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In Defense of Pay-as-Bid Auctions: A Divisible-Good Perspective

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In Defense of Pay-as-Bid Auctions:
A Divisible-Good Perspective

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Economics

by

Kyle Leland Woodward

2015
Abstract of the Dissertation

In Defense of Pay-as-Bid Auctions:
A Divisible-Good Perspective

by

Kyle Leland Woodward

Doctor of Philosophy in Economics
University of California, Los Angeles, 2015
Professor Marek Pycia, Chair

Pay-as-bid auctions are commonly used to sell treasury securities, purchase government debt, distribute electrical generation, and allocate emissions credits. In spite of the practical relevance of the format, the pay-as-bid auction is a fundamentally misunderstood object. As compared to other popular auction formats used in the sale of homogeneous goods, the pay-as-bid auction is often incorrectly assumed to have comparatively bad incentive properties and to induce suboptimal outcomes. In this dissertation I model the pay-as-bid auction as a set of strategic bidders competing for units of a perfectly-divisible commodity. This model allows surprisingly tractable representations of bidder behavior, which provide novel evidence uniformly in favor of the pay-as-bid auction format.

Chapters 2 and 3 consider the theoretical implications of this model for existing empirical work. I demonstrate that when bidders have private information, equilibrium behavior is well-defined and there is an equilibrium in pure strategies. I additionally show that, through a process I call strategic ironing, bidders face strong incentives to submit bids substantially below their true values. The nature of these incentives implies that previous intuition—the pay-as-bid auction is strategically similar to a single-unit first-price auction—is ill-founded. Although
strategic ironing is shown to dramatically reduce equilibrium bids below an intuitive benchmark, the current misunderstanding of the properties of the auction format may perversely imply that the pay-as-bid format generates comparatively higher revenues than has been inferred.

Chapters 4 and 5 engage in a direct comparative investigation of the pay-as-bid format when bidders have no private information. I obtain results on the existence and uniqueness of equilibria in this setting, and show that optimally-parameterized auction formats are revenue equivalent. When the auctions are nonoptimally parameterized, I demonstrate that the pay-as-bid auction revenue-dominates alternate formats. I show computationally that this dominance continues to hold within a broader class of randomized mechanisms. The computational results suggest large-market revenue equivalence within this class.

The findings in this dissertation emphatically support the implementation of the pay-as-bid mechanism. It is well-behaved, potentially tractable, and smoothly revenue-dominant when compared to alternate auction formats.
The dissertation of Kyle Leland Woodward is approved.

Sushil Bikhchandani

Moritz Meyer-ter-Vehn

Joseph Ostroy

Marek Pycia, Committee Chair

University of California, Los Angeles

2015
To Shabnam.

...I’m just thinking of the right words to say.
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To the members of econ-grads2009@econ.ucla.edu, thanks for the camaraderie, commiseration, and general emotional reliability. Godspeed and good luck; I’m sure we’ll continue crossing paths. Carlos, Greg, Jesper, Kenny, Omer, Sean, Tiago, and Box Sean: stay RadLab. Chad, Devin, Owen, and Siwei: stay as you are, but keep coming to visit.

I owe the entire UCLA theory group an exceptional debt of gratitude. Thanks for the time, feedback, funding, and wisdom you’ve shared over the last half decade. To my committee, thank you for the invaluable advice given, as well as for being at least as committed to my success as I was. Thank you for taking the long view and for encouraging me to control my excitable (some might say flighty) nature.

At dinner following a flyout presentation, I had a conversation which included the following statement.1

“You really have a great advisor. It is incredible the amount of effort he is putting into finding you a job.”

Marek, this captures only a fraction of the assistance and guidance you have provided. Thank you for pushing me when I needed pushing, and for not when I didn’t. That Bengt referred to me as his “grandstudent” is not directly attributable to your efforts, but I took immense pleasure from it all the same.

1I am doing my best to accurately capture an entirely unbidden quote, so there may be some discrepancies with reality.
To my family: I am fairly certain that you don’t know what I’ve spent the last half-decade studying, and for that, among other things, I love you. Thank you for your support, helpful distractions, and trust that my graduate education would inevitably, as all things, end. We can talk about research (work!) now, but I’d still prefer just about anything else.

Shabnam, there is some well-documented antipathy between neuroscientists and economists, but we make it work. What is a marriage but an excuse to share a MATLAB license? You are my inspiration and my place of refuge. Let’s keep doing everything together—I’m glad we exist.

Official matters

Some Chapters in this dissertation are adapted from earlier work. Chapters 2 and 3 present work from my job-market paper, Strategic Ironing in Pay-as-Bid Auctions. Chapter 4 presents results from joint work with Marek Pycia, under submission to the Journal of Political Economy as Pay-as-Bid: Selling Divisible Goods to Uninformed Bidders.

I gratefully acknowledge the financial support provided by UCLA’s Dissertation Year Fellowship. The freedom provided by not having to struggle for income greatly assisted the completion of this dissertation.

The colors used in the Figures in this document are derived from the UCLA color palette; those colors which are not officially sanctioned are gradients between accepted colors.² Thank you to UCLA Marketing and Special Events for answering my many questions over the years.

²See Figures 5.1 and 5.2.
Vita

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Southwest Economic Theory XVII (2015): *Strategic Ironing in Pay-as-Bid Auctions*
Pay-as-bid auctions

Pay-as-bid, or discriminatory, auctions are employed when a single seller seeks to allocate multiple units of a homogeneous good. He solicits demand functions from a set of bidders, then computes the market-clearing price at which reported demand equals the available supply of goods. He allocates to each bidder her stated demand at the market-clearing price, then perfectly price-discriminates against each bidder according to her submitted demand curve.

The pay-as-bid auction format is frequently implemented in practice, and is used to sell treasury securities, purchase government assets, distribute electricity generation, and allocate emissions credits; summing over all implementations, hundreds of billions of dollars of assets are transferred via pay-as-bid auctions each year. In spite of its practical importance, surprisingly little is known about the theoretical features of the pay-as-bid auction: its comparative revenue and efficiency properties are inherently ambiguous, and except in special circumstances the current literature is unable to even compute best-response strategies. Without explicit, or even implicit, descriptions of best-response behavior, much of the policy analysis surrounding the implementation of auctions for homogeneous units has leaned on well-understood results in single-unit auctions. Although these analogies appear to provide a useful tool for circumventing the complexity involved in analyzing pay-as-bid auctions and their kin, they are fatally flawed: when agents must submit bids for multiple units, there exist inter-unit incentives which have no analogue in single-unit auctions. Chapter 3 discusses this in greater depth.
The failure of the intuition derived from single-unit auctions has not been fully understood or appreciated, and carries potentially dramatic policy implications.

In this dissertation I contribute to the literature on pay-as-bid auctions in several ways. In Chapter 2, I prove the existence of a pure-strategy equilibrium in the pay-as-bid auction when bidders have private information; the question of existence in this setting had previously been unresolved, and has stood in the way of developing a proper theory of equilibrium behavior. Chapter 3 builds on this existence result, and I demonstrate the presence of a surprising bid-reduction effect which I call strategic ironing. In Chapter 4 I\(^1\) show that there is at most one equilibrium in the pay-as-bid auction without private information, and I give conditions for its existence. From the unique implied representation of bidders’ equilibrium strategies, I demonstrate that the pay-as-bid auction is revenue-dominant when parameterized near-optimally. Chapter 5 builds on this model to consider the possibility that slight modifications to the mechanism might improve seller revenue; in particular, I consider the revenue effects of randomizing between two possible auctions. In simulations of such randomizations, I find that the pay-as-bid auction is smoothly revenue-dominant—indeed, expected revenues are monotonically increasing in the probability that the pay-as-bid auction is implemented.

It is my hope that these results help to further the progress of research into bidder behavior in pay-as-bid auctions. Importantly, taken individually each of the results in this dissertation suggests that previous research may have understated the revenue generated by this auction format; taken as a whole they provide strong evidence that this is the case. As concerns situations in which the seller desires to maximize revenue, this is of obvious importance.\(^2\) Future research should be

\(^1\)Chapter 4 is taken from joint work with Marek Pycia. Nonetheless, I will use the first-person singular throughout this document.

\(^2\)I do not consider that the seller’s objective in a particular auction setting might be something other than revenue maximization, such as buyer-surplus maximization or some alternate criterion. In the case of certain government-sponsored auctions, this is an obvious objection to my claim that the pay-as-bid auction is strictly preferable. For example, the U.S. EPA suggests that the market for sulfur dioxide emission permits “allows sources in cap and trade programs
able to leverage these results to clarify the generality of the dominance of the pay-as-bid format, and to compute bidder behavior in more exotic settings.

In the remainder of this Chapter, I define the pay-as-bid mechanism in terms of standard economic principles, and introduce another common auction format employed in the sale of homogeneous goods. I describe some examplar implementations of the pay-as-bid format as used in practice. I then mathematically define the exact form of the pay-as-bid auction that I investigate in the remainder of this dissertation. I briefly survey the current literature in this area, and conclude with a preview of the results obtained in subsequent Chapters.

1.1 The pay-as-bid mechanism

Consider the problem faced by a single seller (henceforth male) attempting to sell a large, exogenous quantity of a perfectly-homogeneous good.\textsuperscript{3} A set of buyers (henceforth female) would like to purchase quantities of the good, and the seller—not knowing the buyers’ true values for the good—must implement a mechanism to allocate the goods and transfer some of the buyers’ surplus to himself. In a traditional paradigm, the seller might post a fixed price and simply ask buyers the quantity that they desire at this price.

A classical posted-price mechanism necessarily has some difficulties: in most cases demand will not equal supply, and the seller will need to either ration the quantities he allocates, or will be left with some supply remaining. This can lead to obvious incentive problems, both in the static and dynamic sense. Additionally, in the general case the process of price discovery is inhibited:\textsuperscript{4} the most the

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\textsuperscript{3}In this dissertation I will only consider the problem of inelastic supply. There has been work on the role that supply elasticity might play in multi-unit auctions; see, e.g., LiCalzi and Pavan (2005).

\textsuperscript{4}In the case with one-dimensional uncertainty, this claim is invalid if bidders submit deterministic bids as a function of their private information.
seller can uncover about a particular bidder is a single dimension of her private information, hence a seller who is interested in learning about buyer types will be left largely disappointed. Perhaps most importantly, though, is the role that such selling mechanisms play in the provisioning of government assets to private firms and citizens. In the face of participation constraints which often exist in these markets, the government faces political pressure to ensure that its methods are transparent and justifiably fair. A posted price with a participation constraint leads directly to extortionary pressures.\(^5\)

It may therefore be preferable to ask bidders to report demand curves—the quantities they demand at various prices.\(^6\) This enables bidders to find the market price themselves according to classical economic principles, endogenously equates supply and demand, and reveals a much larger extent of the bidders’ private information.\(^7\)

With bidders’ demand curves in hand, the seller must implement an allocation and solicit payments from the bidders. The natural allocation is the one which exactly clears the market: the seller finds a price such that the sum of all bidders’ demands is equal to the quantity available, and then allocates to each bidder her demand at this quantity.\(^8\) The problem of payments is somewhat more ambiguous. In the pay-as-bid auction, the seller perfectly price-discriminates according to the demand submitted, and he charges each bidder exactly her stated willingness to

\(^5\)It is natural to question the value of the participation constraint. Since, for example, treasury security auctions are used to supply liquidity necessary to fund government projects, lacking a reliable source of funding acts against social interests (to the extent that the government’s interests match those of society). Participation constraints address this liquidity problem.

\(^6\)Or price in terms of quantity; the intuition and mathematical results of the model are unaffected by this technical assumption.

\(^7\)Consider a simple situation in which bidders demand at most two units of a good. By posting a price, the seller can obtain information only about demand at the posted price. By soliciting demand functions, the seller can possibly obtain information about demand for both units. This is not a guarantee: pooling equilibria and randomization may cloud the information conveyed. Nonetheless, mechanisms which rely on reported demand curves offer at least the possibility of improving the seller’s information acquisition.

\(^8\)The formal model specified in Section 1.3 will handle the nondeterminacies which arise from nonuniqueness of market price, or nonuniqueness of allocation conditional on market price.
pay for each unit she receives.

Conditional on the demand curves submitted, this is a revenue-maximizing mechanism; indeed, if demand curves were truthfully reported the seller could not extract more buyer surplus. However, bidders’ strategic incentives imply that this mechanism cannot extract all bidder surplus. Truthful reporting can be easily seen to be nonoptimal: if a bidder reports her true demand to the seller, she obtains zero margin on any unit and hence zero total surplus; this strategy is weakly dominated by any other, and in general will be strictly dominated by submitting a bid curve strictly below her true value.

Helpfully, the pay-as-bid auction may be stated quite simply using terms defined in introductory economics courses: a seller solicits strategic demand functions from a group of bidders. He then aggregates these demand functions to find the market-clearing price, allocates to each bidder her demand at this price, and perfectly-price discriminates against her, charging her entire stated willingness to pay up to the quantity she is allocated. The simplicity of this definition is at least partially responsible for the mechanism’s popularity.

1.1.1 The uniform-price auction

Were the pay-as-bid auction the only format employed in the provisioning of large homogeneous quantities, theoretical investigations would be of descriptive value but of no immediate normative importance. Fortunately for this dissertation and for the study of pay-as-bid auctions more generally, there is another popular auction format employed in these settings: the uniform-price auction.

The uniform-price auction is nearly identical to the pay-as-bid auction, differing only in the payment rule. In a uniform-price auction, the seller solicits strategic demand curves from bidders, and uses these to compute the market-

\[9\] This statement need hold only ex ante, unless it is feasible for buyers to leave the mechanism upon announcement of the resultant allocation and transfers. See also footnote 5.
clearing price as in a traditional demand paradigm (and also as in a pay-as-bid auction). Bidders receive their stated demands at the market-clearing price, and then pay the constant market-clearing price for each unit they obtain.

Where the pay-as-bid auction looks like first-order price discrimination against reported demand, the uniform-price auction resembles a classic posted-price mechanism: all bidders pay the same price for each unit they obtain, and this price is found at the intersection of demand and supply. Because the final price is determined by the bid for the ultimate quantity allocation and not—as in the pay-as-bid auction—by the bids for all lesser quantities, the incentive to misreport demand differs from that in the pay-as-bid auction. That this incentive to misreport is distinct from that which arises in the pay-as-bid auction has sometimes been taken to mean that there is no incentive to misreport in the uniform-price auction; this is emphatically not the case.

The theoretical literature comparing these two auction formats has studiously avoided comparisons to basic price-discrimination paradigms: because the bidders transfer their entire stated surplus in the pay-as-bid auction, and only the market price per-quantity in the uniform-price auction, if bidders are nonstrategic the pay-as-bid auction will trivially generate greater revenue than the uniform-price auction. Thus strategic behavior, and in particular, the differing incentives between the two auction formats, has been viewed as essential to modeling these auctions appropriately.

In light of the apparent intractability of strategic behavior in these auctions, the policy debate over the preferential mechanism has often turned on analogies to simpler strategic mechanisms. To an approximation, the pay-as-bid auction looks like a generalization of a traditional first-price auction to many units—in each case the bidder pays her stated willingness to pay—and the uniform-price auction looks like a generalization of a traditional second-price auction to many units—

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10Absent strategic incentives this may alternatively be viewed as a classical Walrasian market.
in each case the bidder pays the highest losing bid.\footnote{Whether bidders pay the highest losing bid or the lowest winning bid is a matter of the design of the particular uniform-price auction; its incentive properties do not differ significantly across the two cases, provided that there are sufficiently-many units.} It is well-understood that truthfully reporting one’s own value is a dominant strategy in the second-price auction.\footnote{In this dissertation I constrain attention to an independent private values paradigm. With correlated values the dominant strategy claim with respect to the single-unit second-price auction is not generally true.} The uniform-price auction has therefore been viewed, by analogy, as encouraging truthful reporting, while the pay-as-bid auction suffers from harmful misreporting, known in this context as demand reduction.

As it turns out, neither of these analogies is correct, or even particularly apt. That a uniform-price auction is not a generalized second-price auction arises from the fact that a given bidder may now determine the price she pays, even though she wins a positive quantity; this is not possible in a single-unit second-price auction. Theoretical models of the uniform-price auction have been careful to address this distinction. That a pay-as-bid auction is not a generalized first-price auction has not yet been fully discussed, and is a main topic of Chapter 3.

\section*{1.2 In practice}

The pay-as-bid auction format is used to sell treasury securities, purchase assets, distribute electricity generation, and allocate pollution credits. Brenner, Galai, and Sade (2009) identify 33 out of 48 nations surveyed as using the pay-as-bid format to sell sovereign debt. The U.S. Federal Reserve implemented a pay-as-bid auction—in procurement form, where bidders submitted supply curves to fulfill the Federal Reserve’s demand—during its Quantitative Easing program. Electricity grid operators, commonly known as Independent System Operators, have run the pay-as-bid procurement auction to cheaply distribute electricity generation across generating units. Since its inception, the U.S. EPA has used a pay-as-bid mechanism to allocation pollution permits in its Acid Rain Program. To motivate,
practically, the investigations in this thesis, in this Section I provide a brief case study of each of these markets.

In each study I present a brief history of the market in question, then describe the particular operations of the mechanism today. I focus largely on the development of the mechanism in the U.S., in particular due to the ready availability of descriptive information and histories. Although two of these examples—treasury security auctions and electricity market auctions—are not currently implemented as pay-as-bid auctions in the U.S., worldwide economic liberalization during the twentieth century suggests that the processes leading to auction mechanisms in the U.S. are instructive when looking to the same processes in other nations. Analyzing the evolution of these institutions in the U.S. is therefore useful, if not directly relevant to the pay-as-bid auctions implemented today.

1.2.1 Treasury security auctions

Until the 1920s, the U.S. Treasury used a posted-price format in its quarterly sales of securities. Over time, these sales became significantly oversubscribed—in classical theory, the posted price was too low—and the relative infrequency of the sales made short-term policy responses difficult without advance planning.\(^\text{13}\) Looking to make its debt sales more effective, the U.S. Treasury followed the lead of the U.K. Treasury, which had been auctioning its debt since 1877.\(^\text{14}\) In 1929, the U.S. Treasury moved from a quarterly, posted-price mechanism to a more-frequent pay-as-bid auction format for the sale of its securities (Garbade, 2008).

\(^{13}\)Of course, the notion of advance planning for short-run policy objectives is inherently contradictory. This further illustrates the problem faced by the quarterly sale schedule.

\(^{14}\)Until the early twentieth century sales the U.K.’s debt sales were relatively small, but government funding requirements increased dramatically during World War I and the auction implementation proved invaluable to providing liquidity for the purchase of wartime materiel. This value of this liquidity in funding the war effort provided positive evidence in favor of frequent auctions of treasury securities as the U.S. Treasury sought a new implementation of its sales mechanism.
Except for a brief period of experimentation in the 1970s, the U.S. Treasury procedure was largely unchanged until the early 1990s. Following a series of incidents in which bidders were accused of market manipulation, the U.S. Treasury again began experimenting with a uniform-price format (Jegadeesh, 1993); it was assumed that the (non-existent) truthful-reporting properties would prevent such manipulation by eliminating strategic behavior (Friedman, 1960; Chari and Weber, 1992). In 1992, the U.S. Treasury formally switched to a uniform-price format (Bikhchandani and Huang, 1993).

Although the U.S. Treasury has moved away from the pay-as-bid auction, the U.K. has continued its reliance upon the pay-as-bid format for selling conventional gilts, government securities which guarantee a rate of return over a fixed period of time. Of the government security auctions which utilize a pay-as-bid format—which constitute a majority of security auctions (Brenner et al., 2009)—the most important, by volume as well as impact on the financial community, is arguably the conventional gilt auction.

In the conventional gilt auction, a fixed quantity of securities is made available. Small retail investors, known as the “approved group,” may submit non-

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15 The U.K. conventional gilt differs from the U.K. index-linked gilt, which has a rate of return which is indexed to the U.K. Retail Prices Index. Index-linked gilts are sold via uniform-price auction.
competitive demands to be fulfilled in full independent of the resulting market price, at the mean fulfillment price of other bidders. For the remaining quantity, a set of pre-qualified “gilt-edged market makers” submits bids as price, quantity pairs and a market-clearing price is computed according to standard principles of supply and demand. Primary dealers are expected to receive at least 2% of the aggregate conventional gilt issuance on a rolling six-month basis (United Kingdom Debt Management Office, 2015). Ties are broken pro-rata on the margin. In financial 2014, nominal £75.25bn was sold in 26 auctions; 21 gilt-edged market makers participated in these auctions.

1.2.2 U.S. Federal Reserve Quantitative Easing program

During the financial market collapse of 2007-2008, the U.S. Federal Reserve engaged in a program of interest rate reduction with the goal of stimulating economic growth. After a year and a half of rate reductions, the Federal Funds rate was reduced to 0.00-0.25% on December 16, 2008. At this low rate, conventional monetary policy was no longer a useful tool for economic stimulus, and deflationary risk became a concern (Blinder, 2010).

The Quantitative Easing program provided an alternative mechanism for stimulating growth. Having arguably observed some amount of success in Japan’s experiment with the policy (Spiegel, 2006), the Federal Reserve began purchasing public and private assets held by non-governmental entities. By injecting money into the economy and removing risk from balance sheets it was hoped that Quantitative Easing would reduce the barriers to lending which were inhibiting economic growth.

As implemented in the U.S., Quantitative Easing was a multi-unit, multi-object, pay-as-bid procurement auction (Song and Zhu, 2014). The Federal Reserve so-

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16This tiebreaking rule is formalized in Section 1.3.
licited bids—in this case, offers—from a set of pre-qualified primary dealers;\footnote{As of 2015 there are 22 primary dealers in U.S. Treasury auctions; the set of pre-qualified dealers in U.S. Treasury auctions was the same as the set of pre-qualified dealers in the Quantitative Easing program. This number varied through the course of the Quantitative Easing program, but was never lower than 16 or higher than 22.} multiple varieties of securities were procured in each auction, which enabled the Federal Reserve to acquire additional information on security values. Individuals were able participate in the auction by routing their orders through primary dealers. Each offer curve was allowed to consist of up to nine bid points, yielding a step function with at most nine steps. Prices were determined by aggregating the submitted bid functions and merging stated supply curves with the Federal Reserve’s own value estimates, which took into account the information conveyed by the submitted bids; that is, the Federal Reserve attempted to use bids to determine its own value for the securities up for auction, and then used its inferred value estimates to determine its demand. By this process the Federal Reserve controlled for exogenous variation in security value, and was able to acquire what it believed to be the most appealing securities. Quantitative Easing took place in three distinct phases, from December 2008 through October 2014, and during this time the Federal Reserve acquired roughly $3.7 trillion on assets.

1.2.3 Electricity markets

In the U.S., the process of generating electricity and distributing it to end consumers began as a competitive, localized industry in the late 1800s. By the 1920s, municipalities recognized the economic benefits of vertical integration in this marketplace, and began treating generators as regulated monopolies (Handmaker, 1989). In particular, the engineering challenges imposed by load-balancing transmission networks implied—and continue to imply—difficult coordination problems, which are more-easily solved when the market is served by a single firm.
This market arrangement continued through the early 1970s, when the oil embargo led to a rapid increase in energy prices (petroleum prices, in particular, rose by an order of magnitude (Smith, 1988)). To address this increase, the U.S. implemented policies favoring energy independence (Hirsh, 1999); in passing the Public Utilities Regulatory Policy Act of 1978 (PURPA), the U.S. became the first nation to begin deregulating the electricity market (Al-Sunaidy and Green, 2006). PURPA required electricity monopolies to allow smaller generators to use their transmission lines, and ultimately this led to technology improvements in both generation and transmission (Hirsh, 1999).

With these technological advances and a slow decentralization of generation capacity, the electricity market quickly became difficult to manage in a centralized “command-and-control” fashion. In the wake of the passage of the (federal) Energy Policy Act of 1992, California was the first state which opted to liberalize its market regulations (White, 1996); in a particularly damning report, the California Public Utilities Commission found, “Increasingly, the customer’s unique circumstances, and competition among suppliers, drive product development. Yet the state relies on an administrative process which projects the date the utility ought to build new generation, as well as the exact type and size of that addition” (California Public Utilities Commission, 1993, page 110, italics in original). Witnessing the success of auction-based market reforms in the U.K. in the early 1990s, California began operating a wholesale electricity market in April 1998, under uniform-price rules (Puller, 2007). Other system operators in the U.S. followed suit (Joskow, 2000). In general, these mechanisms ran—and con-

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18It has been a common misconception that economies of scale in electricity generation lead to natural monopolies in electricity markets. While larger turbines are more efficient to an extent, there is a capacity beyond which the generation process suffers from dis-economies of scale (Huettner and Landon, 1978). The natural monopoly status of electricity production arises instead from ownership of the transmission network, which—except under vertical integration—is not tied to the generation technology.
continue to run—frequent “spot” auctions to clear immediate demand for electricity, alongside “day-ahead” auctions to provide a planning tool for generating firms. Because there are substantial fixed costs to bringing a power plant on- or off-line, day-ahead auctions have proved essential to the proper functioning of the market.

Following a series of well-publicized failures in the electricity market in California, the uniform-price format was called into question. When considering a possible shift to a pay-as-bid format, it was frequently claimed that the pay-as-bid format does not imply unilateral savings,\textsuperscript{19} as supplier strategies are markedly different in the two auction formats. As in other settings, the debate often relied on the faulty assumption that the uniform-price auction induces approximately truthful reporting (Kahn, Cramton, Porter, and Tabors, 2001; Tierney, Schatzki, and Mukerji, 2008). Ultimately, the debate resulted in no change to the status quo.

As of 2015, only a handful of countries rely upon pay-as-bid auctions to distribute electrical generation, including Iran, Mexico, Panama, and Peru (Ghazizadeh, Sheikh-El-Eslami, and Seifi, 2007; Maurer and Barroso, 2011). Developing Latin American economies have brought further innovation to electricity market auctions by moving beyond spot and day-ahead auctions, to begin auctioning long-term contracts for new generation.

\textbf{1.2.4 U.S. EPA Acid Rain Program}

In 1990, the U.S. Congress passed amendments to the Clean Air Act; Title IV of the 1990 Amendments addresses the issue of acid rain. In addition to accelerating structural decay, acid rain can be responsible for soil degradation and stream acidification, the latter of which is associated with an increase in the aluminum content of the water, and is harmful to aquatic life (U.S. Environmental Protection

\textsuperscript{19}As a procurement auction, what is investigated elsewhere in this dissertation as revenue improvement manifests in this case as cost savings.
Agency, 2006). Human-caused sources of acid rain include sulfur dioxide and nitrogen oxides, and approximately two-thirds of domestic sulfur dioxide arises from the process of electricity generation, particularly from power plants which are fueled primarily by coal. To control sulfur dioxide emissions, the Amendments instituted a cap-and-trade system with the goal of reducing power plant emissions at minimum cost. A single emissions allowance grants 1 ton of sulfur dioxide emissions in the year in which is it used (U.S. Environmental Protection Agency, 2015). Allowances were initially distributed to generators on the basis of the amount of electricity they produced; unused allowances may be “banked” for use in future years (Hitaj and Stocking, 2014). Allowances are freely tradeable among any entities—common traders include not only power generators, but also environmental groups seeking to “retire” permits in order to reduce emissions beyond the EPA’s stated objectives.

In addition to free trade among market participants, a small number of permits are annually allocated by pay-as-bid auction: in 2015, 8.6 million current-year allowances were held by generators, and 125 thousand allowances were allocated in a spot-market auction; a further 125 thousand were allocated in a seven-year advance auction. Participation is open to all entities, and frequently generates interest from electricity suppliers, environmental groups, and undergraduate economics courses. Auction competition is commonly thin: in 2015, 15 bids were submitted for 125 thousand units in the spot market. Five bidders submitted an average of three bids apiece, and 99.6% of the permits were allocated to a single participant.

The centralized market operates in two stages, the first of which is a standard pay-as-bid auction. In the first stage, bidders submit an unconstrained number

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20^Over the period of 2006–2015 market-clearing prices fell from $860.07 to $0.11 in the spot auction, and from $241.67 to $0.03 in the advance auction. The continuing trend of falling prices is imperceptible on a standard linear scale, thus prices are plotted instead on a logarithmic scale.
of point bids; allowances are allocated in decreasing order of bid price, the bidder pays her stated per-unit bid for the quantity received, and the quantity is removed from the supply pool. This process continues until no permits remain in the central pool. Once the supply of EPA permits is exhausted, the auction enters a second stage, in which a (possibly distinct) set of bidders may offer permits for sale. The mechanism then transfers permits from the lowest-price supplier to the highest-price buyer at the buyer’s per-unit bid. The second stage continues until the lowest supply price exceeds the highest buy price. In 2015, there were no supply-side participants, and no allowances were allocated in the second stage.

Given the relatively small number of permits issued in the centralized auction—in 2015, roughly 2.9% of the total number of allowances passed through the auction mechanisms—the importance of the auction mechanism has been questioned (Joskow, Schmalensee, and Bailey, 1996). In spite of this, the allowance auction is credited with facilitating price discovery and jump-starting participation in the

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21 There is a degenerate endogenous constraint in this auction: as there are only 125 thousand units, no bidder will submit more than 125 thousand bids. Technically, however, the auction is unconstrained, as the formal rules of the auction allow bidders to submit in excess of 125 thousand bids if they choose to do so.

22 This is a form of a double-auction, which (in this case) induces truthful reporting by agents offering permits for sale.
early days of the allowance market (Schmalensee, Joskow, Ellerman, Montero, and Bailey, 1998). The auction mechanism also provides a predictable source of allowances for new market entrants (Zhou, 2014).

### 1.3 Share auction model

I now present my model of the pay-as-bid auction, the basics of which are consistent throughout all subsequent Chapters. Because the model defined here is overly general, each subsequent Chapter will refine it as necessary.

I analyze pay-as-bid auctions according to the “share auction” model first defined by Wilson (1979). In this model, there are \( n \geq 2 \) agents, \( i \in \{1, \ldots, n\} \). Agent \( i \) has marginal value function \( v^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) which is decreasing and in its first parameter (quantity) and increasing and in its second parameter (signal); thus \( v^i(q; s) \) is agent \( i \)'s marginal value for the \( q^{th} \) unit when her signal is \( s \).

Agent \( i \)'s signal \( s_i \) is distributed according to distribution function \( F \), which admits density function \( f \). The signals of any two agents \( i \neq j \) are statistically independent. In this independent, private-values paradigm there is no common component, reflected in the fact that \( v^i \) depends only on \( s_i \).

Agent \( i \) submits a positive, weakly-decreasing bid function \( b^i \) to the auctioneer. A possibly-random market quantity \( Q \) is realized; if the quantity \( Q \) is random, its distribution is denoted \( F^Q \). The auctioneer then uses the reported bid functions to determine the market-clearing price \( p \). As is standard, the market-

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23 This model does allow for a particular form of correlation: there can be common knowledge of some random economic fundamental, and then independent deviations around this knowledge; see, e.g., Branco (1996).

24 The restriction to weakly-decreasing bid functions is without loss of generality, assuming that the seller accepts bids in decreasing order.

25 Where there is no ambiguity (that is, no other source of randomness) the distribution of quantity will be denoted by undecorated \( F \).
clearing price is the highest price at which there is no excess supply,

\[ p = \sup \left\{ p' : \exists (q_i)_{i=1}^n \geq 0 \text{ s.t. } \forall i, b^i(q_i) \leq p, \sum_{i=1}^n q_i \geq Q \right\}. \]

The agent’s submitted bid function gives rise to two inverse correspondences; the quantity inverse \( \varphi^i \) and the signal inverse \( \psi^i \) are defined by

\[
\varphi^i(p; s) = \left\{ q : \lim_{q' \downarrow q} b^i(q'; s) \leq p \leq \lim_{q' \uparrow q} b^i(q'; s) \right\},
\]

\[
\psi^i(q; p) = \left\{ s : \lim_{s' \downarrow s} b^i(q; s') \leq p \leq \lim_{s' \uparrow s} b^i(q; s') \right\}.
\]

Note that for any \((p, s)\), \( \varphi^i(p; s) \) may be empty or many-valued; for any \((q, p)\), \( \psi^i(q; p) \) may be empty or many-valued. To handle multiplicities, where the correspondences are nonempty I define

\[
\varphi^i(p; s) = \sup \varphi^i(p; s), \quad \varphi^i(p; s) = \inf \varphi^i(p; s);
\]

\[
\psi^i(q; p) = \sup \psi^i(q; p), \quad \psi^i(q; p) = \inf \psi^i(q; p).
\]

I will take care to ensure that these functions are well-defined at the points they are used in model analysis.

If agent \( i \)'s inverse demand is continuous and single-valued at the market-clearing price \( p \), then she is allocated \( q_i = \varphi^i(p) \). Otherwise the auctioneer will need to consider rationing some agent’s demand. The form of the auctioneer’s rationing rule will not be of particular importance to the analyses in this dissertation, nevertheless it is helpful for intuition to fix a particular form.\(^{26}\) The rationing rule I employ is known as pro-rata on the margin, and is commonly employed in practice.

To define the rationing rule, first notice that for rationing to be necessary it must be that \( \sum_{i=1}^n \varphi^i(p) \leq Q \leq \sum_{i=1}^n \varphi^i(p) \), and at least one of these inequalities

\(^{26}\)Quantity rationing is constant-sum, hence any tiebreaking rule that encourages one agent to tie will dissuade one of her opponents. In equilibrium, this implies that the specific rationing rule is irrelevant, up to probability-zero outcomes.
is strict. Pro-rating on the margin removes from each agent’s allocation the proportional extent to which her submitted bid accounts for excess demand at the market-clearing price. The quantity allocated will be

\[ q_i = \varphi_i (p) - \left( \frac{\varphi_i (p) - \varphi_j (p)}{\sum_{j=1}^{n} \varphi_j (p) - \varphi_j (p)} \right) \left( \sum_{j=1}^{n} \varphi_j (p) - Q \right). \]

This has an equivalent representation in which agents are awarded their minimum demand, plus the extent to which they cover excess supply.

Any agent \( i \) has an “equilibrium” quantity distribution conditional on the strategies played by her opponents.\(^{27}\) When agent \( i \) submits bid function \( b \), her equilibrium quantity distribution \( G^i \)—as determined by the allocation rule given above—is

\[ G^i (q; b) = \Pr (q_i \leq q | b). \]

Thus \( G^i (q; b) \) is the probability that agent \( i \)’s marginal quantity allocation is no greater than \( q \). Note that if an agent receives quantity \( q \), she also receives quantity \( q' < q \), so \( G^i (q; b) \) is equivalent to the probability that the agent is ever allocated quantity \( q \), whether as a marginal or inframarginal unit.

Each agent \( i \) is risk-neutral, and her expected utility is ultimately quasilinear in her payment to the seller. When her type is \( s \), her action is \( b \), and her opponents’ strategies are \( b^j, j \neq i \), her expected utility is denoted

\[ U^i (b, b^{-i}; s) = \mathbb{E}_o \left[ \int_0^{q^i} v^i (x; s) - b (x) \, dx \bigg| b, b^{-i} \right]. \quad (1.1) \]

That is, for each infinitesimal unit she obtains she receives the difference between her marginal value and the bid she submits for this quantity; these marginal profits are integrated over all quantities she is receives in a given allocation \( q_i \). In the definition of expected utility, the expectation is taken over whatever sources

\(^{27}\)Formally, a distribution over quantity allocations will arise from any profile of strategies, not just mutual best-respondes. The term “equilibrium” here is meant to serve as a persistent reminder that the distribution arises from patterns of play, and is not exogenous.
of randomness are present in the model being analyzed; this will vary between Chapters.

I make an effort to remain as consistent as possible in the notation applied in this dissertation. As observed in the definitions above, superscripts represent functions, while subscripts (generally) represent scalar values: $b^i$ is the bid function of bidder $i$, and $s_i$ is her signal. When both a superscript and subscript appear, a partial derivative is implied unless a definition suggests otherwise; thus $b^i_q$ is the partial derivative of agent $i$'s bid with respect to quantity $q$. When superscripts or subscripts appear, they will generally imply (putative) actual behavior, while the same variables undecorated denote free parameters. Thus $b^i$ is agent $i$'s putative best-response bid function, or a bid function which is the object of analysis, and $b$ is parameter to an objective optimization which might yield $b^i$. Where notation becomes unwieldy and there is no ambiguity induced by mathematical shorthand, I occasionally drop function arguments.

1.4 Literature

The formal economic study of auctions via game theory began with the seminal work of Vickrey (1961), which examined a setting with a single, indivisible unit and private values. This thread of literature led to the celebrated revenue equivalence results of Riley and Samuelson (1981) and Myerson (1981), as well as extensions to more general single-unit settings. While single-unit auctions are undoubtedly relevant to real-world applications, a central feature of their analysis is fundamentally limiting: agents need solve only a one-dimensional optimization problem. The move beyond the single-unit framework has given rise to two distinct threads of auction-theoretic work: multi-object and multi-unit auctions. In a multi-object auction, agents face a problem of determining bids for multiple

\begin{footnotesize}
\footnote{Although the analysis becomes somewhat complicated, the same tools may be applied to auctions for many goods, provided the goods are sold in a sequence of single-unit auctions.}
\end{footnotesize}

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possible packages of commodities. In a multi-unit auction, agents face a problem of determining bids to obtain shares of a set of homogeneous goods.

Analyses of multi-unit auctions have taken two approaches. It is possible to model multi-unit auctions explicitly as they generally arise in practice, where goods are, at some fundamental level, indivisible. Or, a divisible-good paradigm can be pursued. The two models have been investigated more or less in parallel, with Smith (1966) being the first to discuss a divisible-good model.\textsuperscript{29} This setup was refined to the canonical share auction model specified in Wilson (1979), where bidders compete for fractional shares of a perfectly-divisible commodity. Back and Zender (1993) impose a reserve price on the divisible-good model with constant marginal values, and find that the pay-as-bid format revenue-dominates the uniform-price format when bidders are risk-neutral; Wang and Zender (2002) extend these results to random supply.\textsuperscript{30} With deterministic supply, Ausubel, Cramton, Pycia, Rostek, and Weretka (2014) find that the revenue and efficiency rankings of the two auctions are fundamentally ambiguous, and can only be determined if the parameterization of agents’ values is known in advance.

As with many continuous analyses, the share auction model is meant to be a tractable representation of a complex economic problem. Bikhchandani and Huang (1993) discuss a number of common assumptions in the multi-unit auction literature and find that, on balance, the bias toward tractable analysis works in favor of the pay-as-bid auction; among the key assumptions worth addressing is the perfect divisibility of the underlying commodity. As a salient example, Kastl (2012) finds that bids for indivisible goods may exceed the bidder’s true value, a feature which is not observed in the divisible-good model. In the indivisible-good

\textsuperscript{29}Whether Smith’s model is intended to present divisible or indivisible goods is unclear. Its integral formulation supports either modeling assumption.

\textsuperscript{30}In U.S. Treasury auctions—as well as many others—noninstitutional bidders may participate by submitting “noncompetitive” bids, which are fulfilled in full at a price determined by the market-clearing price of the auction. In cases such as this, it is not unreasonable to assume that such demand is random from the perspective of the large, institutional bidders. See the discussion of U.K. conventional gilt auctions in Section 1.2.1.
context, Maskin and Riley (1989) define a revenue-optimal mechanism, which is neither pay-as-bid nor uniform-price.\footnote{Maskin and Riley’s optimal mechanism is not observed in practice. This is at least partially due to its complex and not-immediately-intuitive representation. As mentioned previously, the ultimate choice of mechanism appears to be between the uniform-price and pay-as-bid auctions.} Lebrun and Tremblay (2003) examine a model in which bidders’ values are step functions where the location of the step is private information, and they compute equilibrium bid functions. Chakraborty (2006) is able to compute closed-form equilibrium strategies when all bidders demand two units, and there are two units for sale. Anwar (2007) demonstrates that with two bidders and affiliated private values, there exists a unique equilibrium in the indivisible-good pay-as-bid auction.

The question of what mechanism is truly optimal has been studied in a number of contexts. In a model closely related to the share auction model, Harris and Raviv (1981b) find that, in the presence bidders’ private information, the asymptotically optimal mechanism—as discrete types become dense—is a posted price and monopolistic supply. When bidders are constrained to unit demand, Matthews (1983) confirms that the optimal selling scheme is equivalent to a deterministic posted-price mechanism, provided buyers are risk-neutral. When attention is constrained to only the pay-as-bid or uniform-price auctions, Milgrom (1989) claims that the uniform-price auction is preferred in common-value settings; this follows from the “linkage principle,” by which agents’ payments are less-tied to their own bids under the uniform-price auction than under the pay-as-bid auction.\footnote{This principle can fail in a dramatic way when values are private. In particular, when goods are perfectly divisible, each bidder sets the market-clearing price. Thus, unlike the single-unit case, as concerns the linkage principle there is no meaningful distinction here between the uniform-price and pay-as-bid auctions.} Branco (1996) assumes that bidders have correlated values and unit demand, and finds revenue equivalence between the uniform-price and pay-as-bid auctions.

For all practical purposes, the seller’s implementation decision is between either the pay-as-bid or uniform-price auctions; these two auction forms constitute the overwhelming share of, for example, treasury security auctions (Brenner et al.,
Given the lack of certainty surrounding which auction format is preferable—as demonstrated by decades of debate, theoretical models pointing in each direction, and finally proved by Ausubel et al. (2014)—the optimal choice of mechanism is fundamentally empirical and context-specific. Unfortunately, it is not regarded as feasible to apply standard reduced-form estimation techniques to these auction formats; natural experiments are not readily available (sellers do not commonly switch auction formats), and, where they are, there are substantial problems regarding the endogeneity of the switch, as well as nonstationarity of the underlying macroeconomic fundamentals. Thus a structural approach is necessary (Binmore and Swierzbinski, 2000).

What structural empirical research exists validates the need for further theoretical guidance. Smith (1967) studies an experimental setting where individuals demand two units and many units are available and finds that the revenue ranking of auction formats depends on the extent of oversubscription. Engelmann and Grimm (2009) extend this experimental work and find that the pay-as-bid format revenue-dominates the uniform-price format. Evidence from real-world auctions is more mixed. In a study of French Treasury auctions, Armantier and Sbaï (2006) find that a counterfactual uniform-price auction outperforms the observed pay-as-bid auction, and improves outcomes for all market participants; Armantier and Sbaï (2009) validate these results and establish that convexifying between the two mechanisms might improve revenue further. Castellanos and Oviedo (2008) study Mexican Treasury auctions, and find that the uniform-price auction revenue-dominates the pay-as-bid auction. Conversely, Kang and Puller (2008) study Korean Treasury auctions and find that the pay-as-bid auction is preferable; the approach taken is arguably more robust, as observed uniform-price and pay-as-

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33Armantier and Sbaï's 2006 work constrains attention to constrained strategic equilibria, as defined in Armantier, Florens, and Richard (2008). In particular, all strategies must be order-two polynomials in quantity and the bidder’s private information. This immediately rules out the strategies defined in Chapter 3, which are shown to be essential to unconstrained equilibrium behavior. Their 2009 work requires only piecewise linearity and therefore does not necessarily fall victim to this problem.
bid auctions are compared to a common baseline counterfactual Vickrey auction. The current state of empirical research in mechanism comparison is perhaps best captured by the study of Turkish Treasury auctions pursued by Hortaçsu and McAdams (2010), which finds no statistically significant difference in outcomes.

