Title

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GRADIENT NLW ON CURVED BACKGROUND IN $4+1$ DIMENSIONS

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Abstract. We obtain a sharp local well-posedness result for the Gradient Nonlinear Wave Equation on a nonsmooth curved background. In the process we introduce variable coefficient versions of Bourgain's $X^{s,b}$ spaces, and use a trilinear multiscale wave packet decomposition in order to prove a key trilinear estimate.

1. Introduction

In this article we are investigating the issue of local well-posedness for a variable coefficient semilinear wave equation in $4+1$ dimensions. To describe the context and motivate the interest in our problem we introduce three related equations. We begin with a generic gradient NLW equation in $\mathbb{R}^{n+1}$,

$$\Box u \,=\, \Gamma(u)(\nabla u)^2 \tag{1}$$

with the nonlinearity

$$\Gamma(u)(\nabla u)^2 \,=\, q^{ij}(u)\partial_i u\, \partial_j u$$

where q^{ij} are smooth functions and the standard summation convention is used.

Then we move on to a similar equation but on a curved background,

$$\Box_g u \,=\, \Gamma(u)(\nabla u)^2 \tag{2}$$

with $\Box_g \,=\, g^{ij}\,\partial_i\partial_j$, where the summation occurs from 0 to n and the index 0 stands for the time variable. To insure hyperbolicity we assume that the matrix g^{ij} has signature $(1, n)$ and the time level sets $x_0 = const$ are space-like, i.e. $g^{00} > 0$. In effect to simplify some of the computations we make the harmless assumption $g^{00} = 1$.

Finally, we consider a corresponding quasilinear equation

$$\Box_{g(u)} u \,=\, \Gamma(u)(\nabla u)^2 \tag{3}$$

with similar assumptions on the matrix g.

In all three cases we are interested in the local well-posedness of the Cauchy problem in Sobolev spaces $H^s(\mathbb{R}^n) \times H^{s-1}(\mathbb{R}^n)$ with initial data

$$u(0, x) \,=\, u_0(x), \qquad \partial_t u(0, x) \,=\, u_1(x) \tag{4}$$

The first equation (1) is the best understood so far, and is known to be locally well-posed for s in the range

$$s > \max\{\frac{n}{2}, \frac{n+5}{4}\}$$

This range is sharp. The $\frac{n}{2}$ obstruction comes from scaling, while the $\frac{n+5}{4}$ is related to concentration along light rays, see Lindblad [9]. The proof of the positive result is fairly straightforward in dimension $2+1$ and $3+1$, where it suffices to rely on the

Strichartz estimates. In $4+1$ dimensions this no longer works and one needs to use instead the $X^{s,\theta}$ spaces, see Foschi-Klainerman [3]. These are multiplier weighted L^2 spaces associated to the wave operator as the Sobolev spaces H^s are connected to the Laplace operator Δ, see Klainerman-Machedon [5]:

$$\|u\|_{X^{s,\theta}} = \|(1+|\xi|^2)^{\frac{s}{2}} \cdot (1+||\tau|-|\xi||^2)^{\frac{\theta}{2}} \cdot |\hat{u}(\tau,\xi)|\|_{L^2} \tag{5}$$

where $\hat{u} = \hat{u}(\tau,\xi)$ is the space-time Fourier transform of function $u = u(t,x)$. Finally, in the most difficult case, $n \geq 5$, this was proved by Tataru [14], using a suitable modification of the $X^{s,\theta}$ spaces, needed in order to control the interaction of high and low frequencies in the multiplicative estimates.

For the quasilinear problem (3) the sharp result is only known to hold in dimensions $n = 2, 3$. This was proved by Smith-Tataru [12] (see also Lindblad's counterexample [10]). The argument there still requires the use of Strichartz estimates. These are derived from a wave packet parametrix construction for a wave equation with very rough coefficients, which in turn is obtained via a very delicate analysis of the Hamilton flow. A different proof of this result in the special case of the Einstein vacuum equation was independently obtained by Klainerman-Rodnianski [6], [8], [7]. In dimensions $n \geq 4$ it is still unclear which is the optimal threshold, the best results so far being contained in the above mentioned paper of Smith-Tataru [12] and in an earlier one, Tataru [15]:

$$\begin{aligned} &n = 4, 5 \qquad s > \frac{n}{2} + \frac{1}{2} \\ &n \geq 6 \qquad\ \ s > \frac{n}{2} + \frac{2}{3} \end{aligned}$$

In the same direction but somewhat closer in spirit to the present paper is Bahouri and Chemin's work [2, 1]. The equation considered there is still quasilinear, but the main estimates are frequency localized versions of the Strichartz estimates for the wave equation on a rough background.

As an intermediate step toward understanding the higher dimensional quasilinear problem, we consider here the semilinear problem on a curved background and we prove the sharp result:

Theorem 1.1. *Let $n = 4$ and assume that the coefficients g^{ij} satisfy $\partial^2 g \in L^2 L^\infty$. Then the Cauchy problem (2), (4) is locally well-posed in $H^s \times H^{s-1}$ for $s > \frac{9}{4}$.*

Here well-posedness is understood in the strongest sense, i.e. the solutions have Lipschitz dependence on the initial data and they exist on a time interval which only depends on the size of the initial data.

One contribution of the present paper is to introduce variable coefficient versions of the $X^{s,b}$ spaces, study their properties and obtain the corresponding Strichartz type embeddings. However, the main novelty, contained in the last two sections, is a new method, based on a trilinear wave packet decomposition, to prove a key trilinear bound which cannot be obtained directly from the Strichartz estimates.

The first step in the proof is to reduce the problem to the case when the initial data is small, using scaling and the finite speed of propagation. This is a routine argument for which we refer the reader to [12]. Once we know that the initial data is small, we can fix the time interval and set it to $[-1, 1]$. This will be the case throughout the rest of the paper.

To solve the problem for small data we use a fixed point argument. Let $S(u_0,u_1)$ and $\Box_g^{-1}$ be respectively the homogeneous and inhomogeneous solution operators

$$\Box_g S(u_0,u_1) = 0, \quad S(u_0,u_1)(0) = u_0, \quad \partial_t S(u_0,u_1)(0) = u_1 \tag{6}$$

$$\Box_g(\Box_g^{-1}H) = H, \qquad (\Box_g^{-1}H)(0) = 0, \qquad \partial_t(\Box_g^{-1}H)(0) = 0 \tag{7}$$

Then a solution u for (2) in $[-1,1]$ is also a fixed point for the functional

$$F(u) = S(u_0,u_1) + \Box_g^{-1}(\Gamma(u)(\nabla u)^2) \tag{8}$$

In order to apply a fixed point argument for F we need to find two Banach spaces X and Y for which the following mapping properties hold:

$$\|S(u_0,u_1)\|_X \lesssim \|(u_0,u_1)\|_{H^s\times H^{s-1}} \tag{9}$$

$$\|\Box_g^{-1}H\|_X \lesssim \|H\|_Y \tag{10}$$

$$\|u\cdot w\|_X \lesssim \|u\|_X\|w\|_X \tag{11}$$

$$\|\Gamma(u)\|_X \lesssim C(\|u\|_{L^\infty})(1+\|u\|_X^5) \tag{12}$$

$$\|u\cdot w\|_Y \lesssim \|u\|_X\|w\|_Y \tag{13}$$

$$\|\nabla v\cdot\nabla w\|_Y \lesssim \|v\|_X\cdot\|w\|_X \tag{14}$$

where $C = C(\|u\|_{L^\infty})$ is a constant that depends solely on $\|u\|_{L^\infty}$. In the flat case (1), for dimension $n=4$, one can make this argument work by choosing

$$X = X^{s,\theta} \qquad Y = X^{s-1,\theta-1}$$

with

$$s = \theta + \frac{3}{2} \qquad \theta > \frac{3}{4} \tag{15}$$

For our problem the challenge is twofold: first we need to find suitable variable coefficient versions for the $X^{s,\theta}$ spaces and then, in this new context, prove the corresponding estimates (9)-(14).

Such spaces were previously introduced by Tataru [13], where they are used in the context of a unique continuation problem. There, for a hyperbolic operator P one defines

$$X^{s,0} = H^s, \quad X^{s,1} = \{u\in H^s | Pu \in H^{s-1}\}$$

Then all the other spaces are defined through interpolation and duality.

In this article we choose to follow a different path based on dyadic decompositions with respect to the spatial frequency and the distance to the characteristic cone. Likely one should be able to prove that the two approaches are equivalent, but we choose not to pursue this here.

Our article is structured as follows. In the next section we define the $X^{s,\theta}$ spaces and prove that they satisfy the linear estimates (9), (10). Our definition of the $X^{s,\theta}$ is slightly different from the standard one (5) in the constant coefficient case. Precisely, in the constant coefficient case our definition gives

$$\|u\|_{X^{s,\theta}} \approx \|(1+|\xi|^2)^{\frac{s}{2}}(1+||\tau|-|\xi||^2)^{\frac{\theta}{2}}\cdot\hat{u}(\tau,\xi)\|_{L^2} + \|\Box u\|_{L^2_tH^{s+\theta-2}_x}, \quad 0<\theta<1 \tag{16}$$

and one can see that the second term above alters the behavior at high modulations $|\tau| \gg |\xi|$. Correspondingly, for negative θ we have

$$\|u\|_{X^{s,\theta}} \approx \|(1+|\xi|^2)^{\frac{s}{2}}(1+||\tau|-|\xi||^2)^{\frac{\theta}{2}}\cdot\hat{u}(\tau,\xi)\|_{L^2} + \|u\|_{L^2_tH^{s+\theta}_x}, \quad -1<\theta<0 \tag{17}$$

This change is consistent with scaling and simplifies somewhat the study of high modulation interactions.

In Section 3 we discuss the Strichartz estimates for $\Box_g$, which translate into embeddings for the $X^{s,\theta}$ spaces. These turn out to suffice for the proof of the algebra properties (11)-(13) and for the high-high frequency interactions in (14).

The difficult part is to study the high-low frequency interactions in (14). For this we first take advantage of the duality relation

$$(X^{s,\theta} + L^2 H^{s+\theta})' \; = \; X^{-s,-\theta} \qquad s \in \mathbb{R}, \quad 0 < \theta < \frac{1}{2} \tag{18}$$

This is consistent with (16) and (17). Using this duality, after factoring out high modulation interactions, the bound (14) is transformed into the trilinear estimate:

$$\left| \int u \cdot v \cdot w \; dx\, dt \right| \lesssim \|u\|_{X^{1-s,1-\theta}} \|v\|_{X^{s-1,\theta}} \|w\|_{X^{s-1,\theta}} \tag{19}$$

with (s, θ) verifying (15). The last section of the paper is devoted to proving this bound. The argument is based on a multiscale trilinear wave packet decomposition for linear waves.

2. The $X^{s,\theta}$ spaces

We first introduce Littlewood-Paley decompositions. As a general rule, all frequency localizations in the sequel are only with respect to the spatial variables. There is a single exception to this. Precisely, the coefficients g^{ij} are truncated using space-time multipliers. In order for these truncations to work, we need for these coefficients to be defined globally in time. Hence we assume they have been extended to functions with similar properties in all of $\mathbb{R}^{n+1}$.

Let ϕ be a smooth function supported in $\{\frac{1}{2} \leq |\xi| \leq 2\}$ with the property that

$$1 = \sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi)$$

We consider a spatial Littlewood-Paley decomposition,

$$1 = \sum_{\lambda=1}^{\infty} S_\lambda(D_x)$$

where for dyadic $\lambda > 1$ we have

$$S_\lambda(\xi) = \phi\left(\frac{\xi}{\lambda}\right)$$

while S_1 incorporates the low frequency contribution in $\{|\xi| \leq 1\}$. Set

$$S_{<\lambda} = \sum_{\mu=1}^{\frac{\lambda}{2}} S_\mu$$

We will also use spatial multipliers $\tilde{S}_\lambda$ with slightly larger support, with $S_\lambda \tilde{S}_\lambda = S_\lambda$. We say that a function u is localized at frequency λ if its Fourier transform is supported in the annulus $\{\frac{\lambda}{8} \leq |\xi| \leq 8\lambda\}$.

For the paradifferential type calculus we also need to truncate the coefficients of $\Box_g$ in frequency. Given $\Box_g$ in (2) we define the modified operators

$$\Box_{g_{<\lambda}} = (S_{<\lambda}(D_x, D_t) g^{\alpha\beta}) \partial_\alpha \partial_\beta$$

In the sequel we omit the space and time variables in our function space notations, i.e. $L^p := L^p_{x,t}$, $L^2H^s := L^2_tH^s_x$, $L^pL^q := L^p_tL^q_x$, etc. We are ready now to define our spaces:

Definition 2.1. *Let $\theta \in (0,1)$ and $s \in \mathbb{R}$. Then $X^{s,\theta}$ is the space of functions $u \in L^2(-1,1;H^s(\mathbb{R}^n))$ for which the following norm is finite:*

$$\|u\|^2_{X^{s,\theta}} = \inf\left\{\sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}};\ u = \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} S_\lambda u_{\lambda,d}\right\} \tag{20}$$

where λ, d take dyadic values and

$$\|u_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}} = \lambda^{2s}d^{2\theta}\|u_{\lambda,d}\|^2_{L^2} + \lambda^{2s-2}d^{2\theta-2}\|\Box_{g_{<\sqrt{\lambda}}}u_{\lambda,d}\|^2_{L^2} \tag{21}$$

We also define the space $X^{s-1,\theta-1}$ of functions for which the following norm is finite:

$$\begin{aligned}\|f\|^2_{X^{s-1,\theta-1}} = \inf\Bigg\{\|f_0\|^2_{L^2H^{s-1}} + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\|f_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}};\\ f = f_0 + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda}\Box_{g_{<\sqrt{\lambda}}}S_\lambda f_{\lambda,d}\Bigg\}\end{aligned} \tag{22}$$

Remark 2.2. *Intuitively d stands for the modulation of the $u_{\lambda,d}$ piece. Indeed, in the constant coefficient case one can easily see that $u_{\lambda,d}$ mainly contributes to u in the region where $||\tau| - |\xi|| \approx d$. The condition $1 \leq d$ is related to the spatial localization on the unit scale in our problem. The condition $d \leq \lambda$ reflects the fact that at high modulation we use a simpler structure, see e.g.* (16), (17).

Remark 2.3. *The cutoff at frequency less than $\sqrt{\lambda}$ for the coefficients $\Box_g$ is related to the regularity of the coefficients, $\partial^2 g \in L^2L^\infty$. This implies that $\Box_{g_{\geq\sqrt{\lambda}}}u_{\lambda,d}$ is an allowable error term.*

We begin our analysis of the $X^{s,\theta}$ spaces with a simple observation, namely that without any restriction in generality one can assume that the functions $u_{\lambda,d}$ and $f_{\lambda,d}$ in Definition 2.1 are localized at frequency λ. Precisely, we have the stronger result:

Lemma 2.4. *The following estimate holds:*

$$\lambda^{s-1}d^\theta\|\nabla S_\lambda v\|_{L^2} + \lambda^{s-1}d^{\theta-1}\|\Box_{g_{<\sqrt{\lambda}}}S_\lambda v\|_{L^2} \lesssim \|v\|_{X^{s,\theta}_{\lambda,d}} \tag{23}$$

Proof. We first bound the time derivatives of v in negative Sobolev spaces,

$$\lambda^s d^\theta(\|\partial_t^2 v\|_{L^2(H^{-2}+\lambda^2L^2)} + \|\partial_t v\|_{L^2(H^{-1}+\lambda L^2)}) \lesssim \|v\|_{X^{s,\theta}_{\lambda,d}} \tag{24}$$

This follows by Cauchy-Schwartz from the interpolation inequality

$$\|\partial_t v\|^2_{L^2(H^{-1}+\lambda L^2)} \lesssim (\|\partial_t^2 v\|_{L^2(H^{-2}+\lambda^2L^2)} + \|v\|_{L^2})\|v\|_{L^2}$$

combined with the bound

$$\|\partial_t^2 v\|_{L^2(H^{-2}+\lambda^2L^2)} \lesssim \lambda^{-2}\|\Box_{g_{<\sqrt{\lambda}}}v\|_{L^2} + \|\partial_t v\|_{L^2(H^{-1}+\lambda L^2)} + \|v\|_{L^2}$$

To prove this last estimate we only use the L^∞ regularity of g together with the condition $g^{00} = 1$. Then we need the fixed time bounds

$$\|g_{<\sqrt{\lambda}}\partial_x\partial_t v\|_{H^{-2}+\lambda^2 L^2} \lesssim \|\partial_t v\|_{H^{-1}+\lambda L^2}$$

$$\|g_{<\sqrt{\lambda}}\partial_x^2 v\|_{H^{-2}+\lambda^2 L^2} \lesssim \|v\|_{L^2}$$

They are similar, so we only discuss the second one. We write

$$g_{<\sqrt{\lambda}}\partial_x^2 v = \partial_x^2(g_{<\sqrt{\lambda}}v) - 2\partial_x(\partial_x g_{<\sqrt{\lambda}}v) + \partial_x^2 g_{<\sqrt{\lambda}}v$$

and use the uniform bounds

$$|g_{<\sqrt{\lambda}}| \lesssim 1, \qquad |\partial_x g_{<\sqrt{\lambda}}| \lesssim \lambda, \qquad |\partial_x^2 g_{<\sqrt{\lambda}}| \lesssim \lambda^2$$

This concludes the proof of (24).

The first term in (23) is directly bounded using (24). For the second it suffices to prove the commutator estimate

$$\|[\Box_{g_{<\sqrt{\lambda}}}, S_\lambda]v\|_{L^2} \lesssim \lambda\|v\|_{L^2} + \|\partial_t v\|_{L^2} \tag{25}$$

We have

$$[\Box_{g_{<\sqrt{\lambda}}}, S_\lambda] = [g_{<\sqrt{\lambda}}, S_\lambda]\partial_t\partial_x + [g_{<\sqrt{\lambda}}, S_\lambda]\partial_x^2$$

and the commutators are localized at frequency λ so the spatial derivatives only contribute factors of λ. Hence (25) follows from the standard commutator estimate

$$\|[g_{<\sqrt{\lambda}}, S_\lambda]\|_{L^2 \to L^2} \lesssim \lambda^{-1}\|\nabla g\|_{L^\infty}$$

□

Applying the above Lemma with S_λ replaced by $\tilde{S}_\lambda$ we obtain

Corollary 2.5. *One can replace the $X^{s,\theta}_{\lambda,d}$ norm in the definition of $X^{s,\theta}$ and $X^{s-1,\theta-1}$ by the norm*

$$\|v\|_{\tilde{X}^{s,\theta}_{\lambda,d}} = \lambda^{s-1}d^\theta\|\nabla v\|_{L^2} + \lambda^{s-1}d^{\theta-1}\|\Box_{g_{<\sqrt{\lambda}}}v\|_{L^2}$$

For the proof of the duality relation (18) it is convenient to work with a selfadjoint operator. Thus we consider the selfadjoint counterpart $\tilde{\Box}_g$ of $\Box_g$

$$\tilde{\Box}_g = \partial_i g^{ij}\partial_j$$

Then for v localized at frequency λ we commute and estimate the frequency localized difference

$$\|\tilde{\Box}_{g_{<\sqrt{\lambda}}}v - \Box_{g_{<\sqrt{\lambda}}}v\|_{L^2} \lesssim \|\nabla v\|_{L^2}$$

This leads directly to

Corollary 2.6. *One can replace the $\Box_{g_{<\sqrt{\lambda}}}$ operator in the definition of $X^{s,\theta}$ and $X^{s-1,\theta-1}$ by the similar operator in divergence form $\tilde{\Box}_{g_{<\sqrt{\lambda}}}$.*

As a consequence of the second part of (23) we have

Corollary 2.7. *The following embedding holds for $-1 < \theta < 0$:*

$$X^{s,\theta} \subset L^2 H^{s+\theta}$$

Another use of this is to establish energy estimates. A direct application of energy estimates for the wave equation yields the bound

$$\|\nabla v\|_{L^\infty L^2}^2 \lesssim \|\nabla v\|_{L^2}^2 + \|\nabla v\|_{L^2} \|\Box_g v\|_{L^2}$$

This leads to

$$\lambda^{s-1} d^{\theta-\frac{1}{2}} \|\nabla S_\lambda v\|_{L^\infty L^2} \lesssim \|v\|_{\tilde{X}^{s,\theta}_{\lambda,d}} \tag{26}$$

Going back to Definition 2.1, this implies

Corollary 2.8. *Assume that* $\theta > \frac{1}{2}$. *Then*

$$\|u\|_{L^\infty H^s} + \|u_t\|_{L^\infty H^{s-1}} \lesssim \|u\|_{X^{s,\theta}} \tag{27}$$

To prove the estimates (9) and (10) in the context of the $X^{s,\theta}$ spaces we need to switch from the frequency truncated coefficients to the full coefficients g^{ij}. The tool needed to do that is contained in the following:

Lemma 2.9. *Assume that* $0 \leq s \leq 3$. *Then the following fixed time estimate holds:*

$$\sum_{\lambda=1}^{\infty} \lambda^{2(s-1)} \|\tilde{S}_\lambda (g_{>\sqrt{\lambda}} u)\|_{L^2}^2 \lesssim (M(\|\partial^2 g\|_{L^\infty}))^2 \|u\|_{H^{s-2}}^2 \tag{28}$$

where M *stands for the maximal function with respect to time. We also have the dual estimate*

$$\|\sum_{\lambda=1}^{\infty} g_{>\sqrt{\lambda}} \tilde{S}_\lambda f_\lambda\|_{H^{2-s}}^2 \lesssim \sum_{\lambda=1}^{\infty} \lambda^{2(1-s)} \|f_\lambda\|_{L^2}^2 \tag{29}$$

Proof. We take a Littlewood-Paley decomposition of both factors,

$$\tilde{S}_\lambda (g_{>\sqrt{\lambda}} u) = \sum_{\mu=1}^{\infty} \sum_{\nu=\sqrt{\lambda}}^{\infty} \tilde{S}_\lambda (g_\nu u_\mu)$$

The (μ, ν) term is nonzero only in the following situations:

(i) $\nu \ll \lambda$, $\mu \approx \lambda$. Then we estimate

$$\|\tilde{S}_\lambda (g_\nu u_\mu)\|_{L^2} \lesssim \|g_\nu\|_{L^\infty} \|u_\mu\|_{L^2} \lesssim \nu^{-2} M(\|\partial^2 g\|_{L^\infty}) \|u_\mu\|_{L^2}$$

and use the square summability with respect to λ together with the relation $\nu^{-2} \lesssim \lambda^{-1}$.

