Title
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Notions and a Passivity Tool for Switched DAE Systems

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Abstract—This paper proposes notions and a tool for passivity properties of non-homogeneous switched Differential Algebraic Equation (DAE) systems and their relationships with stability and control design. Motivated by the lack of results on input-output analysis (such as passivity) for switched DAE systems and their interconnections, we propose to model non-homogeneous switched DAE systems as a class of hybrid systems, modeled here as hybrid DAE systems with linear flows. Passivity and its variations are defined for switched DAE systems and methods relying on storage functions are proposed. The main contributions of this paper are: 1) passivity and detectability concepts for switched DAE systems, 2) links of the aforementioned passivity and detectability properties to stabilization via static output-feedback. Our results are illustrated in a power system, namely, the DC-DC boost converter, whose model involves DAEs and requires feedback control.

I. INTRODUCTION

The characterization of a system behavior based on the relationship between the energy injected and dissipated by a system is known as passivity. A system that stores and dissipates energy without generating energy on its own is said to be passive. The physical interpretation of energy makes passivity an intuitive tool to assert the stability properties of any system with inputs and outputs. There is plenty of literature that documents dissipativity and passivity, from definitions, sufficient conditions for stability, to passivity based control [1]. Passivity using storage functions, which is the transpose operation. Given a function \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \), its domain of definition is denoted by \( \text{dom} \ f \), i.e., \( \text{dom} \ f := \{ x \in \mathbb{R}^m \mid f(x) \text{ is defined} \} \). The range of \( f \) is denoted by \( \text{rge} \ f \), i.e., \( \text{rge} \ f := \{ f(x) \mid x \in \text{dom} \ f \} \). The right limit of the function \( f \) is defined as \( f^{-1}(r) := \lim_{x \to 0^+} f(x + r) \) if it exists. The notation \( f^{-1}(r) \) stands for the \( r \)-level set of \( f \) on \( \text{dom} \ f \), i.e., \( f^{-1}(r) := \{ z \in \text{dom} \ f \mid f(z) = r \} \). Given two functions \( f : \mathbb{R}^m \rightarrow \mathbb{R}^n \) and \( h : \mathbb{R}^m \rightarrow \mathbb{R}^n \), \( (f(x), h(x)) \) denotes the inner product between \( f \) and \( h \) at \( x \). We denote the distance from a vector \( y \in \mathbb{R}^n \) to a closed set \( A \subset \mathbb{R}^n \) by \( |y|_A \), which is given by \( |y|_A := \inf_{x \in A} |x - y| \). Given a matrix \( P \in \mathbb{R}^{n \times n} \), the determinant of \( P \) is denoted by \( \det P \). Given \( n \in \mathbb{N} \), the matrix \( 0_n \in \mathbb{R}^{n \times n} \) denotes the zero \( n \times n \) matrix, while \( I_n \in \mathbb{R}^{n \times n} \) denotes the \( n \times n \) identity matrix.

The remainder of this paper is organized as follows. In Section III the required modeling background is presented.

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1It is important to clarify that, for a solution to exist, at each change in \( \sigma \) it is required to map the state previous to the switching instant to a point in the space defined by the algebraic conditions of the subsequent mode. These resets of the state can be computed by the so-called consistency projectors in Definition [3,4, Definition 3.7].
In Section IV, a description of hybrid DAE systems with inputs/outputs is presented, which is followed in Section V by the introduction of the passivity and stability definitions for such systems. Also Section V-B revisits the motivational Example III, where the definitions and the results in Sections V through V-B are exercised.

II. MOTIVATIONAL EXAMPLE

We consider the DC-DC boost converter shown in Figure 1(a) and model it as a switched DAE as in (1). More interestingly, using concepts of passivity, a given set-point for the voltage, and an appropriate selection of inputs and outputs, we will show that a passivity-based control law renders a set of interest asymptotically stable.

