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Essays on Treatment Effects and Network Treatment Allocation

By

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Abstract

The first chapter considers optimal treatment allocation in the presence of spillover effects in both take-up decisions and outcomes. When there are two layers of spillover effects, I allow an individual's outcome to depend on other individuals' compliance types in the neighborhood. I construct estimators of the welfare function and the budget constraint, then based on the characteristics of the target population, I propose treatment allocations that maximize the estimated welfare subject to the estimated budget constraint, or minimize the estimated cost subject to a welfare target. I show that the proposed allocations achieve asymptotic welfare efficiency and weak feasibility. In an anti-conflict experiment in public middle schools in New Jersey, I estimate the minimum allocation required to achieve various targets for the average expected outcome. The results show that failing to account for both layers of spillover effects can lead to a suboptimal level of treatment allocation.

When there is more than one treatment group in a randomized controlled trial (RCT) with noncompliance, local average treatment effects (LATEs) are not comparable because each treatment has different compliers. In the second chapter, I propose a partial-identification approach to compare two treatments in the presence of noncompliance. In particular, under extra dependence assumptions on the copula of unobserved heterogeneity, I derive pairwise sharp bounds on the average difference between the treatment effects for those who would comply with both treatments, as well as bounds on the causal estimands for other subpopulations. In an experiment evaluating the impacts of different depression treatments on economic outcomes in India, I provide bounds on the average difference between two depression treatments for always compliers.

In the third chapter, we use an influence function approach to propose a variance estimator for the Marginal Treatment Effect (MTE) estimator. With the characterization

of the asymptotic linear representation of the MTE estimator, we obtain a plug-in variance estimator that captures estimation errors from all steps of the procedure. We study the finite-sample performance of the proposed method using Monte Carlo simulations which show that our method had better coverage under moderate sample sizes, compared to the method that ignores estimation errors from intermediate steps.

Chapter 1

Optimal Treatment Under Spillover Effects and Noncompliance

1.1 Introduction

When researchers analyze how to allocate a treatment in a target population, the Stable Unit Treatment Value Assumption (SUTVA) is often imposed. Under SUTVA, an individual's outcome is unaffected by other individuals' treatment status. However, this assumption may fail when individuals are connected through a network. In such cases, outcomes may depend not only on one's own treatment but also on the treatments received by others in the network. For example, if we consider a large community as a network, when the treatment is a vaccine, an individual's health outcome may be positively affected by vaccinated individuals in the community, and negatively affected by unvaccinated ones. In this case, there are spillover effects in the health outcome since one's health also depends on others' treatment status.

The situation becomes even more complex when compliance is imperfect. In the vaccine setting, one's take-up decisions may depend on the treatment assignments of others. For instance, an individual who is assigned to take the vaccine may decide whether or not to comply after observing the assignments of family or friends in the community. Thus, in

addition to spillover effects in outcomes, there can be spillover effects in compliance behaviors. In practice, when spillovers in outcomes and compliance coexist in a network, policymakers seeking to optimally allocate a treatment need to account for both. Ignoring these spillover effects may lead to suboptimal policies.

In this paper, I consider the problem of optimally allocating a treatment, taking into account spillover effects in both take-up decisions and outcomes. By optimal allocation, I mean that based on the characteristics of the target population, the treatment allocation should maximize a welfare function subject to a budget constraint, or minimize the cost of treatment subject to a welfare target specified by policymakers. Incorporating budget constraints and treatment costs is important in practice, as treatments are often costly and only a limited number can be provided. The analysis in this paper is restricted to randomized treatment rules, which can be considered as Bernoulli offers of a treatment.

To incorporate spillover effects into the welfare maximization or cost minimization problem, I explicitly model the structure of spillovers in both compliance and outcomes. For compliance, I assume that each individual's take-up decision depends on her own assignment and the assignment of a fixed peer in the network, and peer groups are disjoint. Under this assumption, spillovers in compliance are only within peer groups. A peer group can be viewed as a pair of "best friends", siblings, roommates, or a couple, among other realistic interpretations. For outcomes, I assume that an individual's outcome depends on her own treatment and a summary statistic of neighbors' treatment status, which can be the number of treated neighbors or the fraction of treated neighbors. An individual's neighbors are those who are connected to her in the network. Here the spillover in outcome is anonymous as one's outcome depends on a summary statistic of neighbors' treatment status, ignoring their identities. This combined peer-plus-neighbor structure allows us to capture two layers of spillover effects while ensuring tractability of the optimization problem.

To estimate optimal treatment allocation, I construct an empirical welfare function using an augmented inverse probability weighting (AIPW) model. Based on the empirical welfare

function with an empirical cost, I propose optimal treatment rules that maximize empirical welfare or minimize empirical cost. Meanwhile, I consider a linear-in-means model for the empirical welfare function. To evaluate the treatment rules proposed, I discuss asymptotic welfare efficiency and asymptotic weak feasibility, which are similar to the theoretical properties of treatment rules from Sun (2024). A treatment rule is asymptotically welfare-efficient if the associated welfare loss is not likely to be positive when network size goes to infinity. Specifically, I focus on convergence in probability of the welfare loss relative to the maximum attainable welfare with known welfare function. I consider a treatment rule as asymptotically weakly feasible if it is not likely to exceed the budget constraint by an arbitrarily small magnitude when network size goes to infinity. I show that the proposed treatment rules based on different empirical welfare functions achieve asymptotic welfare efficiency and weak feasibility under different restrictions on the growth of the network.

This paper builds on the traditional literature on statistical treatment rules, including Manski (2004), Dehejia (2005), Hirano and Porter (2009), Stoye (2009), Chamberlain (2011), Bhattacharya and Dupas (2012), Stoye (2012), Tetenov (2012), Armstrong and Shen (2015), Kasy (2016), Kitagawa and Tetenov (2018, 2021), Athey and Wager (2021), Mbakop and Tabord-Meehan (2021), and Zhou et al. (2023), among others. A critical assumption in this literature is SUTVA. The key difference between this paper and the existing literature is that this paper studies optimal treatment rules in the presence of spillover effects which make SUTVA fail. Thus this paper can be viewed as an extension of the discussion of optimal treatment rules from i.i.d. settings to network settings with spillover effects.

This paper mainly contributes to the growing literature on optimal treatment rules in networks, including Laber et al. (2018), Su et al. (2019), Ananth (2020), Jiang et al. (2023), Kitagawa and Wang (2023), Viviano (2024), and Park et al. (2024). One important point shared by all of these network papers is that they assume perfect compliance, which implies that there are no spillover effects in take-up decisions. Thus, the main contribution of this paper is to explicitly account for spillover effects in take-up decisions when analyzing optimal

treatment rules in a network. In this literature, Park et al. (2024) is most closely related to this paper. They focus on the minimum treatment level needed to achieve a welfare target and consider randomized treatment rules at the cluster level, assuming partial interference such that spillover effects in outcomes occur only within clusters. In contrast, I allow spillover effects to occur in a single large network such that there can be spillovers within and across clusters.

Additionally, this paper connects to the literature on the identification of spillover effects under noncompliance in networks including Kang and Imbens (2016), Kang and Keele (2019), Imai et al. (2021), DiTraglia et al. (2023), Vazquez-Bare (2023), and Hoshino and Yanagi (2024). Among these papers, some focus on partial interference, some only consider one layer of spillover effects, or provide identification results that are not directly applicable for welfare maximization or cost minimization problems for the whole target population. In contrast, the treatment rules in this paper are built based on an identification result that allows spillovers to occur in a single large network. The identification result also incorporates two layers of spillover effects and is on the welfare of the whole target population.

Taken together, this paper is the first to propose optimal treatment rules considering spillovers in both take-up decisions and outcomes in a single large network. The combination of two layers of spillovers has several implications that are new to the network literature. First, an individual’s outcome can be correlated with the outcomes of individuals that they do not share any neighbors with, while existing papers usually assume that an individual’s outcome is correlated with that of another individual only if they are neighbors of each other or share a common neighbor. For example, a couple may have two groups of friends that do not completely overlap. Suppose individual $F1$ is husband’s friend but not wife’s friend, $F2$ is wife’s friend but not husband’s friend, and $F1$ and $F2$ are not friends of each other. In this case, $F1$ and $F2$ ’s outcomes are allowed to be correlated. More importantly, our framework is novel in allowing an individual’s outcome to depend not only on her own compliance type but also on the vector of individual compliance types in the neighborhood.

Next, existing papers usually assume that there is no uncertainty or spillover in the cost of treatment. In this paper, by contrast, the cost of a treatment allocation is uncertain due to noncompliance. Further, we assume that a cost is incurred only when an individual assigned to treatment actually complies; there is no cost when noncompliance occurs. This feature implies that spillovers also show up in the cost of a treatment allocation, as compliance is subject to spillovers. In a non-network setting, Sun (2024) shows that there exist data distributions such that optimal treatment rules cannot achieve asymptotic welfare efficiency when cost is uncertain due to noncompliance. In our setting, I propose sufficient conditions under which asymptotic welfare efficiency and weak feasibility can be achieved. This paper is also the first to discuss simultaneous asymptotic welfare efficiency and weak feasibility of treatment rules in a network setting, especially when the cost function contains uncertainty and spillovers.

I revisit the social network experiment from Paluck et al. (2016), which was designed to encourage anti-conflict norms and behaviors among students in public middle schools in New Jersey. In the experiment, students assigned to the treatment were invited to attend program meetings regularly. Previous literature focuses on identification of treatment and spillover effects and do not properly account for spillover effects in compliance. Considering that there can be both layers of spillover effects in the social network of students, I apply the proposed treatment rules to the experiment and find that not incorporating both layers of spillover effects results in an overestimation of maximum welfare given a budget constraint, or an insufficient level of assignment given a welfare target. I also show that the optimal treatment allocation is driven by characteristics of students.

The remainder of the paper is structured as follows. Section 2 describes the setting and the optimization problem. Section 3 develops the estimation procedure. Section 4 presents theorems on the theoretical properties of the proposed treatment rules. Section 5 provides an empirical application. Section 6 concludes.

1.2 Setup

Consider a target population of size n in an undirected network represented by an $n \times n$ symmetric adjacency matrix A , with $A_{ij} = 1$ if individual i and j are connected and $A_{ij} = 0$ otherwise. Individual j is called i 's neighbor if $A_{ij} = 1$. I assume that there are no self-links, so that $A_{ii} = 0$ for all i . Let γ_i denote the number of neighbors of individual i , referred to as i 's network degree. Let $\gamma_n^{max} = \max\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ indicate the maximum network degree within the target population .

I assume that our data is from an experiment on the whole target population. Let $Z_i \in \{0, 1\}$ indicate whether individual i is assigned to treatment and $D_i \in \{0, 1\}$ indicate treatment receipt. Let Y_i denote i 's outcome and X_i denote i 's characteristics. In the pilot experiment, policymakers observe $\{(Y_i, D_i, Z_i, X_i)\}_{i=1, \dots, n}$ and the network A , where $\{Z_i\}_{i=1, \dots, n}$ are independently generated by random assignment. Later I show that to find the optimal treatment allocation, we can manipulate the marginal distribution of $\{Z_i\}_{i=1, \dots, n}$ by varying the probability of random assignment.

When there are both spillover effects and imperfect compliance, individual i 's take-up decision can depend not only on her own assignment. I make the following assumption on the structure of spillovers in compliance. Specifically, I assume that individual i 's treatment receipt depends only on both i 's own assignment Z_i and the assignment of a fixed peer Z_i^P , and this connection is observed by the researchers. Peer groups are disjoint in the sense that if $A_{ij}^P = 1$, then $A_{ik}^P = 0$ for $j \neq k$. In other words, each individual can be the peer of only one other individual. In practice, this peer group structure applies to settings where there is spillover in decision-making within pairs. Vazquez-Bare (2023) considers an experiment on voting behaviors in which take-up of the treatment can depend on the spouse's assignment. Let A^P denote an $n \times n$ symmetric matrix, with $A_{ij}^P = 1$ if j is i 's peer. It is assumed that a peer is also a neighbor, so $A_{ij} = 1$ whenever $A_{ij}^P = 1$. The target population is characterized by P , the joint distribution of $(Y_i, D_i, Z_i, X_i, A, A^P)$. In addition to $\{(Y_i, D_i, Z_i, X_i)\}_{i=1, \dots, n}$ and network A , policymakers also observe A^P .

Assumption 1.2.1. (*Interference I*)

(i) $D_i = D_i(Z_i, Z_i^P) \quad \forall i$;

(ii) $\{\{D_i(z, z^P)\}_{z, z^P \in \{0,1\}}\}_{i=1, \dots, n}$ are i.i.d. conditional on A and A^P .

In Assumption 1.2.1, condition (i) states that unit i 's treatment status depends on her own assignment and her peer's assignment. This is an exposure mapping for treatment, that is, how an individual's treatment status depends on other individuals' treatment assignments. Condition (ii) states that potential treatment status is i.i.d. conditional on A and A^P . Note that $\{D_1, D_2, \dots, D_n\}$ are not i.i.d. as D_i and D_j for $i \neq j$ depend on (Z_i, Z_j) .

Assumption 1.2.2. (*Monotonicity*)

(i) $D_i = 0$ if $Z_i = 0$;

(ii) $D_i(z, z^P)$ is nondecreasing in z^P .

Assumption 2.2.2 (i) is a common assumption in randomized experiments, it implies that individuals who are not assigned to treatment do not have access to the treatment. In this case, there are no always-takers or defiers. This is plausible when treatments are costly and only a limited number of treatments are available. Compared to Assumption 2.2.2 (i), Assumption 2.2.2 (ii) is novel in the framework of potential treatments. It requires that there should be no “anti-social” behaviors, which can be described as: “comply when the fixed peer is not assigned” and “do not comply when the fixed peer is assigned”. Combining (i) and (ii) in Assumption 2.2.2, treatment D_i is monotone in z_i and z_i^P . Vazquez-Bare (2023) makes a similar assumption on spillovers in compliance, which allows the existence of always takers.

Under Assumptions 1.2.1 and 2.2.2, each individual can be categorized into one of the following compliance types:

$$\tilde{T}_i = \begin{cases} \text{(SC) Self-complier:} & D_i(1, 0) = 1, \quad D_i(1, 1) = 1, \\ \text{(SOC) Social-complier:} & D_i(1, 0) = 0, \quad D_i(1, 1) = 1, \\ \text{(NT) Never-taker:} & D_i(1, 0) = 0, \quad D_i(1, 1) = 0, \end{cases}$$

where let $\tilde{T}_i$ indicate i 's compliance type.

Given these individual compliance types, we can characterize i 's neighbors' compliance types as a vector. Let $\tilde{T}_i^P$ denote the compliance type of i 's peer, and let $\tilde{\mathbf{TP}}_i^N$ denote the ordered vector of compliance types in i 's neighborhood excluding i 's peer:

$$\tilde{\mathbf{TP}}_i^N = (\tilde{T}_j)_{A_{ij}^P=0, j \in N_i} \quad (1.1)$$

where N_i is the set of i 's neighbors with $N_i = \{j \mid A_{ij} = 1\}$. An example of $\tilde{\mathbf{TP}}_i^N$ is $(SC, SOC, \dots, NT)$ with length $\gamma_i - 1$, which describes the compliance types of i 's neighbors excluding her peer. The ordered structure of $\tilde{\mathbf{TP}}_i^N$ means that it distinguishes identities.

At the same time, let $\mathbf{TP}_i^N$ denote the unordered vector that counts the number of each compliance type in i 's neighborhood, excluding i 's peer:

$$\mathbf{TP}_i^N = \left(\sum_j \mathbf{1}\{\tilde{T}_j = SC\}, \sum_j \mathbf{1}\{\tilde{T}_j = SOC\}, \sum_j \mathbf{1}\{\tilde{T}_j = NT\} \right)_{A_{ij}^P=0, j \in N_i}.$$

Compared to the ordered $\tilde{\mathbf{TP}}_i^N$ in (1.1), $\mathbf{TP}_i^N$ does not distinguish identities, since it only counts the number of each compliance type. Note that given some value of $\tilde{\mathbf{TP}}_i^N$, we can pin down $\mathbf{TP}_i^N$. This means that multiple different $\tilde{\mathbf{TP}}_i^N$ can be mapped to the same $\mathbf{TP}_i^N$.

In addition to spillovers in compliance, I assume that there are also spillovers in outcomes and make the following assumption:

Assumption 1.2.3. (*Interference II*)

$$Y_i = r(D_i, T_i, X_i, \gamma_i, \epsilon_i), \quad (1.2)$$

where r is an unknown real-valued function, $T_i = T(\mathbf{D}_{-i}, A)$ is a summary statistics of neighbors' treatment status with exposure mapping $T : \{0, 1\}^{n-1} \times \mathcal{A} \rightarrow \mathcal{T} \subseteq \mathbb{R}$, and $X_i \in \mathbb{R}^q$ denotes individual characteristics.

Under Assumption 1.2.3, individual i 's outcome depends on individual treatment receipt D_i ,

a summary statistic T_i that captures how i 's outcome depends on others' treatment status, observed individual characteristics X_i , network degree γ_i , and unobservables ϵ_i . For T_i , I consider two cases: (i) $\sum_{j \in N_i} D_j$, (ii) $(\sum_{j \in N_i} D_j)/\gamma_i$. The first option requires i 's outcome to depend on the number of treated neighbors, and the second option states that i 's outcome depends on the fraction of treated neighbors. For both cases, the spillover in outcomes is anonymous. These two summary statistics are common choices in the network literature, for example, Cai et al. (2015), Aronow and Samii (2017), Ananth (2020), Leung (2022), Viviano (2024), among others. Here the fraction of treated neighbors is considered as a discrete variable.

Example 1.2.1. Bhattacharya and Dupas (2012) consider an application on optimal treatment allocation in Kenya. To control malaria in certain regions of the country, the government subsidizes the purchase of insecticide-treated bed nets (ITNs), but only a small fraction of the target population is eligible to receive the subsidy. They maintain the SUTVA assumption when analyzing optimal allocations of bed nets and focus on adoption as the primary outcome variable. We may extend the discussion to our framework, incorporating spillover effects in both adoption and health outcomes. On the one hand, malaria is infectious and adoption of bednets can have positive spillover effects on neighbors. On the other hand, adoption is not compulsory and we may assume that adoption decisions depend on the assignment of the closest household. In this case, we assume that spillovers in adoption decisions are narrow, whereas health outcomes may depend on bed net adoption across the entire neighborhood or further, since malaria is transmitted by mosquitoes and spillover effects on health can occur broadly.

To summarize the two layers of spillovers, I use Figure 1 as a visualization of the structure. The first layer on the left is due to noncompliance: unit i decides whether to take the treatment after observing her own assignment and her peer's assignment. The second layer on the right shows that unit i 's outcome is affected by the treatments of her neighbors, regardless of her own treatment status. In Figure 1, Z_1 and Z_2 form a peer group, and Z_3

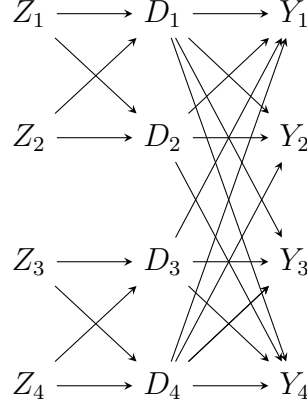


Figure 1.1: Spillovers in Compliance and Outcome

and Z_4 form another peer group. Actual treatment receipt depends only on within-group assignments, so unit 2's treatment receipt is not affected by the assignments of units 3 or 4. For the spillover in outcomes, unit 1's outcome depends on D_1 , D_2 , D_3 , and D_4 , since unit 1's neighbors include units 2, 3, and 4.

Each of these two layers is a direct spillover effect. There are also indirect spillovers. For example, unit 3 is not a neighbor of unit 2, but unit 3's assignment affects unit 4's treatment, which in turn affects unit 2's outcome, since unit 4 is one of unit 2's neighbors. This implies that when analyzing unit i 's outcome, we must take into account all units that affect it either directly or indirectly. For convenience, define

$$G_i = N_i \cup G_i^{L2} \cup G_i^{L3} \quad (1.3)$$

where

$$\begin{aligned} N_i &= \{j \mid A_{ij} = 1\} \\ G_i^{L2} &= \{j \mid \max_k A_{ik} A_{kj} = 1, A_{ij} = 0, i \neq j \neq k\} \\ G_i^{L3} &= \{j \mid \max_{k,l} A_{ik} A_{kl}^P A_{lj} = 1, \max_k A_{ik} A_{kj}^P = 0, A_{ij} = 0\}. \end{aligned}$$

In (1.3), G_i contains units j for which i 's and j 's outcomes depend on the same Z_k for some

k , and is the union of three disjoint sets: N_i , G_i^{L2} , and G_i^{L3} . Units i and j are “correlated” if any one of the following three conditions holds: (i) they are directly connected, $A_{ij} = 1$; (ii) j is not i ’s neighbor, but i and j share a common neighbor k , $\max_k A_{ik}A_{kj} = 1$, i and j are two links away from each other; (iii) neither (i) nor (ii) holds, but j is a neighbor of l , l and k form a peer group, and k is a neighbor of i . In this case, i and j ’s neighborhoods share some peer group so that $\max_{k,l} A_{ik}A_{kl}^P A_{lj} = 1$. In this case, i and j are three links apart.

In summary, if units i ’s and j ’s outcomes depend on the same Z_k for some k , then they cannot be more than three links apart in the network. When there is perfect compliance, previous literature usually has that two units are “correlated” only if they share a common neighbor — they are no more than two links apart. For convenience, let G_i^P denote the set of units that are not i ’s neighbors but are peers of i ’s neighbors:

$$G_i^P = \{j \mid \max_k A_{ik}A_{kj}^P = 1, A_{ij} = 0, i \neq j\}. \quad (1.4)$$

Thus, $j \in G_i^P$ if i and j are not directly connected but j ’s peer is a neighbor of i .

Define $\mathbf{Z}_i^{NP}$ as the vector of ordered assignments of i ’s neighbors and their peers, excluding i ’s peer:

$$\mathbf{Z}_i^{NP} = (Z_j, Z_j^P)_{A_{ij}^P=0, j \in N_i}.$$

Since $\mathbf{Z}$ is the ordered assignment vector including all units, $\mathbf{Z}_i^{NP}$ is a subvector of $\mathbf{Z}$.

Under the spillover structure discussed above, I also make the following exogeneity assumptions on the pilot data:

Assumption 1.2.4. (*Exogeneity*)

- (i) $\mathbf{Z} = \{Z_1, \dots, Z_n\}$ are *i.i.d.*, $\mathbf{Z} \perp A, A^P$, and $\mathbf{Z} \perp \{\epsilon_i, D_i(z, z^P), X_i, \gamma_i\}_{z, z^P \in \{0,1\}, i=1, \dots, n} \mid A, A^P$;
- (ii) $\{D_i(z, z^P)\}_{z, z^P \in \{0,1\}, i=1, \dots, n}, \{\epsilon_i\}_{i=1, \dots, n} \perp \mathbf{X} \mid A, A^P$;
- (iii) $\epsilon_i \perp \{\epsilon_j\}_{j: j \notin G_i} \mid A, A^P, \{X_i\}_{i=1, \dots, n}$;
- (iv) $\{X_1, \dots, X_n\}$ are *i.i.d.* conditional on A, A^P .

In Assumption 1.2.4, condition (i) requires the assignment to be independent of potential treatment status, network degrees, covariates, and unobservables conditional on the network. This condition is plausible if we assume that our data is from a pilot experiment. It also requires the network to be exogenous such that treatment assignment does not change the network. Note that when there is perfect compliance, the i.i.d. treatment receipt assumption is usually made. Here, we assume that assignments are i.i.d. and allow actual treatments to be correlated. Condition (ii) states that potential treatment status and unobservables are independent of covariates. Condition (iii) allows i 's unobservable to be correlated with those of correlated units. Condition (iv) is a common assumption in the network literature requiring individual characteristics to be i.i.d. conditional on the network.

Next, I discuss the planner's problem, in which policymakers use the pilot data to determine the optimal treatment rule based on the network A , A^P , and observed characteristics $\mathbf{X}$ of the target population.

1.2.1 Planner's Problem

I consider the social planner's problem as a mapping $\mathcal{X}^n \times (\mathcal{A}_n, \mathcal{A}_n^P) \rightarrow \Theta = [0, 1]$, where $\mathcal{A}_n$ and $\mathcal{A}_n^P$ are sets of symmetric and unweighted adjacency matrices with size $n \times n$, $\mathbf{X} \in \mathcal{X}^n$. Policymakers choose θ based on the characteristics of the target population and the associated network. The policy θ is the probability of random assignment and can be interpreted as a Bernoulli offer of treatment. This can also be considered as a random peer encouragement design where policymakers determine the intensity of the encouragement. Kang and Imbens (2016) consider personalized encouragement such that every individual's treatment only depends on her own encouragement, that is, her own treatment assignment. In our case, the encouragement is not personalized as one's treatment does not only depend on own encouragement.

Given the network and characteristics of the target population, the utilitarian social

welfare associated with a policy θ is defined as the average expected outcome:

$$W(\theta, \mathbf{X}, A, A^P) = \frac{1}{n} \sum_{i=1}^n \sum_{\mathbf{z} \in \mathbb{Z}^n} E[Y_i \mid \mathbf{Z} = \mathbf{z}, \mathbf{X}, A, A^P] \pi(\mathbf{z}, A, A^P; \theta), \quad (1.5)$$

where $\mathbb{Z}^n = \{0, 1\}^n$, and $\pi(\mathbf{z}, A, A^P; \theta)$ is the probability of the realization $\mathbf{z}$ under θ . Under a policy θ , units are independently assigned to treatment with probability θ , and each individual's outcome depends on the realized assignments of the population. Hence the expected outcome is a weighted average of outcomes across different realizations of the assignment vector $\mathbf{Z}$. For any given θ , we can compute $\pi(\mathbf{z}, A, A^P; \theta)$ for any $\mathbf{z}$. Note that in the pilot experiment, we observe outcomes for the target population under only one realization of $\mathbf{Z}$.

Associated with the welfare function, I assume that for each individual assigned to treatment there is a constant cost, and the cost occurs only when the individual complies with the assignment. Under the one-sided compliance assumption, there is no cost when noncompliance occurs. Given the network and characteristics of the target population, the expected average cost associated with policy θ is defined by:

$$C(\theta, A^P) = \frac{1}{n} \sum_{i=1}^n \sum_{\mathbf{z} \in \mathbb{Z}^n} E[D_i \mid \mathbf{Z} = \mathbf{z}, A^P] \pi(\mathbf{z}, A^P; \theta), \quad (1.6)$$

where the constant cost per unit is omitted for simplicity. Note that spillovers exist in both the welfare function and the cost function, since they exist in both compliance and outcomes.

To find the optimal treatment rule θ , we can either maximize the welfare function subject to a cost constraint, or minimize the cost subject to a welfare target, depending on specific scenarios. The latter is common when policymakers first set a target for the average expected outcome and then find the treatment rule that achieves the target at minimum cost. When the goal is to maximize welfare under a budget constraint, I assume that the average expected cost cannot exceed $B \in (0, 1)$. Since the cost minimization problem is similar, I focus on

welfare maximization. An optimal treatment rule for welfare maximization is defined as:

$$\theta_n^*(\mathbf{X}, A, A^P) \in \arg \max_{\theta \in \Theta} W(\theta, \mathbf{X}, A, A^P) \quad s.t. \quad C(\theta, A^P) \leq B. \quad (1.7)$$

In the literature on Empirical Welfare Maximization (EWM), empirical maximization of $W(\theta)$ requires estimates of $E[Y_i \mid \mathbf{Z} = \mathbf{z}, \mathbf{X}, A]$ for all $\mathbf{z} \in \mathbb{Z}^n$. However, in our data we observe only one realization of $\mathbf{Z}$. To proceed, I rely on assumptions about the interference structure to reduce the number of distinct conditional means.

Assumption 1.2.5. (*Unordered Mean*)

$$\begin{aligned} E[r(d, t, x, \gamma_i, \epsilon_i) \mid \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{TP}}_i^N = \tilde{\mathbf{tp}}^N, X_i = x, A, A^P] \\ = E[r(d, t, x, \gamma_i, \epsilon_i) \mid \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{TP}}_i^N = \tilde{\mathbf{tp}}^{N'}, X_i = x, A, A^P] \end{aligned}$$

for all (d, t) whenever $\mathbf{tp}^N = \mathbf{tp}^{N'}$.

Given the same treatment receipt D_i , summary statistic T_i , i 's and i 's peer's compliance types $\tilde{T}_i$ and $\tilde{T}_i^P$, and the same counts of compliance types in i 's neighborhood, the expected outcome does not depend on the order of compliance types in the neighborhood. In other words, the expected outcome depends on $\mathbf{TP}_i^N$ rather than on $\tilde{\mathbf{TP}}_i^N$. This can be interpreted as anonymity of compliance types in i 's neighborhood, excluding i 's peer. The anonymity here is similar to the common stratified interference assumption in the network literature. Later I show that the anonymity of dependence on neighbors' compliance types is beneficial for the identification of (1.5). Assumption 1.2.5 is a relatively weak assumption, in the sense that we still allow i 's potential outcome to depend on the compliance types of neighbors, not only on i 's own compliance type. For example, in an education training program, for the same treated student, I allow her expected scores to be different when she has the same treatment status, same compliance type and same number of treated friends, but different numbers of friends who are self-compliers.

The following proposition shows that we can simplify the expected outcome in (1.5), which depends on the ordered assignment vector $\mathbf{Z}$.

Proposition 1.2.1. *Under Assumptions 1.2.1–1.2.5, the following equality holds:*

$$E[Y_i \mid \mathbf{Z}, \mathbf{X}, A, A^P] = E[Y_i \mid Z_i, Z_i^P, Z_i^{1,0}, Z_i^{1,1}, X_i, A, A^P], \quad (1.8)$$

where Z_i is i 's assignment, Z_i^P is i 's peer's assignment, $Z_i^{1,0}$ is the number of i 's neighbors assigned to treatment while their peers are not, and $Z_i^{1,1}$ is the number of i 's neighbors assigned to treatment together with their peers.

