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ABSTRACT

A technique for the solution of the Helmholtz equation together with associated boundary conditions is described. This method is based on a generalization of that used for the solution of the Dirichlet problem of potential theory, in which a dipole distribution is introduced on the boundary of a region to generate the potential inside. In order that the boundary conditions be satisfied, the distribution must be found as the solution of an integral equation. If the boundary is smooth, the equation is of Fredholm type, but if it has a corner the equation is singular. The problem of a sharp corner is analyzed, and properties of the solution are developed using the theory of singular integral equations. A few results are given for the numerical evaluation of eigenvalues of the Laplacian for some polygons which can also be obtained analytically. It is observed that eigenvalues of the integral equation can produce non-trivial distributions which generate "null" solutions of the original Helmholtz equation.

I. INTRODUCTION

In modelling a wide variety of physical wave phenomena, one is often faced with the problem of finding a wave motion in a medium which is inhomogeneous overall, but in which the medium is locally homogeneous and has discontinuities across various boundaries. In such cases one must typically find a solution to the equation

$$(\nabla^2 + \kappa_i^2)\, \psi(\vec{r}) = 0, \qquad (1)$$

where κ_i will be a different constant in each region, i, together with certain matching conditions for ψ at the boundaries of the region.

Aside from a few special cases which can be treated analytically, such problems must be solved numerically, usually with the aid of a high-speed computer. Commonly applied techniques of wide applicability in such calculations are the finite difference and finite element techniques. The former directly approximates the derivatives in Eq. (1) by finite differences, and the latter is best based on a Lagrangian variational principle from which Eq. (1) can be deduced. These techniques are generally applicable independently of whether subregions are homogeneous or not. On the other hand, although boundary conditions at finite distances are easily treated, boundary conditions "at infinity", which arise in scattering problems, are difficult to impose.

In this paper we consider a different method for the solution of such problems which is closely related to the classical solution of the Neumann and Dirichlet problems of potential theory. In this technique, the solution of Eq. (1) in a given region is achieved by the introduction of a monopole or dipole distribution on the boundary of that region. The boundary conditions can then be determined in terms of that distribution and one is led to integral equations for the boundary condition which must be solved. The method has two immediate advantages over the previously noted methods: 1. boundary conditions "at infinity" are easily introduced; and, 2. it is only necessary to consider points on the boundary to obtain the solution, thereby reducing the dimensionality of the problem by one. The storage requirements for a computer can thereby be reduced significantly. On the other hand, the method has the disadvantage that the matrices which are generated have relatively few nonzero elements as compared to the former techniques which can produce matrices with small "band-widths", and the elements of the matrices typically require calculation of more complicated functions. It is also true, of course, that the boundary distribution method can only be applied if individual regions are homogeneous. Thus for differing problems different techniques may be most efficacious. Although we believe that the method can be developed for use in fully three-dimensional problems, in this paper we will only consider the two-dimensional case, so that we will deal with a one-dimensional distribution on the boundary.

The integral equation which arises in this method is of Fredholm type if the boundary is smooth, but it becomes singular if sharp corners are introduced. In the last few years a number of applications of the method have been made to acoustic and electromagnetic radiation problems(1), but in these a detailed analysis of the complications arising from sharp corners has not been made. In this paper, in Section II, we first develop the general integral equation for the boundary distribution, and then in Section III we

give a reasonably complete analysis of the properties of the solution in the vicinity of a corner. Finally, in Section IV we give results for the numerical solution of a few problems. Although we wish to apply the technique to scattering problems, we know of no analytic solutions to scattering problems with corners. The examples have been chosen only to illustrate the capability of the method, and so we have determined eigenvalues of the Laplacian for a few special shapes. Each of these can be obtained analytically and so the error obtained provides an indication of the potential accuracy attainable. It will be seen that excellent results can be achieved.

II. THE DIPOLE DISTRIBUTION INTEGRAL EQUATION

The famous Dirichlet problem of potential theory is the determination of a solution of Laplace's equation in a region in which the potential takes a given value on the boundary. This problem has been solved for the inside of a closed region by the introduction of a Green's function and a continuous dipole distribution on the boundary. Thus one writes

$$\phi(\vec{r}) = \oint_{S_V} D(\vec{r}\,')\nabla' G(\vec{r}, \vec{r}\,')\cdot d\vec{\sigma}\,'. \qquad (2)$$

In this expression $D(\vec{r})$ is the dipole distribution at $\vec{r}, G(\vec{r},\vec{r}\,')$ is the potential at $\vec{r}$ owing to a unit charge at $\vec{r}\,'$ and $d\vec{\sigma}\,'$ is the surface element directed along the outward normal. The integral is carried over the surface S_V which encloses the volume V, and $\phi(\vec{r})$ is thereby determined throughout V. In the two-dimensional case, in which we shall be interested in this paper,

$$G(\vec{r}, \vec{r}\,') = \frac{1}{2\pi} \log |\vec{r} - \vec{r}\,'|, \qquad (3)$$

and of course the volume, V, becomes an area, and S_V is its bounding contour. The solution of the Dirichlet problem is then reduced to finding the solution of an integral equation for $D(\vec{r})$.

If we introduce the $G(\vec{r}, \vec{r}')$ of Eq. (3) into Eq. (2), we find:

$$\phi(\vec{r}) = \frac{1}{2\pi} \oint_{S_V} D(\vec{r}') \frac{(\vec{r}' - \vec{r})\cdot d\sigma'}{|\vec{r}' - \vec{r}|^2} .$$

This expression would be useful for determining the potential at internal points of the region V, but if one wishes $\phi(\vec{r})$ on the boundary the limit must be taken from the inside, since the $\phi(\vec{r})$ obtained from this expression is discontinuous across the boundary. In the limit in which $\vec{r}$ approaches a point on the boundary where it is smooth, one can write

$$\lim_{\vec{r} \to S_V} \phi(\vec{r}) = \frac{D(\vec{r})}{2} + P \oint_{S_V} D(\vec{r}') \frac{(\vec{r}' - \vec{r})\cdot d\sigma'}{2\pi|r' - r|^2} .$$

In this relation, the integrand is in general singular as $\vec{r}' \to \vec{r}$ but the integral can be defined as a principal value integral. If the side containing $\vec{r}$ is straight, the contribution from that side will vanish and the integral is then regular, but in any case, if the boundary satisfies a Liapunov smoothness condition, it can then be shown that the integral is in fact well-defined as a principal value integral for $\vec{r}' \to \vec{r}$.[2]

Thus, for a smooth boundary, a solution of the Dirichlet problem is obtained if one can solve the integral equation

$$f(\vec{r}) = \frac{D(\vec{r})}{2} + P \oint_{S_V} D(\vec{r}') \frac{(\vec{r}' - \vec{r})\, d\sigma'}{2\pi|\vec{r}' - \vec{r}|^2} .$$

It can be shown[3] that this equation does in fact have a unique solution and so the problem is solved.

In our approach we extend the preceding technique to apply to the Helmholtz equation:

$$(\nabla^2 + \kappa^2)\, \psi(\vec{r}) = 0. \qquad (4)$$

In this case, we must choose $G(\vec{r}, \vec{r}')$ to be a solution of Eq. (4), with the result that:

$$G(\vec{r},\vec{r}') = A\, J_0(\kappa|\vec{r} - \vec{r}'|) + B\, Y_0(\kappa|\vec{r} - \vec{r}'|),$$

where J_0 and Y_0 are the usual regular and irregular Bessel functions of order zero, and, A and B must be determined using the limiting condition as $\vec{r} \to \vec{r}'$, and, if applicable, the condition as $\vec{r} \to \infty$. If $\vec{r} \to \vec{r}'$, G will approach the same limit as for $\kappa = 0$, and so, since $Y_0(x) \sim (2/\pi) \log x$, as $x \to 0$, we find that $B = 1/4$. On the other hand, A will be determined for the specific problem considered: If one is dealing with the interior of a closed region, A can be chosen as zero. If, however, the region is open and $\vec{r}$ can approach infinity, A will then be chosen in such a way as to satisfy the asymptotic condition on $\psi(\vec{r})$.

In order to find the asymptotic condition, it is helpful to consider the time-dependent equation from which the Helmholtz equation typically arises. In the case of wave propagation, we would have

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \phi(\vec{r}, t) = 0 , \qquad (5)$$

where c is the propagation velocity for waves in the region. If we then assume $\phi(\vec{r}, t) = \psi(\vec{r}) \exp(-i\omega t)$, we get Eq. (4), where $\kappa = \omega/c$. For a scattering situation, we consider as a typical case that a plane wave is incident on the scattering region, and write

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + \psi_{sc}(\vec{r}) .$$

Here, $\psi_{sc}(\vec{r})$ is the scattered wave and we require that it must have only "outgoing" parts. Further, we introduce the distribution $D(\vec{r})$ on the boundaries where $\vec{r}$ is finite and they will then be used to generate ψ_{sc} only.