Thus there is both theoretical and empirical ambiguity about whether or not the pay-as-bid auction is preferable to the uniform-price auction. The model I examine in this dissertation suggestively supports the use of the pay-as-bid auction when the seller is concerned with maximizing revenue, and although these results are novel the model itself is relatively similar to those examined before. Importantly, it suggests that empirical studies may have overstated counterfactual revenues inferred from observed pay-as-bid auctions.

Subsequent Chapters will detail the literature of particular relevance to their content.

1.5 Chapter summary

Chapters 2 and 3 consider a model of the pay-as-bid auction in which quantity is known beforehand to be equal to some value $Q$; that is, $F^Q(x) = 1[x = Q]$. In this model, Chapter 2 establishes the existence of a monotone pure-strategy equilibrium. I demonstrate that this equilibrium can be very close to nearby discrete-unit auctions when the number of units for auction is large. The existence of a pure-strategy equilibrium allows the pursuit of Chapter 3, which establishes a particular qualitative feature of optimum bidding behavior—bids are occasionally determined by nonlocal incentives, leading to bid reduction beyond what might be expected. In particular, the “generalized first-price auction” intuition which is both reasonable and present in the literature fails in a dramatic way, due to bids being determined noncompetitively.

Chapters 4 and 5 take a different approach, and allow for quantity randomiza-
tion while eliminating the presence of private information. It is further assumed that all bidders are completely symmetric; that is, there is some \( v \) such that, for all \( i \) and all \( s \), \( v^i(\cdot; s) = v(\cdot) \). In this model, Chapter 4 establishes that there can be at most one pure-strategy equilibrium, and gives a tight sufficient condition for its existence. Conditional on existence, the unique equilibrium is computed in an explicit form. The precise knowledge of equilibrium bids permits an investigation of revenue-maximizing design principles in the pay-as-bid auction, and I show that randomized supply is uniformly dominated by deterministic supply. This result gives rise to a surprising revenue equivalence between optimally-parameterized pay-as-bid and uniform-price auctions. Chapter 5 builds on the analytical tools of Chapter 4 to study the effects of mechanism randomization on seller revenue. I find that the pay-as-bid auction is smoothly revenue dominant in a theoretical model, and provide computational evidence that this dominance extends to a broader class of parameterizations.

Chapter 6 concludes, summarizing the totality of evidence the preceding Chapters present in favor of the pay-as-bid auction.
CHAPTER 2

Equilibrium existence

In spite of their ubiquity, relatively little is concretely known about the behavior of bidders in pay-as-bid auctions. Beyond the apparent theoretical difficulty of computing fully general revenue and efficiency rankings, progress in the analysis of parameterized models has been hampered by the inability to efficiently compute equilibrium strategies in the case when goods, as in practice, are imperfectly divisible. Although there are meaningful results in certain settings—see, e.g., Engelbrecht-Wiggans and Kahn (2002), Ausubel et al. (2014), and Lotfi and Sarkar (2015)—the general state of the art is best captured by Hortacsu and Kastl (2012), who state, “Unfortunately, computing equilibrium strategies in (asymmetric) discriminatory multi-unit auctions is still an open question.”

It has been long recognized that, where discrete problems appear intractible, continuous approximations may offer sound and available economic insights (Geoffrion, 1976; Hall, 1986). For example, the literature on single-unit auctions frequently relies upon the assumption that the set of available prices is dense. In the case of multi-unit auctions, there is another opportunity for continuous modeling: bids may be submitted at a dense set of quantities. Wilson (1979) was the first to apply this approximation method in the context of multi-unit auctions; since then, results have been found for parameterized models, but in the general case it has not even been known if an equilibrium exists. Without a sound basis

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1 There has been work in building approximate equilibria for multi-unit auctions; see, e.g., Arman-tier et al. (2008).
2 The assumption that the type space is dense is also made, but is not generally disprovable.

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for the existence of equilibrium strategies, it has been problematic to meaningfully apply the divisible-good model to policy debates. In this Chapter, I establish the existence of a pure-strategy equilibrium, that this equilibrium has mathematically nice monotonicity properties, and that it may approximate equilibrium in large multi-unit auctions.

Although existence in this particular mechanism has remained an open problem, the question has been addressed in many related models. A thread of literature beginning with Athey (2001) examines equilibrium existence in models with private information and continuous payoffs. McAdams (2003) extends this result to include multidimensional private information, and Reny (2011) generalizes to the case of arbitrary lattices. These results cannot be directly applied because, as is common in auction models, payoff discontinuities cannot be ruled out by assumption; as Chapter 3 demonstrates, particular payoff discontinuities will almost surely arise. Reny (1999) allows for discontinuous utility functions, but does not permit private information; his results have been extended by McLennan, Monteiro, and Tourky (2011) and Barelli and Meneghel (2013). The results of this Chapter lie at the intersection of these properties.

In this Chapter I extend these existence results and prove that there is, in fact, a pure-strategy equilibrium in the pay-as-bid auction for a perfectly divisible commodity. Aside from certain technical assumptions, all that is required here is that supply not be too large.\(^3\) My proof approach builds directly on the existence results of McAdams (2003), and presents two novel features: first, there is a natural profile of “limiting strategies.” I then show how these limiting strategies can be used to define equilibrium strategies for all bidders.

I establish that in any equilibrium, bidders’ actions are monotonic in their

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\(^3\) If aggregate marginal value for the entire market quantity is nonpositive, bidders are competing for fully-rationed quantities, and nonnegativity constraints on bids imply that there is no best response. Jackson, Simon, Swinkels, and Zame (2002) explore a related issue in an example of a single-unit auction.
private information; additionally, the equilibrium I prove to exist must be close, in observed outcomes, to equilibrium in similarly-parameterized large multi-unit auctions for discrete goods.\footnote{I do not prove the converse of this statement. That is, there may be equilibria of the divisible-good pay-as-bid auction which do not approximate any equilibrium of the discrete-good pay-as-bid auction. There may additionally be equilibria of the discrete-good auction which do not converge to an equilibrium of the divisible-good auction.} This is of particular importance for the empirical value of the pay-as-bid share auction model, and is in stark contrast to the uniform-price auction; it has been observed that the uniform-price share auction model can admit a large set of “collusive-seeming” equilibria which cannot arise in a discretized setting (Fabra, Von der Fehr, and Harbord, 2002; Kremer and Nyborg, 2004).\footnote{This does not invalidate the share auction approach. Empirical work has been careful to address this feature of the uniform-price auction by inferring upper bounds for revenue comparison. Chapter 5 of this dissertation utilizes the same strategy.}

In a direction opposite the results of this Chapter, the macroeconomic literature has long recognized issues regarding applying discrete estimation techniques to continuous problems (Sims, 1971; Phillips, 1974; Geweke, 1978). The key distinction here is that such models assume that the underlying economic fundamentals operate continuously, while in the pay-as-bid auction the true model is fundamentally discrete. More-closely related are the results of Boutilier, Goldszmidt, and Sabata (1999), which posit an equilibrium for a discrete combinatorial auction. This equilibrium is intractable and is approximated by a continuous equilibrium, which is then discretized for computational approaches. Bontemps, Robin, and Van den Berg (2000) develop a continuous search model which renders tractable discrete search problems.

One might hope that, knowing that an equilibrium exists and that it is in a neighborhood of equilibria of other mechanisms, a full characterization is close at hand. This is not the case. Chapter 3 explores issues in analytic characterization of equilibrium strategies, and provides an example in which behavior is tractable.
2.1 Model

There are \( n \geq 2 \) agents, \( i \in \{1, \ldots, n\} \), competing for units of a perfectly-divisible commodity. The available market supply is deterministic and equal to a constant \( Q \). Agent \( i \) has private information \( s_i \sim F \), and for all agents \( j \neq i \), \( s_i \) and \( s_j \) are independent; \( F \) has convex support \([0, 1]\) and admits density \( f \). Conditional on her signal \( s_i \), agent \( i \) has marginal value \( v^i(q; s_i) \) for quantity \( q \). This marginal value is strictly decreasing in \( q \) and strictly increasing and continuous in \( s_i \).

Finally, I assume that the quantity available for auction is not too large. In particular, if all bidders \( i \) receive their lowest-possible signals \( s_i = 0 \), there is an allocation of the good such that each bidder receives positive marginal utility at their allocation. That is,

\[
\exists (q_i)_{i=1}^n \text{ s.t. } \sum_{i=1}^n q_i = Q, \quad \forall i \; q_i \geq 0, \quad \text{and } \forall i \; v^i(q_i; 0) > 0.
\]

The remainder of the model is as given in Section 1.3. I will address the role of each of the additional assumptions made here when it is applicable to the results at hand.

2.2 Results

Note that, since quantity allocations are weakly positive, there are necessarily some quantities which an agent can never win, regardless of the bid function she submits. As a particularly rough bound, the agent can never receive more than \( q_i = Q \) units, hence there are no optimality conditions to determine bids for quantities \( q_i > Q \). In general, similar logic will hold for some \( \overline{q} < Q \).\(^6\) I therefore distinguish between quantities which an agent can win with some probability, which I term relevant quantities, and quantities which can never be obtained, which are never obtained the agent’s bid function may be bounded by the no-deviation constraints of her opponents.

\(^6\)As will be briefly explored in Chapter 4, at some quantities which are never obtained the agent’s bid function may be bounded by the no-deviation constraints of her opponents.
which I term irrelevant quantities.

**Definition 2.1.** Quantity \( q \) is a relevant quantity if \( q \in \text{Supp}(G^i(\cdot; b)) \).

Recalling that \( G^i \) is the equilibrium distribution of agent \( i \)'s allocated quantity, there is an important qualification to Definition 2.1: whether or not a quantity is relevant is not exogenous, but depends on both the bid \( b^i \) of agent \( i \) and on the bids \( \langle b^j \rangle_{j \neq i} \) of her opponents. Although this is a static model, there is some a sense in which some irrelevant quantities may be considered off-path; while pointwise optimality conditions cannot determine behavior for these units, no-deviation constraints for agent \( i \)'s opponents may restrict the values that her bids may take. This will be explored more fully in Chapter 4.

I now present my first result.

**Lemma 2.1 (Best-response monotonicity).** Let \( \langle b^j \rangle_{j \neq i} \) be the profile of strategies played by agents other than \( i \), and suppose that \( b^i \) is a best-response. Then for almost all relevant quantities, \( b^i \) is weakly monotonic in agent \( i \)'s private information \( s_i \).

Note that Lemma 2.1 holds only for almost all relevant quantities. If left- or right-continuity was imposed on the submitted bid function this qualification could be widened to include all relevant quantities;\(^7\) it serves only to account for behavior at quantity-discontinuities in the bid function, which may be resolved in either direction without affecting the agent’s interim utility.

For a proof Lemma 2.1, see Section A.1. Intuitively, the result is not substantially different from the monotonicity results found throughout the auction literature: a bidder with a higher signal has a higher value, hence more to gain by winning a unit. It follows that if a lower-signal agent is indifferent between raising

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\(^7\)Due to the tiebreaking rule, enforcing left- or right-continuity is without loss of generality. Nevertheless, in this Chapter I remain agnostic on this front.
her bid and not, the higher-signal agent would prefer, on the margin, to obtain a higher winning probability in exchange for a slightly lower per-unit surplus.

That this is true of all best responses depends crucially on the independence of agents’ types, and on the fact that values are strictly increasing in signal. If the latter were not the case, agents might be indifferent across an interval of bids for a particular quantity, and there might be some inversion of the type order with regard to the bids submitted. If signals are not independent, an agent can take, for example, a high signal as a sign that she should collude with her opponent, again leading to possible inversion of bids with respect to information; see, e.g., McAdams (2007c), or McAdams (2007b). As the following existence result builds on a sequence of monotone equilibria, the results employed would imply the existence of a monotone equilibrium, but would not independently guarantee that any equilibrium must be monotone.

Lemma 2.1 allows me to focus on monotone strategies. This is particularly useful since Theorem 2.1 obtains “limiting” strategies at only a countable subset of possible signals; the remainder are found constructively, subject to the monotonicity result of Lemma 2.1. Assurance of monotonicity enables the proof to focus attention on a relatively constrained set of possible actions for an agent receiving any particular signal.

I now state the main result of this Chapter.

**Theorem 2.1** (Equilibrium existence). *There is a Bayesian-Nash equilibrium in pure strategies.*

Theorem 2.1 relies on a set of Lemmas which are, themselves, relatively standard. Therefore I present the novel portions of the proof here, and relegate the remainder to Appendix A.

To pursue the proof, I need to define an auxiliary model $\mathcal{M}^\epsilon$. $\mathcal{M}^\epsilon$ is derived from the base model. Bids in $\mathcal{M}^\epsilon$ are constrained to be constant over intervals
of the form \((t\varepsilon, t\varepsilon + \varepsilon]\), where \(t \in \mathbb{N}\), and for each \(q\) there exists \(k \in \mathbb{N}\) such that 
\[ b^i(q) = k\varepsilon. \]
Additionally, quantity allocations must be of the form \(q_i = q\varepsilon\) for some \(q \in \mathbb{N}\); rationing will occur pro-rata on the margin, up to \(\varepsilon\)-indivisibility. Issues of indivisibility in allocations are addressed by equiprobable lotteries for the problematic units. It is useful to view this as a discrete-unit auction on a finite price grid, with \(Q/\varepsilon\) discrete units available for auction, hence I will refer to \(\mathcal{M}^\varepsilon\) as the \(\varepsilon\)-discrete model. Because the proof strategy used below works by constructing actions rather than strategies, I will make regular reference to “agent-types,” a tuple of a particular agent \(i\) and the signal \(s_i\) that she receives.

**Proof.** This proof is divided into individual steps. First, I show that for a particular subset of agent-types, the concept of “limiting actions” with respect to a refining sequence of \(\mathcal{M}^\varepsilon\) has a well-defined basis. Second, I show that for these agent-types, the limiting actions can be easily extended to a well-defined action in the divisible-good model; these actions can be extended to putative equilibrium actions for all agent-types. Third, I select an agent \(i\) and show that her outcomes are almost-unaffected by a transformation which renders her opponents’ strategies well-behaved. Fourth, considering the same transformation for agent \(i\), I show that her utility is well-behaved with respect to particular deviations, and that this transformation does not adversely affect her utility. Fifth, I show that agent \(i\) is best-responding at the transformed version of her limiting strategy, and that this holds for all possible signals she might receive. Thus the transformation is (at worst) utility-neutral, and agents who play the transformed strategies are mutually best-responding.

**Limiting strategies.** Lemma A.3 establishes that for any \(\varepsilon > 0\), \(\mathcal{M}^\varepsilon\) admits a monotone pure-strategy equilibrium \(\langle b^i,\varepsilon \rangle\). Without loss of generality, suppose that \(b^i,\varepsilon(q; \cdot)\) is left-continuous: 
\[ b^i,\varepsilon(q; s) = \sup_{s' < s} b^i,\varepsilon(q; s') \] whenever \(s > 0\).\(^8\) Be-

\(^8\)That this is without loss of generality arises from continuity of \(v^i(q; \cdot)\), and the fact that \(s'\) near \(s\) must be best-responding with \(b^i,\varepsilon_1(\cdot; s')\).
cause \( b^i(\cdot; \cdot) \) is monotonic and bounded,\(^9\) this function is well-defined. That these functions constitute an equilibrium follows from Lemma A.5 and the fact that any equilibrium can be modified on a set of measure zero to meet this criterion. Let \( \varepsilon_i = Q / 2^t \), so that \( \langle \varepsilon_i \rangle_{i=1}^{\infty} \to 0.\(^{10}\)

Let \( Q = [0, Q] \cap \mathbb{Q} \) and \( S = [0, 1] \cap \mathbb{Q} \) be signal and quantity sets constrained to the rationals.\(^{11}\) By standard selection results (Widder, 1941) there is a subsequence of \( \langle \langle b^i, \varepsilon_i \rangle_n \rangle_{i=1}^{\infty} \) which, for each agent, converges pointwise on \( Q \times S \). Denote the limit of such a subsequence by \( \langle \beta^i, \square \rangle_n \).

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Extension of strategies. Let \( W_1^- \) be the set of monotone decreasing functions with domain \([0, Q]\) and range \([0, \max_j v^j(0; 1)]\).\(^{12}\) Given agent \( i \) and signal \( s \in S \), let \( \hat{B}^i(s) \) be the set of functions in \( W_1^- \) which equal \( \beta^i, \square(\cdot; s) \) on \( Q \),

\[
\hat{B}^i(s) = \left\{ \hat{\beta}(\cdot; s) \in W_1^- : \forall q \in Q, \hat{\beta}(q; s) = \beta^i, \square(q; s) \right\}.
\]

Let \( \overline{\beta}^i(q; s) = \sup_{\beta \in B^i(s)} \hat{\beta}(q; s) \) and \( \underline{\beta}^i(q; s) = \inf_{\beta \in B^i(s)} \hat{\beta}(q; s) \); it may be seen that \( \overline{\beta}^i(\cdot; s) \) is left-continuous and \( \underline{\beta}^i(\cdot; s) \) is right-continuous. For \( q \in Q \), \( \overline{\beta}^i(q; s) = \underline{\beta}^i(q; s) \), and \( \{ \overline{\beta}^i(\cdot; s), \underline{\beta}^i(\cdot; s) \} \subseteq \hat{B}^i(s) \).

Since \( \beta^i, \square(\cdot; s) \) is monotone decreasing on a compact range, any \( \beta \in \hat{B}^i(s) \) has at most countably-many discontinuities. Moreover, these discontinuities occur at exactly the \( q \) at which \( \overline{\beta}^i(q; s) > \underline{\beta}^i(q; s) \). This set is of measure zero, hence the area under each curve is identical; additionally, this area is the same as the area under any other bid function \( \beta \in \hat{B}^i(s) \).

Since \( \overline{\beta}^i(q; s) = \underline{\beta}^i(q; s) \) for almost all \( q \), their implied quantity distributions

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\(^9\)For boundedness, see Lemma A.4.

\(^{10}\)The particular construction of \( \varepsilon_i \) is not important, so long as the sequence goes to zero. The assumption that \( \varepsilon_i \) factors \( Q \) is made for analytical convenience with regard to the results of Section A.2.

\(^{11}\)For the purposes of this proof, it is sufficient to use any countable dense subset of \([0, Q] \times [0, 1]\). It is convenient that the set have a product form. Because all bidders are assumed to behave rationally in an economic sense, the use of “rational” here will only refer to a number with a fractional representation.

\(^{12}\)Lemma A.4 establishes that this range is sufficient for limiting actions. Although this point may appear trivial, Kastl (2012) provides an example equilibrium in which bids exceed values.
\( G^i \) and \( G^i \) are everywhere-equal, hence the quantity allocations are equal with probability one. Since payments are identical under either bid function, the utility generated by either bid function is the same.

Now consider any other bid function \( \beta \in \hat{B}^i(s) \). Fixing a quantity allocation, the payment under \( \beta(\cdot; s) \) is identical to that under \( \bar{\beta}(\cdot; s) \) and \( \bar{\beta}(\cdot; s) \). Moreover, the equilibrium quantity allocation under \( \beta(\cdot; s) \) lies weakly between \( \bar{\beta}(\cdot; s) \) and \( \bar{\beta}(\cdot; s) \). As these two have been shown to be equal, the bidder is indifferent across all bid functions in \( \hat{B}^i(s) \).

Let \( W_{1-2+} \) be the set of functions on \([0,Q] \times [0,1]\) which are monotone decreasing in their first argument and monotone increasing in their second. Let \( B^i \) be the set of signal-monotonicity constrained bid functions for agent \( i \) which equal her limiting strategy on \( Q \times S \),

\[
B^i = \left\{ \hat{\beta} \in W_{1-2+} : \forall (q,s) \in Q \times S, \quad \hat{\beta}(q,s) = \beta^{i,\square}(q,s) \right\}.
\]

Note that for any agent \( i \), any \( s \in S \), and any \( \hat{\beta} \in B^i, \hat{\beta}(\cdot; s) \in \hat{B}^i(s) \). By definition, for any agent \( i \) and any \((q,s) \in Q \times S\) there is a unique value which may be taken by any \( \hat{\beta} \in B^i \) at \((q,s)\). Then given any action \( \hat{\beta}(\cdot; s_i) \in \hat{B}^i(s_i) \) for agent \( i \) and any profile of strategies \( \hat{\beta}^{-1} \in \times_{j \neq i} B^j \), \( \hat{q}(s_i, s_{-i}) \) is uniquely determined for all \( s_{-i} \in S^{n-1} \). Since a monotone bounded function on a compact multidimensional domain may have at most a measure-zero set of discontinuities (Lavrič, 1993), arguments similar to those posed above imply that the selection of \( \hat{\beta}^{-1} \in \times_{j \neq i} B^j \) is irrelevant to agent \( i \)'s utility.

Together, these results show that it is without loss to constrain attention to the strategy profile \( \langle \hat{\beta}^j \rangle_{j=1}^n \), where

\[
\hat{\beta}^j(q,s) = \begin{cases} 
\beta^j(q,s) & \text{if } s \in S, \\
\sup_{s' \in [0,s) \cap S} \beta^j(q,s') & \text{otherwise.}
\end{cases}
\]

From \( \langle \hat{\beta}^j \rangle_{j \neq i} \), the above arguments may be iterated any finite number of times without affecting agent \( i \)'s incentives. Therefore let \( \langle \beta^j \rangle_{j=1}^n \) be strategies defined
by

$$
\beta^i(q; s) = \begin{cases} 
\tilde{\beta}^i(q; s) & \text{if } s = 0, \\
\sup_{s' < s} \tilde{\beta}^i(q; s') & \text{otherwise.}
\end{cases}
$$

Intuitively, $\beta^i(q; \cdot)$ is a left-continuous version of $\tilde{\beta}^i(q; \cdot)$.

As discussed above, agent $i$’s interim utility is identical across all $\tilde{\beta}^j \in (\times_{j \neq i} B^j) \cup \{ (\beta^j)_{j \neq i} \}$. In what follows, I will fix an agent $i$ and assume that all agents $j \neq i$ play strategy $\beta^j$, and I will refer to this as agent $j$’s transformed limiting strategy. I will consider both agent $i$’s limiting strategy $\beta^i$ and her transformed limiting strategy $\beta^i$.

Lastly, this same flavor of argument implies that for all $j$, $(b^{\xi^j})_{t=1}^{\infty} \rightarrow \beta^j$ in the $L^1$ norm. This does not imply that for all $s$, $(b^{\xi^j}(\cdot; s))_{t=1}^{\infty} \rightarrow \beta^j(\cdot; s)$; this distinction will be crucial in the analysis of agent $i$’s incentives, and is the chief reason that the ensuing analysis is careful to distinguish between $\beta^i$ and $\beta^i$.

Interim utility is continuous with respect to upward deviations from transformed actions. I now show that interim utility is continuous with respect to the $L^1$ norm, considering upward deviations of agent $i$’s own action and general deviations of her opponents’ strategies from the population profile of transformed strategies.\footnote{I prefer the term “sup-sup strategy,” but it is silly.} This argument does not presume that agents play their transformed limiting strategies as best responses. That these are indeed best responses will be derived from this continuity result.

Before demonstrating this point, note that agent $i$’s ex post payment for a given quantity is independent of her opponents’ strategies; further, her payment for a given quantity is continuous in her own action. Thus if her interim utility is discontinuous it must be that her interim distribution of quantity is discontinuous in her own action and her opponents’ strategies. Rigorously, I first establish...
necessary properties for discontinuity of the expectation of quantity with respect to agent \(i\)'s own action. I then pursue a set of conditions necessary for \(i\)'s interim utility to be discontinuous in her own action. The same logic employed in this argument will hold in the analysis of discontinuities with respect to the strategies of her opponents, which I do not explicitly pursue here. As mentioned above, I restrict attention to the case in which all agents \(j \neq i\) play their transformed limiting strategies \(\beta_j\).

To examine discontinuities, it will be sufficient to analyze the case of \(\beta^\delta(\cdot; s) = \beta^i(\cdot; s) + \delta\), where \(\delta \downarrow 0\). That deviations of the form \(\beta^\delta(\cdot; s)\) are sufficient can be observed by contradiction: suppose that there is a sequence \(\hat{\beta}^t(\cdot; s)\) such that
\[
\lim_{t \to \infty} \mathbb{E}_{s \to_i} \left[ q_i \mid \hat{\beta}^t(\cdot; s) \right] > \mathbb{E}_{s \to_i} \left[ q_i \mid \beta^i(\cdot; s) \right].
\]
Since \(\hat{\beta}^t(\cdot; s)\) and \(\beta^i\) are monotonic bounded functions on a compact domain, that \(\hat{\beta}^t(\cdot; s)\) converges to \(\beta^i\) in the \(L^1\) norm implies that it converges uniformly almost everywhere. Then for any \(\delta > 0\), there is a \(t(\delta)\) such that for all \(t \geq t(\delta)\) and almost all \(q_i\), \(|\hat{\beta}^t(q; s) - \beta^i(q; s)| < \delta\); without loss of generality, I will assume that \(t(\cdot)\) is monotonically increasing in \(\delta\). By definition, \(\beta^\delta(\cdot; s) > \hat{\beta}^t(\cdot; s)\) almost everywhere. It follows that the quantity allocated under \(\beta^\delta(\cdot; s)\) is weakly greater than that allocated under \(\hat{\beta}^t(\cdot; s)\). Then
\[
\lim_{\delta \downarrow 0} \mathbb{E}_{s \to_i} \left[ q_i \mid \beta^\delta(\cdot; s) \right] \geq \lim_{\delta \downarrow 0} \mathbb{E}_{s \to_i} \left[ q_i \mid \hat{\beta}^{t(\delta)}(\cdot; s) \right] = \lim_{t \to \infty} \mathbb{E}_{s \to_i} \left[ q_i \mid \hat{\beta}^t(\cdot; s) \right] > \mathbb{E}_{s \to_i} \left[ q_i \mid \beta^i(\cdot; s) \right].
\]
Thus \(\beta^\delta(\cdot; s)\) generates at least as great an expected quantity while retaining the feature that additional costs are going to zero, and the deviation \(\beta^\delta(\cdot; s)\) is sufficient to analyze discontinuities in the quantity distribution.\(^{16}\)

\(^{16}\)Although this argument is posed with regard to the entire domain \([0, Q]\), it may be applied with equal validity to any nondegenerate subinterval.

The first condition for discontinuity is that \(\beta^i(\cdot; s)\) cannot be strictly decreasing
in quantity.\textsuperscript{17} Consider the transformed limiting allocation \( q^i(s, s_{-i}) \) obtained when agent \( i \)'s opponents receive signal profile \( s_{-i} \); by deviating to \( \hat{\beta}^\delta \), the quantity she obtains is bounded by

\[ q^i(s, s_{-i}) \leq q^\delta(s, s_{-i}) \leq \varphi^i(\beta^i(q^i(s, s_{-i}); s) - \delta; s). \tag{2.1} \]

The upper bound arises from the fact that the agent’s bid under \( \hat{\beta}^\delta \) for unit \( \varphi^i(\beta^i(q^i(s, s_{-i}); s); s) \) is now \( \beta^i(q^i(s, s_{-i}); s) \); as all bid functions are monotonically decreasing, she can never receive more than this quantity under deviation \( \hat{\beta}^\delta \) when her opponents' signals are \( s_{-i} \). Since \( \beta^i(\cdot; s) \) is strictly decreasing, \( \varphi^i(\cdot; s) \) is continuous in price. Then as \( \delta \downarrow 0 \), \( q^\delta(s, s_{-i}) \downarrow q^i(s, s_{-i}) \) for all \( s_{-i} \), and agent \( i \)'s ex post allocation varies continuously in her deviation. It follows that for agent \( i \)'s utility to be discontinuous in her action at her transformed limiting action, there must be a nondegenerate interval \( I^i_q = (q_i, \overline{q}_i) \) such that \( \beta^i(\cdot; s)|_{I^i_q} \) is constant. Let this interval be defined maximally on \( Q \): there does not exist \( q' \notin \overline{C}I^i_q \) such that for \( q \in I^i_q \), \( \beta^i(q'; s) = \beta^i(q; s) \).\textsuperscript{18}

The second condition for discontinuity is that there must be some \( j \) with \( s_j \) such that \( \beta^j(\cdot; s_j) \) is flat at the same price level as \( \beta^i(\cdot; s)|_{I^i_q} \), and that there must be \( s_{-ij} \) such that both agents’ allocations are simultaneously within these flat intervals. Suppose that \( s_{-i} \) is such that \( q^i(s, s_{-i}) \in \text{Int}I^i_q \); that is, the signals of agent \( i \)'s opponents are such that the quantity she obtains is strictly within the interval \( I^i_q \). Suppose further that for all opponents \( j \), \( \beta^j(\cdot; s_j) \) is strictly decreasing in a neighborhood of \( q^j(s_j, (s, s_{-j})) \). By deviating to \( \hat{\beta}^\delta \), the quantity agent \( i \) obtains is bounded by

\[ q^i(s, s_{-i}) \leq q^\delta(s, s_{-i}) \leq Q - \sum_{j \neq i} \varphi^j(\beta^i(q^i(s, s_{-i}); s) + \delta; s_j). \]

\textsuperscript{17}Recall that this statement is made only with respect to the set of relevant quantities.

\textsuperscript{18}Later, it will be necessary to consider subintervals of the maximal \( I^i_q \). In particular, little can be directly said about convergence to \( \beta^i(\cdot; s) \) on maximal \( I^i_q \) because the analysis constrains attention to transformed limiting strategies. The maximal \( I^i_q \) provides a useful benchmark for defining necessary subintervals.
The upper bound arises from reasoning similar to that given when justifying inequality (2.1). Since $\beta^j(\cdot; s_j)$ is strictly decreasing in a neighborhood of $q^j(s_j, (s, s_{-j}))$, $\varphi^j(\cdot; s_j)$ is continuous at $\beta^i(s, s_{-i})$. Then as $\delta \downarrow 0$, $\varphi^j(\beta^i(q^i(s, s_{-i}); s) + \delta; s_j) \uparrow q^j(s_j, (s, s_{-ij}))$, and agent $i$’s ex post allocation varies continuously in her deviation. It follows that for agent $i$’s utility to be discontinuous in her action at her transformed limiting action, there must be intervals $I^j_q = [q^j, q^j]$ for all $j \neq i$ such that $\beta^j(\cdot; s_j)|_{I^j_q} = \beta^i(q^i(s, s_{-i}); s)$ is constant, and

$$\lim_{q \downarrow q^j} \beta^j(q; s_j) \leq \beta^i(q^i(s, s_{-i}); s) \leq \lim_{q \uparrow q^j} \beta^j(q; s_j).$$

For at least one $j \neq i$, the interval $I^j_q$ is nondegenerate, and when summed over all agents it must be that

$$\sum_{j=1}^n q^j_j \leq Q \leq \sum_{j=1}^n \overline{q}_j.\text{19}$$

The third condition for discontinuity is that there must be some agent $j \neq i$ such that $I^j_q$ defined above is nondegenerate, and there is nondegenerate, closed $J^j_q \subseteq I^j_q$ such that $\beta^j(q; \cdot)$ is constant for all $q \in J^j_q$. In particular, there is a nondegenerate interval $I^i_q = [s_j, \overline{s}_j]$ such that a small deviation by agent $i$ will yield a discrete quantity improvement when, among other things, agent $j$’s signal is in $I^j_q$. This essentially repeats the analysis of the second condition; in particular, if all such $I^j_q$ are degenerate then deviation $\hat{\beta}^\delta$ will, in the limit, affect at most a probability-zero set of outcomes. Let $S^i(\delta, \xi)$ be the set of opponents’ signals for which deviating to $\hat{\beta}^\delta$ results in an additional quantity of at least $\xi > 0$. Formally,

$$S^i(\delta; \xi) = \{s_{-i} : q^i(s, s_{-i}) \in I^i_q, \text{ and } q^\delta(s, s_{-i}) > q^i(s, s_{-i}) + \xi\}.$$

It must be that, for some nondegenerate $I^j_q$ as above, and some $\xi > 0$, $\lim_{\delta \downarrow 0} \Pr(s_{-i} \in S^i(\delta; \xi)) > 0$. I show that this implies a nondegenerate $I^j_q$ for some $j$ with nondegenerate $I^j_q$; the desired result follows.\text{20} As in the analysis of the second condition,

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\text{19}Nondegeneracy implies that at least one of these inequalities is strict.

\text{20}A nondegenerate set of signals can yield improvements of at least some $\xi > 0$ only if some of the sets overlap.
if there are only countably-many $s'_j$ with nondegenerate intervals $I^j_q$ such that $\beta^j(\cdot; s'_j)|_{I^j_q} = \beta^j(q'(s, s_{-i}); s)$, then the event that agent $i$ obtains any additional $\xi > 0$ has zero probability in the limiting deviation. It follows that there must be a nondegenerate $I^j_s$ such that $\beta^j(\cdot; s'_j)$ is constant on appropriately-defined intervals for all $s'_j \in I^j_s$.

Finally, it must be that $\Pr(q_i \in I^j_q \mid \sup I^j_q | \beta_i) > 0$, otherwise no deviation will result in a positive-probability change to agent $i$’s allocation on $I^j_q$.

I now show that for each agent $j$ with nondegenerate intervals $J^j_q$ and $I^j_s$, convergence to $\beta^j$ is uniform on $J^j_q \times I^j_s$. Let signal $s_j \in \text{Int}\, I^j_s$. Because bids are monotonic in quantity and $\beta^j(\cdot; s_j)$ is constant on $J^j_q$, for any $\varepsilon_t$ the value $|b^{j,\varepsilon_t}(q; s_j) - \beta^j(q; s_j)|$ will be maximized—not necessarily uniquely—at either $q_j = \min J^j_q$ or $\overline{q}_j = \max J^j_q$.

Since $(b^{j,\varepsilon_t}(q'; s_j))_{t=1}^\infty \to \beta^j(q'; s)$ for $q' \in \{q_j, \overline{q}_j\}$, it follows that convergence on $J^j_q$ is uniform for signal $s_j$.

Now let $J^j_q \subset I^j_q$ be a nondegenerate, closed proper subset of $I^j_q$. The same argument as above establishes that $|b^{j,\varepsilon_t}(q; s) - \beta^j(q; s)|$, considered on $J^j_q \times I^j_s$, will be maximized at either $(q_j; \overline{s}_j)$ or $(\overline{q}_j; \underline{s}_j)$, where $\underline{s}_j = \min J^j_s$ and $\overline{s}_j = \max J^j_s$. Since $(b^{j,\varepsilon_t}(q'; s'))_{t=1}^\infty \to \beta^j(q'; s')$ at either of these points, convergence is uniform on $J^j_q \times J^j_s$.

It is immediate to see then that, given some $\delta > 0$, there is $t(\delta) \in \mathbb{N}$ such that for all $t > t(\delta)$,

$$|b^{j,\varepsilon_t}(q; s) - \beta^j(q; s)| < \delta, \quad \forall (q, s) \in J^j_q \times J^j_s.$$

A handful of cases arise; in each, either agent $i$ or $j$ will have a profitable deviation near the limit. Because the analysis is common across all cases—find a low-signal agent who can profit near the transformed limiting strategies—I expose only one such case.

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21Issues of convergence of $(b^{j,\varepsilon_t})_{t=1}^\infty$ to $\beta^j$—rather than $\beta^j, \square$—are avoided due to the fact that $s_j$ lies strictly within $I^j_q$.

22See footnote 21 above.
Suppose that for any $\delta > 0$, there is $s' \in [0, s) \cap S$ such that

$$\beta^i (\overline{q}_i; s) - 3\delta \leq \beta^i (\overline{q}_i; s') \leq \beta^i (\overline{q}_i; s) - 2\delta.$$  

Due to monotonicity of bids in signal and quantity, the uniform convergence argument may be extended to include the bids that $s'$ submits for quantities $\underline{q}_i$ and $\overline{q}_i$. For $t$ sufficiently large, $b^{i,\varepsilon_t}$ will be such that

$$\beta^i (\overline{q}_i; s) - 4\delta \leq b^{i,\varepsilon_t} (\overline{q}_i; s') \leq \beta^i (\overline{q}_i; s) - \delta.$$  

Additionally, all $s_j \in J^j_i$ will be such that

$$\beta^i (\overline{q}_i; s) - \delta \leq b^{j,\varepsilon_t} (\overline{q}_j; s_j) \leq \beta^i (\overline{q}_i; s) + \delta.$$  

It follows that by a deviation costing no more than $5Q\delta$, type $s'$ can achieve the same discontinuous utility gain as type $s$; hence for $\delta$ sufficiently small type $s'$ is not playing an equilibrium action in $M^\varepsilon$ for $\varepsilon$ small.\(^{23}\)

The other cases assume that agent $i$ has a signal-mass in her bid on a subinterval of $I^i_q$, and show that agent $j$ will have a profitable deviation—if $\langle b^{j,\varepsilon_t} \rangle_{t=1}^\infty$ converges “from below”—or that agent $i$ will have a profitable deviation—if $\langle b^{j,\varepsilon_t} \rangle_{t=1}^\infty$ converges “from above;” as mentioned above, the same basic argument applies. Since $\beta^i$ is defined as a supremum, these analyses cover all possible cases. In each case, if agent $i$’s interim utility is discontinuous in her own action at $\beta^i(\cdot; s)$, a related type of the same agent has a profitable deviation near the limit. It can be shown that if her interim utility is discontinuous in her opponents’ strategies at the transformed limiting strategies, the same analysis applies.

To complete this step of the proof, I show that $\beta^i$ generates at least as much utility as $\beta^{i,\square}$. Suppose that there is $\delta > 0$ with $U^i(\beta^{i,\square}(\cdot; s), \beta^{-i}; s) > U^i(\beta^i(\cdot; s), \beta^{-i}; s) + 4\delta$. Then for agents $s' < s$ sufficiently close to $s$,

$$U^i (\beta^{i,\square}(\cdot; s), \beta^{-i}; s') > U^i (\beta^i(\cdot; s'), \beta^{-i}; s') + 3\delta.$$  

\(^{23}\)This last point relies on the fact that $\langle b^{j,\varepsilon_t} \rangle_{t=1}^\infty \to \beta^j$ in the $L^1$ norm.
This follows from the supremum definition of $\beta^i$. This gives, additionally,

$$U^i(\beta^i(\cdot; s'), \beta^{-i}; s') > U^i(\beta^{i, □}(\cdot; s'), \beta^{-i}; s') - \delta.$$ 

Putting all the parts together,

$$U^i(\beta^{i, □}(\cdot; s), \beta^{-i}; s') > U^i(\beta^{i, □}(\cdot; s'), \beta^{-i}; s') + 2\delta.$$ 

$U^i$ cannot have an upward jump at the limit except under conditions which may be ruled out in the same manner as the above establishing of continuity. It follows that for $t$ sufficiently large,

$$U^i(b^{i, ε_t}(\cdot; s), b^{-i, ε_t}; s') > U^i(b^{i, ε_t}(\cdot; s'), b^{-i, ε_t}; s') + \delta.$$ 

As before, this contradicts the fact that $b^{i, ε_t}(\cdot; s')$ is a best-response action in $M^{ε_t}$. It follows that $U^i(\beta^i(\cdot; s), \beta^{-i}; s) \geq U^i(\beta^{i, □}(\cdot; s), \beta^{-i}; s)$.

Since $\beta^i$ generates at least as much utility as $\beta^{i, □}$, it is safe to replace any agent $j$’s limiting strategy with her transformed strategy.

**Optimality of transformed limiting strategies.** Consider the limiting strategy of agent $i$ when she receives signal $s \in S$; it has already been seen that $\beta^{i, ε_t} \rightarrow \beta^{i, □}$ in the $L^1$ norm. Suppose that when other bidders play $\beta^{-i} = (\beta^j)_{j \neq i}$, agent $i$ has an action $\tilde{β}$ such that, for some $\delta > 0$,

$$U^i(\tilde{β}, \beta^{-i}; s) > U^i(\beta^i(\cdot; s), \beta^{-i}; s) + 3\delta > U^i(\beta^{i, □}(\cdot; s), \beta^{-i}; s) + 3\delta.$$ 

Because agent $i$’s interim payoffs are locally continuous in her action and her opponents’ strategies, and the latter are converging in the $L^1$ norm to $\beta^{-i}$, there is $t_1 \in \mathbb{N}$ such that for all $t > t_1$,

$$|U^i(\beta^{i, ε_t}, \beta^{-i}; s) - U^i(\beta^i, \beta^{-i}; s)| < \delta.$$ 

Moreover, it may be assumed that$^{25}$ that there is $t_2 \in \mathbb{N}$ such that, for all $t > t_2$,

$$|U^i(\tilde{β}, \beta^{-i, ε_t}; s) - U^i(\beta^i, \beta^{-i}; s)| < \delta.$$ 

---

$^{24}$Recall that agent $i$’s utility cannot discontinuously jump upward at the limit $\beta^{i, □}(\cdot; s)$.

$^{25}$This is nontrivial but can be verified.
Then for all \( t > \max_i \ell_i \),

\[
U^i \left( \tilde{\beta}, \beta^{-i}; s \right) > U^i \left( \beta^i, \beta^{-i}; s \right) + \delta.
\]

By Lemma A.6, it must be that \( \delta = O(\varepsilon_t) \), a contradiction. Thus the limiting strategy \( \beta^i \) is a best response.

**Best responses for irrational-signal agents.** Let \( s \in [0,1] \setminus \mathcal{S} \). Let \( \tilde{B}^i(s) \) be the set of actions available to agent \( i \) when her type is \( s \),

\[
\tilde{B}^i(s) = \left\{ \tilde{\beta} : \exists \beta' \in B^i \text{ s.t. } \beta'(\cdot; s) = \tilde{\beta} \right\}.
\]

Suppose that there is some \( \tilde{\beta} \in \tilde{B}^i(s) \) such that there is \( \delta > 0 \) with \( U^i(\tilde{\beta}, \beta^{-i}; s) > U^i(\beta^i(\cdot; s), \beta^{-i}; s) + 2\delta \). Since rational-signal agents are best-responding and best responses are monotonic in the agent’s signal,\(^{26}\) it must then be that, for all \( s' \in \mathcal{S} \),

\[
U^i \left( \tilde{\beta}, \beta^{-i}; s \right) > U^i \left( \beta^i(\cdot; s'), \beta^{-i}; s \right) + 2\delta.
\]

With \( v^i(q; \cdot) \) continuous, there must be \( \overline{s} \in \mathcal{S} \) such that for all \( s'' \in (s, \overline{s}) \cap \mathcal{S} \) and all \( \beta' \),

\[
\left| U^i \left( \tilde{\beta}', \beta^{-i}; s'' \right) - U^i \left( \tilde{\beta}', \beta^{-i}; s \right) \right| < \delta.
\]

Putting the two inequalities together, it follows that

\[
U^i \left( \tilde{\beta}, \beta^{-i}; s'' \right) > U^i \left( \beta^i(\cdot; s''), \beta^{-i}; s'' \right) + \delta.
\]

This contradicts the fact that agent \( i \) is best-responding with \( \beta^i(\cdot; s'') \) when she receives signal \( s'' \). Then the payoffs of type \( s \) generated by bids within \( \tilde{B}^i(s) \) are weakly dominated by the payoffs generated by \( \beta^i(\cdot; s) \); since \( \tilde{B}^i(s) \) represents all actions which satisfy the quantity monotonicity constraint shown necessary by Lemma 2.1, \( \beta^i(\cdot; s) \) is a best response for agent \( i \) when she receives signal \( s \notin \mathcal{S} \).