There is a fair amount of literature related to the control of DC-DC boost converters from many perspectives. The authors in [6] follow an energy-based hybrid control approach to design controllers for impulsive dynamical systems. In [5], the authors propose a Control Lyapunov Function (CLF) approach for the control of the DC-DC boost converter. Following the models therein, the converter is composed by an inductor \( L \), a capacitor \( C \), a resistor \( R \), a voltage source \( v_{cc} \), a switch \( S \), and a diode \( d \). The voltage across the inductor, diode, and capacitor are denoted as \( v_L \), \( v_D \), and \( v_C \), respectively. The current through the inductor, switch, and diode are denoted as \( i_L \), \( i_S \), and \( i_D \), respectively. In this example, we consider the model of the diode depicted in Figure 1(b) where \( \lambda \) is the forward bias voltage of the diode. Now, consider the switching signal \( \sigma : [0, \infty) \to \Sigma \), where each element in \( \Sigma \) represents a mode of operation of the boost converter.

The state conditions where each one of the modes is valid are as follows:

- **Mode 1:** (Switch is open and diode is conducting) In this mode the current through the diode is positive. This requires the voltage \( v_D \) to be larger than the threshold voltage \( \lambda > 0 \).
- **Mode 2:** (Switch is closed and diode is blocking) In this mode the voltage in the capacitor is larger or equal than the forward bias voltage, i.e., \( v_C \geq -\lambda \). In this case, the voltage across the diode is always smaller than \( \lambda \). Hence the diode is modeled as an open circuit.
- **Mode 3:** (Switch is closed and diode is blocking) When the switch is open, the voltage in the diode could become smaller than \( \lambda \), in which case the current through the diode is zero.
- **Mode 4:** (Switch is closed and diode is conducting) In this mode the voltage in the capacitor is smaller than the forward bias voltage of the diode, i.e., \( v_C \leq -\lambda \).

More precisely, given the nature of the switch and the diode, the four different modes can be described by differential-algebraic equations as follows:

\[
\begin{align*}
\text{Mode 1 (}\sigma = 1\text{)} & : \frac{d}{dt}v_{cc} = 0 \\
& \quad c \frac{d}{dt}v_C = i_D - \frac{1}{R}v_C \\
& \quad L \frac{d}{dt}i_L = v_L \\
& \quad 0 = i_L - i_S - i_D \\
& \quad 0 = i_S \\
& \quad 0 = v_{cc} - v_L - v_D - v_C \\
& \quad 0 = \frac{d}{dt}v_D = 0
\end{align*}
\]

\[
\begin{align*}
\text{Mode 2 (}\sigma = 2\text{)} & : \frac{d}{dt}v_{cc} = 0 \\
& \quad c \frac{d}{dt}v_C = i_D - \frac{1}{R}v_C \\
& \quad L \frac{d}{dt}i_L = v_L \\
& \quad 0 = i_L - i_S - i_D \\
& \quad 0 = i_D \\
& \quad 0 = v_{cc} - v_L - v_D - v_C \\
& \quad 0 = \frac{d}{dt}v_D = 0
\end{align*}
\]

\[
\begin{align*}
\text{Mode 3 (}\sigma = 3\text{)} & : \frac{d}{dt}v_{cc} = 0 \\
& \quad c \frac{d}{dt}v_C = i_D - \frac{1}{R}v_C \\
& \quad L \frac{d}{dt}i_L = v_L \\
& \quad 0 = i_L - i_S - i_D \\
& \quad 0 = i_S \\
& \quad 0 = v_{cc} - v_L - v_D - v_C \\
& \quad 0 = \frac{d}{dt}v_D = 0
\end{align*}
\]

\[
\begin{align*}
\text{Mode 4 (}\sigma = 4\text{)} & : \frac{d}{dt}v_{cc} = 0 \\
& \quad c \frac{d}{dt}v_C = i_D - \frac{1}{R}v_C \\
& \quad L \frac{d}{dt}i_L = v_L \\
& \quad 0 = i_L - i_S - i_D \\
& \quad 0 = i_D \\
& \quad 0 = v_{cc} - v_L - v_D - v_C \\
& \quad 0 = \frac{d}{dt}v_D = 0
\end{align*}
\]