A simple way of determining A so that only "outgoing" scattered waves occur, with the assumed time dependence, is to require that

$$G(\vec{r}, \vec{r}\,') = -\frac{i}{4} H_0^{(1)}(\kappa|\vec{r} - \vec{r}\,'|), \qquad (6)$$

where

$$H_0^{(1)}(x) \equiv J_0(x) + i\, Y_0(x)$$

is the usual Hankel function[4]. This clearly gives the correct B, and if $\vec{r} \to \infty$:

$$G(\vec{r}, \vec{r}\,') \underset{r \to \infty}{\sim} -\frac{i}{4} \left(\frac{2}{\pi r}\right)^{1/2} e^{i(\kappa r - \pi/4)},$$

which clearly represents outgoing waves, since asymptotically

$$\phi_{sc}(\vec{r}, t) \underset{r \to \infty}{\sim} \exp[i(\kappa r - \omega t)].$$

If we now introduce Eq. (6) for $G(\vec{r}, \vec{r}\,')$ into Eq. (2), we have:

$$\phi_{sc}(\vec{r}) = -\frac{i}{4} \oint_{S_V} D(\vec{r}\,')\nabla' H_0^{(1)}(\kappa|r' - r|)\cdot d\sigma'$$

or, since $H_0^{(1)}(x)' = -H_1^{(1)}(x)$,

$$\psi_{sc}(\vec{r}) = \frac{i\kappa}{4} \oint_{S_V} \frac{D(\vec{r}\,')H_1^{(1)}(\kappa|\vec{r}\,' - \vec{r}|)(\vec{r}\,' - \vec{r})\cdot d\sigma'}{|\vec{r}\,' - \vec{r}|} \qquad (7)$$

Clearly the scattered wave given by Eq. (7) automatically gives only outgoing waves, and in fact we have

$$\psi_{sc}(\vec{r}) \underset{r \to \infty}{\sim} \frac{-ie^{i(\kappa r - 3\pi/4)}}{(8\pi r)^{1/2}} \oint_{S_V} D(\vec{r}\,')\, e^{-i\kappa\vec{r}\,'\cdot\hat{e}_r}\, e_r\cdot d\sigma' ,$$

where $\hat{e}_r \equiv \vec{r}/r$.

If we let $\vec{r}$ approach the boundary, the resulting equation has the same small $(\vec{r}\,' - \vec{r})$ behavior as in the Dirichlet case, so that we may write:

$$f(\vec{r}) = \frac{D(\vec{r})}{2} + \frac{i\kappa}{4} P \oint_{S_V} \frac{H_1^{(1)}(\kappa|r' - r|)}{|r' - r|} D(\vec{r}\,')(\vec{r}\,' - \vec{r})\cdot d\sigma' . \qquad (8)$$

The same considerations as in the Dirichlet case with regard to the singular nature of the equation apply. It is this integral equation which we propose to investigate.

There is one important difference between the potential problem and the Helmholtz problem that should be mentioned. Although the interior Dirichlet problem has a unique solution, the exterior problem does not, and in fact will only have a solution at all if

$$\oint f(\vec{r})\, |d\vec{\sigma}\,'| = 0 .$$

has a solution. (It is easily seen that a constant satisfies the homogeneous equation.) Consequently, as follows from the Fredholm theory of this self-adjoint equation, only if $f(\vec{r})$ is orthogonal to the solution of the homogeneous equation is there a solution of the inhomogeneous equation. In the Helmholtz case, the homogeneous equation will not generally have a solution, so both the inner and outer problems have a unique solution.

III. ANALYSIS OF THE PROBLEM OF A BOUNDARY WITH SHARP CORNERS

Although the boundary distribution technique can be applied directly to cases in which the boundary is smooth, i.e., satisfies a Liapunov condition, some additional analysis must be given if the boundary has sharp corners. In the former case, the kernel of the equation can be shown to be completely continuous and so the usual Fredholm theorems apply. On the other hand, if there are corners the kernel is singular.

To deal with this situation, we will consider a corner in a boundary and for simplicity we will assume that the two sides of the corner are straight. The angle between these two sides will be called α. Further, since the singular nature of the equation comes about because of the small-distance behavior of the kernel, we divide the kernel into a leading term which includes the most singular part, and a remainder which is completely continuous. Thus we write:

$$H_1^{(1)}(x) \equiv -\frac{2i}{(\pi x)} + R(x) , \tag{9}$$

and we will focus attention principally on the first term.

If Eq. (9) is now introduced into Eq. (8), we find:

$$f(\vec{r}) = \frac{D(\vec{r})}{2} + P \oint \frac{D(\vec{r}\,')(\vec{r}\,' - \vec{r})\cdot d\sigma'}{2\pi|\vec{r}\,' - \vec{r}|^2}$$
$$+ \frac{i\kappa}{4} P \oint \frac{R(\kappa|\vec{r}\,' - \vec{r}|)}{|\vec{r}\,' - \vec{r}|} D(\vec{r}\,')(\vec{r}\,' - \vec{r})\cdot d\sigma' .$$

Let us now introduce the notation that $D_1(s)$ is $D(\vec{r})$ on side 1 of the corner, where s is the distance from the corner, and $D_2(s)$ is $D(\vec{r})$ on side 2. With this notation, the equation can be explicitly written for $\vec{r}$ on side 1 as:

$$f_1(s) = \frac{D_1(s)}{2} + \frac{s \sin\alpha}{2\pi} \int_0^{\ell_2} \frac{D_2(s')ds'}{(s'^2 - 2s' s\cos\alpha + s^2)}$$
$$+ \frac{i\kappa}{4} s \sin\alpha \int_0^{\ell_2} \frac{R\left[\kappa(s'^2 - 2s' s\cos\alpha + s^2)^{1/2}\right]}{(s'^2 - 2s' s\cos\alpha + s^2)^{1/2}} D_2(s')ds'$$
$$+ \frac{i\kappa}{4} \int_{C'} \frac{H_1^{(1)}(\kappa|\vec{r}\,' - \vec{r}|)}{|\vec{r}\,' - \vec{r}|} D(\vec{r}\,')(\vec{r}\,' - \vec{r})\cdot d\sigma' ,$$

where $f_1(s)$ is the boundary value of $f(\vec{r})$ on side 1. For a straight side there is no contribution from the distribution $D_1(s)$ to the potential on that side except for the term $D_1(s)/2$, because the vector $\vec{r}\,' - \vec{r}$ is perpendicular to the surface element. The length of side 2 is ℓ_2. The integral over C' is the contribution from the distribution other than the part on sides 1 and 2. This last integral is analytic as a function of s, since it is a finite integral and $|\vec{r}\,' - \vec{r}| > 0$ for $\vec{r}\,'$ on C' and $\vec{r}$ on side 1.

Similarly, for side 2 we have:

$$f_2(s) = \frac{D_2(s)}{2} + \frac{s \sin\alpha}{2\pi} \int_0^{\ell_1} \frac{D_1(s')ds'}{(s'^2 - 2s' s\cos\alpha + s^2)} + \cdots ,$$

where the $\cdots$ indicates terms similar to the R, C' terms for f_1.

To analyze the corner singularity, we introduce $D_\pm(s) = D_1(s) \pm D_2(s)$, and we then obtain

$$\frac{D_\pm(s)}{2} \pm \frac{s \sin\alpha}{2\pi} \int_0^{\ell_m} \frac{D_\pm(s')ds'}{(s'^2 - 2s' s\cos\alpha + s^2} = F_\pm(s) ,$$

where ℓ_m is the lesser of ℓ_1 and ℓ_2, and $F_\pm(s)$ includes the contributions of $f_i(s)$ and the remainder of the equations coming from R, C', and the integral for the larger ℓ_i beyond ℓ_m. Obviously, these integral equations have a singular kernel as $s, s' \to 0$, and so some care must be used in dealing with them, either for analytic or numerical purposes.

To proceed, we make a Mellin transformation of the equations to obtain

$$\frac{\Delta_\pm(\xi)}{2} \pm \frac{\sin\alpha}{(2\pi)^2 i} \int_{c-i\infty}^{c+i\infty} d\xi' \, \Delta_\pm(\xi') \int_0^{\ell_m} (s')^{-\xi'} ds'$$

$$\times \int_0^\infty \frac{s^\xi \, ds}{s^2 - 2s\, s' \cos\alpha + s'^2} = \Phi_\pm(\xi) \, . \tag{10}$$

In this equation, $\Delta(\xi) \equiv \int_0^\infty D(s)\, s^{\xi-1} ds$. To obtain Eq. (10), we have made the direct Mellin integration and have used the inverse relations:

$$D(s) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \Delta(\xi) s^{-\xi} ds \quad .$$

The choice of the constant c will be discussed later. The transform of the function $F_\pm(s)$ is $\Phi_\pm(\xi)$. In arriving at this equation we have interchanged the order of integrations, but this can be justified à posteriori. Next we can evaluate[5] the right-most integral in Eq. (10):

$$\int_0^\infty \frac{s^\xi \, ds}{s^2 - 2s\, s' \cos\alpha + s'^2} = \pi (s')^{\xi-1} \frac{\sin\left[(\pi - \alpha)\xi\right]}{\sin\alpha \quad \sin(\pi\xi)} \quad ,$$

where $0 < \alpha < 2\pi$, and $-1 < \mathrm{Re}\,\xi < 1$. Then we can carry out the next integral:

$$\int_0^{\ell_m} (s')^{\xi-\xi'-1} ds' = \frac{(\ell_m)^{\xi-\xi'}}{\xi - \xi'} \quad ,$$

where we require that $\mathrm{Re}(\xi - \xi') > 0$. Thus the equations become:

$$\frac{\Delta_\pm(\xi)}{2} \pm \frac{r(\xi)}{4\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(\ell_m)^{\xi-\xi'} \, \Delta_\pm(\xi') d\xi'}{(\xi - \xi')} = \Phi_\pm(\xi) \tag{11}$$

where

$$r(\xi) \equiv \frac{\sin(\pi - \alpha)\xi}{\sin \pi \xi} \quad .$$

This equation is in standard singular integral equation form, and thus may be treated using known techniques[6]. We begin by considering the homogeneous equation, and introduce a function

$$H(\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(\ell_m)^{\xi-\xi'} \, \Delta^{(0)}(\xi') d\xi'}{(\xi - \xi')} \quad ;$$

where $\Delta^{(0)}$ is a solution of the homogeneous equation. Clearly $H(\xi)$ is an analytic function in the finite half-planes defined by $\mathrm{Re}(\xi) \gtrless c$, and it has a discontinuity in crossing the contour of integration. If we define the $H^{(\pm)}(\xi)$ to be the functions obtained from the integral in which $\mathrm{Re}(\xi) \gtrless c$, respectively, together with their analytic continuations, we then easily find that

$$\Delta^{(0)}(\xi) = H^{(+)}(\xi) - H^{(-)}(\xi) \, ,$$

and so

$$\frac{1}{2}\left[H_\pm^{(+)}(\xi) - H_\pm^{(-)}(\xi) \pm r(\xi) H_\pm^{(+)}(\xi)\right] = 0 \tag{12}$$

This equation can be used to deduce the analytic structure of $\Delta_{\pm}^{(0)}(\xi)$.

