Decentralized Optimal Load Scheduling using Extremum Seeking-Based Optimization

Maojiao Ye, Guoqiang Hu and Costas J. Spanos

***Abstract*— In this paper, we address an optimal electricity load scheduling problem for consumers in industrial parks. The industrial consumers are regarded as price-anticipating users since they have large electricity demand. A non-model based online distributed optimization scheme is proposed to achieve optimal load scheduling by which the social cost is minimized. In the proposed method, the social optimum is derived without the coordination of the utility company. Since the explicit expression on the pricing function is usually not available for the consumers due to privacy and flexibility concerns of the utility company, an extremum seeking scheme is used to handle the unknown pricing function. To provide incentives for the industrial consumers to participate in solving the social cost minimization problem, a benefit sharing scheme is designed such that their electricity cost derived by using the proposed method is always not greater than the cost of playing non-cooperative games. An example for industrial buildings with heating ventilation and air conditioning (HVAC) systems is used to verify the effectiveness of the proposed method.**

***Index Terms*— Optimal load scheduling; demand response; energy control; extremum seeking; distributed optimization**

I. Introduction

Demand response is a mechanism designed to make the users adjust their electricity usage to the desired profile based on supply conditions (see [18]–[20] and the references therein). Normally, there are two categories of users considered in the existing literature: price takers and price-anticipating users (e.g., see [22] and the references therein). The price takers do not suppose that their electricity consumption will affect the price while the price-anticipating users consider the effect of their load consumption on electricity price. Industrial consumers with large electricity loads are typical examples of price-anticipating users. Our interest in this paper, focuses on designing a demand response scheme for industrial users in industrial parks.

Considering the effect of loads on electricity price will result in coupled optimization problem for each user. To handle the interactions among the users, game theory is a useful tool for the analysis of the demand response mechanisms (e.g., see [1], [21]–[27], [31] and the references therein). In [21], the authors considered peak-to-average ratio (PAR) minimization and energy cost minimization problem by designing a non-cooperative energy consumption game. A distributed algorithm was proposed to reach the Nash equilibrium. However, the players need to communicate with each other which leads to privacy concerns and the convergence is based on an asynchronous strategy updating mechanism which challenges convergence speed. The authors in [27] studied the interactive behavior among the electric vehicles and aggregators using a non-cooperative game approach. The Nash equilibrium is shown to be the optimal solution by using specific pricing policy. In [22], the authors considered both price takers and price-anticipating users in which the Nash equilibrium coincides with the social optimal solution by choosing the price values carefully. Optimality and fairness issues were considered in [23] where the authors developed a smart electricity billing mechanism to achieve the objective. In [26], the authors proposed a dynamic potential game approach for demand response. The best responses were derived analytically and the efficiency of the equilibrium was studied. Stackelberg game is employed to describe the interactions between multiple utility companies and end-users [24]. A distributed method was proposed to ensure the convergence to the Stackelberg equilibrium but the efficiency of the equilibrium is not discussed. Multiple utility companies were also considered in [25] where the authors used a two level game approach to manage demand response. The competition on the utility companies' level is solved via a non-cooperative game and the users are modelled as evolutionary game players. Though, Nash equilibrium may result in a social optimum solution (e.g., the results in [21]–[23], [27]), the efficiency of the Nash equilibrium depends highly on the game structure and therefore, relies highly on billing policy design. For general resource allocation problems, being non-cooperative usually yields inefficient Nash equilibrium from a system level perspective [28]. Motivated by this fact, we intend to solve social cost minimization problem for electricity consumers in industrial parks with general pricing mechanism designs.

Several works studied social optimum seeking in power grid [7]–[9]. However, these methods either require the participation of utility companies or precise model information when seeking the social optimum solution. Different from these methods, in this paper, we develop a distributed optimization method based on extremum seeking (e.g., see [10]–[16]) to schedule the electricity loads for consumers in industrial parks. Compared with existing works, the main contributions of this paper are summarized as follows: 1). We address the optimal electricity load scheduling problem for a network of industrial users in industrial parks under general pricing mechanism designs. 2). We propose a continuous time distributed updating

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strategy for the users based on extremum seeking in which the users only need to communicate with their trustful neighbors. The proposed updating strategy is completely decentralized and doesn't require the coordination of the utility company. 3). The proposed updating law does not require explicit expression on the pricing function which protects the privacy and flexibility of the utility company. 4). The overall mechanism ensures that every user yields not more cost than that of playing Nash equilibrium.

The rest of this paper is organized as follows: some preliminaries are provided in Section II and system model is developed in Section III. The main results are presented in Section IV where we firstly formulate the centralized problem and then propose a distributed method to solve it. Furthermore, we compare the solution of the social cost minimization problem with the corresponding Nash solution and provide a benefit sharing protocol which ensures that every user will benefit from the proposed coordination process. A numerical example is provided in Section V to verify the proposed method and brief conclusions are given in Section VI.

II. Preliminaries

Throughout this paper, we use R to represent the set of real numbers. R_{++} stands for the set of positive real numbers and R_+ is the set of non-negative real numbers, $\otimes$ denotes the Kronecker product and $Df(x)$ denotes $\frac{\partial f(x)}{\partial x}$.

Theorem 1: ([14]) For a non-autonomous system represented by

$$\dot{x} = f_0(x,t) + \sum_{i=1}^{r} f_i(x,t)\sqrt{\omega}\varphi_i(t,\omega t), \tag{1}$$

with $x(t_0) = x_0 \in \mathbb{R}^n$ and $\omega > 0$. Suppose that $f_i(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \in C^2$, $Df_i, D^2 f_i, \frac{\partial f_i}{\partial t}, \frac{\partial}{\partial t} Df_i, \frac{\partial}{\partial t} D^2 f_i$ be continuous and bounded for all x belongs to any compact set; φ_i, $i \in 1, 2, \cdots, r$ are measurable functions and there exist M_i, N_i such that $|\varphi_i(t_1,\theta) - \varphi_i(t_2,\theta)| \le M_i |t_1 - t_2|$ for all t_1, $t_2 \in \mathbb{R}$ and $\sup_{t,\theta\in\mathbb{R}} |\varphi_i(t_1,\theta)| \le N_i$. Furthermore, $\varphi_i(t,\cdot)$ is T periodic with zero average. Then, if a compact set is globally uniformly asymptotically stable for (1), it is semi-globally practically uniformly asymptotically stable for the following system

$$\dot{x} = f_0(x,t) + \sum_{\substack{i=1 \\ j=i+1}}^{r} [f_i, f_j](t,z) v_{ji}(t), \tag{2}$$

where

$$v_{ji}(t) = \frac{1}{T}\int_0^T \varphi_j(t,\theta)\int_0^\theta \varphi_i(t,\tau)d\tau d\theta. \tag{3}$$

This trajectory approximation process is denoted as Lie bracket approximation and we call the system in (2) the approximated system in the rest of the paper.

In the following, we provide some definitions about practical stability. Readers are referred to [6], [13], [14] for more details on the concepts of practical stability.

Let $x(\cdot) := x(\cdot; t_0, x_0, \omega)$ denote the solution of

$$\dot{x} = f_\omega(t,x) \tag{4}$$

through $x(t_0) = x(0)$.

Definition 1: (Practically uniformly stable) A compact set $\mathcal{S} \subseteq R^n$ is practically uniformly stable for (4) if for every $\epsilon \in (0,\infty)$, there exists a $\delta \in (0,\infty)$ and $\omega_0 \in (0,\infty)$ such that for all $t_0 \in R$ and for all $\omega \in (\omega_0,\infty)$,

$$x(t_0) \in \mathcal{U}_\delta^{\mathcal{S}} \Longrightarrow x(t) \in \mathcal{U}_\epsilon^{\mathcal{S}}, t \in [t_0,\infty).$$

Definition 2: ($\delta-$practically uniformly attractive) Let $\delta \in (0,\infty)$. A compact set $\mathcal{S} \subseteq R^n$ is $\delta-$practically uniformly attractive for (4) if for every $\epsilon \in (0,\infty)$, there exists a $t_f \in [0,\infty)$ and $\omega_0 \in (0,\infty)$ such that for all $t_0 \in R$ and for all $\omega \in (\omega_0,\infty)$,

$$x(t_0) \in \mathcal{U}_\delta^{\mathcal{S}} \Longrightarrow x(t) \in \mathcal{U}_\epsilon^{\mathcal{S}}, t \in [t_0 + t_f,\infty).$$

Definition 3: (Practically uniformly bounded) Let $\mathcal{S} \subseteq R^n$ be a compact set. The solutions of (4) is practically uniformly bounded if for every $\delta \in (0,\infty)$, there exists an $\epsilon \in (0,\infty)$ and $\omega_0 \in (0,\infty)$ such that for all $t_0 \in R$, and for all $\omega \in (\omega_0,\infty)$,

$$x(t_0) \in \mathcal{U}_\delta^{\mathcal{S}} \Longrightarrow x(t) \in \mathcal{U}_\epsilon^{\mathcal{S}}, t \in [t_0,\infty).$$

Definition 4: (Semi-globally practically uniformly asymptotically stable) A compact set $\mathcal{S} \subseteq R^n$ is semi-globally practically uniformly asymptotically stable for (4) if it is practically uniformly stable and for every $\delta \in (0,\infty)$, it is $\delta-$practically uniformly attractive. Furthermore, the solutions for (4) must be practically uniformly bounded.

