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Uniform Exponential Growth in Algebras

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The dissertation of Christopher Alan Briggs is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

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DEDICATION

To my father and mother.

EPIGRAPH

The universe never did make sense. I suspect it was built on government contract.

—Robert A. Heinlein

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ABSTRACT OF THE DISSERTATION

Uniform Exponential Growth in Algebras

by

Christopher Alan Briggs
Doctor of Philosophy in Mathematics

University of California San Diego, 2013

Professor Efim Zelmanov, Chair

We consider uniform exponential growth in algebras. We give conditions for the uniform exponential growth of descending-filtered algebras and prove that an $\mathbb{N}-$graded algebra has uniform exponential growth if it has exponential growth. We use this to prove that Golod-Shafarevich algebras and group algebras of Golod-Shafarevich groups have uniform exponential growth. We prove that the twisted Laurent extension of a free commutative polynomial algebra with respect to an endomorphism with some eigenvalue of norm not 1 must have uniform exponential growth. We prove that the group algebra of a (free abelian)-by-cyclic group has polynomially-bounded or uniform exponential group. We prove that the uniform exponential growth of the universal enveloping algebra U of a Lie algebra L implies uniform exponential growth of L, and contrariwise should L be $\mathbb{N}-$graded, and prove the same result for restricted Lie algebras. We use this to give several conditions equivalent to the uniform exponential growth of graded algebra associated to a group algebra filtered by powers of its fundamental ideal.

Chapter 1

Background

1.1 Conventions

Groups are assumed finitely-generated, and generating sets are assumed finite, unless otherwise stated. We write $G = <S>$ to indicate that the group G is generated by the finite set S. Whenever we write $S = \{x_1, \ldots, x_n\}$ we always assume $x_i^{-1} \in S$ for each i; put another way, we consider a word in S to be a word in elements of S and inverses of elements of S.

Algebras are likewise assumed to be finitely-generated (as algebras over some given field). We write $A = F\!<\!S\!>$ to denote that A is generated by (algebra operations on) the set S over the field F. Algebras are assumed to be associative unless otherwise stated.

Whenever we have an algebra A over a field F and a specified basis B for A as a vector space over F, we refer to elements of the form αu, $\alpha \in F,\ u \in B$ as the monomials of A. For an element $a = \alpha_i u_i \in A$, $\alpha_i \neq 0,\ u_i \in B$ we refer to the $\alpha_i u_i$ as the monomial summands of a. Sometimes the basis is obvious and implicit, as in the case of a group algebra FG with basis the image of G in FG under the natural embedding.

Sometimes we define an algebra by generators and relators. Given an alphabet $X = \{x_1, \ldots, x_n\}$ and a subset R of the free $F-$algebra $F < X >$, by $F < X|R >$ we mean $F<X>/id(R)$ where $id(R)$ is the ideal generated by R in $F<X>$.

Fields are assumed to be of characteristic zero unless otherwise stated.

1.2 Associative Algebras

Definition 1.2.1: An *(associative) $F-$algebra* R is a vector space over the field F together with a multiplication operation, usually indicated by juxtaposition, such that

i) $(x + y)z = xz + yz$

ii) $x(y + z) = xy + xz$

iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

for all $x, y, z \in R$, $\alpha \in F$.

Definition 1.2.2: An *ascending filtered $F-$algebra* is an $F-$algebra R with an increasing sequence of ($F-$vector) subspaces $R_0 \subseteq R_1 \subseteq \ldots$ such that

i) $R = \bigcup_{n=0}^{\infty} R_n$

ii) $R_i R_j \subseteq R_{i+j}$ for each $i,\ j \geq 0$.

Definition 1.2.3: A *descending filtered $F-$algebra* is an $F-$algebra R with a decreasing sequence of ($F-$vector) subspaces $R = R_0 \supseteq R_1 \supseteq \ldots$ such that $R_i R_j \subseteq R_{i+j}$ for each $i,\ j \geq 0$.

Definition 1.2.4: Let S be a commutative semigroup. An *S-graded algebra* is an algebra R which can be written as a direct sum $R = \bigoplus_{s\in S} R_s$ where the R_s are (vector) subspaces of R with $R_sR_{s'} \subseteq R_{ss'}$ for all $s,\ s' \in S$. We only consider the cases $S = \mathbb{N}$, $S = \mathbb{Z}_{\geq 0}$.

Definition 1.2.5: Let R be an descending filtered $F-$algebra with descending sequence $R_0 \supseteq R_1 \supseteq \ldots$. We construct an $\mathbb{N}$-graded algebra $Gr(R)$ related to R, called the *associated graded algebra*. Let $B_n = R_n/R_{n+1}$ for $n \geq 0$. As an $F-$vector space, $Gr(R) = \bigoplus_{i=0}^{\infty} B_i$. We define multiplication by $(x+B_m)(y+B_n) = xy + B_{m+n+1}$ for all $x + B_m \in B_{m+1},\ y \in B_{n+1}$. So $Gr(R)$ has the structure of a graded algebra.

Graded algebras associated to algebras with ascending filtration are similarly defined, but we will not have occasion to make use of them.

Definition 1.2.6: Given a group G and a field F we construct an algebra denoted FG and called the *group algebra* of G over F as follows: let FG be the $F-$vector space with basis G, define a product on elements of the form $1 \cdot g, 1 \cdot h \in FG$ by the product gh in G, and extend multiplication linearly.

Definition 1.2.7: Given a group algebra FG we define the map $f : FG \to F$ by $f(\sum \alpha_i g_i) = \sum \alpha_i$ where $\alpha_i \in F,\ g_i \in G$. This is called the *fundamental mapping*, and its kernel is called the emphfundamental ideal of FG; its customary symbol is Δ.

1.3 Lie Algebras

Definition 1.3.1: A *Lie algebra* over a field F is a vector space L over F with multiplication, usually indicated by a bracket $[\cdot\ ,\cdot]$ subject to:

i) Bilinearity: $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$

ii) Anticommutativity: $[x, y] = -[y, x]$

iii) The Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

where $\alpha, \beta \in F$ and $x, y, z \in L$.

Given any associative algebra A, we may define a bracket product on A by $[x, y] = xy - yx$ for all $x,\ y \in A$ where juxtaposition denotes the associative multiplication. Then

i)$[\alpha x + \beta y, z] = \alpha xz + \beta yz - \alpha zx - \beta zy = \alpha[x, z] + \beta[y, z]$

ii) $[x, y] = xy - yx = -(yx - xy) = -[y, x]$

iii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = x(yz - zy) + y(zx - xz) + z(xy - yx)$

$= xyz - xzy - yzx + zyx + yzx - yxz - zxy + xzy + zxy - zyx - xyz + yxz = 0,$

so that A under the bracket operation is a Lie algebra. We denote the Lie algebra so derived as $A^{(-)}$.

Given a Lie algebra L, choose a basis $\{e_i\}_{i \in I}$. Write $[e_i, e_j] = \sum_k \gamma_{ij}^{(k)} e_k$. Let $X = \{x_i\}_{i \in I}$ and let $F{<}X{>}$ be the free algebra on the alphabet X over F. Let $R = \{x_i x_j - x_j x_i - \sum_k \gamma_{ij}^{(k)} x_k\}_{i,j \in I}$, (the $\gamma_{i,j}^{(k)} \in F$ are called *structural constants*), $id(R)$ be the ideal of $F{<}X{>}$ generated by R, and $U = <X \mid R>$. Map $u : L \to U$ by $u(e_i) + id(R)$. Because $u([e_i, e_j]) = [u(e_i), u(e_j)]$ for all $i,\ j \in I$, u extends to a homomorphism of Lie algebras $u : L \to U^{(-)}$.

Definition 1.3.2: With L, U as in the preceding discussion, we call $U = U(L)$ the *universal enveloping algebra* of L.

The well-known Poincaré-Birkhoff-Witt theorem relates the structures of L and $U(L)$, and will be of frequent use later when we to relate the growth functions of the two.

Theorem 1.3.3: *(Poincaré-Birkhoff-Witt) Let* L, U, $u : L \to U$ *be as defined above. Impose a total order on* I*; the set* $\{x_{i_1}x_{i_2}\dots x_{i_r} \mid i_j \in I,\ i_j \le i_{j+1}\}$ *is then a basis for* U*, and the map* u *is an injection.*

1.4 Growth in Groups

Definition 1.4.1: Given a finitely-generated group G and a finite generating set $S = \{g_1^{\pm 1}, \dots, g_r^{\pm 1}\}$ of G, for each element $g \in G$ we may write $g = g_{i_1}^{n_1} \dots g_{i_k}^{n_k}$. We define the *length* of g with respect to S to be the minimal nonnegative integer k for which such an expression of g is possible (the identity is considered to be an empty word, hence of length 0). We define the *growth function* $\gamma_{G,S}(n) : \mathbb{Z}_{\ge 0} \to \mathbb{N}$ of G with respect to S to be the number of elements of G of length at most n with respect to S. We sometimes write $\gamma_{G,S}(n) = \gamma_S(n)$ if G is understood.

Definition 1.4.2: We say G has *polynomially-bounded growth (of degree* r*)* with respect to S if there is a polynomial p (of degree r) in n with $\gamma_{G,S}(n) \le p(n)$ for all n. We say G has *exponential growth* with respect to S if there is a number $c > 1$ such that $\gamma_{G,S}(n) \ge c^n$ for all n; otherwise we say G has *subexponential growth* with respect to S. If the growth of G with respect to S is subexponential but not polynomially-bounded, we say G has *intermediate growth* with respect to S.

To use growth to study groups, we desire properties of growth functions independent of the chosen generating set.

Proposition 1.4.3: *Let G be a finitely-generated group with generating set S. If $\gamma_S(n)$ is polynomially-bounded (exponential, subexponential), then $\gamma_T(n)$ is polynomially-bounded (resp. exponential, subexponential) for any other generating set T of G.*

Proof: Suppose $\gamma_S(n) \leq p(n)$ for each n and some polynomial p. Fix any other generating set T of G. Each element of T, being an element of G, must have an expression in the elements of S; let k be the maximal length of elements of T with respect to S. Then $\gamma_T(n) \leq \gamma_S(kn) \leq p(kn)$ which is again polynomial (of the same degree as p).

If G has exponential growth with respect to S we may write $lim_{n\to\infty}\gamma_S(n)^{\frac{1}{n}} > 1$ (this is equivalent to the above definition of exponential growth). Let T be another generating set of G and k the maximal length of elements of S with respect to T. We have $\gamma_T(kn) \geq \gamma_S(n)$, and so $\lim_{n\to\infty} \gamma_T(n)^{\frac{1}{n}} > 1$.

The generating-set independence of subexponential growth follows from the generating-set independence of polynomially-bounded and exponential growth. □

In light of Proposition 1.4.3, we are justified in referring to a group as being of polynomially-bounded, exponential, or subexponential growth. For brevity we sometimes say a group is polynomially-bounded, exponential, or subexponential. Some examples are in order.

Example 1.4.4 The free abelian group on r generators is polynomially-bounded (of degree r.)

Proof: Let $S = \{x_1, \ldots, x_r\}$ be a free generating set for such a group G. The number of ways of selecting n objects from a set of r, allowing repetition, is

$\binom{r+n-1}{n}$, which is bounded above by a polynomial in n (of degree r). □

Example 1.4.5: The free (nonabelian) group on $m > 1$ generators F_m is of exponential growth.

Proof: Let $F_m =< S >$, $S = \{x_1, \ldots, x_m\}$. There are $2m$ elements of F_m of length 1 with respect to S; assume inductively there are $2m \cdot (2m-1)^{n-1}$ elements of length exactly $n > 0$. Any element g of length $n+1$ with respect to S is the product of the form hx where h is of length $n-1$ with respect to S and $x \in S$. Writing $h = x_{i_1} \ldots x_{i_n}$ we must have $x \neq x_{i_n}^{-1}$, else g is of length shorter than $n+1$. So there are $2m-1$ choices for x, thus there are $(2m) \cdot (2m-1)^{n-1} \cdot (2m-1) > (2m-1)^{n+1}$ elements of length exactly $n+1$. □

The notion of growth in groups was introduced independently by A. Schwarz in 1955 and J. Milnor in 1968 ([19], [13]). In 1968 Milnor and Wolf proved that a finitely-generated solvable group cannot have intermediate growth ([14], [23]), and in 1981 Gromov proved that polynomial growth is equivalent to virtual nilpotence [8]. Milnor asked whether a group of intermediate growth exists [15]; the question was settled positively by Grigorchuk ([5], [6], [7]) in 1983.