We eventually wish to deduce the analytic structure of $D(s)$, which will require using the inverse transform on $\Delta(\xi)$. For the latter step, in the limit $s \to 0$, the contour in the inverse transform can be closed on the left, and so the behavior of $D(s)$ is determined by singularities on the left of the contour. In this region $H^{(-)}(\xi)$ is clearly analytic, and so we can solve for $H^{(+)}(\xi)$ in terms of $H^{(-)}(\xi)$ using Eq. (12) to analytically continue $H^{(+)}(\xi)$ to the left of the contour. Thus we find:

$$H_{\pm}^{(+)}(\xi) = (1 \pm r(\xi))^{-1} H_{\pm}^{(-)}(\xi).$$

A solution of this equation can be obtained by taking the logarithm of the equation and then noting that $\log H(\xi)$ is a function with a given discontinuity on the contour. The solution of this problem (the "Hilbert problem") then can be written(7):

$$H_{\pm}(\xi) = \exp\left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\ln|1 \pm r(\xi')|}{\xi' - \xi} d\xi' \right\}$$

assuming that the integral converges. We then see that $H_{\pm}^{(-)}(\xi)$ is analytic and nonzero on the left of the contour, and if we use Eq. (12) to analytically continue $H_{\pm}^{(+)}(\xi)$, it is evident that $H_{\pm}^{(+)}(\xi)$ will also be analytic unless

$$1 \pm r(\xi) = 0$$

At such points, $H_{\pm}^{(+)}(\xi)$ will generally have poles. Thus $\Delta_{\pm}^{(0)}(\xi)$ also has poles at such points.

The solution of Eq. (11) may now be obtained by introducing

$$\mathcal{H}(\xi) \equiv \frac{1}{2\pi i} \oint_{c-i\infty}^{c+i\infty} \frac{(\ell_m)^{\xi-\xi'} \Delta(\xi')d\xi'}{(\xi' - \xi)} \,.$$

Then it is easily seen that

$$\mathcal{H}_{\pm}^{(+)}(\xi) - \mathcal{H}_{\pm}^{(-)}(\xi) = -\Delta_{\pm}(\xi),$$

so that

$$(1 \pm r(\xi))\mathcal{H}_{\pm}^{(+)}(\xi) - \mathcal{H}_{\pm}^{(-)}(\xi) = -2\Phi_{\pm}(\xi). \qquad (13)$$

Using

$$(1 \pm r(\xi)) = H_{\pm}^{(-)}(\xi)/H_{\pm}^{(+)}(\xi) \,,$$

this equation can be written:

$$\frac{\mathcal{H}_{\pm}^{(+)}(\xi)}{H_{\pm}^{(+)}(\xi)} - \frac{\mathcal{H}_{\pm}^{(-)}(\xi)}{H_{\pm}^{(-)}(\xi)} = -\frac{2\Phi_{\pm}(\xi)}{H_{\pm}^{(-)}(\xi)} \,.$$

Again we have a discontinuity equation to satisfy and we obtain as the formal solution:

$$\mathcal{H}_{\pm}(\xi) = -\frac{H_{\pm}(\xi)}{i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\Phi_{\pm}(\xi')}{H_{\pm}^{(-)}(\xi')} \frac{d\xi'}{\xi' - \xi} \,.$$

Since $\mathcal{H}_{\pm}^{(-)}(\xi)$ is analytic on the left of the contour, if we use Eq. (13) to obtain the analytic continuation of $\mathcal{H}_{\pm}^{(+)}(\xi)$, we finally find that

$$\Delta_{\pm}(\xi) = \frac{\pm r(\xi)\mathcal{H}_{\pm}^{(-)}(\xi) + 2\Phi(\xi)}{1 \pm r(\xi)} \,.$$

Thus, we can generally expect poles in $\Delta(\xi)$ in the left half plane wherever $1 \pm r(\xi) = 0$ on the left of the contour.

To complete the discussion, it is necessary to specify the contour; i.e., to determine c. In the first place, from the restriction on $\text{Re}(\xi)$, we require that $-1 < c < 1$. In addition, the preceding development will only give a meaningful expression for $H(\xi)$ if $\ell_n[1 \pm r(\xi)] \to 0$ as $|\mathrm{Im}\,\xi| \to \infty$. It is easily seen that $r(\xi) \sim \exp\left[(|\pi - \alpha| - \pi)|\mathrm{Im}\,\xi|\right]$ as $|\mathrm{Im}\,\xi| \to \infty$, so $r(\xi) \to 0$. Thus the logarithm will approach zero at ∞, unless it has an imaginary part of the form $i\pi n$. To guarantee that this does not happen, we can choose $c = 0$, since $r(\xi)$ is real and nonzero on the imaginary axis. Any other c satisfying the limit restriction is equally acceptable as long as the contour would not thereby be distorted from the imaginary axis by going past a zero[8] of $1 \pm r(\xi)$, since in such a case the logarithm would acquire an imaginary part at ∞.

We now can conclude that $D(s)$ will behave as $\sim s^{-\xi_n}$ as $s \to 0$, where ξ_n is a pole in the transform, $\Delta(\xi)$. Such poles will appear if

$$1 = \mp \frac{\sin(\pi - \alpha)\xi_n}{\sin \pi\,\xi_n} \quad ,$$

or

$$\sin \pi\,\xi_n = \mp \sin(\pi - \alpha)\xi_n$$

if $\xi \neq 0$. In the case of Δ_+, the solutions of this equation are

$$\xi_n^{(+)} = -\frac{(2n-1)\pi}{\alpha}\,, \quad -\frac{2n\pi}{2\pi - \alpha}\,;$$

and in the case of Δ_-,

$$\xi_n^{(-)} = -\frac{2n\pi}{\alpha}\,, \quad -\frac{(2n-1)\pi}{2\pi - \alpha}\,,$$

where n is any positive integer.

In addition to these poles, we must consider other possible singularities in $\Delta(\xi)$. Since $\mathcal{H}^{(-)}(\xi)$ is analytic, the only other possibility would be singularities in $\Phi(\xi)$. In fact, $\Phi(\xi)$ in part

comes from contributions to $f(s)$ arising from distributions on the other boundaries, C', and since these contributions will be analytic near $s = 0$, this part of $\Phi_\pm(\xi)$ will be the transform of functions which have power series expansions; i.e., they have poles at the negative integers[9]. There will also be a pole in $\Phi_+(\xi)$ at $\xi = 0$, but because $f_-(0) = 0$, $\Phi_-(\xi)$ has no such pole. Thus, to the poles already given, we have additional poles at the integers.

Finally, we must consider singularities related to the $H_1^{(1)}(x)$, aside from the most singular part which has already been treated. For this we assume that $D(s) \sim s^{-\xi}$, and then deduce the form of

$$I_\xi(s) = \frac{i\kappa s \sin\alpha}{4} \int_0^{\ell} \frac{H_1^{(1)}(\kappa w)}{w} (s')^{-\xi}\, ds' \,,$$

where $w = (s'^2 - 2s' s \cos\alpha + s^2)^{1/2}$. This integral can be evaluated using Gegenbauer's addition theorem[10]:

$$I_\xi(s) = 2\sin\alpha \sum_{m=0}^{\infty} (m+1)\, C_m^{(1)}(\cos\alpha) \Bigg\{ H_{M+L}^{(1)}(\kappa s) \times \int_0^s (s')^{-\xi-1} J_{M+L}(\kappa s')\,ds' + J_{m+1}(\kappa s) \int_s^{\ell} (s')^{-\xi-1} H_{m+1}^{(1)}(\kappa s')\,ds' \Bigg\} \,,$$

where $C_m^{(1)}(\cos\alpha)$ is a Gegenbauer polynomial. The Hankel function can be divided into two parts:

$$H_{m+1}^{(1)}(\kappa s) = \frac{2i}{\pi} \log s - J_{m+1}(\kappa s) + \psi_{m+1}(s),$$

where

$$\psi_{m+1}(s) = \sum_{n=0}^{\infty} a_n\, s^{-m-1+2n} \,.$$

If the series for $\psi_{m+1}(s)$ and $J_{m+1}(\kappa s)$ are introduced into the expression for $I_\xi(s)$, it is then easily found that log s does not occur in $I_\xi(s)$, and that the powers of s which occur are $\xi + 2n$, and $n + 1$, where n is an integer ≥ 0. Thus each of the poles, ξ_i, generates a series of poles spaced at even integers from ξ_i, or, equivalently, an even series of powers of s of the form $s^{-\xi + 2n}$. This completes the determination of the analytic form of the solutions of the boundary integral equation at a sharp corner.

