarsely Lipschitz. We take a moment to prove that this map is coarsely equivariant.

Lemma 5.4. *Let $\Upsilon : \mathcal{M}(S) \to \mathcal{T}(S)$ be as above, and let $H < \mathrm{Mod}(S)$ be finitely generated. Then there exists a constant $K \geq 0$ such that, for any marking $\mu \in \mathcal{M}$ and for any $h \in H$,*

$$d_{\mathcal{T}(S)}(h\Upsilon(\mu), \Upsilon(h\mu)) \leq K.$$

Proof. Let $\mu = \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\} \in \mathcal{M}(S)$ and $h \in H$ be arbitrary. Let $\sigma \in \mathcal{T}(S)$ so that μ is a short marking on σ. (Equivalently, let $\sigma = \Upsilon(\mu)$.) Since the action of H on $\mathcal{T}(S)$ permutes the lengths of curves, the length of each pair (α_i, β_i) with respect to σ is the same as the length of the pair $(h\alpha_i, h\beta_i)$ with respect to $h\sigma$. Therefore as μ was a short marking for σ, this shows that $h\mu$ is a short marking for $h\sigma = h\Upsilon(\mu)$. However, by definition of Υ, $h\mu$ is also a short marking for $\Upsilon(h\mu)$. As Υ was a coarsely well defined function, this shows that $d_{\mathcal{T}(S)}(h\Upsilon(\mu), \Upsilon(h\mu)) \leq K$ for some $K \geq 0$, as desired. □

We now prove Theorem 5.2, using the above lemma and several tools from [Cor17] to show that points in conical neighborhoods in $\mathcal{M}(S)$ end up in conical neighborhoods of $\mathcal{T}(S)$.

Proof. Fix $\mu_0 \in \mathcal{M}(S)$. Let $x \in \partial\mathcal{M}(S)_{\mu_0}$ be a conical limit point of $H\mu_0$. Put $\sigma_0 = \Upsilon(\mu_0)$. We shall show that $f_\infty(x)$ is a conical limit point of $H\sigma_0$ by verifying the condition in

Proposition 4.4. Let $T \geq 0$ be arbitrary, and let $\lambda : [0, \infty) \to \mathcal{T}(S)$ be an arbitrary Morse geodesic ray with $\lambda(0) = \sigma_0$ and $\lambda(\infty) = f_\infty(x)$.

Let $\alpha : \mathbb{N} \to \mathcal{M}(S)$ be an N-Morse geodesic with $\alpha(0) = \mu_0$ and $\alpha(\infty) = x$. By [Cor17, Lemma 4.9], $\Upsilon(\alpha)$ is an N'-Morse (A, B)-quasi-geodesic, for some A, B, and N' depending only on N. Put $\beta = \Upsilon(\alpha)$. Notice that $\beta(0) = \sigma_0$ and, by the construction of f_∞, we have $\beta(\infty) = f_\infty(x)$. (For details on the construction of f_∞, we refer to [Cor17], specifically Proposition 4.11, Theorem 4.12, and Proposition 4.14.)

Now let $\gamma_n = [\sigma_0, \beta(n)]$. Then each γ_n is N''-Morse for N'' depending on N, and by Arzelá-Ascoli and [Cor17, Lemma 2.10], a subsequence of the γ_n converges to a geodesic ray β which is N''-Morse, and by [Cor17, Lemma 4.9], β is bounded Hausdorff distance from γ, where the bound only depends on N. Say that $d_{Haus}(\beta, \gamma) \leq K_1$ for $K_1 \geq 0$. By [Cor17, Corollary 2.6], $d_{Haus}(\gamma, \lambda) \leq K_2$ where $K_2 \geq 0$ depends only on N. Choose $S \geq 0$ so that, for all $s \geq S$, $d_{\mathcal{T}(S)}(\beta(s), \lambda_{[T,\infty)}) \leq K_1 + K_2$.

By Proposition 4.4, there exists $L \geq 0$ where, for all $r \geq 0$, $d_{\mathcal{M}(S)}(h\mu_0, \alpha|_{[r,\infty)}) \leq L$ for some $h \in H$. Since $\beta = \Upsilon(\alpha)$ and Υ is coarse Lipschitz, there exits $K_3 \geq 0$ and $h \in H$ so that $d_{\mathcal{T}(S)}(\Upsilon(h\mu_0), \beta|_{[S,\infty)}) \leq K_3$. Let $s_0 \in [S, \infty)$ so that $d_{\mathcal{T}(S)}(\Upsilon(h\mu_0), \beta(s_0)) \leq K_3$. By Lemma 5.4, there exists $K_4 \geq 0$ such that $d_{\mathcal{T}(S)}(\Upsilon(h\mu_0), h\Upsilon(\mu_0)) \leq K_4$.

By the triangle inequality, we have

$$
\begin{aligned}
d_{\mathcal{T}(S)}(h\sigma_0, \lambda|_{[T,\infty)}) &\leq d(h\Upsilon(\mu_0), \Upsilon(h\mu_0)) + d(\Upsilon(h\mu_0), \beta(s_0)) + d(\beta(s_0), \lambda|_{[T,\infty)}) \\
&\leq K_4 + K_3 + K_2 + K_1.
\end{aligned}
$$

By Proposition 4.4, $\lambda(\infty) = f_\infty(x)$ is a conical limit point of $H\sigma_0$. ☐

Chapter 6

Minimality of Sublinearly Morse Boundaries

In this chapter, we present work with Yulan Qing and Elliott Vest, and we provide the proof of Theorem 1.8, i.e. that the sublinearly Morse boundary is minimal. Before diving into the proofs of these statements, we take a moment to introduce this boundary. Just as the Morse boundary in Definition 2.18 was defined using Morse geodesic rays, we define the sublinearly Morse boundary using sublinearly Morse rays. We direct the reader to [QR22] and [QRT] for a more detailed description of the sublinearly Morse boundary.