In 1981 Gromov formulated the following concept [9]: should there exist a constant $c > 1$ such that for any generating set S of a group G we have $\gamma_S(n) \geq c^n$, we say G is of *uniform exponential growth*. For brevity we sometimes say uniform growth. In the same paper Gromov asked whether a group of exponential growth must be of uniform exponential growth. The question was answered affirmatively for hyperbolic groups by Koubi [12] in 1996, for polycyclic groups by Alperin [1] in 2002, for solvable groups by Osin [16] in 2003, for linear groups over a field of characteristic zero by Eskin, Moses and Oh [4] in 2005, and for linear groups over a field of arbitrary characteristic by Breuillard and Gelander [2] in 2008. In 2002 Wilson gave a negative answer to the question by constructing a group of exponential but not uniform exponential growth [21].

We occasionally use the following characterization of uniform exponential growth: G has uniform exponential growth if $inf_S\{lim_{n\to\infty}\gamma_S(n)^{\frac{1}{n}}\} = c > 1$. We refer to c as the *base rate of growth* of G. As demonstrated in [20][1], this limit always exists. To see so, observe that $\gamma_S(n)$ is nondecreasing and $\gamma_S(m+n) \leq \gamma_S(m)\gamma_S(n)$. For any $0 < n$ and $0 < m < n$ write $n = mk + l$ with $0 \leq k$ and $0 \leq l < m$. Then $\gamma_S(n)^{\frac{1}{n}} \leq \gamma_S(mk)^{\frac{1}{n}}\gamma_S(l)^{\frac{1}{n}} \leq \gamma_S(m)^{\frac{k}{n}}\gamma_S(l)^{\frac{1}{n}} \leq \gamma_S(m)^{\frac{1}{m}}\gamma_S(m)^{\frac{1}{n}}$. Taking $\gamma_S(m)^{\frac{1}{m}}$ close to $\liminf_{n\to\infty}\gamma_S(n)^{\frac{1}{n}}$ shows that $\lim_{n\to\infty}\gamma_S(n)^{\frac{1}{n}} \leq \liminf_{n\to\infty}\gamma_S(n)^{\frac{1}{n}}$.

The following simple example of uniform exponential growth is found in [9]. The proof is original.

Example 1.4.6: The free nonabelian group F_r on $r > 1$ generators has uniform exponential growth.

Proof: Let $F_r =< S >$. The Nielsen-Schreier theorem states that every subgroup of a free group is free, so each pair $x,\ y \in S$ generates either $\mathbb{Z}$ or F_2. If $x,\ y$ generate $\mathbb{Z}$, they commute. If each pair $x,\ y \in S$ commute then F_r is commutative. Hence some pair $x,\ y$ does not commute. They generate a free group of rank 2. A free group of rank r is always freely generated by any set of r generators which generate it [10], so in fact $\{x,\ y\}$ are free generators of F_2. Then $\gamma_{F_r,S}(n) \geq 3^n$. $\square$

1.5 Growth in Algebras

Fix a field F. We call an $F-$algebra R finitely-generated if it is generated as an algebra over F by a finite set. We write $R = F < S >$ to denote that S generates R as an $F-$algebra. We may consider the dimensions as vector spaces over F of spans of products in S. For unital R set $V_n = \sum_{i=0}^{n} FS^i$ where the empty word is 1; if R is nonunital set $V_n = \sum_{i=1}^{n} FS^i$. We define the growth function $\gamma_{R,S}$

[1]The proof in the reference is in the context of algebras, but carries over without change to groups.

of R with respect to S by $\gamma_{R,S}(n) = dim_F V^n$. Occasionally we use the notations $l_{R,S}(r) = inf_n\{r \in V^n\}$ to denote the length of an element $r \in R$ with respect to S and $\lambda_{R,S}(0) = \gamma_{R,S}(0)$, $\lambda_{R,S}(n) = \gamma_{R,S}(n) - \gamma_{R,S}(n-1)$ for $n > 1$ to denote the number of elements of R of length exactly n with respect to S.

Proposition 1.5.1: *Let $R = F < S >$ be an $F-$algebra. If γ_S is polynomially-bounded (exponential, intermediate), then for any other generating set T, γ_T is polynomially-bounded (resp. exponential, intermediate).*

Proof: We find k such that $T \subset \sum_{i=0}^{k} FS^i$. Then for any $n \geq 0$ we have $T^n \subset \sum_{i=0}^{nk} FS^i$ so that $\gamma_T(n) \leq \gamma_S(nk)$, and $\gamma_T(n)$ is polynomially-bounded (subexponential) if $\gamma_S(n)$ is polynomially-bounded (resp. subexponential). If $\lim_{n\to\infty} \gamma_S(n)^{\frac{1}{n}} = c > 1$, find instead k such that $S \subset \sum_{i=0}^{k} FT^i$. Then for $n \geq 0$ we have $S^n \subset \sum_{i=0}^{nk} FT^i$, so that $lim_{n\to\infty}\gamma_T(nk)^{\frac{1}{n}} > 1$, and $\gamma_T(n)$ is exponential with base rate of growth at least $\sqrt[k]{c} > 1$.

In light of Proposition 1.5.1 we may speak of algebras of polynomially-bounded, intermediate, exponential, and uniform exponential growth without referring to a generating set.

We discuss some examples of growth in algebras.

Smith proved that the universal enveloping algebra of any infinite-dimensional Lie algebra of subexponential growth is of intermediate growth [20]. We record Smith's proof here as it illustrates techniques of general use in investigating the growth of algebras.

Theorem 1.5.2: *If L is an infinite-dimensional Lie algebra, its universal enveloping algebra U has growth which is not polynomially-bounded. If L has subexponential growth, then U has subexponential growth. In particular, there exist algebras of intermediate growth realized as universal enveloping algebras of infinite-*

dimensional Lie algebras of subexponential growth.

Proof: The *generating function* of algebra R with respect to generating set S is defined as the series $\sum_{n=0}^{\infty} \gamma_{R,S}(n)t^n$.

The algebra R with generating set S has subexponential growth if and only if $\limsup_{n\to\infty} \lambda_S(n)^{\frac{1}{n}} \leq 1$. Necessity is clear; for sufficiency, note $\limsup_{n\to\infty} \lambda_S(n)^{\frac{1}{n}} \leq 1$ implies that the generating function $F(t) = \sum_{n=0}^{\infty} \lambda_S(n)t^n$ converges for $|t| < 1$. So then does

$$(\textstyle\sum_{m=0}^{\infty} \lambda_S(m)t^m)(\sum_{n=0}^{\infty} t^n) = \sum_{m=0}^{\infty} \lambda_S(m) \sum_{n=0}^{\infty} t^{m+n}$$

$$= \textstyle\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \lambda_S(m)t^n = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \lambda_S(m)t^n = \sum_{n=0}^{\infty} \gamma_S(n)t^n$$

which implies that $\limsup_{n\to\infty} \gamma_S(n)^{\frac{1}{n}} = \lim_{n\to\infty} \gamma_S(n)^{\frac{1}{n}} \leq 1$.

Let L be a Lie algebra and $U = U(L)$ its universal enveloping algebra. Let X be a linearly independent generating set for L as a Lie algebra over F (given any generating set, we may always delete some elements and get a linearly independent generating set). By the Poincaré-Birkhoff-Witt theorem (Theorem 1.3.3) we identify L with its image in U; then X generates U as an associative algebra. We may extend $X = \{u_1, \ldots, u_{\lambda_{L,X}(1)}\}$ to a basis $\{u_i\}_{i\in I}$ for L. The monomials $u_1^{n_1} \ldots u_r^{n_r}$, $r \in \mathbb{N}$, $n_i \geq 0$ form a basis for U.

We have $l_{U,X}(u_i) = l_{L,X}(u_i)$, and for $y = u_{i_1} \ldots u_{i_r} \in U$ we have

$$l_{U,X}(y) = \textstyle\sum_{j=1}^{r} l_{L,X}(u_{i_j}).$$

We therefore write without ambiguity $l(u_i)$, $l(y)$. Now

$$\lambda_U(n) = |\{(\mu_1, \ldots, \mu_{\gamma_L(n)}) \mid \textstyle\sum \mu_i l(u_i) = n, \mu_i \in \mathbb{Z}_{\geq 0}\}|.$$

In particular, if L is infinite-dimensional, $\lambda_U(n)$ is the number of partitions of n which is not bounded by any polynomial.

Now consider the generating function $F(t)$ of U:

$$\sum_{n=0}^{\infty} \lambda_U(n)t^n = (1 + t^{l(u_1)} + t^{2l(u_1)} + \ldots)(1 + t^{l(u_2)} + t^{2l(u_2)} + \ldots)\cdots$$

$$= \Pi_{i=1}^{\infty} \sum_{j=0}^{\infty} t^{jl(u_i)} = \Pi_{i=1}^{\infty}(1 - t^{l(u_i)})^{-1} = \Pi_{i=1}^{\infty}(1 + \frac{t^{l(u_i)}}{1-t^{l(u_i)}})$$

Thus $F(t)$ converges for $|t| < 1$ if and only if $\sum_{i=1}^{\infty} \frac{t^{l(u_i)}}{1-t^{l(u_i)}}$ converges for $|t| < 1$. If L has subexponential growth, then $\sum_{i=1}^{\infty} \lambda_L(n)t^n = \sum_{i=1}^{\infty} t^{l(u_i)}$ converges for $|t| < 1$. By comparison so does $F(t)$:

$$\lim_{i\to\infty} \frac{t^{l(u_i)}}{1-t^{l(u_i)}} \frac{1}{t^{l(u_i)}} = \lim_{i\to\infty} \frac{1}{1-t^{l(u_i)}} = 1 \text{ for } |t| < 1.$$

So U has subexponential growth. □

A group has a natural embedding in its group algebra, and under this embedding any generating set of the group is a generating set of the group algebra with identical growth function. We thereby obtain examples of algebras of polynomially-bounded, intermediate, and exponential growth. The group algebra of Wilson's group of nonuniform exponential growth [21] also has nonuniform exponential growth.

The case of uniform exponential growth is less clear. Let G be a group and FG its group algebra over field F with inclusion $\iota(G) \hookrightarrow FG$. If FG has uniform exponential growth, then in there is some $c > 1$ such that $\gamma_{FG,\iota(S)}(n) > c^n$ for each $<S>= G$ so that $\gamma_{G,S}(n) > c^n$. So G is of uniform exponential growth. If we suppose instead that G is of uniform exponential growth, FG must be of exponential growth. Because FG admits generating sets other than those which are images of generating sets of G, it is not clear that FG should be of uniform exponential growth. It is unknown to the author whether the group algebra of a group of uniform exponential growth must be of uniform exponential growth.

As with groups, the free algebra on $r > 1$ generators affords a simple example of uniform exponential growth.

Example 1.5.3: Let R be the (unital or nonunital) $F-$algebra freely generated by $\{x_1, \ldots, x_r\}$ $(r > 1)$. Then R has uniform exponential growth.