\(^{26}\)Monotonicity of best responses with respect to signal ensures that any \( \beta' \notin \tilde{B}^i(s) \) cannot be a best-response.
Summary. I have shown that equilibrium strategy profiles in the $\varepsilon$-discrete model $\mathcal{M}_\varepsilon$ have a pointwise convergent subsequence for all rational-valued quantities $q$ and all rational-valued signals $s$; for agents receiving rational-valued signals, all extensions to the reals of their limiting schedule on the rationals generate the same payoff. Transforming the left-continuous extension so that it is left-continuous in signal, the interim payoffs of agents receiving rational signals are continuous in their own actions and in the strategies of other agents; hence the suggested transformation is valid. Because of strategic monotonicity, the interim payoffs of agents receiving rational-valued signals are independent of the particular strategies implemented by irrational-signal agents. As bid functions must be converging in the $L^1$ norm for all rational-signal agents, this implies that the extension of the agent $i$’s limiting strategy is in fact a best response at the limit. Agents receiving irrational-valued signals are shown to have best-responses at the limit, and these best responses must satisfy strategic monotonicity. This completes a construction of an equilibrium in the divisible-good case with private information.

Continuity and strict monotonicity of marginal value with respect to signal are crucial to the proof of Theorem 2.1. These properties together ensure that when I build actions for agents when they receive signals which are not within the “limiting” set, these actions are best responses.

The existence result of McAdams (2003) is central to the proof of Theorem 2.1. Largely this is because the antecedents it imposes for equilibrium existence are relatively straightforward to check. Employing the more high-powered existence result of Reny (2011) would give the same result in the Theorem, and would also imply the following remark.

**Remark 2.1** (Symmetric equilibrium). *When all bidders have symmetric marginal values, $v^i = v$, there is a symmetric pure-strategy Bayesian-Nash equilibrium of*
the divisible-good pay-as-bid auction with private information.\textsuperscript{27}

Remark 2.1 establishes that the symmetric models can imply symmetric behavior—although they may also admit asymmetric equilibria. Because Reny (2011) generalizes the results of McAdams (2003), its application in the course of the proof of Theorem 2.1 is equally valid. Reny’s Theorem 4.5—establishing the existence of a symmetric pure-strategy equilibrium—is then sufficient to establish Remark 2.1.

The method of proof applied to demonstrate the existence of a pure-strategy equilibrium suggests the following useful result.

**Theorem 2.2** (Equilibrium approximation). Let $\langle M^ε_r \rangle_{r=1}^\infty$ be a refining sequence of $ε$-discrete models such that $\langle b^i,ε_r \rangle_{r=1}^n$ is a pure-strategy equilibrium of $M^ε_r$. Suppose that each $\langle b^i,ε_r \rangle_{r=1}^n$ converges pointwise on some dense set $C \subseteq [0,Q] \times [0,1]$. Then there is an equilibrium of the divisible-good pay-as-bid auction such that:

(i) The quantities allocated to agent $i$ in the discretized models, $q^i,ε_r$, converge in probability to the quantities allocated to agent $i$ in the divisible-good model, $q^i$; and,

(ii) The market-clearing price implied by $\langle b^i,ε_r \rangle_{r=1}^n$ is converging in probability to the market-clearing price of the divisible-good limit.

**Proof.** Consider the equilibrium $\langle β^i \rangle_{i=1}^n$ of the divisible-good model which is suggested by the proof of Theorem 2.1.

I first establish that for any $i$, $q^i,ε_r \to q^i$ almost everywhere. Suppose to the contrary that there is a positive-measure set $S^i$ such that, for $s \in S$, $q^i,ε_r(s) \not\to q^i(s)$. By market clearing, it is without loss to assume that for all $s \in S^i$, $\lim_{r \to \infty} q^i,ε_r(s) < q^i(s)$. From pointwise convergence of $\langle b^i,ε_r \rangle_{r=1}^\infty$ to $β^i$ on $C$, this implies a discontinuous upward jump in utility at the limit; for reasons similar to those given in the proof of Theorem 2.1 agent $i$ will then have a utility-improving

\textsuperscript{27}I would like to thank Jeffrey Mensch for this Remark.
deviation for $r$ sufficiently large, violating the assumption that $b^{i,\epsilon r}$ is a best response in $\mathcal{M}^{i,\epsilon r}$. Since the distribution of private information is massless, this establishes point (i).

I now demonstrate that if $\beta^i$ is continuous at $(q; s_i)$, then $\langle b^{i,\epsilon r}(q; s_i) \rangle_{r=1}^{\infty} \to \beta^i(q; s_i)$. Because $\beta^i$ is monotonic in both dimensions, if $\beta^i$ is continuous at $(q; s_i)$ it is continuous at all $(q'; s'_i)$ in a neighborhood of $(q; s_i)$. Let $\xi > 0$ be small; by monotonicity and density of $C$, there are $(q_L, s_L), (q_R, s_R) \in C$ with

$$q - \xi < q_L < q < q_R < q + \xi,$$

$$s_i + \xi > s_L > s_i > s_R > s_i - \xi.$$

The distance between any $b^{i,\epsilon r}$ and $\beta^i$ may be written as

$$\left| b^{i,\epsilon r}(q; s_i) - \beta^i(q; s_i) \right| \leq \left| b^{i,\epsilon r}(q_L; s_L) - \beta^i(q_R; s_R) \right|$$

$$\leq \left| b^{i,\epsilon r}(q_L; s_L) - \beta^i(q_L; s_L) \right| + \left| \beta^i(q_L; s_L) - \beta^i(q_R; s_R) \right| .$$

Since $\langle b^{i,\epsilon r}(q_L; s_L) \rangle_{r=1}^{\infty} \to \beta^i(q_L, s_L)$ and $\beta^i$ is continuous, letting $r \uparrow \infty$ and $\xi \downarrow 0$ gives $\langle b^{i,\epsilon r}(q; s_i) \rangle_{r=1}^{\infty} \to \beta^i(q; s_i)$.

Now, consider the difference $||p^{\epsilon r} - p||$; appealing to the definition of market clearing and the triangle inequality,

$$||p^{\epsilon r} - p|| = \int_{[0,1]^n} \left| b^{i,\epsilon r}(q_i^{\epsilon r}(s); s_i) - \beta^i(q^i(s); s_i) \right| ds$$

$$\leq \int_{[0,1]^n} \left| b^{i,\epsilon r}(q_L^{i,\epsilon r}(s); s_i) - \beta^i(q_L^{i,\epsilon r}(s); s_i) \right| ds$$

$$+ \int_{[0,1]^n} \left| \beta^i(q_L^{i,\epsilon r}(s); s_i) - \beta^i(q^i(s); s_i) \right| ds .$$

Point (i) establishes that the right-hand integral converges to zero. The left-hand integrand is zero whenever $\beta^i$ is continuous at $(q^i(s); s_i)$, which is true almost everywhere Lavrič (1993). Hence both integrals are zero, and $p^{\epsilon r}(s) \to p(s)$ for almost all $s$. Point (ii) follows immediately.

The proof of Theorem 2.1 establishes the existence of such sequences as required by the antecedent of Theorem 2.2. The chief implication of Theorem 2.2
is that, given a sequence of convergent equilibria of the discretized pay-as-bid auction, the divisible-good pay-as-bid auction admits a pure-strategy equilibrium such that observed outcomes of the discretized auction converge in probability to outcomes in the divisible-good auction.

**Remark 2.2.** There is a pure-strategy equilibrium of the divisible-good pay-as-bid auction which outcome-approximates a pure-strategy equilibrium of the discrete-good pay-as-bid auction with sufficiently-fine quantities and prices.

Density of the convergence set and the assumption the a particular series of observed auction equilibria converges to the divisible-good auction equilibrium are relatively strong requirements, but these antecedents cannot be weakened. Nonetheless, these particular issues have been discussed at length when applied to other asymptotic settings. The content of Theorem 2.2 is intuitive and informative: the divisible-good pay-as-bid auction admits an equilibrium which approximates outcomes in the multi-unit auction.

There is a second caveat that this approximation only holds with respect to a particular equilibrium of the divisible-good auction. While there are no known issues of equilibrium multiplicity in the pay-as-bid setting (Pycia and Woodward, 2015), uniqueness is not guaranteed. Moreover, a nearby discrete-good auction might not be present in the convergent sequence used to construct equilibrium in the divisible-good auction, hence the nature of convergence will play an important role in determining whether observed discrete-auction behavior is approximated by the divisible-good model.

### 2.3 Conclusion

In this Chapter I have shown that the divisible-good pay-as-bid auction with private information admits a pure-strategy Bayesian-Nash equilibrium, and that when bidders are symmetric there must exist a symmetric pure-strategy Bayesian-
Nash equilibrium. Helpfully, equilibrium strategies are monotone in each agent’s private information; this holds with respect to best responses more generally. The proof strategy I use to establish existence suggests a natural approximation result which I am able to verify: equilibrium outcomes in the divisible-good auction may be near outcomes in discrete pay-as-bid auctions with large numbers of units.

These results provide novel evidence in favor of the pay-as-bid auction. In particular, the format has commonly been believed to be intractable with respect to equilibrium analysis; although the results here do not establish that equilibrium may be easily computed—Chapter 3 discusses this problem in greater depth—that there is an equilibrium at all is suggestive that the model might be tractably analyzed. It is worth noting that although the divisible-good uniform-price auction may not suffer from the problem of equilibrium nonexistence with \( n \geq 3 \) bidders, it is not yet known how to generally compute equilibrium strategies in the format. In light of this fact and the results of this Chapter, tractability arguments in favor of the uniform-price auction hold equally with respect to the pay-as-bid auction.

Monotonicity ensures that empirical investigations have access to ready intuition about higher bids indicating higher valuations.

While the existence and monotonicity results most directly address concern with regard to the utility of the divisible-good model, the approximation result in Theorem 2.2 is arguably the most useful to researchers arguing from counterfactuals generated by this model. It is often taken as given that continuous models offer approximations of the discretized situations they attempt to simplify. By establishing probabilistic convergence, I have shown that this assumption is well-founded in the case of the pay-as-bid auction. This provides suggestive evidence that theoretical outcome analysis from the divisible-good model may in many cases successfully predict real-world discrete outcomes to within a reasonable tolerance.

\[^{28}\text{It is worth noting that although the divisible-good uniform-price auction may not suffer from the problem of equilibrium nonexistence with } n \geq 3 \text{ bidders, it is not yet known how to generally compute equilibrium strategies in the format. In light of this fact and the results of this Chapter, tractability arguments in favor of the uniform-price auction hold equally with respect to the pay-as-bid auction.}\]
CHAPTER 3

Strategic ironing

The existence result of Chapter 2 empowers the discussion of qualitative features of equilibrium in the pay-as-bid auction. Previously, there have been no known equilibrium constructions in the divisible-good pay-as-bid model; I discuss a bid-flattening feature common to all equilibria, and illustrate this feature by contributing an explicit equilibrium in a two-bidder model.

I demonstrate that the pay-as-bid format induces potentially dramatic bid reduction through a process which I term “strategic ironing.” This process distinguishes the pay-as-bid auction from the intuitive generalization of a first-price auction to many units. While it has been understood that the uniform-price auction is not simply a generalized second-price auction, the manner in which the pay-as-bid auction differs from a first-price auction has not previously been clarified.

Explicit computation of equilibrium strategies in the pay-as-bid auction is complicated by two factors. First, there is an inherent asymmetry when the bidders’ marginal values are strictly decreasing in quantity: one bidder bidding for a high-value quantity will be competing against other bidders for relatively low-value quantities. As presented in Maskin and Riley (2000), such asymmetric auctions can be solved as a system of differential equations, but generally lack a closed-form analytical solution. Second, I show here that there are quantities for which the bidder’s optimization problem is non-local; that is, the bid for a particular quantity is determined not by incentives for this quantity, but by
incentives over an interval of quantities. This leads to strategic ironing: bidders subject their idealized bid functions to a monotonicity constraint, similar to the principal’s process described in Myerson (1981). Crucially, this is where the first-price auction intuition breaks down: in single-object auctions there can be no non-local incentives, hence there is no analogy for this effect.

Somewhat ironically, for such non-local incentives the best analogy may be the uniform-price auction. I show that over these intervals, a small increase in bid must increase the payment made for all units on the interval, just as in the uniform-price auction; elsewhere on the bid curve, a small increase in bid will increase only the payment for that unit.¹

Bid flattening has been observed in certain pay-as-bid auction models. Kastl (2012) constructs a divisible-good model in which bidders are constrained to submit step functions with a bounded number of steps. This is meant to capture behavior in certain real-world auctions, where bidders are often constrained to submit no more than a certain (small) number of bid points; see, for example, the discussion of Quantitative Easing in Section 1.2.2. By contrast, I show here that strategic ironing implies that bids will be flat, to a certain extent, regardless of the constraints imposed by the particular mechanism. Engelbrecht-Wiggans and Kahn (1998) examine a model of a two-unit auction in which bidders demand up to two units, and find that bids for the two units may be flat with positive probability. Engelbrecht-Wiggans and Kahn (2002) extend these results to find bids which are flat—and indeed, ironed—for all bidders, with probability one. Their particular model is degenerate in the divisible-good case, and relies crucially on demand only barely exceeding supply. I show that this flattening behavior persists

¹It is important to maintain a crucial distinction here: incentives within an ironed interval are distinct from those implied by the uniform-price format. In particular, raising slightly a bid within such an interval will affect the payment for larger quantities. This forward effect is not present in uniform-price auctions, where bids for lower quantities are independent of payments for higher quantities. Nonetheless, incentives on ironed intervals look like a combination of incentives from the non-ironed pay-as-bid and uniform-price formats.
with private information, and indeed is more general than almost-entire fulfillment might suggest. Without private information, Gresik (2001) obtains bids which are almost flat, in that bids are only submitted at the clearing price, plus or minus one bid increment.\footnote{In a sense, these results extend models in which bidders have unit demand and may submit only a single bid; for a seminal example, see Harris and Raviv (1981a).}

I continue now in Section 3.1 by explicitly defining the share auction model I apply. Section 3.2 introduces the concept of strategic ironing and demonstrates its necessity. Section 3.3 computes an equilibrium and calculates the magnitude of the effect of ironing on revenue. Section 3.4 concludes.

### 3.1 Model

There are \( n \geq 2 \) agents, \( i \in \{1, \ldots, n\} \), competing for units of a perfectly-divisible commodity. The available market supply is deterministic and equal to a constant \( Q \). Agent \( i \) has private information \( s_i \sim F \), and for all agents \( j \neq i \), \( s_i \) and \( s_j \) are independent; \( F \) has convex support \([0, 1]\) and admits density \( f \).

Conditional on her signal \( s_i \), agent \( i \) has marginal value \( v^i(q; s_i) \) for quantity \( q \). This marginal value is continuous and strictly increasing and continuous in \( s_i \), and strictly decreasing and Lipschitz bi-continuous in \( q \): there is \( M > 0 \) such that \( v^i \) is Lipschitz continuous in \( q \) with modulus \( M_v \), and \( [v^i]^{-1} \) is Lipschitz continuous in \( v \) with the same modulus.\footnote{I do not require that this modulus is tight, only that it exists. This condition is also known as bi-Lipschitz continuity.}

#### 3.1.1 Optimality conditions\footnote{The derivation of the agents’ optimality conditions can be found in Hortaçsu (2002) and Février, Préget, and Visser (2002). It is included here because the process will be augmented in Section 3.3.}
To characterize bidding incentives, it is useful to state the agents’ first-order conditions. Agent $i$’s expected utility is

$$U^i (b^i, b^{-i}; s_i) = \mathbb{E}_{q^i} \left[ \int_0^{q^i} v^i (x; s_i) - b^i (x; s_i) \, dx \bigg| b^i \right]. \tag{3.1}$$

Written in full form in terms of the distribution of quantity $G^i$, the expectation is

$$U^i (b^i, b^{-i}; s_i) = \int_0^Q \int_0^{q^i} v^i (x; s_i) - b^i (x; s_i) \, dx \, dG^i (q_i; b^i (:; s_i)).$$

This expression can be integrated by parts; when $q_i = 0$, $\int_0^{q^i} v^i (x; s_i) - b^i (x; s_i) \, dx = 0$, and when $q_i = Q$, $1 - G^i (q_i; b^i (:; s_i)) = 0$. It follows that

$$U^i (b^i, b^{-i}; s_i) = \int_0^Q \left( v^i (q_i; s_i) - b^i (q_i; s_i) \right) (1 - G^i (q; b^i (:; s_i))) \, dq. \tag{3.2}$$

Because agents optimize their interim expected utility, is it sufficient to drop the $s_i$ arguments from the bid function; the agent solves

$$\max_b \int_0^Q \left( v^i (q; s_i) - b (q) \right) (1 - G^i (q; b)) \, dq$$

$$\text{s.t. } q' > q \implies b (q') \leq b (q).$$

For now it will suffice to ignore the monotonicity constraint. Standard tools from the calculus of variations (see, e.g., Intriligator (1971)) then imply a degenerate form for the Euler equation associated with this maximization problem: pointwise almost-everywhere, $\frac{d}{db}[] = 0$. In particular,

$$- \left( 1 - G^i (q; b) \right) - \left( v^i (q; s_i) - b (q) \right) G_b^i (q; b) = 0. \tag{3.3}$$

At the optimum, this may be helpfully arranged into a canonical form,

$$b^i (q; s_i) = v^i (q; s_i) + \frac{1 - G^i (q; b^i (:; s_i))}{G_b^i (q; b^i (:; s_i))}. \tag{3.4}$$

Although the meaning of $G_b^i$ may appear ambiguous, the value itself is well-defined. In particular, when $\varphi^j$ is well-behaved for all $j \neq i$,

$$G^i (q; b) = \Pr (q_i \leq q | b) = \Pr \left( q_i + \sum_{j \neq i} \varphi^j (b (q); s_j) \geq Q \right).$$
Ultimately this expression is an integral over at least \( n - 2 \) signals.\(^5\) Nonetheless, although a closed-form expression in terms of model primitives is impractical, differentiating this expression with respect to \( b \), evaluated at \( q \), is a well-defined action.\(^6\)

### 3.2 Strategic ironing

Strategic ironing is a process by which bidders flatten the initial intervals of their bid functions. To observe the role that ironing plays in best-response strategies, it is helpful to engage in a thought experiment; I will describe intuitive “equilibrium” strategies, then discuss the manner in which these strategies fail to be best responses. Although this argument presents what is, by the end of the thought experiment, an obvious straw man, that the literature has frequently regarded the pay-as-bid auction as a generalization of a first-price auction implies that this straw man is well-motivated.

Suppose that there are \( n = 2 \) bidders, each with linear marginal values. Suppose furthermore that the bidders are playing have a symmetric, pure-strategy equilibrium \( \langle b^1, b^2 \rangle \), where \( b^1 = b^2 = b \); Lemma 2.1 establishes that these strategies must be monotone in signal. Subject to strategic monotonicity, bidders receiving higher signals receive higher quantities, hence conditional on the signal \( s_2 \) received by agent 2, agent 1’s quantity is maximized when she receives signal \( s_1 = 1 \). Contrariwise, the quantity that agent 1 receives is decreasing in the signal \( s_2 \) that agent 2 receives, so conditional on her own signal the quantity she receives is minimized when \( s_2 = 1 \).

Now consider the quantity that agent 1 receives when her signal is \( s_1 = 1 \),

\(^5\)As shown in Appendix B, the \( n = 2 \) bidder case must be separated from the \( n \geq 3 \) bidder case for precisely this reason.

\(^6\)Writing an explicit form for \( G^i \) is more difficult when some \( \phi^j \) are not well-behaved, as occurs when bid functions are flat over particular intervals. When the analysis considers such intervals it will not rely on the first-order conditions, so ultimately this concern is a nonissue.
q^1(1, s_2). Because this quantity is minimized when s_2 = 1, in equilibrium she will receive at least q^1(1, 1). Because the proposed strategies are symmetric and the tiebreaking rule is pro rata on the margin, it must be that q^1(1, 1) = Q/2: when the agent receives her highest-possible signal, she obtains at least the per capita average supply.

If the bid function has a well-behaved form, it will be strictly decreasing to the right of her minimum quantity, Q/2. Consider the incentives that this gives with regard to bidding for quantities q' < Q/2: if agent 1 wins q', then agent 2 must win Q − q' > Q/2. Since—by assumption, in this thought experiment—agent 2’s highest submitted bid, b^2(·; 1), is strictly decreasing to the right of Q/2, it must be that b^2(Q − q'; 1) < b^2(Q/2; 1); and since bids are monotonic in signal, b^2(Q − q'; s_2) < b^2(Q/2; 1) for all s_2. This implies that if agent 1 bids anywhere between b^2(Q − q'; 1) and b^2(Q/2; 1), she will win quantity q' for sure. It follows that bidding b^1(q'; 1) = b^1(Q/2; 1) is strictly dominated by submitting any bid b^1(q'; 1) ∈ (b^2(Q/2; 1), b^2(Q − q'; 1)).

A more rigorous argument here would be careful about the fact that the payment for the unit q' will never matter, as payments are integrals over bids. The same thread of logic applied in this thought experiment holds with equal validity when considering instead lowering the bid for all units [0, q'), but the intuition is less clean.
The bidder then faces a distinct incentive to reduce bids for quantities $q' < Q/2$ so that they are below the bid for quantity $Q/2$. This complicates the requirement that bids must be weakly decreasing in quantity: the idealized best-response bid function is strictly increasing for low quantities, then strictly decreasing for high quantities. The bidder might attempt to meet the monotonicity constraint by what I term “nonstrategic flattening,” setting $b^i(q'; 1) = b^i(Q/2; 1)$, as illustrated in Figure 3.1. However, this does not properly respect incentives: the bidder is now overbidding for all quantities $q' < Q/2$, and correctly bidding for quantities $q' \geq Q/2$, thus there will be strong downward pressures on the initial bids which are not counterbalanced by upward pressures anywhere else. The correct response is what I term “strategic ironing:” the bidder increases bids for low quantities while decreasing bids for middle quantities, rendering the bid function flat for initial quantities extending past $Q/2$. The use of the term “ironing” here is intended to be reminiscent of Myerson (1981); in both cases a function is being flattened in the face of a monotonicity constraint. In the classical case of Myerson, the principal is flattening an incentive curve, while here the agent is flattening her own bid function. The balance of incentives is illustrated in Figure 3.2. It is important to note that while the concept described by Myerson admits a clean geometrical interpretation when information is uniformly distributed, there is no such analogue here. The additional costs—the left wedge—are incurred with probability one, while the sacrificed gains—the right wedge—are given up with small, nonconstant probabilities.

As a final note on this thought experiment, it is important to see that this logic is largely independent of whether or not the agent receives her highest signal, $s_1 = 1$. The reliance on the high-signal agent is necessary only to establish that the minimum quantity $q^i(1, 1) = Q/2 > 0$. The same logic will generally apply when agent 1 receives any signal $s_1$ such that the minimum quantity $q^i(s_1, 1)$ is
Figure 3.2: Nonstrategic flattening implies strong downward pressures on the agent’s bid function: the wedge between the idealized bid function and the non-strategically flattened bid function is not counterbalanced by any wedge above the constrained bid function. Under strategic ironing, downward incentives for low quantities are balanced by upward incentives for larger (but still low) quantities.\footnote{As will be shown in Section 3.2.1, ironing is only strictly necessary for the highest-signal bidder. Nonetheless, the situations in which it does not arise for lower-signal bidders (with $\bar{q}(s) > 0$) are in some sense pathological.}

### 3.2.1 Formalism and necessity

To formalize the concept of strategic ironing, it is useful to introduce three definitions.

**Definition 3.1.** The minimum-possible quantity that agent $i$ can receive contingent on her signal, $\bar{q}^i(s_i)$, is $\bar{q}^i(s_i) = q^i(s_i, 1-i)$.

Note that the minimum-possible quantity is a function of the bid function submitted by agent $i$ and the strategies played by her opponents. Because in any equilibrium strategies must be weakly monotonic in signal, the quantity the agent receives is weakly decreasing in the signals of her opponents, hence it is lowest when her opponents all receive their highest signals, $s_j = 1$.

**Definition 3.2.** The end of the initial flat of agent $i$’s bid function, $\tilde{q}^i(s_i)$, is...
\( \bar{q}^i(s_i) = \sup \{ q : \; b^i(q; s_i) = b^i(0; s_i) \} \).

In the previous subsection I argued that the agent’s bid function should be flat over a nontrivial interval of low quantities (including zero). The end of the initial flat therefore captures the length of this flat portion of the bid function.$^9$

**Definition 3.3.** A bid function exhibits **strategic ironing** if \( \bar{q}^i(s_i) < \bar{q}(s_i) \).

The definition of strategic ironing captures the fact that the bid function may be flat beyond the minimum-possible quantity, as argued above. This succinctly describes the deviation from standard first-price logic: the bid at, e.g., \( \bar{q}^i(s_i) \) is not determined by the optimality conditions at this quantity, but rather by the quantity-monotonicity constraint.$^{10}$ I obtain the following result.

**Theorem 3.1 (Strategic ironing).** If \( \bar{q}^i(1) \in (0, Q) \), \( b^i(\cdot ; 1) \) exhibits strategic ironing.

**Corollary 3.1 (Strategic ironing with symmetry).** In a symmetric equilibrium, all highest-signal bidders’ bid functions exhibit strategic ironing.

Provided the agent, when she receives her highest-possible signal, is guaranteed at least a positive quantity and cannot corner the entire market for sure, her bid function must exhibit strategic ironing. In the symmetric case, cornering cannot occur with certainty, and each highest-signal agent is guaranteed at least a strictly-positive share of the aggregate allocation—in particular, \( \bar{q}^i(1) = Q/n \). Because best-response strategies are monotonic in signal, this happens whenever the agent’s signal is significantly high.$^{11}$

---

$^9$In terms of the definitions given in Section 1.3, it must be the case that \( \bar{q}^i(s_i) = \varphi^i(b^i(0; s_i); s_i) \).

$^{10}$It is possible that optimizing according to the agent’s unconstrained first-order conditions yields a bid function which is nonmonotonic even for quantities \( q > \bar{q}^i(s) \). In this case the bid function will need to be flattened according to a process similar to strategic ironing. Nonetheless, when I discuss strategic ironing here, I refer only to bid flattening for intervals of quantities containing zero.

$^{11}$If one agent \( i \) has \( q^i(s) \in (0, Q) \), there is some \( j \neq i \) such that \( \bar{q}^j(1) \in (0, Q) \). Thus if one agent strategically iron, at least two agents strategically iron.
Although the intuition given above is relatively clean, formal proof of ironing is intricate. In particular, that ironing applies to all equilibria (subject to the no-cornering constraint) requires checking a number of technical corner cases. For this reason, the proof is given in Appendix B.

Theorem 3.1 establishes the necessity of ironing only for agents receiving their highest-possible signals, \( s_i = 1 \). The intuition behind strategic ironing, as discussed previously, plainly applies to agents receiving any signal. However, the particular intricacies of the proof of necessity permit for technical corner cases when \( s_i < 1 \). Necessity may be easily generalized to any \( s_i \) with \( q^i(s_i) > 0 \) if inverse bid functions \( \varphi^i(\cdot; s_i) \) are continuous at prices \( p < b^i(\tilde{q}^i(s_i); s_i) \), or if there are only two bidders.\(^{12}\) In particular, the proof of Theorem 3.1 given in Appendix B builds from an inherent symmetry in beliefs: when strategies are monotonic in information, agent \( i \) receives \( q^i(1) \) when her signal is \( s_i = 1 \) and all signals for agents \( j \neq i \) are \( s_j = 1 \), thus the “view” from the perspective of any agent is in some sense identical.\(^{13}\) When \( s_i < 1 \), agent \( i \) obtains \( q^i(s_i) \) when \( s_j = 1 \) for all \( j \neq i \), but her opponents’ allocations may contain very little information about her signal.

Lastly, it is worth addressing a small concern for equilibrium existence that might be raised by the ironing process. In a symmetric equilibrium with two bidders, the minimum quantity that an agent receiving the highest-possible signal can obtain is \( Q/2 \). When the bid function is ironed, the bid function is flat to \( \tilde{q}^i(1) > Q/2 \); this means that by a small upward deviation, an opponent receiving her highest-possible signal can obtain \( \tilde{q}^j(1) > Q/2 \) when agent \( i \) receives \( s_i = 1 \). Although this deviation looks profitable, this outcome arises with probability zero, hence Bertrand-style arguments do not affect the existence of a pure-strategy

\(^{12}\)When \( q^i(s_i) > 0 \), there is trivially a discontinuity in any realization of \( \varphi^i(\cdot; s_i) \) at \( b^i(q^i(s_i); s_i) \). Thus if it was necessary to require continuity everywhere, the extension would have no value.

\(^{13}\)If strategies are asymmetric, this may not hold absolutely, but the intuition of knowing-one’s-opponents at \( q^i(1) \) remains valid.
equilibrium. There is therefore no contradiction between Theorems 2.1 and 3.1.

3.2.2 The ironing equation

To capture the equilibrium effect of strategic ironing, it is necessary to pin down an (implicit) expression for the optimized bid function. I consider the agent’s optimization as a two-step process: the agent first selects a bid function according to equation (3.4), under the assumption that the result will satisfy the monotonicity constraint. If the bid function does satisfy the monotonicity constraint, no further optimization is necessary. Otherwise, the agent then selects \( \hat{q} \) such that the bid function is monotonic and expected utility is maximized.

Recall from equation (3.2) that the agent’s expected utility is

\[
U^i (b^i, b^{-i}; s_i) = \int_0^Q \left( v^i (q; s_i) - b^i (q; s_i) \right) \left( 1 - G^i (q; b^i) \right) dq.
\]

By strategically ironing to \( \hat{q} \), the agent’s bid function is partitioned into two regions: for \( q \leq \hat{q} \), \( b^i (q; s_i) = b^i (\hat{q}; s_i) \), and for \( q > \hat{q} \) the bid function is unchanged. Expected utility may be split and expressed in terms of these regions,

\[
U^i (b^i, b^{-i}; s_i) = \int_0^{\hat{q}} \left( v^i (q; s_i) - b^i (\hat{q}; s_i) \right) \left( 1 - G^i (q; b^i (\hat{q}; s_i)) \right) dq + \int_{\hat{q}}^Q \left( v^i (q; s_i) - b^i (q; s_i) \right) \left( 1 - G^i (q; b^i (\cdot; s_i)) \right) dq.
\]

Importantly, when \( q > \hat{q} \), the distribution of \( q \) is independent of \( \hat{q} \).

First-order conditions capture the agent’s optimization over \( \hat{q} \). Because the integrands of both integrals are equal when \( q = \hat{q} \), the derivatives with respect to the bounds of integration will cancel. The first-order conditions then reduce to

\[
- \int_0^{\hat{q}} \left( 1 - G^i (q; b^i (\hat{q}; s_i)) \right) + \left( v^i (q; s_i) - b^i (\hat{q}; s_i) \right) G^i_b (q; b^i (\hat{q}; s_i)) \ dq b^i_q (\hat{q}; s_i) = 0.
\]
Because in general it will not be the case that $b'_q(\tilde{q}; s_i) = 0$, this equation becomes
\[
\int_{\tilde{q}}^{\tilde{q}} \left( 1 - G^i \left( q; b^i (\tilde{q}; s_i) \right) \right) + \left( v^i (q; s_i) - b^i (\tilde{q}; s_i) \right) G^i_b \left( q; b^i (\tilde{q}; s_i) \right) dq = 0. \tag{3.5}
\]
I refer to equation (3.5) the “ironing equation.”

Note that the integrand of the ironing equation is identical to the agent’s first-order condition from the calculus of variations for the weakly-decreasing portion of the bid function, as given in equation (3.3). In this sense, the ironing equation is simply an integrated-up version of the agent’s first-order condition for these quantities. This corresponds to the claim of Kastl (2012) that the first-order condition must hold “on average” over flat intervals.\textsuperscript{15,16}

### 3.3 simulations

To illustrate the effect of strategic ironing, I now parameterize the model as having $n = 2$ symmetric bidders, each with linear demand,
\[
v^i (q; s) = \alpha_0 + s \alpha_s - q \alpha_q.
\]
I additionally assume that private information is uniformly distributed, $s_i \sim U(0, 1)$.

I constrain attention to symmetric equilibria with a particular form: agents will bid competitively against all opponents who receive lower signals, and will obtain allocations on their initial flats against all opponents who receive higher

\textsuperscript{14}Appendix B deals with this possibility more explicitly. Intuitively, if $b'_q(\tilde{q}; s_i) = 0$, for any $\varepsilon > 0$ there is $q' \in (\tilde{q}, \tilde{q} + \varepsilon)$ such that $b'_q(q'; s_i) < 0$. If the integral’s value were nonzero, the first-order condition would be strictly positive for $\tilde{q}$ larger than, but sufficiently close to, $q'_i(s_i)$.

\textsuperscript{15}The model in Kastl (2012) presupposes that bid functions are flat by issuing an exogenous constraint on how many steps were allowed in a submitted bid function. The approach here differs in that ironing is a result, not an assumption, and the ironing equation arises from comparison to an optimal bid function.

\textsuperscript{16}While this argument is applied to the initial flat, the same equation must hold—but with both endpoints varying—for any smoothed-over nonmonotonicity in an idealized bid function. Ironing is conceptually identical over such intervals.
signals. This implies that an agent receiving the lowest-possible signal will submit a perfectly flat bid function, and that an agent receiving the highest-possible signal will obtain a quantity strictly within her initial flat only when she faces an opponent who also receives the highest-possible signal.

This equilibrium has convenient analytical properties. First, the market price depends only on the lower of the two signals received by the agents; thus the market price may be written as \( p(s_1 \land s_2) \). Second, for reasons similar to those given in Lemma C.7 the agent’s bid will equal her marginal value at the maximum quantity she can receive. By monotonicity, this will equal the bid placed by the lowest-signal agent for the residual quantity. Since the bid function submitted by the lowest-signal agent must satisfy the same property and is assumed to be perfectly flat, the maximum quantity received by an agent receiving any signal, \( \overline{q}(s) \), is such that

\[
\alpha_0 + s\alpha_s - \overline{q}(s)\alpha_q = \alpha_0 - (Q - \overline{q}(s))\alpha_q \quad \implies \quad \overline{q}(s) = \frac{1}{2}Q + \frac{\alpha_s}{2\alpha_q} s.
\]

That the maximum-possible quantity must be weakly less than \( Q \), and that the bid submitted by any agent must be weakly positive, gives constraints on the allowable \( Q \) is this model. In the former case,

\[
\overline{q}(1) \leq Q \quad \implies \quad \alpha_s \leq Q\alpha_q.
\]

In the latter case,

\[
v^i\left(\frac{1}{2}Q; 0\right) \geq 0 \quad \implies \quad Q\alpha_q \leq 2\alpha_0.
\]

Then I require \( \alpha_s \leq Q\alpha_q \leq 2\alpha_0 \).\(^{17}\)

---

\(^{17}\)Theorem 2.1 establishes that a pure-strategy equilibrium exists in this model, provided—in this particular case—that \( Q\alpha_q \leq 2\alpha_0 \); that is, bidders have positive values for all but the possibly-last marginal unit. The lower bound on \( Q \) is necessary only to establish the feasibility of the particular form of equilibrium I investigate in this Section.
Appendix B.3 computes a system of differential equations which must be satisfied in equilibrium. Equation (B.2) gives the necessary conditions as

\[
\left( Q - \frac{1}{2} \left( \frac{\alpha_s}{\alpha_q} \right) (1 - s) \right) (1 - s) + (2s - 1) \tilde{q} \right) p_s \\
= \left( \frac{\alpha_s}{\alpha_q} - \tilde{q} \right) \left( \alpha_0 + \frac{1}{2} (1 + s) \alpha_s - (Q - \tilde{q}) \alpha_q - p \right) (1 - s) \\
+ (2\tilde{q} - Q) \left( \alpha_0 + s\alpha_s - \frac{Q}{2} \alpha_q - p \right).
\]

(Eq. 3.6)

Here, \( p \) is the market-clearing price, which depends only on the lower-signal agent, and \( \tilde{q} \) is the end of the initial flat for this agent. Bid functions are illustrated in Figure 3.3.

### 3.3.1 Nonstrategic flattening

If strategic ironing is ignored, or not accounted for, it is plausible that agents might rely on the intuition that this is simply a first-price auction for many units; this baseline will be elaborated upon in Section 3.3.2.

Appendix B.3 computes strategies for agents who do not iron, but nonstrategically flatten their bid functions. In the linear model of this Section, these bid functions are given by

\[
b^i (q; s) = \begin{cases} 
(\alpha_0 - \frac{1}{3} Q \alpha_q) + \frac{1}{2} s \alpha_s - \frac{1}{3} q \alpha_q & \text{if } q \geq \frac{3\alpha_s}{4 \alpha_q} (s - 1) + \frac{1}{2} Q, \\
(\alpha_0 - \frac{1}{2} Q \alpha_q) + \frac{1}{4} (1 + s) \alpha_s & \text{otherwise}.
\end{cases}
\]

Figure 3.3 illustrates these bid functions as they compare to equilibrium bid functions, which must reflect strategic ironing.

Figure 3.3 demonstrates an important implication of strategic ironing. Although the definition of strategic ironing—and its equilibrium characterization—is constrained to the initial flat of the bid function, it is clear that bids are reduced across the domain, including at points where the bid function is determined by local optimality constraints.
Figure 3.3: Marginal values, nonstrategically flattened bids, and strategically ironed bids for agents receiving various signals. The left panel represents $s_i = 0$, the middle panel represents $s_i = 1/2$, and the right panel represents $s_i = 1$. Agents’ marginal values are given by $v(q; s) = 4 + 2s - 2q$. $Q = 5$ units are available for auction.

The rationale behind this is straightforward: when a bidder is on the ironed portion of her bid function, she is receiving a relatively low quantity. Thus other bidders are competing for a relatively large residual quantity, and their incentives for relatively large quantities will be determined partially by the low-quantity agent’s bid reduction via strategic ironing.

In the two-agent case, when agent 1 receives quantity $q < \bar{q}_1(s)$, by market clearing agent 2 must receive $Q - q > Q - \bar{q}_1(s)$. Agent 2’s bid for this quantity is not determined “competitively” against agent 1, but rather against agent 1’s monotonicity constraint. That agent 1 has reduced her bid below her pointwise optimum for this quantity implies that agent 2 would like to reduce her bid, as well. Strategic ironing therefore has strong nonlocal implications. This can lead to the substantial gap between nonstrategically flattened bid functions and equilibrium, strategically ironed bid functions.
3.3.2 Calibration and magnitude

As seen in Section 3.3.1, the effects of ironing—when equilibrium is compared to direct application of “first-price” intuition augmented by nonstrategic flattening of the bid function—on the level of bids can be quite dramatic, and these effects are nonlocal. I now consider the magnitude of overall equilibrium effects.

There are several reasons to consider nonstrategic flattening as a baseline for establishing the magnitude of the effect of strategic ironing. Most obviously, the economic literature has as yet failed to distinguish between the strategic incentives present in a first-price auction and those present in a pay-as-bid auction; if economists have not successfully grokked the intuition governing behavior in this complex model, it is questionable that agents will have done so. Additionally, strategic ironing leads to bid reduction throughout the bid function. Intuitively, it follows that ironing should uniformly reduce revenues below those obtained by nonstrategic flattening. Then the revenues of nonstrategic flattening represent an upper bound on the revenue obtained from pseudo-strategic agents.\(^{18}\)

I now calibrate summary statistics of U.S. Treasury bill auctions from 2007 to match the expected revenue generated by nonstrategic flattening in the \(n = 2\) bidder model. As discussed in Section 1.2.1, the U.S. Treasury does not use a pay-as-bid auction to allocate its securities; nonetheless, that empirical studies have demonstrated rough revenue equivalence between the uniform-price auction—which the U.S. Treasury implements—and the pay-as-bid auction allows calibration of pay-as-bid revenue to observed uniform-price statistics.

In this calibration, there are \(n = 2\) bidders and quantity \(Q = 210\) million available for auction.\(^{19}\) Agents have an outside investment that they value at

\(^{18}\)An early version of Pycia and Woodward (2015) established the uniqueness of the solution to the nonstrategic flattening problem, hence the implied upper bound is well-defined. Importantly, nonstrategically-flattened ranges do not factor into the optimization problems of an agent’s opponents—she flattens her bid function only for units which are never marginal.

\(^{19}\)While other parameters are calibrated, that there are only \(n = 2\) auction participants is an obvious simplification.
\[ \alpha_s / \alpha_0 = 0.1 \]

\[ R = \$96.50 \]  
\[ \$1.19bn (10.63\%) \]  
\[ \$2.23bn (22.80\%) \]  
\[ \$1.73bn (19.78\%) \]

\[ \alpha_s / \alpha_0 = 0.3 \]

\[ R = \$96.75 \]  
\[ \$1.10bn (10.60\%) \]  
\[ \$2.10bn (23.08\%) \]  
\[ \$1.71bn (21.06\%) \]

\[ \alpha_s / \alpha_0 = 0.5 \]

\[ R = \$97.00 \]  
\[ \$1.02bn (10.67\%) \]  
\[ \$1.91bn (22.77\%) \]  
\[ \$1.74bn (23.19\%) \]

Table 3.1: Annualized revenue overestimates when bids are flattened instead of ironed; parenthetical percentages represent percentage of revenue overestimate, normalized by outside option.

constant \( R < 100 \). Marginal values for Treasury securities are linear, and are represented as the marginal improvement of purchasing a Treasury bill over \( R \); a bidder receiving the highest-possible signal values the zeroth bill at par, \( \alpha_0 + \alpha_s + R = 100 \). Because Appendix B.3 presents an explicit form for expected revenues under bid flattening, this is a tractable problem. Subject to the sufficient-demand and large-supply constraints issued above, the parameters vary; revenue simulation results are depicted in Table 3.3.2, computed by summarizing \( S = 100 \) simulated auctions in each parameterization.