(ii) $\nu \approx \lambda$, $\mu \ll \lambda$. Then

$$\|\tilde{S}_\lambda (g_\nu u_\mu)\|_{L^2} \lesssim \|g_\nu\|_{L^\infty} \|u_\mu\|_{L^2} \lesssim \lambda^{-2} M(\|\partial^2 g\|_{L^\infty}) \|u_\mu\|_{L^2}$$

This is tight only when $s = 3$ and $\mu = 1$, otherwise there is a gain which insures the summability in λ, μ.

(iii) $\nu \approx \mu \gtrsim \lambda$. Then

$$\|\tilde{S}_\lambda (g_\nu u_\mu)\|_{L^2} \lesssim \|g_\nu\|_{L^\infty} \|u_\mu\|_{L^2} \lesssim \mu^{-2} M(\|\partial^2 g\|_{L^\infty}) \|u_\mu\|_{L^2}$$

This is always stronger than we need. The proof of the lemma is concluded. □

We now establish some simple properties of the linear equation

$$\Box_g u = f, \qquad u(0) = u_0, \qquad u_t(0) = u_1. \tag{30}$$

Then

Lemma 2.10. *The linear equation* (30) *is well-posed in* $H^s \times H^{s-1}$ *for* $0 \le s \le 3$.

The proof follows easily from energy estimates, see [16].
We use this to prove (9), namely

Lemma 2.11. *Assume that* $0 \le s \le 3$ *and* $\theta > 0$. *Then the solution* u *to* (30) *verifies*

$$\|u\|_{X^{s,\theta}} \lesssim \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \|f\|_{L^2H^{s-1}}$$

Proof. We decompose the solution u as

$$u = \sum_{\lambda=1}^{\infty} S_\lambda \tilde{S}_\lambda u$$

and think of this as a part of the sum in (20) which corresponds to $d = 1$. Then

$$\begin{aligned}\|u\|^2_{X^{s,\theta}} &\lesssim \sum_{\lambda=1}^{\infty} \|\tilde{S}_\lambda u\|^2_{X^{s,\theta}_{\lambda,1}} \\ &\approx \sum_{\lambda=1}^{\infty} \lambda^{2s}\|\tilde{S}_\lambda u\|^2_{L^2} + \lambda^{2(s-1)}\|\Box_{g_{<\sqrt{\lambda}}}\tilde{S}_\lambda u\|^2_{L^2} \\ &\lesssim \|u\|^2_{L^2H^s} + \sum_{\lambda=1}^{\infty} \lambda^{2(s-1)}\|\Box_{g_{<\sqrt{\lambda}}}\tilde{S}_\lambda u - \tilde{S}_\lambda \Box_g u\|^2_{L^2} + \|f\|^2_{L^2H^{s-1}}\end{aligned}$$

The first term is easily controlled by energy estimates. The second is decomposed as follows:

$$\Box_{g_{<\sqrt{\lambda}}}\tilde{S}_\lambda u - \tilde{S}_\lambda \Box_g u = [\Box_{g_{<\sqrt{\lambda}}}, \tilde{S}_\lambda]u - \tilde{S}_\lambda \Box_{g_{>\sqrt{\lambda}}} u$$

For the commutator we use the fixed time bound (25) along with square summability in λ. The second part is controlled by (28).

□

The result in the next Lemma implies the estimate (10) for the spaces X, Y:

Lemma 2.12. *Assume that* $0 \le s \le 3$ *and* $\frac{1}{2} < \theta < 1$. *Then the operator* $\Box_g^{-1}$ *has the mapping property*

$$\Box_g^{-1} : X^{s-1,\theta-1} \to X^{s,\theta}$$

Proof. Let $f \in X^{s-1,\theta-1}$. We use the representation in (22),

$$f = f_0 + \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} \Box_{g_{<\sqrt{\lambda}}} S_\lambda f_{\lambda,d}$$

By Definition (2.1) the function

$$u = \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} S_\lambda f_{\lambda,d}$$

belongs to $X^{s,\theta}$. The difference $v = u - \Box_g^{-1} f$ solves

$$\Box_g v = \Box_g u - f, \qquad v(0) = u(0), \qquad v_t(0) = u_t(0)$$

To estimate it we use Lemma 2.11. The initial data is controlled due to Corollary 2.8, so it remains to bound the inhomogeneous term in $L^2 H^{s-1}$. Thus we need to show that

$$\| \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \Box_{g_{>\sqrt{\lambda}}} S_\lambda f_{\lambda,d} \|^2_{L^2 H^{s-1}} \lesssim \sum_{\lambda,d} \| f_{\lambda,d} \|^2_{X^{s,\theta}_{\lambda,d}} \tag{31}$$

Considering the trace regularity result in Corollary 2.8 this would follow from

$$\| \sum_{\lambda=1}^{\infty} \Box_{g_{>\sqrt{\lambda}}} S_\lambda f_\lambda \|^2_{L^2 H^{s-1}} \lesssim \sum_{\lambda} \| \nabla f_\lambda \|^2_{L^\infty H^{s-1}}, \qquad f_\lambda = \sum_{d=1}^{\lambda} f_{\lambda,d}$$

which in turn is a consequence of the fixed time bound (29).

□

We finish this section by proving a key duality relation between $X^{s,\theta}$ spaces with positive, respectively negative θ.

Lemma 2.13. *For $0 < \theta < \frac{1}{2}$ we have the duality relation*

$$X^{-s,-\theta} = (X^{s,\theta} + L^2 H^{s+\theta})' \tag{32}$$

Proof. a) We first show that

$$X^{-s,-\theta} \subset (X^{s,\theta} + L^2 H^{s+\theta})'$$

From Corollary 2.7 we obtain $X^{-s,-\theta} \subset (L^2 H^{s+\theta})'$. It remains to prove the bound

$$\left| \int u \cdot f \; dx\, dt \right| \lesssim \|u\|_{X^{s,\theta}} \|f\|_{X^{-s,-\theta}}$$

We consider Littlewood-Paley decompositions of u and v as in Definition 2.1,

$$u = \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} S_\lambda u_{\lambda,d}, \qquad f = f_0 + \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \tilde{\Box}_{g_{<\sqrt{\lambda}}} S_\lambda f_{\lambda,d}$$

with $\tilde{\Box}_{g_{<\sqrt{\lambda}}}$ in divergence form, see Corollary 2.6. The summation with respect to λ is essentially diagonal therefore it follows by orthogonality. To handle the d summation it suffices to obtain the off-diagonal decay

$$\left| \int S_\lambda u_{\lambda,d_1} \cdot \tilde{\Box}_{g_{<\sqrt{\lambda}}} S_\lambda f_{\lambda,d_2} dxdt \right| \lesssim \min \left\{ \left(\frac{d_2}{d_1} \right)^{\theta}, \left(\frac{d_1}{d_2} \right)^{\frac{1}{2}-\theta} \right\}$$
$$\|u_{\lambda,d_1}\|_{X^{s,\theta}_{\lambda,d_1}} \|f_{\lambda,d_2}\|_{X^{1-s,1-\theta}_{\lambda,d_2}}$$

If $d_2 < d_1$ then this follows directly from (21) and (23). Otherwise we integrate by parts

$$\int S_\lambda u_{\lambda,d_1} \cdot \tilde{\Box}_{g_{<\sqrt{\lambda}}} S_\lambda f_{\lambda,d_2} dxdt = \int \tilde{\Box}_{g_{<\sqrt{\lambda}}} S_\lambda u_{\lambda,d_1} \cdot S_\lambda f_{\lambda,d_2} dxdt$$
$$+ \int (S_\lambda u_{\lambda,d_1} \cdot g^{0\alpha}_{<\sqrt{\lambda}} \partial_\alpha S_\lambda f_{\lambda,d_2} - g^{0\alpha}_{<\sqrt{\lambda}} \partial_\alpha S_\lambda u_{\lambda,d_1} \cdot S_\lambda f_{\lambda,d_2}) dx \Big|_{-1}^{1}$$

For the first term we use (23) and (21). For the second we use the trace regularity result in (26).

b) We now show that

$$(X^{s,\theta} + L^2 H^{s+\theta})' \subset X^{-s,-\theta}$$

Let T be a bounded linear functional on $X^{s,\theta} + L^2 H^{s+\theta}$. Due to the second term we can identify T with a function $u \in L^2 H^{-s-\theta}$.

On the other hand, we can apply it to functions $v \in X^{s,\theta}$ of the form

$$v = \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} S_\lambda v_{\lambda,d}$$

Then we must have the bound

$$|Tv|^2 \lesssim \|v\|^2_{X^{s,\theta}} \lesssim \sum_{\lambda,d} \|v_{\lambda,d}\|^2_{X^{s,\theta}_{\lambda,d}} \lesssim \sum_{\lambda,d} \left(\lambda^{2s} d^{2\theta} \|v_{\lambda,d}\|^2_{L^2} + \lambda^{2s-2} d^{2\theta-2} \|\tilde{\Box}_{g_{<\sqrt{\lambda}}} v_{\lambda,d}\|^2_{L^2} \right)$$

Given the definition of the $X^{s,\theta}_{\lambda,d}$ norms, using succesively the Hahn-Banach theorem and Riesz's theorem it follows that we can find functions $f_{\lambda,d}$ and $h_{\lambda,d}$ with

$$\sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \lambda^{-2s} d^{-2\theta} \|f_{\lambda,d}\|^2_{L^2} + \lambda^{2(1-s)} d^{2(1-\theta)} \|h_{\lambda,d}\|^2_{L^2} = M < \infty \tag{33}$$

so that

$$Tv = \sum_{\lambda=1}^{\infty} \sum_{d=1}^{\lambda} \int f_{\lambda,d}\, v_{\lambda,d} + h_{\lambda,d} \cdot \tilde{\Box}_{g_{<\sqrt{\lambda}}} v_{\lambda,d}\, dxdt$$

In particular this must hold for v of the form $v = S_\lambda v_{\lambda,d}$,

$$\int u\, S_\lambda v_{\lambda,d} dxdt = \int f_{\lambda,d}\, v_{\lambda,d} + h_{\lambda,d} \cdot \tilde{\Box}_{g_{<\sqrt{\lambda}}} v_{\lambda,d}\, dxdt$$

For each λ, d this yields

$$S_\lambda u = f_{\lambda,d} + \tilde{\Box}_{g_{<\sqrt{\lambda}}} h_{\lambda,d}$$

Then we can represent $S_\lambda u$ in the form

$$S_\lambda u = f_{\lambda,1} + \sum_{d=1}^{\frac{\lambda}{2}} \tilde{\Box}_{g_{<\sqrt{\lambda}}} u_{\lambda,d} + \tilde{\Box}_{g_{<\sqrt{\lambda}}} h_{\lambda,\lambda} \qquad u_{\lambda,d} = h_{\lambda,d} - h_{\lambda,2d} \tag{34}$$

This yields for u the representation

$$u = \sum_{\lambda=1}^{\infty} \tilde{S}_\lambda \left(f_{\lambda,1} + \sum_{d=1}^{\frac{\lambda}{2}} \tilde{\Box}_{g_{<\sqrt{\lambda}}} u_{\lambda,d} + \tilde{\Box}_{g_{<\sqrt{\lambda}}} h_{\lambda,\lambda} \right) \tag{35}$$

This is very close to but not exactly the form in (22). However the multipliers $\tilde{S}_\lambda$ can be easily replaced by S_λ by reapplying the Paley-Littlewood decomposition on the right, and then S_λ can be commuted to the right of $\tilde{\Box}_{g_{<\sqrt{\lambda}}}$ due to the Corollary 2.5 and the commutator bound (25). Hence we have

$$\|u\|^2_{X^{-s,-\theta}} \lesssim \sum_{\lambda=1}^{\infty} \left(\lambda^{-2s} \|f_{\lambda,1}\|^2_{L^2} + \sum_{d=1}^{\lambda/2} \|u_{\lambda,d}\|^2_{X^{1-s,1-\theta}_{\lambda,d}} + \|h_{\lambda,\lambda}\|^2_{X^{1-s,1-\theta}_{\lambda,\lambda}} \right)$$

and due to (33) it remains to bound the right hand side by

$$M + \|u\|^2_{L^2 H^{-s-\theta}}$$

There is nothing to do for the $f_{\lambda,1}$ term. On the other hand we can bound

$$\begin{aligned}\|u_{\lambda,d}\|^2_{X^{1-s,1-\theta}_{\lambda,d}} &\lesssim \lambda^{2(1-s)} d^{2(1-\theta)} \|u_{\lambda,d}\|^2_{L^2} + \lambda^{-2s} d^{-2\theta} \|\tilde{\Box}_{g_{<\sqrt{\lambda}}} u_{\lambda,d}\|^2_{L^2} \\ &= \lambda^{2(1-s)} d^{2(1-\theta)} \|h_{\lambda,d} - h_{\lambda,2d}\|^2_{L^2} + \lambda^{-2s} d^{-2\theta} \|f_{\lambda,d} - f_{\lambda,2d}\|^2_{L^2} \\ &\lesssim \lambda^{-2s} d^{-2\theta} (\|f_{\lambda,d}\|^2_{L^2} + \|f_{\lambda,2d}\|^2_{L^2}) \\ &\quad + \lambda^{2(1-s)} d^{2(1-\theta)} (\|h_{\lambda,2d}\|^2_{L^2} + \|h_{\lambda,d}\|^2_{L^2})\end{aligned}$$

Finally, for the last term we have

$$\begin{aligned}\|h_{\lambda,\lambda}\|^2_{X^{1-s,1-\theta}_{\lambda,\lambda}} &\lesssim \lambda^{2(1-s)} \lambda^{2(1-\theta)} \|h_{\lambda,\lambda}\|^2_{L^2} + \lambda^{-2s} \lambda^{-2\theta} \|\tilde{\Box}_{g_{<\sqrt{\lambda}}} h_{\lambda,\lambda}\|^2_{L^2} \\ &= \lambda^{2(1-s)} \lambda^{2(1-\theta)} \|h_{\lambda,\lambda}\|^2_{L^2} + \lambda^{-2s} \lambda^{-2\theta} \|S_\lambda u - f_{\lambda,\lambda}\|^2_{L^2} \\ &\lesssim \lambda^{2(1-s)} \lambda^{2(1-\theta)} \|h_{\lambda,\lambda}\|^2_{L^2} + \lambda^{-2s} \lambda^{-2\theta} \|f_{\lambda,\lambda}\|^2_{L^2} + \|S_\lambda u\|^2_{L^2 H^{-s-\theta}}\end{aligned}$$

The proof is concluded. □

3. Strichartz estimates and applications.

The Strichartz estimates for the variable coefficient wave equation, as proved in [15], have the form:

Theorem 3.1. *(Tataru [15]) Assume that the coefficients g^{ij} of $\Box_g$ satisfy $\partial^2 g^{ij} \in L^1 L^\infty$. Then the solutions to the wave equation in $n+1$ dimensions satisfy the bounds*

$$\|D^\sigma \nabla u\|_{L^p L^q} \lesssim \|u(0)\|_{H^1} + \|u_t(0)\|_{L^2} + \|\Box_g u\|_{L^1 L^2} \tag{36}$$

where

$$\sigma = -\frac{n}{2} + \frac{1}{p} + \frac{n}{q}, \qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}, \qquad 2 \leq p \leq \infty, \quad 2 \leq q < \infty \tag{37}$$

Applying this bound on an interval I of size ϵ^2 we obtain by Cauchy-Schwartz

$$\|D^\sigma \nabla u\|_{L^p(I;L^q)} \lesssim \frac{1}{\epsilon} \|u\|_{H^1(I\times\mathbb{R}^n)} + \epsilon \|\Box_g u\|_{L^2(I\times\mathbb{R}^n)}, \qquad \epsilon \leq 1$$

Summing up over small intervals this extends to intervals of arbitrary lengths. Optimizing over ϵ yields

$$\|D^\sigma \nabla u\|^2_{L^p L^q} \lesssim \|u\|^2_{H^1} + \|u\|_{H^1} \|\Box_g u\|_{L^2} \tag{38}$$

We want to apply this result to the functions $S_\lambda u_{\lambda,d}$ in Definition 2.1. By (23) we obtain

Corollary 3.2. *a) Let (σ, p, q) verifying*

$$\sigma = -\frac{n}{2} + \frac{1}{p} + \frac{n}{q}, \qquad 2 \leq p \leq \infty, \quad 2 \leq q < \infty$$

Then for (σ, p, q) as in (37) we have

$$\|S_\lambda \nabla u\|_{L^p L^q} \lesssim \lambda^{1-s-\sigma} d^{\frac{1}{2}-\theta} \|u\|_{X^{s,\theta}_{\lambda,d}}$$

If additionally $\theta > \frac{1}{2}$ then

$$\|S_\lambda \nabla u\|_{L^p L^q} \lesssim \lambda^{1-s-\sigma} \|u\|_{X^{s,\theta}}$$

b) If instead

$$\frac{2}{p} + \frac{n-1}{q} \geq \frac{n-1}{2}$$

then

$$\|S_\lambda \nabla u\|_{L^pL^q} \lesssim \lambda^{1-s-\sigma+\frac{1}{2}(\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2})} d^{\frac{1}{2}-\theta-\frac{1}{2}(\frac{2}{p}+\frac{n-1}{q}-\frac{n-1}{2})} \|u\|_{X^{s,\theta}_{\lambda,d}}$$

The interesting triplets of indices for (σ, p, q) in $4+1$ dimensions are

$$(0, \infty, 2)(\text{energy}) \qquad (-\frac{1}{2}, \frac{10}{3}, \frac{10}{3})(\text{Strichartz}) \qquad (-\frac{5}{6}, 2, 6)(\text{Pecher})$$

In addition, we can also use the index $q = \infty$. Thus we obtain the triplets

$$(-2, \infty, \infty), \qquad (-\frac{3}{2}, 2, \infty)$$

For the case when $\theta < \frac{1}{2}$, we rely on the additional triplets

$$(-\frac{1}{6}, 2, 3), \qquad (\frac{1}{4}, 4, 2)$$

For convenience we summarize the bounds we need for $\tilde{X}^{s,\theta}_{\lambda,d}$:

Corollary 3.3. *For* $0 < \theta < 1$ *we have*

$$\lambda^{s-1}\|S_\lambda \nabla u\|_{L^\infty L^2} + \lambda^{s-\frac{5}{2}}\|S_\lambda \nabla u\|_{L^2L^\infty} + \lambda^{s-3}\|S_\lambda \nabla u\|_{L^\infty} \lesssim d^{\frac{1}{2}-\theta}\|u\|_{\tilde{X}^{s,\theta}_{\lambda,d}}$$

$$\lambda^{s-\frac{17}{12}}\|S_\lambda \nabla u\|_{L^2L^3} + \lambda^{s-1}\|S_\lambda \nabla u\|_{L^4L^2} \lesssim d^{\frac{1}{4}-\theta}\|u\|_{\tilde{X}^{s,\theta}_{\lambda,d}}$$

The reason we include the gradient is to have also bounds for u_t. Because of the frequency localization, if we drop the gradient the same bounds hold with one less power of λ.