We begin by first recalling the definition of a sublinear function, and we set notation for a common inequality found in the rest of this chapter.

Definition 6.1. *A function* $\kappa : [0, \infty) \to [1, \infty)$ *is* ***sublinear*** *if* $\lim_{x\to\infty} \frac{\kappa(x)}{x} = 0$.
A quantity $D \geq 0$ ***is small compared to*** $r > 0$ *if* $D \leq \frac{r}{2\kappa(r)}$.

Although not required, it is often helpful to additionally assume that sublinear functions are monotone increasing and concave, see [QR22, Remark 3.1] for details. Given a proper, geodesic metric space X and a basepoint $\mathfrak{o}$, we adopt the following conventions

$$||x|| = d(x, \mathfrak{o}), \qquad \kappa(x) = \kappa(||x||)).$$

Definition 6.2. *For a (quasi-)geodesic $\alpha : I \to X$ and a constant $n \geq 0$, we define the (κ, n)-**neighborhood** of α to be $\mathcal{N}_\kappa(\alpha, n) = \{x : d(x, \alpha) \leq n\kappa(x)\}$.*

Comparing this definition to the definition of a M-neighborhood shows a striking difference: the M-neighborhood of a geodesic α is a set of "constant width" centered around α, whereas the (κ, n)-neighborhood of α is a set whose width grows at a sublinear rate, see Figure 6.1. However, these definitions agree exactly in the case that κ is a constant function.

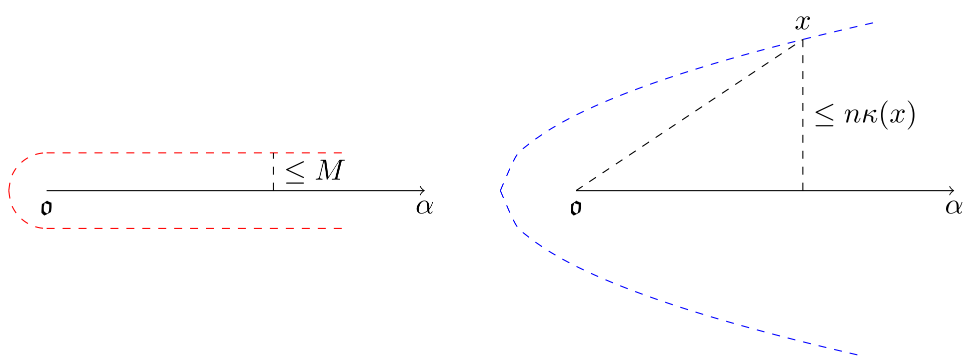

Figure 6.1: A comparison of $\mathcal{N}_M(\alpha)$ and $\mathcal{N}_\kappa(\alpha, n)$. On the left is $\mathcal{N}_M(\alpha)$, a neighborhood of constant width. On the right is $\mathcal{N}_\kappa(\alpha, n)$, a neighborhood of α whose width grows sublinearly.

Now that we have the notion of a sublinear neighborhood, we can use these to define a sublinearly Morse (quasi-)geodesic. Recall from Definition 2.17 that a Morse geodesic α keeps (K, C)-quasi geodesics that begin and end on α within a neighborhood of constant width, where the width depends only on K and C. The following definition modifies this familiar setting by instead using a neighborhood of sublinearly growing width.

Definition 6.3 (Sublinearly Morse Geodesic)**.** *A (quasi-)geodesic α is κ-**Morse** if there is a Morse gauge $N : [1,\infty) \times [0,\infty) \to [0,\infty)$ so that, for any (K,C)-quasi-geodesic $\varphi : [a,b] \to X$ with $\varphi(a), \varphi(b) \in \alpha$, we have $\varphi \subseteq \mathcal{N}_\kappa(\alpha, N(K,C))$.*

*We say α is **sublinearly Morse** if there exists a sublinear κ so that α is κ-Morse.*

Remark 6.4. *This definition is called κ-weakly Morse in the literature, see [QRT, Definition 3.9]. We note that in the special case where κ is a constant function, Definition 6.3 exactly reconstructs the definition of a Morse geodesic found in Definition 2.17.*

Another definition classifies sublinear geodesics by describing how far away the "middle" of a quasi-geodesic is allowed to get from the geodesic.

Definition 6.5. *Let $\alpha : [0,\infty) \to X$ be a (quasi)-geodesic with $\alpha(0) = \mathfrak{o}$, and let κ be a concave sublinear function. We say that α is κ-**strongly Morse** if there exists a Morse gauge $N : [1,\infty)\times[0,\infty) \to [0,\infty)$ such that for any sublinear function κ' and for any $r > 0$, there exists R such that for any (K,C)-quasi-geodesic ray $\beta : [0,\infty) \to X$ with $\beta(0) = \mathfrak{o}$ and $N(K,C)$ small compared to r,*

$$d(\beta(t_R), \alpha) \le \kappa'(R) \quad \Longrightarrow \quad \beta|[0,t_r] \subset \mathcal{N}_\kappa\big(\alpha, N(K,C)\big),$$

where $t_r = \inf\{t \in [0,\infty) : d(\mathfrak{o}, \beta(t)) = r\}$ and $t_R = \inf\{t \in [0,\infty) : d(\mathfrak{o}, \beta(t)) = R\}$, i.e. the first time $\beta(t)$ is distance r or distance R from $\varphi(0) = \mathfrak{o}$, respectively.