Under Proposition 1.2.1, $E[Y_i \mid \mathbf{Z} = \mathbf{z}, \mathbf{X}, A, A^P]$ is equal to $E[Y_i \mid \mathbf{Z} = \mathbf{z}', X_i, A, A^P]$ for $\mathbf{z} \neq \mathbf{z}'$ whenever $(z_i, z_i^P, z_i^{1,0}, z_i^{1,1}) = (z'_i, z_i^{P'}, z_i^{1,0'}, z_i^{1,1'})$. Note that $(Z_i^P, Z_i^{1,0}, Z_i^{1,1})$ can be viewed as a vector summary statistic of treatment assignments in i 's neighborhood, excluding i 's peer. Under perfect compliance, we usually have one-dimensional summary statistics. For convenience, define

$$\mu(\tilde{\mathbf{z}}_i, X_i, A, A^P) = E[Y_i \mid \tilde{\mathbf{Z}}_i = \tilde{\mathbf{z}}_i, X_i, A, A^P],$$

where $\tilde{\mathbf{Z}}_i = (Z_i, Z_i^P, Z_i^{1,0}, Z_i^{1,1})$ and $\tilde{\mathbb{Z}}_i$ denotes the support of $\tilde{\mathbf{Z}}_i$. We can rewrite the welfare function as

$$W(\theta, \mathbf{X}, A, A^P) = \frac{1}{n} \sum_{i=1}^n \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathbb{Z}}_i} \pi(\tilde{\mathbf{z}}_i, A, A^P; \theta) \mu(\tilde{\mathbf{z}}_i, X_i, A, A^P). \quad (1.9)$$

Similarly, under Assumptions 1.2.1, 2.2.2, and 1.2.5, the cost function can be written as

$$C(\theta, A^P) = \sum_{z, z' \in \{0,1\}} E[D_i \mid Z_i = z, Z_i^P = z', A^P] \theta^{z+z'} (1-\theta)^{2-z-z'}. \quad (1.10)$$

1.2.2 Partial Interference

The optimization problem and structure on spillover effects are also applicable for setups where spillover effects occur only within clusters. In this case, we observe many networks rather than a single large network, and we have separate optimization problems for different networks. Since the optimal θ is driven by the characteristics of the target population, the optimal θ can vary across clusters. At the same time, cost and budget can also vary across clusters. Instead of imposing the same policy on the whole population, policymakers may set different welfare targets for different clusters, then obtain the optimal θ for each cluster; or set the same welfare target for all clusters, then get different minimum levels of assignment for different clusters. Later in the empirical application, I show that for some clusters with certain characteristics, it can be more or less costly to achieve the same welfare target, compared to other clusters.

Example 1.2.2. Park et al. (2024) consider an application to Senegal’s water, sanitation, and hygiene (WASH) policy with perfect compliance. Setting the diarrhea-free target threshold around 70%, policymakers face high installation costs in small, rural clusters and low costs in well-developed clusters. At the same time, Park et al. (2024) show that the minimum level of assignment is driven by the average household size within each cluster. In this scenario, clusters with different characteristics and costs are assigned to adopt the WASH facility with varying assignment probabilities. When installation of such facilities is difficult to enforce and noncompliance arises, estimating the optimal θ for each cluster requires accounting for spillovers in compliance. In this case, network A and A^P may be measured using geographic distances or survey data.

1.3 Estimation

The optimal treatment allocation solves for an estimate of (1.9) subject to an estimate of (1.10). Let W_n denote the empirical welfare function. I first construct W_n using the following

augmented inverse probability weighting (AIPW) model:

$$W_n(\theta, \mathbf{X}, A, A^P) = \frac{1}{n} \sum_{i=1}^n \left\{ \frac{\pi(\tilde{\mathbf{Z}}_i, A, A^P; \theta)}{\pi(\tilde{\mathbf{Z}}_i, A, A^P; \theta^{pilot})} (Y_i - \hat{\mu}_{K_i}(\tilde{\mathbf{Z}}_i, X_i, A, A^P)) + \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathcal{Z}}_i} \pi(\tilde{\mathbf{z}}_i, A, A^P; \theta) \hat{\mu}_{K_i}(\tilde{\mathbf{z}}_i, X_i, A, A^P) \right\} \quad (1.11)$$

where $\pi(\tilde{\mathbf{Z}}_i, A, A^P; \theta^{pilot})$ denotes the probability for $\tilde{\mathbf{Z}}_i$ in the pilot experiment, and θ^{pilot} is the probability of random assignment in the pilot experiment which is known. Here, W_n is constructed with cross-fitting: $\mu(\tilde{\mathbf{z}}_i, X_i, A, A^P)$ is estimated by $\hat{\mu}_{K_i}(\tilde{\mathbf{z}}_i, X_i, A, A^P)$, which uses data from K_i with $K_i \subseteq (i \cup G_i)^c$. $\hat{\mu}_{K_i}(\tilde{\mathbf{z}}_i, X_i, A, A^P)$ is from an outcome model such as the regression adjustment in (1.14). Given any θ , W_n is unbiased for W defined in (1.9). When θ^{pilot} is unknown, it can be consistently estimated under completely random design of the pilot experiment.

For the empirical cost, let C_n denote the empirical analog of (1.10). To construct C_n , we estimate $E(D_i \mid Z_i = z, Z_i^P = z', A^P)$ using its empirical analog:

$$\hat{E}(D_i \mid Z_i = z, Z_i^P = z', A^P) = \frac{\sum_{i=1}^n \mathbf{1}\{D_i = 1, Z_i = z, Z_i^P = z'\}}{\sum_{i=1}^n \mathbf{1}\{Z_i = z, Z_i^P = z'\}}.$$

Then we have the following empirical cost function:

$$C_n(\theta, A^P) = \sum_{z, z' \in \{0,1\}} \frac{\sum_{i=1}^n \mathbf{1}\{D_i = 1, Z_i = z, Z_i^P = z'\}}{\sum_{i=1}^n \mathbf{1}\{Z_i = z, Z_i^P = z'\}} \theta^{z+z'} (1 - \theta)^{2-z-z'}. \quad (1.12)$$

Combining the empirical welfare and cost functions, I consider the following optimization problem:

$$\hat{\theta}(\mathbf{X}, A, A^P) \in \arg \max_{\theta \in \Theta} W_n(\theta, \mathbf{X}, A, A^P) \quad s.t. \quad C_n(\theta, A^P) \leq B. \quad (1.13)$$

At the same time, I consider the following linear-in-means model when constructing the

empirical welfare function:

$$E[Y_i \mid \tilde{\mathbf{Z}}_i, X_i, A, A^P] = \beta_0 + \tilde{\mathbf{Z}}_i' \beta_1 + X_i' \beta_2 + \gamma_i \beta_3. \quad (1.14)$$

This is similar to the common linear-in-means models in the network literature where a summary statistic of neighbors' treatment status, individual network degree, and individual characteristics are included as covariates. In this framework, the parameters in the linear-in-means model may yield ITT style causal interpretations for treatment effects or spillover effects.

Let $\widetilde{W}_n(\theta, \mathbf{X}, A, A^P)$ denote the empirical welfare function constructed using (1.14), define $\tilde{\theta}(\mathbf{X}, A, A^P)$ as follows:

$$\tilde{\theta}(\mathbf{X}, A, A^P) \in \arg \max_{\theta \in \Theta} \widetilde{W}_n(\theta, \mathbf{X}, A, A^P) \quad s.t. \quad C_n(\theta, A^P) \leq B. \quad (1.15)$$

With $W_n(\theta, \mathbf{X}, A, A^P)$, $\widetilde{W}_n(\theta, \mathbf{X}, A, A^P)$ and $C_n(\theta, A^P)$ constructed above, I obtain $\hat{\theta}(\mathbf{X}, A, A^P)$ and $\tilde{\theta}(\mathbf{X}, A, A^P)$ by implementing a dense grid search on $[0, 1]$ for the maximization problems.

1.4 Theoretical Properties of the Optimal Policy

To evaluate the performance of the proposed $\hat{\theta}(\mathbf{X}, A, A^P)$ and $\tilde{\theta}(\mathbf{X}, A, A^P)$, I let the network size go to infinity, and discuss asymptotic welfare efficiency and feasibility of $\hat{\theta}(\mathbf{X}, A, A^P)$ and $\tilde{\theta}(\mathbf{X}, A, A^P)$. Compared to regret, which is usually considered when evaluating statistical treatment rules, asymptotic welfare efficiency is stronger in the sense that it is about convergence in probability of the welfare loss associated with the proposed treatment rule, rather than the expectation of welfare loss. First, we have the following definitions:

Definition 1.4.1. (i) *The Bernoulli treatment rule $\hat{\theta}(\mathbf{X}, A, A^P)$ is asymptotically welfare-*

efficient if for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} P\left(W(\hat{\theta}, \mathbf{X}, A, A^P) - W(\theta_n^*, \mathbf{X}, A, A^P) < -\epsilon\right) = 0;$$

(ii) and is asymptotically weakly feasible if for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} P\left(C(\hat{\theta}(\mathbf{X}, A, A^P), A^P) - B > \epsilon\right) = 0.$$

This definition is similar to those in Sun (2024). Here the asymptotic feasibility above is weak in the sense that when n goes to infinity, given arbitrarily small ϵ , the cost is allowed to float around the budget B , compared to restricting the cost to be strictly smaller than B . I then analyze the asymptotic welfare efficiency and weak feasibility of $\hat{\theta}(\mathbf{X}, A, A^P)$ when $n \rightarrow \infty$. First, I make the following assumptions:

Assumption 1.4.1. (*Boundedness and Overlap*)

- (i) There exists some finite $M > 0$ such that $|Y_i| \leq M$, and constants $C_k > 0$ such that $|X_{ik}| < C_k$ for $k = 1, \dots, K, \forall i$;
- (ii) $\theta^{pilot} \in [\delta, 1 - \delta]$ with some $\delta \in (0, 1/2)$, $\forall n$.

Assumption 1.4.1 includes the standard boundedness and overlap conditions. In particular, it is assumed that the assignment probability in the pilot data θ^{pilot} is in the interior of $[0, 1]$. However, when n goes to infinity, Assumption 1.4.1(ii) does not guarantee that $\pi(\tilde{\mathbf{Z}}_i, A, A^P; \theta^{pilot})$ in (1.11) is bounded away from 0 when there is no strong restriction on the growth of network degree. Below I discuss a set of assumptions on the network degree.

Assumption 1.4.2. (*Network Degree*)

- (i) $\sup_n \gamma_n^{max} = O(1)$;
- (ii) $\frac{1}{n} \sum_{i=1}^n \gamma_i^q$ has a finite probability limit for $q \in \{1, 2, 3\}$;
- (iii) $E[\frac{1}{n} \sum_{i=1}^n \sum_{j \in G_i} \gamma_i^2 \gamma_j^2] = O(1)$.

Assumption 1.4.2 restricts the growth of neighborhood and requires the network to be

sparse enough when n goes to infinity. Later I show that $\hat{\theta}(\mathbf{X}, A, A^P)$ and $\tilde{\theta}(\mathbf{X}, A, A^P)$ achieve asymptotic welfare efficiency results under different parts of this assumption. Part (i) imposes a uniform bound on γ_i as n grows, it means that there exists a constant such that no individual's network degree can exceed it. Note that under this condition, the number of correlated individuals $|G_i|$ in the network is also uniformly bounded. Meanwhile, Assumption 1.4.1(ii) together with Assumption 1.4.2 (i) ensure overlap when we discuss the asymptotic welfare efficiency of $\hat{\theta}(\mathbf{X}, A, A^P)$. Compared to part (i), part (ii) and (iii) are much weaker in the sense that they do not require the network degree or $|G_i|$ to be bounded. These conditions are similar to those imposed on the linear-in-means model from Leung (2020).

Assumption 1.4.3. (*Cross-Fitting*)

$\max_{i=1, \dots, n} |K_i| \leq \bar{K} n^{1-\bar{\epsilon}}$ for some $0 < \bar{K} < \infty$ and $0 < \bar{\epsilon} < 1$.

As $\hat{\mu}_{K_i}(\tilde{\mathbf{z}}_i, X_i, A, A^P)$ uses data from K_i , Assumption 1.4.3 imposes a restriction on the selection of training data for the estimation of the outcome model. It requires individual i to use at most $\bar{K} n^{1-\bar{\epsilon}}$ number of individuals outside G_i as the training data. This restriction together with Assumption 1.4.2 (i) helps control the covariance terms in the empirical welfare function in (1.11) by ensuring that, for each individual i , the number of summands with which it has nonzero covariance grows at rate $O(n^{1-\bar{\epsilon}})$.

Theorem 1.4.1. *Under Assumptions 1.2.1–1.2.5, and Assumptions 1.4.1, 1.4.2 (i), if $\sqrt{n}(B - C(\theta_n^*, A^P)) \xrightarrow{p} \infty$, the optimal treatment rule defined in (1.13) is asymptotically welfare-efficient and asymptotically weakly feasible. Specifically, for any $\epsilon > 0$ we have (i)*

$$\lim_{n \rightarrow \infty} P\left(W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) < -\epsilon\right) = 0$$

(ii)

$$\lim_{n \rightarrow \infty} P\left(C(\hat{\theta}(\mathbf{X}, A, A^P), A^P) - B > \epsilon\right) = 0.$$

In Theorem 1.4.1, for $\sqrt{n}(B - C(\theta_n^*, A^P)) \xrightarrow{p} \infty$ to hold, either of the following two

conditions is sufficient: (i) there exists some $\bar{\delta} > 0$ such that $\sup_n C(\theta_n^*) \leq B - \bar{\delta}$; (ii) $B - C(\theta_n^*, A^P) \xrightarrow{p} 0$ at a rate such that $\sqrt{n}(B - C(\theta_n^*, A^P)) \xrightarrow{p} \infty$. The first condition states that the budget constraint needs to be always nonbinding such that B minus the cost associated with θ_n^* is bounded away from 0 by a positive constant. When such a constant does not exist, the second condition states that it also suffices if $B - C(\theta_n^*, A^P) \xrightarrow{p} 0$ at a slow enough rate such that $\sqrt{n}(B - C(\theta_n^*, A^P)) \xrightarrow{p} \infty$.

In practice, it may not be easy to verify the condition imposed on the budget constraint discussed above. As an alternative, I show that by using a new budget in the empirical welfare maximization, we can obtain asymptotic welfare efficiency and weak feasibility without checking the condition proposed in Theorem 1.4.1 .

Theorem 1.4.2. *Under Assumptions 1.2.1–1.2.5, Assumptions 1.4.1, and 1.4.2 (i), if $B + b_n$ is set as the budget in (1.13) with $\sqrt{n}b_n \rightarrow \infty$, the optimal treatment rule defined in (1.13) is then (i) asymptotically welfare-efficient, and (ii) asymptotically weakly feasible.*

The condition in Theorem 1.4.2 states that in the empirical welfare maximization, we need to set the budget above B . This is similar to the trade-off rule from Sun (2024). We can consider b_n as the degree of tolerance for exceeding budget B . The choice of b_n depends policymakers, and can converge to 0 as fast as possible, eg. $\sqrt{\frac{\log \log n}{n}}$, as long as $\sqrt{n}b_n \rightarrow \infty$ is satisfied. In this case, although there is overspending in the empirical welfare maximization, it converges to 0 when n goes to infinity. Note that $\theta_n^*(\mathbf{X}, A, A^P)$ is still from the optimization problem with budget B . The key idea is that, by making $\hat{\theta}(\mathbf{X}, A, A^P)$ have a budget higher than that for $\theta_n^*(\mathbf{X}, A, A^P)$ by an arbitrarily small magnitude, we are able to achieve asymptotic welfare efficiency and weak feasibility conveniently.

Theorem 1.4.3. *Under Assumptions 1.2.1–1.2.5, Assumptions 1.4.1, and 1.4.2 (ii)(iii), if $B + b_n$ is set as the budget in (1.15) with $\sqrt{n}b_n \rightarrow \infty$, the optimal treatment rule defined in (1.15) is then (i) asymptotically welfare-efficient, and (ii) asymptotically weakly feasible.*

Theorem 1.4.3 presents the asymptotic results based on the empirical welfare function

constructed using the linear-in-means model.

1.5 Empirical Application

1.5.1 Experiment and Network

I apply the proposed optimal treatment allocations to the social network experiment from Paluck et al. (2016). The experiment was designed to encourage anti-conflict norms and behaviors among students in public middle schools. During the 2012-2013 school year (September to June), 28 out of 56 public middle schools in New Jersey were randomly assigned to be eligible for an anti-conflict treatment. In each eligible school, a group of non-randomly selected students was identified as eligible for inclusion in the experiment. Among this group, 50% were randomly assigned to treatment, and the remaining students were assigned to the control group. Students assigned to the treatment group were invited to attend a meeting every other week. In each meeting, students participated in activities where they identified common conflict behaviors at their school and discussed anti-conflict strategies.

I restrict the analysis to eligible schools and consider students eligible for inclusion in the experiment as the target population. Let $Z_i \in \{0, 1\}$ indicate whether student i was assigned to attend the anti-conflict meetings, and let $D_i \in \{0, 1\}$ indicate actual treatment receipt. Here, $D_i = 1$ if student i attended at least one meeting. Previous literature, Aronow and Samii (2017), Leung (2022), and Hoshino and Yanagi (2024) examine treatment effects and spillover effects, either without taking into account spillovers in noncompliance or only focusing on subpopulations. In our framework, I focus on the whole target population, and assume that a student’s take-up decision does not only depend on her own assignment. To consider the two layers of spillover effects, we first measure A and A^P .

The social network A was measured by asking students to nominate up to 10 peers (with order) with whom they “decided to spend time in the last few weeks” at their school; $A_{ij} = 1$

if either student i nominated j or j nominated i , as in Aronow and Samii (2017), Leung (2022), and Hoshino and Yanagi (2024). Students were also asked to nominate two best friends (with order) at their school.

To construct the network A^P , we calculate the mutual distance between any two students. For student i , the subjective distance from i to j is defined as j 's ranking in i 's nominations for best friends and for peers with whom she decided to spend time: the first best friend nominated is considered the closest, and the last peer nominated is considered the furthest. For student j , the subjective distance from j to i is defined analogously. The mutual distance between i and j is the sum of these two subjective distances. In the comprehensive list of pairwise distances, A^P is formed by repeatedly finding a pair (i, j) with the smallest mutual distance and setting $A_{ij}^P = 1$. Once a pair is formed, it is removed from the list.

For the outcome variable, following Aronow and Samii (2017), Leung (2022), and Hoshino and Yanagi (2024), I focus on self-reports of wearing an anti-conflict wristband. Wristbands were given to students observed engaging in friendly or anti-conflict behaviors. Let $Y_i \in \{0, 1\}$ indicate whether student i reported: "I wore an orange Roots wristband this past week." For each student i , we let covariates X_i include age, gender, and whether i had moved in the past few years.

Restricting the population to the students eligible to be included in the experiment, the data contains $n = 548$ individuals, 84% of those with $Z^{1,0} = 1$ complied with the treatment assignment and 91% of those with $Z^{1,1} = 1$ complied. This indicates that most of the individuals in the experiment are self-compliers and we have a small fraction of social-compliers and never-takers. For the budget constraint, B indicates the maximum fraction of the target population that can be treated. At the same time, I set the degree of overspending equal to 0.001, then allow B to vary on a grid on $[0, 1]$ and examine changes of optimal policy. For empirical welfare, a linear-in-means model is used here. Given the choice of outcome variable, empirical welfare is the estimated average probability of wearing an orange Roots wristband. Policymakers can find out the optimal assignment probability and maximum

achievable welfare given B ; or they may first set a goal for the fraction of students wearing the wristband, then find out the minimum level of assignment needed to achieve the goal, as treatment cost is often a concern in practice.

1.5.2 Results

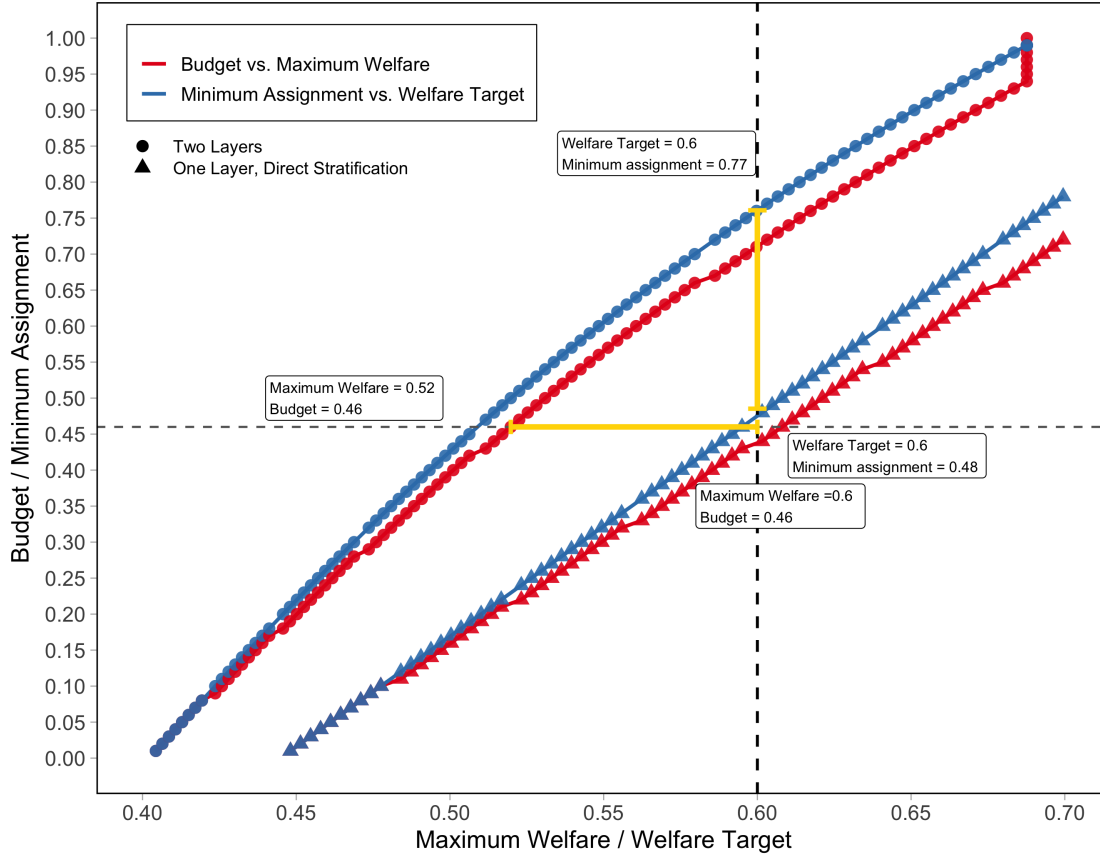


Figure 1.2: The red lines shows the maximum welfare that can be achieved as the budget (the cap on the fraction of the target population that can be treated) increases $[0, 1]$. The blue lines shows the minimum assignment probability needed to achieve different levels of welfare. Since the outcome is binary, the welfare target varies between 0 and 1.

Figure 2.19 shows both the maximum welfare achievable for different budget levels and the minimum assignment probability needed for different welfare targets. The two upper curves present the results from our proposed treatment assignment rule $\tilde{\theta}$. In Figure 2.19, if the welfare target is 0.6, then the minimum assignment probability is 0.77, and by estimation 72% of targeted students will be treated. Conversely, if only 72% of the targeted students can be

treated, then the maximum achievable welfare is 0.6 and the optimal assignment probability is 0.77. Due to noncompliance, the optimal assignment probability exceeds the restriction on the fraction of treated students. We have a duality (after rounding) of maximization and minimization in this setting, while this may not hold in other settings.

Figure 2.19 also shows results based on another common model in the network literature. The two lower curves in Figure 2.19 illustrate the results of welfare maximization and cost minimization with a direct stratification on the number of assigned neighbors. For direct stratification, I set $Y_i = \beta_0 + \beta_1 Z_i + \beta_2 \sum_{j \in N_i} Z_j + \beta_3 \gamma_i + X_i' \beta_4 + u_i$, which has been used in many scenarios when compliance is perfect and treatment receipts are i.i.d.. This specification corresponds to the case where there are no spillovers in take-up decisions and one's outcome depends on the number of treated neighbors. Comparing the upper and lower sets of curves, I find gaps between the minimum level assignments estimated using the two different models. For example, when the target for the fraction of students wearing a wristband is 0.6, for the minimum assignment probability, our method gives an estimate equal to 0.77, while the estimate using direct stratification is 0.48. The implication in this scenario is that not incorporating both layers of spillover effects can result in an insufficient level of assignment given some welfare target, or an overestimation of level of attainable welfare given some budget constraint. In general, the sign of misallocation depends on the specific application.

1.5.3 Treatment Allocation for School Blocks

I also consider treatment allocation for schools with different characteristics. In our data, since friendship in the network is only within schools, we may divide all schools in the data into several blocks and analyze welfare maximization or cost minimization separately. This can be considered as an application in a scenario with partial interference. When we implement optimization for all schools as a whole, it is possible that blocks that we are especially interested in do not reach the welfare target. Dividing schools into blocks can be interesting if policymakers would like to ensure that blocks with certain characteristics reach

the welfare target. Currently, X_i contains age, gender, and whether i had moved in the

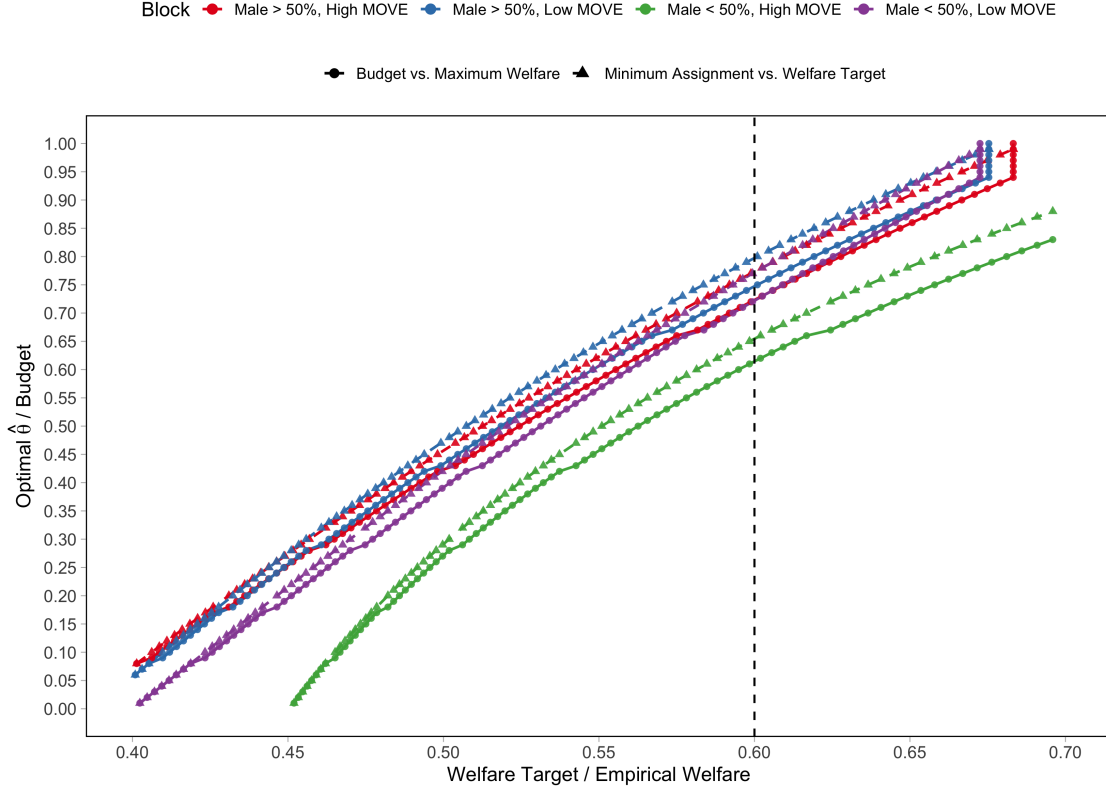


Figure 1.3: The four sets of colored lines show the welfare maximization and cost minimization results for four school blocks in different categories defined using mean values of gender and MOVE (whether a student has moved in the past few years).

past few years. Since students' ages are similar across middle schools, we focus on the other two variables. For gender, we use 50% as the threshold: among the students included in the experiment, if over half of the students are male, then we categorize the school as “male > 50%”. For moving status, if over 25% of the students included in the experiment have moved during the past few years, then we categorize the school as “high MOVE”. Using these two thresholds, we have four schools blocks in total.

Figure 1.3 presents the welfare maximization and cost minimization results for the four school blocks. Among the four school blocks, the one with more female students and more students who have moved in the past few years has the lowest minimum level of assignment in general. For example, the minimum level of assignment is 0.66 when we have a welfare target equal to 0.6. Other groups have similar results, while the group with more male students

and fewer students who have moved in the past few years has the highest minimum level of assignment equal to 0.8, conditional on the welfare target equal to 0.6. These results imply that for school blocks with more female students and more students who have moved in the past few years, it is less costly to achieve a range of welfare targets; and if policymakers aim at ensuring the average outcome of “Male > 50%, Low Move” reach the welfare target, then a high level of treatment assignment is required. To summarize, there is heterogeneity in the optimal treatment intensity across schools. To summarize, there can be substantial heterogeneity in the optimal treatment intensity across schools.

1.6 Conclusion

This paper proposes optimal randomized treatment allocation based on the covariates of the target population, considering the presence of spillover effects in both take-up decisions and outcomes. The optimal treatment allocations are estimated by maximizing empirical welfare subject to the empirical budget constraint or minimizing empirical cost subject to a welfare target. I show that under some sufficient conditions, the proposed treatment allocations are asymptotically welfare-efficient and weakly feasible. The methods in this paper can be applied to various problems related to health, development, and education topics where spillover effects are likely to occur. I apply the proposed treatment rules to an anti-conflict experiment in public schools in New Jersey and show that not incorporating both layers of spillover effects can lead to a suboptimal level of treatment allocation. For future research direction, we may identify and estimate a range of treatment effects and spillover effects based on the framework of spillovers in this paper.

1.7 Appendix

1.7.1 Proof of Proposition 1.2.1

Proof. For T_i we consider (i) $\sum_{j \in N_i} D_j$, and (ii) $(\sum_{j \in N_i} D_j)/\gamma_i$. For convenience, we focus on the first case, the proof for the second case is immediate after we prove for the first one since there is a one-to-one relationship between $\sum_{j \in N_i} D_j$ and $\sum_{j \in N_i} D_j/\gamma_i$. Here define $T_i = \sum_{j \in N_i} D_j$.

Define $\phi : \tilde{\mathbf{T}} \times \tilde{\mathbf{T}}^P \times \tilde{\mathbf{T}}\mathbf{P}^N \times \mathbf{Z} \times \mathbf{Z}^P \times \mathbf{Z}^{NP} \rightarrow \{0, 1\} \times \{0, 1, \dots, \gamma_i\}$, which is a mapping from the space of compliance type vector for i and i 's neighbors and assignment vector to the space of i 's treatment status and number of treated neighbors:

$$\phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP}) = (d_i, t_i).$$

With ϕ , given the compliance types and assignments of i and i 's neighborhood, we can pin down i 's treatment receipt D_i and i 's number of treated neighbors T_i .