In our estimates later on we also need to work with $X^{s,b}$ functions which are concentrated into a smaller modulation range. For this we introduce the additional norm

$$\|u\|^2_{\tilde{X}^{s,\theta}_{\lambda,<d}} = \inf\left\{\sum_{h=1}^{d}\|u_h\|^2_{\tilde{X}^{s,\theta}_{\lambda,h}};\ u = \sum_{h=1}^{d} u_h\right\}$$

If $d = \lambda$ we simply write $\tilde{X}^{s,\theta}_{\lambda}$. A simple argument leads to

$$\|u\|^2_{X^{s,\theta}} = \inf\left\{\sum_{\lambda=1}^{\infty}\|S_\lambda u_\lambda\|^2_{\tilde{X}^{s,\theta}_{\lambda}};\ u = \sum_{\lambda=1}^{\infty} S_\lambda u_\lambda\right\}$$

We also have

Corollary 3.4. *a) Assume that* $\theta > \frac{1}{2}$. *Then*

$$\lambda^{s-1}\|S_\lambda \nabla u\|_{L^\infty L^2} + \lambda^{s-\frac{5}{2}}\|S_\lambda \nabla u\|_{L^2L^\infty} + \lambda^{s-3}\|S_\lambda \nabla u\|_{L^\infty} \lesssim \|u\|_{\tilde{X}^{s,\theta}_{\lambda,<d}}$$

b) Assume that $\theta < \frac{1}{2}$. *Then*

$$\lambda^{s-1}\|S_\lambda \nabla u\|_{L^\infty L^2} + \lambda^{s-\frac{5}{2}}\|S_\lambda \nabla u\|_{L^2L^\infty} + \lambda^{s-3}\|S_\lambda \nabla u\|_{L^\infty} \lesssim d^{\frac{1}{2}-\theta}\|u\|_{\tilde{X}^{s,\theta}_{\lambda,<d}}$$

In preparation for proving bilinear estimates for the $X^{s,\theta}$ spaces we first investigate which multiplications leave the $\tilde{X}^{s,\theta}_{\lambda,d}$ space unchanged. For this we define the algebras M_d, $M_{<d}$ with the norms

$$\|f\|_{M_d} = \|f\|_{L^\infty} + d^{-1}\|f_t\|_{L^\infty} + d^{-\frac{1}{2}}\|f_t\|_{L^2L^\infty} + d^{-\frac{3}{2}}\|f_{tt}\|_{L^2L^\infty}$$
$$\|f\|_{M_{<d}} = \|f\|_{M_d} + d^{\frac{1}{2}}\|f\|_{L^2L^\infty}$$

Then we have the multiplicative properties

Lemma 3.5. *Assume that f is localized at frequency $d \leq \lambda$. Then we have*

$$\|fS_\lambda u\|_{\tilde{X}^{s,\theta}_{\lambda,d}} \lesssim \|f\|_{M_d}\|u\|_{\tilde{X}^{s,\theta}_{\lambda,d}} \tag{39}$$

respectively

$$\begin{aligned} &\|fS_\lambda u\|_{\tilde{X}^{s,\theta}_{\lambda,d}} \lesssim \|f\|_{M_{<d}}\|u\|_{\tilde{X}^{s,\theta}_{\lambda,<d}}, \qquad \theta < \frac{1}{2} \\ &\|fS_\lambda u\|_{\tilde{X}^{s,\theta}_{\lambda,d}} \lesssim d^{\theta-\frac{1}{2}}\|f\|_{M_{<d}}\|u\|_{\tilde{X}^{s,\theta}_{\lambda,<d}}, \qquad \theta > \frac{1}{2} \end{aligned} \tag{40}$$

The proof is straightforward, using Leibnitz's rule and the energy estimate (26). To bound functions in the M_d, respectively $M_{<d}$ norms we use Corollary 3.4 with $d = \lambda$ to obtain:

Lemma 3.6. *a) Assume that $\theta > \frac{1}{2}$. Then*

$$\|S_\lambda u\|_{M_{<\lambda}} \leq \lambda^{2-s}\|u\|_{\tilde{X}^{s,\theta}_{\lambda}} \qquad \|S_{<\lambda} u\|_{M_{\lambda}} \leq \max\{1, \lambda^{2-s}\}\|u\|_{X^{s,\theta}}$$
$$\|S_{<\lambda} u\|_{M_{<\lambda}} \leq \max\{\lambda^{\frac{1}{2}}, \lambda^{2-s}\}\|u\|_{X^{s,\theta}}$$

b) Assume that $\theta < \frac{1}{2}$. Then

$$\|S_\lambda u\|_{M_{<\lambda}} \leq \lambda^{\frac{5}{2}-\theta-s}\|u\|_{\tilde{X}^{s,\theta}_{\lambda}} \qquad \|S_{<\lambda} u\|_{M_{\lambda}} \leq \max\{1, \lambda^{\frac{5}{2}-\theta-s}\}\|u\|_{X^{s,\theta}}$$
$$\|S_{<\lambda} u\|_{M_{<\lambda}} \leq \max\{\lambda^{\frac{1}{2}}, \lambda^{\frac{5}{2}-\theta-s}\}\|u\|_{X^{s,\theta}}$$

Using the above property we prove the algebra property (11) for the space X.

Proposition 3.7. *Assume that $s > 2$ and $\frac{1}{2} < \theta < s - \frac{3}{2}$. Then $X^{s,\theta}$ is an algebra.*

Proof. Let $u, v \in X^{s,\theta}$. For both we consider the decomposition in Definition 2.1,

$$u = \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} S_\lambda u_{\lambda,d}, \qquad v = \sum_{\lambda=1}^{\infty}\sum_{d=1}^{\lambda} S_\lambda v_{\lambda,d},$$

For the terms in the decomposition we use the $\tilde{X}^{s,\theta}_{\lambda,d}$ norms, as allowed by Corollary 2.5. We denote

$$u_\lambda = \sum_{d=1}^{\lambda} u_{\lambda,d}, \qquad u_{\lambda,<d} = \sum_{h=1}^{d} u_{\lambda,h}$$

Then we write

$$uv = \sum_{\mu=1}^{\infty} S_\mu(uv) = \sum_{\mu=1}^{\infty}\sum_{\lambda_1=1}^{\infty}\sum_{\lambda_2=1}^{\infty} S_\mu(S_{\lambda_1}u_{\lambda_1}S_{\lambda_2}v_{\lambda_2})$$

There are two cases when the above summand is nonzero, namely if $\lambda_1 \approx \lambda_2 \gtrsim \mu$ and if $\max\{\lambda_1, \lambda_2\} \approx \mu$. We consider them separately.

Case 1, $\lambda_1, \lambda_2 \approx \lambda \gtrsim \mu$. In this case the summability with respect to λ is trivial, so it suffices to look at the product $S_\lambda u_\lambda S_\lambda v_\lambda$ for fixed λ. This is localized at frequency $\leq \lambda$. Combining the $L^\infty L^2$ and the $L^2 L^\infty$ bounds in Corollary 3.4 we obtain

$$\|S_\lambda u_\lambda S_\lambda v_\lambda\|_{L^2} + \lambda^{-1}\|\partial_t(S_\lambda u_\lambda S_\lambda v_\lambda)\|_{L^2} \lesssim \lambda^{-2s+\frac{3}{2}}\|u_\lambda\|_{\tilde{X}^{s,\theta}_\lambda}\|v_\lambda\|_{\tilde{X}^{s,\theta}_\lambda} \tag{41}$$

Using the equation we can also bound the second time derivative,

$$\lambda^{-2}\|\partial_t^2(S_\lambda u_\lambda S_\lambda v_\lambda)\|_{L^2} \lesssim \lambda^{-2s+\frac{3}{2}}\|u_\lambda\|_{\tilde{X}^{s,\theta}_\lambda}\|v_\lambda\|_{\tilde{X}^{s,\theta}_\lambda} \tag{42}$$

The three bounds above allow us to estimate for $\mu \leq \lambda$

$$\|S_\lambda u_\lambda S_\lambda v_\lambda\|_{X^{s,\theta}_{\mu,\mu}} \lesssim \mu^{s+\theta-2}\lambda^{-2s+\frac{7}{2}}\|u_\lambda\|_{\tilde{X}^{s,\theta}_\lambda}\|v_\lambda\|_{\tilde{X}^{s,\theta}_\lambda}$$

This suffices provided that $\theta < s - \frac{3}{2}$, which is insured by our hypothesis.

Case 2. Here we consider products of the form $S_\mu v_\mu S_\lambda u_\lambda$ where $\mu \ll \lambda$. Then the product is localized at frequency λ. The summation with respect to λ is trivial, but not the one with respect to μ. We write

$$S_\mu v_\mu S_\lambda u_\lambda = S_\mu v_\mu S_\lambda u_{\lambda,<\mu} + \sum_{d=\mu}^{\lambda} S_\mu v_\mu S_\lambda u_{\lambda,d}$$

Using Lemma 3.5 and Lemma 3.6 we obtain

$$\begin{aligned}
\|S_\lambda u_\lambda S_\mu v_\mu\|^2_{\tilde{X}^{s,\theta}_\lambda} &\lesssim \|S_\mu v_\mu S_\lambda u_{\lambda,<\mu}\|^2_{\tilde{X}^{s,\theta}_{\lambda,\mu}} + \sum_{d=\mu}^{\lambda}\|S_\mu v_\mu S_\lambda u_{\lambda,d}\|^2_{\tilde{X}^{s,\theta}_{\lambda,d}} \\
&\lesssim \mu^{2\theta-1}\|S_\mu v_\mu\|^2_{M_{<\mu}}\|u_{\lambda,<\mu}\|^2_{\tilde{X}^{s,\theta}_{\lambda,<\mu}} + \|S_\mu v_\mu\|^2_{M_\mu}\sum_{d=\mu}^{\lambda}\|u_{\lambda,d}\|^2_{\tilde{X}^{s,\theta}_{\lambda,d}} \\
&\lesssim \mu^{2\theta-1}\|S_\mu v_\mu\|^2_{M_{<\mu}}\sum_{d=1}^{\lambda}\|u_{\lambda,d}\|^2_{\tilde{X}^{s,\theta}_{\lambda,d}} \\
&\lesssim \mu^{3+2\theta-2s}\|v_\mu\|^2_{\tilde{X}^{s,\theta}_\mu}\sum_{d=1}^{\lambda}\|u_{\lambda,d}\|^2_{\tilde{X}^{s,\theta}_{\lambda,d}}
\end{aligned}$$

The summation with respect to μ is trivial since $\theta < s - \frac{3}{2}$.

□

We next prove (13).

Proposition 3.8. *Assume that $s > 2$ and $\frac{1}{2} < \theta < s - \frac{3}{2}$. Then we have the multiplicative estimate*

$$X^{s,\theta} \cdot X^{s-1,\theta-1} \subset X^{s-1,\theta-1}$$

Proof. By duality this reduces to the multiplicative estimate

$$X^{s,\theta} \cdot (X^{1-s,1-\theta} + L^2 H^{2-s-\theta}) \subset X^{1-s,1-\theta} + L^2 H^{2-s-\theta}$$

Since $s > 2$ we have the fixed time multiplication

$$H^s \cdot H^{2-s-\theta} \subset H^{2-s-\theta}$$

which implies the space-time bound

$$L^\infty H^s \cdot L^2 H^{2-s-\theta} \subset L^2 H^{2-s-\theta}$$

Due to the energy estimate for $X^{s,\theta}$ it remains to show that

$$X^{s,\theta} \cdot X^{1-s,1-\theta} \subset X^{1-s,1-\theta} + L^2 H^{2-s-\theta}$$

We consider a product $S_\lambda u_\lambda S_\mu v_\mu$ which we decompose as in the previous proof. Because of the lack of symmetry we now need to consider three cases.

Case 1. Here we estimate $S_\mu(S_\lambda u_\lambda S_\lambda v_\lambda)$ where $\mu \lesssim \lambda$. By Corollary 3.4 we obtain

$$\|S_\lambda u_\lambda S_\lambda v_\lambda\|_{L^2 L^{\frac{3}{2}}} \lesssim \|S_\lambda u_\lambda\|_{L^4 L^3} \|S_\lambda v_\lambda\|_{L^4 L^3} \lesssim \lambda^{\theta - \frac{2}{3}} \|u_\lambda\|_{\tilde{X}^{s,\theta}_\lambda} \|v_\lambda\|_{\tilde{X}^{1-s,1-\theta}_\lambda}$$

Using then Sobolev embeddings we obtain

$$\|S_\mu(S_\lambda u_\lambda S_\lambda v_\lambda)\|_{L^2 H^{2-s-\theta}} \lesssim \mu^{\frac{8}{3}-s-\theta} \lambda^{\theta-\frac{2}{3}} \|u_\lambda\|_{\tilde{X}^{s,\theta}_\lambda} \|v_\lambda\|_{\tilde{X}^{1-s,1-\theta}_\lambda}$$

Case 2. Here we bound $S_\mu u_\mu S_\lambda v_\lambda$, $\mu \ll \lambda$. The product is localized at frequency λ, and the analysis is almost identical to Case 2 in Proposition 3.7.

Case 3. Here we bound $S_\lambda u_\lambda S_\mu v_\mu$, $\mu \ll \lambda$. The same argument applies, the only difference is that we gain some extra μ/λ factors.

□

We continue with the Moser estimates in (12), which follow from

Proposition 3.9. *Assume that $s > 2$ and $\frac{1}{2} < \theta < s - \frac{3}{2}$. Let Γ be a smooth function. Then*

$$\|\Gamma(u)\|_{X^{s,\theta}} \lesssim C(\|u\|_{L^\infty})(1 + \|u\|^5_{X^{s,\theta}})$$

Proof. We write

$$\Gamma(u) - \Gamma(v) = (u-v) f(u,v)$$

and

$$f(u,v) - f(x,y) = (u-x) g_1(u,v,x,y) + (v-y) g_2(u,v,x,y)$$

where f, g_1 and g_2 are smooth functions. Then we have

$$\begin{aligned}
\Gamma(u) =& \Gamma(u_1) + \sum_{\lambda=1}^{\infty} \Gamma(u_{\leq 2\lambda}) - \Gamma(u_{\leq \lambda}) \\
=& \Gamma(u_1) + \sum_{\lambda=1}^{\infty} u_{2\lambda} f(u_{\leq 2\lambda}, u_{\leq \lambda}) \\
=& \Gamma(u_1) + \sum_{\lambda=1}^{\infty} u_{2\lambda} [f(u_{\leq 2}, u_1) + \sum_{\mu=2}^{\lambda} (f(u_{\leq 2\mu}, u_{\leq \mu}) - f(u_{\leq \mu}, u_{\leq \mu/2}))] \\
=& \Gamma(u_1) + \sum_{\lambda=1}^{\infty} u_{2\lambda} [f(u_{\leq 2}, u_1) + \sum_{\mu=2}^{\lambda} (u_{2\mu}\, g_1(u_{\leq 2\mu}, u_{\leq \mu}, u_{\leq \mu/2}) \\
&+ u_\mu\, g_2(u_{\leq 2\mu}, u_{\leq \mu}, u_{\leq \mu/2}))]
\end{aligned}$$

Hence we need to bound expressions of the form

$$S_\lambda u_\lambda\, S_\mu v_\mu\, h(S_{<\mu} w), \qquad \mu \leq \lambda$$

There are two different cases to consider:

Case 1. $\mu \approx \lambda$. Then the product has the form

$$S_\lambda u_\lambda\, S_\lambda v_\lambda\, h(S_{<\lambda} w)$$

The first product is localized at frequency λ and can be estimated as in (41), (42). For the nonlinear expression we use Lemma 3.6 to obtain

$$\|S_{<\lambda} w\|_{M_\lambda} \lesssim \|w\|_{X^{s,\theta}}$$

On one hand by the chain rule we obtain

$$\|h(S_{<\lambda} w)\|_{M_\lambda} \lesssim C(\|w\|_{L^\infty})(1+\|w\|^3_{X^{s,\theta}}) \tag{43}$$

On the other hand because of the frequency localization we also have the improved high frequency bound

$$\|\tilde S_\mu h(S_{<\lambda} w)\|_{M_\mu} \lesssim C(\|w\|_{L^\infty}) \left(\frac{\lambda}{\mu}\right)^N (1+\|w\|^3_{X^{s,\theta}}), \qquad \mu \gg \lambda \tag{44}$$

Taking this into account and repeatedly using Leibnitz's rule we get

$$\begin{aligned}
&\|S_\lambda u_\lambda\, S_\lambda v_\lambda\, h(S_{<\lambda} w)\|^2_{X^{s,\theta}} \\
&\lesssim \sum_{\mu=1}^{\infty} \|S_\mu(S_\lambda u_\lambda\, S_\lambda v_\lambda\, h(S_{<\lambda} w))\|^2_{\tilde X^{s,\theta}_\mu} \\
&\lesssim \sum_{\mu \lesssim \lambda} \|S_\lambda u_\lambda\, S_\lambda v_\lambda\, h(S_{<\lambda} w)\|^2_{\tilde X^{s,\theta}_\mu} + \sum_{\mu \gg \lambda} \|S_\lambda u_\lambda\, S_\lambda v_\lambda\, \tilde S_\mu h(S_{<\lambda} w))\|^2_{\tilde X^{s,\theta}_\mu} \\
&\lesssim C(\|w\|_{L^\infty}) \left(\sum_{\mu \lesssim \lambda} \mu^{2s+2\theta-4} \lambda^{-4s+7} + \sum_{\mu \gg \lambda} \lambda^{2\theta+3-2s} \left(\frac{\lambda}{\mu}\right)^N \right) \|u_\lambda\|^2_{\tilde X^{s,\theta}_\lambda} \|v_\lambda\|^2_{\tilde X^{s,\theta}_\lambda} (1+\|w\|^6_{X^{s,\theta}}) \\
&\lesssim C(\|w\|_{L^\infty})\, \lambda^{2\theta+3-2s}\, \|u_\lambda\|^2_{\tilde X^{s,\theta}_\lambda} \|v_\lambda\|^2_{\tilde X^{s,\theta}_\lambda} (1+\|w\|^6_{X^{s,\theta}})
\end{aligned}$$

This is trivially summable with respect to λ.