Definition 5: (*Saddle point*) The saddle point (x^*, y^*) of function $F(x,y)$ is a point on which the following is satisfied:

$$F(x^*,y) \le F(x^*,y^*) \le F(x,y^*).$$

More generally, $(x^*,y^*,z^*) \in R^n \times R^n \times R_+^m$ is the saddle point of $F(x,y,z)$

$$F(x^*,y,z) \le F(x^*,y^*,z^*) \le F(x,y^*,z^*)$$

where $F(x,y,z) : R^n \times R^n \times R_+^m \to R$.

Definition 6: (*Nash equilibrium*) Nash equilibrium is a strategy profile on which no player can reduce its cost by unilaterally changing its own strategy, i.e., a strategy profile $(x_i^*, \mathbf{x}_{-i}^*)$ is Nash equilibrium if for all the players

$$C_i(x_i^*, \mathbf{x}_{-i}^*) \le C_i(x_i, \mathbf{x}_{-i}^*),$$

for $\forall\ x_i \neq x_i^*$, where $\mathbf{x}_{-i}$ denotes all the players' strategies other than player i.

For a graph defined as $G = (V,E)$ in which E is the edge set $E \subset V \times V$, $V = \{1,2,\cdots,N\}$ is the set of nodes in the network, it is undirected if for every $(v_i,v_j) \in E$, $(v_j,v_i) \in E$. An undirected graph is connected if there exists a path between any pair of distinct vertices. The elements in the adjacent matrix A are defined as $a_{ij} = 1$ if node j is at the neighborhood of node i, else, $a_{ij} = 0$. The Laplacian matrix for the graph L is defined as $L = D - A$ where matrix D is the degree matrix of the graph and is defined as a diagonal matrix whose ith diagonal element is equal to the out degree of node i. The out degree of node i is equal to $\sum_{j=1}^{N} a_{ij}$.

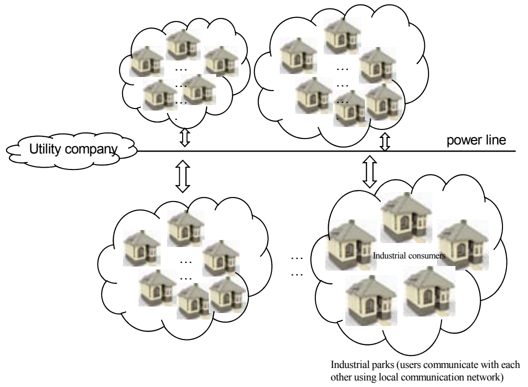

Fig. 1: The demand response model.

III. System Model for Users in Industrial Parks

We consider an industrial park with N networked industrial users whose electricity loads have dominant effect on electricity price. The model is shown in Fig. 1. The users in industrial parks are supposed to be equipped with an energy-management controller (EMC) and an advanced metering infrastructure (AMI). The EMC is able to schedule the usage of electricity for the corresponding user. The AMI enables two way communication among users and utility company. For privacy and flexibility concerns of the utility company who may change the pricing curve according to necessity, we only suppose that the output of the pricing function is measurable to the users but the explicit expression on the pricing function is unknown. To reserve privacy to some extent, the users only communicate with their trustful neighbors in real time instead of every users in the network. The communication graph among the users in the same local communication network is assumed to be undirected and connected and is under ideal transmission conditions, i.e., transmission problems such as loss of data, transmission delay, are not considered for short distance communications. In the rest of this section, we provide a mathematical model for the electricity users in industrial parks.

The cost of user i is

$$C_i(l_i, \mathbf{l}_{-i}) = \mu_i V_i(l_i) + P(\sum_{j=1}^{N} l_j) l_i, \tag{5}$$

where l_i denotes the electricity load of user i, $\mathbf{l}_{-i}$ is all the users' loads except for user i and in the rest of the paper, we use $\mathbf{l}$ as the concatenated vector of l_i. Furthermore, $V_i(l_i)$ is the cost of consuming load l_i. Quadratic functions and logarithmic functions can be used to quantify this cost [25]. For example, it denotes discomfort cost in [1] and in this case, μ_i is a positive parameter which determines the importance of comfort for the users. The term $P(\sum_{j=1}^{N} l_j) l_i$ corresponds to the billing payment for the usage of load l_i. The marginal price $P(\sum_{j=1}^{N} l_j)$ is closely related with the total electricity loads $\sum_{j=1}^{N} l_j$.

The optimal electricity load scheduling is achieved if the electricity loads are such that the social cost is minimized and the social cost is defined as the sum of the users' costs, i.e.,

$$C(\mathbf{l}) = \sum_{i=1}^{N} C_i(l_i, \mathbf{l}_{-i}).$$

Since overloaded electricity demands burden the electricity generation system and may even collapse the system in practice, we put a restriction on the total electricity loads,

$$\sum_{j=1}^{N} l_j \leq L_{\max}, \tag{6}$$

where $L_{\max}$ is a constant. This constraint reveals the electricity generation capacity. From another aspect, $L_{\max}$ can be regarded as the total electricity load available from the utility company [24], in this case, this constraint means that total demand can't exceed the total available load.

If the objective is to meet demand with supply, the constraint should be

$$\sum_{j=1}^{N} l_j = S,$$

where S is the electricity supply. This constraint is equivalent to

$$\begin{aligned} \sum_{j=1}^{N} l_j &\leq S \\ \text{and} \quad -\sum_{j=1}^{N} l_j &\leq -S, \end{aligned}$$

which are of the same form of (6). Therefore, we only consider the linear inequality constraint in the rest of the paper and the proposed method can be easily adjusted if the objective is to meet the demand with supply.

For each user, the electricity load should be within its acceptable range, i.e., $l_i \in [l_i^{\min}, l_i^{\max}]$ where $l_i^{\min}$, $l_i^{\max}$ are the minimal and maximal acceptable electricity load for user i, respectively.

IV. Main Results

A. Centralized Optimization of Electricity Loads for Industrial Parks

We suppose that the goal of the utility company is to induce the users' electricity loads to minimize the total cost (to achieve social optimum), i.e.,

Problem 1: (Social cost minimization) The total cost minimization problem is defined as

$$\begin{aligned} \min \ C(\mathbf{l}) &= \sum_{i=1}^{N} C_i(l_i, \mathbf{l}_{-i}) \\ \text{subject to } &\sum_{i=1}^{N} l_i \leq L_{\max} \\ l_i^{\min} &\leq l_i \leq l_i^{\max}. \end{aligned} \tag{7}$$

The analysis for this problem is based on the following condition.

Assumption 1: The function $C(\mathbf{l})$ is continuous differentiable and convex. Furthermore, the constraint set of Problem 1 is non-empty.

By this assumption and Weierstrass' theorem [30], the set of solutions to Problem 1 is non-empty and compact.

B. Distributed Coordination of Electricity Loads for Industrial Parks

1) Problem Analysis: Solving Problem 1 by utility company in a centralized fashion is inefficient since it requires two way long distance communication among the users with the utility company at each iteration during solution seeking. In this paper, we provide a completely decentralized method to solve it. Since the number of consumers in each industrial park is not that large in practice, the proposed method does not aim to distribute the computational burden but to avoid long distance frequent communication among the users with the utility company. Furthermore, by using completely decentralized method, the single-node congestion problem can be avoided [32].

If the cost function is completely known by the users, the methods in [2]–[5] can be adapted to solve Problem 1 in a distributed way. However, since the explicit expression on the pricing function is not available for the users, the methods in [2]–[5] can not be directly implemented. In the following analysis, we firstly present two lemmas that are essential in the subsequent analysis by adapting the method in [2]–[4]. In the next subsection, we will design a model-free continuous time method to deal with the unknown terms in the cost function.

Same as [2]–[4], we introduce some additional variables to analyze Problem 1. Define

$$\bar{\mathbf{l}}_i = [\bar{l}_{i1}, \bar{l}_{i2}, \cdots, \bar{l}_{iN}]^T,$$

where $\bar{l}_{ij}$ denotes user i's estimation on user j's electricity load. Furthermore,

$$\bar{\mathbf{l}} = [\bar{\mathbf{l}}_1, \bar{\mathbf{l}}_2, \cdots, \bar{\mathbf{l}}_N]^T,$$

where

$$\bar{\mathbf{l}}_1 = \bar{\mathbf{l}}_2 = \cdots = \bar{\mathbf{l}}_N,$$

which is equivalent to

$$\mathbf{L}\bar{\mathbf{l}} = \mathbf{0},$$

under undirected and connected graph. Furthermore, $\mathbf{L} = L \otimes \mathbf{I_N}$, L is the Laplacian matrix of the communication graph and $\mathbf{I_N}$ is a N dimensional identity matrix. Using these variables, we can transform Problem 1 to the following problem.

Problem 2:

$$\begin{aligned} \min\ C(\bar{\mathbf{l}}) &= \sum_{i=1}^{N} C_i(\bar{\mathbf{l}}_i) \\ &= \sum_{i=1}^{N} \left(\mu_i V_i(\bar{l}_{ii}) + P(\sum_{j=1}^{N} \bar{l}_{ij}) \bar{l}_{ii} \right) \\ \text{subject to } & \sum_{j=1}^{N} \bar{l}_{ij} \leq L_{\max} \\ & l_i^{\min} \leq \bar{l}_{ii} \leq l_i^{\max} \\ & \mathbf{L}\bar{\mathbf{l}} = \mathbf{0}, i \in \mathcal{N}, \end{aligned} \tag{8}$$

where $\mathcal{N} = \{1, 2, \cdots, N\}$.

Different from Problem 1, Problem 2 is solvable using only local information. Next lemma states the equivalence of Problem 1 and Problem 2.

Lemma 1: Problem 1 and Problem 2 are equivalent.