The DC-DC boost converter is designed (and controlled) to deliver a desired DC voltage at its output, which is typically the voltage of the capacitor \( v_C \). The set-point for the voltage is denoted as \( u \in \mathbb{R} \) and is treated as a new input. To proceed with a CLF approach, we add an extra state \( e \) that is reset to the value of \( u \) at switching instants. The dynamics of the state \( e \) are given by \( \dot{e} = 0 \) during flows and by \( \dot{e}^+ = u + \tilde{v}_C \) at switches, where \( \tilde{v}_C \in \mathbb{R}_{\geq 0} \) is a constant such that \( \tilde{v}_C \geq v_{cc} + \lambda \). Considering the state of the system given by \( x = [v_{cc}, v_L, v_D, v_C, i_L, i_S, i_D, e]^T \), the data of the switched DAE system in (1) is given by \( \Sigma = \{1, 2, 3, 4\} \),

\[
E_x = \begin{bmatrix} \hat{E}_x & 0 \\ 0 & 1 \end{bmatrix}, \quad A_x = \begin{bmatrix} \hat{A}_x & 0 \\ 0 & 0 \end{bmatrix}, \quad B_x = \begin{bmatrix} \hat{B}_x \end{bmatrix}^T
\]

where \( \hat{E}_x, \hat{A}_x, \) and \( \hat{B}_x \) are given by the differential-algebraic equations describing each mode. We will restrict the state space of this system to the set \( \{x \in \mathbb{R}^8 : v_C \geq 0, i_L \geq 0, v_{cc} \leq v_{cc,max} \} \), where \( v_{cc,min} > 0 \).

To show that a passivity property holds for this system, consider the functions \( V_c : \mathbb{R}^2 \to \mathbb{R}_{\geq 0} \) and \( \tilde{V} : \mathbb{R}^8 \to \mathbb{R}_{\geq 0} \)

\[2\text{Notice that the voltage source } v_{cc} \text{ does not change, thus it can be either modeled as a state or a constant. We choose to model it as a state to maintain the switched DAE model structure.}\]
More precisely, we have given as in [7, Theorem 6.4.4],
\[ \tilde{\xi} \]
which describes a passivity property of the system during

\[ V_\varepsilon(e, v_{cc}) = \frac{1}{2} (e - \hat{v}_C)^2 + \frac{1}{2} \left( \frac{e^2}{v_{cc} R} - \frac{\hat{v}_C^2}{v_{cc} R} \right) \]

Notice that for each \((\xi, \sigma^*)\) such that the algebraic constraints on \(\xi\) for the mode of operation \(\sigma^*\) are met, we can always pick \(\sigma\) such that \(\langle \nabla \tilde{V}(\xi), \hat{f}(\xi, u) \rangle =: \gamma_\sigma(\xi) \leq 0\), where the equivalent vector field \(\hat{f}(\xi, u) := \Pi_u A_\sigma \tilde{\xi} + \Pi_w B_\sigma u\) is given as in [7, Theorem 6.4.4]. \(\Pi_\sigma\) is given in Definition 3.3

More precisely, we have:

- if \(\sigma = 1\), \(\langle \nabla \tilde{V}(\xi), \hat{f}(\xi, u) \rangle = (v_{cc} - e) \left( \frac{\hat{v}_C}{e} - \frac{\hat{v}_L}{e} \right) + \left( i_L - \frac{e^2}{v_{cc} R} \right) \left( \frac{v_{cc} - \hat{v}_C}{v_{cc} R} - \frac{\hat{v}_L}{v_{cc} R} \right) =: \gamma_1(\xi).
- if \(\sigma = 2\), \(\langle \nabla \tilde{V}(\xi), \hat{f}(\xi, u) \rangle = (v_{cc} - e) \left( -\frac{\hat{v}_L}{e} \right) + \left( i_L - \frac{e^2}{v_{cc} R} \right) \left( \frac{v_{cc} - \hat{\xi}}{v_{cc} R} + \frac{\hat{v}_C}{v_{cc} R} \right) =: \gamma_2(\xi).
- if \(\sigma = 3\), \(\langle \nabla \tilde{V}(\xi), \hat{f}(\xi, u) \rangle = (v_{cc} - e) \left( -\frac{\hat{v}_L}{e} \right) + \left( i_L - \frac{e^2}{v_{cc} R} \right) \left( \frac{v_{cc} - \hat{\xi}}{v_{cc} R} \right) =: \gamma_3(\xi).
- if \(\sigma = 4\), \(\langle \nabla \tilde{V}(\xi), \hat{f}(\xi, u) \rangle = (i_L - \frac{e^2}{v_{cc} R} \right) \left( \frac{v_{cc} - \hat{\xi}}{v_{cc} R} \right) =: \gamma_4(\xi).