We can write the expected outcome as a weighted average of the outcomes of different

compliance type vectors:

$$\begin{aligned}
& E[Y_i | \mathbf{Z}, \mathbf{X}, A, A^P] \\
&= E \left[r \left(D_i, \sum_{j \in N_i} D_j, X_i, \gamma_i, \epsilon_i \right) \middle| Z_i, Z_i^P, \mathbf{Z}_i^{NP}, \mathbf{Z}_i^{-NP}, X_i, \mathbf{X}_{-i}, A, A^P \right] \\
&= E \left[r \left(D_i(Z_i, Z_i^P), \sum_{j \in N_i} D_j(Z_j, Z_j^P), X_i, \gamma_i, \epsilon_i \right) \middle| Z_i, Z_i^P, \mathbf{Z}_i^{NP}, \mathbf{Z}_i^{-NP}, X_i, \mathbf{X}_{-i}, A, A^P \right] \\
&= E \left[r \left(D_i(Z_i, Z_i^P), \sum_{j \in N_i} D_j(Z_j, Z_j^P), X_i, \gamma_i, \epsilon_i \right) \middle| Z_i, Z_i^P, \mathbf{Z}_i^{NP}, X_i, A, A^P \right] \\
&= \sum_{d,t} \sum_{\substack{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N): \\ \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \\ \mathbf{z}^{NP}) = (d,t)}} E \left[r(d, t, X_i, \gamma_i, \epsilon_i) \middle| \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}^N, Z_i, Z_i^P, \mathbf{Z}_i^{NP}, X_i, A, A^P \right] \\
&\times P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}^N \mid Z_i, Z_i^P, \mathbf{Z}_i^{NP}, A, A^P)
\end{aligned} \tag{1.16}$$

where the first equality is by partitioning the conditioning set, the second equality is by rewriting D as a function of the assignments within peer groups, the fourth equality is by Law of total expectation, and the last equality is by Assumption 2.4 (i) that assignment is random and hence is independent of potential treatment status and outcome jointly. The third equality holds if we have

$$\{D_i(z, z^P), D_j(z, z^P)\}_{j \in N_i, \epsilon_i \perp \mathbf{Z}_i^{-NP}, \mathbf{X}_{-i} \mid Z_i, Z_i^P, \mathbf{Z}_i^{NP}, X_i, A, A^P}.$$

This holds if

$$\{D_i(z, z^P), D_j(z, z^P)\}_{j \in N_i, \epsilon_i \perp \mathbf{Z}_i^{-NP}, \mathbf{X}_{-i} \mid X_i, A, A^P} \tag{1.17}$$

as under Assumption 2.4 (i) we have

$$Z_i, Z_i^P, \mathbf{Z}_i^{NP} \perp \{D_i(z, z^P), D_j(z, z^P)\}_{j \in N_i, z, z^P \in \{0,1\}}, \epsilon_i, \mathbf{Z}_i^{-NP}, \mathbf{X}_{-i} \mid A, A^P$$

and

$$Z_i, Z_i^P, \mathbf{Z}_i^{NP} \perp X_i \mid A, A^P.$$

Note that for (1.17) to hold, under Assumption 2.4 (i) we have

$$\{D_i(z, z^P), D_j(z, z^P)\}_{j \in N_i, z, z^P \in \{0,1\}}, \epsilon_i, \mathbf{X}_{-i}, X_i \perp \mathbf{Z}^{-NP} \mid A, A^P, \quad (1.18)$$

and under Assumption 2.4 (ii) we have

$$\{D_i(z, z^P), D_j(z, z^P)\}_{j \in N_i, z, z^P \in \{0,1\}}, \epsilon_i \perp \mathbf{X}_{-i}, X_i \mid A, A^P. \quad (1.19)$$

In (1.18), by symmetry and weak union we have

$$\{D_i(z, z^P), D_j(z, z^P)\}_{j \in N_i, z, z^P \in \{0,1\}}, \epsilon_i \perp \mathbf{Z}^{-NP} \mid \mathbf{X}_{-i}, X_i, A, A^P$$

then together with (1.19), by contraction we have

$$\{D_i(z, z^P), D_j(z, z^P)\}_{j \in N_i, z, z^P \in \{0,1\}}, \epsilon_i \perp \mathbf{Z}^{-NP}, \mathbf{X}_{-i}, X_i \mid A, A^P$$

then (1.17) holds by the weak union.

The goal is to show that for $\mathbf{z}$ and $\mathbf{z}'$ with $(z_i, z_i^P, z_i^{1,0}, z_i^{1,1}) = (z'_i, z_i^{P'}, z_i^{1,0'}, z_i^{1,1'})$, we have $E[Y_i | \mathbf{Z} = \mathbf{z}, \mathbf{X}, A, A^P] = E[Y_i | \mathbf{Z} = \mathbf{z}', \mathbf{X}, A, A^P]$. After inspecting the last equality in (1.16), it is easy to see that for $\mathbf{z}$ and $\mathbf{z}'$ with $(z_i, z_i^P, z_i^{1,0}, z_i^{1,1}) = (z'_i, z_i^{P'}, z_i^{1,0'}, z_i^{1,1'})$, we would have

$E[Y_i|\mathbf{Z} = \mathbf{z}, X_i, A, A^P] = E[Y_i|\mathbf{Z} = \mathbf{z}', X_i, A, A^P]$ if for all (d, t) the following equality holds:

$$\begin{aligned}
& \sum_{\substack{\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N): \\ \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP}) = (d, t)\}}} E[r(d, t, X_i, \gamma_i, \epsilon_i) \mid \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^N, X_i, A, A^P] \\
& \quad \times P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^N \mid A, A^P) \\
& = \sum_{\substack{\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N): \\ \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP'}) = (d, t)\}}} E[r(d, t, X_i, \gamma_i, \epsilon_i) \mid \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^N, X_i, A, A^P] \\
& \quad \times P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^N \mid A, A^P)
\end{aligned} \tag{1.20}$$

Note that the first summation is a weighted average of the expected outcomes for ordered types $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP}) = (d, t)\}$, and the second summation is a weighted average of the expected outcomes for ordered types $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP'}) = (d, t)\}$. As $(z_i, z_i^P) = (z'_i, z_i^{P'})$, here we replace $z'_i, z_i^{P'}$ with z_i, z_i^P for convenience. Intuitively, $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP}) = (d, t)\}$ includes all of the ordered vectors of compliance types in the neighborhood such that given assignment $\mathbf{z}^{NP}$, t neighbors would take the treatment. Similarly for $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP'}) = (d, t)\}$, given $\mathbf{z}^{NP'}$.

To show that the two weighted averages are the same, we show that we can find a bijection between $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP}) = (d, t)\}$ and $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP'}) = (d, t)\}$ such that for each pair $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N)$ and $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^{N'})$ from the bijection, they have the same weight and expected outcome. Then the two weighted averages are the same, and as a result we have $E[Y_i|\mathbf{Z} = \mathbf{z}, X_i, A, A^P] = E[Y_i|\mathbf{Z} = \mathbf{z}', X_i, A, A^P]$.

In other words, for (1.20) to hold, it suffices to show that for all (d, t) , when $\tilde{\mathbf{z}}_i = \tilde{\mathbf{z}}'_i$, we can find a bijection between $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP}) = (d, t)\}$ and $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}^{NP'}) = (d, t)\}$ such that for each bijection pair $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N)$

and $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^{'N})$, we have

$$P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N}) = P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N'})$$

and

$$\begin{aligned} & E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N}, X_i, A, A^P] \\ &= E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N}, X_i, A, A^P] \end{aligned}$$

and hence

$$\begin{aligned} & E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N}, X_i, A, A^P] \\ &= P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N} | A, A^P) \\ &= E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N}, X_i, A, A^P] \\ &= P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N} | A, A^P). \end{aligned}$$

Then summing over the bijection pairs we would have

$$\begin{aligned} & \sum_{\substack{\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^{'N}): \\ \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^{'N}, z_i, z_i^P, \mathbf{z}^{NP}) = (d, t)\}}} E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N}, X_i, A, A^P] \\ & \times P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N} | A, A^P) \\ &= \sum_{\substack{\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^{'N}): \\ \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^{'N}, z_i, z_i^P, \mathbf{z}^{NP'}) = (d, t)\}}} E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N}, X_i, A, A^P] \\ & \times P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^{'N} | A, A^P). \end{aligned}$$

For simplicity, we restrict to the case where $z_i = z'_i = 1$ and $z_i^P = z_i^{P'} = 0$, and $d = 1$. Other cases are similar. $z_i^P = z_i^{P'} = 0$ means that for $T_i = t$, t is a summation of neighbors' treatments excluding the peer. For the case where the peer is treated, we replace the t

discussed below with $t - 1$.

When $z_i = z'_i = 1$, $z_i^P = z_i^{P'} = 0$, and $d = 1$, for $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N) \in \{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP}) = (d, t)\}$ and $(\tilde{t}', \tilde{t}'^P, \tilde{\mathbf{t}}\mathbf{p}_i^{N'}) \in \{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP'}) = (d, t)\}$ we have $\tilde{t} = \tilde{t}' = SC$ and $\tilde{t}^P, \tilde{t}'^P \in \{SC, SOC, NT\}$.

Before showing that we can find the bijection discussed above, first note that with $(\tilde{\mathbf{T}}\mathbf{P}_i^N, \mathbf{Z}_i^{NP})$, unit i 's neighbors excluding i 's peer can be randomly categorized into sets such that neighbors with the same value of $(Z_j, Z_j^P, \tilde{T}_j)$ belong to the same set. Define random set V_1 such that $j \in V_1$ if $\mathbf{1}\{\tilde{T}_j = SC, Z_j = 1, Z_j^P = 0\} = 1$. Other random sets are defined similarly. Let neighbors with $Z_j = 0$ and the same compliance type be in the same set. In total, we have nine random sets $\{V_k\}_{k=1,\dots,9}$. The numbering is shown in the following table.

$j \in N_i, A_{ij}^P = 0$	$Z_j=1, Z_j^P=0$	$Z_j=1, Z_j^P=1$	$Z_j=0$
$\tilde{T}_j = SC$	V_1	V_4	V_7
$\tilde{T}_j = SOC$	V_2	V_5	V_8
$\tilde{T}_j = NT$	V_3	V_6	V_9

Let V be the mapping from $(\tilde{\mathbf{t}}\mathbf{p}_i^N, \mathbf{z}_i^{NP})$ to its correspondent $(v_1, \dots, v_9)$. Note that $(v_1, \dots, v_9)$ is the realization of the random sets $(V_1, \dots, V_9)$ given $(\tilde{\mathbf{t}}\mathbf{p}_i^N, \mathbf{z}_i^{NP})$. In other words, with $(v_1, \dots, v_9)$, neighbors' group memberships are confirmed.

$$V(\tilde{\mathbf{t}}\mathbf{p}_i^N, \mathbf{z}_i^{NP}) = (v_1, \dots, v_9)$$

Now we start showing that we can find a bijection between $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP}) = (d, t)\}$ and $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP'}) = (d, t)\}$ with the properties desired so that (1.20) holds.

Instead of directly finding the bijection, we first show that each of $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP}) = (d, t)\}$ and $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP'}) = (d, t)\}$ can be written a union of disjoint sets described below, then focus on finding a bijection between

their component sets.

$$\begin{aligned}
& \{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP}) = (d, t)\} \\
&= \bigcup_{\substack{\tilde{t}^*=SC, \\ \tilde{t}^{P*} \in \{SC, SOC, NT\}}} \bigcup_R \{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i^N) : |V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}
\end{aligned} \tag{1.21}$$

where $|V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP})|$ denotes $(|v_1|, \dots, |v_9|)$, the cardinality of the groups that $(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP})$ maps to, and

$$\begin{aligned}
R = & \{(r_1, \dots, r_9) : r_1 + r_4 + r_5 = t, \\
& r_1 + r_2 + r_3 = z^{1,0}, \\
& r_4 + r_5 + r_6 = z^{1,1}, \\
& r_7 + r_8 + r_9 = z^0\},
\end{aligned}$$

$r_1, \dots, r_9$ are nonnegative integers. Similarly, $\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP'}) = (d, t)\}$ can be written as follows

$$\begin{aligned}
& \{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP'}) = (d, t)\} \\
&= \bigcup_{\substack{\tilde{t}^*=SC, \\ \tilde{t}^{P*} \in \{SC, SOC, NT\}}} \bigcup_{R'} \{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i^N) : |V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP'})| = (r_1, \dots, r_9)\}
\end{aligned} \tag{1.22}$$

where

$$\begin{aligned}
R' = & \{(r_1, \dots, r_9) : r_1 + r_4 + r_5 = t, \\
& r_1 + r_2 + r_3 = z^{1,0'}, \\
& r_4 + r_5 + r_6 = z^{1,1'}, \\
& r_7 + r_8 + r_9 = z^{0'}\}
\end{aligned}$$

We show that the equality in (1.21) holds. For any $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) \in RHS$, we have

$$(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) \in \{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) \in: |V(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}$$

for some $(r_1, \dots, r_9) \in R$. Since $(r_1, \dots, r_9) \in R$, $(r_1, \dots, r_9)$ satisfies $|v_1| + |v_4| + |v_5| = t$, and hence there is $\phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP}) = (d, t)$. Thus $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) \in LHS$.

For any $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) \in LHS$, as $\phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP'}) = (d, t)$, $|V(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP'})|$ automatically satisfies the restrictions in R , thus $|V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP'})| \in R$ and $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N) \in RHS$. Similarly, the equality in (1.22) holds.

Next we show $R = R'$. In other words, we show that the two unions in (1.21) and (1.22) are over the same conditions for $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}_i^N)$. This is by $z^{1,0} = z^{1,0'}$, $z^{1,1} = z^{1,1'}$, and $z^0 = z^{0'}$.

We then show that fixing $(\tilde{t}^*, \tilde{t}^{P*}) \in SC \times \{SC, SOC, NT\}$, for each $(r_1, \dots, r_9) \in R$, there is a bijection between $\{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i^N) : |V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}$ and $\{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i^N) : |V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP'})| = (r_1, \dots, r_9)\}$ such that: for each pair $(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i^N) \in \{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i^N) : |V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}$ and $(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i'^N) \in \{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i^N) : |V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP'})| = (r_1, \dots, r_9)\}$ from the bijection, we have

$$P(\tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^N) = P(\tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i'^N)$$

and

$$\begin{aligned} & E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i^N, X_i, A, A^P] \\ &= E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \tilde{\mathbf{T}}_i^N = \tilde{\mathbf{t}}_i'^N, X_i, A, A^P]. \end{aligned}$$

First, $\{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i^N) : |V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}$ and $\{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}_i'^N) : |V(\tilde{\mathbf{t}}_i^N, \mathbf{z}_i^{NP'})| = (r_1, \dots, r_9)\}$ have the same cardinality as

follows

$$\begin{aligned}
& |\{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}\mathbf{p}_i^N) : |V(\tilde{\mathbf{t}}\mathbf{p}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}| \\
&= \binom{z^{1,0}}{r_1} \binom{z^{1,0} - r_1}{r_2} \binom{z^{1,1}}{r_4} \binom{z^{1,1} - r_4}{r_5} \binom{z^0}{r_7} \binom{z^0 - r_7}{r_8} \\
&= |\{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}\mathbf{p}_i^N) : |V(\tilde{\mathbf{t}}\mathbf{p}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}|
\end{aligned}$$

Let the bijection be as follows. For each element in $\{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}\mathbf{p}_i^N) : |V(\tilde{\mathbf{t}}\mathbf{p}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}$, we map it to a unique element in $\{(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}\mathbf{p}_i^N) : |V(\tilde{\mathbf{t}}\mathbf{p}_i^N, \mathbf{z}_i'^{NP})| = (r_1, \dots, r_9)\}$. The uniqueness is by that the two finite sets have the same cardinality as shown above. Note that the mapping can be arbitrary as long as it is one-to-one.

For each bijection pair $(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}\mathbf{p}_i^N)$ and $(\tilde{t}^*, \tilde{t}^{P*}, \tilde{\mathbf{t}}\mathbf{p}_i^{N'})$, since they have the same $(r_1, \dots, r_9)$, they have the same numbers of individual compliance types in the neighborhood and hence there is $tp_i^N = tp_i^{N'}$, then we have

$$\begin{aligned}
& P(\tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}_i^N) \\
&= P(\tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}_i^{N'}) \\
&= P(\tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, TP_i^N = tp_i^N)
\end{aligned}$$

as individual compliance types are i.i.d., and as a result permutations of the same number of individual types in the neighborhood have the same probability.

At the same time, as $tp_i^N = tp_i^{N'}$, by Assumption 1.2.5, the unordered mean assumption, we also have

$$\begin{aligned}
& E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}_i^N, X_i, A, A^P] \\
&= E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}_i'^N, X_i, A, A^P]
\end{aligned}$$

for each bijection pair. Combining the above, we have

$$\begin{aligned}
& E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \mathbf{\tilde{T}P}_i^N = \mathbf{\tilde{t}p}^N, X_i, A] \\
& P(\tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \mathbf{\tilde{T}P}_i^N = \mathbf{\tilde{t}p}^N | A, A^P) \\
& = E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \mathbf{\tilde{T}P}_i^N = \mathbf{\tilde{t}p}^{N'}, X_i, A] \\
& P(\tilde{T}_i = \tilde{t}^*, \tilde{T}_i^P = \tilde{t}^{P*}, \mathbf{\tilde{T}P}_i^N = \mathbf{\tilde{t}p}^{N'} | A, A^P)
\end{aligned}$$

for each bijection pair from $\{(\tilde{t}^*, \tilde{t}^{P*}, \mathbf{\tilde{t}p}_i^N) : |V(\mathbf{\tilde{t}p}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}$ and $\{(\tilde{t}^*, \tilde{t}^{P*}, \mathbf{\tilde{t}p}_i^N) : |V(\mathbf{\tilde{t}p}_i^N, \mathbf{z}_i^{NP'})| = (r_1, \dots, r_9)\}$.

Now we have shown that in (1.21) and (1.22), for each $(\tilde{t}^*, \tilde{t}^{P*}) \in SC \times \{SC, SOC, NT\}$ and $(r_1, \dots, r_9) \in R$, there is a bijection between $\{(\tilde{t}^*, \tilde{t}^{P*}, \mathbf{\tilde{t}p}_i^N) : |V(\mathbf{\tilde{t}p}_i^N, \mathbf{z}_i^{NP})| = (r_1, \dots, r_9)\}$ and $\{(\tilde{t}^*, \tilde{t}^{P*}, \mathbf{\tilde{t}p}_i^N) : |V(\mathbf{\tilde{t}p}_i^N, \mathbf{z}_i^{NP'})| = (r_1, \dots, r_9)\}$ with the properties desired. Thus we have found a bijection between $\{(\tilde{t}, \tilde{t}^P, \mathbf{\tilde{t}p}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \mathbf{\tilde{t}p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP}) = (d, t)\}$ and $\{(\tilde{t}, \tilde{t}^P, \mathbf{\tilde{t}p}_i^N) : \phi(\tilde{t}_i, \tilde{t}_i^P, \mathbf{\tilde{t}p}_i^N, z_i, z_i^P, \mathbf{z}_i^{NP'}) = (d, t)\}$ with the properties desired for all (d, t) such that

$$\begin{aligned}
& E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \mathbf{\tilde{T}P}_i^N = \mathbf{\tilde{t}p}^N, X_i, A, A^P] \\
& P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \mathbf{\tilde{T}P}_i^N = \mathbf{\tilde{t}p}^N | A, A^P) \\
& = E[r(d, t, X_i, \gamma_i, \epsilon_i) | \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \mathbf{\tilde{T}P}_i^N = \mathbf{\tilde{t}p}^{N'}, X_i, A, A^P] \\
& P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \mathbf{\tilde{T}P}_i^N = \mathbf{\tilde{t}p}^{N'} | A, A^P)
\end{aligned}$$

for each bijection pair $(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N), (\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i'^N)$. As a result, (1.20) holds for all (d, t)

$$\begin{aligned}
& \sum_{\substack{\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N): \\ \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}^{NP})=(d, t)\}}} E[r(d, t, X_i, \gamma_i, \epsilon_i) \mid \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}^N, X_i, A, A^P] \\
& \quad \times P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}^N \mid A, A^P) \\
& = \sum_{\substack{\{(\tilde{t}, \tilde{t}^P, \tilde{\mathbf{t}}\mathbf{p}_i^N): \\ \phi(\tilde{t}_i, \tilde{t}_i^P, \tilde{\mathbf{t}}\mathbf{p}_i^N, z_i, z_i^P, \mathbf{z}^{NP'})=(d, t)\}}} E[r(d, t, X_i, \gamma_i, \epsilon_i) \mid \tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}^N, X_i, A, A^P] \\
& \quad \times P(\tilde{T}_i = \tilde{t}, \tilde{T}_i^P = \tilde{t}^P, \tilde{\mathbf{T}}\mathbf{P}_i^N = \tilde{\mathbf{t}}\mathbf{p}^N \mid A, A^P)
\end{aligned}$$

when $\tilde{\mathbf{z}}_i = \tilde{\mathbf{z}}'_i$. So $E[Y_i | \mathbf{Z} = \mathbf{z}, X_i, A, A^P] = E[Y_i | \mathbf{Z} = \mathbf{z}', X_i, A, A^P]$ for $\mathbf{z} \neq \mathbf{z}'$ when $\tilde{\mathbf{z}}_i = \tilde{\mathbf{z}}'_i$. $\square$

1.7.2 Proof of Theorem 1.4.1

Proof. To evaluate the asymptotic welfare efficiency and weak feasibility of $\hat{\theta}(\mathbf{X}, A, A^P)$, we use the following fact: if $C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) \leq B$, then

$$W_n(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) \geq W_n(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P).$$

Thus, if $C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) \leq B$, then

$$\begin{aligned} & W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) \\ &= W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W_n(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) \\ &\quad + W_n(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) \\ &\leq W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W_n(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) \\ &\quad + W_n(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) \\ &\leq 2 \sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)|. \end{aligned}$$

With the inequality above, we have

$$\begin{aligned} & P\left(W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) < -\epsilon\right) \\ &= P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B, W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - \right. \\ &\quad \left. W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) < -\epsilon\right) \\ &+ P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) \leq B, W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - \right. \\ &\quad \left. W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) < -\epsilon\right) \\ &\leq P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B\right) \\ &\quad + P\left(2 \sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| > \epsilon\right) \end{aligned} \tag{1.23}$$

and

$$\begin{aligned}
& P\left(W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) > \epsilon\right) \\
& \leq P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B\right) \\
& + P\left(2 \sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| > \epsilon\right).
\end{aligned}$$

It implies we only need to show $\lim_{n \rightarrow \infty} P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B\right) = 0$ and $\sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| = o_P(1)$ to demonstrate the first statement of Theorem 1.4.1. First, we show

$$\sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| = o_P(1). \quad (1.24)$$

In the following expression for $W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)$,

$$\begin{aligned}
& W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P) \\
& = \frac{1}{n} \sum_{i=1}^n \left\{ \frac{\pi(\tilde{\mathbf{Z}}_i, A, A^P; \theta)}{\pi^{pilot}(\tilde{\mathbf{Z}}_i, A, A^P)} (Y_i - \hat{\mu}_{K_i}(\tilde{\mathbf{Z}}_i, X_i, A, A^P)) \right. \\
& \quad \left. + \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathbb{Z}}_i} \pi(\tilde{\mathbf{z}}_i, A, A^P; \theta) [(\hat{\mu}_{K_i}(\tilde{\mathbf{z}}_i, X_i, A, A^P) - \mu(\tilde{\mathbf{z}}_i, X_i, A, A^P))] \right\},
\end{aligned}$$

let

$$\begin{aligned}
\Phi_i(\theta) & = \frac{\pi(\tilde{\mathbf{Z}}_i, A, A^P; \theta)}{\pi^{pilot}(\tilde{\mathbf{Z}}_i, A, A^P)} (Y_i - \hat{\mu}_{K_i}(\tilde{\mathbf{Z}}_i, X_i, A, A^P)) \\
& + \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathbb{Z}}_i} \pi(\tilde{\mathbf{z}}_i, A, A^P; \theta) [(\hat{\mu}_{K_i}(\tilde{\mathbf{z}}_i, X_i, A, A^P) - \mu(\tilde{\mathbf{z}}_i, X_i, A, A^P))],
\end{aligned}$$

then we have

$$(W_n - W)(\theta, \mathbf{X}, A, A^P) = \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta).$$

To show $\sup_{\theta \in \Theta} |\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)| = o_P(1)$, we first show that $\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) = o_P(1)$ holds for any θ . Note that for any θ we have $E[\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)] = 0$, then we show $Var(\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)) = o(1)$ so that $\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) = o_P(1)$. For the variance, we have

$$Var(\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)) = \frac{1}{n^2} \sum_{i=1}^n Var(\Phi_i(\theta)) + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \neq i} Cov(\Phi_i(\theta), \Phi_j(\theta)).$$

We analyze $\frac{1}{n^2} \sum_{i=1}^n Var(\Phi_i(\theta))$ and $\frac{1}{n^2} \sum_{i=1}^n \sum_{j \neq i} Cov(\Phi_i(\theta), \Phi_j(\theta))$ separately. As $\hat{\mu}(\tilde{\mathbf{z}}_i, X_i, A, A^P)$ is any outcome model, for convenience we truncate it such that $\hat{\mu}(\tilde{\mathbf{z}}_i, X_i, A, A^P) \leq M \ \forall \ \tilde{\mathbf{z}}_i, X_i, A, A^P$. Let $\tilde{\delta} := \inf_{n \geq 1} \min_{1 \leq i \leq n} \min_{\tilde{\mathbf{z}} \in \tilde{\mathcal{Z}}_i} \pi^{pilot}(\tilde{\mathbf{z}}, A, A^P)$, intuitively it indicates the smallest value that the probability of $\mathbf{z}_i$ can take. $\tilde{\delta}$ exists under Assumption 1.4.1(ii) and 1.4.2(i). For $\frac{1}{n^2} \sum_{i=1}^n Var(\Phi_i(\theta))$, we have

$$\begin{aligned} \frac{1}{n^2} \sum_{i=1}^n Var(\Phi_i(\theta)) &= E[\frac{1}{n^2} \sum_{i=1}^n Var(\Phi_i(\theta)|A, A^P)] \\ &= E[\frac{1}{n^2} \sum_{i=1}^n E[\Phi_i(\theta)^2|A, A^P]] \\ &= E[\frac{1}{n^2} \sum_{i=1}^n E[\left\{ \frac{\pi(\tilde{\mathbf{Z}}_i, A, A^P; \theta)}{\pi^{pilot}(\tilde{\mathbf{Z}}_i, A, A^P)} (Y_i - \hat{\mu}_{K_i}(\tilde{\mathbf{Z}}_i, X_i, A, A^P)) \right. \\ &\quad \left. + \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathcal{Z}}_i} \pi(\tilde{\mathbf{z}}_i, A, A^P; \theta) [(\hat{\mu}_{K_i}(\tilde{\mathbf{z}}_i, X_i, A, A^P) - \mu(\tilde{\mathbf{z}}_i, X_i, A, A^P))]^2 \right\} |A, A^P]] \\ &\leq 2E[\frac{1}{n^2} \sum_{i=1}^n E[\frac{\pi(\tilde{\mathbf{Z}}_i, A, A^P; \theta)^2}{\pi^{pilot}(\tilde{\mathbf{Z}}_i, A, A^P)^2} (Y_i - \hat{\mu}_{K_i}(\tilde{\mathbf{Z}}_i, X_i, A, A^P))^2 \\ &\quad + (\sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathcal{Z}}_i} \pi(\tilde{\mathbf{z}}_i, A, A^P; \theta) ((\hat{\mu}_{K_i}(\tilde{\mathbf{z}}_i, X_i, A, A^P) - \mu(\tilde{\mathbf{z}}_i, X_i, A, A^P)))^2 |A, A^P]] \\ &\leq 2E[\frac{1}{n^2} \sum_{i=1}^n \frac{4}{\tilde{\delta}^2} M^2 + [4(\gamma_i + 1)^2 M]^2] \\ &= o(1) \end{aligned} \tag{1.25}$$

where the last inequality and equality follow from Assumption 1.4.1 and 1.4.2(i). For $\frac{1}{n^2} \sum_{i=1}^n \sum_{j \neq i} Cov(\Phi_i(\theta), \Phi_j(\theta))$, first note that $Cov(\Phi_i(\theta), \Phi_j(\theta)) = 0$ for $K_j \cap (i \cup G_i) = \emptyset$ or

$K_i \cap (j \cup G_j) = \emptyset$, then under Assumption 1.4.1 1.4.2(i), and 1.4.3, following (1.25), we have

$$\begin{aligned}
& E\left[\frac{1}{n^2} \sum_{i=1}^n \sum_{j \neq i} Cov(\Phi_i(\theta), \Phi_j(\theta) | A, A^P)\right] \\
&= E\left[\frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j: K_j \cap (i \cup G_i) \neq \emptyset \\ K_i \cap (j \cup G_j) \neq \emptyset \\ \text{or } j \in G_i}} Cov(\Phi_i(\theta), \Phi_j(\theta) | A, A^P)\right] \\
&= E\left[\frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j: K_i \cap (j \cup G_j) \neq \emptyset \\ \text{or } j \in G_i}} E[\Phi_i(\theta) \Phi_j(\theta) | A, A^P]\right] \\
&\leq E\left[\frac{1}{n^2} \sum_{i=1}^n \sum_{\substack{j: K_i \cap (j \cup G_j) \neq \emptyset \\ \text{or } j \in G_i}} \sqrt{E[\Phi_i(\theta)^2 | A, A^P] E[\Phi_j(\theta)^2 | A, A^P]}\right] \\
&\leq E\left[\frac{1}{n^2} \sum_{i=1}^n (|\{j : K_i \cap (j \cup G_j) \neq \emptyset\}| + |G_i|) \right. \\
&\quad \left. \sqrt{\left(\frac{4}{\tilde{\delta}^2} M^2 + [4(\gamma_i + 1)^2 M]^2\right) \left(\frac{4}{\tilde{\delta}^2} M^2 + [4(\gamma_j + 1)^2 M]^2\right)}\right] \\
&= o(1), \tag{1.26}
\end{aligned}$$

Combining (1.25) and (1.26), we have $\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) = o_P(1)$. Now we show