While more technical than Definition 6.3, these two definitions are equivalent in the setting of proper, geodesic metrics spaces, as shown in the following result.

Proposition 6.6. *([QRT, Proposition 3.10]) Let X be a proper, geodesic metric space, and let α be a (quasi)-geodesic. Then α is κ-Morse if and only if α is κ-strongly Morse.* □

Now looking towards defining a boundary using these geodesics, we define the following relaxed version of fellow-travelling.

Definition 6.7 (κ Fellow Travelling)**.** *Given two (quasi-)geodesic rays α, β based at $\mathfrak{o}$, we say that $\alpha \simeq \beta$ if they **κ-fellow travel** each other: i.e. if*

$$\lim_{r\to\infty} \frac{d_X(\alpha(r), \beta(r))}{r} = 0.$$

Equivalently, we say that $\alpha \simeq \beta$ if α is contained in a κ-neighborhood of β and β is contained in a κ-neighborhood of α. We denote the equivalence class of α by $\alpha(\infty)$, and we denote an equivalence class without a specified representative using Gothic letters such as $\mathfrak{a}$.

Definition 6.8. *Let κ be a sublinear function and let X be a proper, geodesic metric space. Then the **κ-Morse boundary** based at $\mathfrak{o}$, denoted by $\partial_\kappa X_{\mathfrak{o}}$, is the set of all equivalence classes of κ-Morse (quasi-)geodesic rays based at $\mathfrak{o}$ up to κ-fellow traveling.*

Remark 6.9. *If X is a proper geodesic metric space, then by [QRT, Lemma 4.2] every equivalence class $\mathfrak{a}$ contains a geodesic representative, i.e. there exists a geodesic ray α so that $\alpha \in \mathfrak{a}$. In particular, Definition 6.8 is well-defined by either equivalence classes of quasi-geodesic rays or by equivalence classes of geodesic rays. In addition, if G acts on X by isometries, then the action $G \curvearrowright \partial_\kappa X_{\mathfrak{o}}$ is well defined.*

The topology on $\partial_\kappa X_{\mathfrak{o}}$ is defined similarly to the topology on the visual boundary for a hyperbolic space, see Definition 2.9. However, while neighborhoods of α in the visual boundary are defined by (constant width) fellow traveling up to a distance of r away from the basepoint, the neighborhoods of α in the κ-Morse boundary are defined by sublinearly fellow traveling up to distance r from the basepoint.

Definition 6.10. *Let X be a proper, geodesic metric space, and let κ be a sublinear function. Let α be a geodesic ray representative of with $\mathfrak{a} \in \partial_\kappa X_\mathfrak{o}$, and let N be a Morse gauge for α. We define $\mathcal{U}(\alpha, r)$ as follows:*

An equivalence class $\mathfrak{b} \in \partial_\kappa X_\mathfrak{o}$ is an element of $\mathcal{U}(\alpha, r)$ if, for any (K, C)-quasi-geodesic $\varphi : [0, \infty) \to X$ with $\varphi \in \mathfrak{b}$ and with $N(K, C)$ small compared to r (in the sense of Definition 6.1), we have

$$\varphi([0, t_r]) \subseteq \mathcal{N}_\kappa(\alpha, N(K, C))$$

where $t_r = \inf\{t \in [0, \infty) : d(\mathfrak{o}, \varphi(t)) = r\}$, i.e. the first time $\varphi(t)$ is distance r from $\varphi(0) = \mathfrak{o}$.

For a proper, geodesic metric space X, the collection of sets $\mathcal{U}(\alpha, r)$ form a neighborhood basis for $\mathfrak{a}$, and in particular $\mathfrak{a} \in \mathcal{U}(\alpha, r)$ [QRT, Lemma 4.2]. This defines a topology on $\partial_\kappa X_\mathfrak{o}$ [QRT, Proposition 4.7], and in fact this topology is metrizable [QRT, Theorem 4.10]. Notice that when $G \curvearrowright X$ by isometries, that $\mathfrak{a} \in \mathcal{U}(\beta, r)$ is equivalent to $g\mathfrak{a} \in \mathcal{U}(g\beta, r)$.

Now that we have defined the topology for $\partial_\kappa X_\mathfrak{o}$, we begin building the results towards a proof of Theorem 1.8. We start by defining what it means for a group to act minimally on a space.