In addition, notice by picking the output of the system as \(y = \sqrt{-\gamma_\sigma(\tilde{\xi})}\) and assigning the input \(u = -ky\), where \(0 < k \leq 1\), we have that \(\langle \nabla \tilde{V}(\xi), \hat{f}(\xi, u) \rangle \leq u_y\), which describes a passivity property of the system during the continuous regime. At switching instants, by making use of the consistency projectors in Definition 3.3, as we show in Example 5.4, we have that \(v_C^* = v_{cc}\) and \(i_L^* = i_L\) for each \(\sigma, \sigma^* \in \Sigma\). By choosing \(u = -ky\) and \(y = \hat{v}_C - e\) it can be shown that the following holds:

\[ \tilde{V}(\tilde{\xi}, v_{cc}^*, v_C^*, v_C^*, i_L^*, i_L^*, i_L^*, i_L^*, i_L^*, i_L^*, i_L^*, i_L^*, i_L^*, i_L^*, i_L^*) \leq \tilde{V}(\xi), \]

This gives \(V(\xi^+) - V(\xi) \leq 0\) and corresponds to a passive property at jumps. This motivates the development of passivity-based control techniques for such kind of switching DAE systems.

The question that remains open is whether the passivity-like property shown above can be exploited to design a static output-feedback control law that renders asymptotically stable a given set of interest. In the upcoming sections, we propose a static output-feedback control law for the switched DAE system in the previous example. Using concepts of passivity, a given set-point for the voltage \(v_{cc}\), and an appropriate selection of inputs and outputs, we will show that such control law renders a set of interest asymptotically stable. To this end, we develop a general set of notions and results for linking the passivity properties presented here with the asymptotic stability of a set of interest.

### III. Preliminaries

In this paper, we consider a class of switched DAE systems with linear flows given by (1), where \(\xi \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^p\) is the input, \(y \in \mathbb{R}^q\) is the output, for each \(\sigma \in \Sigma, h_\sigma : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^q\) is the output map, \(I : \mathbb{R}_{\geq 0} \rightarrow \Sigma\) is the switching signal, \(\Sigma\) is a finite discrete set, \(E_\sigma, A_\sigma \in \mathbb{R}^{n \times n}\), and \(B_\sigma \in \mathbb{R}^{n \times p}\). Solutions to (1a) are typically given by (right or left) continuous functions (see [8] and references therein). Definition 3.5 below introduces the notion of solution to (1a) employed here.

**Definition 3.1:** (DAE regularity [9, Definition 1-2.1]) The collection \((E_\sigma, A_\sigma)\) is regular if for each \(\sigma \in \Sigma\) the matrix pencil \(sE_\sigma - A_\sigma \in \mathbb{R}^{n \times n}\) is regular. The matrix pencil \(sE_\sigma - A_\sigma\) is called regular if there exists a constant \(s \in \mathbb{C}\) such that \(\det(sE_\sigma - A_\sigma) \neq 0\), or \(\det(sE_\sigma - A_\sigma)\) is not the zero polynomial.

To define a switched DAE system as in [4], we recall first some concepts regarding the linear subspaces where solutions to (1a) belong. Due to the algebraic constraints in (1a), the solutions to (1a) evolve within a linear subspace (or manifold) called the consistency space.

**Definition 3.2:** (Consistency set) Given \(\sigma \in \Sigma\), the consistency set for (1a) is given by

\[ \Omega_\sigma := \{ \xi_0 \in \mathbb{R}^n \mid \exists \text{ Lebesgue measurable functions } \xi : [0, \tau) \rightarrow \mathbb{R}^n, u : [0, \tau) \rightarrow \mathbb{R}^p \text{ s.t.} \]

\[ E_\sigma(t) = A_\sigma \xi(t) + B_\sigma u(t) \forall t \in [0, \tau), \xi(0) = \xi_0, \tau > 0 \}

For a linear switched DAE system as in (1a), for each \(\sigma \in \Sigma\), the consistency set is given by a linear subspace (see more in [4]).