$\sup_{\theta \in \Theta} |\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)| = o_P(1)$. Since W_n and W are differentiable with respect to θ , by the mean value theorem we have

$$\begin{aligned}
W_n(\theta, \mathbf{X}, A, A^P) - W_n(\theta', \mathbf{X}, A, A^P) &\leq \sup_{\theta \in \Theta} |W_n'(\theta'', \mathbf{X}, A, A^P)| |\theta - \theta'| \leq M_1 |\theta - \theta'| \\
W(\theta, \mathbf{X}, A, A^P) - W(\theta', \mathbf{X}, A, A^P) &\leq \sup_{\theta \in \Theta} |W'(\theta'', \mathbf{X}, A, A^P)| |\theta - \theta'| \leq M_2 |\theta - \theta'| \tag{1.27}
\end{aligned}$$

where M_1 and M_2 are the uniform bounds on W_n' and W' discussed below. We have

$$\begin{aligned}
|W_n'(\theta, \mathbf{X}, A, A^P)| &= \left| \frac{1}{n} \sum_{i=1}^n \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathbf{Z}}_i} \pi'(\tilde{\mathbf{z}}_i, A, A^P; \theta) \frac{\mathbf{1}^{pilot}\{\tilde{\mathbf{Z}}_i = \tilde{\mathbf{z}}_i\}}{\pi^{pilot}(\tilde{\mathbf{Z}}_i = \tilde{\mathbf{z}}_i, A, A^P)} Y_i \right| \\
&\leq \frac{1}{n} \sum_{i=1}^n |\pi'(\tilde{\mathbf{z}}_i, A, A^P; \theta)| M \\
&\leq M_1
\end{aligned}$$

for some $M_1 > 0$. Such M_1 exists since $\pi'(\tilde{\mathbf{z}}_i, A, A^P; \theta)$ is a polynomial of θ on $[0, 1]$ that only depends on $\tilde{\mathbf{z}}_i \in \tilde{\mathbb{Z}}_i$ and γ_i besides θ . Under Assumption 1.4.2 (i), γ_i is bounded, thus $\tilde{\mathbb{Z}}_i$ does not grow beyond a finite set. Let M_1 denote a constant larger than the maximum $|\pi'(\tilde{\mathbf{z}}_i, A, A^P; \theta)|$ under maximum network degree. Similarly, under Assumption 1.4.2 (i), we have $|W'(\theta, \mathbf{X}, A, A^P)| \leq 2M_1 M_n^{\frac{1}{n}} \sum_{i=1}^n (\gamma_i + 1)^2 \leq M_2$ for some $M_2 > 0$. Then following (1.27), we have

$$\begin{aligned} & |(W_n - W)(\theta, \mathbf{X}, A, A^P) - (W_n - W)(\theta', \mathbf{X}, A, A^P)| \\ & \leq \sup_{\theta \in \Theta} |W'(\theta, \mathbf{X}, A, A^P)| |\theta - \theta'| + \sup_{\theta \in \Theta} |W'_n(\theta, \mathbf{X}, A, A^P)| |\theta - \theta'| \\ & \leq (M_1 + M_2) |\theta - \theta'|. \end{aligned}$$

Next we show stochastic equicontinuity of $\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)$. For any $|\theta_1 - \theta_2| < \delta$, by the mean value theorem we have

$$\sup_{|\theta_1 - \theta_2| < \delta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_1) - \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_2) \right| \leq (M_1 + M_2) \delta.$$

Then for any $\epsilon, \eta > 0$ we have

$$\limsup_{n \rightarrow \infty} P\left(\sup_{|\theta_1 - \theta_2| < \delta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_1) - \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_2) \right| > \epsilon\right) \leq \limsup_{n \rightarrow \infty} P(M_1 + M_2 > \frac{\epsilon}{\delta}).$$

By choosing a small enough $\delta > 0$ such that $\frac{\epsilon}{\delta} > M_1 + M_2$, we have found a δ such that the following inequality holds.

$$\limsup_{n \rightarrow \infty} P\left(\sup_{|\theta_1 - \theta_2| < \delta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_1) - \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_2) \right| > \epsilon\right) \leq \limsup_{n \rightarrow \infty} P(M_1 + M_2 > \frac{\epsilon}{\delta}) < \eta. \quad (1.28)$$

With (1.28), $\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)$ is stochastically equicontinuous. Eventually, to show

$\sup_{\theta \in \Theta} |\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)| = o_P(1)$, we show that for arbitrary $\epsilon > 0$ and $\eta > 0$, there exists N such that for $n \geq N$ we have $P(\sup_{\theta \in \Theta} |\frac{1}{n} \sum_{i=1}^n \Phi_i(\theta)| > \epsilon) < \eta$. Discretizing $[0, 1]$ into a grid

$\Theta_M = \{0, 1/M_g, \dots, 1\}$ with size equal to $1/M_g$, then we use the following inequality

$$\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) \right| \leq \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) - \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_j) \right| + \max_{\theta_j \in \Theta_M} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_j) \right|$$

where θ_j is the closest point to θ in Θ_M (for any θ such θ_j exists). By stochastic equicontinuity, there exists a discretization $[0, 1]$ with a large enough number of points such that there is

$$\limsup_{n \rightarrow \infty} P\left(\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) - \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_j) \right| > \epsilon/2\right) < \frac{\eta}{2}.$$

where $\theta - \theta_j < 1/M_g$. With the discretization Θ_M that satisfies the inequality above, there exists N_1 such that for $n \geq N_1$ there is

$$P\left(\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) - \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_j) \right| > \epsilon/2\right) < \frac{\eta}{2}.$$

There also exists N_2 such that for $n \geq N_2$ we have

$$P\left(\max_{\theta_j \in \Theta_M} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_j) \right| > \epsilon/2\right) < \frac{\eta}{2},$$

since $\max_{\theta_j \in \Theta_M} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_j) \right| = o_P(1)$. For $n \geq \max\{N_1, N_2\}$ we have

$$\begin{aligned} P\left(\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) \right| > \epsilon\right) &\leq P\left(\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_1) - \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_2) \right| > \epsilon/2\right) \\ &\quad + P\left(\max_{\theta_j \in \Theta_M} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta_j) \right| > \epsilon/2\right) < \eta. \end{aligned}$$

As ϵ and η arbitrary, we have shown $\sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \Phi_i(\theta) \right| = o_P(1)$.

Next, we show

$$\lim_{n \rightarrow \infty} P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B\right) = 0. \quad (1.29)$$

Let $f(\theta) = (\theta(1 - \theta), \theta^2)$, $p = (p^{(1,0)}, p^{(1,1)})'$, where $p^{(1,0)} = P(D_i = 1 | Z_i = 1, Z_i^P = 0, A^P)$ and

$p^{(1,1)} = P(D_i = 1|Z_i = 1, Z_i^P = 1, A^P)$, then $C(\theta, A^P) = f(\theta)p$.

$$\begin{aligned}
P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B\right) &= P\left(f(\theta_n^*)\hat{p} > B\right) \\
&= P\left(f(\theta_n^*)(\hat{p} - p) > B - f(\theta_n^*)p\right) \\
&\leq P\left(\sqrt{n}\|f(\theta_n^*)\|\|\hat{p} - p\| > \sqrt{n}(B - f(\theta_n^*)p)\right) \\
&\rightarrow 0
\end{aligned}$$

as long as $\hat{p} - p = O_p(n^{-1/2})$, since $\sqrt{n}(B - C(\theta_n^*(\mathbf{X}, A, A^P), A^P)) \xrightarrow{p} \infty$. Hence we only need to show $\hat{p} - p = O_p(n^{-1/2})$ so that (1.29) holds. We show $\hat{p}^{(z,z')} - p^{(z,z')} = O_p(n^{-1/2})$ for $(z, z') \in \{(1, 0), (1, 1)\}$. For

$$\hat{p}^{(z,z')} = \frac{1/n \sum_{i=1}^n \mathbf{1}\{D_i = 1, Z_i = z, Z_i^P = z'\}}{1/n \sum_{i=1}^n \mathbf{1}\{Z_i = z, Z_i^P = z'\}},$$

let the numerator of $\hat{p}^{(z,z')}$ be $\hat{p}_{nu}^{(z,z')}$ and denominator be $\hat{p}_{de}^{(z,z')}$. By Taylor expansion we have

$$\begin{aligned}
\hat{p}^{(z,z')} - p^{(z,z')} &= \frac{1}{P(Z_i = z, Z_i^P = z')} (\hat{p}_{nu}^{(z,z')} - P(D_i = 1, Z_i = z, Z_i^P = z')) \\
&\quad - \frac{P(D_i = 1, Z_i = z, Z_i^P = z')}{P(Z_i = z, Z_i^P = z')^2} (\hat{p}_{de}^{(z,z')} - P(Z_i = z, Z_i^P = z')) \\
&\quad + O_P((\hat{p}_{nu}^{(z,z')} - P(D_i = 1, Z_i = z, Z_i^P = z'))^2) \\
&\quad + (\hat{p}_{de}^{(z,z')} - P(Z_i = z, Z_i^P = z'))^2) \\
&= O_p(n^{-1/2}) + O_p(n^{-1/2}) + O_p(n^{-1}) \\
&= O_p(n^{-1/2}),
\end{aligned}$$

where $P(Z_i = z, Z_i^P = z')$ is bounded away from 0 by Assumption 1.4.1, and the second last

equality holds as $Var(\hat{p}_{nu}^{(z,z')}) = O(n^{-1})$ and $Var(\hat{p}_{de}^{(z,z')}) = O(n^{-1})$, which we show below.

$$\begin{aligned}
Var\left(\frac{1}{n} \sum_{i=1}^n \mathbf{1}\{Z_i = z, Z_i^P = z'\}\right) &= \frac{1}{n^2} \sum_{i=1}^n Var(\mathbf{1}\{Z_i = z, Z_i^P = z'\}) \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n Cov(\mathbf{1}\{Z_i = z, Z_i^P = z'\}, \mathbf{1}\{Z_j = z, Z_j^P = z'\}) \\
&= \frac{3}{n^2} \sum_{i=1}^n E[P(Z_i = z)P(Z_i^P = z'|A^P) - P(Z_i = z)^2 \\
&\quad \quad \quad P(Z_i^P = z'|A^P)^2] \\
&\leq \frac{3}{n} = O(n^{-1})
\end{aligned}$$

$$\begin{aligned}
Var\left(\frac{1}{n} \sum_{i=1}^n \mathbf{1}\{D_i = 1, Z_i = z, Z_i^P = z'\}\right) &= \frac{1}{n^2} \sum_{i=1}^n Var(\mathbf{1}\{D_i = 1, Z_i = z, Z_i^P = z'\}) \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n Cov(\mathbf{1}\{D_i = 1, Z_i = z, Z_i^P = z'\}, \\
&\quad \quad \quad \mathbf{1}\{D_j = 1, Z_j = z, Z_j^P = z'\}) \\
&= \frac{1}{n^2} \sum_{i=1}^n E[(P(Z_i = z)P(Z_i^P = z')p^{(z,z')}) \\
&\quad \quad \quad - P(Z_i = z)^2 P(Z_i^P = z'|A^P)^2 p^{(z,z')^2}] \\
&\quad + \frac{2}{n^2} \sum_{i=1}^n E[(p^{(z,z')}p^{(z',z)}P(Z_i = z)P(Z_i^P = z'|A^P) \\
&\quad \quad \quad - P(Z_i = z)^2 P(Z_i^P = z'|A^P)^2 p^{(z,z')}p^{(z',z)})] \\
&= O(n^{-1})
\end{aligned}$$

Combining (1.24) and (1.29), we have (1.23) converging to 0, hence the first statement in

Theorem 1.4.2 holds. For asymptotic weak feasibility, we have

$$\begin{aligned}
C(\hat{\theta}, A^P) - B &= f(\hat{\theta})p - f(\hat{\theta})\hat{p} + f(\hat{\theta})\hat{p} - B \\
&= f(\hat{\theta})(p - \hat{p}) + f(\hat{\theta})\hat{p} - B \\
&\leq \hat{\theta}(1 - \hat{\theta})(p^{(1,0)} - \hat{p}^{(1,0)}) + \hat{\theta}^2(p^{(1,1)} - \hat{p}^{(1,1)}) \\
&\leq |p^{(1,0)} - \hat{p}^{(1,0)}| + |p^{(1,1)} - \hat{p}^{(1,1)}| \\
&= o_P(1)
\end{aligned} \tag{1.30}$$

as $\hat{p} - p = o_P(1)$, $f(\hat{\theta})\hat{p} \leq B$ and $0 \leq \hat{\theta} \leq 1$. Thus the second statement of Theorem 1.4.1 holds by

$$P(C(\hat{\theta}, A^P) - B > \epsilon) \leq P(|p^{(1,0)} - \hat{p}^{(1,0)}| + |p^{(1,1)} - \hat{p}^{(1,1)}| > \epsilon) \rightarrow 0$$

□

1.7.3 Proof of Theorem 1.4.2

Proof. For asymptotic welfare efficiency, if $C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) \leq B + b$, then

$$W_n(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) \geq W_n(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P).$$

Therefore, if $C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) \leq B + b$, then

$$\begin{aligned}
&W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) \\
&\leq 2 \sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)|.
\end{aligned}$$

With the inequality above, we have

$$\begin{aligned}
& P\left(W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) < -\epsilon\right) \\
& \leq P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B + b\right) \\
& + P\left(2 \sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| > \epsilon\right)
\end{aligned}$$

and

$$\begin{aligned}
& P\left(W(\hat{\theta}(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) - W(\theta_n^*(\mathbf{X}, A, A^P), \mathbf{X}, A, A^P) > \epsilon\right) \\
& \leq P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B + b\right) \\
& + P\left(2 \sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| > \epsilon\right).
\end{aligned}$$

That $P\left(C_n(\theta_n^*(\mathbf{X}, A, A^P), A^P) > B + b\right) \rightarrow 0$

and $P\left(2 \sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| > \epsilon\right) \rightarrow 0$ can be shown similarly as in the proof of Theorem 1.4.1. Asymptotic weak feasibility can be shown similarly as in (1.30) since $b_n = o_P(1)$. $\square$

1.7.4 Proof of Theorem 1.4.3

Proof. The proof here is different from that of Theorem 1.4.2 only in showing

$\sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| = o_P(1)$. Following the linear-in-means model, we

have

$$\begin{aligned}
& \sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| \\
&= \sup_{\theta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathcal{Z}}_i} (\tilde{\mathbf{z}}'_i, \gamma_i)(\hat{\beta}^d - \beta^d) + X'_i(\hat{\beta}^c - \beta^c) \right| \\
&\leq \frac{1}{n} \sum_{i=1}^n \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathcal{Z}}_i} \|(\tilde{\mathbf{z}}'_i, \gamma_i)\| \|\hat{\beta}^d - \beta^d\| + \frac{1}{n} \sum_{i=1}^n \sum_{\tilde{\mathbf{z}}_i \in \tilde{\mathcal{Z}}_i} \|X_i\| \|\hat{\beta}^c - \beta^c\| \\
&\leq 2\sqrt{6} \frac{1}{n} \sum_{i=1}^n \gamma_i(\gamma_i + 1)^2 \|\hat{\beta}^d - \beta^d\| + 2\sqrt{K} C_K^{max} \frac{1}{n} \sum_{i=1}^n (\gamma_i + 1)^2 \|\hat{\beta}^c - \beta^c\| \\
&= O_P(1) \|\hat{\beta}^d - \beta^d\| + O_P(1) \|\hat{\beta}^c - \beta^c\|
\end{aligned}$$

where $\beta = (\beta^d, \beta^c)'$, indicating the slopes for the discrete parts and continuous parts. The last equality holds by Assumption 1.4.2. To proceed, we have $\sup_{\theta \in \Theta} |W_n(\theta, \mathbf{X}, A, A^P) - W(\theta, \mathbf{X}, A, A^P)| = o_P(1)$ as long as $\|\hat{\beta} - \beta\| = o_P(1)$. To show $\|\hat{\beta} - \beta\| = o_P(1)$, we use $\hat{\beta} - \beta = (\frac{1}{n} \tilde{X} \tilde{X}')^{-1} (\frac{1}{n} \tilde{X}' u)$, and show $\frac{1}{n} \tilde{X} \tilde{X}' \xrightarrow{P} V$ which is positive definite, and $\frac{1}{n} \tilde{X}' u = o_P(1)$. Here $u_i = Y_i - E[Y_i | \tilde{Z}_i, X_i, A, A^P]$.

First, to show $\frac{1}{n} \tilde{X} \tilde{X}' \xrightarrow{P} V$, we show $E[\frac{1}{n} \tilde{X} \tilde{X}' | A, A^P]$ has a finite probability limit V , assuming V is positive definite, then show $Var(\frac{1}{n} \tilde{X} \tilde{X}' | A, A^P) = o_P(1)$. We consider each element in $\frac{1}{n} \sum_{i=1}^n \tilde{X}_i \tilde{X}'_i$. In the following matrix, $X_i \tilde{X}'_i$ and $\tilde{X}_i X'_i$ are the parts that involve

i.i.d. covariates X_i .

$$\frac{1}{n} \sum_{i=1}^n \tilde{X}_i \tilde{X}_i' = \frac{1}{n} \sum_{i=1}^n \begin{pmatrix} 1 & Z_i & Z_i^P & Z_i^{1,0} & Z_i^{1,1} & \gamma_i & \\ Z_i & Z_i Z_i & Z_i Z_i^P & Z_i Z_i^{(1,0)} & Z_i Z_i^{(1,1)} & Z_i \gamma_i & \\ Z_i^P & Z_i^P Z_i & Z_i^P Z_i^P & Z_i^P Z_i^{(1,0)} & Z_i^P Z_i^{(1,1)} & Z_i^P \gamma_i & \tilde{X}_i X_i' \\ Z_i^{(1,0)} & Z_i^{(1,0)} Z_i & Z_i^{(1,0)} Z_i^P & Z_i^{(1,0)} Z_i^{(1,0)} & Z_i^{(1,0)} Z_i^{(1,1)} & Z_i^{(1,0)} \gamma_i & \\ Z_i^{(1,1)} & Z_i^{(1,1)} Z_i & Z_i^{(1,1)} Z_i^P & Z_i^{(1,1)} Z_i^{(1,0)} & Z_i^{(1,1)} Z_i^{(1,1)} & Z_i^{(1,1)} \gamma_i & \\ \gamma_i & \gamma_i Z_i & \gamma_i Z_i^P & \gamma_i Z_i^{(1,0)} & \gamma_i Z_i^{(1,1)} & \gamma_i^2 & \\ & & & X_i \tilde{X}_i' & & & \end{pmatrix}$$

$$E\left[\frac{1}{n} \sum_{i=1}^n Z_i^{(1,1)} Z_i^P | A, A^P\right] = E\left[\frac{1}{n} \sum_{i=1}^n \left(\sum_{\substack{j \in N_i, \\ A_{ij}^P = 0}} Z_j Z_j^P \right) Z_i^P | A, A^P\right] \leq \frac{1}{n} \sum_{i=1}^n \gamma_i$$

$$\begin{aligned} E\left[\frac{1}{n} \sum_{i=1}^n Z_i^{(1,1)} Z_i^{(1,0)} | A, A^P\right] &= E\left[\frac{1}{n} \sum_{i=1}^n \left(\sum_{\substack{j \in N_i, \\ A_{ij}^P = 0}} Z_j Z_j^P \right) \left(\sum_{\substack{j \in N_i, \\ A_{ij}^P = 0}} Z_j - Z_j Z_j^P \right) | A, A^P\right] \\ &= E\left[\frac{1}{n} \sum_{i=1}^n \left(\sum_{\substack{j \in N_i, \\ A_{ij}^P = 0}} Z_j Z_j^P \right) \left(\sum_{\substack{k \in N_i, A_{ij}^P = 0, \\ k \neq j, A_{jk}^P = 0}} Z_k - Z_k Z_k^P \right) | A, A^P\right] \\ &\leq \frac{1}{n} \sum_{i=1}^n \gamma_i^2 \end{aligned}$$

$$E\left[\frac{1}{n} \sum_{i=1}^n Z_i^{(1,1)} X_{ik} | A, A^P\right] = E\left[\frac{1}{n} \sum_{i=1}^n \left(\sum_{\substack{j \in N_i, \\ A_{ij}^P = 0}} Z_j Z_j^P \right) X_{ik} | A, A^P\right] \leq E[X_{ik}] \frac{1}{n} \sum_{i=1}^n \gamma_i$$

By Assumption 1.4.2 (ii) (iii), $\frac{1}{n} \sum_{i=1}^n \gamma_i$ and $\frac{1}{n} \sum_{i=1}^n \gamma_i^2$ have finite probability limits. We also have $E[X_i] < \infty$ by Assumption 1.4.1 (ii) (iii). The boundedness of other elements in

$E[\frac{1}{n} \sum_{i=1}^n \tilde{X}_i \tilde{X}_i' A, A^P]$ can be shown similarly. So that $E[\frac{1}{n} \sum_{i=1}^n \tilde{X}_i \tilde{X}_i' | A, A^P] \rightarrow_p V$, assuming that V is positive definite. Next we show $Var(\frac{1}{n} \sum_{i=1}^n \tilde{X}_i \tilde{X}_i' | A, A^P) = o_P(1)$. Note that under Assumption 1.4.2 (ii) (iii), we have $\frac{1}{n} \sum_{i=1}^n \gamma_i = O_P(1)$, $\frac{1}{n} \sum_{i=1}^n \sum_{j \in G_i} \gamma_i \gamma_j = O_P(1)$, and $\frac{1}{n} \sum_{i=1}^n \sum_{j \in G_i} \gamma_i^2 \gamma_j^2 = O_P(1)$, then we have

$$\begin{aligned}
Var(\frac{1}{n} \sum_{i=1}^n Z_i^{(1,1)} Z_i^P | A, A^P) &= \frac{1}{n^2} \sum_{i=1}^n Var(Z_i^{(1,1)} Z_i^P | A, A^P) \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \in G_i} \{E[(Z_i^{(1,1)} Z_j^{(1,1)} Z_i^P Z_j^P | A, A^P) \\
&\quad - E[Z_i^{(1,1)} Z_i^P | A] E[Z_j^{(1,1)} Z_j^P | A, A^P]\} \\
&\leq \frac{1}{n^2} \sum_{i=1}^n Var(Z_i^{(1,1)} Z_i^P | A, A^P) \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \in G_i} E[Z_i^{(1,1)} Z_j^{(1,1)} Z_i^P Z_j^P | A, A^P] \\
&\leq \frac{1}{n^2} \sum_{i=1}^n \gamma_i + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \in G_i} \gamma_i \gamma_j = o_P(1)
\end{aligned}$$

$$\begin{aligned}
Var(\frac{1}{n} \sum_{i=1}^n Z_i^{(1,1)} Z_i^{(1,0)} | A, A^P) &= \frac{1}{n^2} \sum_{i=1}^n Var(Z_i^{(1,1)} Z_i^{(1,0)} | A, A^P) \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \in G_i} \{E[Z_i^{(1,1)} Z_j^{(1,1)} Z_i^{(1,0)} Z_j^{(1,0)} | A, A^P] \\
&\quad - E[Z_i^{(1,1)} Z_i^{(1,0)} | A] E[Z_j^{(1,1)} Z_j^{(1,0)} | A, A^P]\} \\
&\leq \frac{1}{n^2} \sum_{i=1}^n Var(Z_i^{(1,1)} Z_i^{(1,0)} | A, A^P) \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \in G_i} E[Z_i^{(1,1)} Z_j^{(1,1)} Z_i^{(1,0)} Z_j^{(1,0)} | A, A^P] \\
&\leq \frac{1}{n^2} \sum_{i=1}^n \gamma_i^2 \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \in G_i} \gamma_i^2 \gamma_j^2 = o_P(1)
\end{aligned}$$

$$\begin{aligned}
\text{Var}\left(\frac{1}{n} \sum_{i=1}^n Z_i^{(1,1)} X_{ik} | A, A^P\right) &= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(Z_i^{(1,1)} X_{ik} | A, A^P) \\
&\quad + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \in G_i} \{E[Z_i^{(1,1)} Z_j^{(1,1)} X_{ik} X_{jk} | A, A^P] \\
&\quad - E[Z_i^{(1,1)} X_{ik} | A] E[Z_j^{(1,1)} X_{jk} | A, A^P]\} \\
&\leq \frac{1}{n^2} C_k \sum_{i=1}^n \gamma_i + \frac{1}{n^2} \sum_{i=1}^n C_k^2 \sum_{j \in G_i} \gamma_i \gamma_j = o_P(1)
\end{aligned}$$

where $|X_{ik}| < C_k$. Thus $\text{Var}(\frac{1}{n} \sum_{i=1}^n \tilde{X}_i \tilde{X}_i' | A, A^P) = o_P(1)$. We obtain

$$\left(\frac{1}{n} \sum_{i=1}^n \tilde{X}_i \tilde{X}_i'\right)^{-1} = V^{-1} + o_P(1) \quad (1.31)$$

by continuous mapping. Next, we show $\frac{1}{n} \sum_{i=1}^n \tilde{X}_i u_i = o_P(1)$ by showing

$\text{Var}(\frac{1}{n} \sum_{i=1}^n \tilde{X}_i u_i | \tilde{\mathbf{X}}, A, A^P) = o_P(1)$. As $E[u_i | \tilde{X}, A, A^P] = 0$, we have

$$\begin{aligned}
\text{Var}\left(\frac{1}{n} \sum_{i=1}^n \tilde{X}_i u_i | \tilde{\mathbf{X}}, A, A^P\right) &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n E[u_i u_j | \tilde{\mathbf{X}}, A, A^P] \tilde{X}_i \tilde{X}_j' \\
&= \frac{1}{n^2} \sum_{i=1}^n E[u_i^2 | \tilde{\mathbf{X}}, A, A^P] \tilde{X}_i \tilde{X}_i' + \frac{1}{n^2} \sum_{i=1}^n \sum_{j \in G_i} E[u_i u_j | \tilde{\mathbf{X}}, A, A^P] \tilde{X}_i \tilde{X}_j'.
\end{aligned} \quad (1.32)$$

Note that all elements in (1.32) are bounded by

$$\begin{aligned}
&\frac{1}{n^2} \max\{C_k\}_{k=1, \dots, K}^2 \sum_{i=1}^n E[u_i^2 | \tilde{\mathbf{X}}, A, A^P] + \frac{1}{n^2} \max\{C_k\}_{k=1, \dots, K}^2 \sum_{i=1}^n \sum_{j \in G_i} E[u_i u_j | \tilde{\mathbf{X}}, A, A^P] \gamma_i \gamma_j \\
&\leq \frac{1}{n^2} \max\{C_k\}_{k=1, \dots, K}^2 \sum_{i=1}^n \tilde{C} + \frac{\tilde{C}}{n^2} \max\{C_k\}_{k=1, \dots, K}^2 \sum_{i=1}^n \sum_{j \in G_i} \gamma_i \gamma_j \\
&= O_P(n^{-1}),
\end{aligned}$$

where the last equality holds by Assumption 1.4.2 (ii) (iii), and let $\tilde{C}$ be such that

$\max\{E[u_i^2|\tilde{\mathbf{X}}, A, A^P], E[u_i u_j|\tilde{\mathbf{X}}, A, A^P]\} < \tilde{C}$. Then we have

$$Var(\frac{1}{n} \sum_{i=1}^n \tilde{X}_i u_i) = O(n^{-1}),$$

and hence obtain

$$\frac{1}{n} \sum_{i=1}^n \tilde{X}_i u_i = o_P(1). \tag{1.33}$$

Combining (1.31) and (1.33), we have $\|\hat{\beta} - \beta\| = o_P(1)$. □

Chapter 2

Causal Inference in RCTs with Multiple Treatment Arms

2.1 Introduction

Randomized controlled trials (RCT) are frequently used when researchers are interested in evaluating the treatment effect of some program or policy. Usually, researchers choose to implement a standard RCT with one treatment group and one control group. However, when researchers would like to examine the treatment effects of more than one treatment, the experiment may include more than one treatment arm. In this case, often the goal is to compare the effects of different treatments.

To compare different treatment effects in an RCT with multiple treatment arms, we can simply estimate the average treatment effect (ATE) of each treatment if there is perfect compliance. However, noncompliance often occurs when treatment is not enforceable. When there is noncompliance, the average treatment effect (ATE) is only partially identified. Previous literature has been focusing on partially identifying ATE, or further, distributions of potential outcomes. For example, Manski (1995), Balke and Pearl (1997), Manski (1990), Kitagawa (2009), Shaikh and Vytlacil (2011), among others. Under noncompliance, some

researchers instead estimate the intent-to-treat (ITT) effect. Since ITT also incorporates the outcomes of those who are assigned but do not comply, ITT estimates can be conservative and may not allow a precise comparison across treatments. Meanwhile, it is also common to estimate each treatment’s local average treatment effect (LATE) under a monotonicity assumption as in Angrist and Imbens (1995). Compared to ATE and ITT which are evaluated for the whole population, each LATE is conditional on the compliers of a specific treatment. As a result, we cannot compare the LATEs of different treatments to obtain a causal comparison since different treatments have different compliers.

To make a causal comparison, we need to compare treatment effects conditional on the same population. In general, ATE and ITT are conditional on the whole population without more specifications. Alternatively, we may divide the whole population into subpopulations and analyze treatment effects within them. ATEs for subpopulations have been studied in the binary-treatment case, as in Chen and Flores (2015); Huber and Mellace (2015); Huber et al. (2017), among others. In the binary-treatment case, besides compliers, examples of subpopulations can be always takers or defiers. When there are multiple treatments, there can be other interesting subpopulations. For example, when there are two treatment arms and one control arm with noncompliance, and if individuals not assigned to a treatment do not have access to it, the population then consists of four subpopulations: “only comply with treatment 1”, “only comply with treatment 2”, “comply with both treatments”, and “comply with neither treatment”. Each of these subpopulations and associated causal estimands are potentially interesting. To compare two treatments causally, for example, we can examine the average difference between the two treatment effects for those who would comply with both treatments. In practice, when there are two poverty-alleviation policies in an RCT, then the individuals who would comply with both policies might be the ones most in need. As another interpretation, when there is a potential new policy but it is uncertain whether the new policy would outperform the existing one, policymakers may run a three-arm RCT where each policy is a treatment. Then individuals who would comply with both policies

are those who would participate in the existing policy and still comply under the new policy. The average difference between the two treatment effects for this subpopulation may reflect the welfare change “on the intensive margin”. To summarize, for causal comparisons across treatments, we can focus on subpopulations, not only is this a way to causally compare different treatments, but the estimands can answer many policy relevant questions which cannot be answered by ATE, ITT, or LATE.