Proof: First suppose R is nonunital. Let $R = F\!<\!S\!>$, $S = \{a_1, \ldots, a_d\}$. Write $a_i = a_i' + a_i''$ where $a_i' = \sum_{j=1}^{r} \alpha_{ij} x_i$, a_i'' is a sum of monomials of degree at least two in the set $T = \{x_1, \ldots, x_r\}$, and $\alpha_{ij} \in F$ (the expression is unique). The span of the set $\{a_1', \ldots, a_d'\}$ must be $r-$dimensional over F, and as a result we may find elements $y = x_1 + y''$, $z = x_2 + z''$ in $\mathrm{span}_F\{S\}$ where y'' and z'' are sums of monomials of degree at least two in T. Now each product of length n in y, z has a unique summand of degree n in the set T. These summands are words of length n in the free generators x_1, x_2. Thus $\gamma_{R,S}(n) \geq \gamma_{R,\{x_1,x_2\}}(n) \geq 2^n$.

Now suppose R is unital, $R = F < S >$, $S = \{s_1, \ldots, s_d\}$. Write $R = F \bigoplus R_1$, R_1 the free nonunital algebra on X, and write $s_i \in S$ as $s_i = \alpha_i \cdot 1 + r_i$, $r_i \in R_1$. Let $T = \{r_i\}$. Let the base rate of growth of R_1 be c. Since T generates R_1, $\gamma_{R_1,T}(n) \geq c^n$. Since $T \subset \sum_{i=0}^{1} FS^i$, we have $\gamma_{R_1,T}(n) \leq \gamma_{R,S}(n)$, and so $\gamma_S(n) \geq c^n$. □

1.6 Uniform Growth in Filtered and Graded Algebras

We establish a sufficient condition for uniform exponential growth of filtered algebras. We also prove that a graded algebra of exponential growth is of uniform exponential growth.

Lemma 1.6.1: *Let A be a $K-$algebra with descending filtration $\{A_i\}$. Let S generate A. Then $\gamma_{A,S}(n) \geq dim_K(A/A_{n+1})$. In particular, A has uniform exponential growth if* $\limsup_n \{dim(A_n/A_{n+1})^{1/n}\} > 1$.

Proof: Suppose A is nonunital; write $A = A_1 \supset A_2 \supset \ldots$. We have

$$S^{n+1} \subset A^{n+1} \subseteq A_{n+1}$$

for each n. Hence $\sum_{i \leq n} KS^i$ must contain a basis for A modulo A_{n+1}, so $\gamma_S(n) \geq$dim(A/A_{n+1}) which is exponential.

Now suppose $A = A_0 \supset A_1 \supset \ldots$ is unital. For $x \in A$ with $x = k_x \cdot 1 + x_1$, $x_1 \in A_1$, set $f(x) = x_1$ and let $T = \{f(u) \mid u \in S\} \subset A_1$. Since span$(K \cdot 1 + T) =$ span$(K \cdot 1 + S)$, we have $\sum_{i=0}^n KT^i = \sum_{i=0}^n KS^i$ so that T generates A as a unital algebra and $\gamma_{A,T}(n) = \gamma_{A,S}(n)$. Moreover, T generates A_1 as a nonunital algebra, and A_1 is of uniform exponential growth by the nonunital case. The claim now follows from $\gamma_{A,S}(n) = \gamma_{A,T}(n) > \gamma_{A_1,T}(n)$. □

Lemma 1.6.2: *Let A be a graded $K-$algebra with either $A = \oplus_{i=1}^{\infty} A_i$ (A nonunital) or $A = K \cdot 1 + \oplus_{i=1}^{\infty} A_i$ (A unital). Let $B_n = \oplus_{i=n}^{\infty} A_i$ and set $a_n =$dim$_K(B_n/B_{n+1})$. The following are equivalent:*

(a) $\limsup(a_n^{1/n}) > 1$.

(b) A has uniform exponential growth.

(c) A has exponential growth.

Proof: That (a) implies (b) follows from Lemma 1.6.1, and that (b) implies (c) is trivial.

To see that (c) implies (a), suppose first A is nonunital. Fix a generating set S of A and $c > 1$ such that $\gamma_S(n) \geq c^n$ for each n. Find k such that $S \subseteq \oplus_{i=1}^{k} A_i$. Then $S^n \subseteq \oplus_{i=1}^{nk} A_i$ for each n. In particular, $\dim(A/B_{nk+1}) \geq \dim \sum_{i=1}^{n} KS^i \geq c^n$, so that $\dim(A/B_n)$ is exponential which implies (a).

If A is unital, fix a generating set S of A and set $T = \{f(u) \mid u \in S\}$ where f is as in the proof of Lemma 1.6.1. We have $B_1 = \sum_{i=1}^{\infty} KT^i$ is of exponential growth, so the claim follows from the nonunital case. $\square$

It is simple but useful that uniform exponential growth lifts from homomorphic images for both groups [1] and algebras; we prove this now.

Lemma 1.6.3: Let $\psi(\Gamma) = \bar{\Gamma}$ be a homomorphic image of Γ and $\phi(R) = \bar{R}$ be a homomorphic image of $F-$algebra R. For any generating sets T of Γ and S of R, $\gamma_{\Gamma,T}(n) \geq \gamma_{\bar{\Gamma},\bar{T}}(n)$ and $\gamma_{R,S}(n) \geq \gamma_{\bar{R},\bar{S}}(n)$. In particular, if $\bar{\Gamma}$ has uniform growth so must Γ; if $\bar{R}$ has (uniform) exponential growth, so must R.

Proof: Words of length at most n in $\bar{\Gamma}$ are homomorphic images of words of length at most n in Γ, so $\gamma_{\Gamma,T}(n) \geq \gamma_{\bar{\Gamma},\psi(T)}(n)$. Likewise,

$$\gamma_{R,S}(n) = dim_F \sum_{i=\epsilon}^{n} FS^i \geq dim_F \sum_{i=\epsilon}^{n} F\phi(S^i) = \gamma_{\bar{R},\bar{S}}(n)$$

where $\epsilon = 0$ or 1 depending on whether R is unital.

Remark 1.6.4: We cannot relax the grading hypothesis of Lemma 1.6.2 to descending filtration. For example, the algebra R of Theorem 2.2.2 below has uniform exponential growth but admits a descending filtration $R = R_0 \supset R_1 \supset R_2 \supset \ldots$, $R_n = F[x^i y^j t^k \mid i + j + k \geq n]$, which has an asymptotically polynomial sequence of relative dimensions.

Chapter 2

Algebras of Twisted Polynomials

Alperin proved that a polycyclic group of exponential growth has uniform exponential growth [1]. The group algebra of such a group then has exponential growth. As a first approximation we study algebras of twisted polynomials.

Unless otherwise defined, the notation set forth here will hold throughout the chapter.

Let K be a field, $X = \{x_1, \ldots, x_d\}$ an alphabet, and $K[X]$ the algebra of commuting polynomials in the x_i. Let σ be an endomorphism of the free semigroup S generated by X. We may associate σ with a matrix $A \in M_d(\mathbb{Z}_{\geq 0})$ according to the action of σ on X. The twisted polynomial algebra $R = K[X]^\sigma[t]$ is, setwise and additively, the free module generated over K by $\{x_1^{n_1} \ldots x_d^{n_d} t^k \mid n_i, k \geq 0\}$. We refer to elements of the form $\alpha x_{i_1} \ldots x_{i_r} \in R$, $\alpha \in K$ as monomials. We equip R with multiplication on monomials $(r_1 t^m)(r_2 t^n) = r_1 r_2^{\sigma^m} t^{m+n}$ and extend multiplication linearly to R. In this section we prove that if A has an eigenvalue of norm not equal to 1 then R has uniform exponential growth.

Throughout the chapter we make use of two degree functions. For a monomial $u = \Pi_{i=1}^d \alpha x_i^{n_i} t^m$ where $\alpha \in K$, $x_i \in X$, n_i and $m \in \mathbb{Z}_{\geq 0}$ we define the *homogeneous degree* to be, as usual, $m + \sum_{i=1}^d n_i$, and we define the degree $d(u)$ to be $\sum_{i=0}^d n_i$. We will only use the notation $d(u)$ to refer to the latter function, and will always say homogeneous degree to refer to the former.

To get accustomed to the computations involved in general, we display some examples with $d = 2$.

2.1 Examples

Example 2.1.1: Let $X = \{x, y\}$, $R = K[X]^{\sigma}[t]$ with σ described by the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

so that $x^{\sigma} = xy^2$, $y^{\sigma} = x^2y$.

Suppose S is a generating set for R. Let $S = \{z_1, \ldots, z_r\}$ and write $z_i = \alpha_i + \beta_i x + \gamma_i y + \delta_i t + z_i'$ where z_i' is a sum of monomials of homogeneous degree at least 2. The set $\{\alpha_i x + \beta_i y + \gamma_i t\}_{1 \leq i \leq r}$ must be three-dimensional over K. Thus we can find in $\text{span}_F S$ elements $u = x + u'$, $v = t + v'$ where u', v' are sums of monomials of homogeneous degree at least 2.

Consider words of the form $w_{\epsilon}(u, v) = u^{\epsilon_0} v u^{\epsilon_1} v \ldots v u^{\epsilon_n}$. Associate $w_{\epsilon}(u, v)$ with $\epsilon = (\epsilon_0, \epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^{n+1}$. Define an order on $\{0, 1\}^{n+1}$: if $\delta \neq \epsilon$, define $\delta < \epsilon$ whenever $sup_i\{\delta_i \neq \epsilon_i\} = j$ we have $\delta_j = 0 < 1 = \epsilon_j$. We claim that if $\delta < \epsilon$ then w_{δ} has a monomial summand of the form rt^n with $r \in K[X]$ such that for any monomial summand $r't^n$ of w_{ϵ}, $d(r) < d(r')$.

Suppose then that $\epsilon, \delta \in \{0, 1\}^{n+1}$, $\epsilon_i = \delta_i$ for $i > k$, and $\epsilon_k > \delta_k$; that is, $\epsilon_k = 1$ and $\delta_k = 0$. Computing, $w_{\epsilon}(u, v)$ has monomial summand of minimal degree

$$z_\epsilon = x^{\epsilon_0}(x^\sigma)^{\epsilon_1}\dots(x^{\sigma^n})^{\epsilon_n}t^n$$

which has degree $d_\epsilon = \epsilon_0 + \epsilon_1 3 + \cdots + \epsilon_n 3^n$.

Similarly, $w_\delta(u,v)$ has monomial summand of minimal degree

$$z_\delta = x^{\delta_0}(x^\sigma)^{\delta_1}\dots(x^{\sigma^n})^{\delta_n}t^n = x^{\delta_0}(x^\sigma)^{\delta_1}\dots(x^{\sigma^{k-1}})^{\delta_{k-1}}(x^{\sigma^{k+1}})^{\epsilon_{k+1}}\dots(x^{\sigma^n})^{\epsilon_n}$$

which has degree $d_\delta = \delta_0 + \delta_1 3 + \ldots + \delta_{k-1}3^{k-1} + \epsilon_{k+1}3^{k+1} + \ldots + \epsilon_n 3^n$.

We have $d_\epsilon - d_\delta \geq 3^k - (3^{k-1} + \ldots + 3^1 + 3^0) > 0$.

This proves the set $\{w_\epsilon(u,v) \mid \epsilon \in \{0,1\}^{n+1}\}$ is linearly independent, so we have $\gamma_T(2n+1) \geq 2^{n+1}$. □

What enables the proof above is the fact that an application of σ to any monomial triples its degree; this comes about from the sums of all columns in the matrix representation of σ being three. We might as well have used y and t instead of x and t. We introduce a subtle complication: again using $d = 2$ but now with $d(x^\sigma) \neq d(y^\sigma)$.

Example 2.1.2: As a second example, let the endomorphism σ be given by the matrix

$$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

As in the previous example, the $d(x^{\sigma^n}) \geq 2d(x^{\sigma^{n-1}})$ for each $n > 0$. Also, $d(x^{\sigma^n}) < d(r^{\sigma^n})$ for all $n > 0$ and any monomial $r \neq x$. The previous proof carries over verbatim until the point of computing the degrees of z_ϵ and z_δ. Beginning from this point, let $d_i = d(x^{\sigma^i})$.

$$d(z'_\epsilon) - d(z'_\delta) \geq d_k - \sum_{i=0}^{k-1} d_i \geq d_k - \sum_{i=0}^{k-1} 2^{-i} d_k > 0,$$

and again R has uniform growth. □

In full generality we cannot use summands of minimal degree as was done in the above examples. This is because we might have $\{d(x_i^{\sigma^n})\}$ subexponential in n for some of the $x_i \in X$. We need tools to distinguish such x_i from the ones for which $\{d(x_i^{\sigma^n})\}$ is exponential in n.