Next, the consistency, differential, and impulse projectors are defined.

**Definition 3.3:** (Consistency, differential, and impulse projectors [7, Definition 6.4.1]) For the quasi-Weierstrass transformation in [10, Theorem 3.4] defined

\[ \Pi_\sigma := \begin{bmatrix} I_n & 0 \\ 0 & 0_{n \times n/2} \end{bmatrix} T_{\sigma}^{-1}, \]

\[ \Pi_d := \begin{bmatrix} I_n & 0 \\ 0 & 0_{n \times n/2} \end{bmatrix} S_{\sigma}, \]

\[ \Pi_i := \begin{bmatrix} 0_{n \times n/2} & I_n \\ 0 & 0_{n \times n/2} \end{bmatrix} S_{\sigma}, \]

Before introducing Definition 3.5, we define a solution to a non-homogeneous switched DAE system, consider the following assumption.

**Assumption 3.4:** For each \(\sigma \in \Sigma, \Pi_{d, \sigma} B_\sigma = 0\).

Now we can define a solution to a switched DAE system.

**Definition 3.5:** (Solution to a non-homogeneous switched DAE system) A solution pair \((\phi, u)\) to the switched DAE system (1a) that satisfies Assumption 3.4, where \(\phi = (\phi_\xi, \sigma)\), consists of a piecewise constant function \(t \mapsto \sigma(t) \in \Sigma\), a piecewise continuously differentiable function \(t \mapsto \phi_\xi(t) \in \mathcal{D}(\sigma(t))\), and \(t \mapsto u(t) \in \mathbb{R}^p\), all right continuous, such that \(E_\sigma(t) \phi_\xi(t) = A_\sigma(t) \phi_\xi(t) + B_\sigma(t) u(t)\) for almost all \(t \in dom \phi_\xi\), with \(dom \phi = dom \phi_\xi = dom \sigma = dom u\).

### IV. Switched DAE Systems With Inputs And Outputs

In this section, we introduce a class of hybrid systems that model switched DAE systems with inputs and outputs, and state-triggered jumps. A simplified version of this model was introduced in [2], [11], and [10]. We build from the...
where $C$ is the state component associated with the switched DAE system and $\sigma$ is the state component associated with the switching signal, where $\Sigma$ is a finite discrete set defined as the consistency set $\Omega_\sigma$, which is given by a linear subspace.

The state vector is given by

$$x = (\xi, \sigma) \in \mathbb{R}^n \times \Sigma,$$

where $\xi$ is the state component associated with the switched DAE system and $\sigma$ is the state component associated with the switching signal, where $\Sigma$ is a finite discrete set as defined in Section III. The input is denoted by $u = (u_c, u_d) \in \mathbb{R}^{m_c} \times \mathbb{R}^{m_d}$, and the output maps $h_{c,\sigma} : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightarrow \mathbb{R}$ and $h_{d,\sigma} : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightarrow \mathbb{R}$.

Then, following [2], a switched DAE system is given by the hybrid inclusion

$$\mathcal{H}_{\text{DAE}}^{SW}(x, u) = \begin{cases} E_\sigma \xi + B_\sigma u \\ \xi^+ \in \bigcup_{\sigma \in \varphi(x, u)} g\left((x, \xi, u)_d\right) \end{cases} (x, u) \in C$$

where

$$C := \bigcup_{\sigma \in \Sigma} (D_\sigma \cap (\mathcal{O}_\sigma \times \{\sigma\} \times \mathbb{R}^{m_c}))$$

$$D := \bigcup_{\sigma \in \Sigma} \left((D_\sigma \cap (\mathcal{O}_\sigma \times \{\sigma\} \times \mathbb{R}^{m_d})) \cup \left((\mathbb{R}^n \times \mathcal{O}_\sigma) \times \{\sigma\} \times \mathbb{R}^{m_d}\right)\right)$$