A few comments are appropriate at this point: In deducing the analytic form of the solution, we have assumed that the unknown functions on the remainder of the boundary away from the corner of interest can be treated as if they were known. The legitimacy of this approach can be rigorously established following the complete treatment of singular equations, but we did not feel that such an approach, which mainly only increases the complexity of notation and the bulk of the equations, was particularly illuminating and so we have chosen the more heuristic approach given above. We refer the interested reader to the rigorous treatments of singular integral equations for a discussion which will indicate the necessary extension of the preceding argument.

In the above analysis, we have assumed also that the poles which appear are simple. While this is generally the case, in special cases, poles may come together. For example, if $\alpha = 2\pi/3$ we find that two poles occur at $\xi = -\frac{3}{2}$ for $\Delta^{(+)}$. In such a situation, the s-space function then has a term of the form $s^{-\xi}\log s$ as well as the usual $s^{-\xi}$.

Finally, we consider the relation between the behavior of D(s) near a corner and that of $f(\vec{r})$, where $\vec{r}$ is not on the boundary. Here:

$$f(\vec{r}) = \frac{i\kappa}{4} \oint \frac{D(\vec{r}\,')H_1^{(1)}(\kappa|\vec{r}\,' - \vec{r}|)(\vec{r}\,' - \vec{r})\cdot d\sigma'}{|\vec{r}\,' - \vec{r}|} .$$

If we represent the point $\vec{r}$ in polar coordinates (r, θ) and assume $D(s) \sim s^{-\xi}$ the leading contribution to f for small r is then

$$f(r, \theta) \sim \frac{1}{2\pi} \int_0^{\ell_1} \frac{(s)^{-\xi} r \sin\theta \, ds}{s^2 + r^2 - 2s\, r\cos\theta}$$

$$\pm \frac{1}{2\pi} \int_0^{\ell_2} \frac{(s)^{-\xi} r \sin(\alpha - \theta) ds}{s^2 + r^2 - 2s\, r\cos(\alpha - \theta)} + \dots ,$$

where the $\pm$ sign is chosen according to whether we have the odd or even part of D, and the remainder is less singular as $r \to 0$. The two integrals then contribute to $f(r, \theta)$ as

$$f(r, \theta) \sim \frac{r^{-\xi}}{2\sin\pi\xi} \{1 \pm 1\} ,$$

and we see that the behavior of $D_+(s)$ is reflected in $f(r, \theta)$ near the corner, but $D_-(s)$ does not contribute a term of the form $r^{-\xi}$.

If the results of this paper are applied to an electric field for which $\vec{E} = -\nabla\phi$, where ϕ is the electric potential, then at a corner, $\phi \sim r^{-\xi}$ and $E_r \sim \xi r^{-\xi-1}$. The electric field energy $(= \vec{E}^2/8\pi)$ is then integrable (as is expected on physical grounds) since $\xi \leq 0$. The condition of integrability of the electromagnetic energy density was introduced by Meixner in order to obtain a unique solution for Maxwell's equations in the case of the diffraction of electromagnetic waves by perfectly conducting screens[11].

IV. NUMERICAL EXAMPLES

To illustrate the effectiveness of the boundary distribution method, we have used it to find approximate eigenvalues for a number of polygons in which the eigensolutions for $\psi(\vec{r})$ and κ are known. Thus we look for solutions of the integral equation in which $f(\vec{r}) = 0$ on the boundary of the region. To our knowledge, the corresponding distributions, $D(\vec{r})$, cannot be obtained analytically in these cases so unfortunately a direct comparison of the numerical results for them cannot be made.

At the outset, it should be noted that we do not feel that the boundary method is necessarily the best choice for finding such eigenvalues, and it is not for such problems that we ultimately wish to use it. An apparent disadvantage as compared with the finite element method, for example, is that it does not seem to satisfy an extremum condition, and, for the lowest eigenvalue, a minimum principle. Thus, by changing certain parameters in the calculation it may be possible to find values for κ which change from below the correct value to above it, and for a suitable choice one could get as accurate a result as desired. In the calculations to be described, variation of parameters with this goal was not carried out and parameters were chosen somewhat arbitrarily. Such a change in κ was observed to occur as the balance between the number of points in the corner regions and the central regions was varied with the total number of points fixed. Thus the accuracy of the calculated κ is not completely satisfying as an indicator of the overall accuracy of the calculation. (It will be seen from the results, however, that in many cases errors in the eigenvalue are very small.)

Another disadvantage of the method for finding eigenvalues is that κ occurs in the kernel of the integral equation so that the approximating matrix must be recalculated for each choice of κ. In the finite element method, such iterative complexity is not necessary. In addition, the kernel is a complicated function, and the ensuing matrix, while having many fewer elements than the finite element method, for example, has few, if any, zero elements. Thus, it is not clear that overall efficiency is obtained. There is generally a trade-off between storage requirements and the complexity involved. On the other hand, for scattering problems it is not necessary to iterate the matrices, and the automatic satisfying of the outgoing scattered wave boundary condition seems to us a great advantage.

We have used the method of this paper to obtain eigensolutions for a square, for an equilateral triangle, for a 45° isosceles triangle, and for a 30°-60°-90° triangle. In each case the eigensolution for κ and $\psi(\vec{r})$ can be obtained analytically. In addition, we have obtained eigensolutions for two other figures which have been treated using the finite element method.

In reducing the integral equation to an approximate finite form we have approximated the integrals in the integral equation in two ways: For $\vec{r}\,'$ near a corner, we have assumed that $D(s)$ can be expanded in a finite series of terms of the form s^{ξ}, in which the ξ's chosen are the lowest values in the set of allowed ξ's. Then the kernel was broken into two parts, of which the first included the most singular terms as $\kappa|\vec{r}\,' - \vec{r}| \to 0$, and the second was the remainder. The first part together with the various (s^{ξ})'s was integrated analytically using various rapidly convergent series, while the second part of the kernel was assumed to be approximated by a quadratic form, and this was then integrated exactly in conjunction with the factors s^{ξ}. The method used for this part of the kernel was quite analogous to that used in obtaining Simpson's rule. On the other hand, for $\vec{r}'$ away from a corner the entire kernel times $D(s')$ was assumed to be approximately a quadratic form in s', and then this function was integrated exactly, again analogously to Simpson's rule.

In the calculations reported here, we are dealing with a closed region, and hence in the kernel no asymptotic condition as $r \to \infty$ is needed. Thus we have chosen the Neumann function Y_1 instead of the Hankel function $H_1^{(1)}$ in the kernel. This has the advantage that the kernel and $D(\vec{r})$ become real. Further, we have chosen to reflect the distribution about one of the sides. This automatically satisfies the $\psi(\vec{r}) = 0$ boundary condition on that side, and no distribution is needed along it. We also find that the results for κ depend on which side is used for reflection in the 30°-60°-90°, and 45° isosceles triangles. In Table I we give some calculated κ's for various triangles. For the results given, we have chosen 26 points on each side of the triangle and in the vicinity of a corner we express $D_+(s)$ using the lowest six terms and $D_-(s)$ using the lowest five, in the series of powers. In Table II, we illustrate the variation of κ for a square, as the number of points per side is varied. The accuracy obtained for the eigenvalues is remarkable, particularly so in the absence of a variational principle. We also give in Table III the values for κ obtained for an "L-shaped" region consisting of three unit squares and of a unit-sided rhombus in which the corners make angles of 60° and 120°. Approximate

eigenvalues for the latter two regions have been obtained using the finite element method[12]. For the rhombus, we have also varied the number of points per side to illustrate the convergence of the eigenvalue. It may be noted that the symmetry of the various figures was not used to reduce the size of the matrices involved, and in the case of the L-shaped region this would have been very desirable to reduce the storage requirements. It is interesting that in comparison with the calculation for a rhombus by J. A. George which was carried out using 225 linear equations for the determination of the solution for which the associated matrix had a band width of 65, our calculation with 79 points on the boundary produced somewhat better results for κ as judged by the convergence of the results. The calculations of Stadter[13] are considerably less accurate, giving $4.98811 \le \kappa \le 4.99770$. For the "L-shaped" region, a very accurate technique developed by Fox, et. al.[14] gives an eigenvalue of $\kappa = 3.1047905$, while the finite element calculation of J.A. George[12] using 210 equations with a bandwidth of 101 and taking advantage of the symmetry, gives $\kappa = 3.1051$; and the finite difference calculation of Reid and Walsh[15], using 360 points and a 5-point formula, gives $\kappa = 3.1102$. One sees that our approach compares very favorably with these results. Furthermore, it is noted by Fox, et. al.[14] that their technique, which is very effective for the "L-shaped" region, is not particularly better than other methods for a rhombus.

A few results are presented in Tables 1-3. A fuller discussion of these calculations will be published elsewhere. The analytic lowest eigenvalues in the various cases are[16]:

κ (square) = $\sqrt{2}\ \pi$,

κ (equilateral triangle) = $4\pi/\sqrt{3}$,

κ (45° isosceles triangle) = $\sqrt{5}$,

and

κ (30°-60°-90° triangle) = $4\pi\sqrt{7}/3$

The corresponding eigensolutions are given in Appendix A. The distributions, $D(s)$, for several representative cases are exhibited in Figs. 1-4.

In obtaining the eigenvalues, it is seen that some configurations produce a considerably better result than do others. For example, in the 45° isosceles triangle the error is much smaller if the side reflected about is the longest one. This is also the case for the 30°-60°-90° triangle. It is not clear as yet why this should be so. In general, the errors in κ seemed to us surprisingly small considering the number of points used per side.