While we would like to point identify the average difference in treatment effects conditional on some subpopulation, it is hard to achieve this goal without strong assumptions. To begin with, noncompliance results in the failure of point identification of various parameters such as ATE and ATT in the binary-treatment setting. In our case, the difficulty of point identification lies in the fact that each observed outcome distribution is a mixture of distributions for subpopulations. This is similar to the binary-treatment setting where the observed outcome distribution under the treatment is a mixture of the distributions for compliers and always takers. ATE for compliers is point identified under the monotonicity assumption in this case. However, we cannot directly apply this strategy when there are multiple arms. Later we show that point identification fails even under some strong monotonicity assumption. Thus in this paper we turn to partial identification of average difference in treatment effects for subpopulations, given the mixtures of outcome distributions for subpopulations.

In this paper, we provide pairwise sharp bounds on the average difference between two treatment effects for always compliers, that is, those who would comply with both treatments. Specifically, we impose a threshold crossing selection model for each treatment arm, as in Angrist and Imbens (1995), Heckman and Vytlacil (2005), Vytlacil (2002). By imposing additional assumptions on the copula linking the unobserved heterogeneity terms in the threshold crossing models, we first characterize sharp bounds on the proportion of the subpopulation of interest within the mixtures of outcome distributions. We then derive sharp bounds on the average difference between two treatment effects for always compliers.

Additionally, given two treatments, we provide bounds on some other causal estimands,

including the ATE of the first or second treatment for those who would comply with only one of the treatments. We then show that the estimation procedure can be cast into a linear programming problem. Lastly, we apply the bounds derived to an experiment evaluating the impacts of different depression treatments on economic outcomes in India.

This paper first contributes to the literature on partial identification of treatment effects in experiments with more than one treatment. Within this small literature, Cheng and Small (2006) study a three-arm setting and provide sharp bounds on the average difference between two treatment effects when the outcome variable is binary. They sharpen the bounds with a one-sided monotonicity assumption: individuals who comply with treatment 1 must comply with treatment 2. We allow the outcome to be continuous and consider one-sided monotonicity only as a special case. At the same time, we impose a threshold-crossing model for compliance and exploit the identifying power of a novel assumption on the unobserved heterogeneity. There are also papers that consider RCTs with two treatment groups but no control group including Roy et al. (2008) and Fischer et al. (2011). Fischer et al. (2011) compare two treatments in a structural mean model, and Roy et al. (2008) focus on the case where individuals always comply with the first treatment.

This paper also contributes to the literature of multivalued treatment effect model with discrete instruments. In this literature, most papers (Heckman and Pinto (2018), Navjeevan et al. (2023)) focus on the identification of weighted averages of compliance types' treatment effects, or the the identification of treatment effects for types not discussed in this paper. For example, in a discrete choice model with multiple treatments and discrete instruments, Heckman and Pinto (2018) introduce a weak monotonicity to identify various treatment effects including the average difference in treatment effects for those who switch across treatments. Another distinction lies in that, papers in this literature usually impose some discrete choice model where individuals can select into each treatment. In our case, we assume that for each treatment, if not assigned, then an individual is not allowed take the treatment. This means that we do not observe individuals' choices under alternative assignments, or preferences

across treatments, and hence we may not directly extend the method under a general discrete choice model with multidimensional unobserved heterogeneity as in Kamat et al. (2023), for example.

The remainder of this paper is organized as follows. The next section describes the econometric framework and derives the bounds on the average difference between the two treatment effects for those who would comply with both treatments. Section 3 discusses estimation procedure. Section 4 applies the bounds to the three-arm RCT evaluating different depression treatments in India. Section 5 concludes.

2.2 Econometric Framework

2.2.1 Setup, Assumptions, and Parameter of Interest

Consider a random sample of size n from a large population. Individuals are randomly assigned to treatment 1, 2, ..., k , or a control group. Let $Z \in \{0, 1, \dots, k\}$ indicate treatment assignment and $D \in \{0, 1, \dots, k\}$ indicate the treatment an individual actually receives. At the same time, let $D(z)$ indicate the potential treatment under assignment z , and $Y(d)$ indicate the potential outcome of treatment d . Let Y denote the observed outcome. From the random sample we observe Z , D , and Y . In this paper, as we focus on pairwise comparisons of the treatments, from now on we focus on RCTs with two treatment arms such that $Z \in \{0, 1, 2\}$ and $D \in \{0, 1, 2\}$. Before discussing our parameters of interest, we first introduce the following assumptions:

Assumption 2.2.1. (*SUTVA*)

$$Y_i(d) \perp D_j \text{ and } D_i(z) \perp Z_j, \forall j \neq i, z = 0, 1, 2, d = 0, 1, 2$$

Assumption 2.2.1 is the stable unit treatment value assumption, which is often imposed in treatment effect models. Under Assumption 2.2.1, the treatment status of individual i is independent of the assignment of other individuals, and the potential outcomes of individual

i are independent of the treatment status of other individuals.

Assumption 2.2.2. (*No Access*)

$$P(D = 1 \mid Z = 0 \text{ or } Z = 2) = P(D = 2 \mid Z = 0 \text{ or } Z = 1) = 0$$

Assumption 2.2.2 states that individuals not assigned to a specific treatment do not have access to it. Under this assumption, there is one-sided noncompliance, and for each treatment there are no always takers or defiers.

For compliance, we assume that individuals assigned to each treatment choose whether or not to take the treatment based on the following threshold-crossing model:

$$D(z) = \begin{cases} z, & \text{if } U_z < \mu_z(X), \quad z \in \{1, 2\} \\ 0, & \text{otherwise,} \end{cases} \quad (2.1)$$

where μ_1 and μ_2 are unknown functions of observed individual characteristics X , U_1 and U_2 are unobserved continuous random variables that affect compliance behaviors. $D(1)$ and $D(2)$ are potential treatments under treatments 1 and 2. Here we assume that the take-up of each treatment is affected by different unobservables and different functions of individual characteristics. Without further assumptions, any correlation between U_1 and U_2 is allowed.

Assumption 2.2.3. (*Random Assignment*)

$$Z \perp Y_0, Y_1, Y_2, U_1, U_2 \mid X.$$

Assumption 2.2.3 states that assignment Z is jointly independent of potential outcomes and unobservables, conditional on X . This assumption holds under random assignment in RCTs.

Suppose we are interested in comparing the treatment effects of treatment 1 and treatment 2. Under the threshold-crossing model above, there is noncompliance when $U_z \geq \mu_z(X)$. With noncompliance, it is hard to compare treatment 1 and 2 by point estimating their average treatment effects separately. Some existing papers focus on the partial identification of average treatment effects. In contrast, we decompose the population into subpopulations and analyze

their treatment effects. First, note that each individual's full potential compliance behavior can be described by potential treatments $D(1)$ and $D(2)$. Under the threshold-crossing model and No-Access assumption, each individual can be randomly categorized into one of the following types:

$$T = \begin{cases} 10 & \text{if } D(1) = 1 \text{ and } D(2) = 0 \\ 02 & \text{if } D(1) = 0 \text{ and } D(2) = 2 \\ 12 & \text{if } D(1) = 1 \text{ and } D(2) = 2 \\ 00 & \text{if } D(1) = 0 \text{ and } D(2) = 0. \end{cases} \quad (2.2)$$

For example, individuals who would only comply with treatment 1 are type “10”, and individuals who would not comply with any treatment belong to type “00”. Note that since T is a function of (U_1, U_2, X) , it follows that assignment Z is also independent of T conditional on X .

In this paper, different from the whole population, we focus on subpopulations characterized by compliance types and individual characteristics. Our main parameter of interest is $E[Y(2) - Y(1) \mid T = 12, X = x]$, the average treatment effect of treatment 2 compared to treatment 1 for those who would comply with both treatments. For convenience, we call them “always compliers”, and we refer to $E[Y(2) - Y(1) \mid T = 12, X = x]$ as the “average difference for always compliers”. There are also other interesting parameters, such as the average treatment effect of treatment 1 for those who would comply with treatment 1. Later we provide partial identification results for a range of parameters. First, we focus on always compliers.

Let $\pi_t(x) = P(T = t \mid X = x)$ for $t \in \{00, 10, 12, 02\}$, then $\pi_{12}(x)$ indicates the proportion of always compliers:

$$\pi_{12}(x) = P(T = 12 \mid X = x).$$

To partially identify $E[Y(2) - Y(1) \mid T = 12]$, the average difference for always compliers,

we analyze $E[Y(1) \mid T = 12]$ and $E[Y(2) \mid T = 12]$ separately. First note that the observed distributions of $Y(1)$ and $Y(2)$ conditional on D and X are weighted averages of the conditional distributions for different types:

$$\begin{aligned} F_{Y|D=1,X=x}(y(1)) &= \frac{\pi_{12}(x)}{\pi_{10}(x) + \pi_{12}(x)} F_{Y(1)|T=12,X=x}(y(1)) + \frac{\pi_{10}(x)}{\pi_{10}(x) + \pi_{12}(x)} F_{Y(1)|T=10,X=x}(y(1)) \\ F_{Y|D=2,X=x}(y(2)) &= \frac{\pi_{12}(x)}{\pi_{02}(x) + \pi_{12}(x)} F_{Y(2)|T=12,X=x}(y(2)) + \frac{\pi_{02}(x)}{\pi_{02}(x) + \pi_{12}(x)} F_{Y(2)|T=02,X=x}(y(2)). \end{aligned} \quad (2.3)$$

At the same time, for the means we have

$$\begin{aligned} E[Y(1) \mid D = 1, X = x] &= \frac{\pi_{12}(x)}{\pi_{10}(x) + \pi_{12}(x)} E[Y(1) \mid T = 12, X = x] \\ &\quad + \frac{\pi_{10}(x)}{\pi_{10}(x) + \pi_{12}(x)} E[Y(1) \mid T = 10, X = x] \\ E[Y(2) \mid D = 2, X = x] &= \frac{\pi_{12}(x)}{\pi_{02}(x) + \pi_{12}(x)} E[Y(2) \mid T = 12, X = x] \\ &\quad + \frac{\pi_{02}(x)}{\pi_{02}(x) + \pi_{12}(x)} E[Y(2) \mid T = 02, X = x]. \end{aligned} \quad (2.4)$$

From (2.3) and (2.4), the distributions or conditional means of potential outcomes for types are not point identified. For example, we only point identify a mixture of $E[Y(1) \mid T = 12, X = x]$ and $E[Y(1) \mid T = 10, X = x]$, not $E[Y(1) \mid T = 12, X = x]$ alone. Note that $\pi_{12}(x)$, the proportion of always compliers, is not point identified either. In summary, without more assumptions, we only point identify mixtures of the parameters which are needed for identifying our parameter of interest.

Next, starting from the mixtures in (2.3), our goal is to provide sharp bounds for $E[Y(2) - Y(1) \mid T = 12, X = x]$. Our method contains two steps. We first partially identify $\pi_{12}(x)$. Given the sharp identified set of $\pi_{12}(x)$, we derive sharp bounds for $E[Y(1) \mid T = 12, X = x]$ and $E[Y(2) \mid T = 12, X = x]$.

2.2.2 Sharp Identified Set of $\pi_{12}(x)$

Define $V_z = F_{U_z}(U_z)$, and $P_z(X) = P(D = z \mid Z = z, X) = P(U_z < \mu_z(X)) = F_{U_z}(\mu_z(X))$, for $z \in \{1, 2\}$, where F_{U_z} is the CDF of U_z . The random variables V_1 and V_2 are uniformly distributed on $[0, 1]$ and unobserved. $P_1(X)$ and $P_2(X)$ are propensity scores for treatment 1 and 2. Following (2.1), always compliers are those characterized by $U_1 < \mu_1(X)$ and $U_2 < \mu_2(X)$, thus we have

$$\pi_{12}(x) = P(V_1 < P_1(X), V_2 < P_2(X) \mid X = x) = P(V_1 < P_1(x), V_2 < P_2(x)), \quad (2.5)$$

which shows that the identification of $\pi_{12}(x)$ depends on the joint distribution of V_1 and V_2 . By Skyler's Theorem, there exists a copula $C : [0, 1]^2 \rightarrow [0, 1]$ with

$$C(v_1, v_2) = P(V_1 \leq v_1, V_2 \leq v_2).$$

Then given (2.5), we have

$$\pi_{12}(x) = C(P_1(x), P_2(x)).$$

For any copula, following the Fréchet–Hoeffding bound there is

$$\max\{P_1(x) + P_2(x) - 1, 0\} \leq \pi_{12}(x) \leq \min\{P_1(x), P_2(x)\}.$$

Meanwhile, $\pi_{12}(x)$ needs to be compatible with the observed data distributions. Let $f(y(0), y(1), y(2) \mid T = t, X = x)$ denote the probability density of $(Y(0), Y(1), Y(2))$ given $T = t$ and $X = x$. For all individuals in the sample, we observe a vector of variables (Y, D, Z) , and distributions $F_{Y(0) \mid D=0, Z=0, X=x}$, $F_{Y(0) \mid D=0, Z=1, X=x}$, $F_{Y(0) \mid D=0, Z=2, X=x}$, $F_{Y(1) \mid D=1, Z=1, X=x}$, $F_{Y(2) \mid D=2, Z=2, X=x}$. The following expressions show that each of these distributions is a

weighted average of unobserved outcome distributions of some types:

$$\begin{aligned}
F_{Y|D=0,Z=0,X=x} &= \pi_{12}(x)F_{Y(0)|T=12,X=x} + [P_1(x) - \pi_{12}(x)]F_{Y(0)|T=10,X=x} + \\
&\quad [P_2(x) - \pi_{12}(x)]F_{Y(0)|T=02,X=x} + [1 - P_1(x) - P_2(x) + \pi_{12}(x)]F_{Y(0)|T=00,X=x} \\
F_{Y|D=0,Z=1,X=x} &= \frac{P_2(x) - \pi_{12}(x)}{1 - P_1(x)}F_{Y(0)|T=02,X=x} + \frac{1 - P_1(x) - P_2(x) + \pi_{12}(x)}{1 - P_1(x)}F_{Y(0)|T=00,X=x} \\
F_{Y|D=0,Z=2,X=x} &= \frac{P_1(x) - \pi_{12}(x)}{1 - P_2(x)}F_{Y(0)|T=10,X=x} + \frac{1 - P_1(x) - P_2(x) + \pi_{12}(x)}{1 - P_2(x)}F_{Y(0)|T=00,X=x} \\
F_{Y|D=1,Z=1,X=x} &= \frac{P_1(x) - \pi_{12}(x)}{P_1(x)}F_{Y(1)|T=10,X=x} + \frac{\pi_{12}(x)}{P_1(x)}F_{Y(1)|T=12,X=x} \\
F_{Y|D=2,Z=2,X=x} &= \frac{P_2(x) - \pi_{12}(x)}{P_2(x)}F_{Y(2)|T=02,X=x} + \frac{\pi_{12}(x)}{P_2(x)}F_{Y(2)|T=12,X=x}. \tag{2.6}
\end{aligned}$$

We characterize the identified set of $\pi_{12}(x)$ as the set of proportions of always compliers given which the Fréchet–Hoeffding bound is satisfied, and there exists a data distribution that reproduces the LHS of (2.6), the observed outcome distributions.

Lemma 2.2.1. *Let ω be a four-dimensional vector of proper PDFs, $\omega = (f(y(0), y(1), y(2)) \mid T = 00, X = x), f(y(0), y(1), y(2)) \mid T = 10, X = x), f(y(0), y(1), y(2)) \mid T = 02, X = x), f(y(0), y(1), y(2)) \mid T = 12, X = x)$, and define*

$$S(x) = \left\{ \pi_{12}(x) \left| \begin{array}{l} \max\{P_1(x) + P_2(x) - 1, 0\} \leq \pi_{12}(x) \leq \min\{P_1(x), P_2(x)\}, \\ \exists \omega \text{ such that } \omega \text{ solves (2.6)} \end{array} \right. \right\}.$$

Then under Assumptions 2.2.1–2.2.3, $S(x)$ is the sharp identified set for $\pi_{12}(x)$.

Next, to further narrow down the bounds for $\pi_{12}(x)$, we impose a further restriction on the dependence between the unobservables U_1 and U_2 . Following Joe (1997), we have the following definition:

Definition 2.2.1. *U_1 and U_2 are positively quadrant dependent (PQD) if $P(U_1 \leq a_1, U_2 \leq a_2) \geq P(U_1 \leq a_1)P(U_2 \leq a_2) \forall a_1, a_2 \in \mathbb{R}$.*

Below we assume that U_1 and U_2 are positively quadrant dependent. This assumption is

plausible if the two treatments compared are similar, in the sense that the unobservables that affect compliance “move in the same direction”.

Assumption 2.2.4. (*Positively Quadrant Dependent (PQD)*)

U_1 and U_2 are positively quadrant dependent.

Let C^* be the true bivariate copula for U_1 and U_2 , there is

$$F_{U_1, U_2}(u_1, u_2) = C^*(F_{U_1}(u_1), F_{U_2}(u_2)).$$

Under Assumption 2.2.4, C^* is positively quadrant dependent. Since copulas are invariant under monotone transformations of the marginals, the copula for V_1 and V_2 is the same as that for U_1 and U_2 . Thus the copula for V_1 and V_2 is positively quadrant dependent as well. Let $\mathcal{C}^{PQD}$ denote the space of bivariate copulas such that copulas in $\mathcal{C}^{PQD}$ satisfy Assumption 2.2.4. Define the set of $\pi_{12}(x)$ that satisfies Assumption 2.2.4 as:

$$S_{PQD}(x) = \{C(P_1(x), P_2(x)) \mid C \in \mathcal{C}^{PQD}\}. \quad (2.7)$$

Lemma 2.2.2. *Under Assumptions 2.2.1–2.2.4, $S(x) \cap S_{PQD}(x)$ is the sharp identified set of $\pi_{12}(x)$.*

2.2.3 Sharp Identified Set of $E[Y(2) - Y(1)|T = 12, X = x]$

With the sharp identified set for $\pi_{12}(x)$, now we bound $E[Y(2) - Y(1)|T = 12, X = x]$. Given some $\pi_{12}(x) \in S(x) \cap S_{PQD}(x)$, we follow the trimming method in Horowitz and Manski (1995) and Imai (2008) to derive the bounds $E[Y(1)|T = 12, X = x]$ and $E[Y(2)|T = 12, X = x]$. Note that in (2.3), $F_{Y(1)|D=1, Z=1, X=x}$ is a mixture of $F_{Y(1)|T=12, X=x}$ and $F_{Y(1)|T=10, X=x}$, and $F_{Y(2)|D=2, Z=2, X=x}$ is a mixture of $F_{Y(2)|T=12, X=x}$ and $F_{Y(2)|T=02, X=x}$.

Below we provide a sketch of the bounding strategy. To find the upper bound of $E[Y(2) - Y(1)|T = 12, X = x]$, let $F_{Y(1)|T=12, X=x}$, the unobserved distribution of $Y(1)$

for always compliers, concentrate on the “left tail” of $F_{Y(1)|D=1,X=x}$, and let $F_{Y(2)|T=12,X=x}$ concentrate on the “right tail” of $F_{Y(2)|D=2,X=x}$. In this way, we try to make $E[Y(1)|T = 12, X = x]$ achieve a value as large as possible, and $E[Y(2)|T = 12, X = x]$ achieve a value as small as possible. Similarly, to find the lower bound of $E[Y(2) - Y(1)|T = 12, X = x]$, we let $F_{Y(1)|T=12,X=x}$ concentrate on the “right tail” of $F_{Y(1)|D=1,X=x}$, and let $F_{Y(2)|T=12,X=x}$ concentrate on the “left tail” of $F_{Y(2)|D=2,X=x}$.

For the “left tail” and “right tail” of the observed distributions, we define the following distributions for always compliers:

$$\begin{aligned} F_{Y(1)|T=12,D=1,X=x}^{L,\pi_{12}(x)}(y(1)) &= \min\left\{ \frac{F_{Y(1)|D=1,X=x}(y(1))}{\pi_{12}(x)/(\pi_{10}(x) + \pi_{12}(x))}, 1 \right\} \\ F_{Y(1)|T=12,D=1,X=x}^{R,\pi_{12}(x)}(y(1)) &= \max\left\{ 0, \frac{F_{Y(1)|D=1,X=x}(y(1)) - \pi_{10}(x)/(\pi_{10}(x) + \pi_{12}(x))}{\pi_{12}(x)/(\pi_{10}(x) + \pi_{12}(x))} \right\}, \end{aligned} \quad (2.8)$$

similar for $F_{Y(2)|T=12,D=2,X=x}^{L,\pi_{12}(x)}(y(2))$ and $F_{Y(2)|T=12,D=2,X=x}^{R,\pi_{12}(x)}(y(2))$. Intuitively, when always compliers occupy the “right tail” of $F_{Y(1)|D=1,X=x}$, their $Y(1)$ has zero density on the left-hand side of $F_{Y(1)|D=1,X=x}^{-1}(\frac{\pi_{10}(x)}{(\pi_{10}(x) + \pi_{12}(x))})$, where $F_{Y(1)|D=1,X=x}^{-1}(t) = \inf\{y(1)|F_{Y(1)|D=1,X=x}(y(1)) \geq t\}$. When always compliers occupy the “left tail” of $F_{Y(2)|D=2,X=x}$, their $Y(2)$ has zero density on the right-hand side of $F_{Y(2)|D=2,X=x}^{-1}(\frac{\pi_{12}(x)}{(\pi_{10}(x) + \pi_{12}(x))})$.

Under this bounding strategy, below we have our main result of the paper.

Proposition 2.2.1. *Under Assumptions 2.2.1-2.2.4, given the sharp identified set of $\pi_{12}(x)$, assuming that $\min S(x) \cap S_{PQD}(x)$ exists, then the average treatment effect of treatment 2 compared to treatment 1 conditional on $X = x$ for always compliers is bounded as below, and the bounds are sharp:*

$$\begin{aligned} E[Y(2) - Y(1) | T = 12, X = x]^{UB} &= \int y(2) dF_{Y(2)|T=12,D=2,X=x}^{R,\pi_{12}^{min}(x)} - \int y(1) dF_{Y(1)|T=12,D=1,X=x}^{L,\pi_{12}^{min}(x)} \\ E[Y(2) - Y(1) | T = 12, X = x]^{LB} &= \int y(2) dF_{Y(2)|T=12,D=2,X=x}^{L,\pi_{12}^{min}(x)} - \int y(1) dF_{Y(1)|T=12,D=1,X=x}^{R,\pi_{12}^{min}(x)}. \end{aligned} \quad (2.9)$$

where $\pi_{12}^{min}(x) = \min S(x) \cap S_{PQD}(x)$.

Besides the sharp bounds of the average difference for always compliers, below we also obtain bounds for some other parameters. For each $\pi_{12}(x) \in S(x) \cap S_{PQD}(x)$, let $\mathcal{F}_{y(0)|t;\pi_{12}(x)}$ denote the set of compatible marginal distributions of $Y(0)$ for type $T = t$. This means that given some $\pi_{12}(x) \in S(x) \cap S_{PQD}(x)$, $\mathcal{F}_{y(0)|t;\pi_{12}(x)}$ contains the marginal distributions compatible with (2.6). Then define

$$E_{inf,\pi_{12}(x)}[Y(0) | T = t, X = x] = \inf\left\{\int y(0)dF \mid F \in \mathcal{F}_{y(0)|t;\pi_{12}(x)}\right\},$$

also define $E_{sup,\pi_{12}(x)}[Y(0) | T = t, X = x]$ similarly.

Proposition 2.2.2. *Under Assumptions 2.2.1-2.2.4, the average treatment effect of treatment 1 conditional on $X = x$ and $T = 10$, and the average treatment effect of treatment 2 conditional on $X = x$ and $T = 02$ have the following bounds:*

$$\begin{aligned} & E[Y(1) - Y(0) | T = 10, X = x]^{UB} \\ &= \sup_{\pi_{12}(x) \in S(x) \cap S_{PQD}(x)} \left(\int y dF_{10|\pi_{12}(x), D=1}^R - E_{inf,\pi_{12}(x)}[Y(0) | T = 10, X = x] \right) \\ & E[Y(1) - Y(0) | T = 10, X = x]^{LB} \\ &= \inf_{\pi_{12}(x) \in S(x) \cap S_{PQD}(x)} \left(\int y dF_{10|\pi_{12}(x), D=1}^L - E_{sup,\pi_{12}(x)}[Y(0) | T = 10, X = x] \right) \end{aligned} \quad (2.10)$$

$$\begin{aligned} & E[Y(2) - Y(0) | T = 02, X = x]^{UB} \\ &= \sup_{\pi_{12}(x) \in S(x) \cap S_{PQD}(x)} \left(\int y dF_{02|\pi_{12}(x), D=2}^R - E_{inf,\pi_{12}(x)}[Y(0) | T = 02, X = x] \right) \\ & E[Y(2) - Y(0) | T = 02, X = x]^{LB} \\ &= \inf_{\pi_{12}(x) \in S(x) \cap S_{PQD}(x)} \left(\int y dF_{02|\pi_{12}(x), D=2}^L - E_{sup,\pi_{12}(x)}[Y(0) | T = 02, X = x] \right) \end{aligned} \quad (2.11)$$

In Proposition 2.2.2, we bound the treatment effect of treatment 1 for those who would only take treatment 1, as well as the treatment effect of treatment 2 for those who would

only take treatment 2. These bounds can be relevant for some policy questions. For example, when bounding the treatment effect of treatment 1 for those who would only take treatment 1, we are bounding the welfare loss resulting from the drop-out of some people under the new policy, if we consider $T = 10$ as those who would only comply with the old policy. Meanwhile, if we would like to bound the welfare increase brought by those who newly join the policy, we can bound the average treatment effect of the new policy for those who would only comply with the new one if we consider this group of individuals as $T = 02$. Different from always compliers, we cannot provide bounds for the average difference between treatment 1 and 2 for these two subpopulations without extrapolation, as we only observe their potential outcomes under one of the treatments.

2.2.4 The Generalized Roy Model

The results in the propositions above can be applied to a generalized Roy model, which has been widely used in the treatment effects literature, as in Heckman and Honore (1990), Heckman and Vytlacil (2005), Eisenhauer et al. (2015). In the generalized Roy model, we have the following potential outcome model:

$$Y(z) = g_z(X) + U_z, \quad z \in \{0, 1, 2\} \quad (2.12)$$

where g_z is an unknown function of X and U_z is an unobserved random variable. The cost of taking a treatment is

$$C(z) = g_z^c(X) + U_{cz}, \quad z \in \{1, 2\} \quad (2.13)$$

where g_z^c is also an unknown function of X , and U_{cz} is unobserved. For compliance, an individual assigned to treatment z would comply if the surplus of taking the treatment is

positive:

$$D(z) = z, \quad \text{if } Y(z) - Y(0) - C(z) > 0, \quad z \in \{1, 2\}. \quad (2.14)$$

Let $\mu_Z(X) = g_Z(X) - g_0(X) - g_Z^C(X)$, and $\tilde{U}_Z = U_0 + U_C - U_Z$, then we have the following threshold-crossing model:

$$D(Z) = Z, \quad \text{if } \tilde{U}_Z < \mu_Z(X), \quad Z \in \{1, 2\}. \quad (2.15)$$

In this case, always compliers are characterized by $\tilde{U}_1 < \mu_1(X)$ and $\tilde{U}_2 < \mu_2(X)$, then we have $E[Y(2) - Y(1) \mid T = 12, X = x] = E[Y(2) - Y(1) \mid \tilde{U}_1 < \mu_1(x), \tilde{U}_2 < \mu_2(x)]$.

Assumption 2.2.5. U_0, U_1, U_2 , and U_{C1}, U_{C2} are mutually independent.

Under Assumption 2.2.5, the dependence among potential outcomes is only through covariates X , and the costs are independent of potential outcomes conditional on X . Note that Assumption 2.2.5 implies that $\tilde{U}_1$ and $\tilde{U}_2$ are positive quadrant dependent. In this case, PQD is an implication of the independence assumption rather than an imposed assumption on the dependence across unobservables. Under PQD, we can return to Assumption 2.2.4 and apply Propositions 2.2.1 and 2.2.2 accordingly.

2.2.5 Bounds of $E[Y(2) - Y(1) \mid T = 12]$ when $U_1 = U_2$

As a special case, treatment 1 and treatment 2 may share the same unobservable in the threshold-crossing model. In this case, the difference in compliance for two treatments are purely driven by observables, and our model is simplified.

Assumption 2.2.6. $U_1 = U_2$.

Under Assumption 2.2.6, let $F_U(u)$ denote the distribution of U_1 and U_2 . Then there is

$$\begin{aligned} E[Y(2) - Y(1) \mid T = 12, X = x] &= E[Y(2) - Y(1) \mid U_1 < \mu_1(X), U_2 < \mu_2(X), X = x] \\ &= E[Y(2) - Y(1) \mid U_1 < \min\{\mu_1(X), \mu_2(X)\}, X = x]. \end{aligned}$$

Since $P(D = 1 \mid Z = 1, X = x) = F_U(\mu_1(x))$ and $P(D = 2 \mid Z = 2, X = x) = F_U(\mu_2(x))$, we observe the order of $\mu_1(x)$ and $\mu_2(x)$ by comparing $P(D = 1 \mid Z = 1, X = x)$ and $P(D = 2 \mid Z = 2, X = x)$. We focus on the case where $P(D = 1 \mid Z = 1, X = x) < P(D = 2 \mid Z = 2, X = x)$, as the other case is similar.

When $\mu_1(X) < \mu_2(X)$, all of those who comply with treatment 1 would comply with treatment 2, thus we have $\pi_{12}^*(x) = P(D = 1 \mid Z = 1, X = x)$ and $\pi_{10}^*(x) = 0$. This corresponds to the monotonicity assumption from Cheng and Small (2006). Now as $\pi_{12}^*(x)$ and $\pi_{10}^*(x)$ are point identified, $\pi_{02}^*(x)$ is point identified as well with $\pi_{02}^*(x) = P(D = 2 \mid Z = 2, X = x) - P(D = 1 \mid Z = 1, X = x)$. Then we have the following identification result for always compliers.

$$\begin{aligned} E[Y(2) - Y(1) \mid T = 12, X = x] &= E[Y(2) - Y(1) \mid U_1 < \min\{\mu_1(X), \mu_2(X)\}, X = x] \\ &= E[Y(2) - Y(1) \mid U_1 < \mu_1(X), X = x] \\ &= E[Y(2) \mid U_1 < \mu_1(X), X = x] \\ &\quad - E[Y(1) \mid U_1 < \mu_1(X), X = x] \\ &= E[Y(2) \mid U_1 < \mu_1(X), X = x] - E[Y(1) \mid D = 1, X = x] \end{aligned} \tag{2.16}$$

where $E[Y(1) \mid D = 1, X = x]$ is point identified, and $E[Y(2) \mid U_1 < \mu_1(X), X = x]$ can be partially identified.