Case 2. $\mu \ll \lambda$. Then the product has the form

$$\begin{aligned}
&S_\lambda u_\lambda\, S_\mu v_\mu\, h(S_{<\mu} w) = \\
&S_\lambda u_{\lambda,<\mu}\, S_\mu v_\mu\, S_{<\mu} h(S_{<\mu} w) + \sum_{\mu \le d \ll \lambda} S_\lambda u_{\lambda,<d}\, S_\mu v_\mu\, S_d h(S_{<\mu} w) \\
&+ \sum_{\mu \le d \ll \lambda} S_\lambda u_{\lambda,d}\, S_\mu v_\mu\, S_{<d} h(S_{<\mu} w) + S_\lambda u_\lambda\, S_\mu v_\mu\, S_\lambda h(S_{<\mu} w) \\
&+ \sum_{\nu \gg \lambda} S_\lambda u_\lambda\, S_\mu v_\mu\, S_\nu h(S_{<\mu} w) = f_1 + f_2 + f_3 + f_4 + f_5
\end{aligned}$$

For f_1 we use Lemma 3.5, Lemma 3.6 and (43) to obtain

$$\begin{aligned}
\|f_1\|_{\tilde X^{s,\theta}_{\lambda,\mu}} &\lesssim \|S_\lambda u_{\lambda,<\mu}\, S_\mu v_\mu\|_{\tilde X^{s,\theta}_{\lambda,\mu}} \|h(S_{<\mu} w)\|_{M_\mu} \\
&\lesssim \mu^{\theta-\frac12} \|u_{\lambda,<\mu}\|_{\tilde X^{s,\theta}_{\lambda,<\mu}} \|S_\mu v_\mu\|_{M_{<\mu}} \|h(S_{<\mu} w)\|_{M_\mu} \\
&\lesssim C(\|w\|_{L^\infty}) \mu^{\theta+\frac32-s} \|u_{\lambda,<\mu}\|_{\tilde X^{s,\theta}_{\lambda,<\mu}} \|v_\mu\|_{\tilde X^{s,\theta}_\mu} (1+\|w\|^3_{X^{s,\theta}})
\end{aligned}$$

The summation with respect to μ is trivial and the square summability with respect to λ is inherited from the first factor.

For f_2 we apply the same argument. There is a loss of a small power of $(d/\mu)^\theta$ from the first product, but this is compensated by the gain of arbitrary powers of

μ/d due to (44). The same works for f_3 but there is no $(d/\mu)^\theta$ loss. In the case of f_4 we need to worry about the λ summability, but the $(\mu/\lambda)^N$ gain in (44) settles this. Finally, for f_5 there is a $(\mu/\nu)^N$ gain which cancels again all the losses.

Summing up the pieces we obtain

$$\|S_\nu(S_\lambda u_\lambda\, S_\mu v_\mu\, h(S_{<\mu}w))\|_{X^{s,\theta}}$$
$$\lesssim C(\|w\|_{L^\infty})\nu^{s+\theta-2}\lambda^{-2s+\frac{7}{2}}(\frac{\mu}{\lambda})^N\|u_\lambda\|_{\tilde{X}^{s,\theta}_\lambda}\|v_\mu\|_{\tilde{X}^{s,\theta}_\mu}(1+\|w\|^3_{X^{s,\theta}})$$

for $\nu \ll \lambda$,

$$\|S_\nu(S_\lambda u_\lambda\, S_\mu v_\mu\, h(S_{<\mu}w))\|_{X^{s,\theta}} \lesssim C(\|w\|_{L^\infty})\mu^{\theta+\frac{3}{2}-s}\|u_\lambda\|_{\tilde{X}^{s,\theta}_\lambda}\|v_\mu\|_{\tilde{X}^{s,\theta}_\mu}(1+\|w\|^3_{X^{s,\theta}})$$

for $\nu \approx \lambda$, respectively

$$\|S_\nu(S_\lambda u_\lambda\, S_\mu v_\mu\, h(S_{<\mu}w))\|_{X^{s,\theta}} \lesssim C(\|w\|_{L^\infty})\mu^{\theta+\frac{3}{2}-s}(\frac{\mu}{\nu})^N\|u_\lambda\|_{\tilde{X}^{s,\theta}_\lambda}\|v_\mu\|_{\tilde{X}^{s,\theta}_\mu}(1+\|w\|^3_{X^{s,\theta}})$$

for $\nu \gg \lambda$.

This concludes the proof of the proposition.

□

Finally, we consider the bilinear estimate in (14), which follows from the next Proposition. Its proof cannot be completed using the type of arguments we have employed so far. Instead, we contend ourselves with reducing it to the trilinear estimate in (50), to the proof of which we devote the rest of the paper.

Proposition 3.10. *Assume that $s > \frac{9}{4}$ and $\frac{3}{4} < \theta < s - \frac{3}{2}$. Then we have the multiplicative estimate*

$$\|\nabla u \nabla v\|_{X^{s-1,\theta-1}} \lesssim \|u\|_{X^{s,\theta}}\|v\|_{X^{s,\theta}} \tag{45}$$

We begin our analysis with a simple observation, namely that

Lemma 3.11. *If $u \in X^{s,\theta}$ then $\nabla u \in \tilde{X}^{s-1,\theta}$ where*

$$\tilde{X}^{s-1,\theta} = X^{s-1,\theta} + (L^2H^{s+\theta-1} \cap H^1H^{s+\theta-2}).$$

Proof. We first consider spatial derivatives, for which we prove the better bound

$$\|\nabla_x u\|_{X^{s-1,\theta}} \lesssim \|u\|_{X^{s,\theta}}$$

By Definition 2.1 and Corollary 2.5 it suffices to show that for functions v localized at frequency λ we have

$$\|\nabla_x v\|_{X^{s-1,\theta}_{\lambda,d}} \lesssim \|v\|_{\tilde{X}^{s,\theta}_{\lambda,d}}$$

But this follows from the straightforward commutator bound

$$\|[\Box_{g_{<\sqrt{\lambda}}}, \nabla]v\|_{L^2} \lesssim \lambda\|\nabla v\|_{L^2} \tag{46}$$

Here we recall that $g^{00} = 1$, therefore every term in the commutator has at least one spatial derivative.

Next we consider time derivatives, where it suffices to show that for functions v localized at frequency λ we can write $v = v_1 + v_2$ where v_1, v_2 have the same frequency localization and

$$\|\partial_t v_1\|_{X^{s-1,\theta}_{\lambda,d}} + \left(\frac{\lambda}{d}\right)^{1-\theta}\|\partial_t v_2\|_{(L^2H^{s+\theta-1}\cap H^1H^{s+\theta-2})} \lesssim \|v\|_{\tilde{X}^{s,\theta}_{\lambda,d}} \tag{47}$$

Roughly speaking v_1 accounts for the low modulation ($\lesssim \lambda$) part of v while v_2 accounts for the high modulation part. We define v_2 as

$$v_2 = (\Delta_{x,t})^{-1}\Box_{g_{<\sqrt{\lambda}}} v$$

This satisfies the bound

$$\|\nabla^2 v_2\|_{L^2} \lesssim \|\Box_{g_{<\sqrt{\lambda}}} v\|_{L^2}$$

which implies both the v_2 bound in (47) and an H^2 bound for v_1 which gives the correct L^2 bound for $\partial_t v_1$,

$$\lambda^{s-1} d^\theta \|\partial_t v_1\|_{L^2} + \left(\frac{\lambda}{d}\right)^{1-\theta} \|\partial_t v_2\|_{(L^2H^{s+\theta-1}\cap H^1H^{s+\theta-2})} \lesssim \|v\|_{\tilde{X}^{s,\theta}_{\lambda,d}}$$

It remains to estimate $\Box_{g_{<\sqrt{\lambda}}}\partial_t v_1$. We have

$$\|\Box_{g_{<\sqrt{\lambda}}}\partial_t v_1\|_{L^2} \le \|[\Box_{g_{<\sqrt{\lambda}}}, \partial_t] v_1\|_{L^2} + \|\partial_t \Box_{g_{<\sqrt{\lambda}}} v_1\|_{L^2}$$

For the first term we use again (46). For the second we compute

$$\Box_{g_{<\sqrt{\lambda}}} v_1 = (-\Box_{g_{<\sqrt{\lambda}}} + \Delta_{x,t})(\Delta_{x,t})^{-1}\Box_{g_{<\sqrt{\lambda}}} v$$

Since the difference $\Box_{g_{<\sqrt{\lambda}}} - \Delta_{x,t}$ contains no second order time derivatives this yields the bound

$$\|\partial_t \Box_{g_{<\sqrt{\lambda}}} v_1\|_{L^2} \lesssim \lambda \|\Box_{g_{<\sqrt{\lambda}}} v\|_{L^2}$$

This allows us to conclude the proof of (47) and therefore the proof of the lemma. □

We now return to the estimate (45). Using the duality in (32), (45) reduces to

$$\left|\int uvw\,dxdt\right| \lesssim \|u\|_{\tilde{X}^{s-1,\theta}} \|v\|_{\tilde{X}^{s-1,\theta}} \|w\|_{X^{1-s,1-\theta}+L^2H^{2-s-\theta}} \tag{48}$$

We do a trilinear Littlewood-Paley decomposition. Due to symmetry, we need to consider two cases.

Case 1. Here we consider high-high-low interactions and bound

$$I = \int S_\lambda u\, S_\lambda v\, S_\mu w\, dxdt, \qquad \mu \lesssim \lambda$$

We have

$$|I| \lesssim \|S_\lambda u\|_{L^\infty L^2} \|S_\lambda v\|_{L^2L^6} \|S_\mu w\|_{L^2L^3}$$

which by the embeddings in Corollary 3.2 give

$$|I| \lesssim \lambda^{\frac{5}{6}-2s+2} \mu^{s+\theta-\frac{4}{3}} \|u\|_{\tilde{X}^{s-1,\theta}} \|v\|_{\tilde{X}^{s-1,\theta}} \|w\|_{X^{1-s,1-\theta}+L^2H^{2-s-\theta}}$$

This suffices since both the exponent of λ and the sum of the two exponents are negative.

Case 2. Here we consider high-low-high interactions and seek to bound

$$I = \int S_\lambda u\, S_\mu v\, S_\lambda w\, dxdt, \qquad \mu \ll \lambda$$

As a first simplification we dispense with the auxiliary L^2 norms. Begin with

$$\begin{aligned} |I| &\lesssim \|S_\lambda u\|_{L^2} \|S_\mu v\|_{L^\infty} \|S_\lambda w\|_{L^2} \\ &\lesssim \lambda^{s-1}\mu^{\frac{3}{4}} \|S_\lambda u\|_{L^2}\ \mu^{\frac{9}{4}-s} \|v\|_{\tilde{X}^{s-1,\theta}}\ \lambda^{1-s} \|S_\lambda w\|_{L^2} \end{aligned}$$

This allows us to dispense not only with the $L^2H^{s+\theta-1}$ part of u, but also with its $X^{s-1,\theta}_{\lambda,>\mu}$ component.

If $v \in L^2H^{s+\theta-1} \cap H^1H^{s+\theta-2}$ then we bound

$$|I| \lesssim \|S_\lambda u\|_{L^\infty L^2}\|S_\mu v\|_{L^2L^\infty}\|S_\lambda w\|_{L^2}$$
$$\lesssim \mu^{3-s-\theta}\|u\|_{\tilde{X}^{s-1,\theta}}\ \|v\|_{L^2H^{s+\theta-1}}\ \lambda^{1-s}\|S_\lambda w\|_{L^2}$$

Finally, if $w \in L^2H^{2-s-\theta}$ then we can also estimate

$$|I| \leq \|S_\lambda u\|_{L^\infty L^2}\|S_\mu v\|_{L^2L^\infty}\|S_\lambda w\|_{L^2}$$
$$\lesssim \mu^{\frac{3}{2}-s+\theta}\|u\|_{\tilde{X}^{s-1,\theta}}\ \|v\|_{\tilde{X}^{s-1,\theta}}\ \lambda^{1-s}\mu^{1-\theta}\|S_\lambda w\|_{L^2}$$

which suffices for both the $L^2H^{2-s-\theta}$ and the $X^{1-s,1-\theta}_{\lambda,>\mu}$ components of w. Hence we have reduced (48) to the bound

$$|I| \lesssim \|S_\lambda u\|_{X^{s-1,\theta}_{\lambda,<\mu}}\|S_\mu v\|_{X^{s-1,\theta}}\|S_\lambda w\|_{X^{1-s,1-\theta}_{\lambda,<\mu}} \qquad \mu \ll \lambda \tag{49}$$

Unfortunately we cannot fully prove this using Strichartz type estimates. However, we can use scaling to simplify this further and reduce it to

$$\left|\int S_\lambda u\, S_\mu v\, S_\lambda w dxdt\right| \lesssim \ln \mu\ \|u\|_{X^{0,1}_{\lambda,1}}\|v\|_{X^{\frac{5}{4},1}_{\mu,1}}\|w\|_{X^{0,\frac{1}{4}}_{\lambda,d}} \qquad \mu \ll \lambda \tag{50}$$

For now we show that (50) implies (49). The remaining sections of the paper are devoted to the proof of (50).

After cancelling the powers of the high frequency the estimate (49) follows after summation with respect to $1 \leq d_1, d_2, d_3 \leq \mu$ from the bounds

$$\left|\int S_\lambda u\, S_\mu v\, S_\lambda w dxdt\right| \lesssim \ln \mu\ d_{min}^{\frac{1}{2}}d_{mid}^{\frac{1}{2}}d_{max}^{\frac{1}{4}}\|u\|_{X^{0,0}_{\lambda,d_1}}\ \|v\|_{X^{\frac{5}{4},0}_{\mu,d_2}}\ \|w\|_{X^{0,0}_{\lambda,d_3}} \tag{51}$$

if $d_2 < d_{max}$, respectively

$$\left|\int S_\lambda u\, S_\mu v\, S_\lambda w dxdt\right| \lesssim \ln \mu\ d_{min}^{\frac{1}{2}}d_{max}^{\frac{3}{4}}\|u\|_{X^{0,0}_{\lambda,d_1}}\ \|v\|_{X^{\frac{5}{4},0}_{\mu,d_2}}\ \|w\|_{X^{0,0}_{\lambda,d_3}} \tag{52}$$

if $d_2 = d_{max}$.

To reduce all these cases to (50) we use scaling combined with a time decomposition argument. Precisely, for $1 < d < \lambda$ we consider a smooth partition of unity in time with respect to time intervals of length d^{-1},

$$1 = \sum \chi_d^j(t)$$

Then a simple commutation argument shows that we can localize the $\tilde{X}^{s,\theta}_{\lambda,d}$ norm to the d^{-1} time intervals while retaining square summability,

$$\|u\|^2_{\tilde{X}^{s,\theta}_{\lambda,d}} \approx \sum_j \|\chi_d^j u\|^2_{\tilde{X}^{s,\theta}_{\lambda,d}} \tag{53}$$

We use such time decompositions in order to carry out the following three reduction steps:

(i) Reduction to $d_{min} = 1$. By (53) all three norms are square summable with respect to time intervals of length d_{min}^{-1}. Hence it suffices to prove the bounds on d_{min}^{-1} time intervals. Rescaling such time intervals back to time 1 we arrive at the case $d_{min} = 1$. The regularity of the coefficients improves after the rescaling, here and below. Also we note that by Duhamel's formula we can replace the factor corresponding to d_{min} by a solution to the homogeneous equation.

(ii) Reduction to $d_{mid} = 1$. By (53) the norms corresponding to d_{max} and d_{mid} are square summable with respect to time intervals of length d_{mid}^{-1}. Hence it suffices to prove the bounds on d_{mid}^{-1} time intervals. Rescaling such time intervals back to time 1 we arrive at the case $d_{mid} = 1$. Again by Duhamel's formula we also replace the factor corresponding to d_{mid} by a solution to the homogeneous equation.

(iii) Here we are in the case where two of the factors are solutions for the homogeneous equation. In the case of (51) the remaining factor is at high frequency λ; then we use directly (50).

In the case of (52) the remaining factor is at low frequency μ, so we need to prove that

$$\left| \int S_\lambda u \, S_\mu v \, S_\lambda w dx dt \right| \lesssim \ln \mu \; d^{\frac{3}{4}} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{\frac{5}{4},0}_{\mu,d}} \|w\|_{X^{0,0}_{\lambda,1}}$$

Partitioning the unit time into about d time intervals of length d^{-1} this would follow from

$$\left| \int \chi_d^i S_\lambda u \, S_\mu v \, S_\lambda w dx dt \right| \lesssim \ln \mu \; d^{\frac{1}{4}} \|u\|_{X^{0,0}_{\lambda,1}} \|v\|_{X^{\frac{5}{4},0}_{\mu,d}} \|w\|_{X^{0,0}_{\lambda,1}}$$

Rescaling the small time intervals to unit size this becomes exactly (50).

4. Half-waves and angular localization operators

We write the symbol for $\Box_g$,

$$p(t,x,\tau,\xi) = \tau^2 - 2g^{0j}\tau\xi_j - g^{ij}\xi_i\xi_j$$

in the form

$$p(t,x,\tau,\xi) = (\tau + a^+(t,x,\xi))(\tau + a^-(t,x,\xi))$$

This leads to a decomposition of solutions to the wave equation into two half-waves:

Proposition 4.1. *(Geba-Tataru [16]) Let u be a solution to the inhomogeneous equation (30) for $\Box_g$. Then there is a representation*

$$\nabla u = u^+ + u^-$$

where

$$\|u^+\|_{L^2} + \|(D_t + A^+(t,x,D))u^+\|_{L^2} + \|u^-\|_{L^2} + \|(D_t + A^-(t,x,D))u^-\|_{L^2} \lesssim \|u\|_{H^1} + \|\Box_g u\|_{L^2}$$

As a consequence, in (50) we are allowed to replace solutions to the $\Box_g$ equation by solutions to the $D_t + A^+$, respectively $D_t + A^-$ equation. We also denote

$$\|u\|_{X_\pm} = \|u\|_{L^2} + \|(D_t + A^\pm(t,x,D))u\|_{L^2}$$
$$\|u\|_{X_{\pm,d}} = d^{\frac{1}{4}}\|u\|_{L^2} + d^{-\frac{3}{4}}\|(D_t + A^\pm(t,x,D))u\|_{L^2}$$

In order to facilitate the use of microlocal analysis tools it is convenient to replace the symbols $a^\pm$ with mollified versions $a^\pm_{<\mu}$ defined by

$$a^\pm_{<\mu}(t,x,\xi) = S_{<\mu}(D_x)a(t,x,\xi)$$

Given an angular scale α we consider the $\pm$ Hamilton flows for $D_t + A^\pm_{<\alpha^{-1}}$.

$$\begin{cases} \frac{d}{dt}x_t^\pm = \partial_\xi a^\pm_{<\alpha^{-1}}(t, x_t^\pm, \xi_t^\pm) \\ \frac{d}{dt}\xi_t^\pm = -\partial_x a^\pm_{<\alpha^{-1}}(t, x_t^\pm, \xi_t^\pm) \end{cases} \qquad \begin{cases} x_0^\pm = x \\ \xi_0^\pm = \xi \end{cases} \tag{54}$$

These are bilipschitz flows, homogeneous with respect to the ξ variable. The angular scale is relevant in that the Hamilton flow for $D_t + A^{\pm}_{<\alpha^{-1}}$ serves as a good approximation to the Hamilton flow for $D_t + A^{\pm}$ up to an $O(\alpha)$ angular difference.

To characterize the higher regularity properties of these flows is convenient to introduce (see [4]) a metric g_α in the phase space, defined by

$$ds^2 = |\xi|^{-4}(\xi d\xi)^2 + |\xi|^{-4}\alpha^{-2}(\xi \wedge d\xi)^2 + \alpha^{-4}|\xi|^{-2}(\xi dx)^2 + |\xi|^{-2}\alpha^{-2}(\xi \wedge dx)^2$$

Then as in [4] we obtain

Lemma 4.2. *The Hamilton flow maps $(x_t^{\pm}, \xi_t^{\pm})$ are g_α-smooth canonical transformations.*

Given a direction $\theta \in S^{n-1}$ at time $t = 0$ we introduce the size α sectors

$$S_\alpha(\theta) = \{\xi;\ \angle(\xi,\theta) < \alpha\}$$

$$\tilde{S}_\alpha(\theta) = \{\xi;\ C\alpha < \angle(\xi,\theta) < 2C\alpha\}$$

where C is a fixed large constant. The images of $\mathbb{R}^n \times S_\alpha(\theta)$, respectively $\mathbb{R}^n \times \tilde{S}_\alpha(\theta)$ along the Hamilton flow for $D_t + A^{\pm}_{<\alpha^{-1}}$ are denoted by $H^{\pm}_\alpha S_\alpha(\theta)$, respectively $H^{\pm}_\alpha \tilde{S}_\alpha(\theta)$.

Let $\xi^\alpha_\theta = \xi^\alpha_\theta(x,t)$ be the Fourier variable which is defined by the $D_t + A^{+}_{<\alpha^{-1}}$ Hamilton flow with initial data $\xi^\alpha_\theta(x,0) = \theta$ (i.e. $\xi^\alpha_\theta(x,t) = \xi^+_t(t)$ is the solution of the flow (54) with initial data $\xi^+_0 = \xi$, for which $x^+_t(t) = x$). This is well defined at least for a short time, precisely for as long as caustics do not occur. From Lemma 4.2 one also sees that ξ^α_θ is a g_α-smooth function of x.