To capture the relevance of strategic ironing, the percentage difference is computed as the revenue difference net of the guaranteed payments from bidders at the price of the outside option: revenues \( QR \) are assumed to be guaranteed, and I compare the additional revenue obtained from bids above \( R \). Table 3.3.2 shows that strategic ironing can account for an expected revenue loss of 10%-20% versus nonstrategic flattening.\(^{20}\)

\[^{20}\] The extent to which auction format matters depends significantly on the value of the outside option \( R \). During the years following the 2008 financial crisis, the market-clearing price for short-term bills was very near par, thus the difference in the “wedge” above the outside option would imply very nearly zero additional revenue. In such cases, the choice of auction format will have very little effect on the revenue obtained by the seller.
3.4 Conclusion

I this Chapter I have built on the model and existence results of Chapter 2, and have demonstrated the necessity of strategic ironing in pure-strategy equilibria of the pay-as-bid auction. Unlike previous studies in which flat bids arose from constraints or indivisibility, I show that bid flattening—strategic ironing—is a robust effect in pay-as-bid auctions, and arises independently of the particular implementation of an auction.

Using data from U.S. Treasury auctions, Section 3.3.2 shows that ironing can account for a substantial expected revenue reduction versus bids placed according to nonstrategic flattening of simple first-price intuition. This strongly suggests that strategic ironing cannot be ignored as a marginal effect, and must be accounted for not only in equilibrium construction, but in any approximating equilibrium construction.

Given the empirical ambiguity surrounding revenue rankings of the pay-as-bid and uniform-price auctions (discussed in Section 1.4), strategic ironing may present an argument in favor of implementing the pay-as-bid format. In particular, there is no current evidence as to whether or not bidders account for ironing pressures when participating in actual auctions. However, nonstrategic flattening represents an intuitive upper bound on the revenues obtained from the pay-as-bid format. If bidders apply first-price intuition to this auction, as to a certain extent the economic literature has, they will systematically overbid for the units they obtain.

This is of relevance to counterfactual revenue comparisons. With rare exceptions, structural studies of counterfactual revenues have taken an existing pay-as-bid auction and compared it to a counterfactual uniform-price auction. If bidders are systematically overbidding in the pay-as-bid auction, there will be

\[ \text{Equation} \]

For example, Kang and Puller (2008) compare observed uniform-price and pay-as-bid auctions to a single Vickrey auction.
a distinct upward bias in the inferred counterfactual revenues of the uniform-price auction. Whether or not bidders, in practice, fully account for ironing pressures—and whether this effect would be sufficient to imply a statistically significant revenue difference between the two formats—merits further attention; although mechanisms are often prized for their strategic simplicity (Borgers and Van Damme, 2004), it is in fact the strategic complexity of the pay-as-bid auction which might give rise to additional seller revenue.
CHAPTER 4

Equilibrium uniqueness

In this Chapter, I develop a theory of equilibrium bidding in pay-as-bid auctions with random supply. This theory has three chief components. First, I establish a sufficient condition for equilibrium existence; this condition is expressed in terms of primitives of the model, and is relatively simple to check. Additionally, this sufficient condition is generically tight, in the sense that its weak-inequality analogue is necessary for the existence of a pure-strategy equilibrium. Usefully, this condition is satisfied in the linear-Pareto settings analyzed by the prior literature. Irrespective of the distribution of supply, the condition is satisfied for linear marginal values if there is a sufficiently-large number of bidders.

Second, I prove that there is a unique pure-strategy Bayesian-Nash equilibrium in this auction, conditional on equilibrium existence. The uniqueness of equilibrium is reassuring for sellers using pay-as-bid format; indeed, there are well-known problems posed by multiplicity of equilibria in other multi-unit auctions. Unique-

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1As mentioned in footnote 1 in Chapter 1, this Chapter is largely taken from Pycia and Woodward (2015). For consistency with the remainder of this dissertation—and for no other reason—I use the first-person singular “I” rather than the first-person plural “we.”

2I establish the uniqueness of bids for relevant quantities, as given in Definition 2.1. My uniqueness result and the subsequent discussion does not apply to these irrelevant bids. Additionally, since I work in a model with continuous quantities I will not distinguish between two bid functions that coincide almost everywhere; an equilibrium bid function may be altered on a measure-zero set of quantities without affecting equilibrium outcomes. Because these qualifications will never factor in to empirical observations, the term uniqueness is sufficient.

3The uniform-price auction can admit multiple equilibria, some of which generate very little revenue. See LiCalzi and Pavan (2005), McAdams (2007a), Kremer and Nyborg (2004), and the discussion of uniform-price below. There is no contradiction here with the revenue equivalence result I present below: my revenue equivalence result proves revenue equivalence between the unique equilibrium in pay-as-bid and the seller-optimal equilibrium in uniform-price; the seller can ensure that the latter equilibrium is unique by judiciously selecting the reserve price.
ness is also important for the empirical study of pay-as-bid auctions. Estimation strategies based on the first-order conditions rely on agents playing comparable equilibria across auctions in the data (Février et al., 2002; Hortaçsu and McAdams, 2010; Hortaçsu and Kastl, 2012; Cassola, Hortaçsu, and Kastl, 2013). Equilibrium uniqueness plays an even larger role in the study of counterfactuals (see Armantier and Sbarbati, 2006). The uniqueness of equilibrium provides a theoretical foundation for these estimation strategies and counterfactual analysis.

The third result in this theory is a bid representation theorem. I show that in the unique pure-strategy Bayesian Nash equilibrium of this auction the bid for any quantity is a weighted average of the bidders’ marginal values for this and larger quantities. The weights in this average are independent of the bidder’s marginal values, and depend only on the distribution of supply and the number of bidders. The tail of the “weighting distribution” is equal to the tail of the distribution of supply scaled by the number of bidders; an increase in the number of bidders shifts the weight away from the tail and towards the unit for which the bid is submitted, hence increasing the bid for that unit.

The bid representation theorem implies several properties of equilibrium bidding: the unique equilibrium is symmetric and the bid functions are strictly decreasing and differentiable in quantities. Rather than assuming them as properties of the model, I prove equilibrium symmetry, strict monotonicity, and differentiability. The analysis allows for all Bayesian-Nash equilibria, including asymmetric ones, and imposes no strict-monotonicity or regularity assumptions on the submitted bids.

The bid representation theorem has other immediate implications. With the distribution of supply concentrated around a target quantity, the representation

\footnote{Maximum likelihood-based estimation strategies (e.g. Donald and Paarsch, 1993) also rely on agents playing comparable equilibria across auctions in the data. Chapman, McAdams, and Paarsch (2005) discuss the requirement of comparability of data across auctions.}

\footnote{See also, in a related context, Cantillon and Pesendorfer (2006)}
of bids implies that bids are nearly flat for units lower than the target, and that
the bidder’s margin on units near the target is low. The representation theorem
also implies that the seller’s revenue increases when bidders’ values increase, or
when more bidders arrive. Finally, the bid representation theorem also plays a
central role in the subsequent analysis of auction design and revenue equivalence
between pay-as-bid and uniform-price auctions.

Building on this theory of equilibrium bidding, I address outstanding questions
surrounding the design of divisible-good auctions. Traditionally a key instrument
in auction design is the reserve price; for divisible goods there is a second nat-
ural instrument: the supply distribution. The first result on design establishes
that every reserve price decision can be replicated by an appropriate supply re-
striction, so that the two choices lead to identical bidding behavior in the unique
Bayesian-Nash equilibrium of the pay-as-bid auction. Thus supply adjustments
can accomplish everything reserve prices can, but—as I also show—the reverse
is not true.6 Second, I address the question of which distributions of supply are
optimal in the pay-as-bid auction. The main result here says that the revenue
in the pure-strategy equilibrium is maximized when supply is deterministic; com-
puting the level of the optimal deterministic supply is equivalent to a standard
monopoly problem. In practice, in many of these auctions the distribution of
supply is partially determined by the demand from non-competitive bidders, and
revenue maximization is not the only objective of the sellers. However, treasuries
and central banks have the ability to influence the distribution of supply, as well
as to release data on non-competitive bids to competitive bidders; in this context
this result provides a revenue-maximizing benchmark.

While the result that deterministic selling strategies are optimal is familiar
from the no-haggling theorem of Riley and Zeckhauser (1983), in multi-object

6In this regard the pay-as-bid auction is different from the uniform-price auction: in uniform-
price auctions reserve prices play an important role. See the discussion below.
settings the reverse has been shown (Thanassoulis, 2004; Pycia, 2006; Manelli and Vincent, 2006, 2007). Furthermore, there is a subtlety specific to pay-as-bid that might suggest the role for randomization: by randomizing supply below the monopoly quantity, the seller forces bidders to compete by bidding more for relatively low quantities, and in a pay-as-bid auction the seller collects the higher bids even when the realized supply is near monopoly quantity. I show that, despite these considerations, committing to deterministic supply is indeed optimal.\(^7\)

Finally, I compare the expected revenue generated by pay-as-bid and uniform-price auctions. The result on the optimality of deterministic supply allows me to easily show that with supply and reserve prices chosen optimally in both auction formats, the two formats are revenue-equivalent. Setting the reserve price has no impact on the revenue in pay-as-bid, but is important in uniform-price. With supply and reserve price set optimally, the uniform-price auction has a unique equilibrium; were the auctioneer to set the supply optimally but ignore the reserve price, the uniform-price auction could have multiple equilibria and the revenue equivalence with the unique equilibrium of pay-as-bid would obtain only for the revenue-maximizing equilibrium of uniform-price.

This divisible-good revenue equivalence result provides a benchmark for the long-standing debate whether pay-as-bid or uniform-price auctions raise higher expected revenues. This debate has attracted substantial attention in empirical structural industrial organization, with Hortaçsu and McAdams (2010) finding no statistically significant differences in revenues, Février et al. (2002) and Kang and Puller (2008) finding slightly higher revenues in pay-as-bid, and Castellanos and Oviedo (2008), Armantier and Sbaï (2006), and Armantier and Sbaï (2009) finding slightly higher revenues in uniform-price. Given the revenue equivalence result obtained in this Chapter, such a pattern is not surprising if sellers set the

\(^7\)This result was presaged by Brimmer (1962). Nevertheless Brimmer’s argument differs substantially in that it presupposes a dynamic model with endogenous noncompetitive demand.
supply and reserve prices in the two auctions near their revenue-maximizing levels.

Prior theoretical work on the pay-as-bid versus uniform price question focused on revenue comparisons for fixed supply distributions. Wang and Zender (2002) find pay-as-bid revenue superior in the equilibria of the linear-Pareto model they consider. Ausubel et al. (2014) show that—with asymmetric bidders—either format can be revenue superior; with symmetric bidders pay-as-bid is revenue superior in all examples they consider. The supply distributions these papers consider are not revenue-maximizing, hence they obtain strict rankings where my results prove revenue equivalence. Swinkels (2001) shows that pay-as-bid and uniform price are revenue-equivalent in large markets; the revenue equivalence result in this Chapter does not rely on the size of the market. ⁸

4.1 Model

There are \( n \geq 2 \) bidders, \( i \in \{1, \ldots, n\} \). Each bidder has a symmetric marginal valuation for quantity \( q \) denoted \( v^i(q) = v(q) \). I assume that \( v \) is strictly decreasing and Lipschitz continuous. Marginal values are sufficiently high, so that \( v(Q/n) > 0 \) for the maximum per-capita supply (defined below). ⁹

The supply \( Q \) is drawn from a non-degenerate distribution \( F \) with density \( f \) and support \([0, Q]\). ¹⁰ I assume that \( f(Q) > 0 \) for all \( Q \in [0, Q] \), and otherwise impose no global assumptions on \( F \). In particular, I allow distributions that are concentrated around some quantity and which take values close to 0 with

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⁸Large market revenue equivalence was also obtained by Ausubel et al. (2014).

⁹The assumption of sufficient demand is made for analytical simplicity and is not necessary. If marginal values are zero for quantities below the per-capita maximum, bids will be determined by letting bids equal values at \( q = \varphi^i(0) \), and \( b^i(q) \in (0, v(q)) \) for all \( q < \varphi^i(0) \). The substantive results of this Chapter will continue to hold although the bid representation will change.

¹⁰For instance, \( Q \) might represent supply net of non-competitive bids as discussed in Back and Zender (1993), Wang and Zender (2002), and subsequent literature. The uniqueness result in this Chapter does not rely on any additional assumptions, and the equilibrium existence theorem specifies a sufficient condition for the existence of a pure-strategy equilibrium.
arbitrarily small probability.\textsuperscript{11} I denote the inverse hazard rate by $H(Q) = \frac{1-F(Q)}{f(Q)}$.

### 4.1.1 Optimality conditions

The same calculus of variations approach pursued in Section 3.1.1 applies in this model. Although there is no private information in the model of this Chapter, and supply is deterministic in the model of Chapter 3, they share a common feature: the equilibrium quantity allocation is endogenous and stochastic. Appendix C provides an explicit derivation of the agent’s first-order conditions. In particular, Lemma C.6 gives the agent’s first-order conditions as

$$- \left( v(q) - b^i(q) \right) G^i_b(q; b^i) = 1 - G^i(q; b^i).$$

### 4.2 Existence, uniqueness, and bid representation

As will be seen in the Theorems and derivations that follow, the concept of a weighting distribution is crucial to the results of this Chapter.

**Definition 4.1.** For any quantity $Q \in [0, Q]$, the $n$-bidder weighting distribution of $F$ is $F^{Q,n}$, where

$$F^{Q,n}(x) = 1 - \left( \frac{1 - F(x)}{1 - F(Q)} \right)^{\frac{n-1}{n}}.$$

Note that for any $(Q, n)$, $F^{Q,n}$ is a proper CDF, with $F^{Q,n}(Q) = 0$ and $F^{Q,n}(Q) = 1$. The auxiliary CDFs $F^{Q,n}$ play a central role in the bid representation theorem and throughout this Chapter. Importantly, these distributions depend only the number of bidders and the distribution of supply, and not on any bidder’s true demand. Note that as the number of bidders increases the weighting distributions put more weight on lower quantities.

The first result is a sufficient condition for equilibrium existence.

\textsuperscript{11}In some results, I also consider the limit as $F$ puts all mass on a single quantity.
**Theorem 4.1 (Equilibrium existence).** There exists a pure-strategy Bayesian-Nash equilibrium if, for all $Q < \overline{Q}$,

$$
\frac{d}{dx} \left[ \ln \frac{1 - F(x)}{f(x)} \right]_{x=Q} > \frac{n-1}{n} \left( \frac{v_q(\frac{Q}{n})}{v(\frac{Q}{n})} - \int_{Q}^{\overline{Q}} v(\frac{x}{n}) dF_{Q,n}(x) \right).
$$

This sufficiency condition is generically tight: its weak-inequality analogue is necessary for the existence of a pure-strategy equilibrium. The proofs of Theorem 4.1 and all subsequent results are given in Appendix C.

Consider some examples. The sufficient condition is satisfied when marginal values, $v$, are linear and $Q$ is drawn from a uniform distribution or a generalized Pareto distribution, $F(x) = 1 - \left(1 - \frac{x}{\theta}\right)^\alpha$ where $\alpha > 0$. With linear marginal values, this condition is also satisfied for any distribution $F$ with $f > 0$, provided there are sufficiently many bidders. Indeed, with linear marginal values the right-hand side of the condition is negative and proportional to $n-1$ while the left-hand side does not depend on $n$. Lastly, the sufficient condition is satisfied whenever the inverse hazard rate $H$ is increasing—hence when the hazard rate is decreasing—irrespective of the marginal value function $v$. This follows since the numerator on the right is negative (the marginal values are decreasing and thus $v_q < 0$) while the denominator is positive (since $v$ is decreasing and the support of the weighting distribution is above $Q$ for $Q < \overline{Q}$). In what follows, I illustrate subsequent results with additional examples in which a pure-strategy equilibrium exists.

Next I establish that the pure-strategy equilibrium is unique.

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12 The existence of equilibrium in the linear/generalized Pareto example has been established by Ausubel et al. (2014). In Section 4.3.1, the results of this Chapter are expanded to include unbounded distributions, including the unbounded Pareto distributions studied by Wang and Zender (2002), Federico and Rahman (2003), and Holmberg (2009); the sufficient condition remains satisfied for unbounded Pareto distributions.

13 The sufficiency of decreasing hazard rate for equilibrium existence was established by Holmberg (2009).

14 In general, the existence condition is closed with respect to several changes of the environment: adding a bidder preserves existence, making the marginal values less concave (or more convex) preserves existence, and imposing a reserve price preserves existence.

15 While the sufficient condition shows that the equilibrium exists in many cases of interest, there are of course situations in which a pure-strategy equilibrium does not exist.
Theorem 4.2 (Uniqueness). The Bayesian-Nash equilibrium is unique.

Two important comments on this and subsequent results are in order:

- From this point onward, all results are stated under the presumption that the bidders’ marginal utilities $v$ and the supply distribution $F$ are such that an equilibrium exists.

- This and all subsequent results constrain attention to bids for relevant quantities. Bids for the quantities that the bidder never wins have to be weakly decreasing and sufficiently competitive, but are not determined uniquely.\(^\text{16}\)

The existence and uniqueness results lead to the main insight of this Chapter, the bid representation theorem:

Theorem 4.3 (Bid representation). In the unique equilibrium, bids are given by

$$b^i(q) = \int_{q_{nq}}^Q v\left(\frac{x}{n}\right) dF^{q_{nq},n}(x).$$

The resulting market price function is given by

$$p(Q) = \int_Q^{\infty} v\left(\frac{x}{n}\right) dF^{Q,n}(x). \quad (4.1)$$

Thus, the equilibrium price $p$ is the appropriately-weighted average of bidders’ marginal values $v$; the same is true of equilibrium bid functions.

Consider three examples. Substitution into the bid representation shows that when marginal values $v$ are linear and the supply distribution $F$ is generalized Pareto, $F(x) = 1 - (1 - \frac{x}{Q})^{\alpha}$ for some $\alpha > 0$, the equilibrium bids are linear in quantity. This case of the general setting in this Chapter has been analyzed by

\(^{16}\)The reason a bidder’s bids on never-won quantities need to be sufficiently competitive is to ensure that other bidders do not want decrease their bids on relevant quantities. In the setting with reserve prices, which will be analyzed in Section 4.4.1, the bids on never-won quantities do not need to be competitive and hence these bids are even less determined, but the equilibrium bids on the relevant quantities remain nevertheless uniquely determined.
Ausubel et al. (2014).\textsuperscript{17} The bid representation remains valid when \( F \) puts all its mass on \( Q \); taking the limit of continuous probability distributions which place increasingly more probability near \( Q \), the representation implies that equilibrium bids are flat, as is intuitive.\textsuperscript{18} Finally, Figure 4.1 illustrates equilibrium bids for ten bidders with linear marginal values who face a distribution of supply that is truncated normal. This and subsequent Figures represent bids, marginal values, and the CDF of supply; it is easy to distinguish between the three curves since bids and marginal values are decreasing (and bids are below marginal values), while the CDF of supply is increasing.\textsuperscript{19}

The preceding three Theorems are proved in Appendix C. The rest of this Chapter builds upon them to establish qualitative properties of the unique equilibrium, and to provide guidance as to how to design divisible-good auctions.

### 4.3 Equilibrium bids and comparative statics

#### 4.3.1 Properties of the unique equilibrium

To start, I recognize some immediate Corollaries of the bid representation theorem. While these results are presented as corollaries, parts of Corollaries 4.1 and 4.3 are among the key Lemmas used in the proofs of the main results of Section 4.2.

\textsuperscript{17}Ausubel et al. (2014) calculate bid functions in terms of the parameters of their model (linear marginal values and Pareto distribution of supply) and do not rely on or recognize the separability property that is crucial to the analysis in this Chapter. While this Chapter focuses chiefly on bounded distributions, Ausubel et al. (2014) look at both bounded and unbounded Pareto distributions, and Wang and Zender (2002) look at unbounded Pareto distributions. The general approach taken in this Chapter remains valid for unbounded distributions, including Pareto, except that uniqueness requires a lower bound on admissible bids, e.g. an assumption that bids are nonnegative. More details are provided in the discussion of reserve prices in Section 4.4.1.

\textsuperscript{18}If bidders know the quantity which will be auctioned and all agents are playing pure strategies, the quantity obtained by a single bidder conditional on her choice of bid function is a deterministic result of that choice, and she will minimize payment wherever possible; hence she will submit a perfectly flat bid function.

\textsuperscript{19}In Figure 4.1 the model is parameterized by a normal distribution of supply with mean 3 and standard deviation 1, truncated to the interval \([0, 6]\).
Figure 4.1: Equilibrium bids when the distribution of supply is truncated normal.

In all such cases Appendix C provides direct proofs of the relevant results.

**Corollary 4.1** (Properties of equilibrium). *The unique equilibrium is symmetric, and its bid functions are strictly decreasing and differentiable in quantities.*

Recall that I have imposed no assumptions on the symmetry, strict monotonicity, or continuity of equilibrium bids; these are derived properties, necessary in equilibrium.

Since the unique equilibrium is symmetric, there is an easy correspondence between the market price $p(Q)$ given supply $Q$ and the bid functions $b^i = b$:

$$b(q) = p(nq).$$

This relationship is embedded in the bid representation theorem.

### 4.3.2 Flat bids, low margins, and concentrated distributions

A case of particular interest arises when the distribution of supply is concentrated near some target quantity.
Figure 4.2: Bids are flatter for more concentrated distributions of supply.

**Definition 4.2.** A distribution $F$ is $\delta$-concentrated near quantity $Q^*$ if $1 - \delta$ of the mass of supply is within $\delta$ of quantity $Q^*$,

$$F(Q^* + \delta) - F(Q^* - \delta) \geq 1 - \delta.$$ 

The bid representation theorem implies that the bids on initial quantities are nearly flat for concentrated distributions.

**Corollary 4.2 (Flat bids).** For any $\varepsilon > 0$ and quantity $Q^*$ there exists $\delta > 0$ such that, if the supply is $\delta$-concentrated near $Q^*$, then the equilibrium bids for all quantities lower than $\frac{Q^*}{n} - \varepsilon$ are within $\varepsilon$ of $\nu\left(\frac{Q^*}{n}\right)$.

Figure 4.2 illustrates the flattening of equilibrium bids; in the three sub-figures ten bidders face a supply distribution that is increasingly concentrated around a maximum supply of 6. Supply concentrates as a Lomax distribution with parameter $\alpha = 0.2^k$ for $k \in \{0, 1, 2\}$.

The bid representation theorem has also implications for bidders’ margins. In Corollary 4.3 below the supremum of quantities the bidder wins with positive probability is referred to as the highest quantity a bidder can win in equilibrium.

**Corollary 4.3 (Low margins).** The highest quantity a bidder can win in equilibrium is $\frac{1}{n}Q^*$, and the bid at this quantity equals the marginal value, $b\left(\frac{1}{n}Q^*\right) = \nu\left(\frac{1}{n}Q^*\right)$. Furthermore, for any $\varepsilon > 0$ and quantity $Q^*$ there exists $\delta > 0$ such
that, if supply is $\delta$-concentrated near $Q^*$, then each bidder’s equilibrium margin $v(\frac{1}{n}Q^* - \delta) - b(\frac{1}{n}Q^* - \delta)$ at quantity $\frac{1}{n}Q^* - \delta$ is lower than $\varepsilon$.

Thus, each bidder’s margin on the last unit they could win is zero; and, if the supply is concentrated around some quantity $Q^*$, then the margin on units just below $\frac{1}{n}Q^*$ is close to zero.

4.3.3 Comparative statics

With the bid representation theorem, it is simple to deduce how bidding behavior changes when the environment changes. First, as one could expect, an increase in marginal values always benefits the seller: higher values imply higher revenue.

**Corollary 4.4 (Higher values).** If bidders’ marginal values increase, the seller’s revenue also increases.

The bid representation theorem further implies that if there is an affine transformation of bidders’ marginal values from $v$ to $\alpha v + \beta$, then the seller’s revenue changes from $\pi$ to $\alpha \pi + \beta$ (hence the seller’s expected revenue changes to $\alpha E_Q[\pi] + \beta E_Q[Q]$). In particular, all additional surplus from raising the value of all bidders by a constant goes to the seller.

Also as one would expect, holding the distribution of supply constant in either absolute or per-capita terms, the bidders’ equilibrium margins are lower and the seller’s revenue is larger when there are more bidders:

**Corollary 4.5 (More bidders).** Bidders submit higher bids and the seller’s revenue is larger when there are more bidders—both when the supply distribution is held constant, and when the per-capita supply distribution is held constant.

Indeed, as the number of bidders increases, $1 - F^{Q,n}(x) = (\frac{1-F(x)}{1-F(Q)})^{n-1}$ decreases, and hence $F^{Q,n}(x)$ increases, thus the probability mass in the weighting
Figure 4.3: Bids increase when more bidders arrive while per capita quantity is kept constant, but not by much: the left panel represents 5 bidders, the middle panel represents 10 bidders, and the right panel represents 5 million bidders. Distribution is shifted towards lower $x$, where the marginal values are higher. At the same time, marginal values at per-capita $x$ either increase in $n$ (if the distribution of supply is held constant) or stay constant (if the per-capita distribution of supply is held constant). Both of these effects point in the same direction, implying that bids and expected revenue increase in the number of bidders. This argument also shows that the seller’s revenue increases if bidders are added while proportionately raising supply, and that a bidder’s profits are decreasing in the number of her competitors even when per-capita supply is held constant.

While bidders raise their bids when facing more bidders even if the per-capita distribution is unchanged, the bid representation theorem implies that the changes are small.\footnote{Notice that if the supply distribution is fixed while more and more bidders participate in the auction, then in the large market limit the revenue converges to average supply times the value on the initial unit. See Swinkels (2001).} This is illustrated in Figure 4.3, in which increasing the number of bidders from 5 bidders to 10 bidders has only a small impact on the bids, as does the further increase from 10 bidders to 5 million bidders. To see analytically what happens for large numbers of bidders, denote the distribution of per-capita supply by $\bar{F}$, and note that $b\left(q\right) = \int_{q}^{\bar{Q}} v\left(x\right) d\bar{F}^{q,n}\left(x\right)$. As $n \to \infty$ the limiting weighting

\[ b\left(q\right) = \lim_{n \to \infty} \int_{q}^{\bar{Q}} v\left(x\right) d\bar{F}^{q,n}\left(x\right) \]
distribution is \( F^{q,n}(x) \rightarrow \frac{F(x) - F(q)}{1 - F(q)} \), and the limiting bids take the form

\[
b(q) = \int_q^{\frac{1}{n}Q} v(x) d\left(\frac{F(x) - F(q)}{1 - F(q)}\right) = \frac{1}{1 - F(q)} \int_q^{\frac{1}{n}Q} v(x) dF(x).
\]

In particular, in large markets the bid for any unit is given by the average marginal value of higher units, where the average is taken with respect to (scaled) per-capita supply distribution.

### 4.4 Designing pay-as-bid auctions

The reserve price and the distribution of supply are two natural elements of the pay-as-bid auction that are at the seller’s discretion. The bid representation theorem can be used to analyze these design choices. In the process I relax the assumption that the distribution of supply is bounded.

#### 4.4.1 Reserve prices

The further analysis of the reserve prices is based on the following:

**Theorem 4.4** (Equilibrium with reserve prices). In the pay-as-bid auction with reserve price \( R \), the equilibrium is unique and it is identical to the unique equilibrium in the pay-as-bid auction with supply distribution

\[
F^R(Q) = \begin{cases} 
1 - F(Q) & \text{if } Q \leq \hat{Q}, \\
1 & \text{otherwise},
\end{cases}
\]

where \( \hat{Q} = n v^{-1}(R) \).

---

21In discussing the design problem I will maintain the assumption that the seller is constrained to implement a pay-as-bid format. The optimal mechanism design for selling divisible goods has been analyzed by, e.g., Maskin and Riley (1989); the optimal mechanism is complex and it is not used in practice. In addition to setting supply and reserve prices, the choice in practice is between pay-as-bid and uniform price auctions—and the question of which of these two auctions to implement is addressed in Section 4.5, as well as Chapters 3 and 5.
Notice that distribution $F^R$ has a probability mass at supply $\hat{Q}$, which is the largest supply under this distribution. While the preceding results have been derived for atomless distributions, the arguments would not change in the presence of an atom at the highest supply. Thus, all equilibrium results remain applicable.\(^{22}\)

The rest of the proof of Theorem 4.4 is then simple. When the distribution of supply is $F^R$, the last relevant bid is exactly $R$ by Corollary 4.3 and hence imposing the reserve price of $R$ does not change bidders’ behavior. Furthermore the equilibrium bids against $F^R$ remain equilibrium bids against $F$ with reserve price $R$, and one direction of the theorem is shown. Consider now equilibrium bids in an auction with reserve price $R$.\(^{23}\) The sum of bidders’ demands is then always weakly lower than $nv^{-1}(R)$ and hence their bids constitute an equilibrium when the supply is distributed according to $F^R$. This establishes the other direction of Theorem 4.4.

The equivalence between reserve prices and particular changes in the supply distribution suggests an immediate Corollary.

**Corollary 4.6** (Reserve price as a supply restriction). *For every reserve price $R$ there is a reduction of supply that is revenue-equivalent to imposing $R$.*

Corollary 4.6 implies that, while reserve prices can be mimicked by supply decisions, not all supply decisions can be mimicked by the choice of reserve prices. In particular, the revenue with optimal supply is typically higher than the revenue with optimal reserve price. Notice also that, with concentrated distributions, the results of Section 4.3 imply that attracting additional bidders is often more profitable than correctly setting the reserve price.

\(^{22}\)More details are provided in Section C.7. Importantly, if marginal values $v$ and the distribution of supply $F$ satisfy the sufficient condition for equilibrium existence, then $F^R$ satisfies this condition as well. Indeed, for $Q < \hat{Q}$, folding the tail of the distribution $F$ into an atom in $F^R$ leaves the left-hand side of this condition unchanged while making the right-hand side more negative (since its numerator is negative and the mass shift makes the positive denominator smaller).

\(^{23}\)Instead of this step of the argument, one could check directly that the equilibrium uniqueness result, Theorem 4.2, remains true in the setting with reserve prices.
Figure 4.4: The equilibrium bid function with a normal distribution of supply
(left), and with an optimal reserve price (right). The bid for the implicit “max-
imum quantity” equals the marginal value for this quantity, and the entire bid
function shifts up.

The analysis of optimum supply in the next subsection further implies that:

Corollary 4.7 (Optimal reserve price). The optimal reserve price $R$ is equal to
bidders’ marginal value at the optimal deterministic supply: $R \in \max_{R'} R'v^{-1}(R')$.

When the reserve price $R$ is binding, the equivalence between reserve price
and supply restriction gives an implicit “maximum supply” of $\overline{Q}^R = nv^{-1}(R)$. At
this quantity, parceled over each agent, each agent’s bid will equal her marginal
value, as at $\overline{Q}$ in the unrestricted case. Since bids fall below values, this bid is
weakly above the bid placed at this quantity when there is no reserve price. For
quantities below $\overline{Q}^R$ the CDF is unchanged, hence the bid representation and
uniqueness theorems combine to imply that the bids submitted with a reserve
price will be higher than without. These effects can be seen in Figure 4.4.

To conclude the discussion of reserve prices, notice that this theory of equilib-
rium bidding has been developed under the assumption that the distribution of
supply is bounded. However, in the presence of a reserve price, any unbounded
distribution is effectively bounded, hence the boundedness assumption may be
easily relaxed.
4.4.2 Optimal supply

Consider first the problem of a seller who has some quantity $Q$ of the good, and would like to design a supply distribution $F$ that maximizes his revenue. For deterministic quantities the problem is simple: offering quantity $\hat{Q} \leq Q$ leads to a unique equilibrium in which all bids are flat.\(^{24}\) The seller’s revenue is thus $\hat{Q}v(\hat{Q}/n)$. Let $Q^*$ be the deterministic monopoly supply; then $Q^*$ is the quantity that maximizes this expression.

However, the seller has the option to offer a stochastic distribution over multiple quantities, and it is plausible that such randomization could increase his expected revenue. Offering randomization over quantities larger than the optimal deterministic supply $Q^*$ may be relatively easily shown to be suboptimal: his profit on the units above $Q^*$ is lower than his profit on deterministically selling $Q^*$, and moreover offering quantities above $Q^*$ suppresses the bids submitted for $Q^*/n$.\(^{25}\) On the other hand, offering quantities lower than $Q^*$ offers the seller a trade-off: he sometimes sells less than $Q^*$, with a direct and negative revenue impact, but when he sells quantity $Q^*$ he will receive higher payments due to the pay-as-bid nature of the auction.\(^{26}\) The answer to this question turns out to be degenerate: deterministically setting supply to $Q^*$ is in fact revenue-maximizing for sellers across all pure-strategy equilibria; for this reason in the sequel $Q^*$ is referred to as optimal supply.\(^{27}\)

\(^{24}\)The standard Bertrand argument suffices. This point was made by Wang and Zender (2002).

\(^{25}\)In spite of this being intuitively clear, the proof is intricate. In particular, the seller is price-discriminating, and a perfectly-discriminating monopolist with zero costs would want to sell as large a quantity as possible. Nevertheless, the proof of this point is substantially similar to that which establishes that, optimally, there is no randomization below $Q^*$.

\(^{26}\)A priori such trade-offs can go either way; see the introduction to this Chapter. The problem is well illustrated in Figure 4.2, in which concentrating supply lowers bids.

\(^{27}\)This result restricts attention to pure-strategy equilibria. A reason a seller may want to ensure that pure-strategy equilibrium is being played is that the symmetry of equilibrium strategies proved in Lemma C.8 implies that every pure-strategy equilibrium in pay-as-bid auctions is efficient, while it is immediate to see that mixed-strategy equilibria are not efficient. Since the pay-as-bid format is largely employed by central banks and governments, the efficiency of allocations is an important concern. Note also that when considering stochastic supply, the global restriction to distributions of supply that have strictly positive density on some interval
Theorem 4.5 (Optimal supply). In any pure-strategy equilibrium, the seller’s expected revenue under non-deterministic supply is strictly lower than his revenue under optimal deterministic supply.

Trivially, when supply is random, the seller evaluates revenue in expectation, and when supply is deterministic the revenue generated by the mechanism is nonrandom. This suggests an additional argument in favor of deterministic supply in all settings: with nominal risk aversion, the deterministic outcome will further dominate the random revenue generated by stochastic supply.

So far I have assumed that the seller has access to quantity $Q$ and is free to design the distribution of supply. This approach can be easily generalized: if the distribution of supply is exogenously given by $F$ and is not directly controlled by the seller, the revenue maximizing-supply reduction by the seller reduces supply to $Q^*$ whenever the exogenous supply is higher than $Q^*$, and otherwise leaves the supply unchanged.

4.5 Divisible-good revenue equivalence

In practice, sellers of divisible goods are not restricted to running pay-as-bid auctions: the pay-as-bid auction and the uniform-price auction are the two most-commonly implemented multi-unit auctions. From a practical perspective, which of these two formats is preferred is a highly important question, and has be extensively studied both in the theoretical and empirical literature on divisible good auctions (see the introduction to this Chapter, as well as Section 1.4).

The results of the previous Section allow for an easy comparison of revenues in the two auctions when the seller optimally sets both the distribution of supply and the reserve price: in the uniform-price auction the optimal supply is then also $Q^*$. In contrast to the pay-as-bid auction, several equilibria are possible in any $[0, Q]$ remains in place.
uniform-price auction. Among them, the equilibrium in which all bidders bid flat at $v(Q^*/n)$ is revenue-maximizing; the seller can assure that this is the unique equilibrium of the uniform-price auction by setting the reserve price at $v(Q^*/n)$. The revenue from the fully-optimized uniform price auction is then exactly the same as in the pay-as-bid auction.\(^{28}\)

**Theorem 4.6** (Revenue equivalence for divisible-good auctions). *With optimal supply and reserve price, the revenue in the unique equilibrium of the pay-as-bid auction is exactly equal to the revenue in the unique equilibrium of the uniform-price auction.*

While the seller’s ability to set a reserve price has no impact on the revenue in pay-as-bid with optimal supply, it plays an important equilibrium-selection role in uniform price. As noted above, without the ability to set reserve prices the two auction formats are revenue-equivalent only with respect to the seller-optimal equilibrium of the uniform-price auction.

**Corollary 4.8** (Pay-as-bid revenue dominance). *With optimal supply, the revenue in the unique equilibrium of the pay-as-bid auction equals the revenue in the revenue-maximizing equilibrium of the uniform price auction; in particular, the revenue in the pay-as-bid auction is always at least as large as the revenue in the uniform-price auction.*

Theorem 4.6 suggests a possible answer to the why the debate over revenue superiority of the two canonical multi-unit auction formats, pay-as-bid and uniform-price, remains unsettled. As captured by the extensive literature on pay-as-bid and uniform-price auctions, sellers are willing to expend significant energy determining which mechanism is preferable; it is reasonable to assume they are just as interested in the particulars of the mechanism they select. Since the mechanisms

\(^{28}\)The equivalence of Theorem 4.6 remains true if the seller is able to set different reserve prices for different units, as then the seller could fully extract bidders’ surplus in both auction formats.
are revenue-equivalent when their parameters are optimally determined, relatively optimized auctions should have similar revenues, independent of the mechanism employed.\textsuperscript{29} And, indeed, as discussed in the introduction to this Chapter and in Chapter 1, this is what the empirical literature finds.

4.6 Conclusion

In this Chapter I have proved that Bayesian-Nash equilibrium is unique in pay-as-bid auction with uninformed, symmetric bidders. I have further provided a sufficient condition for equilibrium existence, and stated a surprisingly tractable bid representation.

These results permit the discussion of design aspects of pay-as-bid auctions. I have shown that in the pay-as-bid auction reserve prices can always be replicated by supply restrictions; this is a property that substantially differentiates the pay-as-bid and uniform-price auction formats. I have also shown that optimal supply is deterministic.

Comparing pay-as-bid and uniform-price auctions, I have established that the two auction formats are revenue-equivalent when supply and reserve prices are set optimally. When the seller has no ability to set the reserve price, pay-as-bid weakly dominates uniform-price. These results support the use of pay-as-bid format, and they may explain why pay-as-bid is indeed the most popular format for selling divisible goods such as treasury securities.

\textsuperscript{29}The results of this Chapter look at the case of symmetric bidders. With asymmetric bidders, Ausubel et al. (2014) show that the revenue comparison can go either way. This is reminiscent of the situation in single-good auctions: with symmetry first price and second price auctions are revenue equivalent, but this equivalence breaks in the presence of asymmetries.
CHAPTER 5

Optimal randomization

In this Chapter I use the tools developed in Chapter 4 to assess further modifications of the pay-as-bid auction mechanism. While Chapter 4 assesses the possibility that randomizing supply might raise the seller’s revenues—a possibility which Theorem 4.5 answers in the negative—this Chapter considers a broader role for randomization in the design of multi-unit auctions. In particular, I explore the possibility that the seller’s revenues might be improved by randomly selecting whether to employ a pay-as-bid or uniform-price transfer rule subsequent to eliciting bids. I provide computational evidence and a simple example which suggest that, in fact, the pay-as-bid mechanism is revenue-maximizing within the class of such mechanisms.

Considering simple alterations of existing mechanisms rather than pursuing a truly optimal mechanism\(^1\) arises from principles of political transparency. As discussed in Section 1.2 the most visible applications of the pay-as-bid auction format occur in the public sector, where a government agency is the auctioneer. In such cases it is advantageous to provide a clear, auditable mechanism for determining allocations. On this front, the pay-as-bid and uniform-price auction formats are highly desirable: although equilibrium strategies may be quite difficult to compute (see Chapter 3), the mechanisms themselves may be described in terms familiar to any economics undergraduate. It is reasonable then to use these mechanisms as a starting point for further development, and to look for suitable

\(^1\)As mentioned previously, Maskin and Riley (1989) gives an explicit characterization of an optimal multi-unit auction, but it is never seen in practice.
augmentations for improving the seller’s outcomes.

It has been suggested that one such modification might be to randomly implement either the pay-as-bid or uniform-price auction, subsequent to soliciting bids from agents; see, for example, Armantier and Sbaï (2009). Helpfully these two mechanisms are identical in their allocation rules, and differ only in the implied transfers. In this Chapter I assess this possibility by applying the symmetric-valuation model of Chapter 4, explicitly solving an example of behavior in a linear model, and by computationally analyzing comparative statics in a broader class of models. In the former case, I prove that the pay-as-bid auction strictly revenue-dominates any randomization between the pay-as-bid and uniform-price auctions; in the latter case I find strong suggestive evidence that this principle holds more generally.\(^3\)

The plausibility of this approach to revenue maximization is grounded in the results of Crémer and McLean (1988), which find that (in a single-unit auction context) the seller can extract all buyer surplus by implementing lotteries, provided values are correlated. Garratt and Pycia (2015) obtain the obverse of this result, showing that efficient bilateral trade may be implemented through lotteries provided agents have slight risk aversion. Although the analogy is imperfect, decreasing marginal values for quantity imply a form of risk aversion in this model. In spite of bidders having symmetric marginal values—hence degenerately perfectly-correlated types—in this model, correlation in values can arise endogenously in the pay-as-bid auction: if bid shading decreases with quantity, values for marginal units will be similar for all agents. Thus even when signals are independent, the

\(^2\)If the randomization were to occur prior to bid solicitation, it is clear from the results of Chapter 4 that the seller’s revenues would fall below those raised by a pure pay-as-bid auction. The computational results in Section 5.3, which suggest that revenue is monotonically increasing in the pay-as-bid implementation probability, provide further evidence that this is the case.

\(^3\)Modeling a divisible-good auction as a randomization has been pursued by, among others, Viswanathan and Wang (2002) and Wang and Zender (2002). Existing results that take this approach model randomization as a probability \(\alpha\)—as I do here—but constrain attention to the boundary cases \(\alpha \in \{0, 1\}\) which imply a for-certain implementation of one format or the other.
values for the marginal units will be positively correlated.\(^4\) This endogenous correlation might prove sufficient to improve revenue through a randomized mechanism.\(^5\) Gresik (2001) discusses optimal rationing rules in the pay-as-bid auction, which represent the ability to randomize quantity in a highly-correlated region of bidders’ marginal values.

Where the tools of Chapter 4 permitted an explicit characterization of conditions under which a pure-strategy exists, stated in Theorem 4.1, no such claim can be easily made in this Chapter. I am able to find strong suggestive evidence that the computed strategies represent equilibrium best responses, but an explicit determination is not directly feasible. A surprising result arises from the process of narrowing the set of equilibria which might exist in a particular randomized mechanism: I show that the set of permissible initial conditions to the equilibrium price function is monotonically shrinking as the implementation moves continuously from pure uniform-price to pure pay-as-bid. This provides a helpful characterization of the multiplicity of equilibria which exist in the uniform-price auction, as compared to the unique equilibrium which exists in the pay-as-bid auction (Theorem 4.2).

Lastly, in spite of the possibility that the strategies computed do not constitute equilibrium best responses, the tools of Chapter 4—and especially those developed in Appendix C—imply that the bid functions I compute represent upper bounds on the bids submitted in any symmetric, pure-strategy equilibrium; I verify this fact in Section 5.3. Since I constrain attention to models in which the sufficient conditions for the existence of a pure-strategy equilibrium in the pure pay-as-bid auction, given in Theorem 4.1, obtain, the revenue dominance of the pure pay-

\(^4\)Obviously this argument carries more weight when agents have private information, as in Chapters 2 and 3. Without private information, the seller can conceivably implement a perfectly-discriminatory mechanism. The results here are meant to be suggestive, hence arguing from the imperfect-information case is valid.