Proof: See Appendix-VII-A for proof. ■

Before we proceed to the updating laws, we firstly present the following Lemma as it will be used in the analysis of the proposed updating law.

Lemma 2: Define

$$\begin{aligned} \tilde{F}(\bar{\mathbf{l}}, \mathbf{y}, \mathbf{z}) &= C(\bar{\mathbf{l}}) + \bar{\mathbf{l}}^T \mathbf{L} \mathbf{y} \\ &+ \frac{1}{2} \bar{\mathbf{l}}^T \mathbf{L} \bar{\mathbf{l}} + \sum_{i=1}^{N} z_{i1} \left(\sum_{j=1}^{N} \bar{l}_{ij} - L_{\max} \right) \\ &+ \sum_{i=1}^{N} z_{i2} (\bar{l}_{ii} - l_i^{\max}) + \sum_{i=1}^{N} z_{i3} (l_i^{\min} - \bar{l}_{ii}), \end{aligned} \tag{9}$$

where $\mathbf{y}$, $\mathbf{z}$ are auxiliary vectors defined as $\mathbf{y} = [\mathbf{y_1}, \mathbf{y_2}, \cdots, \mathbf{y_N}]^{\mathbf{T}}$, $\mathbf{y_i} = [y_{i1}, y_{i2}, \cdots, y_{iN}]$, $\mathbf{z} = [\mathbf{z_1}, \mathbf{z_2}, \cdots, \mathbf{z_N}]^{\mathbf{T}}$, $\mathbf{z_i} = [z_{i1}, z_{i2}, z_{i3}]$. Then, the $\bar{\mathbf{l}}$ component of the saddle point $(\bar{\mathbf{l}}^*, \mathbf{y}^*, \mathbf{z}^*) \in R^{N^2} \times R^{N^2} \times R_+^{3N}$ of $\tilde{F}(\bar{\mathbf{l}}, \mathbf{y}, \mathbf{z})$ coincides with the solution of Problem 2 under Assumption 1.

Proof: The Lagrangian function [30] for Problem 2 is

$$\begin{aligned} L(\bar{\mathbf{l}}, \mathbf{y}, \mathbf{z}) \ = \ & C(\bar{\mathbf{l}}) + \bar{\mathbf{l}}^T \mathbf{L} \mathbf{y} + \sum_{i=1}^{N} z_{i1} \left(\sum_{j=1}^{N} \bar{l}_{ij} - L_{\max} \right) \\ & + \sum_{i=1}^{N} z_{i2} (\bar{l}_{ii} - l_i^{\max}) + \sum_{i=1}^{N} z_{i3} (l_i^{\min} - \bar{l}_{ii}), \end{aligned}$$

which is similar to $\tilde{F}$ and the proof of this Lemma follows the proof of saddle point theorem for Lagrangian functions. Since $\tilde{F}$ is not exactly equal to the Lagrangian function, we provide the proof for the convenience of the readers. Readers are referred to Appendix-VII-B for the details. ■

By this Lemma, we can conclude that to solve Problem 2, we can seek for the saddle point of $\tilde{F}(\bar{\mathbf{l}}, \mathbf{y}, \mathbf{z})$ instead.

2) Updating Strategy for Social Optimum Seeking: During the solution seeking process, the users communicate with their neighbors via a local communication network (e.g., Fig. 2). Since the users only communicate with their neighbors instead of every user in the network, their privacies are protected to some extent. In the following analysis, we proceed to design an updating law for the users to achieve the optimal load scheduling by using only local information.

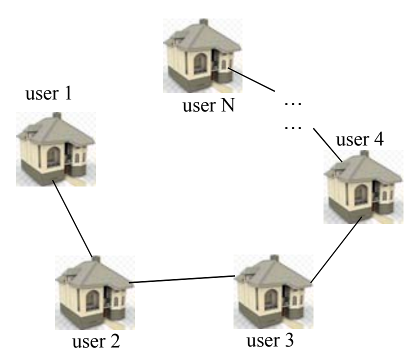

Fig. 2: Local communication network during the solution seeking process.

To seek the saddle point of $\tilde{F}(\bar{\mathbf{l}},\mathbf{y},\mathbf{z})$, a typical choice is to use the saddle point dynamics, i.e.,

$$\begin{aligned}
&\dot{\bar{\mathbf{l}}}=-\boldsymbol{\beta}_1\tilde{\mathbf{F}}_{\bar{\mathbf{l}}}(\bar{\mathbf{l}},\mathbf{y},\mathbf{z}) \qquad (10)\\
&\dot{\mathbf{y}}=\boldsymbol{\beta}_2\tilde{\mathbf{F}}_{\mathbf{y}}(\bar{\mathbf{l}},\mathbf{y},\mathbf{z})\\
&\dot{z}_{i1}=\beta_3\left[\sum_{j=1}^{N}\bar{l}_{ij}-L_{\max}\right]^{+}_{z_{i1}}\\
&\dot{z}_{i2}=\beta_4\left[\bar{l}_{ii}-l_i^{\max}\right]^{+}_{z_{i2}}\\
&\dot{z}_{i3}=\beta_5\left[l_i^{\min}-\bar{l}_{ii}\right]^{+}_{z_{i3}}\\
&\text{for all } i\in\mathcal{N},
\end{aligned}$$

where $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ are two positive diagonal parameter matrices, β_3, β_4, β_5 are positive gains to be determined and $[\Phi]^+_\chi$ is a projection. Furthermore, $[\Phi]^+_\chi=\Phi$ if $\Phi>0$ or $\chi>0$, and $[\Phi]^+_\chi=0$ otherwise. However, the gradient of the pricing curve is not available for the users since the explicit expression on the pricing function is unknown. Hence, the updating law in (10) can not be directly used. And here, we employ an extremum seeking method [14] to deal with the unmodelled function via the measurement of the pricing value. We design the online updating law as

$$\begin{aligned}
\dot{\bar{l}}_{ij} &= -k_{ij}\sqrt{\omega_{ij}}\sin(\omega_{ij}t)\left(P(\sum_{j=1}^{N}\bar{l}_{ij})+z_{i2}-z_{i3}\right)\bar{l}_{ii} \qquad (11)\\
&\quad+\alpha_{ij}\sqrt{\omega_{ij}}\cos(\omega_{ij}t)-\frac{k_{ij}\alpha_{ij}}{2}\left(\mu_i\frac{\partial V_i(\bar{l}_{ii})}{\partial\bar{l}_{ij}}+z_{i1}\right)\\
&\quad-\frac{k_{ij}\alpha_{ij}}{2}\sum_{k=1}^{N}a_{ik}(\bar{l}_{ij}-\bar{l}_{kj})-\frac{k_{ij}\alpha_{ij}}{2}\sum_{k=1}^{N}a_{ik}(y_{ij}-y_{kj})\\
\dot{y}_{ij} &= \sum_{k=1}^{N}a_{ik}(\bar{l}_{ij}-\bar{l}_{kj})\\
\dot{z}_{i1} &= z_{i1}\left(\sum_{j=1}^{N}\bar{l}_{ij}-L_{\max}\right)\\
\dot{z}_{i2} &= z_{i2}\left(\bar{l}_{ii}-l_i^{\max}\right)\\
\dot{z}_{i3} &= z_{i3}\left(l_i^{\min}-\bar{l}_{ii}\right),\ i,j\in\mathcal{N},
\end{aligned}$$

where k_{ij},α_{ij} are positive control gains to be determined. The frequency ω_{ij} are distinct from each other and are large enough and a_{ik} is the element on the ith row and kth column of the adjacent matrix of the communication graph. The concatenated form of (11) is

$$\begin{aligned}
&\dot{\bar{\mathbf{l}}}=\mathbf{diag}(\frac{\mathbf{k_{ij}}\boldsymbol{\alpha}_{\mathbf{ij}}}{\mathbf{2}})\left(-\mathbf{L}\bar{\mathbf{l}}-\mathbf{L}\mathbf{y}-\frac{\partial\boldsymbol{\mu}\mathbf{V}(\bar{\mathbf{l}})}{\partial\bar{\mathbf{l}}}-\mathbf{z}^{\mathbf{1}}\right)+\boldsymbol{\Pi}-\boldsymbol{\Xi}\\
&\dot{\mathbf{y}}=\mathbf{L}\bar{\mathbf{l}} \qquad (12)\\
&\dot{z}_{i1}=z_{i1}\left(\sum_{j=1}^{N}\bar{l}_{ij}-L_{\max}\right)\\
&\dot{z}_{i2}=z_{i2}\left(\bar{l}_{ii}-l_i^{\max}\right)\\
&\dot{z}_{i3}=z_{i3}\left(l_i^{\min}-\bar{l}_{ii}\right),i\in\mathcal{N},
\end{aligned}$$

where $\boldsymbol{\Pi}\in R^{N^2}$, $\boldsymbol{\Xi}\in R^{N^2}$ are both vectors defined as

$$\begin{aligned}
\boldsymbol{\Pi} &= [\Pi_1,\Pi_2,\cdots,\Pi_N]^T\\
\boldsymbol{\Xi} &= [\boldsymbol{\Xi}_1,\boldsymbol{\Xi}_2,\cdots,\boldsymbol{\Xi}_N]^T,
\end{aligned}$$

where

$$\begin{aligned}
\Pi_i &= [\alpha_{i1}\sqrt{\omega_{i1}}\cos(\omega_{i1}t),\alpha_{i2}\sqrt{\omega_{i2}}\cos(\omega_{i2}t),\\
&\quad\cdots,\alpha_{iN}\sqrt{\omega_{iN}}\cos(\omega_{iN}t)],
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\Xi}_i=&\left[k_{i1}\sqrt{\omega_{i1}}\sin(\omega_{i1}t)\left(\left(P(\sum_{j=1}^{N}\bar{l}_{ij})+z_{i2}-z_{i3}\right)\bar{l}_{ii}\right),\right.\\
&k_{i2}\sqrt{\omega_{i2}}\sin(\omega_{i2}t)\left(\left(P(\sum_{j=1}^{N}\bar{l}_{ij})+z_{i2}-z_{i3}\right)\bar{l}_{ii}\right),\cdots\\
&\left.,k_{iN}\sqrt{\omega_{iN}}\sin(\omega_{iN}t)\left(\left(P(\sum_{j=1}^{N}\bar{l}_{ij})+z_{i2}-z_{i3}\right)\bar{l}_{ii}\right)\right].
\end{aligned}$$