$$g(x, \sigma, u_d) := g_D(x, \sigma, u_d) \cup g_D(x, \sigma, u_d)$$

for all $(x, u) \in \mathbb{R}^n \times \Sigma$, where $\mathcal{O}_\sigma$ is the consistency sets and $\varphi(x, u)$ is the consistency projectors in Definition 3.2 and Definition 3.3, respectively, $g^1_D := (\mathbb{R}^n \times \sigma) \times \mathbb{R}^{m_d}$, and $g^2_D := D_\sigma \cap (\mathcal{O}_\sigma \times \{\sigma\} \times \mathbb{R}^{m_d})$. At jumps, the map $\varphi$ determines the changes of $\xi$, where $g_r$ models changes not related to algebraic restrictions. The set-valued map $\varphi$ determines the changes of $\sigma$. For each $\sigma \in \Sigma$, the sets $C_\sigma$ and $D_\sigma$ are subsets of $\mathbb{R}^n \times \mathcal{O}_\sigma \times \mathbb{R}^{m_c}$ and $\mathbb{R}^n \times \mathcal{O}_\sigma \times \mathbb{R}^{m_d}$, respectively, that, together with the consistency sets $\mathcal{O}_\sigma$, define where the evolution of the system according to the continuous and discrete dynamics is possible. Namely, the sets $C$ and $D$ model where the state of the system can change according to the differential algebraic equation or the difference inclusion in [5], respectively. The model in [5] is actually a hybrid DAE system [2] and its data can be defined to model switched DAE systems under a variety of switching signals. For example, a switched DAE system under arbitrary switching signals can be captured by a hybrid DAE system with inputs $u_c := u_d := u$, outputs $y_c := y_d := y$, output maps $h_{c,\sigma} := h_{d,\sigma} := h$, $C_\sigma = D_\sigma = \mathbb{R}^n \times \{\sigma\} \times \mathbb{R}^p$, $g_\sigma(\xi, u_d) = \xi$, and $\varphi(\xi, u_d) = \Sigma \setminus \{\sigma\}$, leading to

$$\mathcal{H}_{\text{DAE}}^{SW}(x, u) = \begin{cases} E_\sigma \xi + B_\sigma u \\ \xi^+ \in \bigcup_{\sigma \in \Sigma \setminus \{\sigma\}} g_\sigma(x, \xi, u_d) \end{cases} (x, u) \in C$$

A switched DAE system under dwell time switching signals can be similarly modeled; for more details and complete examples, see [2] and [11].

As in the framework in [12], we define solutions to hybrid DAEs using hybrid time domains. Therefore, during flows, solutions are parametrized by $t \in \mathbb{R}_{\geq 0}$, while at jumps they are parametrized by $j \in \mathbb{N}$.

V. PASSIVITY AND STABILITY FOR SWITCHED DAEs

Building from previous results on passivity for hybrid systems [3], [13], which employ smooth storage functions, we consider storage functions that are locally Lipschitz functions as those permit analysis of passivity properties in hybrid DAE systems. Given a locally Lipschitz function $V$, to be able to take derivatives, we employ the generalized directional gradient (in the sense of Clarke) of $V$ at $x$ in the direction $v$, which is given by $V^\circ(x, v) = \max_{\xi \in \partial V(x)} \langle \xi, v \rangle$, where $\partial V(x)$ is a closed, convex, and nonempty set equal to the convex hull of all limits of sequences $\nabla V(x_i)$ with $x_i$ converging to $x$. For more information about Clarke’s derivative, see [14].

A. Passivity Notions

For the class of hybrid DAE systems in [5], inspired by [3], we consider the following concept of passivity. Below, the functions $h_{c,\sigma}$, $h_{d,\sigma}$, and a compact set $A \subset \mathbb{R}^n \times \Sigma$ satisfy $h_{c,\sigma}(\xi, 0) = h_{d,\sigma}(\xi, 0) = 0$ for each $(\xi, \sigma) \in A$. For each $\sigma \in \Sigma$, there exist a continuous and locally Lipschitz function $V_\sigma : \mathbb{R}^n \rightarrow \mathbb{R}_\geq 0$ and functions $\omega_{c,\sigma} : \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $\omega_{d,\sigma} : \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}$, such that