2.2 Loops

Although the language is purely algebraic, the notions and results in this section were created, as suggested by the terminology, with a multi-edge directed graph in mind. The reader may find such a visualization useful.

Throughout the section we take S to be a semigroup generated by $X = \{x_1, \ldots, x_d\}$ and σ to be an endomorphism of S.

Definition 2.2.1: For $x = \Pi x_i^{n_i} \in S$, define *the degree of* x to be $\sum_i n_i$ and write $d(x)$ for this quantity. Define *the degree in* i *of* x to be n_i and write $d_i(x)$ for this quantity.

Definition 2.2.2: Let $x_i, x_j \in X$. If there exists some $n \geq 1$ such that $d_j(x_i^{\sigma^n}) > 0$, we say there is *a path (under* σ*) of length* n *from* x_i *to* x_j.

Definition 2.2.3: For $x_i \in X$ we say x_i *is involved in a loop (under* σ*) of length* n if there is a path from x_i to x_i. Then we may find a sequence

$x_{i_0}, x_{i_1}, \ldots, x_{i_n}$ in X such that

i) $x_i = x_{i_0} = x_{i_n}$
ii) $deg_{i_j}(x^{\sigma}_{i_{j-1}}) \geq 1$ for each $j = 1, \ldots, n$

and for notation we write $(x_{i_0}, x_{i_1}, \ldots, x_{i_n})$ *is a loop (under* σ*) of length* n *involving* x_i.

In case $x_{i_j} = x_i$ only if $i_j = 0$ or $i_j = n$ we call the loop *simple* and we say the loop has *multiplicity* $\Pi^n_{j=1} deg_{i_j}(x^{\sigma}_{i_{j-1}})$. In case $n = 1$ we call the loop *trivial.* We say two loops involving x, $(x_{i_0}, \ldots, x_{i_n})$ and $(x_{j_0}, \ldots, x_{j_m})$, are equal if $n = m$ and $i_k = j_k$ for each $k = 0, \ldots, n$. We say the loops are distinct otherwise.

Examples 2.2.4: Consider the semigroup $S =< X >$, $X = \{x_1, x_2, x_3\}$, and the endomorphism σ given by $\sigma(x_1) = x_2 x_3^2$, $\sigma(x_2) = x_1 x_3$, $\sigma(x_3) = x_3^5$. Here x_1 is involved in one simple loop (x_1, x_2, x_1) of length 2 and multiplicity 1. Also x_3 is involved in one simple loop; it is trivial and of multiplicity 5.

As a second example consider again $X = \{x_1, x_2, x_3\}$. Let $x_1^{\sigma} = x_2$, $x_2^{\sigma} = x_1$, and $x_3^{\sigma} = x_1$. Then x_1 is involved in two simple loops, each of length 2 and multiplicity 1. Also x_2 is involved in exactly one simple loop which is of multiplicity 1, and x_3 is involved in no loops.

Lemma 2.2.5: *If* $x \in X$ *is involved in exactly one simple loop* $L_1 = (x_{i_0}, \ldots, x_{i_n})$*, then any other loop* L_2 *in which* x *is involved is a multiple concatenation of* L_1*; that is,* $L_2 = (x_{i_0}, \ldots, x_{i_m})$ *where* $n < m$*,* $n|m$*, and* $x_{i_k} = x_{i_l}$ *whenever* $k \equiv l (mod\ n)$*.*

Proof: Write $L_2 = (x_{j_0}, \ldots, x_{j_m})$. If $x_{i_k} \neq x_{j_k}$ for some $k < n$ then x is involved in two distinct simple loops. For every occurrence of $x_{j_l} = x$ with $l < m$ we must have $x_{j_{l+r}} = x_{i_r}$ for $0 \leq r \leq k$, or again x is involved in a simple loop distinct from L_1. □

Definition 2.2.6: For $x_i \in X$ we say x_i *is of type 1* if there exists some n such that $d_i(x_i^{\sigma^n}) > 1$. Otherwise we say x_i *is of type 2.*

Proposition 2.2.7: *For $x \in X$, x is of type 2 if and only if either x is involved in no loops or x is involved in exactly one simple loop and that loop has multiplicity 1.*

Proof: If x is involved in no simple loops, then x is involved in no loops, and it is clear that x is of type 2. If x is involved in a simple loop of multiplicity greater than 1, it is clear that x is of type 1.

If $x = x_i$ is involved in two distinct simple loops, say $(x_{i_0}, \ldots, x_{i_k})$ and $(x_{j_0}, \ldots, x_{j_l})$, set r to be a common multiple of k and l and let m be the first index such that $x_{i_m} \neq x_{j_m}$. Then $deg_i(x_i^{\sigma^r}) \geq deg_i(x_{i_m}^{\sigma^{r-m}}) + deg_i(x_{j_m}^{\sigma^{r-m}}) \geq 2$ so that x is of type 1.

Now suppose $x = x_i$ is involved in exactly one simple loop, say $(x_{i_0}, x_{i_1}, \ldots, x_{i_k})$, and that the loop has multiplicity 1. By Lemma 2.2.4, for each $j = 1, \ldots, k$, x_{i_j} is the only element of X satisfying both: there is a path from x_j to x_i and $d_j(x_{i_{j-1}}^{\sigma}) > 0$. Specifically, $d_j(x_{i_{j-1}}^{\sigma}) = 1$. For any $n > 0$, write $n \equiv r(mod\ k)$ with $0 < r \leq k$. We have $d_i(x_i^{\sigma^n}) = d_i(x_i^{\sigma^r}) \leq 1$ (with equality if and only if $r = k$). □

Recall that the L_1 norm, sometimes called the taxicab norm, is defined on $\mathbb{C}^r$ by $L_1(\mathbf{x}) = |\mathbf{x}|_1 = \sum_{i=1}^r |x_i|$ where $\mathbf{x} = (x_1, \ldots, x_r)$.

Lemma 2.2.8: *If $A \in M_r(\mathbb{C})$ has all eigenvalues of norm at most 1, then for $x \in \mathbb{C}^r$, the sequence $\{|A^n x|\}_{n\in\mathbb{N}}$ is polynomially-bounded.*

Proof: Write A in Jordan canonical form with block diagonal matrices $A_1, \ldots, A_k$ with A_i corresponding to the eigenvalue λ_i. Let $y_{i_1}, \ldots, y_{i_{n_i}}$ be a full set of generalized eigenvectors corresponding to λ_i with $Ay_{i_j} = y_{i_{j-1}} + \lambda y_{i_j}$. Let $p(n)$ be a polynomial bound on $\binom{n}{l}$ for all $l = 0, \ldots, \max_i\{n_i\}$ and $c = max_i\{\sum_{j=1}^{n_i} |y_{i_j}|_1\}$. We have

$$A^n y_{i_j} = \binom{n}{0}\lambda^n y_{i_j} + \binom{n}{1}\lambda^{n-1} y_{i_{j-1}} + \ldots + \binom{n}{j-1}\lambda^{n-j+1} y_{i_1}$$

$$|A^n y_{i_j}|_1 \leq p(n)(|\lambda|^n |y_{i_j}|_1 + \ldots + |\lambda|^{n-j+1}|y_{i_1}|_1) \leq p(n)\sum_{l=1}^{j} |y_{i_l}|_1 \leq cp(n)$$

Now choose $x \in \mathbb{C}^r$, and let $x = \sum_{i=1}^{k}\sum_{j=1}^{n_i} \alpha_{i_j} y_{i_j}$. We have

$$A^n x = \sum_{i=1}^{k}\sum_{j=1}^{n_i} \alpha_{i_j} A^n y_{i_j}$$

so that

$$|A^n x|_1 \leq \sum_{i=1}^{k}\sum_{j=1}^{n_i} |\alpha_{i_j} A^n y_{i_j}|_1 \leq \sum_{i=1}^{k}\sum_{j=1}^{n_i} |\alpha_{i_j}| cp(n) = Cp(n)$$

where $C = c\sum_{i=1}^{k}\sum_{j=1}^{n_i} |\alpha_{i_j}|$. □

Proposition 2.2.9: *If $x_i \in X$ is of type 1 or there is a path from x_i to $x_j \in X$ and x_j is of type 1, then the sequence $d(x_i^{\sigma^n})_{n\in\mathbb{N}}$ is exponential. Otherwise the sequence is polynomially-bounded.*

Proof: If x_i is of type 1, let $d_i(x_i^{\sigma^m}) > 1$. Then $d(x_i^{\sigma^{mn}}) \geq d_i(x_i^{\sigma^{mn}}) \geq 2^n$.

If there is a path, say of length k, from x_i to x_j and x_j is of type 1, let $d_j(x_j^{\sigma^m}) > 1$. Then $d(x_i^{\sigma^{k+mn}}) \geq d_j(x_i^{\sigma^{k+mn}}) \geq 2^n$.

Now write $B = \{y_1, \ldots, y_t\} = \{x_i\} \bigcup \{x \in X |$ there is a path from x_i to $x\}$ and assume each $y \in B$ is of type 2. By Proposition 2.2.7, each $y \in B$ is involved in at most one loop and each such loop is simple and of multiplicity 1. Let m be the product of the lengths of these loops (with $m = 1$ in case the product is empty) and set $\phi = \sigma^m$. Now each $y \in B$ involved in a loop under ϕ is involved in a trivial loop of multiplicity 1 under ϕ. By Lemma 2.2.4, there are no nontrivial loops under ϕ.

We claim there is some $y_i \in B$ such that $d_i(y_j^{\phi}) = 0$ for $i \neq j$. If not, we may find a sequence $y_{i_0}, y_{i_1}, \ldots$ such that $y_{i_j} \in B$, $y_{i_{j+1}} \neq y_{i_j}$, and there is a path of length 1 from $y_{i_{j+1}}$ to y_{i_j} for each $j \geq 0$. The sequence may only have t entries before some $y \in B$ must appear twice, but then we have a nontrivial loop under ϕ.

So after perhaps relabelling, there is no path from y_j to y_1 except perhaps if $j = 1$. Should we express $\phi \in End(S)$ in matrix form $A = (a_{ij}) \in M_t(\mathbb{Z}_{\geq 0})$, we have $a_{11} = 0$ or 1 and $a_{1j} = 0$ for $j = 2, \ldots, t$. By induction, and after perhaps relabelling $\{y_2, \ldots, y_t\}$, A is lower-triangular with $a_{ii} = 0$ or 1 for $i = 1, \ldots, t$. Hence A has all eigenvalues 0 or 1 and by Lemma 2.2.8 $d(y^{\sigma^n})_{n \in \mathbb{N}}$ is polynomially-bounded for each $y \in B$. □

Lemma 2.2.10: *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences. Suppose $\{a_n\}$ is exponential and $\{b_n\}$ is exponential or polynomially-bounded. Then there exists some k such that $a_{k(n+1)} + b_{k(n+1)} \geq 2(a_{kn} + b_{kn})$ for each n.*

Proof: If $\{b_n\}$ is exponential we can find some k such that $b_{k(n+1)} \geq 2b_{kn}$ and $a_{k(n+1)} \geq 2a_{kn}$ for each n. If $\{b_n\}$ is polynomially-bounded then there exists some k such that $a_{kn} \geq b_{kn}$ and $a_{k(n+1)} \geq 4a_{kn}$ for each n. Then

$$a_{k(n+1)} + b_{k(n+1)} \geq a_{k(n+1)} \geq 4a_{kn} \geq 2a_{kn} + 2b_{kn}.$$ □

2.3 Uniform Growth in Twisted Polynomial Algebras

We return to R as defined at the start of the chapter.