Furthermore, $\mathbf{diag}(\frac{\mathbf{k_{ij}}\boldsymbol{\alpha}_{\mathbf{ij}}}{\mathbf{2}})$ is defined as the diagonal matrix whose diagonal elements are $\frac{k_{ij}\alpha_{ij}}{2}$, $\frac{\partial\boldsymbol{\mu}\mathbf{V}(\bar{\mathbf{l}})}{\partial\bar{\mathbf{l}}}$ is the concatenated vector of $\frac{\partial\mu_iV_i(\bar{l}_{ii})}{\partial\bar{l}_{ij}}$ and $\mathbf{z}^{\mathbf{1}}=[z_{11},z_{21},\cdots,z_{N1}]^T$.

Alternatively, if we regard the explicit expression on the whole cost function to be unknown, the following updating law can be used as well,

$$\begin{aligned}
\dot{\bar{l}}_{ij} &= -k_{ij}\sqrt{\omega_{ij}}\sin(\omega_{ij}t)\left(C_i(\bar{\mathbf{l}}_i)+(z_{i2}-z_{i3})\bar{l}_{ii}\right)\\
&\quad+\alpha_{ij}\sqrt{\omega_{ij}}\cos(\omega_{ij}t)-\frac{k_{ij}\alpha_{ij}}{2}\sum_{k=1}^{N}a_{ik}(\bar{l}_{ij}-\bar{l}_{kj})\\
&\quad-\frac{k_{ij}\alpha_{ij}}{2}z_{i1}-\frac{k_{ij}\alpha_{ij}}{2}\sum_{k=1}^{N}a_{ik}(y_{ij}-y_{kj})\\
\dot{y}_{ij} &= \sum_{k=1}^{N}a_{ik}(\bar{l}_{ij}-\bar{l}_{kj}) \qquad (13)\\
\dot{z}_{i1} &= z_{i1}\left(\sum_{j=1}^{N}\bar{l}_{ij}-L_{\max}\right)\\
\dot{z}_{i2} &= z_{i2}\left(\bar{l}_{ii}-l_i^{\max}\right)\\
\dot{z}_{i3} &= z_{i3}\left(l_i^{\min}-\bar{l}_{ii}\right),i,j\in\mathcal{N}.
\end{aligned}$$

Theorem 2: Suppose that Assumption 1 is satisfied. Then the set of solution to Problem 1 is semi-globally practically uniformly asymptotically stable under the updating law in (11) or (13) with respect to $R^{N^2}\times R^{N^2}\times R_+^{3N}$, i.e., the electricity loads are distributed among the industrial users optimally.

Proof: Firstly, we calculate the Lie bracket approximated system for the closed-loop system

$$\begin{aligned}
&\dot{\bar{\mathbf{l}}}=\mathbf{diag}(\frac{\mathbf{k_{ij}}\boldsymbol{\alpha}_{\mathbf{ij}}}{\mathbf{2}})\left(-\mathbf{L}\bar{\mathbf{l}}-\mathbf{L}\mathbf{y}-\frac{\partial\boldsymbol{\mu}\mathbf{V}(\bar{\mathbf{l}})}{\partial\bar{\mathbf{l}}}-\mathbf{z}^{\mathbf{1}}\right)+\boldsymbol{\Pi}-\boldsymbol{\Xi}\\
&\dot{\mathbf{y}}=\mathbf{L}\bar{\mathbf{l}}\\
&\dot{z}_{i1}=z_{i1}\left(\sum_{j=1}^{N}\bar{l}_{ij}-L_{\max}\right)\\
&\dot{z}_{i2}=z_{i2}\left(\bar{l}_{ii}-l_i^{\max}\right)\\
&\dot{z}_{i3}=z_{i3}\left(l_i^{\min}-\bar{l}_{ii}\right),i\in\mathcal{N},
\end{aligned}$$

by using Theorem 1.

Since $\Phi_{ij\omega}=\int_0^t\sqrt{\omega_{ij}}\cos(\omega_{ij}\xi)d\xi=\frac{1}{\sqrt{\omega_{ij}}}\sin(\omega_{ij}t)$, $\hat{\Phi}_{ij\omega}=\int_0^t\sqrt{\omega_{ij}}\sin(\omega_{ij}\xi)d\xi=$

$-\frac{1}{\sqrt{\omega_{ij}}}\cos(\omega_{ij}t)+\frac{1}{\sqrt{\omega_{ij}}}$, and for all $\omega_i \neq \omega_j$,

$$\begin{aligned}
&\lim_{\omega_i\to\infty}\int_0^t \cos^2(\omega_i\xi)d\xi \\
=\ &\lim_{\omega_i\to\infty}\int_0^t \frac{1+\cos(2\omega_i\xi)}{2}d\xi \\
\overset{2\omega_i\xi=\tau}{=}\ &\lim_{\omega_i\to\infty}\int_0^{2\omega_i t}\frac{\left(\frac{1}{2}+\frac{\cos(\tau)}{2}\right)}{2\omega_i}d\tau \\
=\ &\frac{1}{2}t=\int_0^t \frac{1}{2}d\xi,
\end{aligned}$$

$$\begin{aligned}
&\lim_{\omega_i\to\infty}\int_0^t \sin^2(\omega_i\xi)d\xi \\
=\ &\lim_{\omega_i\to\infty}\int_0^t \frac{1-\cos(2\omega_i\xi)}{2}d\xi \\
\overset{2\omega_i\xi=\tau}{=}\ &\lim_{\omega_i\to\infty}\int_0^{2\omega_i t}\frac{\left(-\frac{1}{2}+\frac{\cos(\tau)}{2}\right)}{2\omega_i}d\tau \\
=\ &\frac{1}{2}t=\int_0^t \frac{1}{2}d\xi,
\end{aligned}$$

$$\begin{aligned}
\lim_{\omega_i\to\infty}\int_0^t \cos(\omega_i\xi)\sin(\omega_i\xi)d\xi &= \lim_{\omega_i\to\infty}\int_0^t \frac{\sin(2\omega_i\xi)}{2}d\xi \\
\overset{2\omega_i\xi=\tau}{=}\lim_{\omega_i\to\infty}\int_0^{2\omega_i t}\frac{\sin(\tau)}{2\omega_i}d\tau &= 0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{\omega_i,\omega_j\to\infty}\int_0^t \sin(\omega_i\xi)\sin(\omega_j\xi)d\xi \\
=\ &\lim_{\omega_i,\omega_j\to\infty}\int_0^t \frac{\cos(\omega_j\xi-\omega_i\xi)-\cos(\omega_j\xi+\omega_i\xi)}{2}d\xi \\
\overset{(\omega_i+\omega_j)\xi=\tau}{=}\ &\lim_{\omega_i,\omega_j\to\infty}\int_0^t \frac{\cos(\frac{\omega_j-\omega_i}{\omega_i+\omega_j}\tau)-\cos(\tau)}{2\left(\omega_i+\omega_j\right)}d\tau \\
=\ &\lim_{\omega_i,\omega_j\to\infty}\left(\frac{1}{\omega_j-\omega_i}\sin(\frac{\omega_j-\omega_i}{\omega_i+\omega_j}\tau)-\frac{\sin(\tau)}{2\left(\omega_i+\omega_j\right)}\right) \\
=\ &0,
\end{aligned}$$

$$\begin{aligned}
&\lim_{\omega_i,\omega_j\to\infty}\int_0^t \cos(\omega_i\xi)\cos(\omega_j\xi)d\xi \\
=\ &\lim_{\omega_i,\omega_j\to\infty}\int_0^t \frac{\cos(\omega_j\xi-\omega_i\xi)+\cos(\omega_j\xi+\omega_i\xi)}{2}d\xi \\
=\ &0,
\end{aligned}$$

$$\begin{aligned}
\lim_{\omega_i\to\infty}\int_0^t \cos(\omega_i\xi)d\xi \overset{\omega_i\xi=\tau}{=} \lim_{\omega_i\to\infty}\int_0^{\omega_i t}\frac{\cos(\tau)}{\omega_i}d\tau &= 0, \\
\lim_{\omega_i\to\infty}\int_0^t \sin(\omega_i\xi)d\xi \overset{\omega_i\xi=\tau}{=} \lim_{\omega_i\to\infty}\int_0^{\omega_i t}\frac{\sin(\tau)}{\omega_i}d\tau &= 0.
\end{aligned}$$

Hence, the value of v_{ij} defined in (3) is $\frac{1}{2}$ for $\lim_{\omega_{ij}\to\infty}\int_0^t \cos(\omega_{ij}\xi)(\cos(\omega_{ij}\xi)-1)d\xi$ and is $-\frac{1}{2}$ for $\lim_{\omega_{ij}\to\infty}\int_0^t -\sin^2(\omega_{ij}\xi)d\xi$ and in other circumstances, $v_{ij}=0$.