We also have found that the series expansion for $D(s)$ does not converge especially well; i.e., the last term in the series for a point at the farthest distance from the corner for which the series is used is typically a few percent of the leading term. The only counter-example is that in which we have a 90° corner at a reflected side. In this case, $D(s)$ is analytic in s and has an odd-integer power expansion which typically converges extremely well. On the other hand, it is found that coefficients of the lower powers in the series are very stable with respect to variations of parameters in the calculation and they seem to converge very well to a precise value as the number of points per side is increased. The higher coefficients in the series tend to be erratic, presumably because

they are called upon to approximate the remainder of a series which is not of the form $s^{-\xi_n}$. It may be noted that the ξ_n's in the expansion are not very well separated and hence the expansion coefficients may not be very precisely obtained, so that the $D(s)$ could be much more accurately given than are its constituent parts.

Finally, in addition to the true eigenvalues discussed above, the boundary distribution integral equation can also produce spurious κ's. In Appendix B, we have applied the boundary distribution technique to the case of a circular boundary, for which an analytic treatment is possible. As is shown there, not only do we obtain the well-known set of eigenvalues for that case, but we also find another set in which $D(s)$ can be finite, but the $\psi(\vec{r})$ which arises from it is identically zero. As is shown in Appendix B, the eigenvalues are given by $J_n(\kappa R)H_n^{(1)}(\kappa R)' = 0$, or, if one is dealing with a finite region since the Green's function can then be constructed using the Neumann function, Y_n, instead of the Hankel function, $H_n^{(1)}$,

$$J_n(\kappa R)\, Y_n(\kappa R)' = 0,$$

or

$$J_n(\kappa R) \left| Y_{n+1}(\kappa R) - \frac{n}{\kappa R} Y_n(\kappa R) \right| = 0 .$$

Thus, if $n = 0$, we have $J_o(\kappa R)Y_1(\kappa R) = 0$. One finds that the roots associated with non-trivial solutions and those which give null solutions are unfortunately quite close together. Thus, for example, the lowest zero of $J_o(x)$ occurs for $x = 2.4048$, while the lowest zero of $Y_1(x)$ occurs at $x = 2.1971$, and higher roots are even closer together.

These spurious solutions also occur in the cases we have evaluated. In Table IV we give some cases of pairs of solutions together with calculated values of $\psi(\vec{r})$ at the center of the figure involved. Since the integral equation is linear, the value of $\psi(\vec{r})$ is arbitrary, but it has been fixed by choosing $D(\vec{r})$ to have a first side maximum of 1. It is seen that for the "null" solutions $\psi(\vec{r})$ is very small, and its size may well be indicative of the accuracy of the treatment of the integral equation.

As in the case of the circular boundary, these pairs of eigenvalues are distressingly close together. Although the proximity of the "null" solutions may not be critical for the eigenvalue determination, the accuracy of the eigensolution, ψ_κ, will be affected adversely. Thus, in the interior of the region we can write:

$$\psi_\kappa(\vec{r}) = -\frac{\kappa}{4} \oint \frac{Y_1(\kappa|\vec{r}\,' - \vec{r}|)}{|\vec{r}\,' - \vec{r}|} D_\kappa(\vec{r}\,')(\vec{r}\,' - \vec{r})\cdot\vec{d\sigma}' , \qquad (14)$$

together with a corresponding equation for the "null" solution:

$$0 = -\frac{\lambda}{4} \oint \frac{Y_1(\lambda|\vec{r}\,' - \vec{r}|)}{|\vec{r}\,' - \vec{r}|} D_\lambda(\vec{r}\,')(\vec{r}\,' - \vec{r})\cdot\vec{d\sigma}' ,$$

where λ is the eigenvalue for the "null" solution. These can be combined to give

$$\psi_\kappa(\vec{r}) = \frac{1}{2} \oint \left\{ \left|\vec{K}_\kappa(\vec{r}, \vec{r}\,') - \vec{K}_\lambda(\vec{r}, \vec{r}\,')\right| \cdot\vec{d\sigma}' \left|D_\kappa(\vec{r}\,') + D_\lambda(\vec{r}\,')\right| \right.$$
$$\left. + \left|\vec{K}_\kappa(\vec{r}, \vec{r}\,') + \vec{K}_\lambda(\vec{r}, \vec{r}\,')\right| \cdot\vec{d\sigma}' \left|D_\kappa(\vec{r}\,') - D_\lambda(\vec{r}\,')\right| \right\} , \qquad (15)$$

where K_κ and K_λ are the corresponding kernels in the equation for ψ. From the numerical solutions, one finds that

$|D_\kappa(\vec{r}) - D_\lambda(\vec{r})| << |D_\kappa(\vec{r})|$, as is shown in Fig. 5 for the 45° triangular case with $\lambda = 9.76196$. Thus, since $K_\lambda \sim K_\kappa$ for $\lambda \sim \kappa$, in case the roots are close together the calculated ψ_κ will be small. On the other hand, errors in its calculation will be generated by D_κ, if Eq. (14) is used, so the relative errors will be large. As an alternative, one could use Eq. (15) to calculate ψ_κ, but one must then have solutions for the original equation for both κ and λ. Another alternative would be to use the Hankel function in the kernel instead of the Neumann function. In the circular case this moves the null-case eigenvalues into the complex plane and these complex roots are typically far removed from the desired ones. The disadvantage with this approach is that one must then deal with a complex kernel and solution, D. On the other hand, if one uses the method for a scattering problem, it is then necessary to use $H_1^{(1)}$ in the kernel in any case, so the spurious cases would be unimportant.

In conclusion, the results presented here clearly show that it is feasible to solve problems involving the Helmholtz equation by making use of distributions on the boundary. Although presence of the "null" solutions produces a larger error in ψ because of cancellations than might otherwise be the case, the cancellations might be avoided with some effort. On the other hand, the eigenvalues calculated seem quite accurate and perhaps they best illustrate the overall accuracy of the technique. Since a variational eigenvalue determination produces a much more accurate eigenvalue than the associated eigenfunction, the method described here appears to give very good solutions of the Helmholtz problem. Whether this method can be an effective competitor to the finite element or finite difference methods in applicable cases can only be determined by experience.

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APPENDIX A

In the numerical analysis we have made comparisons with various analytic eigensolutions of Helmholtz' equation. The solutions for the square are well known. For the various triangles they are:

Equilateral triangle

$$\psi(x,y) = \sin\frac{2\pi}{\sqrt{3}}(\sqrt{3}\,x + y) - \sin\frac{2\pi}{\sqrt{3}}(\sqrt{3}\,x - y) - \sin\frac{4\pi}{\sqrt{3}}y,$$

Isosceles right triangle

$$\psi(x,y) = \sin m\pi x \sin n\pi y - (-1)^{m+n}\sin n\pi x \sin m\pi y ,$$

and

$(30^\circ\text{-}60^\circ\text{-}90^\circ)$ triangle

$$\psi(x,y) = \cos\frac{2\pi}{3}(5x + \sqrt{3}\,y) - \cos\frac{2\pi}{3}(5x - \sqrt{3}\,y)$$

$$+ \cos\frac{2\pi}{3}(-x - 3\sqrt{3}\,y) - \cos\frac{2\pi}{3}(-x + 3\sqrt{3}\,y)$$

$$+ \cos\frac{2\pi}{3}(-4x + 2\sqrt{3}\,y) - \cos\frac{2\pi}{3}(-4x - 2\sqrt{3}\,y) .$$

APPENDIX B

Although the boundary integral equation, Eq. (8), must generally be solved numerically, in the case of a circular boundary the equation can be treated analytically, and some insight can thereby be obtained.

If the radius of the circular boundary is R, Eq. (8) can be written as:

$$f(\theta) \equiv \frac{D(\theta)}{2} + \frac{i\kappa R}{4}\int_0^{2\pi} H_1^{(1)}\left(2\kappa R \sin\left|\frac{\theta' - \theta}{2}\right|\right) D(\theta') \sin\left|\frac{\theta' - \theta}{2}\right| d\theta'$$

If $D(\theta)$ is expanded in a Fourier series,

$$D(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} , \qquad \text{(B1)}$$

we then have the integral, I_n, to evaluate:

$$I_n(\kappa R) = \int_0^{2\pi} H_1^{(1)}\left(2\kappa R \sin\left|\frac{\theta' - \theta}{2}\right|\right) \sin\left|\frac{\theta' - \theta}{2}\right| e^{in\theta'} d\theta' .$$

This integral may be evaluated(17) to give

$$I_n(\kappa R) = -\pi\frac{d}{d(\kappa R)}\left[H_n^{(1)}(\kappa R) J_n(\kappa R)\right] e^{in\theta}$$

Thus we find:

$$f(\theta) = \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n\left\{1 - \frac{i\pi\kappa R}{2}\left[H_n^{(1)}(\kappa R) J_n(\kappa R)\right]'\right\} e^{in\theta}$$

and, on making use of the Wronskian relation between $H_n^{(1)}$ and J_n, we get:

$$f(\theta) = -\frac{i\pi\kappa R}{2}\sum_{n=-\infty}^{\infty} a_n J_n(\kappa R) H_n^{(1)}(\kappa R)' e^{in\theta} .$$

This result can be used to solve specific problems. For example, if one wishes the eigenmodes for the interior of a circular region in which $\phi(r, \theta)$ is zero on the boundary, one immediately obtains the relation:

$$J_n(\kappa R)\, H_n^{(1)}(\kappa R)' = 0$$

The modes associated with $J_n(\kappa R)$ are well known, but the apparent modes for $H_n^{(1)}(\kappa R)'$ are not[18], and we will now demonstrate that for such κ's, even though the dipole distribution does not vanish, the associated $\psi(\vec{r})$ is zero everywhere inside the circle, so such solutions of the integral equation in this case are not useful. On the other hand, such solutions could arise in a numerical calculation of the integral equation, and one must be careful not to confuse them with nontrivial solutions. The distinction between solutions ψ would only be noticeable away from the boundary.