We consider a maximal set O_α of α-separated directions and a partition of unity at time 0

$$1 = \sum_{\theta \in O_\alpha} \chi^{\pm,\alpha}_\theta(0,x,\xi)$$

consisting of 0-homogeneous symbols supported in $S_\alpha(\theta)$ which are smooth on the corresponding scale. Transporting these symbols along the $\pm$ Hamilton flows by

$$\chi^{\pm,\alpha}_\theta(0,x,\xi) \ = \ \chi^{\pm,\alpha}_\theta(t, x^\pm_t, \xi^\pm_t)$$

produces a time dependent partition of unity

$$1 = \sum_{\theta \in O_\alpha} \chi^{\pm,\alpha}_\theta(t,x,\xi) \tag{55}$$

so that the support of $\chi^{\pm,\alpha}_\theta(t,x,\xi)$ is contained in $H^{\pm}_\alpha S_\alpha(\theta)$.

The regularity of these symbols is easily obtained from the transport equations (see again [4]):

Lemma 4.3. *The symbols $\chi^{\pm,\alpha}_\theta(t,x,\xi)$ belong to the class $S(1,g_\alpha)$*[1].

We use the above partition of unity in the phase space to produce a corresponding pseudodifferential partition of unity. Given a frequency $\lambda > \alpha^{-2}$ we define the symbols

$$\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,\xi) = S_{<\lambda/8}(D_x)\chi^{\pm,\alpha}_\theta(t,x,\xi)\tilde{s}_\lambda(\xi)$$

[1]Throughout this paper we will use the standard notation $S(m,g)$, while in [4] we used for $S(1,g)$ the shorter one: $S(g)$.

These are used in order to split general frequency localized waves into square summable superpositions of directionally localized waves,

$$S_\lambda u = \sum_{\theta \in O_\alpha} \chi_{\theta,\lambda}^{\pm;\alpha}(t,x,D) S_\lambda u$$

This decomposition is closely related to a wave packet decomposition, see [11], [12], [16], and [4]. The difference is that here we skip the spatial localization part since it brings no additional benefit. The above localization at spatial frequencies less than $\lambda/8$ insures that the output of the operators $\chi_{\theta,\lambda}^{\pm,\alpha}(t,x,D)S_\lambda$ is still localized at frequency λ. This localization is otherwise harmless:

Lemma 4.4. *The symbols $\chi_{\theta,\lambda}^{\pm;\alpha}(t,x,\xi)$ belong to the class $S(1,g_\alpha)$. In addition, we have similar bounds for the Poisson bracket*

$$\{\tau + a^{\pm}_{<\alpha^{-1}}(t,x,\xi), \chi_{\theta,\lambda}^{\pm;\alpha}(t,x,\xi)\} \in S(1,g_\alpha) \tag{56}$$

Proof. The fact that $\chi_{\theta,\lambda}^{\pm;\alpha}(t,x,\xi) \in S(1,g_\alpha)$ is straightforward since the multiplier $S_{<\lambda/8}$ is a mollifier on the λ^{-1} spatial scale, which is less that the spatial scale of the g_α balls.

Since $\chi_\theta^{\pm,\alpha}$ is transported along the $a^{\pm}_{<\alpha^{-1}}(t,x,\xi)$ flow, the Poisson bracket is expressed in the form

$$\{a^{\pm}_{<\alpha^{-1}}(t,x,\xi), \tilde{s}_\lambda(\xi)\}\chi_{\theta,\lambda}^{\pm;\alpha}(t,x,\xi) + \tilde{s}_\lambda(\xi)[H_{a^{\pm}_{<\alpha^{-1}}}, S_{<\lambda/8}(D_x)]\chi_\theta^{\pm,\alpha}(t,x,\xi)$$

Here $H_{a^{\pm}_{<\alpha^{-1}}}$ is the Hamiltonian operator associated to the $a^{\pm}_{<\alpha^{-1}}(t,x,\xi)$ flow. It is easy to see that the first term belongs to $S(1,g_\alpha)$, therefore it remains to consider the commutator term. We have

$$[H_{a^{\pm}_{<\alpha^{-1}}}, S_{<\lambda/8}(D_x)] = [\partial_\xi a^{\pm}_{<\alpha^{-1}}, S_{<\lambda/8}(D_x)]\partial_x - [\partial_x a^{\pm}_{<\alpha^{-1}}, S_{<\lambda/8}(D_x)]\partial_\xi$$

The commutator of a scalar function g with $S_{<\lambda/8}$ can be expressed as a rapidly convergent series of the form

$$[g, S_{<\lambda/8}] = \lambda^{-1} \sum_j S^{1,j}_{<\lambda/8} \nabla g S^{2,j}_{<\lambda/8}$$

where the multipliers $S^{1,j}_{<\lambda/8}$ and $S^{2,j}_{<\lambda/8}$ have the same properties as $S_{<\lambda/8}$ and decay rapidly with respect to j. Then the above commutator term is expressed as

$$[H_{a^{\pm}_{<\alpha^{-1}}}, S_{<\lambda/8}(D_x)] = \lambda^{-1} \sum_j S^{1,j}_{<\lambda/8} \left(\partial_x \partial_\xi a^{\pm}_{<\alpha^{-1}} \partial_x - \partial_x^2 a^{\pm}_{<\alpha^{-1}} \partial_\xi\right) S^{j,2}_{<\lambda/8}$$

At this stage the effect of the mollifiers is negligible and we can use the regularity properties of $a^{\pm}$ and $\chi_\theta^{\pm,\alpha}$ to directly compute

$$\tilde{s}_\lambda(\xi)[H_{a^{\pm}_{<\alpha^{-1}}}, S_{<\lambda/8}(D_x)]\chi_\theta^{\pm,\alpha}(t,x,\xi) \in S(\frac{1}{\alpha^2\lambda}, g_\alpha)$$

□

To better understand the phase space localization provided by $\chi_{\theta,\lambda}^{\pm,\alpha}$ consider some point (x_0,t_0) and the corresponding center direction $\xi_\theta^\alpha(x_0,t_0)$. A spatial unit g_α ball $B_\theta^\alpha(x_0,t_0)$ centered at (x_0,t_0) has dimensions[2] $\alpha^2 \times \alpha^{n-1}$ with the long

[2]Here n stands for the space dimension

sides normal to $\xi_\theta^\alpha(x_0,t_0)$. Within the ball $B_\theta^\alpha(x_0,t_0)$, $\chi_{\theta,\lambda}^{\pm,\alpha}\tilde{S}_\lambda$ localizes frequencies to a sector of angle α centered at $\xi_\theta^\alpha(x_0,t_0)$. Thus the frequencies are localized to a radial rectangle centered at $\lambda\xi_\theta^\alpha(x_0,t_0)$ of size $\lambda \times (\alpha\lambda)^{n-1}$. In this picture, angle α wave packets correspond to a spatial localization on the scale of the above ball $B_\theta^\alpha(x_0,t_0)$, constructed along a fixed ray of the Hamilton flow.

The g_α metric restricted to frequency λ is slowly varying and temperate at frequencies [3] $\lambda \geq \alpha^{-2}$, and in our analysis we will always be above this threshold. Hence there is a good pseudodifferential calculus for operators with $S(1,g_\alpha)$ symbols. The semiclassical parameter $h = h(\alpha,\lambda)$ in the $S(1,g_\alpha)$ calculus at frequency λ is given by

$$h(\alpha,\lambda) = (\alpha^2\lambda)^{-1}$$

The $S(1,g_\alpha)$ symbols at frequency λ satisfy the bounds

$$\left|(\xi_\theta^\alpha\partial_x)^\sigma(\xi_\theta^\alpha\wedge\partial_x)^\beta\partial_\xi^\nu(\xi\partial_\xi)^\gamma q(t,x,\xi)\right| \lesssim \alpha^{-2\sigma-|\beta|}(\alpha\lambda)^{-\nu} \tag{57}$$

Due to the L^2 in time regularity of the second order derivatives of the coefficients we also introduce the space of symbols $L^2S(1,g_\alpha)$ which at frequency λ satisfy

$$\left|(\xi_\theta^\alpha\partial_x)^\sigma(\xi_\theta^\alpha\wedge\partial_x)^\beta\partial_\xi^\nu(\xi\partial_\xi)^\gamma q(t,x,\xi)\right| \lesssim \alpha^{-2\sigma-|\beta|}(\alpha\lambda)^{-\nu}f(t) \tag{58}$$

for some $f \in L^2$. In all the operators we consider here, the function f is the same:

$$f(t) = M(\|\nabla^2 g(t)\|_{L^\infty}) \tag{59}$$

In some of our estimates we need to deal with two distinct scales at a given frequency λ, namely the angular scale α and the $\lambda^{\frac{1}{2}}$ scale at which the coefficients are truncated. Correspondingly we introduce additional symbol classes $C_\lambda^k S(1,g_\alpha)$ of symbols q localized at frequency λ which satisfy the $S(1,g_\alpha)$ bounds (57) for $\sigma + |\beta| \leq k$, respectively the weaker estimate

$$\left|(\xi_\theta^\alpha\partial_x)^\sigma(\xi_\theta^\alpha\wedge\partial_x)^\beta\partial_\xi^\nu(\xi\partial_\xi)^\gamma q(t,x,\xi)\right| \lesssim (\alpha^{-2\sigma-|\beta|} + \alpha^{-k}\lambda^{\frac{\sigma+|\beta|}{2}})(\alpha\lambda)^{-\nu} \tag{60}$$

for $\sigma + |\beta| > k$. There is still a calculus for such symbols, since the above bounds are stronger than the $S(1,g_{\lambda^{\frac{1}{2}}})$ bounds. The related classes of symbols $L^2C_\lambda^kS(1,g_\alpha)$ are defined in a manner which is similar to (58).

Using the calculus for the above symbol classes one can prove that the partition of unity in (55) yields an almost orthogonal decomposition of functions, namely

Proposition 4.5. *Fix a frequency λ and let $\alpha > \lambda^{-\frac{1}{2}}$. Then for each function u which is localized at frequency λ we have*

$$\sum_{\theta\in O_\alpha}\|\chi_{\theta,\lambda}^{\pm,\alpha}(t,x,D)u\|_{X_\pm}^2 \approx \|u\|_{X_\pm}^2 \tag{61}$$

Proof. We only outline the proof, since this result is essentially contained in [16]. There are two bounds to prove. The first

$$\sum_{\theta\in O_\alpha}\|\chi_{\theta,\lambda}^{\pm,\alpha}(t,x,D)u\|_{L^2}^2 \approx \|u\|_{L^2}^2 \tag{62}$$

[3] This corresponds to the classical wave packets which are localized on the scale of the uncertainty principle. Above this threshold we are dealing with generalized wave packets, which may have a more complex structure, see [16] and [4]

follows from the almost orthogonality of the operators $\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)$. This in turn is due to the almost disjoint supports[4] of $\chi^{\pm,\alpha}_{\theta,\lambda}$ and to the $S(1,g_\alpha)$ calculus.

Consider now the second bound

$$\sum_{\theta\in O_\alpha} \|(D_t + A^\pm)\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|^2_{L^2} \approx \|(D_t + A^\pm)u\|^2_{L^2} + O(\|u\|^2_{L^2}) \tag{63}$$

We first establish it with $A^\pm$ replaced by $A^\pm_{<\lambda^{\frac{1}{2}}}$,

$$\sum_{\theta\in O_\alpha} \|(D_t + A^\pm_{<\lambda^{\frac{1}{2}}})\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|^2_{L^2} \approx \|(D_t + A^\pm_{<\lambda^{\frac{1}{2}}})u\|^2_{L^2} + O(\|u\|^2_{L^2}) \tag{64}$$

Due to (62) and the energy bound

$$\|u\|^2_{L^\infty L^2} \lesssim \|u\|^2_{L^2} + \|u\|_{L^2}\|(D_t + A^\pm_{<\lambda^{\frac{1}{2}}})u\|_{L^2}$$

it suffices to prove the commutator estimate

$$\sum_{\theta\in O_\alpha} \|[D_t + A^\pm_{<\lambda^{\frac{1}{2}}}, \chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)]u\|^2_{L^2} \lesssim \|u\|^2_{L^\infty L^2} \tag{65}$$

which we split into two components.

For the low frequency part of the coefficients we use a second order commutator

$$\sum_{\theta\in O_\alpha} \|[D_t + A^\pm_{<\alpha^{-1}}, \chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)]u\|^2_{L^2} \lesssim \|u\|^2_{L^\infty L^2} \tag{66}$$

For this it suffices to prove that

$$[D_t + A^\pm_{<\alpha^{-1}}, \chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)] \in OPL^2S(1,g_\alpha) \tag{67}$$

The summation with respect to $\theta \in O_\alpha$ follows by orthogonality since the symbols for the above commutators will retain the rapid decay away from the support of $\chi^{\pm,\alpha}_{\theta,\lambda}$. Here it is important that (59) applies uniformly.

Due to the Poisson bracket bound in (56) it suffices to show that

$$[A^\pm_{<\alpha^{-1}}, \chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)] + i\{a^\pm_{<\alpha^{-1}}, \chi^{\pm,\alpha}_{\theta,\lambda}\}(t,x,D) \in OPL^2S(1,g_\alpha)$$

Due to the frequency localization of $\chi^{\pm,\alpha}_{\theta,\lambda}$, only the values of $a^\pm(t,x,\xi)$ in the region $|\xi| \approx \lambda$ can affect the above operator. At this point it is no longer important that $a^\pm_{<\alpha^{-1}}$ and $\chi^{\pm,\alpha}_{\theta,\lambda}$ are related. We consider a rapidly convergent spherical harmonics expansion of $a^\pm$,

$$a^\pm(t,x,\xi) = \sum_j b_j(t,x)\phi_j(\xi)$$

where b_j have the same regularity as the coefficients g^{ij} while $\phi_j(\xi)$ are homogeneous of order 1. It suffices to consider a single term $b(t,x)\phi(\xi)$ in this expansion and show that

$$[b_{<\alpha^{-1}}\phi(D), \chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)] + i\{b_{<\alpha^{-1}}\phi, \chi^{\pm,\alpha}_{\theta,\lambda}\}(t,x,D) \in OPL^2S(1,g_\alpha) \tag{68}$$

To see this we consider the commutators with b and with ϕ. The commutator term with b has the form

$$C_b = ([b_{<\alpha^{-1}}, \chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)] + i\{b_{<\alpha^{-1}}, \chi^{\pm,\alpha}_{\theta,\lambda}\}(t,x,D))\phi(D)$$

[4]modulo tails which are rapidly decreasing on the g_α scale

Since $\partial_x^2 b_{<\alpha^{-1}} \in L^2 S(1, g_\alpha)$, $\partial_\xi^2 \chi_{\theta,\lambda}^{\pm,\alpha} \in S(\alpha^{-2}\lambda^{-2}, g_\alpha)$ and $\phi \in S(\lambda, g_\alpha)$, the $S(g_\alpha)$ calculus at frequency λ yields the better result $C_b \in OPL^2S(\alpha^{-2}\lambda^{-1}, g_\alpha)$, which is tight only when $\alpha = \lambda^{-\frac{1}{2}}$.

The commutator term with ϕ has the form

$$C_\phi = b_{<\alpha^{-1}}([\phi(D), \chi_{\theta,\lambda}^{\pm,\alpha}(t,x,D)] + i\{\phi, \chi_{\theta,\lambda}^{\pm,\alpha}\}(t,x,D))$$

The $b_{<\alpha^{-1}}$ factor belongs to $S(1, g_\alpha)$ and can be neglected. The argument for the remaining part is somewhat more delicate since it hinges on the homogeneity of ϕ. With $b = 1$ denote by ξ the input frequency for C_ϕ and by η the output frequency. Due to the homogeneity of ϕ we have the representation

$$\phi(\eta) - \phi(\xi) = (\eta - \xi)\nabla\phi(\xi) + \psi(\xi, \eta)(\xi \wedge (\xi - \eta))^2 \tag{69}$$

where ψ is a smooth and homogeneous of order -3 matrix valued function. For $|\xi|, |\eta| \approx \lambda$ we can separate variables in ψ and express it as a rapidly convergent series

$$\psi(\xi, \eta) = \lambda^{-3} \sum_j \psi_j^1(\eta)\psi_j^2(\xi)$$

This gives a representation for C_ϕ of the form

$$C_\phi = \lambda^{-3} \sum_j \psi_j^1(D)((\xi \wedge \partial_x)^2 \chi_{\theta,\lambda}^{\pm,\alpha})(t,x,D)\psi_j^2(D)$$

Since $\chi_{\theta,\lambda}^{\pm,\alpha}(x,D) \in S(1, g_\alpha)$ we obtain $(\xi \wedge \partial_x)^2 \chi_{\theta,\lambda}^{\pm,\alpha} \in S(\lambda^2\alpha^{-2}, g_\alpha)$ which shows that $C_\phi \in OPS(\alpha^{-2}\lambda^{-1}, g_\alpha)$. This concludes the proof of (68) and thus the proof of (67).

For the intermediate frequency part of the coefficients we have a first order commutator estimate

$$\sum_{\theta \in O_\alpha} \|[A^\pm_{\alpha^{-1} < \cdot < \lambda^{\frac{1}{2}}}, \chi_{\theta,\lambda}^{\pm,\alpha}(t,x,D)]u\|_{L^2}^2 \lesssim \|u\|_{L^\infty L^2}^2 \tag{70}$$

Together with (66) this implies (65).

This follows from first order commutator estimate

$$[A^\pm_{\alpha^{-1} < \cdot < \lambda^{\frac{1}{2}}}, \chi_{\theta,\lambda}^{\pm,\alpha}(t,x,D)] \in OPL^2C_\lambda^1 S(1, g_\alpha) \tag{71}$$

Indeed, for a scalar function b we can estimate

$$\alpha^{-2}\|b_{\alpha^{-1} < \cdot < \lambda^{\frac{1}{2}}}\|_{L^2L^\infty} + \alpha^{-1}\|\partial_x b_{\alpha^{-1} < \cdot < \lambda^{\frac{1}{2}}}\|_{L^2L^\infty} \lesssim \|\partial^2 b\|_{L^2L^\infty}$$

Applied to the the symbol $a^\pm$ as a function of x this shows that

$$a^\pm_{\alpha^{-1} < \cdot < \lambda^{\frac{1}{2}}} \in L^2C_\lambda^2 S(\alpha^2\lambda, g_\alpha)$$

Since $\chi_{\theta,\lambda}^{\pm,\alpha} \in S(1, g_\alpha)$, the estimate (71) follows by pdo calculus. The square summability with respect to θ is again due to the almost disjoint supports of the symbols $\chi_\theta^{\pm,\alpha}$.