Therefore, according to (2) and (3), we can get that the approximated system is

$$\begin{aligned}
&\dot{\bar{\mathbf{l}}}=-\mathbf{diag}(\frac{\mathbf{k_{ij}}\boldsymbol{\alpha}_{\mathbf{ij}}}{\mathbf{2}})\tilde{F}_{\bar{\mathbf{l}}}(\bar{\mathbf{l}},\mathbf{y},\mathbf{z}) \\
&\dot{\mathbf{y}}=\mathbf{L}\bar{\mathbf{l}} \\
&\dot{z}_{i1}=z_{i1}\left(\sum_{j=1}^N \bar{l}_{ij}-L_{\max}\right) \\
&\dot{z}_{i2}=z_{i2}\left(\bar{l}_{ii}-l_i^{\max}\right) \\
&z_{i3}=z_{i3}\left(l_i^{\min}-\bar{l}_{ii}\right), i\in\mathcal{N}.
\end{aligned} \tag{14}$$

If we implemented the above analysis to (13), we can get that the Lie bracket approximated system of (13) is the same as in (14).

Let

$$\begin{aligned}
\tilde{V}=&\frac{1}{2}(\bar{\mathbf{l}}-\bar{\mathbf{l}}^*)^T\mathbf{diag}(\frac{\mathbf{k_{ij}}\boldsymbol{\alpha}_{\mathbf{ij}}}{\mathbf{2}})^{-1}(\bar{\mathbf{l}}-\bar{\mathbf{l}}^*) \\
&+\frac{1}{2}\left(\mathbf{y}-\mathbf{y}^*\right)^T\left(\mathbf{y}-\mathbf{y}^*\right) \\
&+\sum_{i=1}^N\sum_{j=1}^3\left(z_{ij}-z_{ij}^*-z_{ij}^*\log(z_{ij})+z_{ij}^*\log(z_{ij}^*)\right),
\end{aligned}$$

which is defined for $z_{ij}\geq 0$. Since

$$\begin{aligned}
&\frac{\partial}{\partial z_{ij}}\left(\sum_{i=1}^N\sum_{j=1}^3\left(z_{ij}-z_{ij}^*-z_{ij}^*\log(z_{ij})+z_{ij}^*\log(z_{ij}^*)\right)\right) \\
=\ &1-\frac{z_{ij}^*}{z_{ij}},
\end{aligned}$$

$$1-\frac{z_{ij}^*}{z_{ij}}\begin{cases}=0 \text{ if } z_{ij}=z_{ij}^* \\ <0 \text{ if } z_{ij}<z_{ij}^* \\ >0 \text{ if } z_{ij}>z_{ij}^*\end{cases},$$

Therefore $\sum_{i=1}^N\sum_{j=1}^3\left(z_{ij}-z_{ij}^*-z_{ij}^*\log(z_{ij})+z_{ij}^*\log(z_{ij}^*)\right)$ has a minimum at $z_{ij}=z_{ij}^*$ which is equal to 0. Hence $\tilde{V}\geq 0$ and the equal sign holds if and only if $\bar{\mathbf{l}}=\bar{\mathbf{l}}^*,\mathbf{y}=\mathbf{y}^*,\mathbf{z}=\mathbf{z}^*$.

Taking the time derivative of $\tilde{V}$, we get that

$$\begin{aligned}
\dot{\tilde{V}}\ =\ &-(\bar{\mathbf{l}}-\bar{\mathbf{l}}^*)^T\tilde{F}_{\bar{\mathbf{l}}}(\bar{\mathbf{l}},\mathbf{y},\mathbf{z})+\left(\mathbf{y}-\mathbf{y}^*\right)^T\mathbf{L}\bar{\mathbf{l}} \\
&+\sum_{i=1}^N(z_{i1}(\sum_{j=1}^N\bar{l}_{ij}-L_{\max})-z_{i1}^*(\sum_{j=1}^N\bar{l}_{ij}-L_{\max})) \\
&+\sum_{i=1}^N\left(z_{i2}\left(\bar{l}_{ii}-l_i^{\max}\right)-z_{i2}^*\left(\bar{l}_{ii}-l_i^{\max}\right)\right) \\
&+\sum_{i=1}^N\left(z_{i3}\left(l_i^{\min}-\bar{l}_{ii}\right)-z_{i3}^*\left(l_i^{\min}-\bar{l}_{ii}\right)\right) \\
\leq\ &\tilde{F}(\bar{\mathbf{l}}^*,\mathbf{y},\mathbf{z})-\tilde{F}(\bar{\mathbf{l}},\mathbf{y},\mathbf{z}) \\
&+\tilde{F}(\bar{\mathbf{l}},\mathbf{y},\mathbf{z})-\tilde{F}(\bar{\mathbf{l}},\mathbf{y}^*,\mathbf{z}^*) \\
=\ &\left(\tilde{F}(\bar{\mathbf{l}}^*,\mathbf{y},\mathbf{z})-\tilde{F}(\bar{\mathbf{l}}^*,\mathbf{y}^*,\mathbf{z}^*)\right) \\
&+\left(\tilde{F}(\bar{\mathbf{l}}^*,\mathbf{y}^*,\mathbf{z}^*)-\tilde{F}(\bar{\mathbf{l}},\mathbf{y}^*,\mathbf{z}^*)\right) \\
\leq\ &0.
\end{aligned}$$

Hence, the set of saddle points of $\tilde{F}(\bar{\mathbf{l}}, \mathbf{y}, \mathbf{z})$ is globally uniformly asymptotically stable under (14). Therefore, it is semi-globally practically uniformly asymptotically stable under (11)/(13). Combining this result with Lemmas 1-2, the analysis of this theorem is completed. ■

This convergence analysis for the social optimum seeking problem using the updating law in (11)/(13) generalizes the results in [17] to solve non-modelled based distributed optimization problems with linear inequality constraint. For more general nonlinear convex inequality constraint, the proposed method can be easily adjusted to solve it. The continuous updating law provides addictive communication noise rejection property.

C. Solution Analysis and Benefit Sharing Protocol

While we provide a method for industrial users to update their loads in real time such that system level objective is derived, the main concern lies in the incentives for the users to coordinate with each other, i.e., the users may choose to act non-cooperatively due to their selfishness. If the users do not coordinate with each other, they would try to solve

$$\begin{aligned} &\min\ C_i(l_i, \mathbf{l}_{-i}) = \mu_i V_i(l_i) + P(\sum_{i=1}^{N} l_i) l_i \\ &\text{subject to } \sum_{i=1}^{N} l_i \le L_{\max} \\ &l_i^{\min} \le l_i \le l_i^{\max}. \end{aligned}$$

The Lagrangian function is

$$\begin{aligned} H_i(l_i, \mathbf{l}_{-i}, \lambda_{ij}) \;=\; & C_i(l_i, \mathbf{l}_{-i}) + \lambda_{i1}(\sum_{i=1}^{N} l_i - L_{\max}) \\ & + \lambda_{i2}(l_i - l_i^{\max}) + \lambda_{i3}(l_i^{\min} - l_i). \end{aligned}$$

Hence according to Karush–Kuhn–Tucker condition, the following condition should be satisfied at the Nash equilibrium,

$$\begin{aligned} &\frac{\partial H_i}{\partial l_i}(l_i^{NE}, \mathbf{l}_{-i}^{NE}, \lambda_{ij}^{NE}) = 0 \\ &\lambda_{ij}^{NE} \frac{\partial H_i}{\partial \lambda_{ij}}(l_i^{NE}, \mathbf{l}_{-i}^{NE}, \lambda_{ij}^{NE}) = 0 \\ &\lambda_{ij}^{NE} \ge 0 \\ &\sum_{i=1}^{N} l_i^{NE} \le L_{\max} \\ &l_i^{\min} \le l_i^{NE} \le l_i^{\max}, \end{aligned}$$

for all $i \in \mathcal{N}$, $j \in \{1, 2, 3\}$. However, Nash equilibrium is not necessarily the social optimal solution, i.e.,

$$\sum_{i=1}^{N} C_i(\mathbf{l}^{NE}) \ge \sum_{i=1}^{N} C_i(\mathbf{l}^*), \tag{15}$$

for a minimization problem. In (15), $\mathbf{l}^{NE}$ denotes the Nash equilibrium solution and $\mathbf{l}^*$ is the social optimal solution. From the users' perspective, there are two cases:

Case 1: $C_i(\mathbf{l}^{NE}) \ge C_i(\mathbf{l}^*)$, for all $i \in \mathcal{N}$. In this case, every user can reduce its cost by solving Problem 1 compared with being a non-cooperative player. Therefore, in this case, all the users have incentives to coordinate to derive the social optimum solution. However, fairness on the quantity of the benefit they get from the coordination behavior remains to be addressed.

Case 2: $C_i(\mathbf{l}^{NE}) \ge C_i(\mathbf{l}^*)$, for $i \in \mathcal{N}_s$, where $\mathcal{N}_s$ is a subset of $\mathcal{N}$ while $C_j(\mathbf{l}^{NE}) \le C_j(\mathbf{l}^*)$, for $j \in \mathcal{N} - \mathcal{N}_s$. That is to say, in this case, some users can reduce their costs while others sacrifice for the coordination behavior and they lack incentives to coordinate.

However, since $\sum_{i=1}^{N} C_i(\mathbf{l}^{NE}) \ge \sum_{i=1}^{N} C_i(\mathbf{l}^*)$ is satisfied for sure, a benefit sharing policy can be established among the users to solve the "incentive" and "fairness" problem. The benefit sharing policy can be implemented via signing a contract among the users before the coordination. Any profit sharing policy can be used if they satisfy the following two criteria.