$$V_\sigma^{\omega}(\xi, \mu) \leq \omega_{c,\sigma}(u_c, \xi) \wedge (\xi, \sigma, u_c) \in C,$$

$$\mu \in \bar{F}(x, u_c)$$

$$V_\sigma(\eta) - V_\sigma(\xi) \leq \omega_{d,\sigma}(u_d, \xi) \wedge (\xi, \sigma, u_d) \in D,$$

$$(\eta, \sigma) \in \bar{G}(x, u_d)$$

holds with $x = (\xi, \sigma) \in \bar{F} : \mathbb{R}^n \times \Sigma \ni (\xi, \sigma) \times \mathbb{R}^p \Rightarrow \mathbb{R}^n$ defined as $\bar{F}(x, u) = \Pi_{\sigma}^{u}_{\bar{F}} A_\sigma \xi + \Pi_{\sigma}^{u}_{\bar{F}} B_\sigma u$, $G : \mathbb{R}^n \times \Sigma \times \mathbb{R}^p \Rightarrow \mathbb{R}^n \times \Sigma$ defined as $\bar{G}(x, u_d) := \bigcup_{\sigma \in \varphi(x, u_d)} g^1_D(x, \sigma, u_d)$, and
Next, we revisit the motivational example in Section IV. We model the switched DAE system capturing the dynamics of the boost converter as a hybrid DAE system as in (5).

In Proposition 5.5 by selecting the appropriate inputs and output functions, we show that the system is flow-passive relative to a compact set of interest.

Example 5.4: (Motivational example revisited) Recalling Example II, first we show how use $V$ as a Control Lyapunov Function (CLF). Later in the example, we use the properties of the CLF $V$ and the passivity properties of the system to design a passivity-based control law using the result in Theorem 5.3. To show that $V$ can be used as a CLF, we need show that for each $\xi \in \mathbb{R}^8$ there exists $\sigma$ such that the derivative of $V$ is negative along the solution to the switched DAE system. To do so, we compute the derivative of $V$ along the equivalent vector field $f(\xi,u)$ for each $\sigma \in \Sigma$. It is easy to see that it is always possible to pick some $\gamma_s(\xi)$ such that $\gamma_s(\xi) \leq 0$, where $s \in \Sigma$. For the sake of simplicity, it is possible to use $\gamma_1^{-1}(0)$ and $\gamma_2^{-1}(0)$ as switching surfaces for modes $\{1,3\}$ and $\{2,4\}$, respectively. This is due to the algebraic restrictions on modes 3 and 4. For a more detailed derivation of this assertion, please consider $\gamma_1$ and $\gamma_2$ and apply [5, Lemma III.1] with $p_{11} = p_{22} = 1$, $v_c^e = u > v_{cc} + \lambda$, $i_s^* = u^2/(vcR)$. We have that for each $(\xi,u) \in \mathbb{R}^8 \times \mathbb{R}_{\geq 0}$, such that $u > v_{cc} + \lambda$, $\gamma_s(\xi) \leq 0$ holds for some $s \in \{1,2\}$.

Defining $\mathcal{X} := \mathbb{R}^8 \times \mathbb{R} \times \mathbb{R}_{\geq v_{cc}+\lambda}$ and the set of interest

$$\mathcal{A} := \{(\xi,\sigma) \in \mathbb{R}^8 \times \mathbb{R} | \tilde{V}(\xi) \leq 2\delta, v = \tilde{v}_C\}.$$ (10)

where $\delta \in \mathbb{R}_{>0}$ and $V_c(e,v_{cc}) = 0$ for each $(\xi,\sigma) \in \mathcal{A}$, consider the control problem of steering each solution to $\mathcal{A}$ in finite time and stay in it. To do so, let us recast the switched DAE system in Example II as in (5) with input $u = u_c = u_d$ and data in (11) and