It remains to pass from (64) to (63). Due to the energy bound

$$\|u\|_{L^\infty L^2}^2 \lesssim \|u\|_{L^2}^2 + \|u\|_{L^2}\|(D_t + A^\pm_{<\lambda^{\frac{1}{2}}})u\|_{L^2}$$

this is a consequence of the estimate

$$\|A^\pm_{>\lambda^{\frac{1}{2}}} u\|_{L^2} \lesssim \|u\|_{L^\infty L^2}$$

applied to both u and $\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u$. Using the spherical harmonics decomposition of the symbols $a^\pm$ as above this reduces to the straightforward bound

$$\|b_{>\lambda^{\frac12}} u\|_{L^2} \lesssim \lambda^{-1} \|\partial^2 b\|_{L^2L^\infty} \|u\|_{L^\infty L^2}$$

$\square$

The frequency localization in $\chi^{\pm,\alpha}_{\theta,\lambda}$ contributes to improved Strichartz type estimates above the critical range of exponents. Begin for instance with the endpoint L^2L^6 Strichartz estimate

$$\|\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|_{L^2L^6} \lesssim \lambda^{\frac56} \|u\|_{X_\pm} \tag{72}$$

Here the angular frequency localization plays no role. However, suppose we want to use Bernstein's inequality to replace this by an L^2L^∞ estimate. Modulo rapidly decaying tails, within each spatial g_α ball $B^\alpha_\theta(x_0,t_0)$ the function $\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u$ is frequency localized in a dyadic sector section of size $\lambda \times (\alpha\lambda)^3$. Then the constant in Bernstein's inequality is

$$[\lambda \times (\alpha\lambda)^3]^{\frac16} = \lambda^{\frac23}\alpha^{\frac12}$$

Hence we obtain the better L^2L^∞ bound

$$\|\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|_{L^2L^\infty} \lesssim \alpha^{\frac12}\lambda^{\frac32} \|u\|_{X_\pm}, \qquad \alpha > \lambda^{-\frac12} \tag{73}$$

A simpler related uniform bound is derived directly from the energy estimates,

$$\|\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|_{L^\infty} \lesssim \alpha^{\frac32}\lambda^{2} \|u\|_{X_\pm}, \qquad \alpha > \lambda^{-\frac12} \tag{74}$$

A similar bound holds for the right hand side of the $\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u$ equation. Indeed, for $u \in X_\pm$ we can write

$$(D_t + A^\pm)\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u = (D_t + A^\pm_{<\lambda^{\frac12}})\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u + A^\pm_{>\lambda^{\frac12}}\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u$$

The first term belongs to L^2 and has a similar frequency localization as $\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u$. The second is estimated directly using (73). This yields

$$\|(D_t + A^\pm)\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|_{L^2L^\infty} \lesssim \alpha^{\frac32}\lambda^{2} \|u\|_{X_\pm}, \qquad \alpha > \lambda^{-\frac12} \tag{75}$$

Another way of taking advantage of the angular localization is in corresponding bounds for derivatives. Consider the differentiation operators $\xi^\alpha_\theta \wedge D$ whose symbol vanishes in the ξ^α_θ direction. Then in the support of $\chi^{\pm,\alpha}_{\theta,\lambda}$ these symbols have size $\alpha\lambda$. Hence from (72) we also obtain

$$\|(\xi^\alpha_\theta \wedge D)\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|_{L^2L^6} \lesssim (\alpha\lambda)\lambda^{\frac56} \|u\|_{X_\pm} \tag{76}$$

We can argue in the same way for the energy estimates or for the L^2L^∞ bound in (73). For convenience we collect several such bounds in a single norm,

$$\begin{aligned}\|v\|_{X^{\lambda,\alpha,\theta}_\pm} = {}& \|v\|_{X_\pm} + \|v\|_{L^\infty L^2} + \lambda^{-\frac56}\|v\|_{L^2L^6} + \alpha^{-\frac12}\lambda^{-\frac32}\|v\|_{L^2L^\infty} + \alpha^{-\frac32}\lambda^{-2}\|v\|_{L^\infty} \\ & + \alpha^{-\frac32}\lambda^{-2}\|(D_t + A^\pm)v\|_{L^2L^\infty} + (\alpha\lambda)^{-1}\|(\xi^\alpha_\theta \wedge D)v\|_{L^\infty L^2} \\ & + (\alpha\lambda)^{-1}(\lambda^{-\frac56}\|(\xi^\alpha_\theta \wedge D)v\|_{L^2L^6} + \alpha^{-\frac12}\lambda^{-\frac32}\|(\xi^\alpha_\theta \wedge D)v\|_{L^2L^\infty})\end{aligned}$$

and use it to state a corresponding version of (61),

$$\sum_{\theta \in O_\alpha} \|\chi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|^2_{X^{\lambda,\alpha,\theta}_\pm} \approx \|\tilde S_\lambda u\|^2_{X_\pm} \tag{77}$$

We want to replace the partition of unity in (55) first with a bilinear one and next with a trilinear one. Given two frequencies $\mu < \lambda$, we denote $\alpha_\mu = \mu^{-\frac{1}{2}}$ and introduce a corresponding bilinear partition of unity which is useful when estimating the frequency μ output of the product of two frequency λ waves. The main contribution corresponds to opposite frequencies ξ and η, therefore we organize the following decomposition based on the dyadic angle $\alpha_\mu \leq \alpha \leq 1$ between ξ and $-\eta$. Precisely, by superimposing the α angular decompositions for α in the above range we obtain

$$\tilde{s}_\lambda(\xi)\tilde{s}_\lambda(\eta) = \sum_{\theta_1,\theta_2\in O_{\alpha_\mu}}^{|\theta_1+\theta_2|\leq 2C\alpha_\mu} \sum_{\theta_3,\theta_4\in O_{2\alpha_\mu}}^{|\theta_3+\theta_4|\leq 4C\alpha_\mu} \chi^{\pm,\alpha_\mu}_{\theta_1,\lambda}(t,x,\xi)\chi^{\mp,\alpha_\mu}_{\theta_2,\lambda}(t,x,\eta)\chi^{\pm,2\alpha_\mu}_{\theta_3,\lambda}(t,x,\xi)\chi^{\mp,2\alpha_\mu}_{\theta_4,\lambda}(t,x,\eta)$$
$$+\sum_{\alpha=\alpha_\mu}^{1} \sum_{\theta_1,\theta_2\in O_\alpha}^{C\alpha\leq|\theta_1+\theta_2|\leq 2C\alpha} \sum_{\theta_3,\theta_4\in O_{2\alpha}}^{|\theta_3+\theta_4|\leq 4C\alpha} \chi^{\pm,\alpha}_{\theta_1,\lambda}(t,x,\xi)\chi^{\mp,\alpha}_{\theta_2,\lambda}(t,x,\eta)\chi^{\pm,2\alpha}_{\theta_3,\lambda}(t,x,\xi)\chi^{\mp,2\alpha}_{\theta_4,\lambda}(t,x,\eta)$$

To shorten this expression we redenote factors and harmlessly simplify the summation notations to

$$1 = \sum_{\theta\in O_{\alpha_\mu}} \phi^{\pm,\alpha_\mu}_{\theta,\lambda}(t,x,\xi)\phi^{\mp,\alpha_\mu}_{-\theta,\lambda}(t,x,\eta) + \sum_{\alpha=\alpha_\mu}^{1}\sum_{\theta\in O_\alpha} \phi^{\pm,\alpha}_{\theta,\lambda}(t,x,\xi)\tilde{\phi}^{\mp,\alpha}_{-\theta,\lambda}(t,x,\eta) \tag{78}$$

where the tilde in $\tilde{\phi}^{\pm,\alpha}_{\theta,\lambda}$ indicates an $O(C\alpha)$ angular separation from θ. The symbols $\phi^{\pm,\alpha}_{\theta,\lambda}$, respectively $\tilde{\phi}^{\pm,\alpha}_{\theta,\lambda}$ retain the same properties as $\chi^{\pm,\alpha}_{\theta,\lambda}$, namely

$$\phi^{\pm,\alpha}_{\theta,\lambda} \in S(1,g_\alpha), \qquad \{\tau + a^{\pm}_{<\alpha^{-1}}(t,x,\xi), \phi^{\pm,\alpha}_{\theta,\lambda}(t,x,\xi)\} \in S(1,g_\alpha) \tag{79}$$

and the same for $\tilde{\phi}^{\pm,\alpha}_{\theta,\lambda}$. In particular the counterpart of (77) is still valid,

$$\sum_{\theta\in O_\alpha} \|\phi^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|^2_{X^{\lambda,\alpha,\theta}_\pm} + \|\tilde{\phi}^{\pm,\alpha}_{\theta,\lambda}(t,x,D)u\|^2_{X^{\lambda,\alpha,\theta}_\pm} \approx \|\tilde{S}_\lambda u\|^2_{X_\pm} \tag{80}$$

Finally, we arrive at the main trilinear symbol decomposition. Its aim is to achieve a simultaneous angular decomposition in trilinear expressions of the form

$$\int uvw\,dxdt$$

We denote the three corresponding frequencies by ξ, η and ζ. We assume that each of the factors has a dyadic frequency localization,

$$|\xi| \approx |\eta| \approx \lambda, \qquad |\zeta| \approx \mu, \qquad 1 \ll \mu \leq \lambda$$

If the trilinear decomposition were translation invariant then only its structure on the diagonal $\xi+\eta+\zeta = 0$ is relevant. However, in our case we are working with variable coefficient operators therefore a neighborhood of the diagonal is relevant. The size of this neighborhood is determined by the spatial regularity of the symbols via the uncertainty principle.

Corresponding to the first term in (78) we consider a decomposition in ζ with respect to the dyadic angle between ζ and θ,

$$\tilde{s}_\mu(\zeta) = \phi^{\pm,\alpha_\mu}_{\theta,\mu}(t,x,\zeta) + \sum_{\alpha>\alpha_\mu} \tilde{\phi}^{\pm,\alpha}_{\theta,\mu}(t,x,\zeta)$$

To understand the ζ decomposition corresponding to the second term in (78) we first identify the location of the diagonal $\xi+\eta+\zeta=0$. Given the above dyadic localization of ξ,η and ζ, if the angle between ξ and $-\eta$ is of order α, then the angle between ξ and $\pm\zeta$ must be of order $\alpha\lambda\mu^{-1}$ which is larger than α. Thus the interesting angular separation threshold for ζ is $\alpha\lambda\mu^{-1}$. It would appear that there are two cases to consider, namely when the angle between ξ and ζ is small, and when the angle between $-\xi$ and ζ is small. However, due to our choice of the $\pm$ signs corresponding to ξ, η and ζ, the latter case leads to nonresonant wave interactions and loses its relevance. Hence, the significant dyadic parameter here is the angle between ξ and ζ, and the ζ decomposition has the form

$$\tilde{s}_\mu(\zeta)=\phi^{\pm,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,\zeta)+\tilde{\phi}^{\pm,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,\zeta)+\sum_{\beta>\alpha\mu^{-1}\lambda}\tilde{\phi}^{\pm,\beta}_{\theta,\mu}(t,x,\zeta)$$

Then the full trilinear decomposition has the form

$$\begin{aligned}\tilde{s}_\lambda(\xi)\tilde{s}_\lambda(\eta)\tilde{s}_\mu(\zeta)=&\sum_{\theta\in O_{\alpha_\mu}}\phi^{\pm,\alpha_\mu}_{\theta,\lambda}(t,x,\xi)\phi^{\mp,\alpha_\mu}_{-\theta,\lambda}(t,x,\eta)\phi^{\pm,\alpha_\mu}_{\theta,\mu}(t,x,\zeta)\\&+\sum_{\theta\in O_{\alpha_\mu}}\phi^{\pm,\alpha_\mu}_{\theta,\lambda}(t,x,\xi)\phi^{\mp,\alpha_\mu}_{-\theta,\lambda}(t,x,\eta)\sum_{\alpha>\alpha_\mu}\tilde{\phi}^{\pm,\alpha}_{\theta,\mu}(t,x,\zeta)\\&+\sum_{\alpha>\alpha_\mu}\sum_{\theta\in O_\alpha}\phi^{\pm,\alpha}_{\theta,\lambda}(t,x,\xi)\tilde{\phi}^{\mp,\alpha}_{-\theta,\lambda}(t,x,\eta)\tilde{\phi}^{\pm,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,\zeta)\\&+\sum_{\alpha>\alpha_\mu}\sum_{\theta\in O_\alpha}\phi^{\pm,\alpha}_{\theta,\lambda}(t,x,\xi)\tilde{\phi}^{\mp,\alpha}_{-\theta,\lambda}(t,x,\eta)\phi^{\pm,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,\zeta)\\&+\sum_{\alpha>\alpha_\mu}\sum_{\theta\in O_\alpha}\phi^{\pm,\alpha}_{\theta,\lambda}(t,x,\xi)\tilde{\phi}^{\mp,\alpha}_{-\theta,\lambda}(t,x,\eta)\sum_{\beta>\alpha\mu^{-1}\lambda}\tilde{\phi}^{\pm,\beta}_{\theta,\mu}(t,x,\zeta)\end{aligned}\tag{81}$$

In the above sum the first three terms are the main ones, as they account for the behavior near the diagonal. The remaining terms have off diagonal support, and their contribution to trilinear forms as above is negligible.

5. Proof of the trilinear estimate (50)

As noted in the previous section, we can replace the spaces $X^{s,\theta}_{\lambda,d}$ in (50) with the $X_\pm$ spaces. Hence we restate (50) in the form

Proposition 5.1. *For any choice of the $\pm$ signs and $1<d<\mu\ll\lambda$ we have*

$$\left|\int S_\lambda u\, S_\lambda v\, S_\mu w dxdt\right|\lesssim \ln\mu\cdot\mu^{\frac{5}{4}}\|S_\lambda u\|_{X_\pm}\|S_\lambda v\|_{X_{\pm,d}}\|S_\mu w\|_{X_\pm}\tag{82}$$

Proof. We begin with several simple observations. First, by localizing to a fixed smaller space-time scale and rescaling back to unit scale we can insure that the coefficients g^{ij} vary slowly inside a unit cube,

$$|\nabla_{x,t}g^{ij}|\ll 1$$

This in turn insures that the Fourier variable does not vary much along the Hamilton flow,

$$|\xi^\alpha_\theta-\theta|\ll 1$$

We can also localize all factors in frequency to angular regions of small size, say $<\frac{1}{20}$. The corresponding localization multipliers are easily seen to be bounded in $X_\pm$ and $X_{\pm,d}$.

If the first two $\pm$ signs are identical then the product $S_\lambda u\, S_\lambda v$ is concentrated at a time frequency of the order of λ which makes it almost orthogonal to $S_\mu w$, hence the estimate above is much easier. Therefore without any restriction in generality we fix the first sign to $+$ and the second one to $-$. Even though the problem is not symmetric with respect to the first two factors, the sign in the third factor plays no role whatsoever, so we fix it to $+$. We denote

$$a(t,x,\xi) \;=\; a^+(t,x,\xi)$$

Then

$$a^-(t,x,\xi) \;=\; -a(t,x,-\xi)$$

We note that, for the purpose of the above estimates, in the definition of $X_\pm$ at frequency λ we can replace the symbols $a(x,\xi)$ with their regularized versions, namely $a_{<\lambda^{\frac12}}(x,\xi)$.

To keep the number of parameters small we first present the argument in the case when $d=1$. Once this is done, we show what changes are necessary for $d>1$.

Case 1: $d=1$. Corresponding to the trilinear symbol decomposition (81) of the identity we consider the corresponding pseudodifferential decomposition of the trilinear expression in (82). The we estimate each of the five terms. We remark that, since $S_\lambda u$, $S_\lambda v$ and $S_\mu w$ are frequency localized in a small angle, so are all the factors in (82).

Case 1, term I:

$$I = \sum_{\theta\in O_{\alpha_\mu}} \int \phi^{+,\alpha_\mu}_{\theta,\lambda}(t,x,D)S_\lambda u\; \phi^{-,\alpha_\mu}_{-\theta,\lambda}(t,x,D)S_\lambda v\; \phi^{+,\alpha_\mu}_{\theta,\mu}(t,x,D)S_\mu w\, dxdt$$

We use the energy estimate for the first two factors and the L^2L^∞ bound for the third to obtain

$$|I| \lesssim \mu^{\frac54} \|\phi^{+,\alpha_\mu}_{\theta,\lambda}(x,D)S_\lambda u\|_{X_+^{\lambda,\alpha_\mu,\theta}} \|\phi^{-,\alpha_\mu}_{-\theta,\lambda}(x,D)S_\lambda v\|_{X_-^{\lambda,\alpha_\mu,\theta}} \|\phi^{+,\alpha_\mu}_{\theta,\mu}(t,x,D)S_\mu w\|_{X_+^{\mu,\alpha_\mu,\theta}}$$

The summation with respect to θ is straightforward due to (80).

Case 1, term II: This is the most difficult term,

$$II = \sum_{\theta\in O_{\alpha_\mu}} \int \phi^{+,\alpha_\mu}_{\theta,\lambda}(t,x,D)S_\lambda u\; \phi^{-,\alpha_\mu}_{-\theta,\lambda}(t,x,D)S_\lambda v \sum_{\alpha>\alpha_\mu} \tilde\phi^{+,\alpha}_{\theta,\mu}(t,x,D)S_\mu w\, dxdt$$

The summation with respect to θ is easily done using (80). Hence, in what follows, we fix θ and redenote

$$u_\theta = \phi^{+,\alpha_\mu}_{\theta,\lambda}(t,x,D)S_\lambda u, \qquad v_\theta = \phi^{-,\alpha_\mu}_{-\theta,\lambda}(t,x,D)S_\lambda v, \qquad w^\alpha_\theta = \tilde\phi^{+,\alpha}_{\theta,\mu}(t,x,D)S_\mu w$$

The factors u_θ and v_θ are frequency localized in small angles around θ, respectively $-\theta$; w^α_θ has a similar localization around $\pm\theta$ provided that $\alpha \ll 1$.

We denote by $\tilde a_{<\mu^{\frac12}}(t,x,\xi)$ the linearization of $a_{<\mu^{\frac12}}(t,x,\xi)$ with respect to ξ around $\xi=\xi^{\alpha_\mu}_\theta(t,x)$. Since $a_{<\mu^{\frac12}}(t,x,\xi)$ is a homogeneous symbol of order 1, we have

$$\tilde a_{<\mu^{\frac12}}(t,x,\xi) = \xi\partial_\xi a_{<\mu^{\frac12}}(t,x,\xi^{\alpha_\mu}_\theta)$$

Consider now the difference

$$e = a_{<\mu^{\frac12}} - \tilde a_{<\mu^{\frac12}}$$

It vanishes of second order on the half line $\mathbb{R}^+\xi_\theta$. Due to the uniform (nonradial) convexity of the characteristic cone $\{\tau + a_{<\mu^{\frac12}}(t,x,\xi) = 0\}$, it follows that e is nonzero when ξ is not collinear with $\xi_\theta^{\alpha_\mu}$. Precisely, we can estimate it in terms of the angle $\angle(\xi,\xi_\theta^{\alpha_\mu})$ as

$$e(t,x,\xi) \approx |\xi||\angle(\xi,\xi_\theta^{\alpha_\mu})|^2$$

In particular in the support of the symbol $\tilde{\phi}^{+,\alpha}_{\theta,\mu}$ the above angle has size α and the frequency has size μ. Hence[5]

$$e(t,x,\zeta) \approx \alpha^2\mu, \qquad (t,x,\zeta) \in \text{supp } \tilde{\phi}^{+,\alpha}_{\theta,\mu}$$

Here it may help to think of the constant coefficient case where $\xi_\theta^{\alpha_\mu} = \theta$, while $a - \tilde{a} = |\xi| - \xi\theta$. We introduce a local inverse for $e(t,x,\zeta)$ in the support of $\tilde{\phi}^{+,\alpha}_{\theta,\mu}$, namely

$$l(t,x,\zeta) = \tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu}(t,x,\zeta)e^{-1}(t,x,\zeta)$$

The cutoff symbol $\tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu}$ is similar to $\tilde{\phi}^{+,\alpha}_{\theta,\mu}$ but has a slightly larger support and equals 1 in a neighbourhood of the support of $\tilde{\phi}^{+,\alpha}_{\theta,\mu}$.