Criterion 1: Share the profits such that $C_i(\mathbf{l}^{NE}) \ge C_i(\mathbf{l}^*)$, for all $i \in \mathcal{N}$.

Criterion 2: The profits they gain from the coordination behavior are allocated relatively fairly.

Here, we provide two benefit sharing protocols. For convenience, we define $U_i(\mathbf{l}) = -C_i(\mathbf{l})$ and $U(\mathbf{l}) = \sum_{i=1}^{N} U_i(\mathbf{l})$.

Algorithm 1: Share the benefit based on the users' utilities at the Nash equilibrium.

Begin
Allocate $U_i(\mathbf{l}) = U_i(\mathbf{l}^{NE})$, for all $i \in \mathcal{N}$
if $\sum_{i=1}^{N} C_i(\mathbf{l}^{NE}) - \sum_{i=1}^{N} C_i(\mathbf{l}^*) = 0$
end
else if $\sum_{i=1}^{N} C_i(\mathbf{l}^{NE}) - \sum_{i=1}^{N} C_i(\mathbf{l}^*) \ne 0$
Let $U_i(\mathbf{l}) = U_i(\mathbf{l}) + \frac{U_i(\mathbf{l}^{NE})\left(\sum_{i=1}^{N} U_i(\mathbf{l}^*) - \sum_{i=1}^{N} U_i(\mathbf{l}^{NE})\right)}{\sum_{i=1}^{N} U_i(\mathbf{l}^{NE})}$
for all $i \in \mathcal{N}$
end
return $U_i(\mathbf{l})$
end

Algorithm 2: Share the benefit based on the users' contribution in the coordination.

Begin
Allocate $U_i(\mathbf{l}) = U_i(\mathbf{l}^{NE})$, for all $i \in \mathcal{N}$
if $\sum_{i=1}^{N} C_i(\mathbf{l}^{NE}) - \sum_{i=1}^{N} C_i(\mathbf{l}^*) = 0$
end
else if $\sum_{i=1}^{N} C_i(\mathbf{l}^{NE}) - \sum_{i=1}^{N} C_i(\mathbf{l}^*) \ne 0$
Let $U_i(\mathbf{l}) = U_i(\mathbf{l}) +$
$\frac{\left(U_{\mathcal{N}}^* - U_{\mathcal{N}\setminus i}^*\right)}{\sum_{i=1}^{N}\left(U_{\mathcal{N}}^* - U_{\mathcal{N}\setminus j}^*\right)} \left(\sum_{i=1}^{N} U_i(\mathbf{l}^*) - \sum_{i=1}^{N} U_i(\mathbf{l}^{NE})\right)$
for all $i \in \mathcal{N}$
end
return $U_i(\mathbf{l})$
end

In Algorithm 2, $U_{\mathcal{N}\setminus i}^*$ denotes the social utility when user i does not participate in the coordination and therefore, $U_{\mathcal{N}}^* - U_{\mathcal{N}\setminus i}^*$ represents for user i's contribution on the social utility. This algorithm is based on Shapley value [29].

V. Numerical Example for Industrial Consumers with HVAC Systems

A. Distributed Coordination for Social Optimum Seeking

Since the number of consumers in each industrial park is usually not large, we use a simple demo to show the simulation results in a clearer way. We consider a network of industrial users equipped with heating ventilation and air conditioning (HVAC) systems. For HVAC systems. the discomfort term can be modelled as [1]

$$V_i(l_i) = \theta_i \gamma_i^2 (l_i - \hat{l}_i)^2,$$

where θ_i is the cost coefficient, γ_i is a parameter that specifies thermal characteristic of the HVAC system and $\hat{l}_i$ is the expected electricity load. Furthermore,

$$P(\sum_{i=1}^{N} l_i) = k(\sum_{i=1}^{N} l_i - L^*) + p_0,$$

where L^* is the expected demand determined by the utility company and k is a pricing parameter. Hence, for user i, the cost function is

$$C_i(l_i, \mathbf{l}_{-i}) = \mu_i \theta_i \gamma_i^2 (l_i - \hat{l}_i)^2 + \left(k(\sum_{j=1}^{N} l_j - L^*) + p_0 \right) l_i.$$

Such a pricing mechanism provides some incentives for the users to coordinate on the electricity loads to meet L^* since it reduces users' electricity billings. For more details on the deviation of the cost functions, readers are referred to [1]. In the simulation, we consider a network of 5 users with large electricity usage in a industrial park. The parameters in the model are setting as $k = 0.5$, $\theta_i \gamma_i^2 = 5 + 0.2i$, $\hat{l}_i = (100 + 20i)\, kWh$, $l_i^{\min} = 80$, $l_i^{\max} = 250$, the discomfort cost parameter $\mu_i = 0.8$, $i \in \{1, 2, \cdots, 5\}$, $p_0 = 10$ ¢$/kWh$, $L^* = 0.8 \sum_{i=1}^{N} \hat{l}_i$. For the constraint, $\sum_{i=1}^{N} l_i \leq L_{\max}$, we choose $L_{\max} = 780kWh$. Therefore, the social cost minimization problem is

$$\min \; C(\mathbf{l}) = \sum_{i=1}^{5} C_i(l_i, \mathbf{l}_{-i}) \tag{16}$$

$$\text{subject to } \sum_{i=1}^{5} l_i \leq 780$$

$$80 \leq l_i \leq 250, i \in \{1, 2, 3, 4, 5\}.$$

By some calculation, we can get that the optimal solution is $l_1 = 83.0074kWh$, $l_2 = 104.3775kWh$, $l_3 = 125.6497kWh$, $l_4 = 146.8342kWh$, $l_5 = 167.9397kWh$, $\sum_{i=1}^{5} l_i = 627.8085kWh$. Furthermore, the users communicate under an undirected, connected graph as shown in Fig. 3.

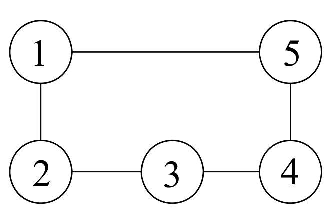

Fig. 3: Communication graph for the users in the network.

The simulation results are shown in Figs. 4-10. From these

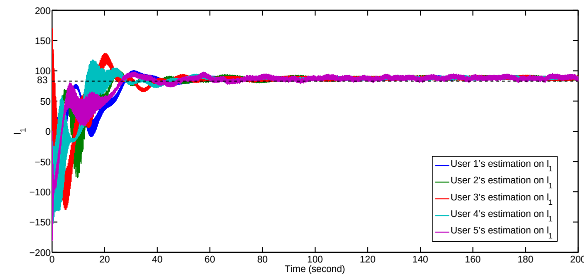

Fig. 4: Estimations on user 1's load.

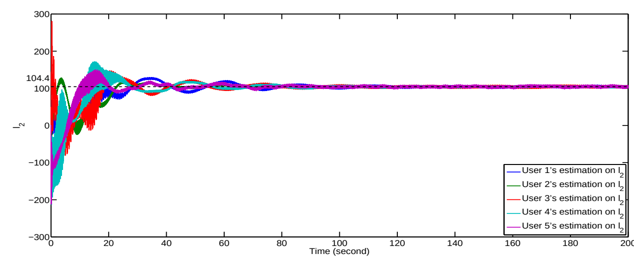

Fig. 5: Estimations on user 2's load.

figures, we see that the load estimations will converge to a small neighborhood of the actual optimal solution.

B. Benefit sharing

From the solution of the above example, we see that at the social optimum, the costs for the users are 6061.8, 5889.4, 5776.7, 5677.1, 5589.4 cents for users 1-5, respectively. Furthermore, the total cost is about 28994 cents. However, the benefit from the coordination process is not distributed fairly and it should be reallocated among the users.

If the users are conducting non-cooperative behaviors, the resulting Nash equilibrium is (107.0136, 126.3741, 145.778, 165.2209, 184.699)kWh with total load $729.0856kWh$. The costs of the users are 6275.3, 7394.1, 8517.5, 9645, 10776 cents for users 1-5, respectively. The total cost at the Nash equilibrium is 42608 cents. By these calculations, we see that the total load reduces from $729.0856kWh$ to $627.8085kWh$ and the total cost reduces from 42608 to 28994. Furthermore, the benefit can be shared according to Algorithm 1 and Algorithm 2. The comparison on the costs is shown in Fig. 11 in which Algorithm 1 is used in the benefit re-allocation. From this figure, we can see that every user reduces its cost by using the proposed method thus providing the users incentives to participate in solving the social cost minimization problem. More fairly, the benefit they get from the coordination process can be shared using Shapley value (Algorithm 2).

VI. Conclusion

In this paper, we solve the social cost minimization problem for industrial parks. The social cost minimization is solved in

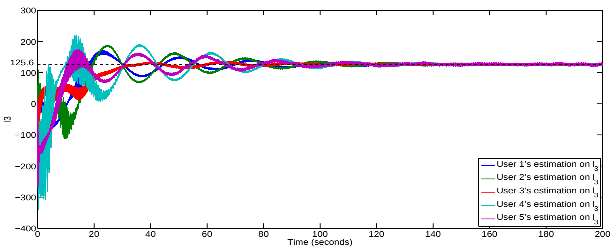

Fig. 6: Estimations on user 3's load.