Proposition 2.2.3. *Under Assumptions 2.2.1-2.2.3 and Assumption 2.2.6, when $P(D = 1 \mid Z = 1, X = x) < P(D = 2 \mid Z = 2, X = x)$, the average treatment effect of treatment 2*

compared to treatment 1 conditional on $X = x$ for always compliers has the following sharp bounds:

$$\begin{aligned} E[Y(2) - Y(1) \mid T = 12, X = x]^{UB} &= \int y(2) dF_{Y(2)|T=12,D=2,X=x}^{R,\pi_{12}^*(x)} - E[Y(1) \mid D = 1, X = x] \\ E[Y(2) - Y(1) \mid T = 12, X = x]^{LB} &= \int y(2) dF_{Y(2)|T=12,D=2,X=x}^{L,\pi_{12}^*(x)} - E[Y(1) \mid D = 1, X = x] \end{aligned} \quad (2.17)$$

Note that now $E[Y(0) \mid T = 02, X = x]$ is point identified with

$$\begin{aligned} E[Y(0) \mid T = 02, X = x] &= \{E[Y \mid D = 0, Z = 1, X = x] - \frac{\pi_{00}(x)}{\pi_{02}(x) + \pi_{00}(x)} \\ &\quad E[Y \mid D = 0, Z = 2, X = x]\} \times \frac{\pi_{02}(x) + \pi_{00}(x)}{\pi_{02}(x)}. \end{aligned}$$

Proposition 2.2.4. *Under Assumptions 2.2.1-2.2.3 and Assumption 2.2.6, the average treatment effect of treatment 2 conditional on $X = x$ and $T = 02$ has the following sharp bounds:*

$$\begin{aligned} E[Y(2) - Y(0) \mid T = 02, X = x]^{UB} &= \int y(2) dF_{Y(2)|T=02,D=2,X=x}^{R,\pi_{12}^*(x)} - E[Y(0) \mid T = 02, X = x] \\ E[Y(2) - Y(0) \mid T = 02, X = x]^{LB} &= \int y(2) dF_{Y(2)|T=02,D=2,X=x}^{L,\pi_{12}^*(x)} - E[Y(0) \mid T = 02, X = x]. \end{aligned} \quad (2.18)$$

2.3 Estimation

In this section, we show that the bounds in (2.9) can be estimated by linear programming. To estimate the bounds in (2.9), we first estimate $S(x) \cap S_{PQD}(x)$, the identified set for $\pi_{12}(x)$. We discretize the support of Y into $\mathcal{Y} = \{y_1, y_2, \dots, y_l\}$ and let $h_t(y_0, y_1, y_2) = P(Y(0) = y_i, Y(1) = y_j, Y(2) = y_k \mid T = t, X = x)$. To estimate $S(x)$, we convert (2.6) into a set of linear constraints and look for feasible solutions to the linear system. For the compatibility

constraint, we have

$$\begin{aligned}
P(Y = y_i \mid D = 0, Z = 0, X = x) &= \pi_{12}(x) \sum_{j=1}^l \sum_{k=1}^l h_{12}(y_i, y_j, y_k) \\
&\quad + (P_1(x) - \pi_{12}(x)) \sum_{j=1}^l \sum_{k=1}^l h_{10}(y_i, y_j, y_k) \\
&\quad + (P_2(x) - \pi_{12}(x)) \sum_{j=1}^l \sum_{k=1}^l h_{02}(y_i, y_j, y_k) \\
&\quad + (1 - P_1(x) - P_2(x)) \sum_{j=1}^l \sum_{k=1}^l h_{00}(y_i, y_j, y_k), \\
&\quad \forall i = 1, \dots, l \\
P(Y = y_i \mid D = 0, Z = 1, X = x) &= \frac{P_2(x) - \pi_{12}(x)}{1 - P_1(x)} \sum_{j=1}^l \sum_{k=1}^l h_{02}(y_i, y_j, y_k) \\
&\quad + \frac{1 - P_1(x) - P_2(x) + \pi_{12}(x)}{1 - P_1(x)} \sum_{j=1}^l \sum_{k=1}^l h_{00}(y_i, y_j, y_k), \\
&\quad \forall i = 1, \dots, l \\
P(Y = y_i \mid D = 0, Z = 2, X = x) &= \frac{P_1(x) - \pi_{12}(x)}{1 - P_2(x)} \sum_{j=1}^l \sum_{k=1}^l h_{10}(y_i, y_j, y_k) \\
&\quad + \frac{1 - P_1(x) - P_2(x) + \pi_{12}(x)}{1 - P_2(x)} \sum_{j=1}^l \sum_{k=1}^l h_{00}(y_i, y_j, y_k), \\
&\quad \forall i = 1, \dots, l \\
P(Y = y_j \mid D = 1, Z = 1, X = x) &= \frac{P_1(x) - \pi_{12}(x)}{P_1(x)} \sum_{i=1}^l \sum_{k=1}^l h_{10}(y_i, y_j, y_k) \\
&\quad + \frac{\pi_{12}(x)}{P_1(x)} \sum_{i=1}^l \sum_{k=1}^l h_{12}(y_i, y_j, y_k), \quad \forall j = 1, \dots, l \\
P(Y = y_k \mid D = 2, Z = 2, X = x) &= \frac{P_2(x) - \pi_{12}(x)}{P_2(x)} \sum_{i=1}^l \sum_{j=1}^l h_{02}(y_i, y_j, y_k) \\
&\quad + \frac{\pi_{12}(x)}{P_2(x)} \sum_{i=1}^l \sum_{j=1}^l h_{12}(y_i, y_j, y_k), \\
&\quad \forall k = 1, \dots, l
\end{aligned} \tag{2.19}$$

At the same time, we need to ensure that the joint distribution of potential outcomes has proper distributions $\forall t \in \{00, 10, 02, 12\}$:

$$\sum_{i=1}^l \sum_{j=1}^l \sum_{k=1}^l h_t(y_i, y_j, y_k) = 1, \quad \forall t \in \{00, 10, 02, 12\}$$

$$h_t(y_i, y_j, y_k) \geq 0, \quad \forall i = 1, \dots, l, \quad \forall j = 1, \dots, l, \quad \forall k = 1, \dots, l, \quad \forall t \in \{00, 10, 02, 12\}. \quad (2.20)$$

For each $\pi_{12}(x) \in \{\max\{P_1(x) + P_2(x) - 1, 0\}, \max\{P_1(x) + P_2(x) - 1, 0\} + 0.01, \dots, \min\{P_1(x), P_2(x)\}\}$, we check whether there exists a solution to the linear system (2.19) and (2.20) through linear programming. Let $\hat{S}(x)$ be the set that contains all $\pi_{12}(x)$ under which there exists a solution to (2.19) and (2.20).

To estimate $S_{PQD}(x)$, we search for copulas that satisfy PQD from the whole space of bivariate copulas. To achieve this, we use a procedure similar to the ones in Chetty et al. (2017) and Frandsen and Lefgren (2021). Specifically, we search over the space of discrete copulas defined on a 100×100 grid, subject to the PQD constraints. Under PQD, we need

$$C_{i,j} - C_{i,100}C_{100,j} \geq 0, \quad \forall i = 1, \dots, 100, \quad j = 1, \dots, 100, \quad (2.21)$$

at the same time, for C to be a proper copula, we need

$$\begin{aligned} 0 &\leq C_{i,j} \leq 1, \\ C_{i,100} &= i/100, \\ C_{100,j} &= j/100, \\ C_{i+1,j} - C_{i,j} &\geq 0, \\ C_{i,j+1} - C_{i,j} &\geq 0, \\ \forall i &= 1, \dots, 100, \quad j = 1, \dots, 100. \end{aligned} \quad (2.22)$$

We look for feasible solutions in the form of discrete copulas to the linear system (2.21) and

(2.22). As $\pi_{12}(x) = C_{P_1(x) \cdot 100, P_2(x) \cdot 100}$, let $\hat{S}_{PQD}(x)$ be the set that contains $\pi_{12}(x)$ which is from a solution to (2.21) and (2.22). Let $\hat{\pi}_{12}^{min}(x) = \min \hat{S}(x) \cap \hat{S}_{PQD}(x)$.

Lastly, given $\hat{\pi}_{12}^{min}(x)$, we obtain an estimate of $\int y(1) dF_{Y(1)|T=12, D=1}^{R, \hat{\pi}_{12}^{min}(x)}$ by taking average of the values above $F_{Y(1)|D=1, X=x}^{-1}\left(\frac{P_1(x) - \hat{\pi}_{12}^{min}(x)}{P_1(x)}\right)$, and an estimate of $\int y(2) dF_{Y(2)|T=12, D=2, X=x}^{L, \hat{\pi}_{12}^{min}(x)}$ by taking average of the values below $F_{Y(2)|D=2, X=x}^{-1}\left(\frac{\hat{\pi}_{12}^{min}(x)}{P_2(x)}\right)$.

2.4 Empirical Application

We apply the bounds in Proposition 2.9 to an experiment from Angelucci and Bennett (2024) that studies the impacts of two different depression treatments on depression and socioeconomic outcomes. The RCT was implemented in Karnataka, India, from 2016 to 2017 based on a sample of 1,000 mildly or moderately depressed individuals across 506 villages. The sampled individuals were randomized into four arms: a control arm, a Psychiatric Care (PC) arm, a Livelihoods Assistance (LA) arm, and a PC/LA arm. The PC treatment provided eight months of free psychiatric care that included diagnosis and medical treatment from a local research hospital. The LA treatment provided two group meetings discussing ways to earn income and deal with on-the-job challenges, and personalized one-on-one assistance about job placements, small loans, or job training. The PC/LA treatment combined the two treatments.

Compliance in the PC arm was defined as attending at least one psychiatric meeting, and compliance in the LA arm was defined as attending at least one livelihoods assistance meeting. Compliance in the PC/LA arm was defined as attending at least one of the two types of meetings. In this scenario, always compliers are those who would comply with both PC and LA treatments if assigned separately. Always compliers differ from the compliers in the PC/LA arm. First, since compliance in the PC/LA arm was defined as attending at least one of any type of meeting, compliers in the PC/LA arm also included those who would comply with only one of the two treatments. Additionally, those who would comply with the

Table 2.1: Bounds with PQD (Rounds 3–5)

Round 3			
Outcomes	Lower Bound	Upper Bound	ITT
PHQ-9 depression scale (0–27)	−1.3618	1.1622	0.0076
Weekly paid and unpaid work hours	−20.8005	34.8306	5.0236*
Weekly earnings (Rs.)	−521.4000	643.1200	−56.3386
Round 4			
Outcomes	Lower Bound	Upper Bound	ITT
PHQ-9 depression scale (0–27)	−1.4153	1.3647	−0.0657
Weekly paid and unpaid work hours	−31.2807	31.9548	1.3426
Weekly earnings (Rs.)	−441.1392	1021.4100	106.9218
Round 5			
Outcomes	Lower Bound	Upper Bound	ITT
PHQ-9 depression scale (0–27)	−1.0329	1.2634	0.0682
Weekly paid and unpaid work hours	−29.9023	31.5693	−1.7208
Weekly earnings (Rs.)	−766.3333	1105.1530	61.3473

Table 2.2: Bounds with $U_1 = U_2$ (Rounds 3–5)

Round 3		
Outcomes	Lower Bound	Upper Bound
PHQ-9 depression scale (0–27)	−0.58	0.41
Weekly paid and unpaid work hours	−1.79	18.33
Weekly earnings (Rs.)	−372.43	82.61
Round 4		
Outcomes	Lower Bound	Upper Bound
PHQ-9 depression scale (0–27)	−0.58	0.50
Weekly paid and unpaid work hours	−10.28	15.90
Weekly earnings (Rs.)	−307.98	406.59
Round 5		
Outcomes	Lower Bound	Upper Bound
PHQ-9 depression scale (0–27)	−0.40	0.56
Weekly paid and unpaid work hours	−12.04	13.99
Weekly earnings (Rs.)	−483.90	269.57

two treatments separately may not comply when they are asked to take the two treatments at the same time. Therefore, we restrict our analysis to the arms excluding the PC/LA arm.

The RCT had 395 participants in the control arm, 207 participants in the PC arm, and 205 participants in the LA arm. Following Angelucci and Bennett (2024), the main outcomes we study include depression severity, weekly work hours, and weekly earnings for participants. Depression severity was measured using the PHQ-9 scale. In total, 5 rounds of surveys were conducted, we focus on rounds 3-5, which were after the implementation of the treatments.

Table 2.1 presents the results for the three outcomes, it shows the average difference between the PC treatment and LA treatment for always compliers. As all of the bounds contain zero, for the three main outcomes, we cannot not conclude the sign of the average difference between the two treatments. In this setting, we may consider always compliers as those who are the most eager to recover from depression. Then the results in Table 2.1 imply

that Psychiatric Care and Livelihoods Assistance treatments do not make a difference for this subpopulation.

Next, rather than PQD, we examine the bounds with $U_1 = U_2$, which means that the unobservables that determine the compliance of PC and LA are the same. Under $U_1 = U_2$, we have monotonicity in compliance as in Cheng and Small (2006). That is, those who comply with treatment PC must comply with treatment LA, as LA has a higher take-up rate. In this RCT, $U_1 = U_2$ would be a plausible assumption if those who are willing to take the Livelihoods Assistance treatment are also willing to take the Psychiatric Care treatment. In this case, let LA be treatment 1 and PC be treatment 2, then $E[Y_2 | T = 12]$ is point identified, and we partially identify $E[Y_1 | T = 12]$. Table 2 shows the bounds for the average difference between the two treatments under $U_1 = U_2$. As in Table 1, from Table 2 we cannot conclude the sign of the average difference between the two treatments. Compared to Table 1, we obtain tighter bounds in Table 2, as $U_1 = U_2$ implies perfect correlation between the two unobservables, which is stronger than PQD.

2.5 Conclusion

In this paper, I define novel subpopulations in RCTs when there is non-compliance and more than one treatment arm. Under the positive quadrant dependence (PQD) assumption, I derive sharp bounds for the average difference between two treatment effects for always compliers, as well as sharp bounds for the average treatment effects of other subpopulations. The bounds in this paper can be widely applied to settings where always compliers or other subpopulations are of interest. I apply the bounds derived to an experiment evaluating the impacts of different depression treatments on economic outcomes in India, and show that we cannot pin down the sign of the average difference between Psychiatric Care and Livelihoods Assistance for always compliers.

2.6 Appendix

2.6.1 Proof of Lemma 2.2.1

Proof. Let P_x be a joint distribution of $(Y(0), Y(1), Y(2), Z, U_1, U_2)$ given $X = x$, and $\mathbb{P}_x$ be the set of P_x such that P_x satisfies Assumptions 2.2.1-2.2.3 and is compatible with the observed distribution of (Y, Z, D) . For $S(x)$ to be the sharp identified set of $\pi_{12}(x)$, we need to show that $\forall \pi_{12}(x) \in S(x)$, there exists $P_x \in \mathbb{P}_x$ which generates π_{12} and the observed distribution, and that $\forall P_x \in \mathbb{P}_x$, there exists some $\pi_{12}(x) \in S(x)$ which can be generated by P_x .

For the first direction, let $\tilde{P}(Z = z|X = x) = a_z$, $z \in \{0, 1, 2\}$ with a_z equal to the probability of random assignment for treatment z in the experiment. Let $\tilde{f}_{u_1, u_2|X=x}$ be any joint distribution of $U_1, U_2 \sim Unif(0, 1)$ with $P(U_1 < \mu_1(X), U_2 < \mu_2(X)|X = x) = \pi_{12}(x)$, $P(U_1 < \mu_1(X)|X = x) = P_1(x)$, and $P(U_2 < \mu_2(X)|X = x) = P_2(x)$. Define $D(1) = \mathbf{1}\{U_1 < \mu_1(x)\}$, $D(2) = 2 * \mathbf{1}\{U_2 < \mu_2(x)\}$, $D = \mathbf{1}\{Z = 1\}D(1) + \mathbf{1}\{Z = 2\}D(2)$, and $Y = Y(D)$. Define T the same way as in (2.2). Lastly, to construct P_x , let

$$\begin{aligned} \tilde{f}(y(0), y(1), y(2), z, u_1, u_2|X = x) &= \tilde{f}_{u_1, u_2|X=x}(u_1, u_2|X = x) \\ &\quad \tilde{f}(y(0), y(1), y(2)|U_1 = u_1, U_2 = u_2, X = x) \\ &\quad \tilde{P}(Z = z|X = x) \end{aligned} \tag{2.23}$$

where let

$$\tilde{f}(y(0), y(1), y(2)|U_1 = u_1, U_2 = u_2, X = x) = f(y(0), y(1), y(2)|T = t, X = x). \tag{2.24}$$

We obtain $f(y(0), y(1), y(2)|T = t, X = x)$ from ω (note that given $\pi_{12}(x) \in S(x)$, by the definition of $S(x)$, there exists ω that satisfies (2.6)).

Now we verify that the distribution constructed in (2.23) satisfies Assumptions 2.2.1-2.2.3

and generates the observed distribution of (Y, Z, D) . SUTVA and No-Access assumptions are satisfied following the definition of D and Y , the Random Assignment assumption is satisfied following (2.23) with

$$P(Z = z|X = x) = P(Z = z|U_1 = u_1, U_2 = u_2, Y(1) = y(1), Y(2) = y(2), Y(0) = y(0), X = x).$$

Following the joint distribution constructed in (2.23) and the definition of D , T , and Y , first note that by the definition of $\tilde{\pi}_{12}(x)$ we have $\tilde{\pi}_{12}(x) = \pi_{12}(x)$, hence $\pi_{12}(x)$ is reproduced. Then given the DGP constructed above, we have the following:

$$\begin{aligned} \tilde{F}_{Y|D=0,Z=0,X=x} &= \tilde{\pi}_{12}(x)\tilde{F}_{Y(0)|T=12,X=x} + [\tilde{P}_1(x) - \tilde{\pi}_{12}(x)]\tilde{F}_{Y(0)|T=10,X=x} + \\ &\quad [\tilde{P}_2(x) - \tilde{\pi}_{12}(x)]\tilde{F}_{Y(0)|T=02,X=x} + [1 - \tilde{P}_1(x) - \tilde{P}_2(x) + \tilde{\pi}_{12}(x)]\tilde{F}_{Y(0)|T=00,X=x} \\ \tilde{F}_{Y|D=0,Z=1,X=x} &= \frac{\tilde{P}_2(x) - \tilde{\pi}_{12}(x)}{1 - \tilde{P}_1(x)}\tilde{F}_{Y(0)|T=02,X=x} + \frac{1 - \tilde{P}_1(x) - \tilde{P}_2(x) + \tilde{\pi}_{12}(x)}{1 - \tilde{P}_1(x)}\tilde{F}_{Y(0)|T=00,X=x} \\ \tilde{F}_{Y|D=0,Z=2,X=x} &= \frac{\tilde{P}_1(x) - \tilde{\pi}_{12}(x)}{1 - \tilde{P}_2(x)}\tilde{F}_{Y(0)|T=10,X=x} + \frac{1 - \tilde{P}_1(x) - \tilde{P}_2(x) + \tilde{\pi}_{12}(x)}{1 - \tilde{P}_2(x)}\tilde{F}_{Y(0)|T=00,X=x} \\ \tilde{F}_{Y|D=1,Z=1,X=x} &= \frac{\tilde{P}_1(x) - \tilde{\pi}_{12}(x)}{\tilde{P}_1(x)}\tilde{F}_{Y(1)|T=10,X=x} + \frac{\tilde{\pi}_{12}(x)}{\tilde{P}_1(x)}\tilde{F}_{Y(1)|T=12,X=x} \\ \tilde{F}_{Y|D=2,Z=2,X=x} &= \frac{\tilde{P}_2(x) - \tilde{\pi}_{12}(x)}{\tilde{P}_2(x)}\tilde{F}_{Y(2)|T=02,X=x} + \frac{\tilde{\pi}_{12}(x)}{\tilde{P}_2(x)}\tilde{F}_{Y(2)|T=12,X=x}. \end{aligned} \quad (2.25)$$

where by (2.24) $\tilde{F}_{Y(0)|T=12,X=x} = F_{Y(0)|T=12,X=x}$, similar for other conditional distributions on the RHS of (2.25). Also by the definitions of $\tilde{P}_1(x)$ and $\tilde{P}_2(x)$ there is $\tilde{P}_1(x) = P_1(x)$ and $\tilde{P}_2(x) = P_2(x)$. As a result, $\tilde{F}_{Y|D=0,Z=0,X=x} = F_{Y|D=0,Z=0,X=x}$, similarly for other distributions on the LHS of (2.25) and (2.6). Thus the LHS of (2.25) is equal to the LHS of (2.6), and the observed distribution is reproduced by the DGP constructed.

For the other direction, when P_x satisfies Assumptions 2.2.1-2.2.3 and is compatible with the observed distribution of (Y, Z, D) , (2.6) is then implied, thus the condition for $S(x)$ is satisfied and $\pi_{12}(x) \in S(x)$. $\square$

2.6.2 Proof of Lemma 2.2.2

Proof. The proof follows from the one for Lemma 2.1 with the additional PQD constraint on U_1 and U_2 . $\square$

2.6.3 Proof of Proposition 2.2.1

Proof. We first show that the bounds in (2.9) are valid, that is,

$$\begin{aligned} E[Y(2) - Y(1)|T = 12, X = x]^{LB} &\leq E[Y(2) - Y(1)|T = 12, X = x] \\ &\leq E[Y(2) - Y(1)|T = 12, X = x]^{UB}. \end{aligned}$$

Given $\pi_{12}(x) \in S(x) \cap S_{PQD}(x)$, we first show that

$$\begin{aligned} \int y(1) dF_{Y(1)|T=12,D=1}^L &\leq E[Y(1)|T = 12, X = x] \leq \int y(1) dF_{Y(1)|T=12,D=1}^R \\ \int y(2) dF_{Y(2)|T=12,D=2}^L &\leq E[Y(2)|T = 12, X = x] \leq \int y(2) dF_{Y(2)|T=12,D=2}^R. \end{aligned} \quad (2.26)$$

First note that from (2.3) and (2.8) we have

$$\begin{aligned} F_{Y(1)|T=12,D=1,X=x}(y(1)) &= (F_{Y(1)|D=1,X=x}(y(1)) - \frac{\pi_{10}(x)}{\pi_{10}(x) + \pi_{12}(x)}) \\ &\quad F_{Y(1)|T=10,X=x}(y(1)) \frac{\pi_{10}(x) + \pi_{12}(x)}{\pi_{12}(x)} \\ F_{Y(1)|T=12,D=1,X=x}^L(y(1)) &= \min\left\{ \frac{F_{Y(1)|D=1,X=x}(y(1))}{\pi_{12}(x)/(\pi_{12}(x) + \pi_{10}(x))}, 1 \right\} \end{aligned}$$

which implies

$$F_{Y(1)|T=12,D=1,X=x}(y(1)) \leq F_{Y(1)|T=12,D=1,X=x}^L(y(1)) \quad \forall y(1),$$

so that $F_{Y(1)|T=12,D=1,X=x}$ stochastically dominates $F_{Y(1)|T=12,D=1,X=x}^L$. Similarly we have that

$F_{Y(2)|T=12,D=2,X=x}$ stochastically dominates $F_{Y(2)|T=12,D=2,X=x}^L$.

At the same time, we have

$$\begin{aligned}
F_{Y(1)|T=12,D=1,X=x}(y(1)) &= (F_{Y(1)|D=1,X=x}(y(1)) - \frac{\pi_{10}(x)}{\pi_{10}(x) + \pi_{12}(x)}) \\
&\quad F_{Y(1)|T=10,X=x}(y(1)) \frac{\pi_{10}(x) + \pi_{12}(x)}{\pi_{12}(x)} \\
F_{Y(1)|T=12,D=1,X=x}^R(y(1)) &= \max\{0, \frac{F_{Y(1)|D=1,X=x}(y(1)) - \pi_{10}(x)/(\pi_{10}(x) + \pi_{12}(x))}{\pi_{12}(x)/(\pi_{12}(x) + \pi_{10}(x))}\},
\end{aligned}$$

which imply

$$F_{Y(1)|T=12,D=1,X=x}^R(y(1)) \leq F_{Y_1|T=12,D=1,X=x}(y(1)) \quad \forall y(1),$$

so that $F_{Y(1)|T=12,D=1,X=x}^R$ stochastically dominates $F_{Y(1)|T=12,D=1,X=x}$. Similarly, we have that $F_{Y(2)|T=12,D=2,X=x}^R$ stochastically dominates $F_{Y(2)|T=12,D=2,X=x}$. As expectation respects stochastic dominance, we have shown (2.26).

Let $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{LB}$ be the lower bound of $E[Y(2) - Y(1)|T = 12, X = x]$ given some $\pi_{12}(x)$, similarly for $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{UB}$. Now we show that $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{LB}$ is monotonically increasing in $\pi_{12}(x)$ and that $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{UB}$ is monotonically decreasing in $\pi_{12}(x)$ so that we obtain

$$\begin{aligned}
E[Y(2) - Y(1)|T = 12, X = x]^{LB} &\leq E[Y(2) - Y(1)|T = 12, X = x] \\
&\leq E[Y(2) - Y(1)|T = 12, X = x]^{UB}.
\end{aligned}$$

We take the derivative of $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{LB}$ and $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{UB}$ with respect to $\pi_{12}(x)$:

$$\frac{\partial E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{LB}}{\partial \pi_{12}(x)} > 0. \quad (2.27)$$

We obtain the sign from (2.27) since there is

$$\frac{\partial E[Y(2)|D = 2, X = x, Y(2) < F_{Y(2)|D=2, X=x}^{-1}(\pi_{12}/P_2(x))]}{\partial \pi_{12}(x)} > 0$$

$$\frac{\partial E[Y(1)|D = 1, X = x, Y(1) > F_{Y(1)|D=1, X=x}^{-1}((P_1(x) - \pi_{12}(x))/P_1(x))]}{\partial \pi_{12}(x)} < 0,$$

and hence we have

$$\frac{\partial E[Y(2)|T = 12, X = x; \pi_{12}(x)]^{LB}}{\partial \pi_{12}(x)} > 0;$$

$$\frac{\partial E[Y(1)|T = 12, X = x; \pi_{12}(x)]^{UB}}{\partial \pi_{12}(x)} < 0.$$

Similar to (2.27), we also have

$$\frac{\partial E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{UB}}{\partial \pi_{12}(x)} < 0. \quad (2.28)$$

Following (2.27) and (2.28), we have

$$E[Y(2) - Y(1)|T = 12, X = x]^{LB} = E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}^{min}(x)]^{LB}$$

and

$$E[Y(2) - Y(1)|T = 12, X = x]^{UB} = E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}^{min}(x)]^{UB}.$$

Our next goal is to show that the bounds in (2.9) are sharp. The proof of sharpness involves three steps. First, we show that $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{LB}$ is monotonically increasing in π_{12} and that $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{UB}$ is monotonically decreasing in π_{12} . We have shown this above.

In the second step, we show that as a result, $E[Y(2) - Y(1)|T = 12, X = x]^{LB}$ and $E[Y(2) - Y(1)|T = 12, X = x]^{UB}$ are attained at π_{12}^{min} with a proper $F(y(0), y(1), y(2)|T = t, X = x)$ which is compatible with the observed distributions and satisfies Assumptions

2.2.1-2.2.4.

Lastly, we show that for any $E[Y(2) - Y(1)|T = 12, X = x] \in [E[Y(2) - Y(1)|T = 12, X = x]^{LB}, E[Y(2) - Y(1)|T = 12, X = x]^{UB}]$, there exists $\pi_{12} \in S(x) \cap S_{PQD}(x)$ and a proper $F(y(0), y(1), y(2)|T = t, X = x)$ which is compatible with the observed distributions and satisfies Assumptions 2.2.1-2.2.4.

Starting with the second step, to show that when bounds are attained at $\pi_{12}^{min}(x)$, there exist proper distributions of potential outcomes for $T = 10, 02$, and 12 , first note that besides compatibility with the observed distributions, there is no restriction on the joint distribution of $Y(0)$, $Y(1)$, and $Y(2)$ for each type. Below we focus on the marginal distributions for $Y(1)$ and $Y(2)$, then obtain a joint distribution through combining marginal distributions assuming independence, and show that the joint distribution is compatible with the observed distributions and satisfies Assumptions 2.2.1-2.2.4. Let

$$\begin{aligned} F_{Y(1)|T=10, X=x}^{LB, \pi_{12}^{min}(x)}(y(1)) &= \min\left\{ \frac{F_{Y(2)|D=1, X=x}(y(1))}{(P_1(x) - \pi_{12}^{min}(x))/P_1(x)}, 1 \right\} \\ F_{Y(2)|T=02, X=x}^{LB, \pi_{12}^{min}(x)}(y(2)) &= \max\left\{ 0, \frac{F_{Y(2)|D=2, X=x}(y(2)) - \pi_{12}^{min}/P_2(x)}{(P_2(x) - \pi_{12}^{min}(x))/P_2(x)} \right\} \\ F_{Y(1)|T=12, X=x}^{LB, \pi_{12}^{min}(x)}(y(1)) &= \max\left\{ 0, \frac{F_{Y(1)|D=1, X=x}(y(1)) - (P_1(x) - \pi_{12}^{min})/P_1(x)}{\pi_{12}^{min}(x)/P_1(x)} \right\} \\ F_{Y(2)|T=12, X=x}^{LB, \pi_{12}^{min}(x)}(y(2)) &= \min\left\{ \frac{F_{Y(2)|D=2, X=x}(y(2))}{\pi_{12}^{min}(x)/P_2(x)}, 1 \right\}. \end{aligned} \quad (2.29)$$

$F_{Y(1)|T=10, X=x}^{LB, \pi_{12}^{min}(x)}$, $F_{Y(2)|T=02, X=x}^{LB, \pi_{12}^{min}(x)}$, $F_{Y(1)|T=12, X=x}^{LB, \pi_{12}^{min}(x)}$, and $F_{Y(2)|T=12, X=x}^{LB, \pi_{12}^{min}(x)}$ are proper CDFs. First note that (2.29) satisfies the last two equations in (2.6). As $\pi_{12}(x) \in S(x) \cap S_{PQD}(x)$, by the definition of $S(x) \cap S_{PQD}(x)$, there exists a compatible $F(y(0), y(1), y(2)|T = t, X = x)$ that satisfies (2.6), to construct our joint distribution, join the associated marginals which do not appear in (2.29) with (2.29), assuming independence. The constructed joint distributions satisfy (2.6) and Assumptions 2.2.1-2.2.4. At the same time, there is

$$E[Y(2) - Y(1)|T = 12, X = x]^{LB} = \int y(2) dF_{Y(2)|T=12, X=x}^{LB, \pi_{12}^{min}(x)} - \int y(1) dF_{Y(1)|T=12, X=x}^{LB, \pi_{12}^{min}(x)},$$

which means that $[E[Y(2) - Y(1)|T = 12, X = x]^{LB}$ is attained at $\pi_{12}^{min}(x)$. Similarly, $[E[Y(2) - Y(1)|T = 12, X = x]^{UB}$ is also attained at $\pi_{12}^{min}(x)$.