As defined, the operator $L(t,x,D)$ is not localized at frequency μ. To remedy this we truncate its output in frequency and set

$$\tilde{L} = \tilde{S}_\mu(D)L(t,x,D)$$

The properties of the operator $\tilde{L}$ are summarized in the following

Lemma 5.2. *The operator $\tilde{L}$ satisfies the following estimates:*

a) fixed time L^p mapping properties:

$$\|\tilde{L}\|_{L^p\to L^p} \lesssim \alpha^{-2}\mu^{-1}, \qquad 1 \le p \le \infty$$

b) fixed time approximate inverse of $A(t,x,D) - \tilde{A}(t,x,D)$:

$$\|(A(t,x,D) - \tilde{A}(t,x,D))\tilde{L} - \tilde{\tilde{\phi}}(t,x,D)\|_{L^p\to L^p} \lesssim \mu^{-\frac12} + \alpha^{-2}\mu^{-1}, \quad 1 \le p \le \infty$$

c) space-time X_+ mapping properties:

$$\|\tilde{L}\|_{X_+\to X_+} \lesssim \alpha^{-2}\mu^{-1}$$

Proof. We first compute the regularity of the symbol $e(t,x,\zeta)$ within the support of l. With respect to ξ this is smooth and homogeneous, therefore we only have to keep track of the order of vanishing when ξ is in the $\xi_\theta^{\alpha_\mu}$ direction. With respect to x there is the dependence coming from the symbol a, as well as the dependence due to the $\xi_\theta^{\alpha_\mu}$ direction occuring in the linearization. Since a is Lipschitz in x and ξ_θ is Lipschitz in x and smooth on the α_μ scale, within the support of l we obtain

$$e \in C^1_\mu S(\alpha^2\mu, g_\alpha) \tag{83}$$

Combining this with the regularity of the symbol $\tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu} \in S(1,g_\alpha)$ we obtain the symbol regularity for l,

$$l \in C^1_\mu S((\alpha^2\mu)^{-1}, g_\alpha) \tag{84}$$

To prove part (a) of the Lemma we observe that for fixed (t,x) the symbol $l(t,x,\xi)$ is a smooth bump function of size $(\alpha^2\mu)^{-1}$ in a rectangle of size $\mu\times(\alpha\mu)^{n-1}$

[5]here we switch to the letter ζ for the frequency, as the following analysis refers to the region at low frequency μ corresponding to the last factor w in the trilinear form.

oriented in the $\xi_\theta^{\alpha_\mu}$ direction. This implies that its kernel $K(t,x,y)$ is bounded by $(\alpha^2\mu)^{-1}$ times an integrable bump function on the dual scale,

$$|K(t,x,y)| \lesssim (\alpha^2\mu)^{-1}\mu(\alpha\mu)^{n-1}(1+\mu|\xi_\theta^{\alpha_\mu}(t,x)(x-y)|+\alpha\mu|\xi_\theta^{\alpha_\mu}(t,x)\wedge(x-y)|)^{-N}$$

This bound is symmetric; indeed, since $\xi_\theta^{\alpha_\mu}(t,x)$ is Lipschitz in x we can replace it by $\xi_\theta^{\alpha_\mu}(t,y)$ in the above bound. Thus integrating we have

$$\sup_x \int |K(t,x,y)|dy \lesssim (\alpha^2\mu)^{-1}, \qquad \sup_y \int |K(t,x,y)|dx \lesssim (\alpha^2\mu)^{-1}$$

The L^p bounds for $L(t,x,D)$ and also for $\tilde{L}$ immediately follow.

For later use in the proof we observe that within the support of l we have

$$|\xi_\theta^{\alpha_\mu}(t,x)\wedge\xi| \lesssim \alpha\mu$$

Then the same argument as above yields the additional bounds

$$\|(\xi_\theta^{\alpha_\mu}(t,x)\wedge D)^\beta \tilde{L}u\|_{L^p} \lesssim (\alpha\mu)^{|\beta|}(\alpha^2\mu)^{-1}\|u\|_{L^p} \tag{85}$$

For part (b) we write

$$(A(t,x,D)-\tilde{A}(t,x,D))\tilde{L} - \tilde{\tilde{\phi}}(t,x,D) = R_1(t,x,D)+R_2(t,x,D)$$

where

$$R_1(t,x,D) = E(t,x,D)\tilde{S}_\mu(D)L(t,x,D) - \tilde{\tilde{\phi}}(t,x,D),$$

respectively

$$R_2(t,x,D) = (A_{>\mu^{\frac12}}(t,x,D) - \tilde{A}_{>\mu^{\frac12}}(t,x,D))\tilde{S}_\mu(D)L(t,x,D),$$

The operator R_1 is localized at frequency μ. The principal part cancels, and since $e \in C^1_\mu S(\alpha^2\mu, g_\alpha)$ and $l \in C^1_\mu S((\alpha^2\mu)^{-1}, g_\alpha)$ by the pseudodifferential calculus it follows that

$$R_1(t,x,D) \in C^0_\mu S((\alpha^2\mu)^{-1}, g_\alpha)$$

In addition, the symbol of R_1 decays rapidly away from the support of $\tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu}$. Hence we obtain the same kernel and L^p bounds as in the case of $L(t,x,D)$.

Consider now the operator R_2. Since $a(t,x,\zeta)$ is Lipschitz in x it follows that $|a_{>\mu^{\frac12}}(t,x,\zeta)| \lesssim \mu^{-\frac12}|\zeta|$. Expanding $a_{>\mu^{\frac12}}(t,x,\zeta)$ in a rapidly decreasing series of spherical harmonics with respect to ζ, we can separate variables and reduce the problem to the simpler case when $a_{>\mu^{\frac12}}(t,x,\zeta) = b(t,x)c(\zeta)$ with $|b| < \mu^{-\frac12}$ and c is smooth and homogeneous of order 1. For the symbol $c-\tilde{c}$ we use the representation

$$c(\zeta) - \tilde{c}(t,x,\zeta) = \psi(\xi_\theta^{\alpha_\mu},\zeta)(\xi_\theta^{\alpha_\mu}(t,x)\wedge\zeta)^2$$

where ψ is smooth in both arguments and homogeneous of order -1 in ζ. Separating variables in ψ we can assume without any restriction in generality that ψ depends only on ζ. Then after some simple commutations we obtain

$$c(D) - \tilde{c}(t,x,D) = (\xi_\theta^{\alpha_\mu}(t,x)\wedge D)^2\psi(D) + O(1)_{L^p\to L^p}$$

To estimate this we use (85). The factor $\psi(D)\tilde{S}_\mu(D)$ yields an extra μ^{-1} factor in the L^p bounds, therefore we obtain

$$\|R_2(t,x,D)\|_{L^p\to L^p} \lesssim \mu^{-\frac12}$$

Finally we prove part (c). By (a), $\tilde{L}$ is L^2 bounded with norm $O(\alpha^{-2}\mu^{-1})$, therefore it remains to prove the commutator estimate

$$\|[D_t + A_{<\mu^{\frac{1}{2}}}(t,x,D), \tilde{S}_\mu L(t,x,D)]\|_{L^\infty L^2 \to L^2} \lesssim \alpha^{-2}\mu^{-1} \tag{86}$$

This is a consequence of the operator bound

$$[D_t + A_{<\mu^{\frac{1}{2}}}(t,x,D), \tilde{S}_\mu L(t,x,D)] \in L^2 C^0_\mu S(\alpha^{-2}\mu^{-1}, g_\alpha)$$

To prove it we use the pdo calculus to represent the commutator as a principal term plus a second order error,

$$[D_t + A_{<\mu^{\frac{1}{2}}}(t,x,D), \tilde{S}_\mu L(t,x,D)] = \tilde{S}_\mu Q(t,x,D) + R(t,x,D)$$

where the principal part q has symbol

$$q(t,x,\xi) = -i\{\tau + a_{<\mu^{\frac{1}{2}}}(t,x,\xi), l(t,x,\xi)\}$$

The remainder R is localized at frequency μ. A direct computation, using (84), shows that its symbol satisfies

$$r \in L^2 C^0_\mu S(\alpha^{-2}\mu^{-1}, g_\alpha)$$

It remains to consider the above Poisson bracket and prove that

$$q \in L^2 C^0_\mu S(\alpha^{-2}\mu^{-1}, g_\alpha) \tag{87}$$

For this we write q in the form

$$iq = -\tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu} q_1 e^{-2} + q_2 e^{-1} + q_3 e^{-1}$$

where

$$q_1(t,x,\xi) = \left\{\tau + a_{<\mu^{\frac{1}{2}}}, e\right\}, \qquad q_2(t,x,\xi) = \left\{\tau + a_{<\alpha^{-1}}, \tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu}\right\}$$

respectively

$$q_3(t,x,\xi) = \left\{a_{\alpha^{-1}<\cdot<\mu^{\frac{1}{2}}}, \tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu}\right\}$$

Within the support of $\tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu}$ we know that $e \in C^1_\mu S(\alpha^2\mu, g_\alpha)$ is an elliptic symbol. Hence for the first term it suffices to show that $q_1 \in C^0_\mu S(\alpha^2\mu, g_\alpha)$. Indeed, by definition q_1 is a homogeneous symbol of order 1 which is continuous in x and homogeneous in ζ. In addition, we know that $e(t,x,\zeta)$ vanishes of second order in ζ at $(t,x,\xi^{\alpha_\mu}_\theta(x,t))$ which is also invariant with respect to the $\tau + a_{<\mu^{\frac{1}{2}}}$ Hamilton flow. Then q must vanish of second order in ζ at $(t,x,\xi^{\alpha_\mu}_\theta(x,t))$. Arguing as in the case of e, this implies that within the support of $\tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu}$ we have $q_1 \in C^0_\mu S(\alpha^2\mu, g_\alpha)$.

As in (67) we know that $q_2 \in S(1, g_\alpha)$. Also we have $a_{\alpha^{-1}<\cdot<\mu^{\frac{1}{2}}} \in L^2 C^2_\mu S(\alpha^2\mu, g_\alpha)$ and $\tilde{\tilde{\phi}}^{+,\alpha}_{\theta,\mu} \in S(1, g_\alpha)$ therefore $q_3 \in L^2 C^1_\mu S(1, g_\alpha)$.

This concludes the proof of (87) and therefore the proof of the lemma.

□

To continue the estimate of term II in Case 1 we define the auxiliary trilinear form

$$\begin{aligned} E(u,v,\tilde{w}) \quad &= \quad \int (D_t + A(t,x,D))u\, v\tilde{w}dxdt + \int u(D_t - A(t,x,-D))v\, \tilde{w}dxdt \\ &+ \quad \int uv\,(D_t + \tilde{A}(t,x,D))\tilde{w}dxdt \end{aligned}$$

With $\tilde{w} = \tilde{L} w_\theta^\alpha$ we write

$$\begin{aligned}\int u_\theta v_\theta w_\theta^\alpha dxdt = & -\int u_\theta v_\theta \left((A(t,x,D) - \tilde{A}(t,x,D))\tilde{L} - 1\right) w_\theta^\alpha dxdt \\ & + \int (D_t + A(t,x,D)) u_\theta \, v_\theta \tilde{w} dxdt \\ & + \int u_\theta (D_t - A(t,x,-D)) v_\theta \, \tilde{w} dxdt \\ & + \int u_\theta v_\theta (D_t + A(t,x,D)) \tilde{w} dxdt \\ & - E(u_\theta, v_\theta, \tilde{w})\end{aligned} \tag{88}$$

We bound each term separately. For the first one we write

$$\begin{aligned}(A(t,x,D) - \tilde{A}(t,x,D))\tilde{L} - 1 = & (A(t,x,D) - \tilde{A}(t,x,D))\tilde{L} - \tilde{\tilde{\phi}}_{\theta,\mu}^{+,\alpha}(t,x,D) \\ & + (\tilde{\tilde{\phi}}_{\theta,\mu}^{+,\alpha}(t,x,D) - 1)\end{aligned}$$

The contribution of the first line is estimated using Lemma 5.2 (b) and (73) for w_θ^α,

$$\begin{aligned}\left| \int u_\theta v_\theta [(A(t,x,D) - \tilde{A}(t,x,D))\tilde{L} - \tilde{\tilde{\phi}}_{\theta,\mu}^{+,\alpha}(t,x,D)] w_\theta^\alpha dxdt \right| & \\ \lesssim (\alpha^{-2}\mu^{-1} + \mu^{-\frac{1}{2}}) \|u_\theta\|_{L^\infty L^2} \|v_\theta\|_{L^\infty L^2} \|w_\theta^\alpha\|_{L^2 L^\infty} & \\ \lesssim (\alpha^{-2}\mu^{-1} + \mu^{-\frac{1}{2}}) \|u_\theta\|_{X_+} \|v_\theta\|_{X_-} \|w_\theta^\alpha\|_{L^2 L^\infty} & \\ \lesssim (\alpha^{-2}\mu^{-1} + \mu^{-\frac{1}{2}}) \alpha^{\frac{1}{2}} \mu^{\frac{3}{2}} \|u_\theta\|_{X_+} \|v_\theta\|_{X_-} \|w_\theta^\alpha\|_{X_+^{\mu,\alpha,\theta}} &\end{aligned}$$

For the contribution of the second line we observe that

$$(\tilde{\tilde{\phi}}_{\theta,\mu}^{+,\alpha}(t,x,D) - 1) w_\theta^\alpha = (\tilde{\tilde{\phi}}_{\theta,\mu}^{+,\alpha}(t,x,D) - 1) \tilde{\phi}_\theta^{+,\alpha} S_\mu w$$

where the symbols $\tilde{\tilde{\phi}}_{\theta,\mu}^{+,\alpha} - 1$ and $\tilde{\phi}_{\theta,\mu}^{+,\alpha} s_\mu$ have disjoint supports. Since they both belong to $S(1, g_\alpha)$, this yields a gain of a factor $(\alpha^2\mu)^{-N}$ in (73), with N arbitrarily large:

$$\sum_\theta \|(\tilde{\tilde{\phi}}_{\theta,\mu}^{+,\alpha}(t,x,D) - 1) w_\theta^\alpha\|_{L^2 L^\infty}^2 \lesssim \mu^{\frac{5}{2}} (\alpha^2\mu)^{-N} \|w_\theta^\alpha\|_{X_+^{\mu,\alpha,\theta}}^2$$

This is more than we need.

For the second term in (88) we use the L^2 bound for $(D_t + A)u_\theta$, the energy bound for v_θ and (73) for $\tilde{w}$. This yields

$$\left| \int (D_t + A(t,x,D)) u_\theta \, v_\theta \tilde{w} dxdt \right| \lesssim \alpha^{-\frac{3}{2}} \mu^{\frac{1}{2}} \|u_\theta\|_{X_+} \|v_\theta\|_{X_-} \|w_\theta^\alpha\|_{X_+^{\mu,\alpha,\theta}}$$

The third term is similar.

For the fourth term in (88) we use the energy for the first two factors combined with Bernstein derived $L^2 L^\infty$ bound for the third,

$$\begin{aligned}\left| \int u_\theta v_\theta \, (D_t + A(t,x,D)) \tilde{w} dxdt \right| & \lesssim \|u\|_{X_+} \|v\|_{X_-} \|(D_t + A(t,x,D))\tilde{w}\|_{L^2 L^\infty} \\ & \lesssim (\alpha^2\mu)^{-1} (\mu(\alpha\mu)^3)^{\frac{1}{2}} \|u_\theta\|_{X_+} \|v_\theta\|_{X_-} \|w_\theta^\alpha\|_{X_+^{\mu,\alpha,\theta}}\end{aligned}$$

It remains to prove the estimate for E. Observe that the time derivatives in E can be integrated out, producing contributions of the form

$$\int u_\theta\, v_\theta\, \tilde{w} dx \tag{89}$$

at the initial and the final time. These are estimated using energy bounds for the first two factors and the pointwise bound arising from Bernstein's inequality for the last factor,

$$\|\tilde{w}\|_{L^\infty} \lesssim (\alpha^2\mu)^{-1}\|w_\theta^\alpha\|_{L^\infty} \lesssim (\alpha^2\mu)^{-1}(\mu(\alpha\mu)^3)^{\frac{1}{2}}\|w_\theta^\alpha\|_{X_+^{\mu,\alpha,\theta}} = (\alpha^2\mu)^{-\frac{1}{4}}\mu^{\frac{5}{4}}\|w_\theta^\alpha\|_{X_+^{\mu,\alpha,\theta}}$$

This leaves us with a purely spatial trilinear form,

$$\int E_0(u_\theta, v_\theta, \tilde{w})dt$$

where

$$E_0(u, v, \tilde{w}) = \int A(t, x, D)u\, v\tilde{w} - uA(t, x, -D)v\, \tilde{w} + uv\, \tilde{A}(t, x, D)\tilde{w}\ dx$$

The main bound for E_0 is provided in the next lemma.

Lemma 5.3. *Let $1 \le \mu \lesssim \lambda$. Assume that ξ_θ is a Lipschitz function of x with $|\xi_\theta - \theta| \ll 1$ and that $a \in C^1 S^1_{hom}$. Then the trilinear form E_0 satisfies the fixed time estimate:*

$$\begin{aligned} |E_0(u, v, \tilde{w})| \lesssim\ & \|u\|_{L^{p_1}}\|v\|_{L^{q_1}}\|\tilde{w}\|_{L^{r_1}} \\ & + \lambda^{-1}\|(\xi_\theta \wedge D)u\|_{L^{p_2}}\|v\|_{L^{q_2}}\|(\xi_\theta \wedge D)\tilde{w}\|_{L^{r_2}} \\ & + \lambda^{-1}\|u\|_{L^{p_2}}\|(\xi_\theta \wedge D)v\|_{L^{q_2}}\|(\xi_\theta \wedge D)\tilde{w}\|_{L^{r_2}} \\ & + \mu\lambda^{-2}\|(\xi_\theta \wedge D)u\|_{L^{p_3}}\|(\xi_\theta \wedge D)v\|_{L^{q_3}}\|\tilde{w}\|_{L^{r_3}} \end{aligned} \tag{90}$$

for all indices

$$\frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{r_i} = 1, \qquad 1 \le p_i, q_i, r_i \le \infty$$

and for all functions u, v localized at frequency λ in a small angular neighbourhood of θ, respectively $-\theta$ and all w localized at frequency μ.

While any choice of L^p norms is allowed in the lemma, in order to conclude the proof of the estimate for E it suffices to use the set of indices $(2, 2, \infty)$. We apply the lemma with $u = u_\theta$, $v = v_\theta$ and $\tilde{w} = \tilde{L}w_\theta^\alpha$ as above. This yields

$$\begin{aligned} \left|\int E_0(u_\theta, v_\theta, \tilde{w})dt\right| \lesssim\ & \|u_\theta\|_{L^\infty L^2}\|v_\theta\|_{L^\infty L^2}\|\tilde{w}\|_{L^2 L^\infty} \\ & + \lambda^{-1}\|(\xi_\theta \wedge D)u_\theta\|_{L^\infty L^2}\|v_\theta\|_{L^\infty L^2}\|(\xi_\theta \wedge D)\tilde{w}\|_{L^2 L^\infty} \\ & + \lambda^{-1}\|u_\theta\|_{L^\infty L^2}\|(\xi_\theta \wedge D)v_\theta\|_{L^\infty L^2}\|(\xi_\theta \wedge D)\tilde{w}\|_{L^2 L^\infty} \\ & + \mu\lambda^{-2}\|(\xi_\theta \wedge D)u_\theta\|_{L^\infty L^2}\|(\xi_\theta \wedge D)v_\theta\|_{L^\infty L^2}\|\tilde{w}\|_{L^2 L^\infty} \end{aligned} \tag{91}$$

Due to the angular localization, the operator $(\xi_\theta \wedge D)$ yields a factor of $\mu^{-\frac{1}{2}}\lambda$ when applied to u_θ or v_θ, respectively a factor of $\alpha\mu$ when applied to $\tilde{w}$. Hence we obtain

$$\left|\int E_0(u_\theta, v_\theta, \tilde{w})dt\right| \lesssim \frac{\mu^{\frac{3}{2}}\alpha^{\frac{1}{2}}}{\alpha^2\mu}(1 + \alpha\mu^{\frac{1}{2}} + \alpha\mu^{\frac{1}{2}} + 1)\|u_\theta\|_{X_+^{\lambda,\alpha,\theta}}\|v_\theta\|_{X_-^{\lambda,\alpha,\theta}}\|w_\theta^\alpha\|_{X_+^{\mu,\alpha,\theta}}$$

which is acceptable since $\alpha^2\mu \ge 1$.

Proof of Lemma 5.3: Since the symbol a is smooth and homogeneous of order 1 with respect to ξ, we can use its representation in terms of the spherical harmonics and reduce the problem to the case when a has the form

$$a(x,\xi) = b(x)c(\xi)$$

where b is Lipschitz continuous.

We denote by ξ, respectively η the frequencies for the u_θ, respectively v_θ factors in E_0. Then ξ and η have size λ and are in a small angular neighbourhood of θ. We expand c around the line generated by ξ_θ into a linear term and a quadratic error,

$$c(\xi) = \xi(\nabla c)(\xi_\theta) + \xi B(\xi,\xi_\theta)\xi$$

where B is homogeneous of order -1 with respect to ξ and can be chosen so that

$$\xi_\theta B(\xi,\xi_\theta) = 0, \qquad B(\xi,\xi_\theta)\xi_\theta = 0$$

To see that this is possible we observe that after a rigid rotation we can assume that $\xi_\theta = e_1$. For $\xi = (1,\xi')$ with $|\xi'| \ll 1$ we write the first order Taylor polynomial with integral remainder

$$\begin{aligned} c(1,\xi') &= c(1,0) + \xi' c_{\xi'}(1,0) + \xi' B(1,\xi')\xi' \\ &= c_{\xi_1}(1,0) + \xi' c_{\xi'}(1,0) + \xi' B(1,\xi')\xi' \end{aligned}$$

where B is given by

$$B(1,\xi') = \int_0^1 (1-h)\nabla^2_{\xi'} a(1,h\xi')dh$$

This extends by homogeneity to all ξ in a small angle around θ.