\(^5\)The lotteries suggested in Crémer and McLean (1988) involve randomization over the quantity allocation. When randomizing between only the pay-as-bid and uniform-price auctions, the quantity allocation is fixed.
as-bid auction versus the upper bound of revenue in randomized implementations implies that the pay-as-bid auction revenue-dominates all possible symmetric, pure-strategy equilibria of the randomized auctions.

5.1 Model

There are \( n \geq 2 \) bidders, \( i \in \{1, \ldots, n\} \). Each bidder has a symmetric marginal valuation for quantity \( q \) denoted \( v^i(q) = v(q) \). I assume that \( v \) is decreasing in \( q \).

The available market quantity \( Q \) is stochastic and drawn from a non-degenerate distribution \( F \) with density \( f \) and support \([0, \bar{Q}]\). All bidders are risk neutral.

Upon soliciting bid functions, the seller implements the pay-as-bid outcome with probability \( \alpha \in [0, 1] \); with probability \( 1 - \alpha \) the seller implements the uniform-price outcome. Note that the seller must have the ability to commit to this randomization scheme: ex post it is always optimal to implement the pay-as-bid outcome—the quantity allocations are identical, while the seller necessarily receives greater payment in the pay-as-bid scheme. Hence without a commitment device a rational buyer will assume that the pay-as-bid outcome will be implemented, and will bid accordingly.

5.1.1 Optimality conditions

Due to the possibility of the uniform-price transfer rule being implemented, a bidder’s expected utility in this model differs somewhat from those in the pure pay-as-bid auction; this small difference in the form of expected utility has substantial implications for the nature of a bidder’s optimality conditions. Conditional on the bid she submits, the agent’s expected utility is

\[
U^i \left( b^i, b^{-i} \right) = \mathbb{E}_{q_i} \left[ \int_0^{q_i} v(x) - \alpha b^i(x) \, dx - (1 - \alpha) q_i b^i(q_i) \right].
\]
This is equation (1.1), modified to eliminate private information, and to account for the \((1 - \alpha)\) probability that the uniform-price transfer rule is implemented.

The agent's optimization problem is then

\[
\max_b \mathbb{E}_q \left[ \int_0^q v(x) - \alpha b(q) \, dx - (1 - \alpha) q_i b(q_i) \right].
\]

Integrating by parts as in Chapter 3 gives

\[
\max_b \int_0^Q (v(q) - \alpha b(q) - (1 - \alpha) (b(q) + q b'(q))) (1 - G^i(q; b)) \, dq.
\]

The calculus of variations implies

\[
- (v(q) - b^i(q) - (1 - \alpha) q_i b^i(q)) G^i_b(q; b^i) - (1 - G^i(q; b^i))
\]

\[
= \frac{d}{dq} \left[- (1 - \alpha) q (1 - G^i(q; b^i)) \right].
\]

This in turn simplifies to

\[
b^i(q) = v(q) + \frac{\alpha (1 - G^i(q; b^i)) + (1 - \alpha) q G^i_b(q; b^i)}{G^i_b(q; b^i)}. \tag{5.1}
\]

In what follows, I will consider only symmetric equilibrium strategies, \(b^i = b\).

Thus in equilibrium \(Q = nq, b(q) = p(Q)\), and \(\varphi_p(p) = 1/n p_Q(Q)\); substituting in for \(G\), the agent’s first-order condition becomes

\[
p(Q) = \hat{v}(Q) + \left( \frac{\alpha (1 - F(Q))}{f(Q)} + \frac{1 - \alpha}{n} Q \right) \left( \frac{n}{n - 1} \right) p_Q(Q).
\]

As in Chapter 4, this may be represented simply in differential form,

\[
p_Q(Q) = (p(Q) - \hat{v}(Q)) \tilde{H}(Q),
\]

\[
\tilde{H}(Q) = (n - 1) \left( \frac{f(Q)}{n \alpha (1 - F(Q)) + (1 - \alpha) Q f(Q)} \right). \tag{5.2}
\]

Note that the function \(\tilde{H}\) is independent of the particular solution to the market-price equation. The differential system in (5.2) has a solution,

\[
p(Q) = \left( C - \int_0^Q \exp \left( - \int_0^x \tilde{H}(y) \, dy \right) \tilde{H}(x) \hat{v}(x) \, dx \right) \exp \left( \int_0^Q \tilde{H}(x) \, dx \right). \tag{5.3}
\]
Unlike Chapter 4, a simple closed-form solution to this equation is not readily available in the generic case, and prices will need to be determined computationally.\textsuperscript{6} Since $\tilde{H} \geq 0$ and $p \leq \hat{v}$, it is clear that any solution to the market price equation is decreasing in quantity.

### 5.2 Linear marginal values

To enable a simple characterization and expose the underpinnings of the revenue-maximization problem, I first examine what occurs in a fully-linear model. Suppose that quantity $Q \sim U(0, \bar{Q})$ and marginal values are $v(q) = v_0 - q\nu_q$.

Further assume that there are at least three bidders, $n \geq 3$, and that marginal valuations are sufficiently high,

$$v \left( \left( \frac{n-1}{n-2} \right) \frac{1}{n} \bar{Q} \right) > 0.$$  

Since this Chapter constrains attention to symmetric equilibria, where the agent’s maximum allocation is $\bar{Q}/n$, this second assumption roughly requires that there is strictly positive demand at this maximum allocation. These assumptions are sufficient, but not necessary, to ensure that the solution obtained is an equilibrium.\textsuperscript{7}

I posit and solve for a linear equilibrium, $p(Q) = p_0 - Qp_Q$. In this equilibrium, the market-clearing price equation is

$$p_0 - Qp_Q = v_0 - \frac{1}{n}Q\nu_q - \left( \frac{1}{n-1} \right) \left( n\alpha \left( \bar{Q} - Q \right) + (1 - \alpha)Q \right) p_Q.$$  

\textsuperscript{6}It can be shown that $\tilde{H}(Q) = d\ln\left[n\alpha(1 - F(Q)) + (1 - \alpha)Qf(Q)\right]$ only when $Q \sim U(0, \bar{Q})$. Hence the market price equation may only be solved using the methods of Chapter 4 when the market quantity is uniformly distributed. See Section 5.2.

\textsuperscript{7}That $n \geq 3$ is required relates to the fact that bid-shading at the maximum allocation is maximized in the uniform-price auction, $\alpha = 0$. It is known that the uniform-price auction may not admit an equilibrium with only two bidders, each of whom has decreasing marginal values.
This implies a pair of equations for the coefficients $p_0$ and $p_Q$:

$$p_0 = \nu_0 - \left(\frac{n\alpha}{n-1}\right)\bar{Q} p_Q,$$

$$-p_Q = -\frac{1}{n}\nu_q - \left(-\frac{n\alpha + (1 - \alpha)}{n-1}\right) p_Q.$$

It follows that

$$p_Q = \left(\frac{n-1}{n}\right)\left(\frac{1}{(n-2) + (n+1)\alpha}\right)\nu_q,$$

$$p_0 = \nu_0 - \left(\frac{\alpha\bar{Q}}{(n-2) + (n+1)\alpha}\right)\nu_q.$$  \hspace{1em} (5.4)

Thus the market price is given by

$$p(Q) = \nu_0 - \left(\frac{n-1}{n}\right)\left(\frac{1}{(n-2) + (n+1)\alpha}\right)\nu_q.$$

5.2.1 Equilibrium verification

I verify two features of this equilibrium price function—and its implied strategies—that suggest that it is indeed an equilibrium.

*Price is positive.* Because price is decreasing in quantity, it suffices to verify that $p(Q) > 0$, or

$$\nu_0 \geq \left(\alpha + \left(\frac{n-1}{n}\right)\right)\left(\frac{1}{(n-2) + (n+1)\alpha}\right)\bar{Q}\nu_q.$$  \hspace{1em} (5.5)

For this inequality to hold for all $\alpha$, it is sufficient show that it holds at the minimizing $\alpha$. To determine which $\alpha$ generates the highest value of the right-hand side, the first derivative gives

$$\frac{d}{d\alpha} \left[\left(\frac{1}{(n-2) + (n+1)\alpha}\right) - \left(\alpha + \left(\frac{n-1}{n}\right)\right)\left(\frac{n+1}{((n-2) + (n+1)\alpha)^2}\right)\bar{Q}\nu_q\right].$$

Because only the sign of the derivative is relevant to maximizing behavior, note that the above expression is signed as

$$\left((n-2) + (n+1)\alpha\right) - \left(\alpha + \left(\frac{n-1}{n}\right)\right)(n+1) = \frac{-2n-1}{n}.$$
The derivative is therefore always negative, hence the right-hand side of inequality (5.5) will be maximized when $\alpha = 0$. When this is the case, the comparison is

$$\nu_0 \geq \left( \frac{n - 1}{n} \right) \left( \frac{1}{n - 2} \right) \frac{Q}{n} \nu_q = \left( \frac{n - 1}{n - 2} \right) \frac{1}{n} \frac{Q}{n} \nu_q.$$ 

Since $v(q) = \nu_0 - q \nu_q$, the price is always positive so long as

$$v \left( \left( \frac{n - 1}{n - 2} \right) \frac{1}{n} \frac{Q}{n} \nu_q \right) \geq 0.$$ 

*Second-order calculus of variations conditions are satisfied.* To ensure that the linear solution is a strong minimizer of the agent’s variational problem, three conditions must be checked. The first two, the Legendre condition and the Weierstrass condition, hold for the same reasons given in Appendix C. Letting $L$ be the integrand in the agent’s objective, the third condition for a strong minimizer—the Jacobi condition—requires that

$$L_{bb} (q, b, b') \neq \frac{d}{dt} L_{bb'} (q, b, b').$$ 

Substituting in for the integrand, this becomes

$$2G_b - (v - b - (1 - \alpha) q b') G_{bb} \neq (1 - \alpha) (G_b + q G_{bq}).$$ 

Replacing $G$ and its derivatives with terms of $F$ and $p$ and their derivatives yields

$$(1 + \alpha) \left( \frac{n - 1}{n} \right) f \frac{1}{pQ}$$

$$\neq (v - p) \left( \left( \frac{n - 1}{n} \right)^2 \frac{1}{pQ} f_Q - \left( \frac{n - 1}{n} \right) \frac{pQ}{pQ} f \right)$$

$$+ (1 - \alpha) Q \left( \frac{n - 1}{n} \right) f_Q \frac{1}{npQ}.$$ 

In the linear solution of the linear model, $pQ = 0$ and $f_Q = 0$, hence the preceding inequality reduces to

$$(1 + \alpha) \left( \frac{n - 1}{n} \right) f (Q) \left( \frac{1}{pQ (Q)} \right) \neq 0.$$ 

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This is trivially satisfied.

Bids are therefore everywhere positive and, for each agent, the computed bid function is a strong maximizer with regard to the utility-maximization problem. The analysis of the linear model then continues under the presumption that bids constitute an equilibrium.\textsuperscript{8,9}

5.2.2 Maximizing revenue

Given a market-clearing price \( p \), the expected revenue generated by the randomized mechanism will be

\[
E_Q [\pi] = E_Q \left[ \alpha \int_0^Q p(x) \, dx + (1 - \alpha) Q \, p(Q) \right].
\]

This expression neatly captures the tradeoffs inherent in randomizing the mechanism: although the uniform-price auction encourages stronger bids—agents are less likely to pay any given bid and are therefore more willing to bid more—implementing the uniform-price auction by lowering parameter \( \alpha \) implies sacrificing some rents found only in the integrand. It turns out that:

**Proposition 5.1** (Revenue maximization). *In the putative linear equilibrium of the linear-uniform model, the pure pay-as-bid auction is expected-revenue-maximizing within the class of mechanisms which randomize over uniform-price and pay-as-bid outcomes.*

Proposition 5.1 formalizes the dominance of the pay-as-bid auction. The proof of this Proposition is given in Appendix D.

\textsuperscript{8}That bids were an equilibrium in Chapter 4 resulted from the uniqueness of the market-clearing price. Since uniqueness in this case is not guaranteed, strong maximization hints that the linear solution may be an equilibrium, but also permits there to be a global maximizer which differs from this solution.

\textsuperscript{9}The linear solution also satisfies the necessary condition later developed in Lemma 5.1.
5.3 Numerical results

The system of market-clearing equations, defined in equation (5.2) may be solved numerically, given a value for $C$. For any given $\alpha$, I parameterize the model so that $p(\bar{Q}) = v(\bar{Q}/n)$, giving an explicit value for $C$ as

$$C = \exp \left( - \int_0^{\bar{Q}} \hat{H}(x) \, dx \right) v \left( \frac{1}{n} \bar{Q} \right) + \int_0^{\bar{Q}} \exp \left( - \int_0^x \hat{H}(y) \, dy \right) \hat{H}(x) \hat{v}(x) \, dx.$$  

I show below that this parameterization will provide an upper bound on the revenue generated by any randomized mechanism parameterized by $\alpha$.

Appendix D.2 provides a number of Lemmas useful to establishing this upper bound on seller revenue. In particular, solutions to the market-price equation are well-behaved in the expected way: solutions must be continuous and they cannot cross. Additionally, bidders will never bid above their marginal values for quantities that they obtain with positive probability.

**Proposition 5.2** (Upper bound on revenue). *The expected revenue generated in a symmetric equilibrium will never exceed that of the solution to the market-clearing price in which $p(\bar{Q}) = v(\bar{Q}/n)$.*

*Proof.* Any symmetric equilibrium must solve the equilibrium market-price equation almost everywhere. Lemma D.1 establishes that, in fact, it must be solved everywhere. Lemma D.2 says that equilibrium market-price functions can be well-ordered, and Lemma D.3 establishes that this ordering may be reduced to the scalar $C$. Lemma D.4 gives an upper bound on $C$, and the result follows. $\square$

Notice that the uniqueness result of Chapter 4 implies that, in fact, the maximum expected revenue in the pure pay-as-bid mechanism is the unique expected revenue generated by any pure-strategy equilibrium of the mechanism. Hence any comparison that suggests that this upper bound of the pure pay-as-bid mechanism expected-revenue dominates an upper bound of a randomized mechanism implies
that the pure pay-as-bid mechanism expected-revenue dominates all equilibria of
the randomized mechanism.\textsuperscript{10}

5.3.1 Simulations

I now consider three models of marginal value, and three distributions of market
quantity. Each parameterization is set to ensure that an equilibrium exists in the
pure pay-as-bid auction, according to the criterion established by Theorem 4.1.

The marginal values I consider are

\[
\begin{align*}
v^1 (q) &= \nu_1 - q\gamma_1, \\
v^2 (q) &= \nu_2 \exp (-q\gamma_2), \\
v^3 (q) &= \sqrt{1 - \gamma_3 q} + \nu_3.
\end{align*}
\]

Thus I examine marginal values which are, respectively, linear, convex, and con-
cave.

The supply distributions I consider each have support \([0, Q]\), and are

\[
\begin{align*}
F^1 (Q) &= \frac{Q}{Q}, \\
F^2 (Q) &= \left( \exp (\lambda Q) - 1 \right) / \left( \exp (\lambda Q) - 1 \right), \\
F^3 (Q) &= \left( \Phi \left( \frac{Q - \mu}{\sigma} \right) - \Phi \left( \frac{-\mu}{\sigma} \right) \right) / \left( \Phi \left( \frac{Q - \mu}{\sigma} \right) - \Phi \left( \frac{-\mu}{\sigma} \right) \right).
\end{align*}
\]

Distribution \(F^1\) is uniform, distribution \(F^2\) is truncated exponential with a bias
toward higher quantities, and distribution \(F^3\) is truncated normal.

Figure 5.1 displays computed price functions in the considered models. Fig-
ure 5.2 displays expected revenue curves as functions of \(\alpha\), for varying parame-
terizations of the underlying models. The specific parameterizations of the two
Figures are given in Appendix D.

\textsuperscript{10}Proposition 5.2 implies that the common bound on uniform-price revenues, established by
assuming that agents report truthfully (see, e.g., Hortaçsu and McAdams, 2010), may dramati-
cally overestimate the expected revenue generated by the mechanism.
Figure 5.1: Market-clearing prices; the plot in row $i$, column $j$ is parameterized with distribution $F^i$ and value $v^j$. Prices fade from orange to gold as $\alpha$ goes from 1 (pure pay-as-bid) to 0 (pure uniform-price); generally, pay-as-bid prices are lower for low $Q$ and higher for high $Q$. 
Figure 5.2: Increase of expected revenue over minimum (pure uniform-price) revenue. Column \( j \) is parameterized by value \( v^j \), and supply is distributed according to \( F^2 \); row 1 shows variation in \( n \), row 2 shows variation in \((v_j, \gamma_j)\), and row 3 shows variation in \( \lambda \). Color represents the average revenue generated in the parameterization, taken from a uniform distribution on \( \alpha \): blue represents the lowest average revenue, fading to pink which represents the highest.
5.3.2 Comparative statics

The plots in Figures 5.1 and 5.2 suggest some intuitive comparative statics. In each of the following implied rules, the variation is smooth in the pay-as-bid implementation probability $\alpha$ as it varies from 0 to 1.

**Bid monotonicity with respect to pay-as-bid implementation probability.** For low quantities $Q$, bids are falling in the probability that the pay-as-bid auction is implemented. In the pay-as-bid auction, there is near certainty that these bids will be paid, hence there is a strong incentive to shade below true values. In the uniform-price auction, truthful reporting of $b(0) = v(0)$ is necessary in equilibrium.

Conversely, for relatively high quantities $Q$ near $Q$, bids are higher in the pay-as-bid auction. This arises from the margins defining incentives in each auction. In the pay-as-bid auction, there is a relatively low probability of making this payment, hence bidding higher is not disincentivized. In the uniform-price auction the probability of making the payment is negligible—marginally equivalent to the actual width of the bid for the quantity in the pay-as-bid auction—but the marginal payment is now made for all units $Q$. The expected cost of bidding is then, in some sense, increasing in $Q$ in the uniform-price auction but decreasing in $Q$ in the pay-as-bid auction.

**As demand increases, the mechanisms become revenue-equivalent.** Increased demand is illustrated in the row 1 of Figure 5.2 as an increase in the number of bidders $n$, holding constant the distribution of quantity. As $n$ increases, bids in each pure auction format approach truthful reporting; since per-capita quantity is decreasing in these plots, at the limit ($n = +\infty$) the unique equilibrium in either auction has truthful reporting and all surplus going to the seller.

The comparative plots for increasing numbers of bidders suggest that this limiting revenue equivalence is approached smoothly, and across all randomizations
between the mechanisms.\footnote{Swinkels (2001) finds that both auctions are efficient and revenue-equivalent in large markets with indivisible units.}

As marginal utilities become more sensitive to quantity, the pay-as-bid auction gains revenue relative to the uniform-price auction. Value sensitivity to quantity is captured in the parameters $\gamma_\ell$. For large $\gamma_\ell$, value is relatively sensitive to quantity, and row 2 of Figure 5.2 shows that the revenue differential between pay-as-bid and uniform-price is higher the more sensitive is demand to quantity.

A separate measure of sensitivity to quantity is reflected in the relative change in marginal value, captured by the parameters $\nu_\ell$; as verification, note that the revenue curves are constant in $\nu_2$ for each $\gamma_2$. A larger $\nu_\ell$ implies weakly smaller marginal utility ratios of small quantities to large quantities, or a reduced sensitivity to quantity. The plots in row 2 of Figure 5.2 show that this sensitivity intuition generates the same comparative statics as that generated by variation in $\gamma_\ell$: increasing value-sensitivity to quantity implies that the pay-as-bid auction becomes more dominant.

As supply is biased rightward, the performance of the pay-as-bid auction increases. This is captured in row 3 of Figure 5.2 as an increase in parameter $\lambda$ of the (inverse) exponential distribution generated by $F^2$. When higher quantities are attained with higher probability, the likelihood of relatively complete price discrimination—against reported demand—increases in the pay-as-bid auction, thus bids fall. In the uniform-price auction, the probability of obtaining anything other than monopolistic profits is going to zero. These two features balance in favor of the pay-as-bid auction.\footnote{The dramatic extent to which the pay-as-bid auction outperforms the uniform-price auction—according to Figure 5.2, up to 50%—results from the fact that value functions $v^1$ and $v^3$ are parameterized so that $v'(Q/n) = 0$, $\ell \in \{1, 3\}$. When the marginal value on the last unit is raised, the revenue difference dissipates but does not disappear; compare the plot of $v^2$ against variations in $\lambda$ for an illustration. Footnote 20 in Chapter 3 discusses a related issue.}
5.4 Equilibrium multiplicity

In comparison to the uniqueness result of Chapter 4, Back and Zender (1993) establish that the uniform-price mechanism can sustain a broad multiplicity of equilibria. I now use the market-clearing price function $p$ and the Lemmas developed in Appendix D to provide suggestive evidence of equilibrium existence to establish bounds on the variety of equilibria which may be sustained by an $\alpha$-randomized mechanism.

The argument proceeds by analyzing a particular condition for best-response bid functions to be undominated. To be clear, I consider only small upward deviations near $\bar{Q}/n$. This is, locally, a more powerful tool than the Jacobi equation from the calculus of variations, which requires only that a certain function lack zeros; here, I gain leverage from understanding the sign of a particular function of the market price and model fundamentals. The bounds on this endpoint incentive criterion place natural limits on the set of permissible equilibrium market-clearing price functions. These sets, taken as functions of $\alpha$, are then shown to be well-behaved and monotone in inclusivity.

**Lemma 5.1** (Equilibrium necessary condition). A necessary condition for a market price function $p$ to represent an equilibrium is

$$
\frac{1 - 2\alpha}{1 - \alpha} \left( p(\bar{Q}) - \hat{v}(\bar{Q}) \right) + \bar{Q}\hat{v}_Q(\bar{Q}) \leq 0.
$$

The proofs for Lemma 5.1 and the following results appear in Section D.3. Note that Lemma 5.1 places only a very weak restriction on the equilibrium price function. It does not guarantee that a particular solution is an equilibrium, only that the response cannot be locally improved-upon from deviations near $\bar{Q}/n$. Nonetheless, it will be sufficient to describe a form of monotonicity on the set of equilibrium bid functions.
Lemma 5.2 ($\alpha$-monotonicity of necessary condition). Suppose that $p$ is a solution to the equilibrium market-price equation for randomization $\alpha$, and satisfies the necessary condition of Lemma 5.1. Then for any $\alpha' < \alpha$, there is a solution to the equilibrium market-price equation $p'$ with $p'(\overline{Q}) = p(\overline{Q})$ which satisfies the necessary condition of Lemma 5.1.

Now, given $\alpha$, let $P(\alpha)$ be the set of endpoints to valid solutions to the market-clearing equation, so that price is everywhere-positive, its implied bids are below the agents’ value functions, and satisfies the inequality of Lemma 5.1. This construction provides the following result.

Theorem 5.1 (Decreasing multiplicity). $P$ is ordered by reverse-inclusion: $\alpha' < \alpha$ implies $P(\alpha') \supseteq P(\alpha)$.

Theorem 5.1 is a direct consequence of Lemma 5.2. Thus the set of permissible market-price functions, according to endpoint conditions, is increasing as $\alpha$ falls, in the sense that the set of available endpoint prices is growing. A stronger claim is not immediately feasible, as $\alpha$ fundamentally alters the shape of the market-price equation.

The inclusion-monotonicity of $P$ with respect to $\alpha$ provides evidence that the multiplicity of equilibria in the uniform-price auction and the uniqueness of equilibrium in the pay-as-bid auction are not knife-edge cases obtained by mechanism degeneracy. Indeed, the set of permissible initial conditions for the market-clearing price function is shrinking smoothly as the mechanism moves from uniform-price to pay-as-bid.

5.5 Conclusion

In this Chapter I consider a model of random mechanism selection, where bidders submit bids and the seller then randomly selects either the pay-as-bid or
uniform-price transfer rule. In a simplified linear model, I show that the revenue-maximizing mechanism is a pure pay-as-bid implementation. This fact generalizes to a broad set of models which I analyze computationally.

Although this Chapter does not explicitly establish the existence of an equilibrium, or that the computed strategies represent equilibrium best responses, I present results which suggest that the computed bid functions satisfy standard second-order requirements. Additionally, the computed strategies are shown to represent upper bounds on the revenue obtained in any pure-strategy equilibrium, hence given the uniqueness result of Chapter 4 all revenue comparisons to the pay-as-bid auction are well-founded.

The analysis of existence conditions leads to Theorem 5.1, which establishes that, under a necessary condition for equilibrium existence, the set of plausible equilibria is monotonically shrinking as the implemented auction moves from pure uniform-price to pure pay-as-bid. This suggests that an auctioneer looking to reduce potential low-revenue “collusive-seeming” outcomes in the uniform-price auction might consider randomly implementing the pay-as-bid auction with small probability.
CHAPTER 6

Conclusion

The pay-as-bid auction, although frequently implemented in practice, is subject to a common misconception that the incentives it presents for strategic underbidding outweigh the benefits to the seller which arise from perfect discrimination according to reported demand. To a large extent, this misconception arises from a faulty application of intuition from single-unit auctions, which suggests that the pay-as-bid auction’s chief competitor, the uniform-price auction, is strategyproof (Binmore and Swierzbinski, 2000). Because of the complexity of analyzing best-response behavior in the mechanism, it has been relatively poorly understood.

In this dissertation I have provided evidence in favor of the pay-as-bid auction. Chapter 2 opens and establishes that the divisible-good model of the pay-as-bid auction admits a solution in the presence of private information. Although computing equilibrium strategies remains, to a large extent, an open problem, that there is an equilibrium in pure strategies puts the pay-as-bid auction on equal footing with the uniform-price auction. To this end, complexity is no longer an argument in favor of the uniform-price format over the pay-as-bid format.

Chapter 3 builds on this existence result and demonstrates how the pay-as-bid auction is distinct from a generalized first-price auction: there are points at which bidders’ incentives are distinctly nonlocal to the unit they are bidding for, rendering irrelevant any direct application of first-price intuition. The necessity of these nonlocal incentives is shown to imply not only dramatic bid flattening for small quantities, but potentially substantial bid reduction across the domains
of the submitted bid functions. While bid reduction, on its face, may appear to be an argument against the pay-as-bid auction, ultimately the opposite is the case. The existing empirical literature has found a rough revenue equivalence—properly, no statistically significant difference—between pay-as-bid and uniform-price auctions. Because natural experiments are not feasible in most multi-unit auction contexts, counterfactual methods are used to establish implicit revenue rankings. Two possibilities arise: first, bidders may already fully-account for strategic ironing, in which case the results of Chapter 3 are agnostic on the revenue difference between uniform-price and pay-as-bid auctions. Or, bidders may not account for strategic ironing, in which case the results of Chapter 3 imply that the pay-as-bid auction may be unexpectedly revenue-dominant. This arises in part from the fact that counterfactual revenue computations have been constrained, thus far, to infer upper bounds on uniform-price outcomes from observed pay-as-bid auctions. When bidders do not account for strategic ironing, this implies a doubly-upward bias in favor of the uniform-price auction: empirical results compare observed pay-as-bid revenue to an upper bound of uniform-price revenue, and bidders who submit bids which are too high will have inferred values which are above their true values. Compounded, it is plausible that existing empirical work has consequentially overstated the revenues of the uniform-price auction and biased policy proposals against the pay-as-bid auction.

In Chapter 4 I switch from a private information paradigm to a completely symmetric paradigm where supply is random. The results in Chapter 4 establish that the pay-as-bid auction in this context is well-behaved with respect to empirical strategies: it admits an easily-verifiable equilibrium existence condition, and submitted bids are uniquely determined by model parameters. These results are used to explore design properties of optimal pay-as-bid auctions, and I find that, optimally, the seller would prefer the pay-as-bid auction to operate as a deterministic posted-price mechanism with monopolistic supply. The optimized mechanism
is revenue-equivalent to the optimal mechanism when the seller is constrained to implement a variant of the uniform-price auction, except that in a uniform-price auction the seller must also use a reserve price, while in the pay-as-bid auction the seller need only adjust the distribution of supply. This provides suggestive evidence in favor of the pay-as-bid auction: if the reserve price determined even slightly suboptimally, the pay-as-bid auction will revenue-dominate the uniform-price auction.

This analysis is furthered in Chapter 5, in which I consider the possibility that the seller randomizes the transfer rule by committing to stochastically implement either the pay-as-bid or uniform-price auction subsequent to bid submission. In a theoretical result in a simplified linear model, as well as in computational results on a broad class of models of bidder demand and the distribution of supply, I find that the seller’s revenue is monotonically increasing in the probability that the pay-as-bid auction is implemented. Additionally, the set of allowable equilibrium market price functions is monotonically shrinking in the probability that the pay-as-bid auction is implemented. This suggests a novel argument in favor of the pay-as-bid auction: not only is it revenue-dominant, but it may smoothly address the equilibrium multiplicity problem which is known to befall the uniform-price auction. Thus a seller who is generally committed to a uniform-price scheme might stochastically implement a pay-as-bid auction with small probability in order to discourage the low-revenue outcomes which can arise in the uniform-price auction.

Altogether, these results provide novel evidence in favor of the pay-as-bid auction: it is well-behaved and potentially tractable, its strategic complexity may have caused previous empirical studies to overstate the revenue generated by counterfactual uniform-price auctions, when nearly-optimally parameterized it subjects the seller to less revenue risk than does the uniform-price auction, it is smoothly revenue-dominant as the implemented auction approaches a pure pay-as-bid format, and small stochastic doses can potentially help sellers avoid low-revenue,
seemingly-collusive equilibria in the uniform-price auction. It stands that this misunderstood auction format merits further considerations on its theoretical and empirical merits.

May 15, 2015
APPENDIX A

Appendix to Chapter 2

A.1 Proof of Lemma 2.1: Best-response monotonicity

Prior to proving Lemma 2.1, I establish a set of auxiliary results which simplify the analysis.

**Lemma A.1** (Stochastic ordering). Suppose that \( b \) and \( b' \) are bid functions, and let \( \bar{b} = b' \lor b \) and \( \underline{b} = b' \land b \). The distribution of allocated quantity conditional on the bids submitted by other agents, \( G^i(\cdot; \cdot) \), satisfies \( G^i(\cdot; \bar{b}) \succeq_{FOSD} G^i(\cdot; b) \) and \( G^i(\cdot; \underline{b}) \succeq_{FOSD} G^i(\cdot; b') \), and \( G^i(\cdot; b) \succeq_{FOSD} G^i(\cdot; b') \) and \( G^i(\cdot; \bar{b}) \succeq_{FOSD} G^i(\cdot; b) \).

**Proof.** I demonstrate only the first inequality, and the rest follow similarly. Let \( S_{-i} \) be defined as

\[
S_{-i}(q; b'') = \{ s_{-i} : q_i(s_{-i}; b'') \leq q \}.
\]

Since \( \bar{b} \geq b \) and bids are strictly decreasing, for any \( s_{-i} \in S_{-i}(q; \bar{b}) \) it must be that \( s_{-i} \in S_{-i}(q; b) \); relatively speaking, were bidder \( i \) to submit \( \bar{b} \) rather than \( b \) the stop-out price would rise and \( q_{-i} \) would fall. It follows that \( S_{-i}(q; \bar{b}) \subseteq S_{-i}(q; b) \) for all \( q \). With \( G^i(q; b'') = \Pr(s_{-i} \in S_{-i}(q; b'')) \), it follows that \( G^i(q; \bar{b}) \leq G^i(q; b) \) for all \( q \), and hence \( G^i(\cdot; \bar{b}) \succeq_{FOSD} G^i(\cdot; b) \).

It is useful to introduce some notation in advance of further analysis. Suppose that \( b' \) is a putative best-response bid function for bidder \( i \), and that \( b \) is some other bid function. Let \( D^+_i(x) = [x, x + \varepsilon) \) and \( D^-_i(x) = (x - \varepsilon, x] \). Let \( X' = \{ q : b'(q) = b(q) \} \cup \{0, Q\} \), and let \( X = \text{Cl}X' \setminus \text{IntCl}X' \) be the set of boundaries of
intersections. There are \( x_L, x_R \in X, x_L < x_R \), such that there is no \( x \in X \) with \( x_L < x < x_R \). Taking any such \( x_L, x_R \), define \( \bar{b} \) and \( \underline{b} \) as

\[
\bar{b}(q) = \begin{cases} 
\max \{ b(q), b'(q) \} & \text{if } q \in [x_L, x_R], \\
\quad b'(q) & \text{otherwise};
\end{cases}
\]

\[
\underline{b}(q) = \begin{cases} 
\min \{ b(q), b'(q) \} & \text{if } q \in [x_L, x_R], \\
\quad b(q) & \text{otherwise}.
\end{cases}
\]

**Lemma A.2 (Incentive inequalities).** Suppose that \( b^i \) is a best-response bidding strategy, and let \( b = b^i(\cdot; s) \) and \( b' = b^i(\cdot; s') \), \( s' \neq s \). The following inequalities must be valid:

\[
\mathbb{E}_q \left[ \int_0^{q_i} v(x; s') - b(x) \, dx \right] \left| \bar{b} \right| \leq \mathbb{E}_q \left[ \int_0^{q_i} v(x; s') - b(x) \, dx \right] \left| b \right| \leq \mathbb{E}_q \left[ \int_0^{q_i} v(x; s') - b'(x) \, dx \right] \left| b' \right|. 
\]

**Proof.** To economize on notation, let \( v^i(\cdot) = v^i(\cdot; s') \), \( G(\cdot) = G^i(\cdot; b) \), \( G(\cdot) = G^i(\cdot; b') \), and \( \mathcal{G}(\cdot) = \mathcal{G}^i(\cdot; \bar{b}) \); further, I will drop the arguments from functions where no confusion is likely to occur. The desired result is therefore

\[
\mathbb{E}_q \left[ \int v' - b' \, dx \left| b \right| \right] \leq \mathbb{E}_q \left[ \int v' - b' \, dx \right] \leq \mathbb{E}_q \left[ \int v' - b' \, dx \left| b' \right| \right]. 
\]

By incentive compatibility, it must be that \( \mathbb{E}_q [\int v' - b \, dx | b] \leq \mathbb{E}_q [\int v' - b' \, dx | b'] \) and \( \mathbb{E}_q [\int v' - b \, dx | b] \leq \mathbb{E}_q [\int v' - b' \, dx | b'] \). Thus it is sufficient to show

\[
\mathbb{E}_q \left[ \int v' - b \, dx \right] \leq \mathbb{E}_q \left[ \int v' - b \, dx \right]. \tag{A.1}
\]

If \( b(q) \leq b'(q) \) for all \( q \in [x_L, x_R] \), then \( b = b \) and the proof is trivial. Let \( x_L, x_R \) be as above and assume that \( b(q) \geq b'(q) \) for all \( q \in [x_L, x_R] \). I first perform some
helpful manipulation of inequality (A.1),
\[
\mathbb{E}_{q_i} \left[ \int v' - b dx \right] = \int \int v' - b dx dG \\
= \int \int v' - b dx dG + \int b - b dx dG \\
= \int_{b=b} v' - b dx dG + \int_{b>b} v' - b dx dG' + \int b - b dx dG \\
= \mathbb{E}_{q_i} \left[ \int v' - b dx \right] - \int_{b>b} v' - b dx dG \\
+ \int_{b>b} v' - b dx dG' + \int b - b dx dG.
\]
Then the desired result is equivalent to
\[
\int_{b>b} v' - b dx dG - \int_{b>b} v' - b dx dG' \leq \int b - b dx dG.
\]
In turn, this is
\[
\int_{b>b} v' - b dx dG - \int_{b>b} v' - b dx dG' \leq \int_{b=b} v' - b dx dG. \quad (A.2)
\]
Looking at \( \bar{b} \), incentive compatibility gives
\[
\mathbb{E}_{q_i} \left[ \int v' - \bar{b} dx \right] \leq \mathbb{E}_{q_i} \left[ \int v' - b' dx \right].
\]
Following the same manipulations as with respect to \( b, \bar{b} \) gives
\[
\int_{b>b} v' - \bar{b} dx dG - \int_{b>b} v' - b' dx dG' \leq \int_{b=b} v' - b dx dG. \quad (A.3)
\]
To obtain the final result, compare the left-hand sides of (A.2) and (A.3). This comparison is
\[
\int_{b>b} v' - b dx dG - \int_{b>b} v' - b dx dG' \geq \int_{b>b} v' - \bar{b} dx dG - \int_{b>b} v' - b' dx dG'.
\]
This comparison is equivalent to
\[
\int_{b>b} \bar{b} - b dx dG \geq \int_{b>b} b' - b dx dG'.
\]
By construction, these relate as
\[
\left[ \int_0^{XL} b' - bdx \right] \left( G^i(x_R; b) - G^i(x_L; b) \right) \geq \left[ \int_0^{XL} b' - bdx \right] \left( G^i(x_R; b') - G^i(x_L; b') \right).
\]
Since these two sides are equivalent, it must be that inequality (A.3) holds with its left-hand side replaced by the left-hand side of inequality (A.2), which is exactly the desired result. \qed

**Lemma 2.1** (Best-response monotonicity). Let \( \langle b^j \rangle_{j \neq i} \) be the profile of strategies played by agents other than \( i \), and suppose that \( b^i \) is a best-response. Then for almost all relevant quantities, \( b^i \) is weakly monotonic in agent \( i \)'s private information \( s_i \).

**Proof.** Let \( s_i < s'_i \), and define \( b = b^i(\cdot; s) \) and \( b' = b^i(\cdot; s') \). By monotonicity of value functions in signal, incentive compatibility, and Lemma A.2, it follows that
\[
\mathbb{E}_{q_i} \left[ \int v - bdx \bigg| b \right] \leq \mathbb{E}_{q_i} \left[ \int v - bdx \bigg| b' \right] \leq \mathbb{E}_{q_i} \left[ \int v' - bdx \bigg| b' \right] \leq \mathbb{E}_{q_i} \left[ \int v' - bdx \bigg| b \right].
\]
Subtraction gives
\[
\mathbb{E}_{q_i} \left[ \int v' - vdx \bigg| b \right] \leq \mathbb{E}_{q_i} \left[ \int v' - vdx \bigg| b' \right].
\]
If there exists \( q \) such that \( b(q) > b(q) \) and \( G^i(q; b') < 1 \), then Lemma A.1 gives \( G^i(\cdot; b) >_{\text{FOSD}} G^i(\cdot; b') \). With this stochastic ordering, the above inequality cannot hold. Thus it must be that for all \( q \) with \( G^i(q; b') < 1 \), \( b(q) = b(q) \), and hence \( b \leq b' \). \qed

### A.2 Auxiliary results for Theorem 2.1: Equilibrium existence

**Lemma A.3** (Satisfaction of assumptions). For \( \varepsilon > 0 \), \( \mathcal{M}^\varepsilon \) satisfies the assumptions of McAdams (2003).
Proof. I verify each of the numbered assumptions made in McAdams (2003).

Assumption 1. Bids may be safely bounded above by the maximum value for quantity \( q = 0, \overline{v} = \max_j v^j(0;1) \), hence it is safe to consider the action space \( \{0, \varepsilon, \ldots, \overline{v}\}^{Q/\varepsilon} \). As a finite subset of \( \mathbb{R}^{Q/\varepsilon} \) constrained only by monotonicity, the action space is a lattice.

Assumption 2. All players have signal \( s_i \sim U(0,1) \), with common support \([0,1]\) and density bounded and bounded away from zero.

Assumption 3. Ex post payoffs are bounded since values are bounded and bids are constrained to be weakly positive.

Assumption 4. Let \( b \) and \( b' \) be bid functions in model \( \mathcal{M}^\varepsilon \), and let \( \bar{b} \equiv b \land b' \) and \( \overline{b} \equiv b \lor b' \). Quasilinearity requires that

\[
U^i (b', b^{-i}; s) \geq U^i (\bar{b}, b^{-i}; s)
\]

\[
\implies U^i (\overline{b}, b^{-i}; s) \geq U^i (b', b^{-i}; s).
\] (A.4)

It is also necessary to verify that strict inequality in the former implies strict inequality in the latter. This will follow immediately from the weak-inequality derivation results.

Let \( DG^i(q; \hat{b}) = G^i(q; \hat{b}) - \lim_{q' \to q} G^i(q'; \hat{b}) \) be the probability that agent \( i \) receives quantity \( q \) in equilibrium. Note that this is no longer a density, since quantities are discrete. Let \( \bar{b} \equiv b' \land b \). Letting \( Q/\varepsilon = T \), the antecedent utility expression in the quasilinearity requirement (A.4) is

\[
\sum_{t=0}^{T} \int_{0}^{t\varepsilon} v^i(x; s) - b'(x) \, dx \, DG^i(t\varepsilon; b') - \sum_{t=0}^{T} \int_{0}^{t\varepsilon} v^i(x; s) - \bar{b}(x) \, dx \, DG^i(t\varepsilon; \bar{b}) \geq 0.
\]
Rewrite this as
\[
\sum_{t=0}^{T} \int_{0}^{t \varepsilon} v^i(x; s) \, dx \left[ DG^i(t \varepsilon; b') - DG^i(t \varepsilon; \bar{b}) \right]
\geq \sum_{t=0}^{T} \int_{0}^{t \varepsilon} b'(x) \, dx DG^i(t \varepsilon; b') - \int_{0}^{t \varepsilon} b(x) \, dx DG^i(t \varepsilon; \bar{b}) .
\]

Noting that the “order of integration” (in this case, summing then integrating) may be changed, this is
\[
\sum_{t=1}^{T} \int_{(t-1) \varepsilon}^{t \varepsilon} v^i(x; s) \, dx \left[ G^i(t \varepsilon; b) - G^i(t \varepsilon; \bar{b}) \right]
\geq \sum_{t=1}^{T} \int_{(t-1) \varepsilon}^{t \varepsilon} b'(x) \, dx \left[ 1 - G^i(t \varepsilon; b') \right] - \int_{(t-1) \varepsilon}^{t \varepsilon} b(x) \, dx \left[ 1 - G^i(t \varepsilon; b) \right] .
\]

Working in parallel, the same series of steps rewrites the implication of quasipermodularity (A.4) as
\[
\sum_{t=1}^{T} \int_{(t-1) \varepsilon}^{t \varepsilon} v^i(x; s) \, dx \left[ G^i(t \varepsilon; b) - G^i(t \varepsilon; \bar{b}) \right]
\geq \sum_{t=1}^{T} \int_{(t-1) \varepsilon}^{t \varepsilon} \bar{b}(x) \, dx \left[ 1 - G^i(t \varepsilon; \bar{b}) \right] - \int_{(t-1) \varepsilon}^{t \varepsilon} b(x) \, dx \left[ 1 - G^i(t \varepsilon; b) \right] .
\]

Since \( G^i(x; b) - G^i(x; \bar{b}) = G^i(x; b) - G^i(x; b') \) for all \( x \), the left-hand terms of these inequalities are equal. Thus it is sufficient to show
\[
\sum_{t=1}^{T} \int_{(t-1) \varepsilon}^{t \varepsilon} b'(x) \, dx \left[ 1 - G^i(t \varepsilon; b') \right] - \int_{(t-1) \varepsilon}^{t \varepsilon} b(x) \, dx \left[ 1 - G^i(t \varepsilon; b) \right]
\geq \sum_{t=1}^{T} \int_{(t-1) \varepsilon}^{t \varepsilon} \bar{b}(x) \, dx \left[ 1 - G^i(t \varepsilon; \bar{b}) \right] - \int_{(t-1) \varepsilon}^{t \varepsilon} b(x) \, dx \left[ 1 - G^i(t \varepsilon; b) \right] .
\]

Since \( \bar{b}(x) + b(x) = b'(x) + b(x) \) for all \( x \), this simplifies to
\[
\sum_{t=1}^{T} \int_{(t-1) \varepsilon}^{t \varepsilon} \bar{b}(x) \, dx G^i(t \varepsilon; \bar{b}) + \int_{(t-1) \varepsilon}^{t \varepsilon} b(x) \, dx G^i(t \varepsilon; b)
\geq \sum_{t=1}^{T} \int_{(t-1) \varepsilon}^{t \varepsilon} b'(x) \, dx G^i(t \varepsilon; b') + \int_{(t-1) \varepsilon}^{t \varepsilon} b(x) \, dx G^i(t \varepsilon; b) .
\]
Bids are for discrete units and are constrained to be constant on $\varepsilon$-intervals, hence for any valid bid function $f$ it is the case that $\int_{(t-1)\varepsilon}^{t\varepsilon} f(x) dx = \varepsilon f(t\varepsilon)$. Then the above inequality can be written

$$\sum_{t=1}^{T} b(t\varepsilon) G^i(t\varepsilon; b) + \hat{b}(t\varepsilon) G^i(t\varepsilon; \hat{b}) \geq \sum_{t=1}^{T} b'(t\varepsilon) G^i(t\varepsilon; b') + b(t\varepsilon) G^i(t\varepsilon; b).$$

Since at any $t\varepsilon$, either $b = \hat{b}$ and $b' = b$ or $b = \hat{b}$ and $b' = \hat{b}$, this weak inequality holds with equality and weak quasisupermodularity is satisfied. Strict quasisupermodularity can be established in the same manner.