To find $\psi(r, \theta)$ once $D(\vec{r})$ is known, we can use

$$\psi(r, \theta) = \frac{i\kappa}{4} \int_0^{2\pi} H_1^{(1)}(\kappa w) D(\theta') \cos \chi \, d\theta'$$

where

$$w = \left[r^2 + R^2 - 2rR \cos (\theta - \theta')\right]^{1/2} ,$$

and χ is the angle between the vector $(\vec{r}' - \vec{r})$ and $\hat{e}_{r'}$. (See Fig. 8.) We again express $D(\theta)$ in a Fourier series, Eq. (B1) and, for $r < R$, we use Graf's addition theorem for Bessel functions[19] to give:

$$H_1^{(1)}(\kappa w) \cos \chi = \sum_{m=-\infty}^{\infty} H_{m+1}^{(1)} (\kappa R)\, J_m(\kappa r) \cos m(\theta - \theta') .$$

Thus one easily finds:

$$\psi(r,\theta) = \frac{i\pi\kappa R}{4} \sum_{n=-\infty}^{\infty} a_n \left\{ H_{n+1}^{(1)} (\kappa R)\, J_n(\kappa r) + H_{-n+1}^{(1)}(\kappa R)\, J_{-n}(\kappa r) \right\} e^{in\theta} .$$

Using the relations[20]:

$$J_{-n}(z) = (-1)^n J_n(z), \quad H_{-n}^{(1)}(z) = (-1)^n H_n^{(1)}(z), \quad \text{and}$$

$$H_{n+1}^{(1)}(z) - H_{n-1}^{(1)}(z) = -2\, H_n^{(1)}(z)' ,$$

we find:

$$\psi(r,\theta) = -\frac{i\pi\kappa R}{2} \sum_{n=-\infty}^{\infty} a_n\, d^{in\theta}\, J_n(\kappa r)\, H_n^{(1)}(\kappa R)' .$$

Thus, for the eigenmodes, we see that

$$\psi(r,\theta) = -\frac{i\pi\kappa R}{2} \left(a_n e^{in\theta} + a_{-n} e^{-in\theta}\right) J_n(\kappa r)\, H_n^{(1)}(\kappa R)' .$$

The a_n, a_{-n} are arbitrary, and, as stated earlier, we see that if $J_n(\kappa R) = 0$ we get the well-known eigenmodes, whereas if $H_n^{(1)}(\kappa R)' = 0$, $\psi(r, \theta) = 0$ for all r.

FOOTNOTES AND REFERENCES

* This work was supported by the U. S. ERDA.

1. See, for example: K. K. Mei, "Scattering of Radio Waves by Rectangular Cylinders", Ph.D. dissertation, University of Wisconsin, Madison, 1962; K. K. Mei and J. G. Van Bladel, IEEE Trans. Antennas Propag. AP-11, 185 (1963); Lawrence G. Copley, J. Acoust. Soc. Am. 44, 41 (1968); Burke, et. al., J. Comput. Phys. 10, 22 (1972); Miller, et. al., Can. J. Phys. 50, 879 (1972); Miller, et. al. J. Comput. Phys. 12, 24 and 210 (1973).
2. W. Pogorzelski, Integral Equations and Their Applications (Pergamon, Long Island City, N. Y., 1966), p. 230, ff.
3. Ref. 1, p. 239 ff.
4. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (U. S. Government Printing Office, Washington, D. C., 1964), p. 358, Eq. (9.13).
5. A. Erdélyi, et. al., Tables of Integral Transforms, Vol. 1 (McGraw-Hill Book Co., Inc., New York, N. Y. 1954), p. 309, Sec. 6.2, Eq. (12).
6. Ref. 2, Chapter XVI.
7. Ref. 2, Chapter XVII.

8. In principle the contour could pass both a zero and a pole in $1 \pm r(\xi)$ and still have a well-behaved integral as $|\mathrm{Im}\ \xi| \to \infty$. This condition cannot be achieved within the restriction on C, however, since $r(\xi)$ has poles at $\xi = n, n \neq 0$.
9. At first sight, one might expect that $f_+(s)$ should only have even terms, and $f_-(s)$ only odd, but except for cases with special symmetries both functions have even and odd powers. The only restriction is that $f_-(0) = 0$.

10. Ref. 4, p. 363, Eq. (.1.80).
11. J. Meixner, Ann. Phys. 6, 2 (1949).
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16. Grove C. Nooney (Dissertation), On the Vibrations of Triangular Membranes, Dept. of Mathematics, Stanford University, Stanford California, Oct. 22, 1953.

17. To obtain the result, we use the relation $H_0^{(1)}(x)' = -H_1^{(1)}(x)$ and the integrals in Ref. 3, p. 485, Eqs. (11.4.8), (11.4.9).
18. We have chosen $H_1^{(1)}$ in the kernel, and so the spurious κ's are not real. On the other hand, for a closed region it is more convenient to use Y_1 instead, and the analysis goes through as before except that the spurious modes occur then for $Y_1(\kappa R) = 0$. In this case the κ's are real.

19. Ref. 3, p. 363, Eq. (9.1.79).
20. Ref. 3, p. 358, Eqs. (9.1.5), (9.1.6); p. 361, Eq. (9.1.27).

TABLE I

Calculated Eigenvalues for Various Triangles

Case	Refl. Side	κ	Error
Equilateral Triangle	Any	7.255218367	2.09×10^{-5}
45°-45°-90°	Long Side	9.934553730	-3.45×10^{-5}
	Short Side	9.933726725	-8.62×10^{-4}
30°-60°-90°	Long Side	11.08250607	8.89×10^{-6}
	Middle Side	11.08234361	-1.54×10^{-4}
	Short Side	11.08187716	-6.20×10^{-4}

TABLE II

Calculated Eigenvalues for a Square

Points/Side	κ	Error
26	4.442863650	-1.93×10^{-5}
36	4.442880014	-2.92×10^{-6}
46	4.442882134	-8.04×10^{-7}

TABLE III

Calculated Eigenvalues for an L-shaped Region and a 30° Rhombus

Case	Points/Side	κ (Calc.)	κ (fin. elem.)
L	28	3.10465	3.10479
Rhombus	26	4.9898337	
Rhombus	36	4.9898439	4.98988
Rhombus	46	4.9898456	

TABLE IV

Comparison of "Null" and Proper Solutions for Various Figures

Case	Refl Side	λ	κ	ϕ_λ(cen)	ϕ_κ(cen)
Square	Any	4.07596	4.44286	$-.74 \times 10^{-5}$	-.17
45°-45°-90°	Long	9.56806	9.93455	$-.15 \times 10^{-4}$	-.13
45°-45°-90°	Short	9.76275	9.93373	$-.20 \times 10^{-3}$	-.042
30°-60°-90°	Long	10.81737	11.08251	$.92 \times 10^{-5}$	-.072
30°-60°-90°	Middle	10.84880	11.08234	$-.92 \times 10^{-4}$	-.048

FIGURE CAPTIONS

Fig. 1. D(s) for a square. From symmetry $D(3 - s) = D(s)$.

Fig. 2. D(s) for an equilateral triangle. From symmetry $D(2 - s) = D(s)$.

Fig. 3. D(s) for a 30°-60°-90° triangle reflected along the long side.

Fig. 4. D(s) for the L-shaped region.

Fig. 5. The solid curve represents $D_\kappa(s)$ for a 45° isosceles triangle reflected along a short side. The dashed curve represents $5 \times (D_\kappa(s) - D_\lambda(s))$.

Fig. 6. Illustration of variables used for the analytic treatment of a circular region.