Lemma 2.3.1: *Suppose $u_1, \ldots, u_n \in R$ have the property that, for each i, there is a monomial summand of u_i which is of smaller degree than any monomial summand of u_j for $j > i$. Then $\{u_1, \ldots, u_n\}$ is linearly independent.*

Proof: Note that there are no nontrivial linear relations among a set of monomial elements of R all of distinct degrees. Let $r_1 t^{n_1}$ be the monomial summand of u_1 of minimal degree. Any dependence relation $\sum \alpha_i u_i$, $\alpha_i \in K$, must have $\alpha_1 = 0$, else $r_1 t^{n_1}$ is involved in a dependence relation with summands of strictly larger degree which is impossible. Inductively, having concluded $\alpha_i = 0$, we have $\alpha_{i+1} = 0$ for the same reason. □

Theorem 2.3.2: *Let K be a field, X a finite alphabet, $K[X]$ the algebra of commuting polynomials in X, σ an endomorphism of the free semigroup on X, and $R = K^\sigma[X, t]$ the twisted polynomial algebra. If the matrix A associated to σ has an eigenvalue λ of norm not equal to 1, then R has uniform exponential growth.*

Proof: R has descending filtration $R_1 \supset R_2 \supset \ldots$ where R_i is the span of homogeneous elements of R of homogeneous degree at least i. Let S be any finite generating set for R over K. Then S must contain a basis for R_1 modulo R_2, so we can find $u_1, \ldots, u_r, v \in$span(S) such that x_i is a monomial summand of u_i and t, x_j are not for $i \neq j$, and t is a monomial summand of v and no x_i is.

Since A is an integer matrix, to say it has an eigenvalue of norm not 1 is to say it has an eigenvalue of norm greater than 1. Then

$$T = \{x_i \mid \{d(x_i^{\sigma^m})\} \text{ is exponential}\}$$

is nonempty. Perhaps after reindexing, $T = \{x_1, \ldots, x_l\}$. Note that, by Proposition 2.2.9 and Lemma 2.2.10, if $l < r$ then

$$R' = \{\Pi_{i=l+1}^{r} x_i^{n_i} t^j \mid n_i,\, j \geq 0\}$$

is a subalgebra of R, and every element in the subalgebra of R generated by $T \bigcup \{t\}$ has as a factor some $x \in T$. By Proposition 2.2.9 and Lemma 2.2.10 we choose k so that $d(x^{\sigma^{k(n+1)}}) \geq 2d(x^{\sigma^{kn}})$ for each $x \in T$. We have $v^k \in \sum_{i=1}^{k} KS^i$, and t^k is the unique monomial summand of v^k of minimal homogeneous degree k.

We will consider words of the form $u^{\epsilon_0} v u^{\epsilon_1} v \ldots v u^{\epsilon_n}$ where

$$\epsilon = (\epsilon_n, \ldots, \epsilon_0) \in \{0,1\}^{n+1}$$

and u is among the u_i. We will show that the dimension of the span of such words is at least 2^{n+1}.

Define $f_\epsilon(x) = d(x^{\epsilon_0} x^{\epsilon_1 \sigma^k} \ldots x^{\epsilon_n \sigma^{kn}})$, and for each $\epsilon \in \{0,1\}^{n+1}$ choose $x_\epsilon \in T$ by the property $f_\epsilon(x_\epsilon) = \inf_{x \in T}\{f_\epsilon(x)\}$ (it may be that x_ϵ is not unique, but then any choice will do). Set y_ϵ to be the u_i with x_ϵ as a summand, and set $w_\epsilon = y_\epsilon^{\epsilon_0} v \ldots v y_\epsilon^{\epsilon_n}$. Order $\{0,1\}^{n+1}$ as in Example 2.1.1. We claim that it $\delta < \epsilon$ then w_δ has a monomial summand of the form mt^{nk} with some $x \in T$ a factor of m, and that m is of smaller degree than any such summand of w_ϵ. Linear independence then follows from Lemma 2.3.1.

Choosing such a δ and ϵ, let j be the largest number with $\delta_j \neq \epsilon_j$. Then we have $\epsilon_j = 1$, $\delta_j = 0$,and

$$d(x_\epsilon^{\epsilon_{j+1}\sigma^{k(j+1)}} x_\epsilon^{\epsilon_{j+2}\sigma^{k(j+2)}} \ldots x_\epsilon^{\epsilon_n \sigma^{kn}}) = d(x_\epsilon^{\delta_{j+1}\sigma^{k(j+1)}} x_\epsilon^{\delta_{j+2}\sigma^{k(j+2)}} \ldots x_\epsilon^{\delta_n \sigma^{kn}}).$$

Since $d(x_\epsilon^{\sigma^{ki}}) \geq 2d(x_\epsilon^{k(i-1)})$ for each i, we have

$$d(x_\epsilon^{\epsilon_j \sigma^{kj}}) > d(x_\epsilon x_\epsilon^{\sigma^k} \dots x_\epsilon^{\sigma^{k(j-1)}}) \geq d(x_\epsilon^{\delta_0} x_\epsilon^{\delta_1 \sigma^k} \dots x_\epsilon^{\delta_{j-1}\sigma^{k(j-1)}}).$$

That is, $f_\epsilon(x_\epsilon) > f_\delta(x_\epsilon)$.

The word w_ϵ has monomial summands of the form mt^{nk}. Among the m with some factor $x \in T$, the one of lowest degree has

$$d(m) = f_\epsilon(x_\epsilon) = d(x_\epsilon^{\epsilon_0} x_\epsilon^{\epsilon_1 \sigma^k} \dots x_\epsilon^{\epsilon_n \sigma^{kn}})$$

$$> d(x_\epsilon^{\delta_0} x_\epsilon^{\delta_1 \sigma^k} \dots x_\epsilon^{\delta_n \sigma^{kn}})$$

$$\geq d(x_\delta^{\delta_0} x_\delta^{\delta_1 \sigma^k} \dots x_\delta^{\delta_n \sigma^{kn}});$$

the inequailities are precisely that $f_\epsilon(x_\epsilon) > f_\delta(x_\epsilon) \geq f_\delta(x_\delta)$. So $\{w_\epsilon \mid \epsilon \in \{0,1\}^{n+1}\}$ is linearly independent by Lemma 2.3.1. $\square$

Chapter 3

The Golod-Shafarevich Constructions

In this section it is proved that Golod-Shafarevich algebras and group algebras of Golod-Shafarevich groups are of uniform exponential growth. The latter implies the result of de la Harpe [11] that Golod-Shafarevich groups are of uniform exponential growth. For the convenience of the reader, we recall the Golod-Shafarevich constructions as presented in ([3], [22], [24]).

3.1 Golod-Shafarevich Algebras: Graded Case

First we consider the graded case. Let K be a field, $X = \{x_1, \ldots, x_r\}$ an alphabet, and $K<X>$ the free algebra generated by X over K. We define the degree of a nonzero monomial element of $K<X>$ as its length with respect to the generating set X, and the degree of a nonzero homogeneous element of $K<X>$ to be the degree of its monomial summands. Let $R \subseteq K<X>$ be a subset consisting of homogeneous elements. If R has an element of degree 1 it amounts to an elimination of some $x_i \in X$, so we assume the elements of R are of degree at least two. Further, R contains finitely many elements of each degree. For each $i > 1$ set r_i to be the number of elements of R of degree i. Let I be the ideal generated by R and define $A = K<X>/I$.

For all $n \geq 0$ let $K < X >_n = \{w \in K < X > \mid \deg(w) \geq n\}$. Let $\phi : K{<}X{>} \to A$ be the natural projection and ϕ_n its restriction to $K{<}X{>}_n$. A is graded by $A = A_1 \supset A_2 \supset \ldots$ where $A_n = \phi_n(K{<}X{>}_n)$. Let $a_i = \dim_K(A_i/A_{i+1})$. Define $H_A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $H_R(t) = \sum_{n=2}^{\infty} r_n t^n$.

The Golod-Shafarevich inequality states that, as a formal series,

$(1 - rt + H_R(t))H_A(t) \geq 1.$

Definition 3.1.1: An algebra as defined above is called graded Golod-Shafarevich if there exists some $0 < t_0 < 1$ such that $H_R(t_0)$ converges and $1 - rt + H_R(t_0) < 0$.

In that case, $H_A(t_0)$ diverges; indeed, $\limsup_n \sqrt[n]{a_n} \geq t_0^{-1} > 1$. Applying Lemma 1.6.2, we have proved:

Corollary 3.1.2: *Graded Golod-Shafarevich algebras have uniform exponential growth.*

We wish now to drop the assumption that R consists of homogeneous elements.

3.2 Golod-Shafarevich Algebras: General Case

Let K be a field, $X = \{x_1, \ldots, x_r\}$ an alphabet, and define $K{<<}X{>>}$ to be the algebra of noncommuting power series in X over K. We define the degree of a nonzero monomial element of $K{<<}X{>>}$ as its length with respect to the generating set X, and the degree of a nonzero homogeneous element of $K{<<}X{>>}$ to be the degree of its monomial monomial summands. For arbitary nonzero

$f \in K{<<}X{>>}$ write $f = \sum \alpha_i u_i$ for some monomials u_i, $u_i \neq u_j$ for $i \neq j$, $\alpha_i \neq 0$ for all i. Set the degree of f to be the smallest of the degrees of the u_i. Set $K{<<}X{>>}_n = \{f \in K{<<}X{>>} \mid \deg(f) \geq n\}$. The $K{<<}X{>>}_n$ form a neighborhood base for 0 in the natural degree topology on $K{<<}X{>>}$. Let $I \subseteq K{<<}X{>>}$ be a closed (with respect to the topology) ideal and $R \subset K{<<}X{>>}$ be a set which generates I as a closed ideal. Let $r_i = |\{r \in R \mid \deg(r) = i\}|$. We assume that $r_1 = 0$, and possibly by performing linear operations on the elements of R, we assume that r_i is finite for each $i > 1$.

Let $A = K{<<}X{>>}/I$, $\phi : K{<<}X{>>} \rightarrow A$ be the natural projection, ϕ_n be the restriction of ϕ to $K{<<}X{>>}_n$, and $A_n = \phi_n(K{<<}X{>>}_n)$. A is filtered by the A_n. Let $a_i = \dim_K(A_i/A_{i+1})$.

Let $H_A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $H_R(t) = \sum_{n=2}^{\infty} r_n t^n$. The Golod-Shafarevich inequality states that, as a formal series,

$$\frac{(1-rt+H_R(t))H_A(t)}{1-t} \geq \frac{1}{1-t}.$$

Definition 3.2.1: An algebra as defined is called Golod-Shafarevich if there is some $0 < t_0 < 1$ such that $1 - rt + H_R(t_0) < 0$.

In this case, the sequence $\{a_n\}$ is exponential, and so by Lemma 1.6.1 we have:

Corollary 3.2.2: *Golod-Shafarevich algebras have uniform exponential growth.*

3.3 Golod-Shafarevich Group Algebras

Let p be a fixed prime and G a finitely-generated pro-p group, $\Omega_p(G)$ the set of normal open subgroups of G which have $p-$power index in G, and let $\mathbb{F}_p[[G]] = \varprojlim_{N\in\Omega_p(G)} F_p[G/N]$.

Definition 3.3.1: We say G is Golod-Shafarevich if $\mathbb{F}_p[[G]]$ satisfies the Golod-Shafarevich condition.

Let Δ be the kernel of the fundamental mapping $\epsilon : \mathbb{F}_p[[G]] \to \mathbb{F}_p$. Then the filtration $\mathbb{F}_p[[G]] = \Delta^0 \supset \Delta \supset \Delta^2 \supset \ldots$ has $\limsup_n(\dim(\Delta^n/\Delta^{n+1})^{1/n}) > 1$ [11], so by Lemma 1.6.1, $\mathbb{F}_p[[G]]$ is of uniform exponential growth. This implies that the group underlying the group algebra is of uniform exponential growth, and so we obtain a strengthening of de la Harpe's result [11]:

Corollary 3.3.2: *The group algebra of a Golod-Shafarevich group has uniform exponential growth.*

Chapter 4

Lie Algebras and Universal Enveloping Algebras

Smith proved that a Lie algebra has exponential growth precisely when its universal enveloping algebra does [20]. It is unknown to the author whether the same holds for uniform exponential growth. In this section it is proved that a Lie algebra L has uniform exponential growth if its universal enveloping algebra does, and that the converse holds if L is graded. We use this to give several conditions equivalent to the uniform exponential growth of a group whose group algebra is filtered by powers of its fundamental ideal.