$g_\sigma(\xi,u_d) := [v_{cc},v_L,v_D,v_C,i_L,i_S,i_D,u_d + \tilde{v}_C]^\top \forall \sigma \in \Sigma,\varphi(x,u_d)$ is given by the outer semicontinuous set-valued map returning 1 at points $x \in \{(x,u_d) \in \mathcal{X} | \sigma \in \{2,4\}, \gamma_2(\xi) \geq \epsilon, \tilde{V}(\xi) - V_c(e,v_{cc}) \geq \delta\}$, 2 at points $x \in \{(x,u_d) \in \mathcal{X} | \sigma \in \{1,3\}, \gamma_1(\xi) \geq \epsilon, \tilde{V}(\xi) - V_c(e,v_{cc}) \geq \delta\}$, 3 at points $x \in \{(x,u_d) \in \mathcal{X} | \sigma = 1,i_D = 0, v_C \geq v_{cc} - \lambda - v_{L}, \gamma_1(\xi) \leq \epsilon\}$, 4 at points $x \in \{(x,u_d) \in \mathcal{X} | \sigma = 2, v_C = \lambda, \gamma_2(\xi) \leq \epsilon\}$, where the constant $\tilde{v}_C$ is such that $\tilde{v}_C \geq v_{cc} + \lambda$ and $-1 \leq \epsilon \leq 0$. Next, by considering the storage function in (11), we choose the inputs and outputs of the switched DAE system such that the passivity properties in Definition 5.2 hold during flows and jumps. To properly define the output of the system, let us define the function $v_p$ which returns $v$ that minimizes the square of the distance between $(v_{cc},i_L)$ and the curve $\{(v_c,i_L) \in \mathbb{R} \times \mathbb{R} : 0 = i_L - v_{cc}^2/(vcR)\}$, namely, $v_p(\xi) := \arg\min_v \left\{ \frac{1}{2} (v_{cc} - v)^2 + \frac{1}{2} \left(i_L - \frac{v^2}{v_{cc}R}\right)^2 \right\}$ (12)

which is equal to

$$v \in \mathbb{R} \left\{ \begin{array}{ll}
\frac{2v^3}{v_{cc}R^2} + v \left(1 - \frac{2i_L}{v_{cc}R}\right) - v_{cc} &= 0
\end{array} \right.$$

Due to space constraints, the proof will be published elsewhere.

Notice that, the storage function $V$ can be explicitly indexed by $\sigma$. For sake of simplicity, the common storage function case is presented.
Given the state space defined for this system with positive \( v_C \) and \( i_L \), and \( v \in \mathbb{R} \), it can be shown that \( v_p \) is well defined and single valued.

**Proposition 5.5:** Consider the switched DAE system modeled as in (5) with the data above. For each \( \tilde{v}_C \geq v_{cc} + \lambda \), let us consider the storage function

\[
V(\xi) := \max \left\{ \tilde{V}(\xi) - \delta, 0 \right\}
\]

where \( \delta \in \mathbb{R}_{>0} \). Also, let us consider the function \( v_p \) defined in (12) and define the function \( d_p : \mathbb{R}^8 \times \mathbb{R} \to \mathbb{R}_{\geq0} \) as

\[
d_p(\xi) := V_e(\nu_p(\xi), v_{cc}) \max \left\{ \tilde{V}(\xi) - V_e(\nu, v_{cc}) - \delta, 0 \right\}
\]

for each \( \xi \in \mathbb{R}^8 \) such that \( \nu_C > 0 \) and \( i_L \geq 0 \). Then, the switched DAE system is flow-passive with respect to the compact set \( A \) in (10) with storage function \( V \) in (13) and the control law using the feedback \( v_C^* = -ky_c \), with \( 0 < k \leq 1 \) with the output map

\[
h_\sigma(x,u) = \begin{cases} 
\min \left\{ \tilde{v}_C - \epsilon, d_p(\tilde{v}_C, \sqrt{-\gamma_\sigma}) \right\} & \text{if } v_p \geq \tilde{v}_C \geq e \\
\min \left\{ \epsilon - \tilde{v}_C, d_p(\xi, \sqrt{-\gamma_\sigma}) \right\} & \text{if } v_p < \tilde{v}_C < e \\
0 & \text{otherwise}
\end{cases}
\]

renders the set \( A \) in (10) pre-asymptotically stable.

In Figure 2, a numerical simulation of the hybrid DAE system is presented. Notice that the level sets \( \gamma_1^{-1}(0) \) and \( \gamma_2^{-1}(0) \) are also depicted at each jump (change in \( \sigma \)). □

**REFERENCES**