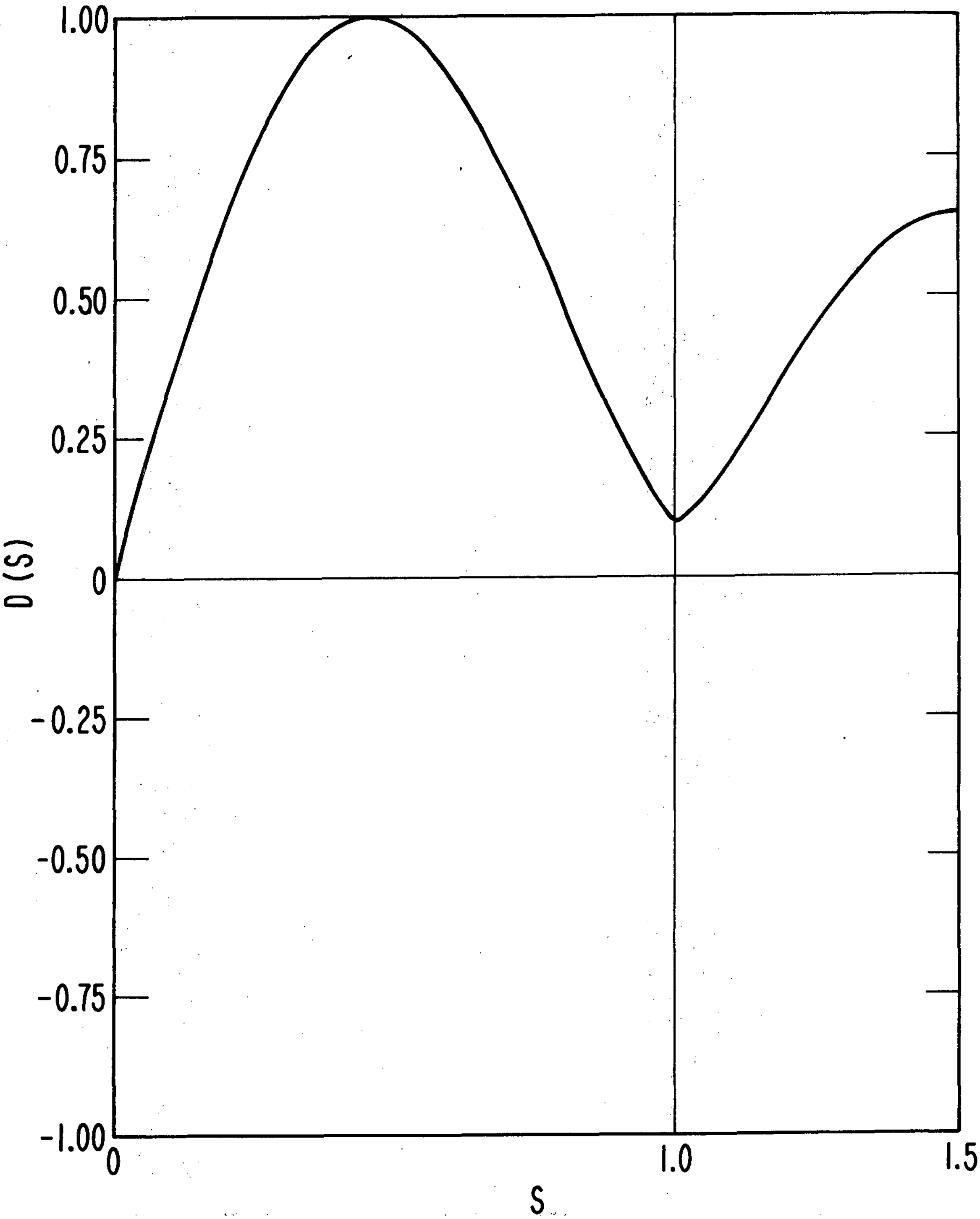


Fig. 1

XBL 763-2544

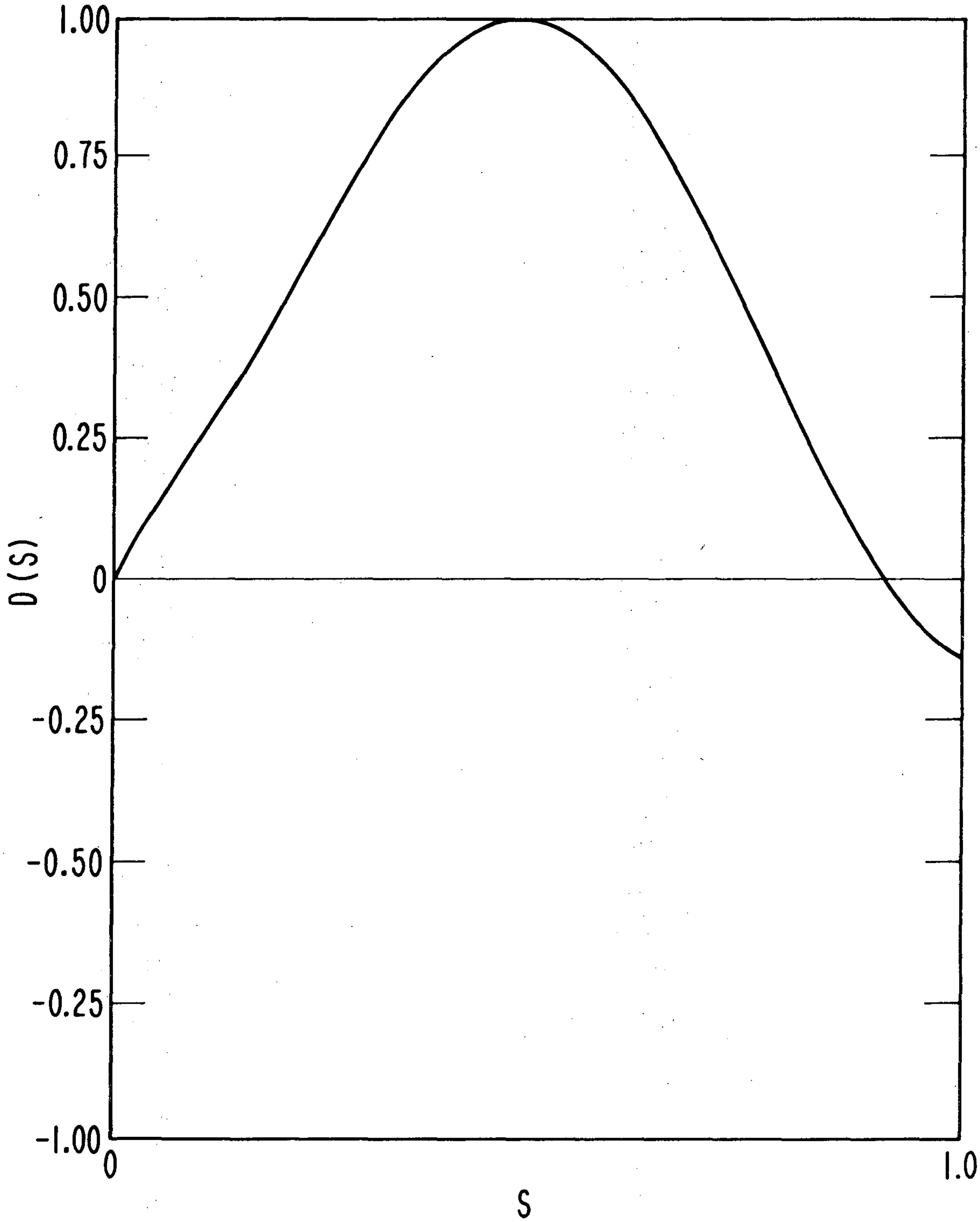
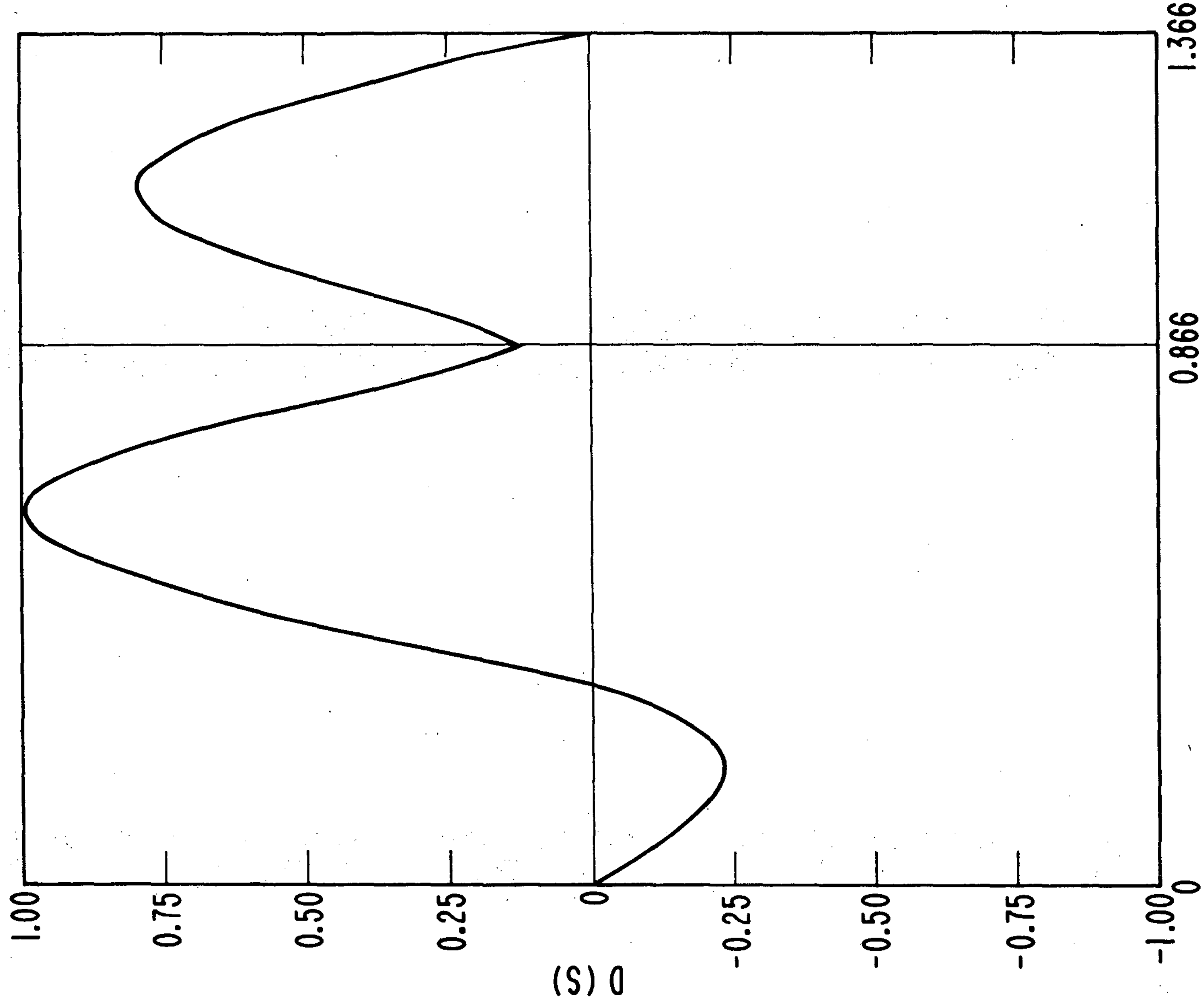