4.1 Relationship of Growth in L and $U(L)$

Proposition 4.1.1: *Let L be a Lie algebra over field K and U the universal enveloping algebra of L. If U has uniform exponential growth, then L does as well.*

Proof: We make use of arguments from [20], also found in Theorem 1.5.2. Fix $c > 1$ so that for all generating sets S of U we have $lim_{n\to\infty}\gamma_{U,S}(n)^{1/n} \geq c$. Fix a set $S = \{u_1, \ldots, u_r\}$ generating L as a Lie algebra over K. We may assume that S is minimal in the sense of being linearly independent over K. Henceforth we write $\gamma_U(n)$ instead of $\gamma_{U,S}(n)$ and $\gamma_L(n)$ instead of $\gamma_{L,S}(n)$.

The generating function $\sum_{n=1}^{\infty}\gamma_U(n)t^n$ diverges for $t > c^{-1}$, so $\sum_{n=1}^{\infty}\lambda_U(n)t^n$ diverges for $t > c^{-1}$.

Extend S to a totally ordered basis $\{u_1, u_2, \ldots, u_{\gamma_L(1)}, u_{\gamma_L(1)+1}, \ldots\}$ for L. We write $l(u)$ to denote the length of u with respect to S. By the Poincare-Birkhoff-Witt theorem (1.3.3),

$$\lambda_U(n) = |\{(\mu_1, \ldots, \mu_{\gamma_L(n)}) \mid \sum \mu_i l(u_i) = n, \mu_i \in \mathbb{Z}_{\geq 0}\}|.$$

The generating function for λ_U is $\sum_{n=1}^{\infty}\lambda_U(n)t^n$ which converges precisely when $\sum_{n=1}^{\infty}\lambda_L(n)t^n$ converges (see the proof of Theorem 1.5.2). So $\limsup_n \lambda_L(n)^{1/n} \geq c$. Now

$$\lim_n \gamma_L(n)^{1/n} = \limsup_n \gamma_L(n)^{1/n} \geq \limsup_n \lambda_L(n)^{1/n} \geq c. \qquad \square$$

Proposition 4.1.2: *Let L be a graded Lie $K-$algebra of exponential growth. Then $U(L)$ has uniform exponential growth.*

Proof: Let S generate $L = \oplus_{i=1}^{\infty} L_i$; $U(L)$ inherits the grading from L. Since L is of exponential growth, $U(L)$ is as well ([20], presented in Theorem 1.5.2), and uniform exponential growth follows from Lemma 1.6.2. $\square$

Corollary 4.1.3: *A graded Lie algebra has uniform exponential growth if and only if its universal enveloping algebra has uniform exponential growth.*

Recall that a restricted Lie algebra is a Lie algebra over a field K of positive characteristic p and a mapping $x \mapsto x^{[p]}$ for each $x \in L$ which satisfies the following properties:

(i) $ad(x^{[p]}) = ad(x)^p, \; x \in L$

(ii) $(tx)^{[p]} = t^p x^{[p]},\ t \in K,\ x \in L$

(iii) $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i},\ x, y \in L$, where $s_i(x,y)$ is the coefficient

of t^{i-1} in the formal expression $ad(tx+y)^{p-1}(x)$.

If L is a restricted Lie algebra over field K of characteristic $p > 0$ and $X = \{u_1, \ldots, u_k\}$ is a linearly independent generating set for L, we may extend X to an ordered basis $\{u_1, \ldots, u_{\gamma_L(1)}, u_{\gamma_L(1)+1}, \ldots\}$ for L, and the Poincare-Birkhoff-Witt theorem for restricted Lie algebras states that

$$\{u_{i_1} u_{i_2} \ldots u_{i_r} \mid i_1 \leq \ldots \leq i_r\}$$

is a basis for the restricted universal enveloping algebra U of L. The growth of U with respect to X is given by

$$\lambda_U(n) = |\{(\mu_1, \ldots, \mu_{\gamma_L(n)}) \mid \sum \mu_i l(u_i) = n, 0 \leq \mu_i < p\}|.$$

Setting $c_n = |\{(\mu_1, \ldots, \mu_{\gamma_L(n)}) \mid \sum \mu_i l(u_i) = n, \mu_i \in \mathbb{Z}_{\geq 0}\}|$, $a_n = \lambda_L(n)$, and $b_n = \lambda_U(n)$ we have $b_n \leq c_n$ for each n. Thus if there is some $t_0 \in (0,1)$ such that $\sum_{n=0}^{\infty} b_n (t_0)^n$ diverges, then $\sum_{n=0}^{\infty} c_n (t_0)^n$ diverges as well. Since$\sum_{n=0}^{\infty} c_n t^n$ diverges for exactly the same t values for which $\sum_{n=0}^{\infty} a_n t^n$ does, the restricted Lie algebra L has uniform exponential growth when the restricted universal enveloping algebra U does. Likewise, should L be graded, the argument of Proposition 4.1.2 applies as well to restricted Lie algebras. We have proved:

Proposition 4.1.4: *A graded restricted Lie algebra has uniform exponential growth if and only if its restricted universal enveloping algebra has uniform exponential growth.*

Definition 4.1.5: A $p-$filtration of a group G is a sequence of subgroups $G = G_1 \supset G_2 \supset \ldots$ such that the following hold:

(i) $[G_r, G_s] \subseteq G_{r+s}$ for all r, s

(ii) For any r, $x^p \in G_{pr}$ for all $x \in G_r$

Let G be a group, K a field of characteristic $p > 0$, and Δ the fundamental ideal of the group algebra KG. Suppose KG is filtered by powers of its fundamental ideal and define $grKG = \oplus_{i=0}^{\infty}\Delta^i/\Delta^{i+1}$. Let $\Gamma_n^p G$ be the subgroup of G generated by elements of the form $[x_1, [\ldots[x_{r-1}, x_r]\ldots]]^{p^s}$ with $x_i \in G$, $rp^s \geq n$, and define $gr^p G = \oplus_{i=0}^{\infty}\Gamma_n^p/\Gamma_{n+1}^p$. Then $gr^p G$ has a restricted Lie algebra structure. Set U to be the restricted universal enveloping algebra of $gr^p G \otimes_{\mathbb{Z}} K$. Quillen proved that $U \cong grKG$ [17].

4.2 Main Result

Proposition 4.2.1: *Let* G, $grKG$, gr^pG, *and* U *be as above. The following are equivalent:*

(a) $\limsup_n(dim(\Delta^n/\Delta^{n+1})^{1/n}) > 1.$

(b) $grKG$ *has uniform exponential growth.*

(c) $U(gr^pG \otimes_{\mathbb{Z}} K)$ *has uniform exponential growth.*

(d) gr^pG *has uniform exponential growth.*

(e) $\limsup_n(dim(\Gamma_n^pG/\Gamma_{n+1}^pG)^{1/n}) > 1.$

(f) G *has a* $p-$*filtration* $\{G_n\}$ *with* $\limsup_n(|G_n/G_{n+1}|^{1/n}) > 1.$

Proof:

$(a) \iff (b)$: This follows from Lemma 1.6.2.

$(b) \iff (c)$: This follows from Quillen's theorem [17].

$(c) \iff (d)$: This follows from Proposition 4.1.4.

$(d) \iff (e)$: This follows from Lemma 1.6.2.

$(e) \iff (f)$: That (e) implies (f) is obvious, and the reverse holds since $\{\Gamma_n^p G\}$ is the fastest-descending $p-$filtration of G [17]. □

Chapter 5

Group Algebras of (Free Abelian)-by-(Infinite Cyclic) Groups

Consider groups G and H and a group homomorphism $\phi : H \to aut(G)$ from H into the automorphism group of G. Write $\phi(h) = \sigma_h$. For $g \in G$ we write g^{σ_h} to mean $\sigma_h(g)$. We may form a new group $G \rtimes H$ which is setwise $\{gh | g \in G, h \in H\}$ and has multiplication $(g_1h_1)(g_2h_2) = g_1 g_2^{\sigma_{h_1}} h_1 h_2$ where juxtaposition of the elements $g_1,\ g_2^{\sigma_{h_1}}$ of G indicates the multiplication of G and juxtaposition of the elements $h_1,\ h_2$ of H indicates the multiplication of H. This is called the extension of G by H.

We may always find such a construction in a normal subgroup and quotient of a given group. Say $N \triangleleft \Gamma$ and $\Gamma/N \cong H$. Then H acts via conjugation on N by inner automorhisms, and $\Gamma \cong N \rtimes H$.

Should N embed into a ring R which is a finite-dimensional vector space over some field (say, if N is linear), we gain the advantage of representing H as a group of matrices which renders the group structure tractable.

We study the case of the extension of a free abelian group by an infinite cyclic group.

5.1 A Reduction

Let G be the free abelian group of rank $d > 1$. Write $< t^{\pm 1} > \cong \mathbb{Z}$ and choose $\sigma \in aut(G)$ of infinite order; construct the extension of G by $< t^{\pm 1} >$ as above.

Fix a generating set $\{g_1, \ldots, g_d\}$ of G. We associate σ with a unimodular (square with integer entries and unit determinant) matrix $[a_{ij}] = A \in M_d(\mathbb{Z})$ with a_{ij} determined by $g_j^\sigma = \sum_{i=1}^d a_{ij} g_i$ (we write G additively). We write a set $\{x_1, \ldots, x_d\}$ of eigenvectors (and possibly generalized eigenvectors) for A. We associate each g_i with the i^{th} standard basis vector e_i of $\mathbb{R}^d$ and write $e_i = \sum_{j=i}^d \alpha_j^{(i)} x_j$. We put A in Jordan canonical form:

$$\text{A=}\begin{pmatrix} A_{\lambda_1} & & & \\ & A_{\lambda_2} & & \\ & & \ddots & \\ & & & A_{\lambda_m} \end{pmatrix}$$

Now x_d is a (possibly generalized) eigenvector corresponding to λ with the property: whenever $Ax_i = \sum_{j=1}^d \beta_j x_j$ and $\beta_d \neq 0$, $i = d$. Let M be the $\mathbb{Z}[\lambda, \lambda^{-1}]$-module generated by $\{\alpha_d^{(1)}, \ldots, \alpha_d^{(d)}\}$.

Because λ satisfies the characteristic polynomial f of A of degree d we may write

$$\lambda^d + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + |A| = 0$$

We may express λ^d as an integer linear combination of $1, \lambda, \ldots, \lambda^{d-1}$. Performing as many substitutions as necessary (and each substitution dropping the exponent by at least one), we can then express every positive integer power of λ as an integer linear combination of $1, \lambda, \dots, \lambda^{d-1}$. Alternatively, we may multiply through the above equation by λ^{-1} to express λ^{-1} as an integer linear combination of $1, \lambda, \dots, \lambda^{d-1}$, and then we can express every negative integer power of λ as in integer linear combination of $1, \lambda, \dots, \lambda^{d-1}$. Now M is finitely generated as a module over $\mathbb{Z}$ by

$$\{\alpha_d^{(i)}\lambda^j | 1 \leq i \leq d, 0 \leq j < d\}$$

so that the group $\bar{\Gamma} = \{(\alpha)t^k \mid \alpha \in M, k \in \mathbb{Z}\}$ with multiplication $(\alpha)t^k(\beta)t^m = (\alpha + \lambda^k\beta)t^{k+m}$ is generated by the set $\{\alpha_d^{(1)}, \dots, \alpha_d^{(d)}, t, t^{-1}\}$. In particular it is finitely-generated. Further, the map $\phi : \Gamma \to \bar{\Gamma}$ given by $\phi(\sum_{i=1}^d \alpha_i x_i)t^k = (\alpha_d)t^k$ is, by choice of x_d, a homomorphism.