**Assumption 5.** Let $b'$ and $b$ be actions available to agent $i$ in model $M^\varepsilon$, and suppose that $b' > b$ and

$$U^i(b', b^{-i}; s) \geq U^i(b; b^{-i}; s).$$

Let $DG^i(q; \hat{b}) = G^i(q; \hat{b}) - \lim_{q' \downarrow q} G^i(q'; \hat{b})$ be the probability that agent $i$ receives quantity $q$ in equilibrium. Letting $Q/\varepsilon = T$, the utility expression above is

$$\sum_{t=0}^{T} \int_{0}^{te} v^i(x; s) dx DG^i(t\varepsilon; b') \geq \sum_{t=0}^{T} \int_{0}^{te} v^i(x; s) dx DG^i(t\varepsilon; b).$$

This expression is rearranged as

$$\sum_{t=0}^{T} \int_{0}^{te} v^i(x; s) dx (DG^i(t\varepsilon; b') - DG^i(t\varepsilon; b)) \geq \sum_{t=0}^{T} \int_{0}^{te} b'(x) dx DG^i(t\varepsilon; b') - \sum_{t=0}^{T} \int_{0}^{te} b(x) dx DG^i(t\varepsilon; b).$$

Fixing $b'$ and $b$, the right-hand side is constant. Therefore restrict attention to the left-hand side, and (as in the analysis of Assumption 4 above) rewrite it as

$$\sum_{t=0}^{T} \int_{0}^{te} v^i(x; s) dx (DG^i(t\varepsilon; b') - DG^i(t\varepsilon; b)) = \sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{te} v^i(x; s) dx \left[ (1 - G^i(t\varepsilon; b')) - (1 - G^i(t\varepsilon; b)) \right]$$

$$= \sum_{t=1}^{T} \int_{(t-1)\varepsilon}^{te} v^i(x; s) dx \left[ G^i(t\varepsilon; b) - G^i(t\varepsilon; b') \right].$$
When \( b' > b \), Lemma A.1 gives us that \( G_i(q; b) \geq G_i(q; b') \) for all \( q \). Hence the left-hand side is increasing in \( v^i \); it follows that the left-hand side is greater under \( s' > s \), establishing single-crossing. Strict single crossing can be shown in the same manner.

**Lemma A.4** (Upper bound on bids). For any \( \varepsilon > 0 \), in any pure-strategy equilibrium \( \langle b^{i,\varepsilon} \rangle_{i=1}^{n} \) of the \( \varepsilon \)-discrete coarsening \( M^\varepsilon \),

\[
b^{i,\varepsilon} (0; s) \leq \max_j v^j (0; 1) + \varepsilon.
\]

**Proof.** Suppose otherwise. Without loss, assume that \( b^{i,\varepsilon} (0; s) \geq b^{j,\varepsilon} (0; s') \) for all \( j, s' \). Let \( \bar{v} = [\max_j v^j (0; 1)/\varepsilon] \varepsilon \), and let \( \hat{q} = \max \{ q : b^{i,\varepsilon} (q; s) \geq \bar{v} \} \). Suppose that the bidder deviates to

\[
\hat{b}(q) = \begin{cases} 
\bar{v} & \text{if } q \leq \hat{q}, \\
 b^{i,\varepsilon} (q; s) & \text{otherwise.}
\end{cases}
\]

The bidder evidently saves payment whenever her allocation is above \( \hat{q} \). However, she also sacrifices quantity allocations whenever her allocation is weakly below \( \hat{q} \) under \( b^{i,\varepsilon} \). Because bids have only been reduced where they strictly exceed the maximum marginal value for the initial unit, hence where they strictly exceed the marginal value of any unit, the bidder only sacrifices quantities which she was previously obtaining at negative margin, hence the deviation is profitable.

**Lemma A.5** (Utility convergence). In any equilibrium \( \langle b^i \rangle_{i=1}^{n} \) of \( M^\varepsilon \), or of the divisible-good model,

\[
\lim_{s' \uparrow s_i} U^i (b^i (\cdot; s'), b^{-i}; s_i) = \lim_{s' \downarrow s_i} U^i (b^i (\cdot; s'), b^{-i}; s_i).
\]

That is, expected utility converges from both sides.\(^2\)

---

\(^2\)This does not imply expected utility equivalence of the limiting strategies themselves.
Proof. Suppose that there is $\varepsilon > 0$ such that

$$\lim_{s' \uparrow s_i} U^i \left( b^i \left( \cdot s' \right), b^{-i}; s_i \right) + \varepsilon = \lim_{s' \downarrow s_i} U^i \left( b^i \left( \cdot s' \right), b^{-i}; s_i \right).$$

Because $v^i(q; \cdot)$ is continuous and strictly increasing, for any $\delta > 0$ there is $\underline{s} < s_i$ such that

$$U^i \left( b^i \left( \cdot s_i \right), b^{-i}; s_i \right) - \delta, \forall s' > \underline{s}.$$  

Since $b^i(\cdot; s_i)$ is a best response when agent $i$ receives signal $s_i$, it follows that

$$U^i \left( b^i \left( \cdot s_i \right), b^{-i}; s_i \right) > \lim_{s' \uparrow s_i} U^i \left( b^i \left( \cdot s' \right), b^{-i}; s_i \right) + \varepsilon - \delta.$$  

For $\delta$ sufficiently small, this is a contradiction.

The case where $\lim_{s' \uparrow s_i} U^i(b^i(\cdot; s'), b^{-i}; s_i) > \lim_{s' \downarrow s_i} U^i(b^i(\cdot; s'), b^{-i}; s_i)$ is analogous. Hence the stated result holds. \hfill \Box

Lemma A.6 (Utility approximation). Suppose that $\langle b^j \rangle_{j \neq i}$ are strategies played by agents other than agent $i$ in $\mathcal{M}^\varepsilon$, and that $b^i$ is a best response for agent $i$ in $\mathcal{M}^\varepsilon$. Let $\hat{b} : [0, Q] \rightarrow \mathbb{R}^+$ be an unconstrained monotonic bid function with $U^i(\hat{b}, b^{-i}; s) > U^i(b^i, b^{-i}; s)$, and let $f : [0, Q] \rightarrow \mathbb{R}^+$ be any monotonic function that agrees with $b^i$ at available units, $f(t \varepsilon) = b^i(t \varepsilon; s)$. Then

$$U^i \left( \hat{b}, b^{-i}; s \right) - U^i \left( f, b^{-i}; s \right) = O \left( \varepsilon \right).$$

Proof. I first show that, given signal $s$, the ex post utility generated by $\hat{b}$ can be approximated by a constrained bid function $f^\varepsilon$ from $\mathcal{M}^\varepsilon$ so that interim utility $U^i$ is such that $U^i(\hat{b}, b^{-i}; s) - U^i(f^\varepsilon, b^{-i}; s) \leq C \varepsilon$ for some constant $C$.\footnote{This inequality will trivially hold in absolute value: $\hat{b}$ is preferred to $b^i$, but is not feasible. Since $f^\varepsilon$ is feasible and $b^i$ is a best response within $\mathcal{M}^\varepsilon$, $\hat{b}$ generates strictly greater utility than $f^\varepsilon$.} I then show that the worst approximation $\hat{f}$ of $b^i$ yields utility which is lower by at worst $O(\varepsilon)$. The result then follows.
Construct $f^\varepsilon$ as

$$f^\varepsilon(q) = \left\lceil \frac{\hat{b}\left(\left\lfloor \frac{q}{\varepsilon} \right\rfloor \varepsilon \right)}{\varepsilon} \right\rceil \varepsilon.$$ 

That is, $f^\varepsilon$ is the maximum of $\hat{b}$ over any discrete unit, rounded up to the nearest available point on the price grid. By construction, $f^\varepsilon(q) \geq \hat{b}(q)$ for all $q$; hence the gross utility generated by $f^\varepsilon$ weakly exceeds that generated by $\hat{b}^i$ (see the proof of Lemma A.1). It will then suffice to bound the extra payment required by $f^\varepsilon$ versus $\hat{b}$. For this, a weak bound is the extra payment required if the agent obtains all possible units, $q_t = Q$. This comparison is

$$\int_0^Q f^\varepsilon(x) - \hat{b}(x) \, dx \leq \sum_{t=1}^{Q/\varepsilon} \left[ f^\varepsilon((t - 1)\varepsilon) - \hat{b}(t\varepsilon) \right] \varepsilon \leq \sum_{t=1}^{Q/\varepsilon} \left[ \hat{b}((t - 1)\varepsilon) + \varepsilon - \hat{b}(t\varepsilon) \right] \varepsilon.$$ 

Because bids are monotonic in quantity, and will not exceed values for the initial unit, this becomes

$$\int_0^Q f^\varepsilon(x) - \hat{b}(x) \, dx \leq \left[ v^i(0; s) + Q \right] \varepsilon.$$ 

Letting $C = v^i(0; s) + Q$, the inequality is

$$U^i\left(\hat{b}, b^{-i}; s\right) - U^i\left(f^\varepsilon, b^{-i}; s\right) \leq C\varepsilon.$$ 

I now show that any unconstrained monotonic bid function which equals $\hat{b}^i$ at the multi-unit grid points must approximate the payoffs to within an order-$\varepsilon$ difference. Let $\bar{b}$ and $\underline{b}$ be defined as $\varepsilon$-unit offsets of $b^i$,

$$\bar{b}(q) = b^i(q - \varepsilon; s), \quad \underline{b}(q) = b^i(q + \varepsilon; s).$$

Let $f$ be any monotonic function satisfying $f(t\varepsilon) = b^i(t\varepsilon; s)$ for all $t \in \mathbb{N}$. Appealing to Lemma A.1, the gross utility obtained under $f$ is weakly dominated by that obtained under $\bar{b}$, and weakly dominates the gross utility obtained under $\underline{b}$. Moreover, the payment made under $f$ is weakly less than that made under $\bar{b}$ and
weakly greater than that made under \( b \). This implies useful bounds with respect to these two functions.

The same logic applied in the first argument of this Lemma gives that the additional payment incurred by \( b \) versus \( b_i \) will not exceed \( v^i(0; s) \varepsilon \); similarly, the savings of \( b \) versus \( b_i \) will not exceed \( v^i(0; s) \varepsilon \). The gross utility obtained by \( b \) can be bounded by noticing that in against any profile of opponents’ signals this bid function gives the agent no more than one \( \varepsilon \)-unit above what she would have obtained under \( b_i \). Since the per-unit margin is bounded by \( v^i(0; s) \), the additional gross utility does not exceed \( v^i(0; s) \varepsilon \). A similar argument establishes that the gross utility sacrificed by \( b \) does not exceed \( v^i(0; s) \varepsilon \).

Since for all \( q \), \( b(q) \leq f(q) \leq \overline{b}(q) \), the utility generated by \( f \) may be bounded below by assuming that quantities arise from \( b \) while payments arise from \( \overline{b} \), and above by assuming that quantities arise from \( b \) while payments arise from \( b_i \). Then the utility obtained under \( f \) must be such that

\[
\left| U^i \left( f, b^i; s \right) - U^i \left( b_i, b^i; s \right) \right| \leq 2v^i \left( 0; s \right) \varepsilon. \tag{A.5}
\]

Now, since \( b_i \) is a best-response to \( b^i \) in \( \mathcal{M}^\varepsilon \), for any other available bid function \( \hat{b} \) in \( \mathcal{M}^\varepsilon \) it must be the case that

\[
U^i \left( \hat{b}, b^i; s \right) - U^i \left( b_i, b^i; s \right) \leq 0.
\]

Hence any unconstrained monotonic bid function \( \hat{b} \) is such that

\[
\left| U^i \left( \hat{b}, b^i; s \right) - U^i \left( b^i, b^i; s \right) \right| \leq [v^i \left( 0; s \right) + Q] \varepsilon. \tag{A.6}
\]

When \( f \) is an unconstrained monotonic bid function which meets \( b_i \) at the \( \varepsilon \)-units, and the unconstrained monotonic bid function \( \hat{b} \) generates strictly greater utility than \( b_i \), equations (A.5) and (A.6) can be summed to obtain

\[
\left| U^i \left( f, b^i; s \right) - U^i \left( \hat{b}, b^i; s \right) \right| \leq [3v^i \left( 0; s \right) + Q] \varepsilon.
\]
Appendix to Chapter 3

B.1 Proof of Theorem 3.1: Strategic ironing (general case)

Demonstrating the necessity of strategic ironing relies on a set of auxiliary Lemmas which establish that equilibrium strategies must satisfy certain technical properties. In this Section I show the necessity of strategic ironing in the case with $n \geq 3$ bidders; for technical reasons the case of $n = 2$ bidders is proved in Section B.2.

**Lemma B.1** (Equal upper bids). There must be at least two bidders $i, j$, $i \neq j$ such that $\lim_{q \downarrow 0} b^i(q; 1) = \lim_{q \downarrow 0} b^j(q; 1) \geq \lim_{q \downarrow 0} b^k(q; 1)$ for all $k$.

**Proof.** This result follows from standard auction logic. Suppose that this is not the case; then there is an agent $k$ and an $\varepsilon > 0$ such that

$$\lim_{q \downarrow 0} b^k(q; 1) = \max_{\ell} \left\{ \lim_{q \downarrow 0} b^\ell(q; 1) \right\} + \varepsilon.$$

Then there is a $\delta > 0$ such that for all $q \in [0, \delta)$ it is the case that $b^k(q; 1) > b^k(0; 1) - \varepsilon$. By reducing the bid for all such units by no more than $\varepsilon$, the agent reduces her payment in all outcomes without affecting the quantity received; hence she should do so.

Bidders of the sort described by Lemma B.1 are the cornerstone of subsequent analysis. In particular, I focus on bidders who submit bids which will tie the highest-submitted bid, and who obtain a strictly positive quantity no matter the signals of their opponents.
**Definition B.1.** Agent $i$ is a maximal bidder if $\bar{q}^i(1) > 0$.

An immediate implication of Definition B.1 is that if $i$ is maximal, then $b^i(0; 1) = \max_j b^j(0; 1)$.

Henceforth I will assume that all bidders are maximal; for the purposes of the argument here, this assumption is without loss. Since only the behavior of maximal agents in a neighborhood of the quantity $\bar{q}^i(1)$ is used in analysis, any bidder who is non-maximal is of no consequence. Should this assumption not be made, in what follows the population of $n$ bidders would need to be replaced with the $m$ maximal bidders; otherwise the results remain unchanged.

**Lemma B.2 (Zero probability of maximal ties).** For all $i$, it must be that $\Pr(q_i \in [\bar{q}^i(1), \bar{q}^i(1)]) = 0$.

*Proof.* Suppose that $\Pr(q_i \in [\bar{q}^i(1), \bar{q}^i(1)]) > 0$. Then $\mathbb{E}_{s \sim \mathcal{S}}[q_i | q_i < \bar{q}^i(1), b^i(\cdot; 1)] < \bar{q}^i(1)$. Given $\varepsilon > 0$, define deviation $\hat{b}^\varepsilon$ by

$$\hat{b}^\varepsilon(q) = \begin{cases} 
\min \{b^i(q; 1) + \varepsilon, v^i(q; 1)\} & \text{if } q \leq \bar{q}^i(1), \\
b^i(q; 1) & \text{otherwise.}
\end{cases}$$

This deviation requires extra payment for all obtained units, no greater than $\varepsilon q^i(1)$. It also yields greater expected utility by shifting expected quantity in this range from $\mathbb{E}_{s \sim \mathcal{S}}[q_i | q_i < \bar{q}^i(1), b^i(\cdot; 1)]$ to $\bar{q}^i(1)$. This difference is strictly positive and independent of $\varepsilon$, and by Lipschitz bicontinuity of value functions the margin per unit is also strictly positive. The gain from this deviation is therefore bounded away from zero while the cost goes to zero, hence the deviation is profitable. \hfill $\square$

**Lemma B.3 (Nondegenerate maximal bids).** For any maximal bidder $i$, it must be that $b^i(0; 1) > 0$.

*Proof.* Suppose otherwise. Since bids are monotonic in value, all bidders must submit $b^j(\cdot; \cdot) = 0$. Then for any bidder $j$ and for almost all $s_j$, the bidder receives
quantities $q_j \in [q_j^l(s_j), q_j^u(s_j)]$ with positive probability. For any such agent there is some $\varepsilon > 0$ such that bidding $\hat{b}^i(q) = \min\{v^i(q; s_j), \varepsilon\}$ yields strictly higher expected utility.

With Lemma B.2 I verify the intuitive result that bidders submitting the highest (ex ante) bid functions cannot be subject to ties for low quantities. This might lead appear to imply that these agents are never rationed; strategic ironing claims only that these agents are almost never rationed. Rationing will still arise when, for example, quantities are discontinuous in opponents’ signals. I now address this point by contradiction: henceforth unless specified otherwise, assume that $q_i^j(1) = \hat{q}_i^j(1)$.

**Lemma B.4.** At least one maximal agent’s bid function must be continuous at $\hat{q}_i^j(1)$.

**Proof.** Note that if $b^i(\cdot; 1)$ is discontinuous at $q = \hat{q}_i^j(1)$, then $G^i(\hat{q}_i^j(1); b^i(\cdot; 1)) > 0$—otherwise the agent should shade her bid slightly on the initial flat. Two cases arise.

Suppose first that $b^i(\hat{q}_i^j(1); 1) = v^i(\hat{q}_i^j(1); 1)$. Let $b^r = \lim_{q \downarrow \hat{q}_i^j(1)} b^i(q; 1) < b^i(\hat{q}_i^j(1); 1)$, and let $\varepsilon > 0$. Define a deviation $\hat{b}^\varepsilon$ by

$$
\hat{b}^\varepsilon(q) = \begin{cases} 
    b^r & \text{if } q \in (\hat{q}_i^j(1) - \varepsilon, \hat{q}_i^j(1)], \\
    b^i(q; 1) & \text{otherwise}. 
\end{cases}
$$

Note that this deviation saves the bidder payment of at least $(b^i(\hat{q}_i^j(1); 1) - b^r)(1 - G^i(\hat{q}_i^j(1); b^i(\cdot; 1)))\varepsilon = O(\varepsilon)$. The utility sacrificed is at most $M_v \varepsilon^2 G^i(\hat{q}_i^j(1); b^i(\cdot; 1)) = O(\varepsilon^2)$. It follows that for $\varepsilon$ sufficiently close to zero, deviation is profitable.

Now suppose that $b^i(\hat{q}_i^j(1); 1) < v^i(\hat{q}_i^j(1); 1)$, and that all agents $j \neq i$ have discontinuities in $b^j(\cdot; 1)$ at $q = \hat{q}_i^j(1)$. By deviating downward by small $\varepsilon > 0$ the

---

1There is no concern here that deviating downward will discontinuously affect agent $i$’s quantity: $q_i^j(1) = \hat{q}_i^j(1)$, and $\hat{q}_i^j(1) > 0$; hence if ties are broken pro-rata on the margin, $q_i^j(1) = \hat{q}_i^j(1)$ for all other maximal bidders $j$ with $\hat{q}_j^j(1) > 0$.

---
agent will sacrifice quantity with zero probability—otherwise she could profit by increasing her bid to the right of \( \tilde{q}'(1) \)—while saving \( \varepsilon q'(1) \). Deviation is therefore profitable.

I now constrain attention to maximal bidders \( i \) such that \( b^i(\cdot; 1) \) is continuous at \( q = \tilde{q}'(1) \). By Lemma B.4 there is at least one such bidder. When referring to agents \( j \neq i \) I will continue to include agents whose bid functions are discontinuous at the ends of their respective initial flats. In all cases, undecorated indexed bid functions \( b^i \) represent putative best responses played in equilibrium.

Let \( \Delta^b \) be the gap between the maximum bid \( b(q'(1); 1) \) and a quantity slightly to the right of \( \tilde{q}'(1) \),

\[
\Delta^b(\delta) = b(\tilde{q}'(1); 1) - b(\tilde{q}'(1) + \delta; 1).
\]

**Lemma B.5.** For any maximal bidder \( i \), it must be that

\[
\lim_{\delta \downarrow 0} \frac{\Delta^b_i(\delta)}{G^i(\tilde{q}'(1) + \delta; b^i)} = 0.
\]

**Proof.** Consider a deviation \( b^\delta \) defined by

\[
b^\delta(q) = \begin{cases} 
  b^i(\tilde{q}'(1) + \delta; 1) & \text{if } q \leq \tilde{q}'(1) + \delta, \\
  b^i(q; 1) & \text{otherwise}.
\end{cases}
\]

By reducing her submitted bid, the agent saves costs on all units won, but sacrifices utility by also reducing the number of units she wins. A lower bound on cost savings from this deviation is \( \tilde{q}'(1) \Delta^b(\delta)(1 - G^i(\tilde{q}'(1); b^i)) \): she saves \( \tilde{q}'(1) \Delta^b(\delta) \) whenever she wins quantity \( q \geq \tilde{q}'(1) + \delta \).

She will lose utility with probability \( G^i(\tilde{q}'(1) + \delta; b^i) \), and the per-unit margin will be bounded above by \( \gamma^i(\delta) = v^i(0; 1) - b^i(0; 1) + \Delta^b(\delta) \). The number of units sacrificed will be bounded by the number of units lost when all other agents are maximal, plus \( \delta \); denoting the quantity lost to maximal opponents by \( \Delta^m(\delta) \), the absolute bound on the quantity lost when facing any profile of opponents’ signals
is $\Delta^q_i(\delta) + \delta$. Since $q^i(1) = \bar{q}^i(1)$ by assumption, it is the case that $\Delta^q_i(\delta) + \delta \to 0$ when $\delta \downarrow 0$. Putting these terms together, the utility loss is bounded above by $\gamma^i(\delta)(\Delta^q_i(\delta) + \delta)G^i(\bar{q}^i(1) + \delta; b^i)$.

Incentive compatibility implies

$$\dot{q}^i(1) \Delta^{b^i}(\delta) \left(1 - G^i(\bar{q}^i(1) + \delta; b^i)\right) \leq \gamma^i(\delta) \left(\Delta^q_i(\delta) + \delta\right)G^i(\bar{q}^i(1) + \delta; b^i).$$

This may be rearranged as

$$\frac{\Delta^{b^i}(\delta)}{G^i(\bar{q}^i(1) + \delta; b^i)} \leq \frac{\gamma^i(\delta) \left(\Delta^q_i(\delta) + \delta\right)}{\dot{q}^i(1) \left(1 - G^i(\bar{q}^i(1) + \delta; b^i)\right)}.$$

As $\delta \downarrow 0$, the right-hand side approaches 0. Since the left-hand side is weakly positive, it must then be that

$$\lim_{\delta \downarrow 0} \frac{\Delta^{b^i}(\delta)}{G^i(\bar{q}^i(1) + \delta; b^i)} = 0.$$

\[ \square \]

**Lemma B.6.** If $i$ is a maximal bidder, then

(i) $v^i(\dot{q}^i(1); 1) > b^i(\dot{q}^i(1); 1)$ implies

$$\limsup_{\delta \downarrow 0} \frac{1}{\delta} G^i(\bar{q}^i(1) + \delta; b^i) = 0,$$

and

(ii) $v^i(\dot{q}^i(1); 1) = b^i(\dot{q}^i(1); 1)$ implies

$$\liminf_{\delta \downarrow 0} \frac{1}{\delta} G^i(\bar{q}^i(1) + \delta; b^i) = +\infty.$$

**Proof.** To demonstrate point (i), let $\delta > 0$ and consider a deviation $b^\delta$ defined by

$$b^\delta(q) = \begin{cases} 
  b^i(0; 1) & \text{if } q \leq \bar{q}^i(1) + \delta, \\
  b^i(q; 1) & \text{otherwise}.
\end{cases}$$

This deviation introduces extra costs, bounded by $\delta \Delta^{b^i}(\delta)$, with probability 1. With probability $G^i(\bar{q}^i(1) + \delta; b^i)$ some quantity is gained, at maximum margin

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\[ v^i(\bar{q}^i(1) + \delta; 1) - b^i(\bar{q}^i(1); 1). \] Letting \( \gamma = v^i(\bar{q}^i(1) + \delta; 1) > 0, \) for sufficiently small \( \delta \) the per-unit margin is at least \( \gamma/2. \) Lastly, the number of units gained will be written as

\[ \Delta q^i(\delta) = \delta - \mathbb{E}_{s_{-i}} \left[ q^i - \bar{q}^i(1) | q^i \leq \bar{q}^i(1) + \delta \right]. \]

Incentive compatibility requires

\[ \delta \Delta b^i(\delta) \geq \left( \frac{\gamma}{2} \right) \Delta q^i(\delta) G^i(\bar{q}^i(1) + \delta; b^i). \]

This may be rearranged as

\[ \frac{\Delta b^i(\delta)}{G^i(\bar{q}^i(1) + \delta; b^i)} \geq \left( \frac{\gamma}{2} \right) \frac{\Delta q^i(\delta)}{\delta}. \]

Lemma B.5 implies that the right-hand side goes to zero as \( \delta \) becomes small; since the left-hand side is positive, it follows that

\[ \lim_{\delta \downarrow 0} \frac{1}{\delta} \Delta q^i(\delta) = 0 \implies \lim_{\delta \downarrow 0} \frac{1}{\delta} \mathbb{E}_{s_{-i}} [q^i - \bar{q}^i(1) | q^i \leq \bar{q}^i(1) + \delta] = 1. \]

This integral can be bounded in a natural way using a two-rectangle approximation, hence the subsequent of the above expression requires

\[ \lim_{\delta \downarrow 0} \frac{G^i(\bar{q}^i(1) + \delta/2; b^i)}{G^i(\bar{q}^i(1) + \delta; b^i)} = 0. \]

That is, for all \( \kappa > 0 \) there is \( \varepsilon_\kappa > 0 \) such that for all \( 0 < \delta < \varepsilon_\kappa, \)

\[ G^i(\bar{q}^i(1) + \delta; b^i) \geq \kappa G^i(\bar{q}^i(1) + \delta/2; b^i). \]

Now suppose that \( \limsup_{\delta \downarrow 0} G^i(\bar{q}^i(1) + \delta; b^i)/\delta = \xi > 0. \) Then for all \( \lambda > 0, \) there is \( \varepsilon_\lambda \) such that there are an infinite number of points \( 0 < \delta_k < \varepsilon_\lambda \) with \( G^i(\bar{q}^i(1) + \delta_k; b^i)/\delta_k > \xi - \lambda. \)

Let \( \lambda = \gamma/2, \kappa = 4, \) and \( \varepsilon = \min\{\varepsilon_\lambda, \varepsilon_\kappa\}. \) Then there are an infinite number of \( \delta_k \) with \( 0 < \delta_k < \varepsilon \) and \( G^i(\bar{q}^i(1) + \delta_k; b^i)/\delta_k > \xi/2; \) but at any such \( \delta_k \) it is also the case that \( G^i(\bar{q}^i(1) + 2\delta_k; b^i/(2\delta_k) > 2\xi. \) For any \( \varepsilon' < 2\varepsilon, \) there must be an infinite
number of such $2\delta^k$, hence $\limsup_{\delta \downarrow 0} G^i(\bar{q}^i(1) + \delta; b^i)/\delta \geq 2\xi$, a contradiction. Thus it must be that either this limit is zero or infinite. The latter can be ruled out by the fact that $G^i(\cdot; b^i) \in [0, 1]$, hence the result is shown.

Point (ii) follows from the fact that bids are constrained above by values, and values are Lipschitz continuous. Hence $\Delta b^i(\delta) \geq \delta/M_v$. Then from Lemma B.5 it follows that

$$\lim_{\delta \downarrow 0} \frac{\delta/M_v}{G^i(\bar{q}^i(1) + \delta; b^i)} = 0.$$ 

This directly implies that

$$\lim_{\delta \downarrow 0} \frac{1}{\delta} G^i(\bar{q}^i(1) + \delta; b^i) = +\infty.$$ 

If $G^i$ is continuous at $\bar{q}^i(1)$, the derivative of $G^i$ with respect to $q$ is well-defined at $\bar{q}^i(1)$, and $G^i_q(\bar{q}^i(1); b^i) = +\infty$. Otherwise, there must be a mass point. \qed

**Corollary B.1.** If $i$ is a maximal bidder, then $\lim_{\delta \downarrow 0} G^i(\bar{q}^i(1) + \delta; b^i)/\delta = G^i_q(\bar{q}^i(1); b^i)$ is well-defined.

**Lemma B.7.** Suppose that agent $i$ is such that $b^i(\bar{q}^i; 1) < v^i(\bar{q}^i; 1)$. Then for all other maximal agents $k \neq i$,

$$\lim_{q \downarrow \bar{q}^i(1)} G^k_b(q; b^i) = -\infty.$$ 

**Proof.** Consider agent $i$’s distribution of quantity. Choosing some $k \neq i$, this function can be written as

$$G^i(q; b^i) = \int_0^1 \cdots \int_0^1 \int_0^1 \psi^k(Q - q - \sum_{j \neq i, k} \phi^i(b^i(q; 1); s_j); b^i(q; 1)) dF(s_k) \prod_{j \neq i, k} dF(s_j).$$ 

Since $s_j \sim U(0, 1)$ for all $j$, this can in turn be written as

$$G^i(q; b^i) = 1 - \int_0^1 \cdots \int_0^1 \int_0^1 \psi^k(Q - q - \sum_{j \neq i, k} \phi^i(b^i(q; 1); s_j); b^i(q; 1)) \, ds_{-i}.$$ 

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Having constrained attention to bounded monotonic functions, all functions of interest are differentiable almost everywhere. Therefore examine the derivative of $G^i$ with respect to $b$ where applicable, and take limits as needed. Where it exists, this derivative is

$$G^i_b (q; b^i) = \int_0^1 \cdots \int_0^1 \psi^k_q \left( Q - q - \sum_{j \neq k} \varphi^j \left( b^j (q; 1); s_j \right); b^i (q; 1) \right) \sum_{j \neq i,k} \varphi^i_j \left( b^i (q; 1); s_j \right)$$

$$- \psi^k_b \left( Q - q - \sum_{j \neq i,k} \varphi^j \left( b^j (q; 1); s_j \right); b^i (q; 1) \right) ds_{-ik}.$$ 

Substituting in for $G^i_q$ and dividing by $b^i_q$ gives

$$\frac{G^i_q (q; b^i)}{b^i_q (q; 1)} = G^i_b (q; b^i)$$

$$+ \int_0^1 \cdots \int_0^1 \psi^k_q \left( Q - \sum_{j \neq k} \varphi^j \left( b^j (q; 1); s_j \right); b^i (q; 1) \right) \varphi^i_b \left( b^i (q; 1); 1 \right) \varphi^i_b \left( b^i (q; 1); \delta \right) ds_{-ik}.$$ 

From Lemma B.5, it must be that

$$\lim_{\delta \to 0} G^i_b (\hat{q}^i (1) + \delta; b^i)$$

$$+ \int_0^1 \cdots \int_0^1 \psi^k_q \left( Q - \sum_{j \neq k} \varphi^j \left( b^j (\hat{q}^i (1) + \delta; 1); s_j \right); b^i \left( \hat{q}^i (1) + \delta; 1 \right) \right)$$

$$\times \varphi^i_q \left( b^i \left( \hat{q}^i (1) + \delta; 1 \right); 1 \right) ds_{-ik} = -\infty.$$ 

Suppose that $\lim_{\delta \to 0} G^i_b (\hat{q}^i (1) + \delta; b^i) \neq -\infty$. Then it must be that for all maximal $k \neq i$,\(^2\)

$$\lim_{\delta \to 0} \int_0^1 \cdots \int_0^1 \psi^k_q \varphi^i_q ds_{-i,k} = -\infty.$$ 

\(^2\)The two-maximal-agent case must be handled separately, but can be dealt with much more simply than this. See Section B.2.
Now, for all other maximal agents $k, j \neq i, k \neq j$, it is the case that

$$G_k^b(q; b^k) = \int_0^1 \cdots \int_0^1 \psi_q^j \sum_{\ell \neq j,k} \varphi^\ell (b^k(q; 1); 1) - \psi_p^j (b^k(q; 1); 1) \, ds_{-k,j}$$

$$< \int_0^1 \cdots \int_0^1 \psi_q^j \sum_{\ell \neq j,k} \varphi^\ell (b^k(q; 1); 1) \, ds_{-k,j}$$

$$< \int_0^1 \cdots \int_0^1 \psi_q^j \varphi^i (b^k(q; 1); 1) \, ds_{-k,j}.$$

The last term has been shown to go to $-\infty$ as $\delta \downarrow 0$, hence $\lim_{\delta \downarrow 0} G_k^b(\tilde{q}^k(1)+\delta; b^k) = -\infty$ for all maximal agents $k \neq i$. \qed

**Lemma B.8.** If $i$ is a maximal agent with $G^i(\tilde{q}^i(1); b^i) = 0$, then $v^i(\tilde{q}^i(1); 1) > b^i(\tilde{q}^i(1); 1)$.

**Proof.** Suppose that $b^i(\tilde{q}^i(1); 1) = v^i(\tilde{q}^i(1); 1)$. Let $\varepsilon > 0$ and consider a deviation $b^\varepsilon$ defined by

$$b^\varepsilon(q) = \begin{cases} 
  b^i(q; 1) & \text{if } q \leq \tilde{q}^i(1) - \varepsilon, \\
  b^i(\tilde{q}^i(1); 1) - \varepsilon & \text{if } q > \tilde{q}^i(1) - \varepsilon \text{ and } b^i(q; 1) > b^i(\tilde{q}^i(1); 1) - \varepsilon, \\
  b^i(q; 1) & \text{otherwise.}
\end{cases}$$

This deviation saves the agent payment of at least $\varepsilon^2$, with probability 1. The costs of the deviation are associated with sacrificed quantities, which are easily bounded: under the deviation, the new minimum quantity $\tilde{q}^\varepsilon$ is such that $\tilde{q}^\varepsilon \geq \tilde{q}^i(1) - \varepsilon$, while the maximum quantity affected is bounded above by $\varphi^i(b^\varepsilon(\tilde{q}^i(1); 1) - \varepsilon; 1) \leq \tilde{q}^i(1) + M_v \varepsilon$, since bids fall weakly below values, and $b^i(\tilde{q}^i(1); 1) = v^i(\tilde{q}^i(1); 1)$ by assumption. The total quantity lost is therefore bounded by $(1 + M_v)\varepsilon$. The per-unit margin lost to the left of $\tilde{q}^i(1)$ is bounded again by $M_v \varepsilon$, while on the right it is bounded by $M_v \varepsilon / b^\varepsilon_q(\tilde{q}^i(1); 1) \leq \varepsilon$. The utility loss is therefore bounded by $[(1 + M_v)\varepsilon]^2$, and this loss occurs with at most probability $G^i(\tilde{q}^i(1) + M_v \varepsilon; b^i)$.\[127]
Incentive compatibility requires that the expected benefits from this deviation be outweighed by the expected losses; that is,

\[ \varepsilon^2 \leq [(1 + M_v)\varepsilon]^2 G^i \left( \tilde{q}^i(1) + M_v\varepsilon; b^i \right). \]

After cancelation, this is \( G^i(\tilde{q}^i(1) + M_v\varepsilon; b^i) \geq 1/(1 + M_v)^2 \) for all \( \varepsilon > 0 \). Since \( G^i \) is right-continuous and \( G^i(\tilde{q}^i(1); b^i) = 0 \), this inequality cannot hold for \( \varepsilon \) sufficiently small, and the deviation will be profitable. \( \square \)

**Lemma B.9.** It cannot be that for all maximal agents \( i \), \( v^i(\tilde{q}^i(1); 1) = b^i(\tilde{q}^i(1); 1) \).

**Proof.** Suppose otherwise. Consider the argument from Lemma B.5, which required

\[ \tilde{q}^i(1) \Delta b^i(\delta) \left( 1 - G^i(\tilde{q}^i(1) + \delta; b^i) \right) \leq \gamma^i(\delta) \left( \Delta \gamma^i(\delta) + \delta \right) G^i(\tilde{q}^i(1) + \delta; b^i). \]

In the proof of Lemma B.6, point (ii) it was demonstrated that \( \Delta b^i(\delta) \geq \delta/M_v \).

Moreover, since all other maximal agents \( j \) have \( v^j(\tilde{q}^j(1); 1) = b^j(\tilde{q}^j(1); 1) \), for small \( \delta \), \( \Delta q^i(\delta) \leq (n - 1)M_v\Delta b^i(\delta) \). These inequalities give

\[ \tilde{q}^i(1) \left( 1 - G^i(\tilde{q}^i(1) + \delta; b^i) \right) \leq \gamma^i(\delta) \left( n - 1 \right) M_v + \frac{\delta}{\Delta b^i(\delta)} \right) G^i(\tilde{q}^i(1) + \delta; b^i) \]

\[ \leq nM_v\gamma^i(\delta) G^i(\tilde{q}^i(1) + \delta; b^i). \]

As \( \delta \downarrow 0 \), the right-hand side goes to zero while the left-hand side goes to \( \tilde{q}^i(1) > 0 \), contradicting the possibility that this equilibrium satisfies incentive compatibility. \( \square \)

**Lemma B.10.** If there are \( m \) maximal agents, it cannot be that for \( m - 1 \) maximal agents \( k \), \( v^k(\tilde{q}^k(1); 1) = b^k(\tilde{q}^k(1); 1) \).

**Proof.** Lemma B.9 establishes that it cannot be the case that all maximal agents \( k \) have \( v^k(\tilde{q}^k(1); 1) = b^k(\tilde{q}^k(1); 1) \). Moreover, Lemma B.7 argued that if there is a single maximal agent \( i \) with \( v^i(\tilde{q}^i(1); 1) > b^i(\tilde{q}^i(1); 1) \), then the remaining \( m - 1 \).
maximal agents $k \neq i$ have $v^k(\tilde{q}^k(1); 1) = b^k(\tilde{q}^k(1); 1)$. From Lemma B.8, this is only possible when all such agents have $G^k(\tilde{q}^k(1); 1) = \pi_k > 0$.

Because signals are independent, this is only possible if, given a particular $j$, all other maximal agents $k \neq j$ are such that $b^k(\tilde{q}^k(1); s_k)$ is constant for $s_k \in (1 - \varepsilon_k, 1]$ for some $\varepsilon_k > 0$. Since this must be true for all agents $j \neq i$, it follows that all maximal agents $k$ satisfy this requirement for some $\varepsilon_k > 0$. By the same token, it must be that $G^i(\tilde{q}^i(1); b^i) \geq \prod_{j \neq i} \varepsilon_j > 0$, contradicting Lemma B.5, point (i). Thus the result is shown.

\[ \square \]

**Theorem B.1.** If $\tilde{q}^i(1) \in (0, Q)$, $b^i(\cdot; 1)$ exhibits strategic ironing.

**Proof.** Assuming that there is no ironing, $\tilde{q}^i(1) = \tilde{q}^i(1)$, I have shown that:

(Lemma B.9) it cannot be that all maximal agents $j$ have $b^j(\tilde{q}^j(1); 1) = v^j(\tilde{q}^j(1); 1)$;

(Lemma B.7) there can be at most one agent $j$ with $b^j(\tilde{q}^j(1); 1) < v^j(\tilde{q}^j(1); 1)$;

(Lemma B.10) it cannot be that $n-1$ maximal agents have $b^j(\tilde{q}^j(1); 1) = v^j(\tilde{q}^j(1); 1)$.

Since these results together contradict the existence of an equilibrium without ironing so long as the market is not cornered, any equilibrium must exhibit non-trivial ironing: $\tilde{q}^i(1) > 0$ implies $\tilde{q}^i(1) > q^i(1)$.

\[ \square \]

**B.2 Proof of Theorem 3.1: Strategic ironing (two-agent case)**

As mentioned in the main text and the preceding Section, the two-agent case requires special care. Because the fundamental results are the same—even if the proofs differ—it is valid to lump this case in with the general $n$-agent case for exposition’s sake. In particular, with two agents the probability expression $G^i(q; b^i)$ is not written as an integral, but is rather $G^i(q; b^i) = 1 - \psi^{-i}(Q - q; b^i(q))$.

Because this is the only meaningful change, all results that do not involve the integral form of $G^i$ continue to go through without modification. It is therefore
only necessary to reestablish Lemma B.7 in the two-agent case.

**Lemma B.11** (Unbounded marginal probability improvement (two agents)). At least one agent has \( \lim_{q \downarrow \tilde{q}^i(1)} G_b^i(q; b^i) = +\infty \).