Definition 6.11. *Let X be a topological space, and suppose G is a group acting on X. The action $G \curvearrowright X$ is called **minimal** if the orbit Gx is dense in X for any $x \in X$.*

Remark 6.12. *Just because a group G has one dense orbit Gx in X, this does not mean every orbit is dense in X. For example, let $X = \mathbb{R}$ and let $G = \mathbb{Q}^{\times}$, the multiplicative group of rational numbers, act on X by multiplication. Then $G \cdot 1 = \mathbb{Q}^{\times}$ is dense in $\mathbb{R}$, but $G \cdot 0 = \{0\}$ is clearly not dense.*

Definition 6.13. *Let X be a proper, geodesic metric space and suppose $G \curvearrowright X$ acts coboundedly with cobounded constant K. Let $\beta : [0, \infty) \to X$ be a geodesic ray. We say a sequence of group elements $(g_i)_i$ **tracks** β if, for every $T \geq 0$, there exists $M \geq 0$ so that, for every $i \geq M$, there exists $t \geq T$ with $d(g_i\mathfrak{o}, \beta(t)) < K$.*

In all the results to follow, X will be a proper, geodesic metric space, and $G \curvearrowright X$ will be a cobounded action by isometries. We first show that when $|\partial_\kappa X_\mathfrak{o}| \geq 3$, for any $\mathfrak{b} \in \partial_\kappa X_\mathfrak{o}$, there exists $\mathfrak{a} \in \partial_\kappa X_\mathfrak{o}$ so that $\mathfrak{a}$ can be translated away from $\mathfrak{o}$ along $\mathfrak{b}$, and vice versa.

Lemma 6.14. *Let $K \geq 0$ be the cobounded constant of $G \curvearrowright X$ and assume $|\partial_\kappa X_\mathfrak{o}| \geq 3$. For any $\mathfrak{b} \in \partial_\kappa X_\mathfrak{o}$, choose a geodesic ray $\beta \in \mathfrak{b}$ and let $(g_i)_i$ be any sequence in G that tracks β. Then, there exists $\mathfrak{a} \in \partial_\kappa X_\mathfrak{o}$ such that for any $\alpha \in \mathfrak{a}$ and $R > 0$, there exists $j \in \mathbb{N}$ such that $g_i\alpha \cap B_R(\mathfrak{o}) = \emptyset$ for all $i \geq j$. In addition, for any sequence $(h_i)_i$ that tracks α, and for any $R > 0$ there exists $j \in \mathbb{N}$ such that $h_i\beta \cap B_R(\mathfrak{o}) = \emptyset$ for all $i \geq j$.*

Proof. Since $(g_i)_i$ tracks β, $d(g_i \cdot \mathfrak{o}) \to \infty$ as $i \to \infty$. Let $\mathfrak{a}, \mathfrak{c} \in \partial_\kappa X_\mathfrak{o}$ so that $\mathfrak{a} \neq \mathfrak{c} \neq \mathfrak{b}$. Let $\alpha \in \mathfrak{a}$ and $\zeta \in \mathfrak{c}$ be geodesic representatives. For each i, let $p_i \in \pi_{g_i\alpha}(\mathfrak{o})$ and $q_i \in \pi_{g_i\zeta}(\mathfrak{o})$. For the sake of contradiction, assume that both sequences $(||p_i||)_i$ and $(||q_i||)_i$ have a subsequence bounded above by some $R > 0$. By passing to a subsequence, we may assume both sequences $(||p_i||)_i$ and $(||q_i||)_i$ are bounded above by R. Note that this implies $d(g_i\mathfrak{o}, p_i) \to \infty$ and

$d(g_i\mathfrak{o}, q_i) \to \infty$, so we get that $\{g_i^{-1}p_i\}$ and $\{g_i^{-1}q_i\}$ are unbounded sequences. For each i, we have $d(p_i, q_i) < 2R$. Thus, $d(g_i^{-1}p_i, g_i^{-1}q_i) < 2R$. This gives two unbounded sequences $\{g_i^{-1}p_i\}$ and $\{g_i^{-1}q_i\}$ such that $d(g_i^{-1}p_i, g_i^{-1}q_i) < 2R$. This implies α and ζ fellow travel which gives $\alpha \simeq \zeta$, a contradiction to $\mathfrak{a} \neq \mathfrak{c}$.

Without loss of generality we may assume $(||p_i||)_i$ is unbounded. Now assume, for contradiction, that there exists an infinite sequence $(x_i)_i$ with $x_i \in h_i\beta \cap B_R(\mathfrak{o})$ for some $R > 0$. Let $k_i\mathfrak{o} \in B_K(x_i)$. Then $\{k_i\}$ is a sequence in G that tracks β, but $d(k_i\mathfrak{o}, \mathfrak{o}) \leq K + R$ for all i, a contradiction. □

We now introduce a lemma which, given a geodesic representative α of $\mathfrak{a}$ and group element g, constructs a uniform quality quasi-geodesic from $g\alpha$ to β.

Lemma 6.15. *Let $K \geq 0$ be the cobounded constant of $G \curvearrowright X$ and assume $|\partial_\kappa X_\mathfrak{o}| \geq 3$. For any $\mathfrak{b} \in \partial X$, choose a geodesic ray $\beta : [0, \infty) \to X$ with $\beta \in \mathfrak{b}$ and $\beta(0) = \mathfrak{o}$, and a sequence $(g_i)_i$ that tracks β. Let $\alpha \in \mathfrak{a}$ be as in Lemma 6.14. For any $p_i \in \pi_{g_i\alpha}(\mathfrak{o})$, there exists a $(27, 3K)$-quasi-geodesic ray that contains $[\mathfrak{o}, p_i]$ whose tail end is β.*

Proof. There exists an $r \geq 0$ such that $d(g_i \cdot \mathfrak{o}, \beta(r)) < K$ by definition of the cocompact action. Thus, by [QR22, Lemma 2.5] $\gamma_i = [\mathfrak{o}, p_i] * [p_i, g_i\mathfrak{o}] * [g_i\mathfrak{o}, \beta(r)]$ is a $(3, K)$-quasi-geodesic. Let $R > 0$ be such that $B_R(\mathfrak{o})$ contains γ_i. Denote $q_i = \pi_{\gamma_i}(\beta(R))$. We have

$$\begin{aligned} ||q_i|| = d(\mathfrak{o}, q_i) &\geq d(\mathfrak{o}, \beta(R)) - d(\beta(R), q_i) \\ &\geq R - d(\beta(r), \beta(R)) \\ &= R - (R - r) = r. \end{aligned}$$

Also, we see that $d(\mathfrak{o}, p_i) \leq r + K$ because $d(\mathfrak{o}, g_i\mathfrak{o}) \leq r + K$ and p_i is a closest point projection. We now break into cases.