Let $\bar{R} = F\bar{\Gamma}$ and extend ϕ linearly to $\psi : R \to \bar{R}$. Since uniform exponential growth lifts from homomorphic images (Lemma 1.6.3), Γ has uniform exponential growth if $\bar{\Gamma}$ does and R has uniform exponential growth if $\bar{R}$ does. Note that if $\lambda \in \mathbb{R}$, then $x = x_d$ may be chosen in $\mathbb{R}^d$, $\alpha_d^{(i)} \in \mathbb{R}$ for each i, and $\bar{\Gamma} \subset \{(\alpha)t^k | \alpha \in \mathbb{R}, k \in \mathbb{Z}\}$. We also define the (not finitely-generated) group $H = \{(z)t^k | z \in \mathbb{C}, k \in \mathbb{Z}\} \supseteq \bar{\Gamma}$ with multiplication as in $\bar{\Gamma}$.

5.2 Structure of the Reduction

We define a total order on $\bar{\Gamma}$. Let $a = (\alpha)t^k$, $b = (\beta)t^m \in \bar{\Gamma}$. We say $a < b$ if $k < m$ or if $k = m$ and $\alpha < \beta$. [1] Elements of $\bar{R}$ of the form ωg with $\omega \in F$ and $g \in \bar{\Gamma}$ are called the *monomials* of $\bar{R}$. Since $\bar{\Gamma}$ taken setwise is a basis for the vector space $\bar{R}_F$, each nonzero element $u \in \bar{R}$ can be uniquely written as a sum

[1] We are only interested in the case $\lambda \in \mathbb{R}$, in which case $\alpha, \beta \in \mathbb{R}$. The order can be defined in general using the standard order on $\mathbb{C}$, but the order does not then behave well with respect to multiplication (see Lemma 5.2.1.)

(with nonzero field coefficients) of distinct monomials; we call these the *monomial summands* or just *monomials* of u. With u so expressed we may totally order its summands according to the order on $\bar{\Gamma}$ and write $u = \sum \omega_i a_i$, $0 \neq \omega_i \in F$, $a_i < a_{i+1} \in \bar{\Gamma}$; with u so expressed we say it is in *standard form*. We denote the greatest monomial of u by $G(u)$ and the least by $L(u)$. Our convention is that $G(0) = L(0) = 0$.

Lemma 5.2.1: *If $\lambda > 0$ the order defined above respects multiplication in that if a, b, u, $v \in \bar{\Gamma}$ with $a < b$ and $u \leq v$, then $au < bv$ and $ua < vb$.*

Proof: Let $a = (\alpha)t^k$, $b = (\beta)t^m$, $u = (\gamma)t^r$, and $v = (\delta)t^s$. If $r < s$ or $k < m$ then $k + r < m + s$ and the claim follows. Otherwise $r = s$, $k = m$, $\alpha < \beta$, and $\gamma \leq \delta$, so that $\alpha + \lambda^k\gamma < \beta + \lambda^k\delta$ and $\gamma + \lambda^r\alpha < \delta + \lambda^r\beta$. Now

$$au = (\alpha + \lambda^k\gamma)t^{k+r} < (\beta + \lambda^k\delta)t^{k+r} = bv, \text{ and}$$

$$ua = (\gamma + \lambda^r\alpha)t^{r+k} < (\delta + \lambda^r\beta)t^{r+k} = vb. \qquad \square$$

Remark 5.2.2: It is false that under the hypotheses of Lemma 5.2.1 we have $ua < bv$ or $au < vb$.

Corollary 5.2.3: *If $u = \sum_{i=0}^{k} a_i$, $v = \sum_{j=0}^{m} b_j \in \bar{R}$ (with the a_i, b_j monomials in $\bar{R}$) are written in standard form then $L(uv) = a_0b_0$ and $G(uv) = a_kb_m$.*

Lemma 5.2.4: *Suppose $|\lambda| \neq 1$. If $a = (\alpha)t^k \in H$ and $k \neq 0$ then $a = c^k$ for some $c \in H$. If also $b = (\beta)t^m \in H$ then $b = dc^m$ for some $d = (\delta)t^0 \in H$.*

Proof: Write $c = (\gamma)t$. If $k > 0$ then

$$c^k = ((\gamma)t)^k = (\gamma(1 + \lambda + \ldots + \lambda^{k-1}))t^k = (\gamma(\tfrac{1-\lambda^k}{1-\lambda}))t^k,$$

so we set $\gamma = \alpha\frac{1-\lambda}{1-\lambda^k}$. If $k < 0$ let $K = -k > 0$. We have

$$((\gamma)t)^{-K} = (((\gamma)t)^{-1})^K = ((-\lambda^{-1}\gamma)t^{-1})^K = (\tfrac{-\gamma(1-\lambda^K)}{\lambda^K(1-\lambda)})t^K$$

so set $\gamma = -\alpha\frac{\lambda^K(1-\lambda)}{1-\lambda^K}$.

In either case, set $\delta = \beta - \gamma\frac{1-\lambda^m}{1-\lambda}$. Then $dc^m = (\delta)t^0c^m = b$. □

Lemma 5.2.5: *Suppose $|\lambda| \neq 1$ and let $a = (\alpha)t^k, b = (\beta)t^m \in H$. Then a and b commute if and only if $k = m = 0$ or there is some $c \in H$ such that $c^k = a$, $c^m = b$, in which case we call a and b mutual powers (of c).*

Proof: The reverse direction is clear. Suppose then $k \neq 0$ and by Lemma 5.2.4 write $a = c^k$, $b = (\gamma)c^m$. We have $ab = (\lambda^k\gamma)c^{k+m}$ and $ba = (\gamma)c^{k+m}$ which are equal precisely when $\gamma = 0$. □

Corollary 5.2.6: *If $|\lambda| \neq 1$, $0 \neq r, s \in \mathbb{Z}$, and $a, b \in H$ do not commute, neither do a^r, b^s.*

Proof: We claim that under the hypotheses a^r, b do not commute. Applying the claim twice proves the result. Write $a = (\alpha)t^k$, $b = (\beta)t^m$ and suppose a^r, b do commute. If $k = m = 0$ then a, b commute. If $k \neq 0$ or $m \neq 0$ by Lemma 5.2.5 we write $a^r = c^{rk}$, $b = c^m$ for some $c \in H$, and then $a = c^k$ commutes with b. □

Corollary 5.2.7: *Suppose $|\lambda| \neq 1$ and a, b, $c \in \bar{\Gamma}$, $a = (\alpha)t^k$, $k \neq 0$. If a, b commute and a, c commute then b, c commute.*

Proof: By Lemma 5.2.5 find u, $v \in H$ such that $a = u^k$, $b = u^m$, $a = v^k$, and $c = v^n$. Then $u = v$ and $c = u^n$ commutes with b. □

Proposition 5.2.8: *Suppose* $|\lambda| \geq 2$ *or* $|\lambda| \leq \frac{1}{2}$. *Let* $a = (\alpha)t^k$, $b = (\beta)t^m \in H$ *with* $|k| > |m|$. *Either* a, b *commute or* a, ba *freely generate a free semigroup.*

Proof: Suppose $ab \neq ba$. By Lemma 5.2.4 write $a = c^k$, $b = (\gamma)c^m$ for some $c \in H$, $\gamma \neq 0$. We consider words of the form $b^{\epsilon_0}a \ldots ab^{\epsilon_n}$, $\epsilon_i \in \{0, 1\}$. If two such words are equal but formally distinct, we may suppose they differ on the right so that

$$b^{\epsilon_0}a \ldots ab^{\epsilon_n} = b^{\delta_0}a \ldots ab^{\delta_n},\ \epsilon_n = 1,\ \delta_n = 0.$$

Let $l = \sum \epsilon_i$. Two elements of H are distinct if they have distinct exponents of t, so $m \sum \epsilon_i = m \sum \delta_i$. In particular $m = 0$ or $\sum \delta_i = l$.

Case 1: Suppose $m = 0$. If $\beta = 0$ then a, b commute. Otherwise we have

$$(\beta(\epsilon_0 + \epsilon_1\lambda^k + \ldots + \epsilon_n\lambda^{nk}))t^{nk} = (\beta(\delta_0 + \delta_1\lambda^k + \ldots + \delta_n\lambda^{nk}))t^{nk}.$$

$$\textstyle\sum_{i=0}^{n}(\epsilon_i - \delta_i)\lambda^{ik} = 0.$$

Since $|\lambda| \leq \frac{1}{2}$ or $|\lambda| \geq 2$ and each $|\epsilon_i - \delta_i| \leq 1$, all coefficients must be 0, contradicting $\epsilon_n - \delta_n = 1$.

Now suppose $m \neq 0$. We have $\sum_{i=0}^{n} \delta_i = l$ and

$$(\gamma(\epsilon_0 + \epsilon_1\lambda^{k+\epsilon_0 m} + \ldots + \epsilon_n\lambda^{nk+(\epsilon_0+\ldots+\epsilon_{n-1})m}))c^{nk+lm}$$

$$= (\gamma(\delta_0 + \delta_1\lambda^{k+\delta_0 m} + \ldots + \delta_{n-1}\lambda^{(n-1)k+(\delta_0+\ldots+\delta_{n-2})m}))c^{nk+lm}.$$

Set $r = nk + m\sum_{i=0}^{n-1} \epsilon_i$. If $k > 0$, then

$$r = nk + (l-1)m = (n-1)k + (l-1)m + k > (n-1)k + (l-1)m + m$$

$$\geq (n-1)k + m\sum_{i=0}^{n-2} \delta_i,$$

so r is the unique largest present exponent of λ. If $k < 0$, since $|k| > |m|$ and $|\sum_{i=0}^{n-2} \delta_i - \sum_{i=0}^{n-1} \epsilon_i| \leq 1$, we have

$$-k - m(\sum_{i=0}^{n-1} \epsilon_i + \sum_{i=0}^{n-2} \delta_i) > 0$$

$$r = nk + m\sum_{i=0}^{n-1} \epsilon_i < (n-1)k + m\sum_{i=0}^{n-2} \delta_i$$

so that $r < 0$ is the unique smallest present exponent of λ. Whether $k > 0$ or $k < 0$ we have an impossible equality of the form $\sum \mu_i \lambda^i = 0$ with each $|\mu_i| = 0$ or 1 and not all $\mu_i = 0$. □

5.3 Uniform Growth in Abelian-by-Cyclic Groups

An alternate proof of a result [1] of Alperin follows easily, and further gives an explicit lower bound independent of $rank(G)$ for the base rate of exponential growth.