We represent B as a rapidly convergent sum of terms of the form

$$\lambda^{-1}F(\xi_\theta)g(\xi)$$

where g is a scalar function which is bounded and smooth on the λ scale and F is a matrix inheriting the above property of B,

$$\xi_\theta F(\xi_\theta) = 0, \qquad F(\xi_\theta)\xi_\theta = 0 \tag{92}$$

So we have

$$c(\xi) = \xi(\nabla c)(\xi_\theta) + \lambda^{-1}\sum \xi F(\xi_\theta)\xi g(\xi)$$

Then we obtain the rapidly convergent series representation

$$\begin{aligned} c(\xi) - c(\eta) \;&=\; (\xi-\eta)(\nabla c)(\xi_\theta) + \lambda^{-1}\sum(\xi-\eta)F(\xi_\theta)\xi g(\xi) \\ &+\; \lambda^{-1}\sum \eta F(\xi_\theta)(\xi-\eta)g(\eta) \\ &+\; \lambda^{-2}\sum \eta F(\xi_\theta)\xi(\xi-\eta)h(\xi)k(\eta) \end{aligned}$$

where h and k are smooth and bounded on the λ dyadic scale.

We use this representation for the first two components in E_0. The contribution of the first term above cancels the principal part of the third component in E_0. We retain the other three terms though, therefore this yields the following rapidly convergent series representation for E_0:

$$E_0(u,v,\tilde{w}) = \int uv\tilde{w}D(b(\nabla c)(\xi_\theta))dx + \sum E_0^1 + \sum E_0^2 + \sum E_0^3$$

The first term is easily estimated since $b(\nabla c)(\xi_\theta)$ is Lipschitz continuous. The first summand has the form

$$E_0^1 = \lambda^{-1} \int F(\xi_\theta) D(Dg(D)uv)\, \tilde{w} dx$$
$$= -\lambda^{-1} \int DF(\xi_\theta) Dg(D)uv\tilde{w} + F(\xi_\theta)Dg(D)uvD\tilde{w}\ dx$$

In the first term $F(\xi_\theta)$ is Lipschitz in x and the u derivative yields a factor of λ. For the second term on the other hand we use (92) to estimate

$$|Dg(D)uF(\xi_\theta)D\tilde{w}| \lesssim |(\xi_\theta \wedge D)g(D)u||(\xi_\theta \wedge D)\tilde{w}|$$

Commuting $g(D)$ with $(\xi_\theta \wedge D)$ we get

$$|(\xi_\theta \wedge D)g(D)u| \leq |g(D)(\xi_\theta \wedge D)u| + |[g(D), \xi_\theta \wedge D]u|$$

with the commutator $[g(D), \xi_\theta \wedge D]$ bounded in all L^p spaces. Hence

$$|E_0^1| \lesssim \|u\|_{L^{p_1}}\|v\|_{L^{q_1}}\|\tilde{w}\|_{L^{r_1}} + \lambda^{-1}\|(\xi_\theta \wedge D)u\|_{L^{p_2}}\|v\|_{L^{q_2}}\|(\xi_\theta \wedge D)\tilde{w}\|_{L^{r_2}}$$

The second summand of E_0 is similar but with the roles of u and v reversed.

Finally,

$$E_0^3 = \lambda^{-2} \int F(\xi_\theta)\, D(Dh(D)u\, Dk(D)v)\, \tilde{w} dx$$
$$= -\lambda^{-2} \int DF(\xi_\theta)\, Dh(D)u\, Dk(D)v\, \tilde{w} + F(\xi_\theta)\, Dh(D)u\, Dk(D)v\, D\tilde{w}\ dx$$

where the matrix $F(\xi_\theta)$ is paired with the u and v derivatives. In the first term the two derivatives on u and v yield a λ^2 factor. In the second term we use as before (92) and commute out the $h(D)$ and $k(D)$ multipliers. We obtain

$$|E_0^3| \lesssim \|u\|_{L^{p_1}}\|v\|_{L^{q_1}}\|\tilde{w}\|_{L^{r_1}} + \mu\lambda^{-2}\|(\xi_\theta \wedge D)u\|_{L^{p_3}}\|(\xi_\theta \wedge D)v\|_{L^{q_3}}\|\tilde{w}\|_{L^{r_3}}$$

Summing up the results we get the conclusion of the Lemma.

□

Case 1, term III: This has the form

$$III = \int \sum_{\alpha > \mu^{-\frac{1}{2}}} \sum_{\theta \in O_\alpha} \phi^{+,\alpha}_{\theta,\lambda}(t,x,D)S_\lambda u\ \tilde{\phi}^{-,\alpha}_{-\theta,\lambda}(t,x,D)S_\lambda v\ \tilde{\phi}^{+,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,D)S_\mu w\, dxdt$$

In this case the summation with respect to θ is accomplished by (77), while for the α summation we simply accept a $\ln \mu$ loss. Fixing α and θ we set

$$u_\theta^\alpha = \phi^{+,\alpha}_{\theta,\lambda}(t,x,D)S_\lambda u, \quad v_\theta^\alpha = \phi^{-,\alpha}_{-\theta,\lambda}(t,x,D)S_\lambda v, \quad w_\theta^\alpha = \tilde{\phi}^{+,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,D)S_\mu w.$$

and repeat the analysis for Case 1, term II. The angular localization of u_θ^α and v_θ^α is not used in the bounds for the first four terms in (88), therefore that part of the argument rests unchanged. The same applies to the bound for the fixed time integral in (89).

It remains to consider the bound for $E(u_\theta^\alpha, v_\theta^\alpha, \tilde{w})$. The α localization angle for w_θ^α is now $\alpha\mu^{-1}\lambda$, therefore part (b) of Lemma 5.2 gives

$$\|\tilde{w}\|_{X_+} \lesssim \frac{\mu}{\alpha^2\lambda^2}\|w_\theta^\alpha\|_{X_+}$$

This is stronger than in the previous case because it gives a high frequency gain. Now we are able to use Lemma 5.3 with exponents $(3,2,6)$ to obtain

$$
\begin{aligned}
|\int E_0(u_\theta^\alpha, v_\theta^\alpha, \tilde{w})dt| \lesssim & \|u_\theta^\alpha\|_{L^2L^3}\|v_\theta^\alpha\|_{L^\infty L^2}\|\tilde{w}\|_{L^2L^6} \\
& + \lambda^{-1}\|(\xi_\theta \wedge D)u_\theta^\alpha\|_{L^2L^3}\|v_\theta^\alpha\|_{L^\infty L^2}\|(\xi_\theta \wedge D)\tilde{w}\|_{L^2L^6} \\
& + \lambda^{-1}\|u_\theta^\alpha\|_{L^2L^3}\|(\xi_\theta \wedge D)v_\theta^\alpha\|_{L^\infty L^2}\|(\xi_\theta \wedge D)\tilde{w}\|_{L^2L^6} \\
& + \mu\lambda^{-2}\|(\xi_\theta \wedge D)u_\theta^\alpha\|_{L^2L^3}\|(\xi_\theta \wedge D)v_\theta^\alpha\|_{L^\infty L^2}\|\tilde{w}\|_{L^2L^6}
\end{aligned}
$$

Due to the angular localization on the α scale for u_θ^α and v_θ^α, respectively on the $\alpha\mu^{-1}\lambda$ scale for w_θ^α, all $(\xi_\theta \wedge D)$ operators above yield $\alpha\lambda$ factors. Hence, taking advantage of the Strichartz estimates, we obtain

$$
\begin{aligned}
|\int E_0(u_\theta^\alpha, v_\theta^\alpha, \tilde{w})dt| &\lesssim \frac{\mu}{\alpha^2\lambda^2}\alpha^2\lambda\ \lambda^{\frac{5}{12}}\mu^{\frac{5}{6}}\|u_\theta^\alpha\|_{X_+^{\lambda,\alpha,\theta}}\|v_\theta^\alpha\|_{X_-^{\lambda,\alpha,\theta}}\|w_\theta^\alpha\|_{X_+^{\mu,\frac{\alpha\lambda}{\mu},\theta}} \\
&= \lambda^{-\frac{7}{12}}\mu^{\frac{11}{6}}\|u_\theta^\alpha\|_{X_+^{\lambda,\alpha,\theta}}\|v_\theta^\alpha\|_{X_-^{\lambda,\alpha,\theta}}\|w_\theta^\alpha\|_{X_+^{\mu,\frac{\alpha\lambda}{\mu},\theta}}
\end{aligned}
$$

which is satisfactory since $\lambda \gtrsim \mu$.

We conclude this case with two remarks. First, in this context the proof of Lemma 5.3 is somewhat of an overkill. In fact, it would suffice to linearize separately $a(t,x,\xi)$ and $a(t,x,\eta)$ around ξ_θ and use the fact that the symbol $a(t,x,\xi)-\tilde{a}(t,x,\xi)$ has size $\alpha^2\lambda$ at frequency λ in $H_\alpha S_\alpha(\theta)$. Secondly, the endpoint Strichartz estimate is only used here for convenience; there is some flexibility in choosing the indices.

Case 1, term IV. This has the form

$$
IV = \int \sum_{\alpha > \mu^{-\frac{1}{2}}} \sum_{\theta \in O_\alpha} \phi_{\theta,\lambda}^{+,\alpha}(t,x,D)S_\lambda u\ \tilde{\phi}_{-\theta,\lambda}^{-,\alpha}(t,x,D)S_\lambda v\ \phi_{\theta,\mu}^{+,\alpha\mu^{-1}\lambda}(t,x,D)S_\mu w\, dxdt
$$

Again the summation with respect to θ is accomplished by (77), while for the α summation we simply accept a $\ln\mu$ loss. This term is better behaved because the symbol

$$
\phi_{\theta,\lambda}^{+,\alpha}(x,\xi)\,\tilde{\phi}_{-\theta,\lambda}^{-,\alpha}(x,\eta)\,\phi_{\theta,\mu}^{+,\alpha\mu^{-1}\lambda}(x,\zeta)
$$

vanishes on $H = \{\xi + \eta + \zeta = 0\}$. Precisely, in the support of the above symbol we have

$$
|\xi| \approx \lambda,\ |\xi \wedge \xi_\theta^\alpha| \lesssim \alpha\lambda, \qquad |\eta| \approx \lambda,\ |\eta \wedge \xi_\theta^\alpha| \approx C\alpha\lambda, \qquad |\zeta| \approx \lambda,\ |\zeta \wedge \xi_\theta^\alpha| \lesssim \alpha\lambda.
$$

This leads to

$$
|(\xi + \eta + \zeta) \wedge \xi_\theta^\alpha| \approx C\alpha\lambda \tag{93}
$$

This can be taken advantage of in a direct computation in the above formula. Including the dyadic frequency localizations into the ϕ's, each term in IV has the integral representation

$$
\int \phi_{\theta,\lambda}^{+,\alpha}(t,x,\xi)\hat{u}(\xi)\ \tilde{\phi}_{-\theta,\lambda}^{-,\alpha}(t,x,\eta)\hat{v}(\eta)\ \phi_{\theta,\mu}^{+,\alpha\mu^{-1}\lambda}(t,x,\zeta)\hat{w}(\zeta)\, e^{ix(\xi+\eta+\zeta)}\, d\xi d\eta d\zeta dx dt
$$

Defining the spatial elliptic operator F with symbol

$$
f(t,x,\xi) = (\xi \wedge \xi_\theta^\alpha)^{2N}
$$

we have

$$
F(t,x,D_x)e^{ix(\xi+\eta+\zeta)} = |(\xi + \eta + \zeta) \wedge \xi_\theta^\alpha|^{2N} e^{ix(\xi+\eta+\zeta)}
$$

Hence integration by parts in the above formula leads to

$$\int \psi(t,x,\xi,\eta,\zeta)\hat{u}(\xi)\,\hat{v}(\eta)\,w(\zeta)e^{ix(\xi+\eta+\zeta)}\,d\xi d\eta d\zeta dx dt$$

where the new symbol ψ is

$$\psi(t,x,\xi,\eta,\zeta) = F^*(t,x,D_x)\left(\frac{\phi^{+,\alpha}_{\theta,\lambda}(t,x,\xi)\tilde{\phi}^{-,\alpha}_{-\theta,\lambda}(t,x,\eta)\phi^{+,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,\zeta)}{|(\xi+\eta+\zeta)\wedge\xi^\alpha_\theta|^{2N}}\right)$$

In the support of the numerator the bound (93) holds. Hence separating the variables we can represent the denominator as a rapidly convergent series with terms

$$(\alpha\lambda)^{-2N}\chi_{<\alpha\lambda}(\xi\wedge\xi^\alpha_\theta)\chi_{C\alpha\lambda}(\eta\wedge\xi^\alpha_\theta)\chi_{<\alpha\lambda}(\zeta\wedge\xi^\alpha_\theta)$$

where each of the χ's above is a unit bump function on the $\alpha\lambda$ scale. Thus they can be included in the corresponding ϕ factors. Due to the $S(g_\alpha)$ regularity of the ϕ factors, each derivative $\xi^\alpha_\theta\wedge D$ applied to them yields an α^{-1} factor. Thus ψ is represented as a rapidly convergent series of products of the form

$$(\alpha^2\lambda)^{-2N}\psi^{+,\alpha}_{\theta,\lambda}(t,x,\xi)\tilde{\psi}^{-,\alpha}_{-\theta,\lambda}(t,x,\eta)\psi^{+,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,\zeta)$$

where the ψ factors have the same support and regularity as the corresponding ϕ's. The integral above is similarly represented as a rapidly convergent series with terms of the form

$$(\alpha^2\lambda)^{-2N}\int\psi^{+,\alpha}_{\theta,\lambda}(t,x,D)S_\lambda u\,\tilde{\psi}^{-,\alpha}_{-\theta,\lambda}(t,x,D)S_\lambda v\,\psi^{+,\alpha\mu^{-1}\lambda}_{\theta,\mu}(t,x,D)S_\mu w\,dxdt$$

Since $\alpha > \mu^{-\frac{1}{2}}$, the factor in front of the above integral allows us to exchange low frequencies for high frequencies. This suffices in order to bound the last integral using Strichartz estimates.

Case 1, term V

This is similar to Case 1, term IV. This time in the support of the symbol

$$\phi^{+,\alpha}_{\theta,\lambda}(t,x,\xi)\,\tilde{\phi}^{-,\alpha}_{-\theta,\lambda}(t,x,\eta)\,\tilde{\phi}^{+,\beta}_{\theta,\mu}(t,x,\zeta)$$

we have

$$|\xi|\approx\lambda,\ |\xi\wedge\xi^\alpha_\theta|\lesssim\alpha\lambda,\qquad |\eta|\approx\lambda,\ |\eta\wedge\xi^\alpha_\theta|\approx C\alpha\lambda,\qquad |\zeta|\approx\lambda,\ |\zeta\wedge\xi^\alpha_\theta|\approx C\beta\lambda.$$

Hence

$$|(\xi+\eta+\zeta)\wedge\xi^\alpha_\theta|\approx C\alpha\lambda$$

therefore the symbol above is supported at distance $\beta\lambda$ from the diagonal H. Hence integrating by parts as in the previous case we gain arbitrary powers of $(\alpha\beta\lambda)^{-1}$. Then we can close the argument using Strichartz type estimates.

Case 2, $1 < d < \mu$. This requires only minor changes, which we describe in what follows. We still consider the five terms in the trilinear decomposition (81), but we replace the smallest localization angle $\mu^{-\frac{1}{2}}$ by $d^{\frac{1}{2}}\mu^{-\frac{1}{2}}$.

Case 2, term I. Here we need (77) to sum expressions of the form

$$I = \int\phi^{+,d^{\frac{1}{2}}\mu^{-\frac{1}{2}}}_{\theta,\lambda}(t,x,D)S_\lambda u\;\phi^{-,d^{\frac{1}{2}}\mu^{-\frac{1}{2}}}_{-\theta,\lambda}(t,x,D)S_\lambda v\;\phi^{+,d^{\frac{1}{2}}\mu^{-\frac{1}{2}}}_{\theta,\mu}(t,x,D)S_\mu w\,dx$$

over $\theta\in O_{d^{\frac{1}{2}}\mu^{-\frac{1}{2}}}$. Each term is bounded by combining the energy estimate for the first factor, the L^4L^2 bound for the second and (73) for the third.

Case 2, term II. Here we use (77) for the summation of expressions of the form

$$II = \int \phi^{+,d^{\frac{1}{2}}\mu^{-\frac{1}{2}}}_{\theta,\lambda}(t,x,D)S_\lambda u \; \phi^{-,d^{\frac{1}{2}}\mu^{-\frac{1}{2}}}_{-\theta,\lambda}(t,x,D)S_\lambda v \sum_{\alpha > d^{\frac{1}{2}}\mu^{-\frac{1}{2}}} \tilde{\phi}^{+,\alpha}_{\theta,\lambda}(t,x,D)S_\mu w dx$$

over $\theta \in O_{d^{\frac{1}{2}}\mu^{-\frac{1}{2}}}$. We use the same operator L, the same function $\tilde{w}$ and the same trilinear form E. In (88) the first, second and fourth terms are estimated in the same way, but using the L^4L^2 bound for the second factor. In the third term we lose a power of d,

$$\begin{aligned}\left|\int u_\theta(D_t - A(t,x,-D))v_\theta \tilde{w} dx\right| &\lesssim \|u_\theta\|_{L^\infty L^2}\|(D_t - A(t,x,-D))v_\theta\|_{L^2}\|\tilde{w}\|_{L^2L^\infty} \\ &\lesssim \|u_\theta\|_{X_+} d^{\frac{3}{4}}\|v_\theta\|_{X_{-,d}} \frac{1}{\alpha^2\mu}\alpha^{\frac{1}{2}}\mu^{\frac{3}{2}}\|w^\alpha_\theta\|_{X^{\mu,\alpha,\theta}_+} \\ &\lesssim \left(\frac{d}{\alpha^2\mu}\right)^{\frac{3}{4}} \mu^{\frac{5}{4}}\|u_\theta\|_{X_+}\|v_\theta\|_{X_{-,d}}\|w_\theta\|_{X^{\mu,\alpha,\theta}_+}\end{aligned}$$

But this is still acceptable due to the reduced range for α, namely $\alpha^2\mu \geq d$.

In the expression (89) there is a $d^{\frac{1}{4}}$ loss in the L^2 bound for v_θ, but this is compensated for by the previously unused $(\alpha^2\mu)^{-\frac{1}{4}}$ factor in the pointwise bound for $\tilde{w}$.

Finally, for the E_0 bounds we reuse (91) but with all the v_θ factors estimated in L^2. This produces an extra $d^{-\frac{1}{4}}$ gain. On the other hand, the angular localization for u_θ and v_θ is worse. Precisely, the operator $(\xi_\theta \wedge D)$ yields a factor of $d^{\frac{1}{2}}\mu^{-\frac{1}{2}}\lambda$ when applied to u_θ or v_θ, respectively a factor of $\alpha\mu$ when applied to $\tilde{w}$. Hence we obtain

$$\left|\int E_0(u_\theta, v_\theta, \tilde{w})dt\right| \lesssim \frac{\mu^{\frac{3}{2}}\alpha^{\frac{1}{2}}}{d^{\frac{1}{4}}\alpha^2\mu}(1+d^{\frac{1}{2}}\alpha\mu^{\frac{1}{2}}+d^{\frac{1}{2}}\alpha\mu^{\frac{1}{2}}+d)\|u^\alpha_\theta\|_{X^{\lambda,\theta,\alpha}_+}\|v^\alpha_\theta\|_{X^{\lambda,\theta,\alpha}_-}\|w^\alpha_\theta\|_{X^{\mu,\theta,\alpha}_+}$$

This is still acceptable since $\alpha^2\mu \geq d$.

Case 2, term III. Compared to the similar argument in Case 1, the following modifications are required:

(i) The third term in (88) is treated as in Case 2, term II.

(ii) In the bound for E_0, the $L^\infty L^2$ norms are replaced by L^4L^2 in all the v_θ factors.

Case 2, terms IV,V. These are identical to Case 1.

□

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