CASE 1: Suppose $q_i \notin [\mathfrak{o}, p_i]$, then let η be the concatenation of γ_i, starting at $\mathfrak{o}$ and ending at q_i, with $[q_i, \beta(R)]$. Note that η is a $(9, K)$-quasi-geodesic. Furthermore, as $||q_i|| \leq R = ||\beta(R)||$, we get that for any $x \in \beta([R, \infty))$, $\pi_\eta(x) = \beta(R)$, via an argument found in [QR22, Lemma 4.3]. Hence $\eta * \beta([R, \infty))$ is an $(27, K)$-quasi geodesic that fellow travels β. See Figure 6.2.

CASE 2: In the case that $q_i \in [\mathfrak{o}, p_i]$, then q_i is within K of p_i. Indeed, since $||q_i|| \geq r$ and $||p_i|| \leq r + K$, the fact that both q_i and p_i are on the geodesic $[\mathfrak{o}, p_i]$ emanating from $\mathfrak{o}$ implies $d(q_i, p_i) \leq K$. Let η' be the path which follows γ_i from $\mathfrak{o}$ to q_i, then follows γ_i from q_i back to p_i, then follows $[p_i, q_i]$, and finally follows $[q_i, \beta(R)]$. Then η' is a $(9, 3K)$-quasi-geodesic. Similar to CASE 1, we find a $(27, 3K)$-quasi-geodesic that fellow travels β. □

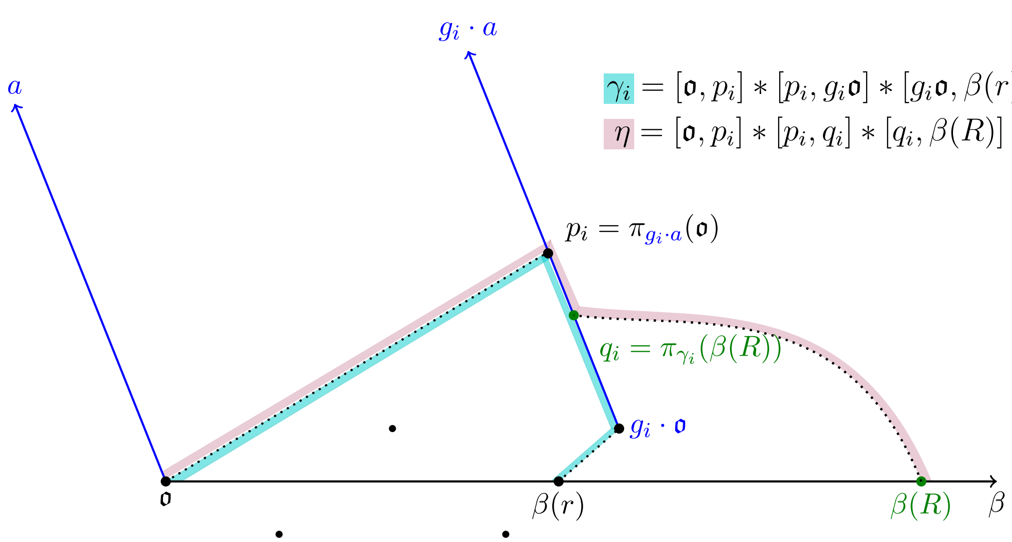

Figure 6.2: A picture of CASE 1. We have η will be a (9,K) quasi-geodesic that contains the geodesic segment $[\mathfrak{o}, p_i]$.

Corollary 6.16. *Given the conditions of Lemma 6.15, if β is κ-Morse, then the $(27, 3K)$-quasi-geodesic ray found in Lemma 6.15 is κ-Morse with Morse gauge depending only on K and the Morse gauge of β.*

Proof. This is immediate from Lemma 6.15 and [QRT, Corollary 3.5]. □

Remark 6.17. *Notice that Lemma 6.14 gives symmetric results: translating the basepoint of α along β leaves every ball of radius R, and vice-versa. Therefore, by just changing letters in the proof of Lemma 6.15, we can prove that there exists a $(27, 3K)$-quasi-geodesic which first projects to an orbit of β, then eventually fellow travels α. The observation that these arguments can change the role of α and β will be important point in proving Theorem 6.23.*

To summarize the above lemma, we have found a quasi-geodesic which first nearest-point projects to $g_i\alpha$ and then, eventually, fellow travels β. In this next lemma, we find a quasi-geodesic λ_i which closest point projects to $g_i\alpha$ and then fellow travels $g_i\alpha$. Using the fact that the quasi-geodesic from Lemma 6.15 is Morse for a Morse gauge depending only on K and β will give us control over the Morse gauge for λ_i.