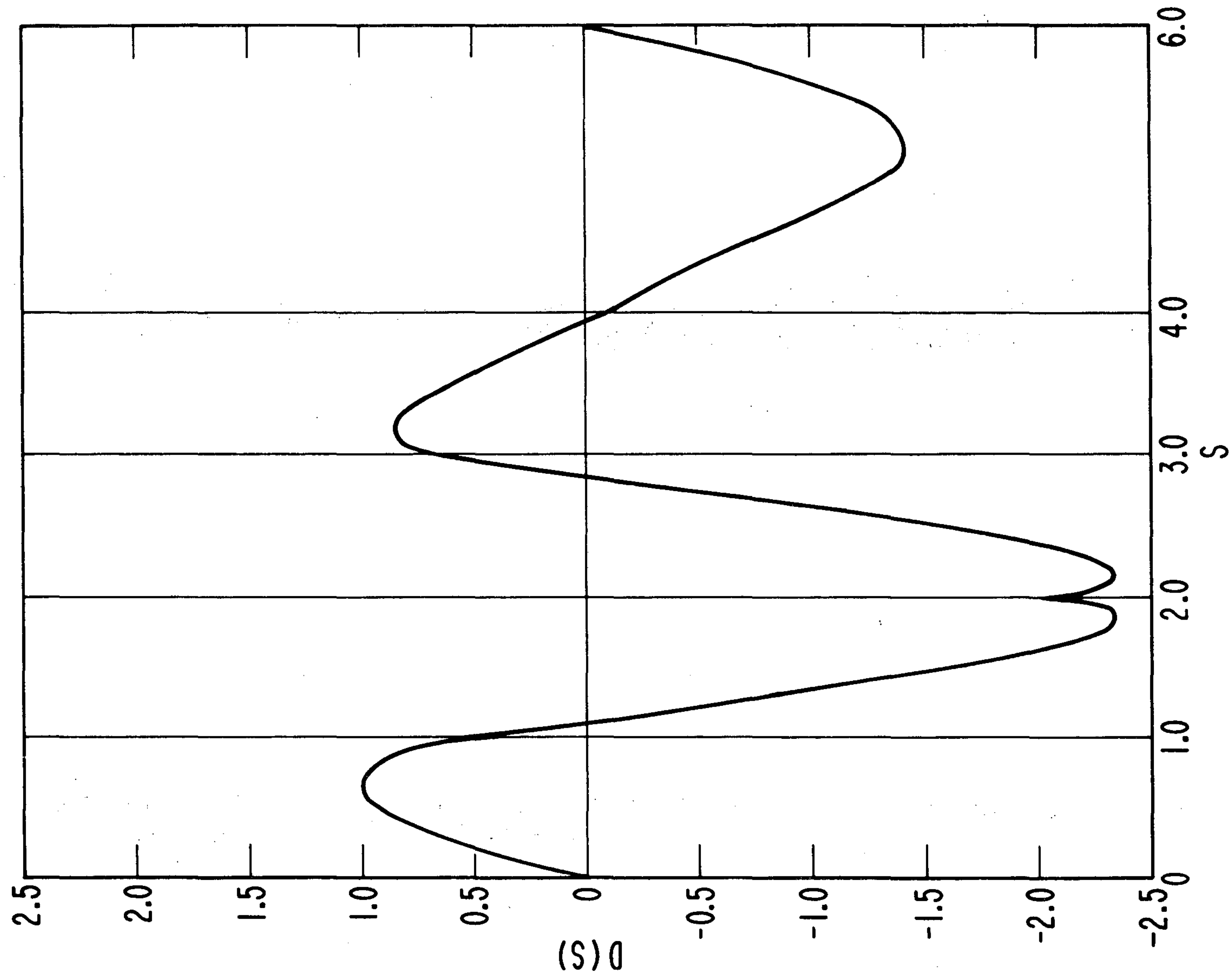
Fig. 2

XBL 763-2545



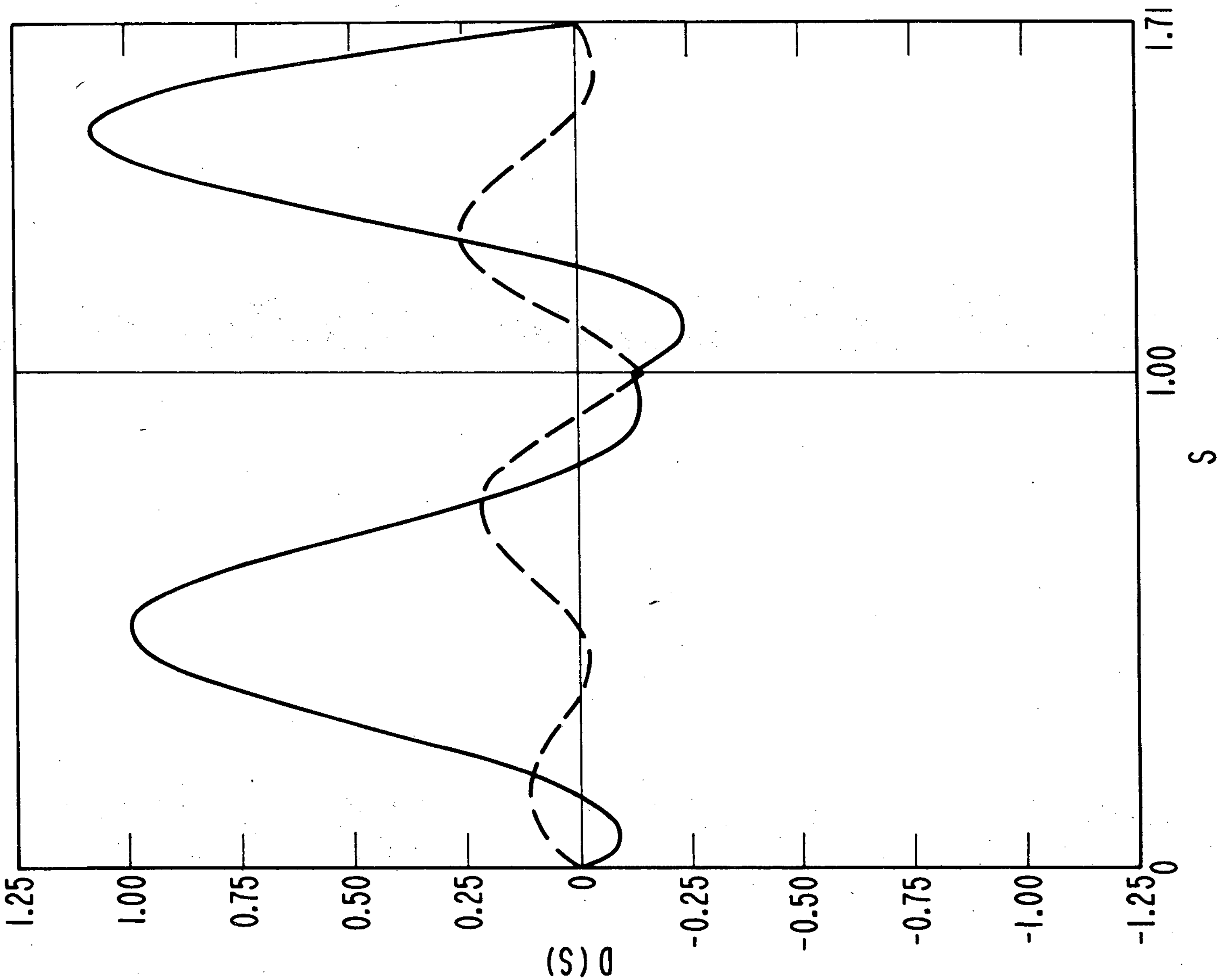
XBL 763-2546

Fig. 3



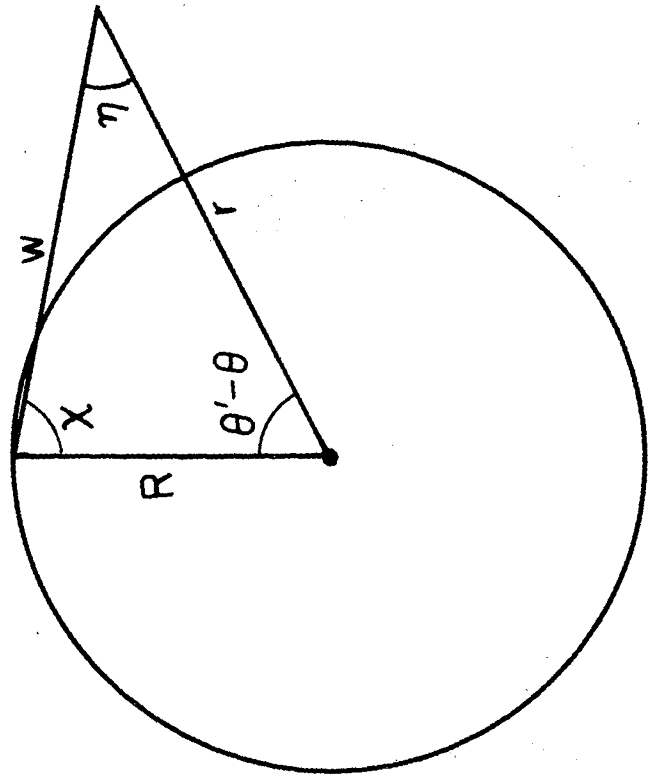
XBL 763-2547

Fig. 4



XBL 763-2548

Fig. 5



XBL747-3688

Fig. 6

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