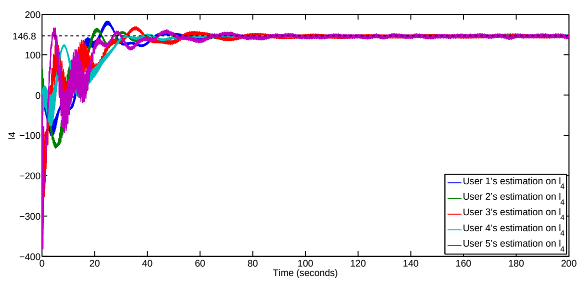

Fig. 7: Estimations on user 4's load.

a distributed fashion by the users. We employ an extremum seeking based method to solve the distributed optimization problem online. Lie bracket approximation and saddle point algorithms are used in the analysis of the updating law. We propose a profit sharing strategy for the proposed scheme to motivate the users to participate in solving the social cost minimization problem. An example for industrial consumers equipped with HVAC systems is used which helps to verify the proposed method.

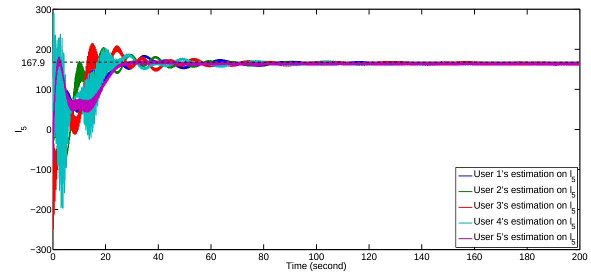

Fig. 8: Estimations on user 5's load.

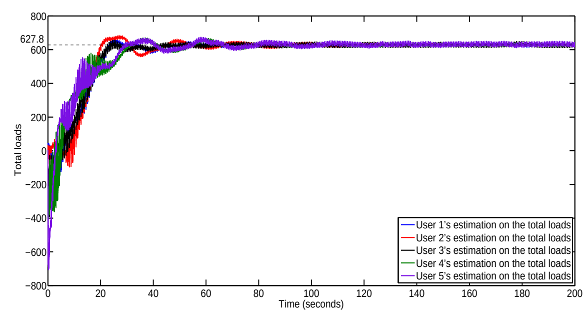

Fig. 9: Users' estimations on the total loads.

References

[1] K. Ma, G. Hu and C. Spanos, "Distributed energy consumption control via real-time pricing feedback in smart grid," *IEEE Transactions on Control Systems Technology,* Vol. 22, pp. 1907-1914, 2014.

[2] J. Wang and N. Elia, "Control approach to distributed optimization," *Allerton Conference on Communications, Control and Computing*, pp. 557-561, 2010.

[3] J. Wang and N. Elia, "A control perspective for centralized and distributed convex optimization," *IEEE Conference on Decision and Control*, pp. 3800-3805, 2011.

[4] B. Gharesifard and J. Cortes, "Distributed continuous-time convex optimization on weight-balanced digraphs," *IEEE Transactions on Automatic Control*, Vol. 59, pp. 781-786, 2014.

[5] M. Kraning, E. Chu, J. Lavaei and S. Boyd, "Dynamic network energy management via proximal message passing," *Foundations and Trends in Optimization*, Vol. 1, pp. 70–122, 2013.

[6] L. Moreau and D. Aeyels, "Practical stability and stabilization," *IEEE Transactions on Automatic Control*, Vol. 45, pp. 1554-1558, 2000.

[7] Z. Tan, P. Yang and A. Nehorai, "An optimal and distributed demand response strategy with electric vehicles in smart grid," *IEEE Transactions on Smart Grid*, Vol. 5, pp. 861-869, 2014.

[8] N. Li, L. Chen and S. Low, "Optimal demand response based on utility maximization in power networks," *IEEE Power and Energy Society General Meeting*, pp. 1-8, 2011.

[9] L. Chen, N. Li, S. Low and J. Doyle, "Two market models for demand response in power networks," *IEEE SmartGridComm*, 397-402, 2010.

[10] M. Krstic and H. Wang, "Stability of Extremum Seeking Feedback for General Nonlinear Dynamic Systems," *Automatica*, Vol. 36, No. 4, pp. 595-601, 2000.

[11] Y. Tan, D. Nesic, I. Mareels and A. Astolfi, "On global extremum seeking in the presence of local extrema," *Automatica*, Vol. 45, No. 1, pp. 245-251, 2009.

[12] K. B. Ariyur and M. Krstic, Real-Time Optimization by Extremum-Seeking Control, A John Wiley & Sons, Inc: Wiley-Interscience, 2003.

[13] Y. Tan, D. Nešić and I. Mareels, "On non-local stability properties of extremum seeking control," *Automatica*, Vol, 42, pp. 889-903, 2006.

[14] H. Durr, M. Stankovic, C. Ebenbauer and K.H. Johansson, "Lie bracket approximation of extremum seeking systems," *Automatica,* Vol. 49, pp. 1538-1552, 2013.

[15] M. Ye and G. Hu, "Distributed seeking of time-varying Nash equilibrium for non-cooperative games," *IEEE Conference on Control and Automation,* pp. 1674-1679, 2013.

[16] M. Ye and G. Hu, "Extremum seeking under input constraint for systems with a time-varying extremum," *IEEE Conference on Decision and Control*, pp. 1708-1713, 2013.

[17] H. Durr, Z. Chen and C. Ebenbauer, "Saddle point seeking for convex optimization problems," *IFAC Symposium on Nonlinear Control Systems*, pp. 540-545, 2013.

[18] C. Gellings, "The concept of demand-side management for electric utilities," *Proceedings of the IEEE*, Vol. 73, pp. 1468-1470, 1985.

[19] P. Palensky, D. Dietrich, "Demand side management: demand response, intelligent energy systems, and smart loads," *IEEE Transactions on Industrial Informatics,* Vol. 7, pp. 381-388, 2011.

[20] P. Vrba, V. Marik, P. Siano, P. Leitao, G. Zhabelova, V. Vyatkin and T. Strasser, "A review of agent and service-oriented concepts applied to intelligent energy systems," *IEEE Transactions on Industrial Informatics,* Vol. 10, pp. 1890-1903, 2014.

[21] H. Mohsenian-Rad, V. Wong, J. Jatskevich, R. Schober and A. Garcia, "Autonomous demand-side management based on game-theoretic energy consumption scheduling for the future smart grid," *IEEE Transactions on Smart Grid*, Vol. 1, pp. 320-331, 2010.

[22] P. Samadi, H. Mohsensian-Rad, R. Schober and V. Wong, "Advanced demand side management for future smart grid using mechanism design," *IEEE Transactions on Smart Grid*, Vol. 3, pp. 1170-1180, 2012.

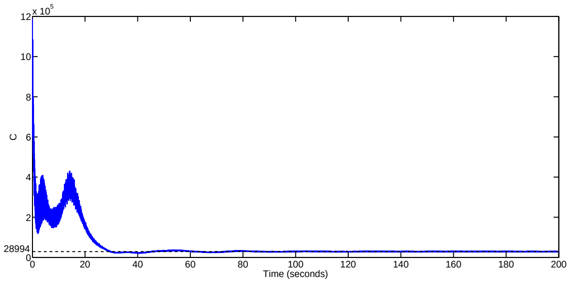

Fig. 10: The social cost produced by the proposed method in the simulation.

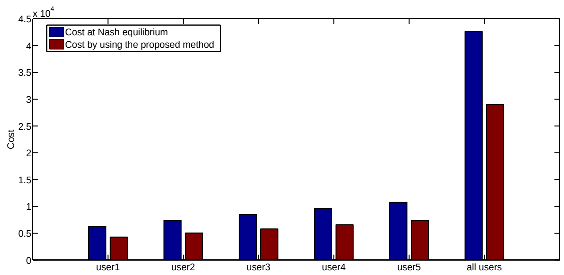

Fig. 11: The users' costs at Nash equilibrium and the costs derived by using the proposed method.

[23] Z. Baharlouei, M. Hashemi, H. Narimani and H. Mohsenian-Rad, "Achieving optimality and fairness in autonomous demand response: benchmarks and billing mechanisms," *IEEE Transactions on Smart Grid*, Vol. 4, pp. 968-975, 2013.

[24] S. Maharjan, Q. Zhu, Y. Zhang, S. Gjessing and T. Basar, "Dependable demand response management in the smart grid: a stackelberg game approach," *IEEE Transactions on Smart Grid*, Vol. 4, pp. 120-132, 2013.

[25] B. Chai, J. Chen, Z. Yang and Y. Zhang, "Demand response management with multiple utility companies: a two level game approach, " *IEEE Transactions on Smart Grid*, Vol. 5, No. 2, 2014.

[26] C. Wu, H. Mohsenian-Rad, J. Huang and A. Wang, "Demand side management for wind power integration in microgrid using dynamic potential game theory," *IEEE GLOBECOM Workshops,* pp. 1199-1204, 2011.

[27] C. Wu, H. Mohsenian-Rad and J. Huang, "Vehicle-to-aggregator interaction game," *IEEE Transactions on Smart Grid*, Vol. 3, pp. 434-442, 2012.

[28] R. Johari, S. Mannor and J. Tsitsiklis, "Efficiency loss in network resource allocation game: the case of elastic supply," *IEEE Transactions on Automatic Control*, Vol. 50, pp. 1712-1724, 2005.

[29] L. Shapley, A value for n-person games, Theory of Games II, H. W. Kuhn and A. W. Tucker, Eds. Princeton, NJ: Princeton Univ. Press, Annals of Mathematics Studies, Vol. 28, pp. 307–317, 1953.

[30] D. Bertsekas, A. Nedic and A. Ozdaglar, Convex analysis and optimization, Athena Scientific, 2003.

[31] R. Deng, Z. Yang, J. Chen, N. R. Asr and M. Chow, "Residential energy consumption scheduling: a coupled-constraint game approach," *IEEE Transactions on Smart Grid*, Vol. 5, pp. 1340-1350, 2014.