Lemma 6.18. *Let $K \geq 0$ be the cobounded constant of $G \curvearrowright X$ and assume $|\partial_\kappa X_\mathfrak{o}| \geq 3$. For any $\mathfrak{b} \in \partial_\kappa X_\mathfrak{o}$, choose a geodesic $\beta : [0, \infty) \to X)$ with $\beta \in \mathfrak{b}$ and $\beta(0) = \mathfrak{o}$, and sequence $(g_i)_i$ that tracks β. Let $\alpha \in \mathfrak{a}$ be as in Lemma 6.14. For some $p_i \in \pi_{g_i\alpha}(\mathfrak{o})$, define λ_i as the path which first follows $[\mathfrak{o}, p_i]$, then follows the ray $g_i\alpha$ forever. Then for every i, λ_i is a $(3, 0)$-quasi-geodesic that is κ-Morse with Morse gauge depending only on α, β and K.*

Proof. See Figure 6.3 for a picture of this proof. Consider any (q, Q)-qusi-geodesic $\xi \in \lambda_i(\infty)$. Let $q_i \in \pi_\xi(p_i)$. Let ω be the path defined by following ξ from $\mathfrak{o}$ to q_i, then following

$[q_i, p_i]$. Then ω is a $(3q, Q)$-quasi-geodesic with endpoints on $[\mathfrak{o}, p_i]$ which is contained in η, where η is as constructed as in the proof of Lemma 6.15. By Definition 6.3,

$$\omega \subset \mathcal{N}_\kappa\big(\eta, m_\eta(3q, Q)\big),$$

where m_η is the Morse gauge of η, and by Lemma 6.15, m_η depends only on β and K. Similarly, defining ω' as the path which first follows $[\mathfrak{o}, q_i]$, and then follws the quasi-geodesic ray ξ to infinity, we have that ω' is a $(3q, Q)$-quasi-geodesic that fellow travels $g_i\alpha$. Thus

$$\omega' \subset \mathcal{N}_\kappa\big(g_i\alpha, m_{g_i\alpha}(3q, Q)\big),$$

where $m_{g_i\alpha}$ is the Morse gauge of $g_i\alpha$. But since g_i acts by isometries, $m_\alpha = m_{g_i\alpha}$ is a Morse gauge for α. Hence, we conclude $\xi \subset \omega \cup \omega' \subset \mathcal{N}_\kappa\big(g_i \cdot a, m_\eta(3q, Q) + m_\alpha(3q, Q)\big)$. $\square$

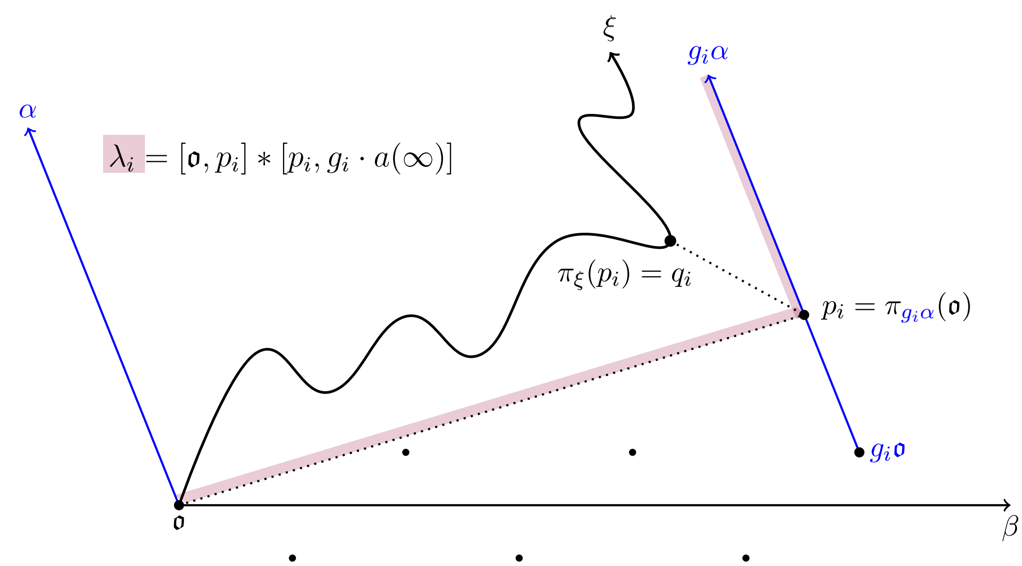

Figure 6.3: Diagram for Lemma 6.18. To show ξ is κ-Morse, we subdivide ξ into two parts. The initial segment can be leveraged by using the κ-Morseness of η in Lemma 6.15. The remaining part of ξ fellow travels $g_i \cdot a$, so we leverage the κ-Morseness of $g_i \cdot a$.

Remark 6.19. *Notably, the Morse gauge for λ_i does not depend on i.*

Notice that the construction of each λ_i begins by projecting to a point on $g_i\alpha$. Recall that we are choosing the sequence g_i so that g_i stays close to β and gets farther and farther from $\mathfrak{o}$. It is therefore not surprising for us to find that the λ_i end up staying sublinearly close to β for longer and longer periods of time, as we show in the next proposition.