For the last step, since $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{LB}$ is monotonically increasing in $\pi_{12}(x)$ and $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{UB}$ is monotonically decreasing in $\pi_{12}(x)$, for $\pi_{12}(x) \geq \pi_{12}^{min}(x)$, upper and lower bounds of $E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]$ are contained in $[E[Y(2) - Y(1)|T = 12, X = x]^{LB}, E[Y(2) - Y(1)|T = 12, X = x]^{UB}]$:

$$\begin{aligned} & [E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{LB}, E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}(x)]^{UB}] \subset \\ & [E[Y(2) - Y(1)|T = 12, X = x]^{LB}, E[Y(2) - Y(1)|T = 12, X = x]^{UB}], \end{aligned}$$

when $\pi_{12}(x) \geq \pi_{12}^{min}(x)$. This means that for any

$$\begin{aligned} E[Y(2) - Y(1)|T = 12, X = x] & \in [E[Y(2) - Y(1)|T = 12, X = x]^{LB}, \\ & E[Y(2) - Y(1)|T = 12, X = x]^{UB}], \end{aligned}$$

$\pi_{12}^{min}(x)$ is compatible. Thus we only need to show that for any value in $[E[Y(2) - Y(1)|T = 12, X = x]^{LB}, E[Y(2) - Y(1)|T = 12, X = x]^{UB}]$, given $\pi_{12}^{min}(x)$, there exists proper $F(y(0), y(1), y(2)|t, z, x)$ which is compatible with the observed distributions and satisfies Assumptions 2.2.1-2.2.4. This can be achieved by the convex combinations of $F_{Y(1)|T=12, X=x}^{LB, \pi_{12}^{min}(x)}$, $F_{Y(2)|T=12, X=x}^{LB, \pi_{12}^{min}(x)}$, $F_{Y(1)|T=12, X=x}^{UB, \pi_{12}^{min}(x)}$ and $F_{Y(2)|T=12, X=x}^{UB, \pi_{12}^{min}(x)}$. Here we also focus on the marginal distributions, similar to the second step. Given $\pi_{12}^{min}(x)$, let

$$\begin{aligned} E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}^{min}(x)]^\alpha & = \alpha E[Y(2) - Y(1)|T = 12, X = x]^{LB} \\ & + (1 - \alpha) E[Y(2) - Y(1)|T = 12, X = x]^{UB}, \\ 0 & < \alpha < 1, \end{aligned}$$

also let

$$\begin{aligned}
F_{Y(1)|T=10,X=x}^\alpha(y(1)) &= \alpha \min\left\{ \frac{F_{Y(1)|D=1,X=x}(y(1))}{(P_1(x) - \pi_{12}^{min}(x))/P_1(x)}, 1 \right\} \\
&\quad + (1 - \alpha) \max\left\{ 0, \frac{F_{Y(1)|D=1,X=x}(y(1)) - \pi_{12}^{min}/P_1(x)}{(P_1(x) - \pi_{12}^{min}(x))/P_1(x)} \right\} \\
F_{Y(2)|T=02,X=x}^\alpha(y(2)) &= \alpha \max\left\{ 0, \frac{F_{Y(2)|D=2,X=x}(y(2)) - \pi_{12}^{min}/P_2(x)}{(P_2(x) - \pi_{12}^{min}(x))/P_2(x)} \right\} \\
&\quad + (1 - \alpha) \min\left\{ \frac{F_{Y(2)|D=2,X=x}(y(1))}{(P_2(x) - \pi_{12}^{min}(x))/P_2(x)}, 1 \right\} \\
F_{Y(1)|T=12,X=x}^\alpha(y(1)) &= \alpha \max\left\{ 0, \frac{F_{Y(1)|D=1,X=x}(y(1)) - (P_1(x) - \pi_{12}^{min})/P_1(x)}{\pi_{12}^{min}(x)/P_1(x)} \right\} \\
&\quad + (1 - \alpha) \min\left\{ \frac{F_{Y(1)|D=1,X=x}(y(1))}{\pi_{12}^{min}(x)/P_1(x)}, 1 \right\} \\
F_{Y(2)|T=12,X=x}^\alpha(y(2)) &= \alpha \min\left\{ \frac{F_{Y(2)|D=2,X=x}(y(2))}{\pi_{12}^{min}(x)/P_2(x)}, 1 \right\} \\
&\quad + (1 - \alpha) \max\left\{ 0, \frac{F_{Y(2)|D=2,X=x}(y(2)) - (P_2(x) - \pi_{12}^{min})/P_2(x)}{\pi_{12}^{min}(x)/P_2(x)} \right\},
\end{aligned}$$

where $F_{Y(1)|T=10,X=x}^\alpha$, $F_{Y(2)|T=02,X=x}^\alpha$, $F_{Y(1)|T=12,X=x}^\alpha$, and $F_{Y(2)|T=12,X=x}^\alpha$ are proper CDFs. Below we show that they are compatible with the observed distributions of $Y(1)$ and $Y(2)$, that is, the last two equations in (2.6).

$$\begin{aligned}
&\frac{P_1(x) - \pi_{12}^{min}(x)}{P_1(x)} F_{Y(1)|T=10,X=x}^\alpha + \frac{\pi_{12}^{min}(x)}{P_1(x)} F_{Y(1)|T=12,X=x}^\alpha \\
&= \alpha \min\left\{ F_{Y(1)|D=1,X=x}(y(1)), \frac{P_1(x) - \pi_{12}^{min}(x)}{P_1(x)} \right\} + (1 - \alpha) \\
&\quad \max\left\{ 0, F_{Y(1)|D=1,X=x}(y(1)) - \frac{\pi_{12}^{min}}{P_1(x)} \right\} \\
&\quad + \alpha \max\left\{ 0, F_{Y(1)|D=1,X=x}(y(1)) - \frac{P_1(x) - \pi_{12}^{min}}{P_1(x)} \right\} \\
&\quad + (1 - \alpha) \min\left\{ F_{Y(1)|D=1,X=x}(y(1)), \frac{\pi_{12}^{min}(x)}{P_1(x)} \right\}
\end{aligned}$$

$$\text{Case 1: } F_{Y(1)|D=1,X=x}(y(1)) \leq \frac{P_1(x) - \pi_{12}^{min}(x)}{P_1(x)} \leq \frac{\pi_{12}^{min}}{P_1(x)}$$

$$\text{Case 2: } \frac{P_1(x) - \pi_{12}^{min}(x)}{P_1(x)} \leq F_{Y(1)|D=1,X=x}(y(1)) \leq \frac{\pi_{12}^{min}}{P_1(x)}$$

$$\text{Case 3: } \frac{\pi_{12}^{min}}{P_1(x)} \leq F_{Y(1)|D=1,X=x}(y(1)) \leq \frac{P_1(x) - \pi_{12}^{min}(x)}{P_1(x)}$$

Case 4: $\frac{P_1(x) - \pi_{12}^{min}(x)}{P_1(x)} \leq \frac{\pi_{12}^{min}}{P_1(x)} \leq F_{Y(1)|D=1, X=x}(y(1))$

In all of the four cases, $F_{Y(1)|T=10, X=x}^\alpha$, $F_{Y(2)|T=02, X=x}^\alpha$, $F_{Y(1)|T=12, X=x}^\alpha$, and $F_{Y(2)|T=12, X=x}^\alpha$ satisfy (2.3), similarly for $Y(2)$. Finally, we also have

$$E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}^{min}(x)]^\alpha = \int y(2) dF_{Y(2)|T=12, X=x}^\alpha - \int y(1) dF_{Y(1)|T=12, X=x}^\alpha,$$

since

$$\begin{aligned} & E[Y(2) - Y(1)|T = 12, X = x; \pi_{12}^{min}(x)]^\alpha \\ &= \alpha E[Y(2) - Y(1)|T = 12, X = x]^{LB} \\ &+ (1 - \alpha) E[Y(2) - Y(1)|T = 12, X = x]^{UB}, 0 < \alpha < 1, \end{aligned}$$

and $E[Y(2) - Y(1)|T = 12, X = x]^{LB}$ is achieved with

$$\begin{aligned} & \max\left\{ 0, \frac{F_{Y(1)|D=1, X=x}(y(1)) - (P_1(x) - \pi_{12}^{min})/P_1(x)}{\pi_{12}^{min}(x)/P_1(x)} \right\} \text{ and } \min\left\{ \frac{F_{Y(2)|D=2, X=x}(y(2))}{\pi_{12}^{min}(x)/P_2(x)}, 1 \right\}, \\ & E[Y(2) - Y(1)|T = 12, X = x]^{UB} \text{ is achieved with } \min\left\{ \frac{F_{Y(1)|D=1, X=x}(y(1))}{\pi_{12}^{min}(x)/P_1(x)}, 1 \right\} \text{ and} \\ & \max\left\{ 0, \frac{F_{Y(2)|D=2, X=x}(y(2)) - (P_2(x) - \pi_{12}^{min})/P_2(x)}{\pi_{12}^{min}(x)/P_2(x)} \right\}. \end{aligned} \quad \square$$

Chapter 3

Influence Function and Variance

Estimation of the MTE Estimator

(with Takuya Ura)

3.1 Introduction

The marginal treatment effect (MTE) framework of Heckman and Vytlacil (1999, 2005) has become an important tool for analyzing heterogeneous treatment effects under selection on unobservables. MTE is considered as a building block for a wide range of parameters, such as the average treatment effect (ATE), the average treatment effect on the treated (ATT), and the local average treatment effect (LATE), which can be integrated from MTE with different weights. Building on this framework, Carneiro and Lee (2009) propose a multi-step estimation procedure that combines semiparametric and nonparametric methods. Their asymptotic theory shows that the distribution of the MTE estimator is driven by the local polynomial derivative component from the final step, while estimation errors from earlier steps are asymptotically negligible. They suggest estimating the asymptotic variance based on this leading term, which can be used to construct confidence intervals for the MTE. While this characterization is valid asymptotically, ignoring estimation errors from intermediate

steps may lead to undercoverage of confidence intervals in finite samples. In practice, existing commands for MTE estimation in Stata including Brave and Walstrum (2014) and Andresen (2018) either ignore the estimation errors from intermediate stages, or suggest researchers use bootstrap.

In this paper, we use an influence function approach to propose a variance estimator for the MTE estimator. In other words, with the characterization of the asymptotic linear representation of the MTE estimator, we can obtain a plug-in variance estimator. This paper is the first to provide a closed-form solution for the standard error of the MTE that captures estimation errors from all steps of the procedure. We study the finite-sample performance of the proposed method using Monte Carlo simulations, and the results show that our method had better coverage under moderate sample sizes.

3.2 Model

In this section, we introduce the model for the marginal treatment effect (MTE). In particular, we follow the model from Carneiro and Lee (2009). For each individual, we can observe an outcome variable Y :

$$Y = DY_1 + (1 - D)Y_0$$

Let X be a vector of observed random variables. Assume there are unknown vectors β_0^* and β_1^* and unobserved random variables U_0 and U_1 such that

$$\mathbb{E}[Y_1 | X, V] = X'\beta_1^* + \mathbb{E}[U_1|V]$$

$$\mathbb{E}[Y_0 | X, V] = X'\beta_0^* + \mathbb{E}[U_0|V].$$

Assume

$$D = 1\{V < P\}$$

where Z is a vector of observed random variables, and P is an unknown function of Z , and V is an unobserved random variable with $V | P \sim U(0, 1)$. In this paper, we are interested in the marginal treatment effect:

$$MTE(x, v) = \mathbb{E}[Y_1 - Y_0 | X = x, V = v].$$

Assumption 3.2.1. (1) (U_1, V) and (U_0, V) are independent of (Z, X) ; (2) P is a nondegenerate random variable conditional on X ; (3) $\mathbb{E}[Y | P = p, D = d]$ is continuously differentiable with respect to p for almost every x , $D = 0, 1$; (4) $E|Y_1|^4 < \infty$ and $E|Y_0|^4 < \infty$; (5) P is continuously distributed, its density is bounded away from zero, and $0 < P(Z) < 1$.

Lemma 3.2.1 (Theorem 1 of Carneiro and Lee (2009)). *Under Assumption 1, we have*

$$\begin{aligned} MTE(x, v) = & x'(\beta_1^* - \beta_0^*) + \mathbb{E}[U_1 | P = v, D = 1] + v \frac{\partial \mathbb{E}[U_1 | P = v, D = 1]}{\partial v} \\ & - \mathbb{E}[U_0 | P = v, D = 0] + (1 - v) \frac{\partial \mathbb{E}[U_0 | P = v, D = 0]}{\partial v}. \end{aligned} \quad (3.1)$$

Proof. Under Assumption 1.1, we have

$$\mathbb{E}[Y | X = x, P = p, D = d] = x' \beta_d^* + \mathbb{E}[U_d | P = p, D = d].$$

$$\begin{aligned} \mathbb{E}[U_1 | V = v] &= \mathbb{E}[U_1 | P = v, D = 1] + v \frac{\partial \mathbb{E}[U_1 | P = v, D = 1]}{\partial v} \\ \mathbb{E}[U_0 | V = v] &= \mathbb{E}[U_0 | P = v, D = 0] - (1 - v) \frac{\partial \mathbb{E}[U_0 | P = v, D = 0]}{\partial v} \end{aligned}$$

□

3.3 Estimation of MTE

To estimate MTE, following (3.1), we provide estimators of the objects on the RHS of (3.1), similar to Carneiro and Lee (2009). To estimate P , we use a parametric model $\rho(z, \vartheta)$ for $\mathbb{P}(D = 1 \mid Z = z)$.¹ Assume there is ϑ^* such that $\mathbb{P}(D = 1 \mid Z = z) = \rho(z, \vartheta^*)$ and that there is an estimator $\hat{\vartheta}$ for ϑ^* . Assume there is an estimator $\hat{\mu}(p, d) = (\hat{\mu}_X(p, d), \hat{\mu}_Y(p, d))$ for $\mu^*(p, d) = (\mu_X^*(p, d), \mu_Y^*(p, d)) = (\mathbb{E}[X \mid P = p, D = d], \mathbb{E}[Y \mid P = p, D = d])$. Our estimator for $MTE(x, v)$ is defined as

$$\widehat{MTE}(x, v) = x'(\hat{\beta}_1 - \hat{\beta}_0) + (e_1 + ve_2)' \hat{\gamma}_{\hat{\vartheta}}(v, 1) - (e_1 - (1 - v)e_2)' \hat{\gamma}_{\hat{\vartheta}}(v, 0),$$

where e_1 and e_2 are the first and second standard unit vectors in $\mathbb{R}^{J+1}$, $\hat{\beta}_d$ and $\hat{\gamma}_{\hat{\vartheta}}(v, d)$ are defined below. Define

$$\hat{\beta}_d = \mathbb{E}_n[\mathcal{G}_X(d, \hat{\eta})]^{-1} \mathbb{E}_n[1\{D = d\}(X - \hat{\eta}_X(Z, d))(Y - \hat{\eta}_Y(Z, d))]$$

where let

$$\hat{\eta}(z, d) = (\hat{\eta}_X(z, d), \hat{\eta}_Y(z, d)) = (\hat{\mu}_X(\rho(z, \hat{\vartheta}), d), \hat{\mu}_Y(\rho(z, \hat{\vartheta}), d))$$

$$\eta^*(z, d) = (\eta_X^*(z, d), \eta_Y^*(z, d)) = (\mu_X^*(\rho(z, \vartheta^*), d), \mu_Y^*(\rho(z, \vartheta^*), d))$$

$$\mathcal{G}_X(d, \eta) = 1\{D = d\}(X - \eta_X(Z, d))(X - \eta_X(Z, d))'$$

$$m_X(d, \eta) = \mathbb{E}[\mathcal{G}_X(d, \eta)].$$

¹Note that Carneiro and Lee (2009) uses the partially linear additive regression. Instead, we consider a Probit model.

Also define

$$\begin{aligned}\hat{\gamma}_{\hat{U}}(v, d) &= \text{diag}(Q(h))^{-1} \mathbb{E}_n[\mathcal{G}_P(v, d, \hat{\vartheta})]^{-1} \mathbb{E}_n[1\{D = d\} \\ &\quad K_h(\rho(Z, \hat{\vartheta}) - v) Q((\rho(Z, \hat{\vartheta}) - v)/h) (Y - X' \hat{\beta}_D)] \\ \gamma_U(v, d) &= \text{diag}(Q(h))^{-1} m_P(v, d, \vartheta^*)^{-1} \mathbb{E}[1\{D = d\} \\ &\quad K_h(\rho(Z, \vartheta^*) - v) Q((\rho(Z, \vartheta^*) - v)/h) (Y - X' \beta_D^*)]\end{aligned}$$

where let $Q(u) = (1 \ u \ \cdots \ u^J)'$, $K_h(u) = K_1(U/h)/h$, and

$$\begin{aligned}\mathcal{G}_P(v, d, \vartheta) &= 1\{D = d\} K_h(\rho(Z, \vartheta) - v) Q((\rho(Z, \vartheta) - v)/h) Q((\rho(Z, \vartheta) - v)/h)' \\ m_P(v, d, \vartheta) &= \mathbb{E}[\mathcal{G}_P(v, d, \vartheta)].\end{aligned}$$

For intermediate steps and our derivation of the influence function, also define the following:

$$\mathcal{G}_{X,Y}(d, t, \eta) = 1\{D = d\} (X - \eta_X(Z, d)) (Y - \eta_Y(Z, d) - (X - \eta_X(Z, d))' t)$$

$$m_{X,Y}(d, t, \eta) = \mathbb{E}[\mathcal{G}_{X,Y}(d, t, \eta)]$$

$$\mathcal{G}_{P,U}(v, d, t, \vartheta) = 1\{D = d\} K_h(\rho(Z, \vartheta) - v) Q((\rho(Z, \vartheta) - v)/h) (Y - X' \beta_D^* - Q(\rho(Z, \vartheta) - v)' t)$$

$$m_{P,U}(v, d, t, \vartheta) = \mathbb{E}[\mathcal{G}_{P,U}(v, d, t, \vartheta)].$$

$$\mathcal{G}_{P,X'}(v, d, t, \vartheta) = 1\{D = d\} K_h(\rho(Z, \vartheta) - v) Q((\rho(Z, \vartheta) - v)/h) (X' - Q(\rho(Z, \vartheta) - v)' t)$$

$$m_{P,X'}(v, d, t, \vartheta) = \mathbb{E}[\mathcal{G}_{P,X'}(v, d, t, \vartheta)]$$

$$\hat{\gamma}_{X'}(v, d) = \text{diag}(Q(h))^{-1} \mathbb{E}_n[\mathcal{G}_P(v, d, \hat{\vartheta})]^{-1} \mathbb{E}_n[1\{D = d\} K_h(\rho(Z, \hat{\vartheta}) - v) Q((\rho(Z, \hat{\vartheta}) - v)/h) X']$$

$$\gamma_{X'}(v, d) = \text{diag}(Q(h))^{-1} m_P(v, d, \vartheta^*)^{-1} \mathbb{E}[1\{D = d\} K_h(\rho(Z, \vartheta^*) - v) Q((\rho(Z, \vartheta^*) - v)/h) X']$$

$$\bar{\gamma}_U(v, d) = \left(\frac{1}{0!} \mathbb{E}[U \mid P = v, D = d] \quad \frac{1}{1!} \frac{\partial \mathbb{E}[U \mid P = v, D = d]}{\partial v} \quad \frac{1}{2!} \frac{\partial^2 \mathbb{E}[U \mid P = v, D = d]}{\partial v^2} \quad \cdots \quad \frac{1}{J!} \frac{\partial^J \mathbb{E}[U \mid P = v, D = d]}{\partial v^J} \right)'$$

$$MTE_h(x, v) = x' \beta_1^* - x' \beta_0^* + (e_1 + v e_2)' \gamma_U(v, d) - (e_1 - (1 - v) e_2)' \gamma_U(v, d).$$

3.4 Influence Function and Variance Estimation

In this section, we first introduce a set of assumptions, then show that under these assumptions we obtain the influence function for $\widehat{MTE}(x, v)$, given which we provide the variance estimator for $\widehat{MTE}(x, v)$.

Assumption 3.4.1. *The data $\{(Y_i, D_i, X_i, Z_i) : i = 1, \dots, n\}$ are independent and identically distributed.*

Assumption 3.4.2. $\mathbb{E}[\|X\|^4] < \infty$, $\mathbb{E} \left[\sup_{\vartheta \in \Theta} \left\| \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \right\|^4 \right] < \infty$, $\mathbb{E} \left[\sup_{\vartheta \in \Theta} \left\| \frac{\partial^2}{\partial \vartheta \partial \vartheta'} \rho(Z, \vartheta) \right\|^2 \right] < \infty$.

Assumption 3.4.3. K_1 is a second-order kernel function with compact support and is Lipschitz continuous, $K_1(u) \leq 1\{u \in \mathcal{K}\}$ with a bounded set $\mathcal{K}$, $\frac{\partial^2}{\partial t^2} K_1(t) < \infty$.

Assumption 3.4.4. *There is a sequence of i.i.d. random variables $(\varphi_1, \varphi_2, \dots)$ with $\mathbb{E}[\varphi] = 0$ and $\mathbb{E}[\varphi^2] < \infty$ such that $\hat{\vartheta} - \vartheta^* = \mathbb{E}_n[\varphi] + o_p(n^{-1/2})$.*

Assumption 3.4.5. *The support of Z , denoted by $\mathcal{Z}$, is a bounded, convex subset of $\mathbb{R}^{\dim(Z)}$ with nonempty interior.*

$$\sup_z \|\hat{\eta}(z, d) - \eta^*(z, d)\| = o_p(n^{-1/4}).$$

There is a constant α with $\alpha > \dim(Z)/2$ and a diminishing sequence M_n such that $\hat{\eta}_X - \eta_X^* \in \mathcal{C}^\alpha(\mathcal{Z})_{M_n}$ with probability approaching one. There is a constant α' with $\alpha' > \dim(Z)/2$ and a diminishing sequence R_n such that $\hat{\eta}_Y - \eta_Y^* \in \mathcal{C}^{\alpha'}(\mathcal{Z})_{R_n}$ with probability approaching one. For any positive value M $\mathcal{C}^\alpha(\mathcal{Z})_M$ is the set defined in van der Vaart and Wellner (2023, p.223).

Assumption 3.4.6. $\mathbb{E}[(X - \mathbb{E}[X | P, D])(X - \mathbb{E}[X | P, D])' | D = d]$ is invertible.

For example, when we use the probit model for $\mathbb{P}(D = 1 \mid Z)$, the influence function for $\hat{\vartheta}$ is $\varphi = -\mathbb{E}[\frac{\partial^2}{\partial t^2} \log \varphi(t) |_{t=(2D-1)Z'\vartheta^*} ZZ']^{-1} \frac{\partial}{\partial t} \log \Phi(t) |_{t=(2D-1)Z'\vartheta^*} (2D-1)Z$.

Assumption 3.4.7. $\frac{nh^3}{\log n} \rightarrow \infty$, $h \rightarrow 0$.

Lemma 3.4.1. *If $f_P(v) > 0$, $\int K_1(u)Q(u)Q(u)'du$ is invertible, $\int |K_1(u)u^j|du < \infty$ for every $j = 0, 1, \dots, 2J+1$ and f_P is twice continuously differentiable with $\sup_p |f_P(p)|, \sup_p |f_P^{(1)}(p)|$, then $m_P(v, d, \vartheta^*)^{-1} = O(1)$.*

Proof. By Taylor's theorem,

$$\begin{aligned} f_P(v+uh)(v+uh)^d(1-(v+uh))^{1-d} &= f_P(v)v^d(1-v)^{1-d} \\ &\quad + \int_v^{v+uh} f_P^{(1)}(p)p^d(1-p)^{1-d}p \\ &\quad + (2d-1) \int_v^{v+uh} f_P(p)p. \end{aligned}$$

Then we have

$$\begin{aligned} m_P(v, d, \vartheta^*) &= \mathbb{E}[P^d(1-P)^{1-d}K_h(P-v)Q((P-v)/h)Q((P-v)/h)'] \\ &= \int K_h(p-v)Q((p-v)/h)Q((p-v)/h)'f_P(p)p^d(1-p)^{1-d}dp \\ &= \int K_1(u)Q(u)Q(u)'f_P(v+uh)(v+uh)^d(1-(v+uh))^{1-d}du \\ &= f_P(v)v^d(1-v)^{1-d} \int K_1(u)Q(u)Q(u)'du \\ &\quad + \int K_1(u)Q(u)Q(u)' \int_v^{v+uh} f_P^{(1)}(p)p^d(1-p)^{1-d}pdu \\ &\quad + \int K_1(u)Q(u)Q(u)'(2d-1) \int_v^{v+uh} f_P(p)pdu. \end{aligned}$$

By the assumptions of this lemma, $m_P(v, d, \vartheta^*) = f_P(v)v^d(1-v)^{1-d} \int K_1(u)Q(u)Q(u)'du + O(h)$. Since $\int K_1(u)Q(u)Q(u)'du$ is invertible, we have the statement of this lemma. $\square$

Lemma 3.4.2. *Under Assumptions 1, 3, and 4, $\mathbb{E}[\mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)] = 0$ and*

$$\mathbb{E}[\|\mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)\|^2] < \infty.$$

Proof. The first part follows from the definitions of $\mathcal{G}_{X,Y}$ and β_d^* . The second part follows from the boundedness assumptions on the kernel function K , $\mathbb{E}[\|X\|^4]$, and $\mathbb{E}[|Y|^4]$. $\square$

Lemma 3.4.3. *Under Assumptions 1, 3, and 4, $m_{P,U}(v, d, \gamma_U(v, d), \vartheta^*) = 0$ and $\mathbb{E}[\|\mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \vartheta^*)\|^2] = O(\frac{1}{h})$.*

Proof. The first part follows from the definitions of $\mathcal{G}_{P,U}$ and $\gamma_U(v, d)$. The second part follows from the boundedness assumptions on the kernel function K , $\mathbb{E}[\|X\|^4]$, and $\mathbb{E}[|Y|^4]$, also on $0 < \rho(z, \vartheta) < 1$. $\square$

Lemma 3.4.4. *Under Assumptions 3 and 6, $\mathbb{E}_n[\mathcal{G}_X(d, \hat{\eta})] = m_X(d, \eta^*) + o_p(1)$ for every $d = 0, 1$.*

Proof. Let c be any $\dim(X)$ -dimensional column vector. By Assumption 3.4.5, it suffices to show that

$$m_X(d, \hat{\eta}) = m_X(d, \eta^*) + o_p(1) \quad (3.2)$$

and that

$$\sup_{\eta \in \mathcal{N}} \sqrt{n} |(\mathbb{E} - \mathbb{E}_n)[c'(\mathcal{G}_X(d, \eta) - \mathcal{G}_X(d, \eta^*))c]| \quad (3.3)$$

converges in probability to zero, where $\mathcal{N} = \{\eta : \eta - \eta^* \in \mathcal{C}^\alpha(\mathcal{Z})_{M_n}\}$.

First, we are going to show (3.2). Since

$$\begin{aligned} |\mathcal{G}_X(d, \eta) - \mathcal{G}_X(d, \eta^*)| &\leq 2\|X\| \|\eta_X(Z, d) - \eta_X^*(Z, d)\| + \|\eta_X(Z, d) - \eta_X^*(Z, d)\|^2 \\ &\quad + 2\|\eta_X^*(Z, d)\| \|\eta_X(Z, d) - \eta_X^*(Z, d)\|, \end{aligned}$$

we have

$$\begin{aligned} |m_X(d, \hat{\eta}) - m_X(d, \eta^*)| &\leq 2\mathbb{E}[\|X\|] \sup_z \|\eta_X(z, d) - \eta_X^*(z, d)\| \\ &\quad + \sup_z \|\eta_X(z, d) - \eta_X^*(z, d)\|^2 + 2 \sup_z \|\eta_X^*(z, d)\| \sup_z \|\eta_X(z, d) - \eta_X^*(z, d)\|. \end{aligned}$$

Therefore, by Assumption 3.4.2 and 3.4.5, we have (3.2).