[32] N. Rahbari-Asr and M. Chow, "Cooperative distributed demand management for community charging of PHEV/PEVs based on KKT conditions and consensus networks," *IEEE Transactions on Industrial Informatics*, Vol. 10, pp. 1907-1916, 2014.

VII. Appendix

A. Proof of Lemma 1

Proof: From Problem 2→Problem 1: Since the communication network is undirected, connected and $\mathbf{L}\bar{\mathbf{l}} = \mathbf{0}$, we have, $\bar{\mathbf{l}} = \mathbf{1}_\mathbf{n}^T \otimes \mathbf{l} = [\bar{\mathbf{l}}_1, \bar{\mathbf{l}}_2, \cdots, \bar{\mathbf{l}}_N]^T$. Therefore, $\sum_{i=1}^N C_i(\bar{\mathbf{l}}_i) = \sum_{i=1}^N \left(\mu_i V_i(\bar{l}_{ii}) + P(\sum_{j=1}^N \bar{l}_{ij})\bar{l}_{ii}\right) = \sum_{i=1}^N \left(\mu_i V_i(l_i) + P(\sum_{j=1}^N l_j)l_i\right) = \sum_{i=1}^N C_i(l_i, \mathbf{l}_{-i})$, $\sum_{j=1}^N \bar{l}_{ij} \leq L_{\max}$ is equivalent to $\sum_{i=1}^N l_i \leq L_{\max}$, $l_i^{\min} \leq \bar{l}_{ii} \leq l_i^{\max}$ is equivalent to $l_i^{\min} \leq l_i \leq l_i^{\max}$.

From Problem 1→Problem 2: For $\bar{\mathbf{l}} = [\bar{\mathbf{l}}_1, \bar{\mathbf{l}}_2, \cdots, \bar{\mathbf{l}}_N]^T$, if $\bar{\mathbf{l}}_1 = \bar{\mathbf{l}}_2 = \cdots = \bar{\mathbf{l}}_N = \mathbf{l}$, then we have $\sum_{i=1}^N C_i(l_i, \mathbf{l}_{-i}) = \sum_{i=1}^N \left(\mu_i V_i(l_i) + P(\sum_{i=1}^N l_i)l_i\right) = \sum_{i=1}^N \left(\mu_i V_i(\bar{l}_{ii}) + P(\sum_{j=1}^N \bar{l}_{ij})\bar{l}_{ii}\right) = \sum_{i=1}^N C_i(\bar{\mathbf{l}}_i)$. And the constraints $\sum_{i=1}^N l_i \leq L_{\max}$ is equivalent to $\sum_{j=1}^N \bar{l}_{ij} \leq L_{\max}$ and $l_i^{\min} \leq l_i \leq l_i^{\max}$ is equivalent to $l_i^{\min} \leq \bar{l}_{ii} \leq l_i^{\max}$. To satisfy $\bar{\mathbf{l}}_1 = \bar{\mathbf{l}}_2 = \cdots = \bar{\mathbf{l}}_N = \mathbf{l}$, we let $\mathbf{L}\bar{\mathbf{l}} = \mathbf{0}$ since $\mathbf{L}\bar{\mathbf{l}} = \mathbf{0}$ if and only if $\bar{\mathbf{l}} = \mathbf{1}_\mathbf{n}^T \otimes \mathbf{l}$. ■

B. Proof of Lemma 2

Proof: According to the definition of saddle point, we have

$$\begin{aligned}
& \sup\ \tilde{F}(\bar{\mathbf{l}}^*, \mathbf{y}, \mathbf{z}) \\
= & \sup\ \Big(C(\bar{\mathbf{l}}^*) + \bar{\mathbf{l}}^{*T}\mathbf{L}\mathbf{y}^* \\
& + \sum_{i=1}^N z_{i1}\left(\sum_{j=1}^N \bar{l}_{ij}^* - L_{\max}\right) + \frac{1}{2}\bar{\mathbf{l}}^{*T}\mathbf{L}\bar{\mathbf{l}}^* \\
& + \sum_{i=1}^N z_{i2}\left(\bar{l}_{ii}^* - l_i^{\max}\right) + \sum_{i=1}^N z_{i3}\left(l_i^{\min} - \bar{l}_{ii}^*\right) \\
= & \tilde{F}(\bar{\mathbf{l}}^*, \mathbf{y}^*, \mathbf{z}^*) \\
= & C(\bar{\mathbf{l}}^*) + \bar{\mathbf{l}}^{*T}\mathbf{L}\mathbf{y}^* + \frac{1}{2}\bar{\mathbf{l}}^{*T}\mathbf{L}\bar{\mathbf{l}}^* \\
& + \sum_{i=1}^N z_{i1}\left(\sum_{j=1}^N \bar{l}_{ij}^* - L_{\max}\right) \\
& + \sum_{i=1}^N z_{i2}\left(\bar{l}_{ii}^* - l_i^{\max}\right) + \sum_{i=1}^N z_{i3}\left(l_i^{\min} - \bar{l}_{ii}^*\right).
\end{aligned} \tag{17}$$

To achieve (17), the followings must be satisfied:

$$\bar{\mathbf{l}}^{*T}\mathbf{L} = \mathbf{0},$$

$$\begin{aligned}
& \sum_{j=1}^N \bar{l}_{ij}^* - L_{\max} \leq 0 \\
& l_i^{\min} \leq l_i^* \leq l_i^{\max},
\end{aligned}$$

and

$$\begin{aligned}
z_{i1}^*\left(\sum_{j=1}^N \bar{l}_{ij}^* - L_{\max}\right) &= 0 \\
z_{i2}^*\left(\bar{l}_{ii}^* - l_i^{\max}\right) &= 0 \\
z_{i3}^*\left(l_i^{\min} - \bar{l}_{ii}^*\right) &= 0.
\end{aligned}$$

Otherwise, if $\mathbf{y} = -ksign(\mathbf{y})$ and $k \to +\infty$ or $z_{ij} \to +\infty$, then $\sup \tilde{F}(\bar{\mathbf{l}}^*, \mathbf{y}, \mathbf{z}) \to \infty$, which contradicts (17). Therefore, if $(\bar{\mathbf{l}}^*, \mathbf{y}^*, \mathbf{z}^*)$ is a saddle point for $\tilde{F}(\bar{\mathbf{l}}, \mathbf{y}, \mathbf{z})$, then, $\bar{\mathbf{l}}^*$ must be within the constraints set Ω of Problem 2. Furthermore,

$$\begin{aligned}
& \inf\ \tilde{F}(\bar{\mathbf{l}}, \mathbf{y}^*, \mathbf{z}^*) = \inf\ \Big(C(\bar{\mathbf{l}}) + \bar{\mathbf{l}}^T \mathbf{L} \mathbf{y}^* \\
& + \sum_{i=1}^{N} z_{i1}^* \left(\sum_{j=1}^{N} \bar{l}_{ij} - L_{\max} \right) + \frac{1}{2} \bar{\mathbf{l}}^T \mathbf{L} \bar{\mathbf{l}} \\
& + \sum_{i=1}^{N} z_{i2}^* \left(\bar{l}_{ii} - l_i^{\max} \right) + \sum_{i=1}^{N} z_{i3}^* \left(l_i^{\min} - \bar{l}_{ii} \right) \Big) \\
= \ & \inf_{\bar{\mathbf{l}} \in \Omega}\ C(\bar{\mathbf{l}}) \\
= \ & C(\bar{\mathbf{l}}^*),
\end{aligned}$$

that is to say, $\bar{\mathbf{l}}^*$ solves the optimization problem.

From the other aspect, if $\bar{\mathbf{l}}^*$ solves Problem 2, then there must exist $(\mathbf{y}, \mathbf{z}) \in R^{N^2} \times R_+^{3N}$ such that

$$\begin{aligned}
& \sup \tilde{F}(\bar{\mathbf{l}}^*, \mathbf{y}, \mathbf{z}) = \sup C(\bar{\mathbf{l}}^*) + \bar{\mathbf{l}}^{*T} \mathbf{L} \mathbf{y} + \\
& \sum_{i=1}^{n} z_{i1} \left(\sum_{j=1}^{N} \bar{l}_{ij}^* - L_{\max} \right) + \frac{1}{2} \bar{\mathbf{l}}^{*T} \mathbf{L} \bar{\mathbf{l}}^* \\
& + \sum_{i=1}^{N} z_{i2} \left(\bar{l}_{ii}^* - l_i^{\max} \right) + \sum_{i=1}^{N} z_{i3} \left(l_i^{\min} - \bar{l}_{ii}^* \right) \\
= \ & C(\bar{\mathbf{l}}^*) = \tilde{F}(\bar{\mathbf{l}}^*, \mathbf{y}^*, \mathbf{z}^*).
\end{aligned}$$

Furthermore, according to the Karush–Kuhn–Tucker condition and the convexity of $\tilde{F}(\bar{\mathbf{l}}, \mathbf{y}, \mathbf{z})$ with respect to $\bar{\mathbf{l}}$, we have $\tilde{F}(\bar{\mathbf{l}}^*, \mathbf{y}^*, \mathbf{z}^*)$ is a global minimizer of $\tilde{F}(\bar{\mathbf{l}}, \mathbf{y}^*, \mathbf{z}^*)$, i.e.,

$$\tilde{F}(\bar{\mathbf{l}}^*, \mathbf{y}^*, \mathbf{z}^*) \leq \tilde{F}(\bar{\mathbf{l}}, \mathbf{y}^*, \mathbf{z}^*).$$

Hence, the proof is completed. ■