Proposition 6.20. *Let $K \geq 0$ be the cobounded constant of $G \curvearrowright X$ and assume $|\partial_\kappa X_\mathfrak{o}| \geq 3$. For any $\mathfrak{b} \in \partial_\kappa X_\mathfrak{o}$, choose a geodesic $\beta : [0, \infty) \to X)$ with $\beta \in \mathfrak{b}$ and $\beta(0) = \mathfrak{o}$, and sequence $(g_i)_i$ that tracks β. Let $\alpha \in \mathfrak{a}$ be as in Lemma 6.14. For some $p_i \in \pi_{g_i\alpha}(\mathfrak{o})$, define λ_i as the path which first follows $[\mathfrak{o}, p_i]$, then follows the ray $g_i\alpha$ forever. For any $r > 0$ there exists an i such that*

$$\lambda_i([0, t_r]) \subset \mathcal{N}_\kappa\big(\beta, m_\beta(9, 0)\big),$$

where t_r is the first time $\lambda_i(t)$ is distance r from $\mathfrak{o}$ and m_β is the Morse gauge of β.

Proof. Set $\kappa' = m_\beta(9, 0)\kappa$. Since β is κ-Morse, there exists an $R = R(3, 0, r, \kappa')$ such that Definition 6.5 holds. By Lemma 6.14, There exists an i such that for λ_i, we have $||p_i|| \geq R$. By Lemma 6.15, $d(\lambda_i(R), \beta) \leq m_\beta(9, 0)\kappa(\lambda_i(R)) = m_\beta(9, 0)\kappa(R)$. Hence, as β is κ-Morse, $\lambda_i([0, t_r]) \subset \mathcal{N}_\kappa\big(\beta, m_\beta(9, 0)\big)$. □

Notice that, referring to Definition 6.10, we have just shown that for every $r > 0$, there exists i so that $\lambda_i \in \mathcal{U}(\beta, r)$. However, in order to satisfy the full conditions of Definition 6.10, we need to show that the entire equivalence class of λ_i is contained in $\mathcal{U}(\beta, r)$. This fact is straightforward: If ξ is in the same equivalence class as λ_i, then ξ and λ_i sublinearly fellow travel in the sense of Definition 6.7. Since λ_i sublinearly follows β up to distance r, and ξ sublinearly follows λ_i for all time, ξ must also sublinearly travel β up to some distance r'. We formalize this argument in the next proposition.

Proposition 6.21. *Let $K \geq 0$ be the cobounded constant of $G \curvearrowright X$ and assume $|\partial_\kappa X_{\mathfrak{o}}| \geq 3$. For any $\mathfrak{b} \in \partial_\kappa X_{\mathfrak{o}}$, choose a geodesic $\beta : [0, \infty) \to X$ with $\beta \in \mathfrak{b}$ and $\beta(0) = \mathfrak{o}$, and sequence $(g_i)_i$ that tracks β. Let $\alpha \in \mathfrak{a}$ be as in Lemma 6.14. For some $p_i \in \pi_{g_i\alpha}(\mathfrak{o})$, define λ_i as the path which first follows $[\mathfrak{o}, p_i]$, then follows the ray $g_i\alpha$ forever. For any $r > 0$, there exists an i such that for any (q, Q)-quasi-geodesic ξ with $\xi \simeq \lambda_i$ and $m_\beta(q, Q)$ small compared to r, we have*

$$\xi([0, t_r]) \subset \mathcal{N}_\kappa\big(\beta, m_\beta(q, Q)\big),$$

where t_r is the first time $\xi(t)$ is distance r from $\mathfrak{o}$ and m_β is the Morse gauge of β.

Proof. Let m_{λ_i} be the Morse gauge for λ_i. Choose $R > 0$ to be sufficiently large and i such that $\lambda_i([0, t_R]) \subset \mathcal{N}_\kappa\big(\beta, m_\beta(9, 0)\big)$, where t_R is the first time $\lambda_i(t)$ is distance R from $\mathfrak{o}$. Specifically, we can choose R to be larger than the $R(3, 0, r, \kappa')$ from Proposition 6.20 and also larger than $2r$. Pick any (q, Q)-quasi-geodesic ξ such that $\xi \simeq \lambda_i$ with $m_\beta(q, Q)$ small compared to r. By being in the same equivalence class, $\xi([0, t_r]) \subseteq \mathcal{N}_\kappa\big(\lambda_i, m_{\lambda_i}(q, Q)\big)$. Since m_{λ_i} is independent of i by Remark 6.19, we denote m_{λ_i} by m_λ. For any $x \in \xi([0, t_r])$, $||\pi_{\lambda_i}(x)|| \leq 2||x|| \leq 2r$ by [QRT, Lemma 2.2]. Since $R > 2r$,

$$d\bigg(\pi_\beta\big(\pi_{\lambda_i}(x)\big), \pi_{\lambda_i}(x)\bigg) \leq m_\beta(9, 0)\kappa(\pi_{\lambda_i}(x)) \leq 2m_\beta(9, 0)\kappa(x).$$

That is, for any $r > 0$, we can find an i such that all $\xi \simeq \lambda_i$ have

$$\xi([0, t_r]) \subset \mathcal{N}_\kappa\big(\beta, 2m_\beta(9, 0) + m_\lambda(q, Q)\big).$$

See Figure 6.4. Note that by Lemma 6.18, $2m_\beta(9,0) + m_\lambda(q,Q)$ is also a Morse gauge for β. By [QRT, Lemma 4.2] and its proof, there will also exist an i such that any ξ with $\xi \simeq \lambda_i$ will also have $\xi([0,t_r]) \subset \mathcal{N}_\kappa\big(\beta, m_\beta(q,Q)\big)$. □