Proposition 5.3.1: *Let G be free abelian of rank at least 2 and $\sigma \in Aut(G)$. If σ has an eigenvalue λ such that $|\lambda^k| \geq 2$ then $\Gamma = G \rtimes_\sigma \mathbb{Z}$ is of uniform exponential growth with base rate of exponential growth at least $2^{\frac{1}{3k}}$.*

Proof: Let $S = S^{-1}$ be a generating set for Γ. Let $\bar{S}$ be the image of S under ϕ as defined above; $\bar{S}$ generates $\bar{\Gamma} = \phi(\Gamma)$. Since $\bar{\Gamma}$ contains elements of the form $(\alpha)t^0$, $\alpha \neq 0$, and $(0)t$ which do not commute, $\bar{\Gamma}$ is of exponential growth by Proposition 5.2.8. Hence there exist $x, y \in \bar{S}$ which do not commute; write $x = (\alpha)t^r$, $y = (\beta)t^m$ with $r \geq |m|$. By Proposition 5.2.8, x^{2k}, $y^k x^{2k}$ are free generators of a free semigroup. Hence $\bar{\Gamma}$ is of uniform exponential growth, and so must be Γ, with base rate of growth at least $2^{\frac{1}{3k}}$. □

Lemma 5.3.2: *If $|\lambda| \neq 1$, then $\bar{R}$ has exponential growth.*

Proof: Choose an integer k such that $|\lambda|^k \geq 2$ (if $|\lambda| < 1$, $k < 0$), and set $a = (0)t^k$, $b = (\beta)t^0$. By Proposition 5.2.8 there are 2^{n+1} distinct words of length at most $2n + 1$ in the elements a, ba. So append a, b to any generating set $\bar{S}$ of $\bar{R}$, say $\bar{T} = \{a,\ b\} \bigcup S$. Then $lim_n(\gamma_{\bar{R},\bar{S}}(n))^{\frac{1}{n}} > 1$. □

Lemma 5.3.3: *Every complex eigenvalue of a unimodular matrix has norm 1.*

Proof: Let α be a complex eigenvalue of unimodular A. Then α is an algebraic integer, so $|\alpha|$ is an integer dividing $det(A) = \pm 1$. □

Lemma 5.3.4: *If $u_1, \ldots, u_n \in \bar{R}$ have linearly independent greatest (least) monomials, then the u_i are linearly independent. Note the hypothesis is true exactly when $\{G(u_i)\} = \{\omega_i a_i\}$ (resp. $\{L(u_i)\} = \{\omega_i a_i\}$) with $0 \neq \omega_i \in F$ for some distinct $a_1, \ldots, a_n \in \bar{\Gamma}$.*

Proof: Label the u_i such that $G(u_i) < G(u_{i+1})$, $1 \leq i < n$. If $\sum \omega_i u_i = 0$, then $\omega_n = 0$ because $G(u_n)$ is greater than every other monomial in the relation. Inductively, having concluded $\omega_{i+1} = 0$, we have $\omega_i = 0$. The case of linearly independent least monomials follows the same logic. □

Theorem 5.3.5: *Let G be the free abelian group on $d \geq 2$ generators, $\sigma \in Aut(G)$, $\Gamma = G \rtimes_\sigma \mathbb{Z}$, and F a field of characteristic 0. Then the group algebra $R = F\Gamma$ has polynomially-bounded growth if and only if all eigenvalues of σ have norm 1, and R has uniform exponential growth otherwise.*

Proof: If σ has all eigenvalues of norm 1, then Γ is of polynomially-bounded growth [1], hence R is as well. Otherwise σ has an eigenvalue λ of norm greater than 1. By Lemma 5.3.3, λ is real. Using this λ, construct $\bar{\Gamma}$, $\bar{R}$ as in Section 5.1.

Fix a generating set $\bar{S} = \{r_1, \ldots, r_s\}$ for $\bar{R}$. Let m be an integer such that $\lambda^m \geq 2$ (if $\lambda < -1$, m will be even). We will find a pair of elements of $\bar{R}$ whose length in $\bar{S}$ is at most 2 and whose greatest (least) terms do not commute. We take an exponent of these depending on m, and then have free generators of a free algebra whose lengths are bounded in terms of m.

If each pair r_i, r_j commute then $\bar{R}$ is commutative, violating Lemma 5.3.2, so assume r_1 and r_2 do not commute. We can then find monomial summands a_1 of r_1 and b_1 of r_2 which do not commute. We may assume a_1 has a nonzero exponent k of t.

Case 1: Suppose $k > 0$. For each $i > 1$ recursively define a_i to be the maximal summand of r_1 which does not commute with b_{i-1}, then let b_i be the maximal summand of r_2 which does not commute with a_i. Since $a_1 \leq a_2 \leq \ldots$, $b_1 \leq b_2 \leq \ldots$, and r_1, r_2 are finite $F-$linear combinations of elements of $\bar{\Gamma}$, we must have $a_l = a_{l+1}$ for some l. Let $a = a_l$ and $b = b_l$. Then a commutes with every summand of r_2 greater than b, and b commutes with every summand of r_1 greater than a.

It is possible there are summands $u > a$ of r_1 and $v > b$ of r_2 with $uv - vu \neq 0$. If so, relabel $a_1 := u$, $b_1 := v$ and repeat the process. At the termination, we have summands a of r_1 and b of r_2 with the properties:

i) a commutes with every summand of r_2 greater than b
ii) b commutes with every summand of r_1 greater than a
iii) every summand of r_1 greater than a commutes with
every summand of r_2 greater than b

We claim that without loss of generality a is the maximal summand of r_1 or b is the maximal summand of r_2. If not, let x be the maximal summand of r_1 and y the maximal summand of r_2. We have x, b commute and x, y commute, and $x > a$ means x has a positive exponent of t, so by Corollary 5.2.7 y, b commute. Also a, y commute and y, b commute, but a, b do not commute. Again applying Corollary 5.2.7, y has zero exponent of t. But then a, y commuting forces $y = \omega(0)t^0$ for some $\omega \in F$. Replace $r_2 := r_2 - \omega(r_2)^0$ (a word of length 1 in the generating set $\bar{S}$). Then b is the maximal summand of r_2.

Case 1a: Suppose a is the maximal summand of r_1. If also b is the maximal summand of r_2, then we have found two elements r_1, r_2 of length 1 in $\bar{S}$ whose greatest monomials do not commute. Suppose then b is not the maximal summand of r_2. We claim that the maximal summand of $r_1r_2 - r_2r_1$ does not commute with a. To see this, write $r_1 = \sum v_i + a$, $r_2 = \sum u_l + b + \sum U_j$ in standard form. If neither ab nor ba is the maximal summand of $r_1r_2 - r_2r_1$, then the maximal summand must be of the form v_iU_j or U_jv_i. Recall a commutes with each U_j and so is mutual powers with each U_j. If any v_iU_j commutes with a, then v_iU_j is also mutual powers with a, forcing v_i is mutual powers with a. Hence v_i and U_j commute, so that $v_iU_j - U_jv_i = 0$ is absent from $r_1r_2 - r_2r_1$.

Case 1b: Suppose b is the maximal summand of r_2 and a is not the maximal summand of r_1. We claim that the maximal summand of $r_1r_2 - r_2r_1$ does not commute with the b. Write $r_1 = \sum v_l + a + \sum V_i$, $r_2 = \sum u_j + b$ in standard form. Observe that a, b noncommuting forces if $b = (\beta)t^0$, $\beta \neq 0$, which then contradicts b, V_i commute. So b has a nonzero exponent of t. If neither ab nor ba is the maximal summand of $r_1r_2 - r_2r_1$, then the maximal summand must be of the form V_iu_j or u_jV_i. If any V_iu_j commutes with b, then V_iu_j is mutual powers with b, and so then is u_j. Then $V_iu_j - u_jV_i$ is absent from $r_1r_2 - r_2r_1$.

Now we have two elements s_1, s_2 each of length at most 2 in $\bar{S}$, s_1 has greatest monomial a, s_2 has greatest monomial b, a has a positive exponent of t, and a, b do not commute. After perhaps a harmless multiplication by a field element, we write $a = (\alpha)t^k$, $b = (\beta)t^l$. If $|k| \geq |l|$, set $x = a$ and $y = b$, otherwise set $x = b$ and $y = a$ and switch the labels s_1 and s_2. By Corollary 5.2.3 x^{2m} is the greatest monomial of s_1^{2m} and y^m is the greatest monomial of s_2^m, by Corollary 5.2.6 x^{2m} and y^m do not commute, and so by Proposition 5.2.8 x^{2m} and $y^m x^{2m}$ freely generate a free algebra. By Lemma 5.3.4, s_1^{2m} and s_2^m freely generate a free algebra. The word $(s_2^m)^{\epsilon_0} s_1^{2m} \dots s_1^{2m} (s_2^m)^{\epsilon_n}$, $\epsilon_i \in \{0,1\}$ has length at most $2m(n+1) + 4m(n) \leq 8mn$ in $\bar{S}$, and so $\inf_S\{\lim_n(\gamma_{\bar{R},S}(n)^{\frac{1}{n}})\} \geq 2^{\frac{1}{8m}}$.

Case 2: Suppose $k < 0$. We proceed as in Case 1. For $i > 1$ let a_i be the least summand of r_1 which does not commute with b_{i-1}, then let b_i be the least summand of r_2 which does not commute with a_i. When the iteration stabilizes at $a_l = a_{l+1}$, set $a = a_l$ and $b = b_l$, and check for summands $u < a$ of r_1, $v < b$ of r_2 which do not commute. If they exist, replace $a_1 := u$ and $b_1 := v$ and repeat the iteration. At the termination, we have summands a of r_1 and b of r_2 with the properties:

i) a commutes with every summand of r_2 less than b
ii) b commutes with every summand of r_1 less than a
iii) every summand of r_1 less than a commutes with
every summand of r_2 less than b

We claim that without loss of generality a is the minimal summand of r_1 or b is the minimal summand of r_2. If not, let x be the minimal summand of r_1 and y the minimal summand of r_2 respectively. We have x, b commute and x, y, commute, and $x < a$ means x has a negative exponent of t, so by Corollary 5.2.7 y, b commute. Also a, y commute and y, b commute, but a, b do not commute. Again applying Corollary 5.2.7, y has zero exponent of t. But then a, y commuting forces $y = \omega(0)t^0$ for some $\omega \in F$. Replace $r_2 := r_2 - \omega(r_2)^0$ (a word of length 1 in the generating set $\bar{S}$). Then b is the minimal summand of r_2.

Case 2a: Suppose a is the minimal summand of r_1. If also b is the minimal summand of r_2, then we have found our two elements r_1, r_2 of length 1 in $\bar{S}$ whose least monomials do not commute. Otherwise b is not the minimal summand of r_2. We claim that the minimal summand of $r_1r_2 - r_2r_1$ does not commute with a. To see this, write r_1, r_2 in standard form: $r_1 = a + \sum v_i$, $r_2 = \sum u_j + b + \sum U_l$. If neither ab nor ba is the minimal summand of $r_1r_2 - r_2r_1$, then the minimal summand must be of the form v_iu_j or u_jv_i. Recall a commutes with each u_j and so is mutual powers with each u_j. If any v_iu_j commutes with a, then v_iu_j is also mutual powers with a, forcing v_i is mutual powers with a. Hence v_i and u_j commute, so that $v_iu_j - u_jv_i = 0$ is absent from $r_1r_2 - r_2r_1$.

Case 2b: Suppose b is the minimal summand of r_2 and a is not the minimal summand of r_1. Write $r_1 = \sum v_l + a + \sum V_i$, $r_2 = b + \sum u_j$ in standard form. If neither ab nor ba is the minimal summand of $r_1r_2 - r_2r_1$, then the minimal summand must be of the form v_iu_j or u_jv_i. If any v_iu_j commutes with b, then v_iu_j is mutual powers with b, and so then is u_j. Then $v_iu_j - u_jv_i$ is absent from $r_1r_2 - r_2r_1$.

Now we have two elements s_1, s_2 each of length at most 2 in $\bar{S}$, s_1 has least monomial a, s_2 has least monomial b, a has a negative exponent of t, and a, b do not commute. After perhaps a harmless multiplication by a field element, we write $a = (\alpha)t^k$, $b = (\beta)t^l$. If $|k| \geq |l|$, set $x = a$ and $y = b$, otherwise set $x = b$ and $y = a$ and switch the labels s_1 and s_2. By Corollary 5.2.3 x^{2m} is the least monomial of s_1^{2m} and y^m is the least monomial of s_2^m, by Corollary 5.2.6 x^{2m} and y^m do not commute, and so by Proposition 5.2.8 x^{2m}, $y^m x^{2m}$ freely generate a free algebra. By Lemma 5.3.4, s_1^{2m} and s_2^m freely generate a free algebra. The word $(s_2^m)^{\epsilon_0} s_1^{2m} \ldots s_1^{2m} (s_2^m)^{\epsilon_n}$, $\epsilon_i \in \{0, 1\}$ has length at most $2m(n+1) + 4m(n) \leq 8mn$ in $\bar{S}$, and so $\inf_S\{lim_n(\gamma_{\bar{R},S}(n)^{\frac{1}{n}})\} \geq 2^{\frac{1}{8m}}$. □

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