**Proof.** With \( G^i(q; b^i) = 1 - \psi^{-i}(Q - q; b^i(q)) \), it is the case that

\[
G_b^i(q; b^i) = -\psi^{-i}_p(Q - q; b^i(q)) ,
\]

\[
G_q^i(q; b^i) = \psi^{-i}_q(Q - q; b^i(q)) + G_b^i(q; b^i) b^i_q(q) .
\]

By implicit differentiation,

\[
\psi^{-i}_p(Q - q; b^i(q)) = 1/b^i_q(Q - q; \psi^{-i}_q(Q - q; b^i(q))) ,
\]

\[
\psi^{-i}_q(Q - q; b^i(q)) = -b^i_q(Q - q; \psi^{-i}_q(Q - q; b^i(q))) \psi^{-i}_p(Q - q; b^i(q)) .
\]

It follows that at all \( q \),

\[
G_q^i(q; b^i) = \left( b^i_q(Q - q; \psi^{-i}_q(Q - q; b^i(q))) + b^i_q(q) \right) G_b^i(q; b^i) .
\]

Thus at all \( q \),

\[
\frac{G_q^i(q; b^i)}{b^i_q(q)} = \left( \frac{b^i_q(Q - q; \psi^{-i}_q(Q - q; b^i(q)))}{b^i_q(q)} + 1 \right) G_b^i(q; b^i) .
\]

As \( q \downarrow \tilde{q}^i(s) \) the left-hand side goes to \(+\infty\), and this holds for both agents. Thus either \( b^i_q/\tilde{q}^i_q \rightarrow +\infty \), or \( G^i_b \rightarrow +\infty \). Note that if \( b^i_q/\tilde{q}^i_q \rightarrow +\infty \), then \( b^j_q/\tilde{q}^j_q \rightarrow 0 \) for \( j \neq i \), and thus \( G^j_b \rightarrow +\infty \). It follows that at least one of \( i \in \{1, 2\} \) has \( \lim_{q \downarrow \tilde{q}^i(s)} G_b^i(q; b^i) = +\infty \). \( \square \)

The remainder of the necessity proof is identical to the generic \( n \)-agent case: it cannot be that all-but-one agent bids her true value at the end of the initial flat, and since in this case all-but-one is one, Lemma B.11 implies that this must be the case when one agent’s bid does not equal her value at the end of the initial flat. Since it also cannot be the case that both agents bid their true values at the end of the initial flat, any pure-strategy equilibrium must involve ironing, \( \tilde{q}^i(1) < \tilde{q}^i(1) \).
B.3 Calculations for simulated models

To obtain a solution to the agent’s first-order conditions under a presumption of nonstrategic flattening, I first consider a relaxed problem without proper constraints on the probability distribution (i.e., it can fall below zero or exceed one). Once a solution is found to the relaxed problem, bounds may be imposed on the probability distribution. In particular, finding the quantity at which the probability of allocation is zero gives the quantity at which nonstrategic flattening should be applied. Incentives for higher quantities are not affected by this augmentation of the bid function, as the probability that quantities were allocated in this range was negative (zero). Thus this procedure will accurately compute behavior in the model with nonstrategic flattening, where bids are determined competitively above \( q^i(s) \) for all agents \( i \) and all signals \( s \), and are flat at bid \( b^i(q^i(s); s) \) for all lower quantities \( q \).

**Relaxed problem**

Consider a discriminatory auction with \( n = 2 \) bidders, with symmetric marginal values

\[
v^i(q; s) = \alpha_0 + s\alpha_s - q\alpha_q.
\]

I posit a symmetric linear solution to the equilibrium necessary first-order conditions,

\[
b^i(q; s) = \beta_0 + s\beta_s - q\beta_q.
\]

**Quantities.** Market clearing dictates that each agent pay the same price, hence

\[
\beta_0 + s_1\beta_s - q_1\beta_q = \beta_0 + s_2\beta_s - q_2\beta_q \implies (q_1 - q_2)\beta_q = (s_1 - s_2)\beta_s.
\]

Since market clearing also requires \( q_1 + q_2 = Q \), this becomes

\[
q_1 = \frac{\beta_s}{2\beta_q} (s_1 - s_2) + \frac{1}{2} Q.
\]
Probabilities. Any bidding strategy is weakly monotonic in the agent’s signal. Assuming $\beta_s \neq 0$ (which will be verified later) this implies that

$$G^i(q; b^i) = \Pr(q_i \leq q | b^i)$$

$$= \Pr(b^i (Q - q; s_{-i}) \geq b^i (q; s))$$

$$= \Pr(s_{-i} \geq \psi^{-i} (b^i (q; s), Q - q))$$

$$= 1 - \psi^{-i} (b^i (q; s), Q - q).$$

Here, $\psi^{-i}$ may be determined explicitly,

$$\psi^{-i} (p, Q - q) = \frac{1}{\beta_s} (p - \beta_0 + (Q - q) \beta_q).$$

It is immediate that $G^i_0(q; b^i) = -1/\beta_s$. By symmetry and market clearing, it must also be that $1 - \psi^{-i}(b^i(q), Q - 1) = s_{-i}$.

First-order conditions. Altogether, this allows the expression of the first-order conditions as

$$b^i (q; s_i) = v^i (q; s_i) + \frac{1 - G^i (q; b^i)}{G^i_b(q; b^i)}$$

$$\beta_0 + s_i \beta_s - q \beta_q = \alpha_0 + s_i \alpha_s - q \alpha_q - s_{-i} \beta_s$$

$$\beta_0 + s_i \beta_s - \frac{1}{2} (s_i - s_{-i}) \beta_s - \frac{1}{2} Q \beta_q = \alpha_0 + s_i \alpha_s - \frac{1}{2} (s_i - s_{-i}) \frac{\alpha_q \beta_s}{\beta_q} - \frac{1}{2} Q \alpha_q - s_{-i} \beta_s.$$  

Since this equation must be valid for all $(s_i, s_{-i})$, linearity implies three equalities:

$$\beta_0 - \frac{1}{2} Q \beta_q = \alpha_0 - \frac{1}{2} Q \alpha_q,$$

$$\frac{1}{2} \beta_s = \alpha_s - \frac{1}{2} \left( \frac{\alpha_q \beta_s}{\beta_q} \right),$$

$$\frac{1}{2} \beta_s = \frac{1}{2} \left( \frac{\alpha_q \beta_s}{\beta_q} \right) - \beta_s.$$  

It is straightforward then to compute

$$\beta_0 = \alpha_0 - \frac{1}{3} Q \alpha_q,$$

$$\beta_s = \frac{1}{2} \alpha_s,$$

$$\beta_q = \frac{1}{3} \alpha_q.$$
Bid functions are then given by

\[ b_i(q; s) = \left( \alpha_0 - \frac{1}{3} Q \alpha_q \right) + \frac{1}{2} s \alpha_s - \frac{1}{3} q \alpha_q. \]

Nonstrategic flattening

As discussed in the Section 3.2, equilibrium bid functions will not be everywhere-decreasing. Since the linear solution obtained implies that bids are strictly monotone, the bidder is overbidding for units she wins for certain. I now consider the application of nonstrategic flattening to the linear bid functions obtained above. In particular, the bidder will reduce her bid for low quantities by binding the monotonicity constraint to the bid for \( q^i(s) \).

**Quantities.** The linear solution above gives quantities as

\[ q^i(s_i, s_{-i}) = \frac{\beta_s}{2 \beta_q} (s_i - s_{-i}) + \frac{1}{2} Q = \frac{3 \alpha_s}{4 \alpha_q} (s_i - s_{-i}) + \frac{1}{2} Q. \]

Agent \( i \)'s equilibrium allocation is minimized when agent \(-i\) receives signal \( s_{-i} = 1 \). Thus

\[ q^i(s_i) = \frac{3 \alpha_s}{4 \alpha_q} (s_i - 1) + \frac{1}{2} Q. \]

**Bids.** With a lower bound on the quantity obtained, nonstrategically-flattened bids are given by

\[
\begin{cases} 
(\alpha_0 - \frac{1}{4} Q \alpha_q) + \frac{1}{2} s \alpha_s - \frac{1}{3} q \alpha_q & \text{if } q \geq \frac{3 \alpha_s}{4 \alpha_q} (s_i - 1) + \frac{1}{2} Q, \\
(\alpha_0 - \frac{1}{4} Q \alpha_q) + \frac{1}{4} (1 + s_i) \alpha_s & \text{otherwise.}
\end{cases}
\]

(B.1)

**Expected revenue.** To compute the baseline calibration in Section 3.3.2, it is necessary to know either the revenue in an equilibrium of the pay-as-bid auction, or the revenue implied in the model with nonstrategic flattening. Because, as found in equation (B.1), bids with nonstrategic flattening admit a relatively straightforward linear form, it is most useful to calibrate observed revenues to the expected revenue computed in the flattened model.
Expected revenue is

\[
E_s [\pi] = E_s \left[ \int_0^{q_1} b^1 (x) \, dx + \int_0^{q_2} b^2 (x) \, dx \right]
\]

\[
= 2E_{s_1} \left[ \int_0^{q_1} b^1 (x) \, dx \right]
\]

\[
= 2E_{s_1} \left[ b^1 (q^1 (s_1, s_2); s_1) q^1 (s_1, s_2) + \frac{1}{6} \alpha_q \left( q^1 (s_1, s_2)^2 - q^1 (s_1)^2 \right) \right].
\]

Straightforward calculation gives

\[
E_s [\pi] = Q \left( \alpha_0 - \frac{1}{2} Q \alpha_q \right) + \frac{3}{8} Q \alpha_s - \frac{1}{32} \left( \frac{\alpha_s^2}{\alpha_q} \right).
\]

Ironing

Structure of equilibrium. I look for a symmetric equilibrium with the following characteristics: (i) an agent receiving signal \( s_i = 0 \) submits a perfectly flat bid function; (ii) \( b^i(q; 0) = v^i(Q/2; 0) \) whenever \( q \leq Q/2 \); (iii) \( \bar{q}^i(s) \) is continuous, and \( \bar{q}^i(s) \geq Q/2 \) for all \( s \). Under the assumptions, strategic monotonicity implies that 

\[
G^i(\bar{q}(s); b^i(\cdot; s)) = 1 - s. \quad \quad (3)
\]

Derivatives. When bids are strictly monotonic and \( \bar{q}^i(s) \geq Q/2 \) the market price \( p(s_1, s_2) \) will be determined by the bid placed along the flat of the agent with the lower of the two signals, \( i \in \text{arg min}_j \{ s_j \} \). Thus the market price equation is written as \( p(s_1, s_2) = b^i(\bar{q}^i(s); \bar{s}) \), where \( \bar{s} = s_i = \text{min}\{s_1, s_2\} \). I will therefore consider \( p \) as a function of only one variable, the lower of the two signals.

Now, let \( s'_1 > s_1 > s_2 = \bar{s} \). Since the market price is determined by \( s_2 \) alone,
it must be that
\[
\begin{align*}
b^1(q^1(s'_1, s_2); s'_1) &= b^1(q^1(s_1, s_2); s_1) \\
\implies v^1(q^1(s'_1, s_2); s'_1) + \frac{s_2}{G_b^1(q^1(s'_1, s_2); b^1(\cdot; s'_1))} &= v^1(q^1(s_1, s_2); s_1) + \frac{s_2}{G_b^1(q^1(s_1, s_2); b^1(\cdot; s_1))}.
\end{align*}
\]

In a strictly monotone equilibrium, \(q^1(s, s_2) > Q/2\) when \(s \in \{s_1, s'_1\}\). Then slightly increasing the bid for unit \(q^1(s, s_2)\) will render this unit pivotal against some higher \(s'_2 > s_2\), but this alternative opponent will also be bidding along her initial flat. Since the opponent’s bid is, by construction, still flat along this region, the change in win probability is the same for either of \(s \in \{s_1, s'_1\}\). It follows that \(G_b^1(q(s, s_2); b^1(\cdot; s))\) is constant for \(s > s_2\). The above equation then becomes
\[
s'_1\alpha_s - q^1(s'_1, s_2)\alpha_q = s_1\alpha_s - q^1(s_1, s_2)\alpha_q.
\]
From this, it follows that \(q^1_s(s, s_2) = \alpha_s/\alpha_q\) whenever \(s > s_2\).

By virtue of both paying the same price, it must also be that
\[
\frac{\partial}{\partial s} \left[ b^1(q^1(s, s_2); s) \right] = 0 = b^1_q q^1_s + b^1_s.
\]
It follows that \(b^1_s(q; s) = -(\alpha_s/\alpha_q)b^1_q(q; s)\) when \(s > s_2\) and \(q = q^1(s, s_2)\).

With respect to the lower of the two signals, it must also be that
\[
\frac{\partial}{\partial s} \left[ b^1(q^1(s_1, s); s_1) \right] = p_s(s) = b^1_q(q^1(s_1, s_2); s_1) q^1_{s_2}(s_1, s_2).
\]

**Intervals.** Let \(i \in \arg\min_j s_j\) as before. It is helpful to define two quantities, \(\underline{q}(s)\) and \(\overline{q}(s)\) as the minimum possible quantity and the maximum rationed quantity along the flat, respectively. In particular,
\[
\underline{q}(s) = q^i(s, 1-i) \quad \text{and} \quad \overline{q}(s) = \lim_{s \downarrow s_i} q^i(s_i, s).
\]
Since \(q^i_{s_i}(s_i, s_{-i}) = \alpha_s/\alpha_q\) when \(s_{-i} > s_i\), it follows that
\[
\underline{q}(s_i) = \overline{q}(s_i) - \frac{\alpha_s}{\alpha_q} (1 - s_i).
\]
When bids are strictly monotonic and bids are continuous in signal, and so is $\bar{q}(\cdot)$, it will be the case that
\[ q^\ell(s_i) = Q - \bar{q}(s_i). \]
That is, when $s_i$ is only slightly smaller than $s_{-i}$, bidder $i$ loses the difference between $\bar{q}(s_i)$ and $Q/2$—as she would when opposing $s_{-i} = s_i$—and then this difference once more.

**Ironing.** The first-order condition with respect to the right endpoint of the initial flat is
\[
 b_q(\bar{q}; s) \int_{0}^{\bar{q}} (1 - G^i(q; b^i)) \, dq = -b_q(\bar{q}; s) \int_{0}^{\bar{q}} (v^i(q; s) - b^i(\bar{q}; s)) \, G_b^i(q; b^i) \, dq.
\]
The $b_q$ terms cancel. This leaves two sides of the equation,
\[
 \text{LHS}(\bar{q}; s) = \int_{0}^{\bar{q}} (1 - G^i(q; b^i)) \, dq,
\]
\[
 \text{RHS}(\bar{q}; s) = \int_{0}^{\bar{q}} \left( v^i(q; s) - b^i(\bar{q}; s) \right) G^i_b(q; b^i) \, dq.
\]
In what follows I will suppress arguments from functions which are determined by a single signal; this is done for space efficiency.

**Left-hand side.** I analyze $\text{LHS}(\bar{q}; s)$ by splitting the integral into parts,
\[
 \text{LHS}(\bar{q}; s) = \bar{q} - \left[ \int_{0}^{2} G^i(q; b^i) \, dq + \int_{2}^{\bar{q}} G^i(q; b^i) \, dq + \int_{\bar{q}}^{\bar{q}^l} G^i(q; b^i) \, dq \right].
\]
When $q \in [0, \bar{q})$, $G^i(q; b^i) = 0$: there is no probability that the agent receives a quantity in this range. When $q \in [\bar{q}, \bar{q}^l)$, $G^i(q; b^i) = 1 - s$: there is no probability of being allocated in this interval (except for the probability-zero event $s_{-i} = s_i$). Then the only interval of interest is $(\bar{q}, \bar{q}^l)$. Since equilibrium quantities are linear in signal equilibrium probabilities must be linear in quantity, therefore
\[
 \int_{\bar{q}}^{\bar{q}^l} G^i(q; b^i) \, dq = \frac{1}{2} (1 + s) (q^l - \bar{q}) = \frac{1}{2} (1 + s) \frac{\alpha_s}{\alpha_q} (1 - s).
\]
Putting all these pieces together gives
\[
\text{LHS} (\tilde{q}; s) = \tilde{q} - \left[ \frac{1}{2} (1 - s) \frac{\alpha_s}{\alpha_q} (1 - s) + (\tilde{q} - q') (1 - s) \right] \\
= \tilde{q} - \left[ \frac{1}{2} \left( \frac{\alpha_s}{\alpha_q} \right) (1 - s) + 2\tilde{q} - Q \right] (1 - s) \\
= \left[ Q - \frac{1}{2} \left( \frac{\alpha_s}{\alpha_q} \right) (1 - s) \right] (1 - s) + (2s - 1) \tilde{q}.
\]

Right-hand side. As with the analysis of LHS(\tilde{q}; s), I analyze RHS(\tilde{q}; s) by splitting the integral into parts,
\[
\text{RHS} (\tilde{q}; s) = \int_0^{\tilde{q}} (\psi^+(x; s) - b^i(\tilde{q}; s)) G^i_b(q; b^i(\cdot; s)) \, dq \\
+ \int_{\tilde{q}}^{q'} (\psi^+(x; s) - b^i(\tilde{q}; s)) G^i_b(q; b^i(\cdot; s)) \, dq \\
+ \int_{q'}^{\tilde{q}} (\psi^+(x; s) - b^i(\tilde{q}; s)) G^i_b(q; b^i(\cdot; s)) \, dq.
\]
It will suffice to understand the behavior of $G^i_b$ on each of these ranges. The analysis of each interval begins with the definitional equality
\[
G^i(q; b) = 1 - \psi^{-i}(Q - q; b) \implies G^i_b(q; b) = -\psi^{-i}_p(Q - q; b).
\]

First, let $q \in (0, \tilde{q})$. Then $G^i(q; b^i(\cdot; s)) = 0$. Since bids are symmetric, $G^i(q; b^i(\tilde{q}; s) + \varepsilon) = 0$ for all $\varepsilon > 0$, and for any such $\varepsilon$ there exists $\delta > 0$ such that $G^i(q'; b^i(\tilde{q}) - \varepsilon) = 0$ for all $q' < q - \delta$. Since bids are continuous, as $\varepsilon \to 0$ it can be assumed that $\delta \to 0$. It follows that an order-$\varepsilon$ change in $b$ does not change $G^i$ on this range, except possibly on a set with measure going to zero with $\varepsilon$. Thus $G^i_b(q; b^i(\tilde{q}; s)) = 0$ on this range.

Second, let $q \in (\tilde{q}, q')$. On this range, the agent is competing against opponents who are in the strictly-decreasing portion of their bid functions. It follows that $-\psi^{-i}_p(Q - q; b) = -1/b^{-i}_s(Q - q; s_{-i})$. Earlier analysis showed $p_s(s) = b_q(\tilde{q}(s); s)\tilde{q}(s) + b_s(\tilde{q}(s); s)$ and $b_q = -\alpha_q b_s/\alpha_s$. Additionally, $b_q(\tilde{q}(s); s) = b_q(q(s'); s'; s')$ for all $s' > s$. It follows that
\[
p_s(s) = \left( 1 - \frac{\alpha_q}{\alpha_s} \tilde{q}_s \right) b_s.
\]
Hence
\[ G^i_b(q; b^i(\cdot; s)) = -\frac{\alpha q}{\alpha s} \hat{q}_s. \]

Lastly, let \( q \in (q^f, \hat{q}) \). Then following any infinitesimal change in bid these quantities are still won “on the gap,” against the agents whose flats the deviation now beats. In this case,
\[ G^i(q; b^i(\cdot; s)) = \Pr (b^{-i}(Q - q; s_{-i}) \geq b^i(q; s)). \]

On the gap, \( b^{-i}(Q - q; s_{-i}) = p(s_{-i}). \) Then \( G^i(q; b^i(\cdot; s)) = 1 - p^{-1}(b) \). It follows that \( G^i_b(q; b^i(\cdot; s)) = -1/p_s(s) \).

Recognizing that \( b^i(\hat{q}; s) = p(s) \) in any solution to this system, RHS may be rewritten as
\[ \text{RHS}(\hat{q}; s) = -\int_{q^f}^{q^e} (v^i(x; s) - p(s)) \left( \frac{\alpha_s - \alpha_q \hat{q}_s(s)}{\alpha_s p_s(s)} \right) dq \]
\[ -\int_{q^e}^{\hat{q}} (v^i(x; s) - p(s)) \left( \frac{1}{p_s(s)} \right) dq. \]

Since \( v^i \) is linear in \( q \) and none of the \( p \) terms depend on \( q \), these integrals become
\[ \text{RHS}(\hat{q}; s) = -\left( \frac{\alpha_s - \alpha_q \hat{q}_s(s)}{\alpha_s p_s(s)} \right) \frac{\alpha_s}{\alpha_q} (1 - s) \]
\[ \times \left( \alpha_0 + s\alpha_s - (Q - \hat{q}(s)) \alpha_q + \frac{1}{2} \left( \frac{\alpha_s}{\alpha_q} \right) (1 - s) \alpha_q - p(s) \right) \]
\[ - \left( \frac{1}{p_s(s)} \right) (2\hat{q}(s) - Q) \left( \alpha_0 + s\alpha_s - \frac{Q}{2} \alpha_q - p(s) \right). \]

**Differential equation.** By construction, equilibrium requires \( \text{LHS}(\hat{q}; s) = -\text{RHS}(\hat{q}; s) \).

Omitting function arguments, this is
\[ \left( Q - \frac{1}{2} \left( \frac{\alpha_s}{\alpha_q} \right) (1 - s) \right) (1 - s) + (2s - 1) \hat{q} \]
\[ = \left( \frac{\alpha_s}{\alpha_q} - \hat{q}_s \right) \left( \alpha_0 + \frac{1}{2} (1 + s) \alpha_s - (Q - \hat{q}) \alpha_q - p \right) (1 - s) \]
\[ + (2\hat{q} - Q) \left( \alpha_0 + s\alpha_s - \frac{Q}{2} \alpha_q - p \right). \] (B.2)
To pin down the second dimension of this two-dimensional differential equation, recall that

\[
\frac{b^i}{(q; s_i)} = v^i (q; s_i) - s_{-i} p_s (s_{-i}).
\]

At \( \tilde{q} \), this is

\[
p(s) = v^i (\tilde{q} (s); s) - s p_s (s).
\]

Thus there is a second equation,

\[
p = \alpha_0 + s \alpha_s - \tilde{q} \alpha_q - sp_s. \tag{B.3}
\]

Equations (B.2) and (B.3) completely define a differential system which may be solved computationally. While equation (B.3) may be easily solved in an integral form—\( p = \int_0^s v^i(\tilde{q}; x)dx/s \)—and substituted into equation (B.2) to obtain a second-order ordinary differential equation, the unavoidable presence of product terms makes explicit analysis intractible no matter the simplification applied.
APPENDIX C

Appendix to Chapter 4

In this Appendix I present the main results of Chapter 4. This begins with a set of key Lemmas which will be used to ensure that certain technical properties of equilibrium behavior are satisfied.

C.1 Auxiliary results

Fix a pure-strategy candidate equilibrium \( \langle b^i \rangle_{i=1}^n \). Recall that bid functions are weakly decreasing, and that it is without loss to assume that they are right-continuous. Conditional on equilibrium bids, the market price (that is, the stop-out price) \( p(Q) \) is a function of realized supply \( Q \). All statements are made for relevant quantities, as given by Definition 2.1: for any bidder \( i \), the results generally ignore quantities above the maximum quantity this bidder can obtain in equilibrium (for instance, in Lemma C.1 all bidders could bid flat for units they never obtain).

**Lemma C.1** (Bids below values). *Bids are below values: \( b^i(q) \leq v(q) \) for all relevant quantities, and \( b^i(q) < v(q) \) for \( q < \varphi^i(p(Q)) \).*

**Proof.** This proof first establishes that agent \( i \) is never subject to “mutual ties,” in the sense that she and another agent are never simultaneously rationed. Suppose that there are \( q^i_\ell < q^i_r \) and some agent \( j \neq i \) with \( q^j_\ell < q^j_r \) such that, for all \( q^i \in [q^i_\ell, q^i_r] \) and \( q^j \in [q^j_\ell, q^j_r] \), \( b^i(q^i) = b^j(q^j) \). Let \( \bar{q}^i = E_Q[Q \mid p(Q) = b(q^i_r)] \), and without loss assume that agent \( i \) is such that \( \bar{q}^i < q^i_r \). If \( v(\bar{q}^i) < b^i(q^i) \), the agent
has a profitable downward deviation. If \( v(\bar{q}^i) \geq b^i(q^i_r) \), the agent has a profitable upward deviation: she can increase her bid slightly by \( \delta > 0 \) on \([q^i_\ell, q^i]\) (enforcing monotonicity constraints as necessary to the left of \( q^i_\ell \)), and submits her true value function on \([\bar{q}^i, q^i_r]\) (enforcing monotonicity constraints to the right of \( q^i_r \)).\(^1\)

Now suppose that there exists \( q \) with \( b^i(q) > v(q) \); because \( b^i \) is right-continuous and \( v \) is continuous, there must exist a range \((q_\ell, q_r)\) of relevant quantities such that \( b^i(q) > v(q) \) for all \( q \in (q_\ell, q_r) \). The agent wins quantities from this range with positive probability, and hence she could profitably deviate to

\[
\hat{b}^i(q) = \min \{ b^i(q), v(q) \}.
\]

By the first part of this proof, this deviation will never affect how she might be rationed, hence it is necessarily utility-improving.

Now consider \( q < \varphi^i(p(Q)) \). If \( b^i(q) = v(q) \) then right-continuity of \( b^i \) and Lipschitz-continuity of \( v \) imply that for small \( \varepsilon > 0 \) winning units \([q, q + \varepsilon] \) brings per-unit profit lower than \( \varepsilon \). By lowering the bid for this quantity to \( \hat{b}^i(q) = \min \{ b^i(q + \varepsilon), v(q + \varepsilon) \} \), the utility loss from losing the relevant quantities is bounded above by \( \varepsilon^2 (G^i(q + \varepsilon; b^i) - G^i(q; b^i)) \). Even if there is a probability mass at \( q \), the probability difference here goes to zero as \( \varepsilon \) goes to zero. At the same time the cost savings from paying lower bids at quantities higher than \( q + \varepsilon \) is of the order \( \varepsilon^2 \). Hence this deviation would be profitable. \( \square \)

**Lemma C.2** (No ties). For no price level \( p \) are there two or more bidders who, in equilibrium, bid \( p \) constant on some non-trivial intervals of relevant quantities.

**Proof.** The proof resembles similar proofs in other auction contexts; a smaller version of this result has already been established in the proof of Lemma C.1. Suppose agent \( i \) bids \( p \) on \((q_\ell, q_r)\) and bidder \( j \) bids \( p \) on \((q'_\ell, q'_r)\). Since the support of supply is \([0, \bar{Q}]\), it must be that \( G^i(q_r; b^i) > G^i(q'_r; b^i) \) and similarly for

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\(^1\)Because the proof conditions on her expected quantity, it is not necessary to directly consider whether quantities are relevant.
Lemma C.1 then implies that, shrinking $q$, if needed, it is safe assume that $v(q_t) < p$ and $q_r < \varphi^i(p(Q))$. But then bidder $i$ would gain by raising her bid on $(q_t, q_r)$ by small $\varepsilon > 0$ (and, if needed, raising the bids on lower units as little as necessary for her bid function to be weakly decreasing). Indeed, the cost of this deviation would be go to zero as $\varepsilon \downarrow 0$ while the quantity of the good the bidder would gain and the per-unit utility gain would be both bounded away from zero.

**Lemma C.3** (Price is strictly decreasing). The market price $p(Q)$ is strictly decreasing in supply $Q$.

**Proof.** Since bids are weakly decreasing in quantity, the market price is weakly decreasing as a direct consequence of the market-clearing equation. If price is not weakly decreasing in quantity at some $Q$, then a small increase in $Q$ will not only increase the price, but will weakly decrease the quantity allocated to each agent. This implies that total demand is no greater than $Q$, contradicting market clearing.

Lemma C.2 is sufficient to further imply that the market price must be strictly decreasing: at every relevant price level at which at least two bidders compete, at least one of their submitted bid functions is strictly decreasing. Furthermore, at no relevant price levels can it be that only one bidder, $i$, submits a flat bid at price $p$: at such a flat bidder $i$ can have no positive incentive to reduce her entire flat, it must be that there is some other agent $j$ whose bid function is right-continuous at price $p$. If $p = 0$, all opponents $j \neq i$ have a profitable deviation. If $p > 0$, the result follows from Lemma C.1. Given that $i$ submits a flat bid and the bids of bidder $j$ are strictly below her values for some non-trivial subset of quantities at which his bid is near $p$, bidder $j$ can then profit by slightly raising her bid; this

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2Recall that this result arises in the model in which marginal utilities on all possible units are strictly positive. This strict positivity assumption could be dispensed with by allowing for negative bids.

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reasoning is similar to that given in the proof of Lemma C.2.

\[\text{Lemma C.4} \text{ (Bids are strictly decreasing). Bid functions are strictly decreasing for relevant quantities.}\]

\[\text{Proof.} \text{ Lemma C.3 establishes that the market price function is strictly decreasing in supply. Suppose bidder } i \text{ bids flat at } b \text{ in a non-trivial interval } (q_\ell, q_r) \text{ of the relevant range of quantities. First note that it cannot be that } b = b^i (\varphi^i (p (Q))) \text{ as then the quantities in question are allocated with zero probability, and hence are irrelevant.}^3 \text{ Thus, } b > b^i (\varphi^i (p (Q))). \text{ Now, if no other bidder bids with positive probability in some interval } [p - \varepsilon, p], \text{ then bidder } i \text{ could profitably lower her bid on quantities } (q_\ell, q_r) \text{ (and possibly some other quantities above } q_r). \text{ If there is another bidder who bids with positive probability in } [p - \varepsilon, p] \text{ for every } \varepsilon > 0, \text{ then by Lemma C.1 this bidder earns strictly positive margin on relevant units and she could profitably raise her bid just above } p.\]

The monotonicity of bid functions established in Lemma C.4 implies that as long as \(\langle b^j \rangle_{j=1}^n\) is an equilibrium, the probability \(G^i (q; b^i)\) depends on \(b^i\) only through the value \(b^i (q)\).

The next Lemma defines the derivative of \(G^i\) with respect to the bid submitted, and establishes that it is well-defined almost everywhere.

\[\text{Lemma C.5 (Marginal probability of increased bid). For each agent } i \text{ and almost every } q,\]

\[G^i_q (q; b^i) = f \left( q + \sum_{j \neq i} \varphi^j (b^j (q)) \right) \cdot \sum_{j \neq i} \varphi^j_p (b^j (q)).\]

\[^3\text{In fact, as long as quantities are rationed in a monotonic way, even conditional on the zero-probability event that } Q = Q, \text{ agent } i \text{ would get no quantity from the flat.}\]
Proof. By definition, \( G^i(q; b^i) = \Pr(q_i \leq q|b^i) \). From market clearing, this is

\[
G^i(q; b^i) = \Pr\left( Q \leq q + \sum_{j \neq i} \varphi^j(b^i(q)) \right) = F\left( q + \sum_{j \neq i} \varphi^j(b^i(q)) \right).
\]

Where the demands \( \varphi^j \) of agents \( j \neq i \) are differentiable, this gives

\[
G^i_b(q; b^i) = f\left( q + \sum_{j \neq i} \varphi^j(b^i(q)) \right) \sum_{j \neq i} \varphi^j_p(b^i(q)) = 1 - G^i(q; b^i).
\]

Since for any agent \( j \) the demand function \( \varphi^j \) must be differentiable almost everywhere, the result follows.

\( \square \)

**Lemma C.6** (First-order conditions). At points where \( G^i_b(q; b^i) \) is well-defined, the first-order conditions for this model are given by

\[
- (v(q) - b^i(q)) G^i_b(q; b^i) = 1 - G^i(q; b^i).
\]

Equivalently, the first-order condition can be written as

\[
- (v(q) - b^i(q)) (Q_p(b^i(q)) - \varphi^i_p(b^i(q))) = H(Q(b^i(q))),
\]

where \( Q(p) \) is the inverse of \( p(Q) \).

Proof. The first equation arises from the same methods used to obtain the optimality condition given in equation (3.3). To derive the second expression, substitute into the above formulae for \( G^i \) and \( G^i_b \) given in Lemma C.5 to obtain

\[
- (v(q) - b^i(q)) f\left( q + \sum_{j \neq i} \varphi^j(b^i(q)) \right) \left( \sum_{j \neq i} \varphi^j_p(b^i(q)) \right) = 1 - F\left( q + \sum_{j \neq i} \varphi^j(b^i(q)) \right).
\]

Now, \( Q(p) \) is well-defined since Lemma C.3 shows that \( p \) is strictly monotonic. By Lemma C.4 bids are strictly monotonic in quantities, hence \( q + \sum_{j \neq i} \varphi^j(b^i(q)) = 144 \).
$Q(b^i(q))$, and

$$- (v(q) - b^i(q)) \left( \sum_{j \neq i} \varphi^j_p (b^j(q)) \right) = H(Q(b^i(q))).$$

Since $\sum_{j \neq i} \varphi^j_p (b^j(q)) = Q_p (b^i(q)) - \varphi^i_p (b^i(q))$, the second expression of the first order condition obtains.

**Lemma C.7** (Initial condition). The market-clearing price for the maximum possible quantity, $p(\overline{Q})$, is uniquely determined in equilibrium. The equilibrium quantities $q^i(\overline{Q})$ are also uniquely determined in equilibrium.

**Proof.** This result is tackled in two steps: first, if bid functions have finite slope for individual agents’ maximum quantities $q^i = q^i(\overline{Q})$, then bids meet values at $\overline{q}^i$; second, this must be the case. While this sequencing is out of proper logical order, the former is easier to demonstrate than the latter.

First, suppose that in a particular equilibrium each agent’s bid function has a finite slope at the maximum-obtainable quantity $\overline{q}^i$: there is $M \in \mathbb{R}_+$ such that $\limsup_{q \uparrow \overline{q}^i} (b^i(q) - b^i(\overline{q}^i))/(\overline{q}^i - q) < M$. This implies that $\liminf_{p \downarrow b^i} (\varphi^i(b^i) - \varphi^i(p))/(p - b^i) > 1/M$, where $b^i = b^i(\overline{q}^i)$. Without loss, assume that $b^i$ is left-continuous at $\overline{q}^i$.\(^4\) To account for the fact that the first-order condition only needs to hold almost everywhere, take the limit of the agent’s first-order condition as $q \uparrow \overline{q}^i$. Then

$$\lim_{q \uparrow \overline{q}^i} b^i(q) = \lim_{q \uparrow \overline{q}^i} \left[ v(q) + H(\sum_j \varphi^j (b^j(q))) \left( \sum_{j \neq i} \varphi^j_p (b^j(q)) \right)^{-1} \right].$$

As $q \uparrow \overline{q}^i$, $\sum_j \varphi^j (b^j(q)) \rightarrow \overline{Q}$. Since $f(Q) > 0$ everywhere, $H(\overline{Q}) = 0$; as $\liminf_{p \downarrow b^i} (\varphi^i(b^i) - \varphi^i(b))/(p - b^i)$ is bounded away from zero, it follows that $b^i(\overline{q}^i) = v(\overline{q}^i)$.

\(^4\) $b^i$ has already been presumed to be right-continuous everywhere, hence this additional assumption is equivalent to $b^i$ being continuous at $\overline{q}^i$. 145
Note that the above argument is valid as long as at least two agents’ bid functions have finite slope at $\bar{q}^i$, as the limit infimum will then be bounded away from zero for all agents. If only one agent’s bid function has finite slope at $\bar{q}^i$, then as demonstrated above all other agents $j \neq i$ must have $v(\bar{q}^j) = b^j(\bar{q}^j)$, and by assumption the slope of their bid functions is infinite. But since $v$ is Lipschitz continuous, this implies that $b^i(q) > v(q)$ for $q$ near $\bar{q}^i$, which cannot occur. Thus the only other possible condition is that all agents’ bid functions have infinite slope at $\bar{q}^i$, and again by Lipschitz continuity this requires that $v(\bar{q}^i) > b^i(\bar{q}^i)$ for all $i$.

It is helpful here to state the approach taken by the remainder of the proof. I first posit a deviation for agent $i$ which “kinks out” her bid function, and renders it flat to the right of some $q$ near $\bar{q}^i$. I then state an incentive compatibility condition which must be satisfied, since in equilibrium this deviation cannot yield a utility improvement. I notice that the ratio of additional bid to profit margin can be made arbitrarily small over the relevant regions of this new flat. Fixing a small deviation, I pick the agent who has the least probability of obtaining quantities affected by the deviation, and then find even smaller deviations for all other agents such that, ceteris paribus, they have an equal probability of their outcome being affected. Since these deviations cannot be profitable in equilibrium, this gives an inequality which must be satisfied by the sum of all agents’ incentives. I then find that, for sufficiently small deviations, this inequality cannot be satisfied, implying that for some agent a small deviation must be profitable. It follows that it cannot be the case that all agents have bids below values.

In this case, for a given agent $i$ and $\delta > 0$, consider a deviation $\hat{b}^i(\cdot; \delta)$ defined

\[5\] In what follows, I assume that all agents receive positive quantities with positive probability. When this is not the case, Lemma C.4 is sufficient to imply that at least two agents receive positive quantities with positive probability, and all subsequent arguments go through when attention is restricted to such agents.

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by

\[
\hat{b} (q; \delta) = \begin{cases} 
    b^i (q) & \text{if } q \leq \bar{q}^i - \delta, \\
    b^i (\bar{q}^i - \delta) \land v (q) & \text{otherwise.}
\end{cases}
\]

For small \( \delta \), this deviation will give the agent the full marginal market quantity for all realizations \( Q \geq \sum_{j=1}^{n} \varphi^j (b^j (\bar{q}^j - \delta)) \). Given this deviation, let \( q^* (q; \delta) \) be the quantity obtained under the deviation when, under the original strategy, the quantity would have been \( q \geq \bar{q}^i - \delta \). Explicitly,

\[
q^* (q; \delta) = \sum_j \varphi^j (b^j (q)) - \sum_{j \neq i} \varphi^j \left( \hat{b} (q; \delta) \right).
\]

This quantity will be used to analyze the additional quantity the deviation yields above baseline \( \bar{q}^i - \delta \). To simplify subsequent expressions, it is helpful to define \( \Delta^i \) functions,

\[
\Delta^i_L (q; \delta) = q - [\bar{q}^i - \delta] , \quad \Delta^i_R (q; \delta) = q^* (q; \delta) - q ,
\]

\[
\Delta^i (q; \delta) = \Delta^i_L (q; \delta) + \Delta^i_R (q; \delta).
\]

Incentive compatibility requires that this deviation cannot be profitable, hence the additional costs must outweigh the additional benefits,

\[
\int_{\bar{q}^i - \delta}^{\bar{q}^i} \int_{\bar{q}^i - \delta}^{q} \hat{b} (x; \delta) - b^i (x) \, dx \, dG^i (q; b^i) \\
\geq \int_{\bar{q}^i - \delta}^{\bar{q}^i} \int_{q}^{q^* (q; \delta)} v (x) - \hat{b} (x; \delta) \, dx \, dG^i (q; b^i) ,
\]

\[
\Rightarrow \int_{\bar{q}^i - \delta}^{\bar{q}^i} \int_{\bar{q}^i - \delta}^{q} b^i (\bar{q}^i - \delta) - b^i (\bar{q}^i) \, dx \, dG^i (q; b^i) \\
\geq \int_{\bar{q}^i - \delta}^{\bar{q}^i} \int_{q}^{q^* (q; \delta)} v (q^* (q; \delta)) - b^i (\bar{q}^i - \delta) \, dx \, dG^i (q; \delta) .
\]

Because in the latter expression the inner integrands are constant, the inequality may be expressed in terms of the \( \Delta^i \) functions defined above,

\[
\int_{\bar{q}^i - \delta}^{\bar{q}^i} \Delta^i_L (q; \delta) (b^i (\bar{q}^i - \delta) - b^i (\bar{q}^i)) \, dG^i (q; b^i) \\
\geq \int_{\bar{q}^i - \delta}^{\bar{q}^i} \Delta^i_R (q; \delta) (v (q^* (\bar{q}^i; \delta)) - b^i (\bar{q}^i - \delta)) \, dG^i (q; \delta) .
\]
Since it is assumed, to derive a contradiction, that \( v(q^i) > b^i(q^i) \) for all \( i \), for all \( \kappa > 0 \) there is a \( \delta^i > 0 \) such that for all \( \delta' \in (0, \delta^i) \) it must be that
\[
 b^i(q^i - \delta') - b^i(q^i) \leq \kappa \left( v(q^i; \delta') - b^i(q^i - \delta') \right).
\]

For any such \((\kappa, \delta')\),
\[
\kappa \int_{\bar{q} - \delta}^{\bar{q}} \Delta_L^i(q; \delta') \, dG^i(q; \bar{b}^i) > \int_{\bar{q} - \delta}^{\bar{q}} \Delta_R^i(q; \delta') \, dG^i(q; \bar{b}^i).
\]
With a finite number of agents, for any \( \kappa > 0 \) there is a \( \delta' \) such that for all \( \delta' < \bar{\delta} \) the above inequality is satisfied. Picking such a \( \bar{\delta} \), let agent \( i \) be such that \( i \in \arg\max_k G^k(q^* - \bar{\delta}; b^k) \), and for each agent \( j \) let \( \hat{\delta}^j \leq \delta \) be defined by
\[
 G^j(q^j - \hat{\delta}^j; b^j) = G^i(q^i - \delta; b^i).
\]
Let \( Q_{\bar{\delta}} = Q - \sum_j \hat{\delta}^j \), and note that \( F(Q_{\bar{\delta}}) = G^j(q^j - \hat{\delta}^j) \) for all \( j \). The above incentive compatibility argument must hold for each agent, hence summing over all agents gives
\[
\kappa \sum_{j=1}^{n} \int_{\bar{q} - \hat{\delta}^j}^{\bar{q}} \Delta_L^j(q; \hat{\delta}^j) \, dG^j(q; \bar{b}^j) / F(Q_{\bar{\delta}}) > \sum_{j=1}^{n} \int_{\bar{q} - \hat{\delta}^j}^{\bar{q}} \Delta_R^j(q; \hat{\delta}^j) \, dG^j(q; \bar{b}^j) / F(Q_{\bar{\delta}}).
\]
Writing this in terms of conditional expectations gives
\[
\kappa \sum_{j=1}^{n} \mathbb{E}_Q \left[ \Delta_L^j(q; \hat{\delta}^j) \mid Q \geq Q_{\bar{\delta}} \right] > \sum_{j=1}^{n} \mathbb{E}_Q \left[ \Delta_R^j(q; \hat{\delta}^j) \mid Q \geq Q_{\bar{\delta}} \right]
= \sum_{j=1}^{n} \mathbb{E}_Q \left[ \Delta^j(q; \hat{\delta}^j) \mid Q \geq Q_{\bar{\delta}} \right] - \mathbb{E}_Q \left[ \Delta_L^j(q; \hat{\delta}^j) \mid Q \geq Q_{\bar{\delta}} \right].
\]
By definition, \( \Delta_L^j(q; \hat{\delta}^j) = q - [q^j - \hat{\delta}^j] \), and it must be that \( \Delta^j(q; \hat{\delta}^j) = Q - [q^j - \hat{\delta}^j] \). Therefore
\[
(\kappa + 1) \sum_{j=1}^{n} \mathbb{E}_Q \left[ q^j - [q^j - \hat{\delta}^j] \mid Q \geq Q_{\bar{\delta}} \right] > \sum_{j=1}^{n} \mathbb{E}_Q \left[ Q - [q^j - \hat{\delta}^j] \mid Q \geq Q_{\bar{\delta}} \right].
\]
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Since $\sum_{j=1}^{n} q^j = Q$, this becomes

$$(\kappa + 1) (\mathbb{E}_Q [Q | Q \geq Q_{\delta}] - Q_{\delta}) > n (\mathbb{E}_Q [Q | Q \geq Q_{\delta}] - Q_{\delta}).$$

Dividing through gives $(\kappa + 1)/n > 1$, and this inequality does not depend on $\delta$; since $\kappa > 0$ may be arbitrarily small, this is a contradiction. Thus it cannot be that $v(q^j) > b^i(q^i)$ for all agents $i$. Having already established that when at least one agent $i$ has $v(q^i) = b^i(q^i)$ is must be the case that $v(q^i) = b^i(q^i)$ for all agents $j$, it follows that all agents submit bid functions which equal their marginal values at their maximum possible quantity.