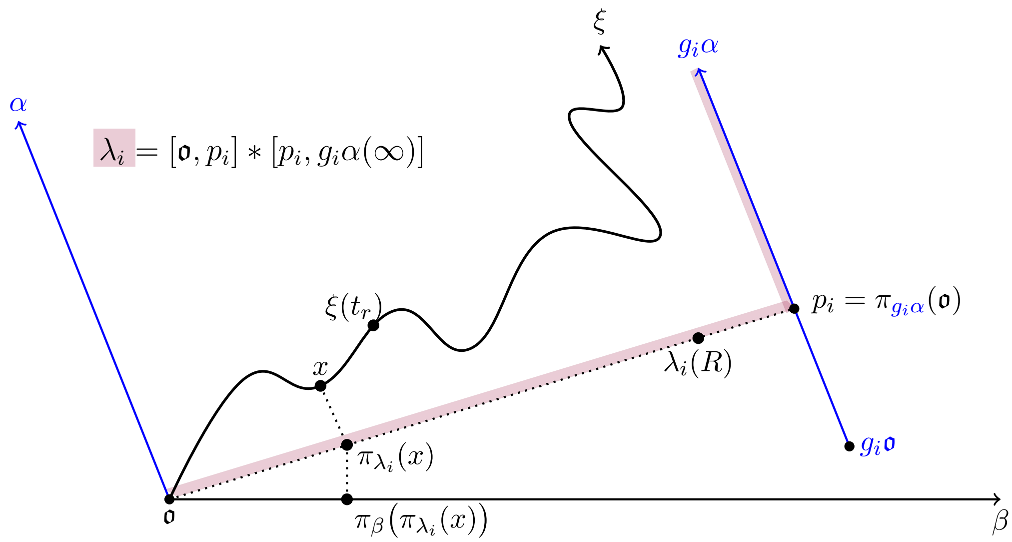

Figure 6.4: Diagram for Proposition 6.21. We choose R large enough for any ξ and any $x \in \xi([0,t_r])$, its projection to λ_i will be within $[\mathfrak{o}, p_i] \subseteq \lambda_i$. This bounds the distance of all $x \in \xi([0,t_r])$ to β in terms of m_β and m_λ.

Corollary 6.22. *Let $K \geq 0$ be the cobounded constant of $G \curvearrowright X$ and assume $|\partial_\kappa X_\mathfrak{o}| \geq 3$. For any $\mathfrak{b} \in \partial_\kappa X_\mathfrak{o}$, choose a geodesic $\beta \in \mathfrak{b}$ and a sequence $(g_i)_i$ that tracks β. Let $\mathfrak{a}$ be as in Lemma 6.14. Then for any $r > 0$, there exists i so that $g_i\mathfrak{a} \in \mathcal{U}(\beta, r)$.*

Proof. This is Proposition 6.21 using the notation of Definition 6.10. □

We now prove the main result of this chapter, minimality of the sublinearly Morse boundary. Notice that, by Lemma 6.14, for any $\mathfrak{b} \in \partial_\kappa X_\mathfrak{o}$, there exists some element $\mathfrak{a}$ so that $\mathfrak{b} \in \overline{G\mathfrak{a}}$. However, we need to show that **any** element of $\partial_\kappa X_\mathfrak{o}$ has a dense orbit.

Theorem 6.23 (Minimality)**.** *Let $K \geq 0$ be the cobounded constant of $G \curvearrowright X$ and assume $|\partial_\kappa X_\mathfrak{o}| \geq 3$. For every $\mathfrak{a} \in \partial_\kappa X_\mathfrak{o}$, the orbit $G\mathfrak{a}$ is dense in $\partial_\kappa X_\mathfrak{o}$.*

Proof. Let $\mathfrak{b}, \mathfrak{c} \in \partial_\kappa X$. If $\mathfrak{b} \in \overline{G\mathfrak{c}}$, we are done. Otherwise, let $\mathfrak{a} \neq \mathfrak{b} \neq \mathfrak{c}$. Let $\alpha \in \mathfrak{a}$ $\beta \in \mathfrak{b}$, and $\zeta \in \mathfrak{c}$ be geodesic ray representatives all with domain $[0, \infty)$. Let $(g_i)_i$ be a sequence in G that tracks β and let $\{h_j\}$ be a sequence in G that tracks α as in Lemma 6.14. Let $r > 0$ be arbitrary. By Corollary 6.22 there exists i so that $g_i\mathfrak{a} \in \mathcal{U}(\beta, r)$. It is clear that, since the group action of $G \curvearrowright X$ is by isometries, $\mathfrak{a} \in \mathcal{U}(g_i^{-1}\beta, r)$. By [QRT, Claim 4.6], there exists $r' > 0$ and $\alpha' \in \mathfrak{a}$ so that $\mathcal{U}(\alpha', r') \subseteq \mathcal{U}(g_i^{-1}\beta, r)$. Again by Corollary 6.22 (and keeping in mind Remark 6.17) there exists j so that $h_j\mathfrak{c} \in \mathcal{U}(\alpha', r')$, and so $h_j\mathfrak{c} \in \mathcal{U}(g_i^{-1}\beta, r)$, i.e., $g_i h_j \mathfrak{c} \in \mathcal{U}(\beta, r)$. ◻

We note that Theorem 1.8 adds an additional assumption that the group action $G \curvearrowright X$ is also proper, and in fact is the special case where X is a Cayley Graph for G, see Definition 2.1 and Lemma 2.5. Therefore, Theorem 1.8 is a special case of Theorem 6.23.

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