With $b^i(q^i) = v(q^i)$ for all $i$, the remainder of the proof is immediate. By market clearing, $p(Q) = v(q^i)$ for all $i$; inverting gives

$$Q = \sum_{i=1}^{n} v^{-1} (p(Q)).$$

Since $v$ is strictly decreasing in $q$, there is a unique solution to this equation. From $q^i = v^{-1}(p(Q))$, it follows that quantities are also unique. \qed

**Lemma C.8.** Every equilibrium is symmetric.

**Proof.** Consider an equilibrium, possibly asymmetric. Let $p(Q)$ be its market price function. By Lemma C.3 this function is strictly decreasing, and hence it has an inverse, denoted $Q(p)$, which is differentiable almost everywhere.

Let $\varphi^i$ be bidder $i$’s demand function. Together with the model’s assumptions, Lemmas C.4, C.6, and C.7 imply that this function is weakly-decreasing, continuous, and that it almost everywhere satisfies

$$-(Q_p(p) - \varphi^i_p(p)) (v(\varphi^i(p)) - p) = H(Q(p)),$$

$v(\varphi^i(p)) - p > 0$ for $p > p(Q)$, and $\varphi^i(p(Q)) = Q/n$. Since these five conditions are identical for all bidders, in order to show that the equilibrium is symmetric it

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6The demand function $\varphi^i$ is continuous because the bid functions are strictly decreasing.
is enough to prove that these five conditions uniquely determine bidder $i$’s demand function.

For an indirect argument, suppose that $\hat{\varphi}^i$ is another function satisfying the above five conditions. Then there is $p' > p(\overline{Q})$ such that $\varphi^i(p') \neq \hat{\varphi}^i(p')$, and by the continuity of $\varphi^i$ the same is true on some small interval around $p'$; in particular it is without loss to assume that $\varphi^i$ and $Q(\cdot)$ are differentiable at $p'$. Suppose that $\varphi^i(p') > \hat{\varphi}^i(p')$; the proof in the other case follows the same steps. The monotonicity of $\nu$ gives $v(\varphi^i(p')) < v(\hat{\varphi}^i(p'))$, hence the above-displayed equation, the strict positivity of the right-hand side, and the strict positivity of the second factor on the left-hand side together imply that $\varphi^i_p(p') > \hat{\varphi}^i_p(p')$. Thus, the difference $\varphi^i(\tilde{p}) - \hat{\varphi}^i(\tilde{p})$ increases as $\tilde{p}$ increases locally near $p'$. This implies that $\varphi^i(\tilde{p}) > \hat{\varphi}^i(\tilde{p})$ for all $\tilde{p} \leq p'$. This contradicts $\varphi^i$ and $\hat{\varphi}^i$ taking the same value at $p(\overline{Q}) < p'$, and concludes the proof. 

C.2 Proof of Theorem 4.1: Equilibrium existence

Although in Chapter 4, Theorem 4.1 is presented prior to all other results, its proof builds on Theorems 4.2 and 4.3. The arrangement in the Chapter 4 is made only to place immediate qualifications on when the investigation of bidding behavior is well-founded. As seen earlier in this Appendix, the proofs of the latter two Theorems do not depend on Theorem 4.1.

Proof. I look to prove that the candidate equilibrium constructed in Theorem 4.3 is in fact an equilibrium. Fix a bidder $i$ to be the target of incentive analysis, and assume that other bidders follow the strategies given in Theorem 4.3 when bidding on quantities $q \leq \overline{Q}/n$ and that they bid $v(\overline{Q}/n)$ for quantities they never win.\footnote{When proving the analogue of Theorem 4.1 in the context of reserve prices, this expression must be adjusted to deal with the induced “maximum” quantity: in the analogue, $\overline{Q}$ becomes $n\nu^{-1}(R)$. The remainder of the argument does not change.} Since bids and values are weakly decreasing, in equilibrium there is no incentive
for bidder $i$ to obtain any quantity $q > \overline{Q}/n$, and it is sufficient to check that bidder $i$ finds it optimal to submit the bid function prescribed by Theorem 4.3 for quantities $q < \overline{Q}/n$. Thus, agent $i$ maximizes

$$\int_0^{\overline{Q}/n} \left( v(q) - b(q) \right) \left( 1 - G^i(q; b(q)) \right) dq$$

over weakly decreasing functions $b(\cdot)$.

To show that the function given by Theorem 4.3 is utility-maximizing, it is sufficient to ignore the monotonicity constraint because the prescribed bid function is strictly monotonic. The problem can then be analyzed by pointwise maximization: for each quantity $q \in [0, \overline{Q}/n]$ the agent finds $b(q)$ that maximizes $(v(q) - b(q))(1 - G^i(q; b(q)))$. Therefore, demonstration of sufficiency can rely on one-dimensional optimization strategies to assert conditions for a maximum. The agent’s first-order condition from Lemma C.6 is

$$- (1 - G^i(q; b)) - (v(q) - b)G^i_{bb}(q; b) = 0.$$ 

This first order condition on $b$ yields the bidding strategy of Theorem 4.3, and it remains to verify that the sufficient condition of Theorem 4.1 implies that the sufficient second-order condition for a maximum,

$$2G^i_{bb}(q; b) - (v(q) - b)G^i_{bb}(q; b) < 0,$$

is satisfied for any $q$ when all bidders submit the bid function of Theorem 4.3.

Aside. To justify the remark following Theorem 4.1, notice that the calculus of variations implies that the weak-inequality counterpart of this sufficient condition is a necessary condition for a bidder to achieve a utility maximum while bidding $b$, and hence for the existence of a pure-strategy equilibrium.

To complete the proof I show that the algebraic expression of the sufficient condition in Theorem 4.1 is equivalent to the sufficient condition derived above.
The following equalities are useful (see Lemma C.6):

\[ G(q; b) = F(q + (n - 1) \varphi(b)), \]
\[ G_b(q; b) = (n - 1) f_q(q + (n - 1) \varphi(b)) \varphi_p(b), \]
\[ G_{bb}(q; b) = (n - 1)^2 f'(q + (n - 1) \varphi(b)) \varphi_p(b)^2 + (n - 1) f(q + (n - 1) \varphi(b)) \varphi_{pp}(b). \]

Notice that, as I am verifying that a profile of symmetric strategies represents mutual best-responding, in these expressions the superscript \( i \) on \( G^i \) is suppressed, and \( \varphi \) denotes the symmetric demand function of any bidder \( j \neq i \), any bidder other than the bidder whose incentives are being analyzed. This demand function common to other bidders is given as the inverse of the bid function \( b(\cdot) \) in Theorem 4.3. Standard rules of differentiation give \( \varphi_p = \frac{1}{b_q} \) and \( \varphi_{pp} = -\frac{b_{qq}}{b_q^2} \). The above equalities allow the second-order condition (C.1) to be equivalently expressed as

\[ 2f b_q^2 < (v - b) ((n - 1) f' b_q - f b_{qq}) . \]

Denoting \( \bar{H}(q) = \frac{1 - F(nq)}{(n - 1)f(nq)} \), the agent’s first order condition may be expressed as

\[ b_q(q) = \frac{b(q) - v(q)}{H(q)}. \]

Differentiation gives

\[ b_{qq} = \frac{b_q - v_q}{H} - \frac{(b - v) \bar{H}_q}{H^2} = \frac{(b - v)(1 - \bar{H}_q)}{H^2} - \frac{v_q}{H}. \]

The second-order condition is imposed at points satisfying the first-order condition; thus substituting in for \( b_q \) and \( b_{qq} \), the second-order condition is further re-expressed as

\[ 2f \left[ \frac{b - v}{H} \right]^2 < (v - b) \left( (n - 1) f' \left[ \frac{b - v}{H} \right] - f \left[ \frac{(b - v)(1 - \bar{H}_q)}{H^2} - \frac{v_q}{H} \right] \right) . \]

Since \( v - b > 0 \) for \( q \in [0, \overline{Q}/n) \), algebraic manipulation gives the expression in the main text. \( \square \)
C.3 Proof of Theorem 4.2: Uniqueness

Proof. From Lemma C.6 and market clearing, bidder $i$’s first-order conditions are given by

$$(p(Q) - v(q)) G^b_i (q; b^i) = 1 - G^i (q; b^i).$$

Since Lemma C.8 shows that agents’ strategies are symmetric, Lemma C.5 allows the first-order condition to be restated as

$$\left( p(Q) - v \left( \frac{Q}{n} \right) \right) (n-1) \varphi_p (p(Q)) = H (Q).$$

From market clearing, $p(Q) = b(Q/n)$; thus $p_Q(Q) = b_q(Q/n)/n$. Additionally, standard rules of inverse functions give $\varphi_p (p(Q)) = 1/b_q(Q/n)$ almost everywhere. Thus

$$\left( p(Q) - v \left( \frac{Q}{n} \right) \right) \frac{n-1}{n} = H (Q) p_Q (Q).$$

Now suppose that there are two solutions, $p$ and $\hat{p}$. From Lemma C.7, $p(Q) = \hat{p}(Q)$ at the maximum quantity. Suppose that there is a $Q$ such that $\hat{p}(Q) > p(Q)$; taking $Q$ near the supremum of $Q$ for which this strict inequality obtains gives $\hat{p}_Q(Q) < p_Q(Q)$. But this implies

$$\hat{p}(Q) > p(Q) = v \left( \frac{Q}{n} \right) + \left( \frac{n}{n-1} \right) H (Q) p_Q (Q)$$

$$> v \left( \frac{Q}{n} \right) + \left( \frac{n}{n-1} \right) H (Q) \hat{p}_Q (Q) = \hat{p}(Q).$$

The right-continuity of bids, and hence of $p$, implies that if $p$ solves the first-order conditions, $\hat{p}$ cannot.

---

8The inequality inversion here from usual derivative-based approaches reflects the fact that bid construction is “working backward” from $\overline{Q}$, while any solution must be weakly decreasing: thus a small reduction in $Q$ should yield $\hat{p}(Q) = p(Q) \leq p < \hat{p}$. 

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C.4 Proof of Theorem 4.3: Bid representation

Proof. From the first order condition established in the proof of uniqueness, the equilibrium price satisfies

\[ p_Q = \hat{p} \hat{H} - \hat{v} \hat{H}, \]

where \( \hat{v}(x) = v(x/n) \), and \( \hat{H}(x) = 1/H(x) \). The solution to this equation has general form

\[ p(Q) = \exp \left( \int_0^Q \tilde{H}(x) \, dx \right) \left( C - \int_0^Q \exp \left( - \int_0^x \tilde{H}(y) \, dy \right) \tilde{H}(x) \hat{v}(x) \, dx \right), \]

parametrized by \( C \in \mathbb{R} \). Define \( \rho = \frac{n-1}{n} \in [\frac{1}{2}, 1) \). Basic manipulation gives \( \tilde{H} = -\rho \frac{d}{dQ} \ln(1 - F) \). The exponential terms may therefore be replaced,

\[ \exp \left( \int_0^t \tilde{H}(x) \, dx \right) = \exp \left( -\rho \int_0^t \frac{\partial}{\partial x} \ln (1 - F(x)) \, dx \right) \]
\[ = \exp \left( -\rho (\ln (1 - F(t)) - \ln 1) \right) \]
\[ = (1 - F(t))^{-\rho}. \]

Substituting and canceling, for \( Q < \overline{Q} \):\(^9\)

\[ p(Q) = \left( C - \rho \int_0^Q f(x) (1 - F(x))^{\rho-1} \hat{v}(x) \, dx \right) (1 - F(Q))^{-\rho}. \quad (C.2) \]

\( C \) may be found by evaluating the price function at \( Q = \overline{Q} \). Since \( 1 - F(\overline{Q}) = 0 \), it must be that \( C = \rho \int_0^Q f(x) (1 - F(x))^{\rho-1} \hat{v}(x) \, dx \). The market price is then given by

\[ p(Q) = \rho \int_0^Q f(x) (1 - F(x))^{\rho-1} \hat{v}(x) \, dx \quad (1 - F(Q))^{-\rho}. \]

Since \( d/dy[F^{Q,n}(y)] = \rho f(y)(1 - F(y))^{\rho-1}(1 - F(Q))^{-\rho} \), the suggested formula for market price obtains; since Lemma C.8 shows that equilibrium bids are symmetric, the formula for bids obtains as well. \( \square \)

\(^9\)At \( Q = \overline{Q} \), \( \ln(1 - F) \) is undefined. The price at this quantity is determined by Lemma C.7, and is equal to the limit of nearby prices for smaller quantities.
C.5 Proof of Theorem 4.5: Optimal supply

To begin, it is necessary to setup the seller’s revenue-maximization problem. Conditional on the distribution of supply, the seller’s expected revenue is

\[ \mathbb{E}_Q[\pi] = \int_0^Q \int_0^Q p(x) \, dx \, dF(Q). \]

Integrating by parts, this is

\[ \mathbb{E}_Q[\pi] = -\left[ \int_0^Q p(x) \, dx \, (1 - F(Q)) \right]_{Q=0}^{Q=Q} + \int_0^Q p(Q) \, (1 - F(Q)) \, dQ. \]

Notice that the left-hand term is 0. Hence

\[ \mathbb{E}_Q[\pi] = \int_0^Q p(Q) \, (1 - F(Q)) \, dQ. \]

This solution is intuitive: the seller obtains payment \( p(Q) \) with exactly the probability that the supply is at least \( Q \).

Substituting in for \( p \) as determined by Theorem 4.3,

\[ \mathbb{E}_Q[\pi] = \int_0^Q \int_0^Q (n - 1) v\left(\frac{x}{n}\right) \left(\frac{f(x)}{1 - F(x)}\right) \left(\frac{1 - F(x)}{1 - F(Q)}\right)^{n-1} dx \, (1 - F(Q)) \, dQ. \]

Letting \( J(Q) = (1 - F(Q))^{(n-1)/n} \), the seller’s problem is

\[ \max \left\{ -n \int_0^Q \int_0^Q v\left(\frac{x}{n}\right) J_Q(x) J(Q)^{\frac{1}{n}} \, dx dQ \right\}. \]

**Lemma C.9** (Deterministic supply is best). Let \( Q^* \) be the optimal monopoly quantity, and let \( F \) be some quantity distribution function. Then the revenue generated by any continuous distribution function \( F \) is weakly dominated by the revenue generated by deterministically selling \( Q^* \) at the monopoly price.

**Proof.** Rewrite the expected revenue form as

\[ \mathbb{E}_Q[\pi] = -n \int_0^Q v\left(\frac{Q}{n}\right) J_Q(Q) \int_0^Q J(x)^{\frac{1}{n}} \, dx dQ. \]
\( J(Q) \leq 1 \) everywhere, and \( 1/(n-1) \leq 1 \) as well. It follows that \( \int_0^Q J(x)^{1/(n-1)} \, dx \leq Q \). Then expected revenue is bounded by

\[
\mathbb{E}_Q [\pi] \leq -n \int_0^Q Q_v \left( \frac{Q}{n} \right) J_Q(Q) \, dQ.
\]

From the monopolist’s problem, for all \( Q \) it is the case that \( Q_v(Q/n) \leq Q^*v(Q^*/n) \).

Then

\[
\mathbb{E}_Q [\pi] \leq -n Q^*v \left( \frac{Q^*}{n} \right) \int_0^Q J_Q(Q) \, dQ.
\]

Since \(-J_Q\) is a valid PMF, it must be that

\[
\mathbb{E}_Q [\pi] \leq n Q^*v \left( \frac{Q^*}{n} \right) .
\]

This is exactly the revenue from the monopolist’s problem, hence the monopolist’s posted-price solution weakly dominates all other quantity distributions. \( \square \)

Lemma C.9 directly implies Theorem 4.5.

### C.6 Calculations for examples

**Derivative of the log of the inverse hazard rate, generalized Pareto distributions.**

For a generic distribution, the derivative of the log of the inverse hazard rate is

\[
\frac{d}{dx} \ln H(x) = -\frac{f(x)^2 + (1 - F(x)) f'(x)}{(1 - F(x)) f(x)}.
\]

For generic generalized Pareto distributions with parameter \( \alpha > 0 \),

\[
1 - F(x) = \left( 1 - \frac{x}{Q} \right)^\alpha,
\]
\[
f(x) = \frac{\alpha}{Q} \left( 1 - \frac{x}{Q} \right)^{\alpha-1},
\]
\[
f'(x) = \frac{\alpha - \alpha^2}{Q^2} \left( 1 - \frac{x}{Q} \right)^{\alpha-2}.
\]

Substituting in gives

\[
\frac{d}{dx} \ln H(x) \big|_{x=Q} = -\frac{1}{Q - Q} .
\]

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Right-hand side of sufficient condition, linear marginal values. Let \( v(q) = \beta_0 - q\beta_q \). Then

\[
\frac{n-1}{n} \left( \frac{v\left(\frac{Q}{n}\right)}{v\left(\frac{Q}{n}\right) - \int_{Q}^{\infty} v\left(\frac{z}{n}\right) dF_{Q,n}(x)} \right)
= -\beta_q \left( \frac{n-1}{n} \right) \left( \beta_0 - \frac{Q}{n} \beta_q - \int_Q^{\infty} \beta_0 - \frac{x}{n} \beta_q dF_{Q,n}(x) \right)^{-1}.
\]

Significant cancelation gives that this is equal to

\[
- (n-1) \left( \int_Q^{\infty} xdF_{Q,n}(x) - Q \right)^{-1}.
\]

(C.3)

Linear marginal values with generalized Pareto distribution of supply. The sufficiency condition given in Theorem 4.1 gives

\[
- \frac{1}{Q - Q} > - (n-1) \left( \int_Q^{\infty} xdF_{Q,n}(x) - Q \right)^{-1}.
\]

This is more clearly stated as

\[
\int_Q^{\infty} xdF_{Q,n}(x) - Q < (n-1) (Q - Q).
\]

Since \( n - 1 \geq 1 \) and \( \int_Q^{\infty} xdF_{Q,n}(x) < Q \) for \( Q < Q \), it follows that this parameterization satisfies the sufficiency condition, and Theorem 4.3 provides equilibrium strategies.

Linear marginal values with large \( n \). The right-hand side of the sufficient condition in this case, equation (C.3), is

\[
- (n-1) \left( \int_Q^{\infty} xdF_{Q,n}(x) - Q \right)^{-1}.
\]

For \( Q < Q \), the right multiplicand is strictly positive. With large \( n \) this expression becomes arbitrarily negative (as discussed in the main text, the effect on \( dF_{Q,n}(x) \) points in the same direction). Then for large \( n \) the sufficient condition will be satisfied, independent of the underlying distribution.
Linear marginal values with generalized Pareto distribution of supply imply linear bids. Recall Theorem 4.3,
\[ b(q) = \int_{nq}^{\overline{Q}} v \left( \frac{x}{n} \right) dF_{nq,n}(x). \]
Integrating by parts gives
\[ b(q) = \beta_0 - q\beta_q - \frac{\beta_q}{n} \int_{nq}^{\overline{Q}} 1 - F_{nq,n}(x) \, dx. \]
With generalized Pareto distributions with parameter \( \alpha > 0 \), this simplifies to
\[ 1 - F_{nq,n}(x) = \left( \frac{\overline{Q} - x}{\overline{Q} - nq} \right)^\alpha (\frac{\alpha - 1}{\alpha}). \]
Integrating, the bid function is
\[ b(q) = \beta_0 - q\beta_q - \frac{\beta_q}{\alpha(n-1) + n(\overline{Q} - nq)}. \]
Bids are therefore linear in \( q \).

C.7 A note on reserve prices

Section 4.4.1 investigates the implementation of reserve prices, and shows that imposing a binding reserve price is equivalent to creating an atom at the quantity at which marginal value equals to the reserve price. In order to extend the main results of Chapter 4 to the setting with reserve prices, it is necessary to extend them to distributions in which there might be an atom at the upper bound of support \( \overline{Q} \). All of the results remain true, and the proofs go through without much change except for the end of the proof of Theorem 4.3, where more care is needed.

The proof of Theorem 4.3 goes through until the claim that \( 1 - F(\overline{Q}) = 0 \); in the presence of an atom at \( \overline{Q} \) this claim is no longer valid. To work around this difficulty, multiply both sides of equation (C.2) by \( (1 - F(Q))^\rho \), giving
\[ p(Q)(1 - F(Q))^\rho = C - \rho \int_0^Q f(x)(1 - F(x))^{\rho - 1} \hat{v}(x) \, dx. \]
Now, let $\tilde{F}(\overline{Q}) \equiv \lim_{Q \uparrow \overline{Q}} F(Q')$. Because the market price and the right-hand integral are continuous as $Q \uparrow \overline{Q}$, this yields

$$p(\overline{Q}) \left( 1 - \tilde{F}(\overline{Q}) \right) = C - \rho \int_{0}^{\overline{Q}} f(x) (1 - F(x))^{\rho-1} \hat{v}(x) \, dx.$$ 

The parameter $C$ is determined by this equation. The market price function is then

$$p(Q) = \left( \frac{1 - \tilde{F}(Q)}{1 - F(Q)} \right)^{\rho} p(\overline{Q}) + \rho \int_{Q}^{\overline{Q}} f(x) (1 - F(x))^{\rho-1} \hat{v}(x) \, dx \left( 1 - F(Q) \right)^{-\rho}.$$ 

Recall from Lemma C.7 that $p(\overline{Q}) = v(\overline{Q}/n)$. Extending the limiting notation applied to $F$ to the auxiliary distribution $F^{Q,n}$ gives

$$F^{Q,n}(Q) - F^{Q,n}(\overline{Q}) = 1 - F^{Q,n}(\overline{Q}) = \left( \frac{1 - \tilde{F}(Q)}{1 - F(Q)} \right)^{\rho}.$$ 

Since $d/dy[F^{Q,n}(y)] = \rho f(y)(1 - F(y))^{\rho-1}(1 - F(Q))^{-\rho}$ for all $Q, y < \overline{Q}$,

$$p(Q) = \left( F^{Q,n}(Q) - F^{Q,n}(\overline{Q}) \right) v\left( \frac{Q}{n} \right) + \int_{Q}^{\overline{Q}} v\left( \frac{x}{n} \right) \frac{d}{dy} \left[ F^{Q,n}(y) \right]_{y=x} \, dx.$$ 

$$= \int_{Q}^{\overline{Q}} v\left( \frac{x}{n} \right) dF^{Q,n}(x).$$

This verifies the formula for the equilibrium market-clearing price in the presence of a reserve price.
D.1 Proof of Proposition 5.1: Revenue maximization

Proof. Recall the seller’s expected revenue,
\[
\mathbb{E}_Q [\pi] = \mathbb{E}_Q \left[ \alpha \int_0^Q p(x) \, dx + (1 - \alpha) Q p(Q) \right].
\]
Since the market-clearing price is linear in quantity, this is
\[
\mathbb{E}_Q [\pi] = \mathbb{E}_Q \left[ Q p_0 + \left( 1 - \frac{\alpha}{2} \right) Q^2 p_Q \right].
\]
Relying on the uniform distribution of market quantity, the expectation simplifies to
\[
\mathbb{E}_Q [\pi] = \frac{1}{2} Q p_0 - \frac{1}{3} \left( 1 - \frac{\alpha}{2} \right) Q^2 p_Q.
\]
When the seller seeks to maximize revenue by optimally setting the randomization parameter \( \alpha \), the objective is therefore
\[
\max_{\alpha} \frac{1}{2} Q p_0 - \frac{1}{3} \left( 1 - \frac{\alpha}{2} \right) Q^2 p_Q \rightarrow \max_{\alpha} 3 p_0 - (2 - \alpha) Q p_Q.
\]
To find attain a maximum, the first-order condition with respect to \( \alpha \) is
\[
\frac{d}{d\alpha} : 3 \frac{dp_0}{d\alpha} - 2 Q \frac{dp_Q}{d\alpha} + Q p_Q + \alpha Q \frac{dp_Q}{d\alpha}.
\]
Equations (5.4) imply
\[
\frac{dp_0}{d\alpha} = - \left( [(n - 2) + (n + 1) \alpha] Q - (n + 1) \alpha Q \right) \left( \frac{1}{(n - 2) + (n + 1) \alpha} \right)^2 \nu_q,
\]
\[
= - (n - 2) \left( \frac{1}{(n - 2) + (n + 1) \alpha} \right)^2 Q \nu_q,
\]
\[
\frac{dp_Q}{d\alpha} = - \left( \frac{n - 1}{n} \right) \left( \frac{1}{(n - 2) + (n + 1) \alpha} \right)^2 (n + 1) \nu_q.
\]
Substituting and removing common positive coefficients, equation (D.1) becomes

\[-3(n-2) + 2 \left( \frac{n-1}{n} \right) (n+1) \]
\[+ \left( \frac{n-1}{n} \right) ((n-2) + (n+1) \alpha) - \alpha \left( \frac{n-1}{n} \right) (n+1) \]
\[= -3(n-2) + 3n \left( \frac{n+1}{n} \right) = 3. \]

Then the derivative of expected revenue with respect to \( \alpha \) is unambiguously positive for all \( \alpha \in [0,1] \), and expected revenue will therefore be maximized when \( \alpha = 1 \). This immediately implies that the pure pay-as-bid auction maximizes seller revenue. \( \square \)

D.2 Auxiliary results for Proposition 5.2: Upper bound on revenue

**Lemma D.1.** In any symmetric pure-strategy equilibrium, the equilibrium price function must be continuous at all \( Q \in (0, \overline{Q}) \).

**Proof.** Suppose that \( p \) has a discontinuity at \( Q \), and define \( \overline{p} = \lim_{x \uparrow Q} p(x) \) and \( \underline{p} = \lim_{x \downarrow Q} p \). Let \( \delta = \overline{p} - \underline{p} \). Note that if \( p \) has a discontinuity, symmetric equilibrium bids also have a discontinuity. For any \( \varepsilon > 0 \), define a deviation \( \hat{b}^\varepsilon \) such that

\[ \hat{b}^\varepsilon (q) = \begin{cases} 
    b(q) & \text{if } nq \notin [Q - \varepsilon, Q], \\
    \underline{p} & \text{otherwise.} 
\end{cases} \]

With probability \( \alpha(1 - F(Q)) \) this deviation yields savings of at least \( \varepsilon \delta \), and with probability \((1 - \alpha)(F(Q) - F(Q - \varepsilon)) \) this deviation yields savings of at least \((Q/n - \varepsilon)\delta\). With probability \( F(Q) - F(Q - n\varepsilon) \) this deviation sacrifices utility of no more than \((v(Q/n - \varepsilon) - \overline{p})\varepsilon\). Comparing benefits to costs gives

\[ \alpha(1 - F(Q)) \varepsilon \delta + (1 - \alpha)(F(Q) - F(Q - \varepsilon)) \left( \frac{1}{n}Q - \varepsilon \right) \delta \]
\[ \geq (F(Q) - F(Q - n\varepsilon)) \left( v \left( \frac{1}{n}Q - \varepsilon \right) - \overline{p} \right) \varepsilon. \]
For small \( \varepsilon > 0 \), the benefits (left-hand side) outweigh the costs (right-hand side) and the deviation is profitable.

**Lemma D.2.** Two continuous solutions to the equilibrium market-price equation cannot strictly cross one another. That is, if \( p^1 \) and \( p^2 \) are solutions to the market-price equation and there is \( Q \) such that \( p^1(Q) > p^2(Q) \), there is no \( Q' \) such that \( p^1(Q') < p^2(Q') \).

**Proof.** Let \( p^1 \) and \( p^2 \) be two equilibrium market-price equations, and suppose that there is \( Q \) such that \( p^1(Q) > p^2(Q) \). Since any solution \( p \) is such that \( p_Q = (p - v)\tilde{H} \), it must be that \( p^1_Q - p^2_Q = (p^1 - p^2)\tilde{H} \) wherever both functions are differentiable. Thus where the functions are differentiable and \( p^1 > p^2 \), \( p^1_Q > p^2_Q \). If the functions strictly cross, without loss there is \( Q' > Q \) such that \( p^1(Q') < p^2(Q') \), and hence by continuity there is \( \bar{Q} \in (Q, Q') \) with \( \bar{\varepsilon} > 0 \) such that for all \( \varepsilon \in (0, \bar{\varepsilon}) \),

\[
p^1(\bar{Q} - \varepsilon) > p^2(\bar{Q} - \varepsilon) > p^2(\bar{Q} + \varepsilon) > p^1(\bar{Q} + \varepsilon).
\]

Since \( p^1 \) and \( p^2 \) must be differentiable almost everywhere, this contradicts the facts that

\[
p^1_Q(\bar{Q} - \varepsilon) > p^2_Q(\bar{Q} - \varepsilon), \quad \text{and} \quad p^1_Q(\bar{Q} + \varepsilon) < p^2_Q(\bar{Q} + \varepsilon).
\]

Recall that the market-price equation (5.3) is parameterized by \( C \), which represents the intercept term of the underlying price equation. In what follows I demonstrate some comparative statics in terms of this parameter.

**Lemma D.3.** In an equilibrium represented by the market-price equation, expected revenue is increasing in \( C \).

\footnote{Note that this allows for two market-price functions to meet at \( Q = 0 \) but diverge elsewhere. This is important in the pure uniform-price auction.}
Proof. Let \( p^1 \) and \( p^2 \) be two equilibrium market-price equations, generated by \( C_1 \) and \( C_2 \), respectively. Without loss of generality, assume that \( C_1 > C_2 \). For any \( Q \),

\[
p^1(Q) - p^2(Q) = (C_1 - C_2) \exp \left( \int_0^Q \tilde{H}(x) \, dx \right) \equiv \Delta(Q).
\]

Note that \( \Delta(Q) \geq 0 \). The difference in expected revenue between the two equilibria is

\[
\mathbb{E}_Q \left[ \alpha \int_0^Q p^1(x) \, dx + (1 - \alpha) Q p^1(Q) - \alpha \int_0^Q p^2(x) \, dx + (1 - \alpha) Q p^2(Q) \right]
= \mathbb{E}_Q \left[ \alpha \int_0^Q p^1(x) - p^2(x) \, dx + (1 - \alpha) Q (p^1(Q) - p^2(Q)) \right]
= \mathbb{E}_Q \left[ \alpha \int_0^Q \Delta(x) \, dx + (1 - \alpha) Q \Delta(Q) \right] \geq 0.
\]

Then the market-price equation \( p^1 \) generates weakly greater expected revenue than \( p^2 \), and expected revenue is increasing in \( C \). \( \square \)

Lemma D.4. In any symmetric pure-strategy equilibrium, \( p(Q) \leq v(Q/n) \) for all \( Q < \overline{Q} \).

Proof. Suppose that there is \( Q < \overline{Q} \) such that \( p(Q) > v(Q/n) \); let \( \delta = \overline{Q}/n - Q/n \).

For an agent \( i \), consider a deviation \( \hat{b} \) defined by

\[
\hat{b}(q) = \begin{cases} 
  b(q) & \text{if } q \leq \frac{1}{n} \overline{Q} - \delta, \\
  b(q) \wedge v(q) & \text{otherwise}.
\end{cases}
\]

Note that the deviation does not alter the agent’s allocation when \( p(nq) = b(q) \leq v(q) \).\(^2\) When \( p(nq) > v(q) \), the deviation ensures that the agent will not sacrifice utility winning units above their value, and therefore improves the agent’s utility.

The original bid function cannot be a best response. \( \square \)

\(^2\)This is not strictly true in the case in which bid functions are possibly flat; however, it has already been seen that a solution to the market-clearing price equation \( p \) must be strictly decreasing in the market quantity, hence in a symmetric equilibrium bids will be strictly decreasing in \( q \).
Corollary D.1. Revenue is maximized when $C$ is such that $p(\overline{Q}) = v(\overline{Q}/n)$.

Corollary D.1 follows from the preceding Lemmas. Lemma D.3 establishes that $C$ should be set as high as possible to maximize seller revenue. Lemma D.4 establishes that $C$ cannot be so high that $p(\overline{Q}) > v(\overline{Q}/n)$. Lemma D.2 establishes that all equilibria must be of the form given in equation (5.3).

D.3 Auxiliary results for Theorem 5.1: Decreasing multiplicity

Lemma 5.1 (Equilibrium necessary condition). A necessary condition for a market price function $p$ to represent an equilibrium is

$$
\frac{1 - 2\alpha}{1 - \alpha} \left( p(\overline{Q}) - \hat{v}(\overline{Q}) \right) + \overline{Q} \hat{v}_Q (\overline{Q}) \leq 0.
$$

Proof. Suppose that $p(\overline{Q}) < v(\overline{Q}/n)$. Then for sufficiently small $\varepsilon > 0$, there is $\delta > 0$ such that for all $q \in (\overline{Q}/n - \delta, \overline{Q}/n)$ it must be that

$$
v \left( \frac{1}{n} \right) > \lim_{q' \uparrow \frac{1}{n} Q} b^i(q') + \varepsilon > b^i(q) \geq \lim_{q' \uparrow \frac{1}{n} Q} b^i(q').
$$

Define the limiting price $\overline{b} = \lim_{q' \uparrow \overline{Q}/n} b^i(q')$, and let $\overline{v} = v(\overline{Q}/n) - \overline{b}$. For $\varepsilon \in (0, \overline{v})$, define $q_\varepsilon = \inf \{ q : b^i(q) \leq \overline{b} + \varepsilon \}$. Define a deviation $\hat{b}_\varepsilon$ such that

$$
\hat{b}_\varepsilon(q) = \begin{cases} b^i(q) & \text{if } q < q_\varepsilon, \\ \overline{b} + \varepsilon & \text{if } q \geq q_\varepsilon. \end{cases}
$$

The costs of the deviation must outweigh the benefits, hence

$$
\int_{nq_\varepsilon}^{\overline{Q}} \int_{\frac{1}{n} Q}^{\overline{Q} - (n - 1)q_\varepsilon} v(y) - \hat{b}_\varepsilon(y) \, dy - \alpha \int_{q_\varepsilon}^{\frac{1}{n} Q} \hat{b}_\varepsilon(y) - b(y) \, dy \\
- (1 - \alpha) \left( \frac{1}{n} Q \right) \left( \hat{b}_\varepsilon \left( \frac{1}{n} Q \right) - b \left( \frac{1}{n} Q \right) \right) dF(Q) \leq 0.
$$
When $\varepsilon = 0$, this holds with equality; the same is true of both the first and second derivatives. Thus to ensure that this deviation is not profitable for small $\varepsilon > 0$, it is necessary that the third derivative is weakly negative.

Evaluated at $\varepsilon = 0$, the third derivative is
\[
- \left[ -2 (n-1) \left( \hat{v}(\bar{Q}) - \bar{v} \right) \frac{d^2 q_\varepsilon}{d\varepsilon^2} + (n-1) (n-2) n \hat{v}_Q(\bar{Q}) \left( \frac{d q_\varepsilon}{d\varepsilon} \right)^2 
+ (2\alpha + 3n - 4) \frac{d q_\varepsilon}{d\varepsilon} \right] n \frac{d q_\varepsilon}{d\varepsilon} \frac{d F(\bar{Q})}{d\varepsilon} \leq 0.
\]

Noting that $\frac{d q_\varepsilon}{d\varepsilon} = 1 / np_Q$, $d^2 q_\varepsilon / d\varepsilon^2 = -p_{QQ} / np_Q^3$, and substituting in for the solution to the agent’s first-order conditions, this is equivalent to
\[
-2 (n-1) \frac{p_{QQ}(\bar{Q})}{\bar{H}(\bar{Q})} + (n-1) (n-2) \hat{v}_Q(\bar{Q}) + (2\alpha + 3n - 4) p_Q(\bar{Q}) \leq 0.
\]

Since $p_Q = (p - \hat{v})\tilde{H}$, substitution into this expression gives
\[
-2 (n-1) (p(\bar{Q}) - \hat{v}(\bar{Q})) \left( \bar{H}(\bar{Q}) + \frac{\tilde{H}_Q(\bar{Q})}{\bar{H}(\bar{Q})} \right) + 2 (n-1) \hat{v}_Q(\bar{Q}) 
+ (n-1) (n-2) \hat{v}_Q(\bar{Q}) + (2\alpha + 3n - 4) (p(\bar{Q}) - \hat{v}(\bar{Q})) \bar{H}(\bar{Q}) \leq 0.
\]

Finally, replacing $\bar{H}$ and $\tilde{H}_Q$ yields
\[
\frac{1}{(1-\alpha)\bar{Q}} (n-2\alpha n) (p(\bar{Q}) - \hat{v}(\bar{Q})) + n \hat{v}_Q(\bar{Q}) \leq 0.
\]

The desired inequality is immediate. \qed

**Lemma 5.2** ($\alpha$-monotonicity of necessary condition). *Suppose that $p$ is a solution to the equilibrium market-price equation for randomization $\alpha$, and satisfies the necessary condition of Lemma 5.1. Then for any $\alpha' < \alpha$, there is a solution to the equilibrium market-price equation $p'$ with $p'(\bar{Q}) = p(\bar{Q})$ which satisfies the necessary condition of Lemma 5.1.*

*Proof.* Note that the existence of a solution is simply a claim that there is a $C'$ that provides $p'(\bar{Q}) = p(\bar{Q})$. This is trivial. It is necessary then only to establish the latter claim in the Lemma.
To recapitulate, the necessary condition of Lemma 5.1 is
\[
\frac{1 - 2\alpha}{1 - \alpha} \left( p(Q) - \hat{v}(Q) \right) + \overline{Q} \hat{v} Q (Q) \leq 0.
\]

If this inequality is increasing in \( \alpha \), the claim will be established. Notice that the right-hand term is constant in \( \alpha \) while, by assumption, \( p(Q) - \hat{v}(Q) \) is constant and negative. Hence it is sufficient to show that
\[
\frac{d}{d\alpha} \left[ \frac{1 - 2\alpha}{1 - \alpha} \right] \leq 0.
\]
Basic algebra reveals the derivative to be
\[
\frac{-2}{1 - \alpha} + \frac{1 - 2\alpha}{(1 - \alpha)^2} = -\left( \frac{1}{1 - \alpha} \right)^2 < 0.
\]
Thus if \( p \) satisfies the endpoint condition of Lemma 5.1 for randomization parameter \( \alpha \), \( p' \) satisfies the endpoint condition for randomization parameter \( \alpha' < \alpha \).

**D.4 Simulation parameterizations**

*Base parameterizations for Figure 5.1.* Recall the functional forms of marginal values and the distribution of supply,

\[
\begin{align*}
v^1(q) &= \nu_1 - q\gamma_1, \\
v^2(q) &= \nu_2 \exp(-q\gamma_2), \\
v^3(q) &= \nu_3 + \sqrt{1 - \gamma_3 q},
\end{align*}
\]

\[
\begin{align*}
F^1(Q) &= \frac{Q}{Q}, \\
F^2(Q) &= \frac{\exp(\lambda Q) - 1}{\exp(Q) - 1}, \\
F^3(Q) &= \frac{\Phi\left(\frac{Q - \mu}{\sigma}\right) - \Phi\left(-\frac{-\mu}{\sigma}\right)}{\Phi\left(\frac{Q - \mu}{\sigma}\right) - \Phi\left(-\frac{-\mu}{\sigma}\right)}.
\end{align*}
\]

The parameterizations of marginal values in Figure 5.1 are

\[
\begin{align*}
v^1(q) &= 1 - q & (\nu_1, \gamma_1) &= (1, 1), \\
v^2(q) &= \exp(-2q) & (\nu_2, \gamma_2) &= (1, 2), \\
v^3(q) &= \sqrt{1 - q} & (\nu_3, \gamma_3) &= (0, 1).
\end{align*}
\]
The parameterizations of distribution in Figure 5.1 are

\[
F^1(Q) = \frac{Q}{10} \quad (\overline{Q}) = (10),
\]

\[
F^2(Q) = \exp \left( \frac{1}{2} Q - \frac{1}{5} \right) \quad (\overline{Q}, \lambda) = \left( 10, \frac{1}{2} \right),
\]

\[
F^3(Q) = \Phi \left( \frac{Q-3}{3} \right) - \Phi \left( -\frac{3}{3} \right) \quad (\mu, \sigma) = (8, 3).
\]

In all cases the maximum quantity available is \( \overline{Q} = 10 \), and there are \( n = 10 \) bidders participating in the auction.

**Comparative parameterizations for Figure 5.2.** The “base” parameterizations in each figure are identical to those given for Figure 5.1. That is, unless otherwise specified the parameterization of the Figure is identical to that in Figure 5.1. In each case, the distribution employed is of the exponential form given in \( F^2 \).

In each row a particular set of parameters is varied. Row 1 varies the number of bidders \( n \): \( n \in \{10, 12, 16, 24\} \), to simulate the effect of additional competition. Row 3 varies the distributional parameter \( \lambda \), the extent to which supply is weighted toward larger quantities: \( \lambda \in \{0.01, 0.05, 0.1, 0.5, 1\} \).

Row 2 varies the demand characteristics \( (\nu, \gamma) \). The displayed variations depend on the demand function employed:

\[
(\nu_1, \gamma_1) \in \{2, 3, 4\} \times \left\{ \frac{1}{2}, 1, 2 \right\},
\]

\[
(\nu_2, \gamma_2) \in \left\{ \frac{1}{2}, 1, 2 \right\} \times \{1, 2, 3\},
\]

\[
(\nu_3, \gamma_3) \in \left\{ 1, \frac{3}{2}, 2 \right\} \times \left\{ \frac{1}{2}, \frac{3}{4}, 1 \right\}.
\]

Note that, except in the case of variation in \( \nu^2 \), the base parameterization is not an element of the variations found in row 2. This reflects the fact that the algorithm used to generate the plots in Chapter 5 takes a simple Cartesian product of suggested models, while the base parameterizations have relatively degenerate—but helpfully illustrative—forms.
In all variations, if the sufficient condition for the existence of a pure-strategy equilibrium in the pure pay-as-bid auction (Theorem 4.1) is not satisfied, the parameterization is not displayed in Figure 5.2.
Bibliography


D. V. Widder. The Laplace Transform. 1941.