Next, we are going to show the convergence of (3.3). Note that

$$\begin{aligned} c'(\mathcal{G}_X(d, \eta) - \mathcal{G}_X(d, \eta^*))c &= 1\{D = d\}(c'(\eta_X(Z, d) - \eta_X^*(Z, d)))^2 \\ &\quad + 2 \cdot 1\{D = d\}(c'(X - \eta_X^*(Z, d)))((c'(\eta_X(Z, d) - \eta_X^*(Z, d))). \end{aligned}$$

We focus on showing

$$\mathbb{E} \left[\sup_{\eta \in \mathcal{N}} \sqrt{n} |(\mathbb{E} - \mathbb{E}_n)[1\{D = d\}(c'(X - \eta_X^*(Z, d)))((c'(\eta_X(Z, d) - \eta_X^*(Z, d))))| \right] = o(1),$$

and we can show the convergence of the other term similarly. Define

$$(\mathcal{F}_1, \mathcal{F}_2) = (\mathcal{C}^\alpha(\mathcal{Z})_1, \{M_n \cdot 1\{D = d\}c'(X - \eta_X^*(Z, d))\}).$$

Define

$$(F_1, F_2) = (1, M_n \|c\| \|X - \eta_X^*(Z, d)\|)$$

and

$$\delta = \sup_{(f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2} \frac{\mathbb{E}[(f_1 f_2)^2]^{1/2}}{\|F_1 F_2\|_{P,2}}.$$

By the maximal inequality (Chernozhukov et al., 2014, Theorem 5.2),

$$\begin{aligned} &\mathbb{E} \left[\sup_{(f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2} |\sqrt{n}(\mathbb{E} - \mathbb{E}_n)[f_1 f_2]| \right] \\ &\leq K_1 \sup_{(f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2} \mathbb{E}[(f_1 f_2)^2]^{1/2} \left(\frac{1}{\delta} \int_0^\delta \sup_Q \sqrt{1 + \log N(u\|F_1 F_2\|_{Q,2}, \mathcal{F}_1 \cdot \mathcal{F}_2, e_Q)} du \right) \\ &\quad + K_1 \frac{\|\sup_{i=1, \dots, n} F_i G_i\|_{P,2}}{\sqrt{n}} \left(\frac{1}{\delta} \int_0^\delta \sup_Q \sqrt{1 + \log N(u\|F_1 F_2\|_{Q,2}, \mathcal{F}_1 \cdot \mathcal{F}_2, e_Q)} du \right)^2, \end{aligned}$$

where $N(u\|F_1 F_2\|_{Q,2}, \mathcal{F}_1 \cdot \mathcal{F}_2, e_Q)$ is the covering number defined in van der Vaart and Wellner (2023, p.132), the supremum is taken over all finitely discrete probability measures on the sample space and e_Q is the $L^2(Q)$ -semimetric. By Chernozhukov et al. (2014, Corollary A.1)

and van der Vaart and Wellner (2023, Theorem 2.7.1), we have

$$\log N(u\|F_1 F_2\|_{Q,2}, \mathcal{F}_1 \cdot \mathcal{F}_2, e_Q) \leq \log N(u\|F\|_{Q,2}/\sqrt{2}, \mathcal{F}_1, e_Q) \leq K_2 \left(\frac{\sqrt{2}}{u} \right)^{\dim(Z)/\alpha}$$

for some constant K_2 . Then

$$\begin{aligned} \frac{1}{\delta} \int_0^\delta \sup_Q \sqrt{1 + \log N(u\|F_1 F_2\|_{Q,2}, \mathcal{F}_1 \cdot \mathcal{F}_2, e_Q)} du &= \int_0^1 \sqrt{1 + K_2 \left(\frac{\sqrt{2}}{u} \right)^{\dim(Z)/\alpha}} du \\ &\leq 1 + O(1) \int_0^1 u^{-\dim(Z)/(2\alpha)} du \\ &= O(1). \end{aligned}$$

Note that

$$\begin{aligned} \left\| \sup_{i=1,\dots,n} F_{1i} F_{2i} \right\|_{P,2} &\leq E \left[\sup_{i=1,\dots,n} F_{2i}^2 \right]^{1/2} \leq E \left[\sup_{i=1,\dots,n} F_{2i}^4 \right]^{1/4} \leq \left(\sum_{i=1}^n E[F_{2i}^4] \right)^{1/4} \\ &= n^{1/4} \|F_2\|_{P,4} = O(n^{1/4} M_n). \end{aligned}$$

Since

$$\sup_{(f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2} \mathbb{E}[(f_1 f_2)^2]^{1/2} \leq \mathbb{E}[(M_n \cdot 1\{D = d\} c'(X - \eta_X^*(Z, d)))^2]^{1/2} = O(M_n),$$

we have

$$\mathbb{E}^* \left[\sup_{(f_1, f_2) \in \mathcal{F}_1 \times \mathcal{F}_2} |\sqrt{n}(\mathbb{E} - \mathbb{E}_n)[f_1 f_2]| \right] = O(M_n).$$

□

Lemma 3.4.5. *Under Assumptions 6, $\mathbb{E}_n[\mathcal{G}_{X,Y}(d, \beta_d^*, \hat{\eta}) - \mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)] = o_p(n^{-1/2})$.*

Proof. By Assumption 3.4.5, it suffices to show that

$$m_{X,Y}(d, \beta_d^*, \hat{\eta}) - m_{X,Y}(d, \beta_d^*, \eta^*) = o_p(n^{-1/2}) \quad (3.4)$$

and that

$$\sup_{\eta \in \mathcal{N}} \sqrt{n} |(\mathbb{E} - \mathbb{E}_n)[\mathcal{G}_{X,Y}(d, \beta_d^*, \eta) - \mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)]| \quad (3.5)$$

converges in outer probability to zero, where $\mathcal{N} = \{\eta : \mathcal{G}_X(d, \eta) - \mathcal{G}_X(d, \eta^*) \in \mathcal{C}^\alpha(\mathcal{Y} \times \mathcal{X} \times \mathcal{Z})_{M_n}\}$.

First, we are going to show (3.4). Since $t \mapsto m_{X,Y}(\beta_d^*, \eta^* + t(\eta - \eta^*))$ is a quadratic function of t , it follows that

$$m_{X,Y}(\beta_d^*, \eta) = m_{X,Y}(\beta_d^*, \eta^*) + \frac{\partial}{\partial t} m_{X,Y}(\beta_d^*, \eta^* + t(\eta - \eta^*))|_{t=0} + \frac{1}{2} \frac{\partial^2}{\partial t^2} m_{X,Y}(\beta_d^*, \eta^* + t(\eta - \eta^*))|_{t=0}.$$

Since

$$\begin{aligned} & \frac{\partial}{\partial t} m_{X,Y}(\beta_d^*, \eta^* + t(\eta - \eta^*))|_{t=0} \\ &= -\mathbb{E}[1\{D = d\}(\eta_X(Z, d) - \eta_X^*(Z, d))(Y - (\eta_Y^*(Z, d)) - (X - \eta_X^*(Z, d))' \beta_d^*)] \\ & \quad - \mathbb{E}[1\{D = d\}(X - \eta_X^*(Z, d))((\eta_Y(Z, d) - \eta_Y^*(Z, d)) - (\eta_X(Z, d) - \eta_X^*(Z, d))' \beta_d^*)] \\ &= -\mathbb{E}[1\{D = d\}(\eta_X(Z, d) - \eta_X^*(Z, d))\mathbb{E}[(Y - (\eta_Y^*(Z, d)) - (X - \eta_X^*(Z, d))' \beta_d^*) \mid (Z, D)]] \\ & \quad - \mathbb{E}[1\{D = d\}\mathbb{E}[X - \eta_X^*(Z, d) \mid (Z, D)]((\eta_Y(Z, d) - \eta_Y^*(Z, d)) - (\eta_X(Z, d) - \eta_X^*(Z, d))' \beta_d^*)] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} & \left\| \frac{\partial^2}{\partial t^2} m_{X,Y}(\beta_d^*, \eta^* + t(\eta - \eta^*))|_{t=0} \right\| \\ &= 2 \|\mathbb{E}[1\{D = d\}(\eta_X(Z, d) - \eta_X^*(Z, d))((\eta_Y(Z, d) - \eta_Y^*(Z, d)) - (\eta_X(Z, d) - \eta_X^*(Z, d))' \beta_d^*)]\| \\ &\leq 2 \sup_z \|\eta_X(z, d) - \eta_X^*(z, d)\| (\sup_z \|\eta_Y(z, d) - \eta_Y^*(z, d)\| + \sup_z \|\eta_X(z, d) - \eta_X^*(z, d)\| \|\beta_d^*\|), \end{aligned}$$

Assumption 3.4.5 implies (3.4). Next, we are going to show the convergence of (3.5).

$$\begin{aligned}
\mathcal{G}_{X,Y}(d, \beta_d^*, \eta) - \mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*) &= 1\{D = d\}(-(\eta_X(Z, d) - \eta_X^*(Z, d))(Y - \eta_Y(Z, d))' \\
&\quad - (X - \eta_X^*)(\eta_Y(Z, d) - \eta_Y^*(Z, d)))' \\
&\quad - (\mathcal{G}_X(d, \eta) - \mathcal{G}_X(d, \eta^*))\beta_d^*
\end{aligned}$$

From Lemma 6, we have that $\sup_{\eta \in \mathcal{N}} \sqrt{n}|(\mathbb{E} - \mathbb{E}_n)[1\{D = d\}(\mathcal{G}_X(d, \eta) - \mathcal{G}_X(d, \eta^*)\beta_d^*)]| = o_p(1)$.

To show (3.5), we show

$$\mathbb{E} \left[\sup_{\eta \in \mathcal{N}} \sqrt{n}|(\mathbb{E} - \mathbb{E}_n)[1\{D = d\}(X - \eta_X^*(Z, d))(\eta_Y(Z, d) - \eta_Y^*(Z, d))]| \right] = o(1)$$

then

$$\mathbb{E} \left[\sup_{\eta \in \mathcal{N}} \sqrt{n}|(\mathbb{E} - \mathbb{E}_n)[1\{D = d\}(\eta_X(Z, d) - \eta_X^*(Z, d))(Y - \eta_Y(Z, d))]| \right] = o(1)$$

can be shown similarly. Define

$$(\mathcal{F}_1, \mathcal{F}_3) = (\mathcal{C}^\alpha(\mathcal{Z})_1, \{R_n \cdot 1\{D = d\}(X - \eta_X^*(Z, d))\}).$$

so that $(\eta_Y(Z, d) - \eta_Y^*(Z, d))/R_n \in \mathcal{C}^\alpha(\mathcal{Z})_1$ and $R_n \cdot 1\{D = d\}(X - \eta_X^*(Z, d)) \in \{R_n \cdot 1\{D = d\}(X - \eta_X^*(Z, d))\}$. Define

$$(F_1, F_3) = (1, R_n \|X - \eta_X^*(Z, d)\|)$$

and

$$\delta' = \sup_{(f_1, f_3) \in \mathcal{F}_1 \times \mathcal{F}_3} \frac{\mathbb{E}[(f_1 f_3)^2]^{1/2}}{\|F_1 F_3\|_{P,2}}$$

In the proof of Lemma 6, replacing M_n with R_n , we obtain

$$\mathbb{E} \left[\sup_{(f_1, f_3) \in \mathcal{F}_1 \times \mathcal{F}_3} |\sqrt{n}(\mathbb{E} - \mathbb{E}_n)[f_1 f_3]| \right] = O(R_n).$$

□

Lemma 3.4.6. *Under Assumptions 1, 3, 4, 5, 8, and Lemma 3, $\mathbb{E}_n[\mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \hat{\vartheta}) - \mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \vartheta^*) - \frac{\partial}{\partial \vartheta'} m_{P,U}(v, d, \gamma_U(v, d), \vartheta)|_{\vartheta=\vartheta^*} \varphi] = o_p(n^{-1/2}h)$.*

Proof. Let c be any $(J+1)$ -dimensional column vector. Define

$$\mathcal{A}(t) = K_1(t) c' Q(t) (Y - X' \beta_D^* - Q(t)' \text{diag}(Q(h)) \gamma_U(v, d)).$$

By Taylor's theorem,

$$\begin{aligned} & \mathcal{A}((\rho(Z, \hat{\vartheta}) - v)/h) - \mathcal{A}((\rho(Z, \vartheta^*) - v)/h) \\ &= \frac{\partial}{\partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h)|_{\vartheta=\vartheta^*} (\hat{\vartheta} - \vartheta^*) \\ &+ (\hat{\vartheta} - \vartheta^*)' \int_0^1 \frac{\partial^2}{\partial \vartheta \partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h)|_{\vartheta=\vartheta^*+t(\hat{\vartheta}-\vartheta^*)} (1-t) dt (\hat{\vartheta} - \vartheta^*). \end{aligned}$$

Therefore,

$$\begin{aligned} & c' \mathbb{E}_n[\mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \hat{\vartheta}) - \mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \vartheta^*) - \frac{\partial}{\partial \vartheta'} m_{P,U}(v, d, \gamma_U(v, d), \vartheta)|_{\vartheta=\vartheta^*} \varphi] \\ &= \mathbb{E}_n \left[\frac{\partial}{\partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h)|_{\vartheta=\vartheta^*} \right] (\hat{\vartheta} - \vartheta^* - \mathbb{E}_n[\varphi]) \\ &+ (\hat{\vartheta} - \vartheta^*)' \mathbb{E}_n \left[\int_0^1 \frac{\partial^2}{\partial \vartheta \partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h)|_{\vartheta=\vartheta^*+t(\hat{\vartheta}-\vartheta^*)} (1-t) dt \right] (\hat{\vartheta} - \vartheta^*) \\ &+ (\mathbb{E}_n - \mathbb{E}) \left[\frac{\partial}{\partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h)|_{\vartheta=\vartheta^*} \right] \mathbb{E}_n[\varphi] \end{aligned}$$

By Assumption 3.4.4 and 3.4.7 it suffices to show

$$\mathbb{E} \left[\left\| \int_0^1 \frac{\partial^2}{\partial \vartheta \partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h) |_{\vartheta = \vartheta^* + t(\hat{\vartheta} - \vartheta^*)} (1-t) dt \right\| \right] = O\left(\frac{1}{h^2}\right) = O(n^{1/2}h) \text{ and} \quad (3.6)$$

$$\mathbb{E} \left[\left\| \frac{\partial}{\partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h) |_{\vartheta = \vartheta^*} \right\|^2 \right]^{1/2} = O\left(\frac{1}{h}\right) = O(n^{1/2}h). \quad (3.7)$$

Note that

$$\begin{aligned} \mathcal{A}(t) &= K_1(t) c' Q(t) (Y - X' \beta_D^*) - K_1(t) c' Q(t) Q(t)' \text{diag}(Q(h)) \gamma_U(v, d) \\ \mathcal{A}^{(1)}(t) &= \left(\frac{\partial}{\partial t} K_1(t) c' Q(t) \right) (Y - X' \beta_D^*) - \left(\frac{\partial}{\partial t} K_1(t) c' Q(t) Q(t)' \right)' \text{diag}(Q(h)) \gamma_U(v, d), \text{ and} \\ \mathcal{A}^{(2)}(t) &= \left(\frac{\partial^2}{\partial t^2} K_1(t) c' Q(t) \right) (Y - X' \beta_D^*) - \left(\frac{\partial^2}{\partial t^2} K_1(t) c' Q(t) Q(t)' \right)' \text{diag}(Q(h)) \gamma_U(v, d). \end{aligned}$$

First, we are going to show (3.6). Define $\Theta_n = \{\vartheta : \|\vartheta - \vartheta^*\| \leq \log(n)n^{-1/2}\}$. we have $\hat{\vartheta} \in \Theta_n$ with probability approaching one. Assume $\hat{\vartheta} \in \Theta_n$ without loss of generality. Also note that $|t| < \frac{1-v}{h}$. Then

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_0^1 \frac{\partial^2}{\partial \vartheta \partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h) |_{\vartheta = \vartheta^* + t(\hat{\vartheta} - \vartheta^*)} (1-t) dt \right\| \right] \\ & \leq \frac{1}{h} \mathbb{E} \left[\sup_t |\mathcal{A}^{(1)}(t)| \sup_{\vartheta \in \Theta_n} \left\| \frac{\partial^2}{\partial \vartheta \partial \vartheta'} \rho(Z, \vartheta) \right\| \right] \\ & \quad + \frac{1}{h^2} \mathbb{E} \left[\sup_t |\mathcal{A}^{(2)}(t)| \sup_{\vartheta \in \Theta_n} \left\| \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \right\|^2 \right] \\ & \leq \frac{1}{h} \sup_t \left| \frac{\partial}{\partial t} K_1(t) c' Q(t) \right| \mathbb{E} \left[|Y - X' \beta_D^*| \sup_{\vartheta \in \Theta_n} \left\| \frac{\partial^2}{\partial \vartheta \partial \vartheta'} \rho(Z, \vartheta) \right\| \right] \\ & \quad + \frac{1}{h} \sup_t \left| \left(\frac{\partial}{\partial t} K_1(t) c' Q(t) Q(t)' \right)' \text{diag}(Q(h)) \gamma_U(v, d) \right| \mathbb{E} \left[\sup_{\vartheta \in \Theta_n} \left\| \frac{\partial^2}{\partial \vartheta \partial \vartheta'} \rho(Z, \vartheta) \right\| \right] \\ & \quad + \frac{1}{h^2} \sup_t \left| \left(\frac{\partial^2}{\partial t^2} K_1(t) c' Q(t) \right) \right| \mathbb{E} \left[|Y - X' \beta_D^*| \sup_{\vartheta \in \Theta_n} \left\| \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \right\|^2 \right] \\ & \quad + \frac{1}{h^2} \sup_t \left| \left(\frac{\partial^2}{\partial t^2} K_1(t) c' Q(t) Q(t)' \right)' \text{diag}(Q(h)) \gamma_U(v, d) \right| \mathbb{E} \left[\sup_{\vartheta \in \Theta_n} \left\| \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \right\|^2 \right]. \end{aligned}$$

By Assumptions 1, 3, 4, and Lemma 3, we have (3.6).

Next, we are going to show (3.7). Note that

$$\begin{aligned}
& \mathbb{E} \left[\left\| \frac{\partial}{\partial \vartheta'} \mathcal{A}((\rho(Z, \vartheta) - v)/h) \Big|_{\vartheta=\vartheta^*} \right\|^2 \right]^{1/2} \\
& \leq \frac{1}{h} \mathbb{E} \left[\left\| \mathcal{A}^{(1)}((\rho(Z, \vartheta^*) - v)/h) \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \Big|_{\vartheta=\vartheta^*} \right\|^2 \right]^{1/2} \\
& \leq \frac{1}{h} \sup_t \left| \frac{\partial}{\partial t} K_1(t) c' Q(t) \right| \mathbb{E} \left[((Y - X' \beta_D^*))^2 \left\| \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \Big|_{\vartheta=\vartheta^*} \right\|^2 \right]^{1/2} \\
& + \frac{1}{h} \sup_t \left| \left(\frac{\partial}{\partial t} K_1(t) c' Q(t) Q(t)' \right) \text{diag}(Q(h)) \gamma_U(v, d) \right| \mathbb{E} \left[\left\| \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \Big|_{\vartheta=\vartheta^*} \right\|^2 \right]^{1/2}.
\end{aligned}$$

By Assumptions 1, 3, 4, and Lemma 3, we have (3.7). $\square$

Lemma 3.4.7. *Under Assumptions 3, 4, 5, 8, and Lemma 3, $\mathbb{E}_n[\mathcal{G}_{P,X'}(v, d, \gamma_{X'}(v, d), \hat{\vartheta}) - \mathcal{G}_{P,X'}(v, d, \gamma_{X'}(v, d), \vartheta^*)] = o_p(1)$.*

Proof.

$$\begin{aligned}
\mathcal{G}_{P,X'}(v, d, \gamma_{X'}(v, d), \vartheta) &= 1\{D = d\} K_h(\rho(Z, \vartheta) - v) Q((\rho(Z, \vartheta) - v)/h) \\
&\quad (X' - Q(\rho(Z, \vartheta) - v)' \gamma_{X'}(v, d))
\end{aligned}$$

$$m_{P,X'}(v, d, t, \vartheta) = \mathbb{E}[\mathcal{G}_{P,X'}(v, d, t, \vartheta)].$$

$$\gamma_{X'}(v, d) = \text{diag}(Q(h))^{-1} m_P(v, d, \vartheta^*)^{-1} \mathbb{E}[1\{D = d\} K_h(\rho(Z, \vartheta^*) - v) Q((\rho(Z, \vartheta^*) - v)/h) X']$$

Define

$$\mathcal{A}^*(t) = 1\{D = d\} c' K_1(t) Q(t) (X' - Q(t)' \text{diag}(Q(h)) \gamma_{X'}(v, d)) c$$

then

$$\begin{aligned} \mathcal{A}^{*(1)}(t) &= 1\{D = d\} \frac{\partial}{\partial t} (c' K_1(t) Q(t) X' c) - 1\{D = d\} \frac{\partial}{\partial t} (c' K_1(t) Q(t) Q(t)' \\ &\quad \text{diag}(Q(h)) \gamma_{X'}(v, d) c) \end{aligned}$$

By Taylor's Theorem,

$$\mathbb{E}_n[c'(\mathcal{G}_{P,X'}(v, d, \hat{v}) - \mathcal{G}_{P,X'}(v, d, v^*))c] = \mathbb{E}_n\left[\int_0^1 \frac{\partial}{\partial v'} c' \mathcal{G}_{P,X'}(v, d, v) c|_{v=v^*+t(\hat{v}-v^*)} dt\right](\hat{v} - v^*)$$

By Assumption 3.4.4 and 3.4.7, it suffices to show

$$E\left|\int_0^1 \frac{\partial}{\partial v'} c' \mathcal{G}_P(v, d, v) c|_{v=v^*+t(\hat{v}-v^*)} dt\right| = O\left(\frac{1}{h}\right) \quad (3.8)$$

$$\begin{aligned} E\left|\int_0^1 \frac{\partial}{\partial v'} c' \mathcal{G}_P(v, d, v) c|_{v=v^*+t(\hat{v}-v^*)} dt\right| &\leq \frac{1}{h} E\left[\sup_t |\mathcal{A}^{*(1)}(t)| \sup_{v \in \Theta_n} \left\| \frac{\partial}{\partial v} \rho(Z, v) \right\|\right] \\ &\leq \frac{1}{h} \sup_t |1\{D = d\} \frac{\partial}{\partial t} (c' K_1(t) Q(t) X' c)| \\ &\quad E\left[\sup_{v \in \Theta_n} \left\| \frac{\partial}{\partial v} \rho(Z, v) \right\|\right] \\ &\quad + \frac{1}{h} \sup_t |1\{D = d\} \frac{\partial}{\partial t} (c' K_1(t) Q(t) Q(t)' \\ &\quad \text{diag}(Q(h)) \gamma_{X'}(v, d) c)| E\left[\sup_{v \in \Theta_n} \left\| \frac{\partial}{\partial v} \rho(Z, v) \right\|\right] \end{aligned}$$

By Assumptions 3, 4, and Lemma 3, we have (3.8). □

Lemma 3.4.8. *Under Assumptions 3, 4, 5, and 8, $\mathbb{E}_n[\mathcal{G}_P(v, d, \hat{v})] - m_P(v, d, v^*) = o_p(1)$.*

Proof. First note that $\mathbb{E}_n \mathcal{G}_P(v, d, v^*) = m_P(v, d, v^*) + o_p(1)$, then we show

$$\mathbb{E}_n[c'(\mathcal{G}_P(v, d, \hat{\vartheta}) - \mathcal{G}_P(v, d, \vartheta^*))c] = o_p(1)$$

$$\mathcal{G}_P(v, d, \vartheta) = 1\{D = d\}K_h(\rho(Z, \vartheta) - v)Q((\rho(Z, \vartheta) - v)/h)Q((\rho(Z, \vartheta) - v)/h)'$$

By Taylor's Theorem,

$$\mathbb{E}_n[c'(\mathcal{G}_P(v, d, \hat{\vartheta}) - \mathcal{G}_P(v, d, \vartheta^*))c] = \mathbb{E}_n\left[\int_0^1 \frac{\partial}{\partial \vartheta'} c' \mathcal{G}_P(v, d, \vartheta) c|_{\vartheta=\vartheta^*+t(\hat{\vartheta}-\vartheta^*)} dt\right](\hat{\vartheta} - \vartheta^*)$$

By Assumption 3.4.4 and 3.4.7, it suffices to show

$$E\left|\int_0^1 \frac{\partial}{\partial \vartheta'} c' \mathcal{G}_P(v, d, \vartheta) c|_{\vartheta=\vartheta^*+t(\hat{\vartheta}-\vartheta^*)} dt\right| = O\left(\frac{1}{h}\right) \quad (3.9)$$

Define

$$\tilde{\mathcal{A}}(t) = 1\{D = d\}c'K_1(t)Q(t)Q(t)'c$$

then

$$\tilde{\mathcal{A}}^{(1)}(t) = 1\{D = d\}\frac{\partial}{\partial t}c'K_1(t)Q(t)Q(t)'c$$

$$\begin{aligned}
E \left| \int_0^1 \frac{\partial}{\partial \vartheta'} c' \mathcal{G}_P(v, d, \vartheta) c|_{\vartheta=\vartheta^*+t(\hat{\vartheta}-\vartheta^*)} dt \right| &\leq \frac{1}{h} \sup_t |\tilde{\mathcal{A}}^{(1)}(t)| E \left[\sup_{\vartheta \in \Theta_n} \left\| \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \right\| \right] \\
&\leq \frac{1}{h} \sup_t |1\{D = d\} \frac{\partial}{\partial t} c' K_1(t) Q(t) Q(t)' c| \\
&\quad E \left[\sup_{\vartheta \in \Theta_n} \left\| \frac{\partial}{\partial \vartheta} \rho(Z, \vartheta) \right\| \right]
\end{aligned}$$

By Assumptions 3 and 4, we have (3.9). $\square$

Theorem 3.4.1. *Suppose the assumptions in Lemma 3.4.1 hold. If $\int |K_1(u)| \|Q(u)\| |u|^{J+1} du < \infty$, $\sup_t f_P(t) < \infty$, $p \mapsto \mathbb{E}[U \mid P = p, D = d]$ is $J + 1$ -times continuously differentiable with bounded $(J + 1)$ th derivative, then $MTE_h(x, v) - MTE(x, v) = O(h^J)$.*

Proof. Note that

$$MTE_h(x, v) - MTE(x, v) = (e_1 + ve_2)'(\gamma_U(v, 1) - \bar{\gamma}_U(v, 1)) - (e_1 - (1-v)e_2)'(\gamma_U(v, 0) - \bar{\gamma}_U(v, 0)).$$

Therefore, it suffices to show $\text{diag}(Q(h))(\gamma_U(v, d) - \bar{\gamma}_U(v, d)) = O(h^{J+1})$. By the law of iterated expectations, we have

$$\begin{aligned}
&\text{diag}(Q(h))(\gamma_U(v, d) - \bar{\gamma}_U(v, d)) \\
&= m_P(v, d, \vartheta^*)^{-1} \mathbb{E}[P^d (1 - P)^{1-d} K_h(P - v) Q((P - v)/h) \\
&\quad (\mathbb{E}[U \mid P, D = d] - Q(P - v)' \bar{\gamma}_U(v, d))].
\end{aligned}$$

By Taylor's theorem,

$$\mathbb{E}[U \mid P, D = d] - Q(P - v)' \bar{\gamma}_U(v, d) = \frac{1}{J!} \int_v^P (P - t)^J \frac{\partial^{J+1} \mathbb{E}[U \mid P = t, D = d]}{\partial t^{J+1}} dt.$$

Therefore,

$$\begin{aligned}
& \left\| \mathbb{E}[P^d(1-P)^{1-d}K_h(P-v)Q((P-v)/h)(\mathbb{E}[U | P, D = d] - Q(P-v)'\tilde{\gamma}_U(v, d))] \right\| \\
&= \left\| \mathbb{E}[P^d(1-P)^{1-d}K_h(P-v)Q((P-v)/h) \int_v^P (P-t)^J \frac{\partial^{J+1} \mathbb{E}[U | P = t, D = d]}{\partial t^{J+1}} dt] \right\| \\
&= \left\| \int p^d(1-p)^{1-d}K_h(p-v)Q((p-v)/h) \int_v^p (p-t)^J \frac{\partial^{J+1} \mathbb{E}[U | P = t, D = d]}{\partial t^{J+1}} dt f_P(p) dp \right\| \\
&\leq \int |K_h(p-v)| |Q((p-v)/h)| |p-t|^{J+1} dp \sup_t \left| \frac{\partial^{J+1} \mathbb{E}[U | P = t, D = d]}{\partial t^{J+1}} \right| \sup_t f_P(t) \\
&\leq \int |K_1(u)| |Q(u)| |uh|^{J+1} du \sup_t \left| \frac{\partial^{J+1} \mathbb{E}[U | P = t, D = d]}{\partial t^{J+1}} \right| \sup_t f_P(t) \\
&= h^{J+1} \int |K_1(u)| |Q(u)| |u|^{J+1} du \sup_t \left| \frac{\partial^{J+1} \mathbb{E}[U | P = t, D = d]}{\partial t^{J+1}} \right| \sup_t f_P(t).
\end{aligned}$$

Together with Lemma 3.4.1, we have the statement of this lemma. $\square$

For the influence function, let

$$\omega(x, v) = \omega_1(x, v) + \omega_2(x, v) + \omega_3(x, v)$$

with

$$\begin{aligned}
\omega_1 &= \frac{1}{h} e'_2 (vm_P(v, 1, \vartheta^*)^{-1} \mathcal{G}_{P,U}(v, 1, \gamma_U(v, 1), \vartheta^*) + (1-v)m_P(v, 0, \vartheta^*)^{-1} \mathcal{G}_{P,U}(v, 0, \gamma_U(v, 0), \vartheta^*)) \\
&\quad + e'_2 (v\gamma_{X'}(v, 1)m_X(1, \eta^*)^{-1} \mathcal{G}_{X,Y}(1, \beta_1^*, \eta^*) + (1-v)\gamma_{X'}(v, 0)m_X(0, \eta^*)^{-1} \mathcal{G}_{X,Y}(0, \beta_0^*, \eta^*)) \\
&\quad + \frac{1}{h} e'_2 (vm_P(v, 1, \vartheta^*)^{-1} \frac{\partial}{\partial \vartheta'} m_{P,U}(v, 1, \gamma_U(v, 1), \vartheta)|_{\vartheta=\vartheta^*} \varphi + (1-v)m_P(v, 0, \vartheta^*)^{-1} \\
&\quad \frac{\partial}{\partial \vartheta'} m_{P,U}(v, 0, \gamma_U(v, 0), \vartheta)|_{\vartheta=\vartheta^*} \varphi) \\
\omega_2 &= e'_1 (m_P(v, 1, \vartheta^*)^{-1} \mathcal{G}_{P,U}(v, 1, \gamma_U(v, 1), \vartheta^*) - m_P(v, 0, \vartheta^*)^{-1} \mathcal{G}_{P,U}(v, 0, \gamma_U(v, 0), \vartheta^*)) \\
&\quad + e'_1 (\gamma_{X'}(v, 1)m_X(1, \eta^*)^{-1} \mathcal{G}_{X,Y}(1, \beta_1^*, \eta^*) - \gamma_{X'}(v, 0)m_X(0, \eta^*)^{-1} \mathcal{G}_{X,Y}(0, \beta_0^*, \eta^*)) \\
&\quad + e'_1 (m_P(v, 1, \vartheta^*)^{-1} \frac{\partial}{\partial \vartheta'} m_{P,U}(v, 1, \gamma_U(v, 1), \vartheta)|_{\vartheta=\vartheta^*} \varphi - m_P(v, 0, \vartheta^*)^{-1} \\
&\quad \frac{\partial}{\partial \vartheta'} m_{P,U}(v, 0, \gamma_U(v, 0), \vartheta)|_{\vartheta=\vartheta^*} \varphi) \\
\omega_3 &= x' (m_X(1, \eta^*)^{-1} \mathcal{G}_{X,Y}(1, \beta_1^*, \eta^*) - m_X(0, \eta^*)^{-1} \mathcal{G}_{X,Y}(0, \beta_0^*, \eta^*)).
\end{aligned}$$

Theorem 3.4.2. *Under Assumptions 3.2.1-3.4.7 and Lemmas 3.2.1-3.9, we have*

$$\widehat{MTE}(x, v) - MTE_h(x, v) = \mathbb{E}_n[\omega(x, v)] + o_p(n^{-1/2})$$

with $\mathbb{E}[\omega(x, v)] = 0$. If in addition, the assumptions in Theorem 3.4.1 hold, then $\widehat{MTE}(x, v) - MTE(x, v) = \mathbb{E}_n[\omega(x, v)] + O(h^J) + o_p(n^{-1/2})$.

Proof of Theorem 3.4.2. Note that we have

$$\widehat{MTE}(x, v) = x'(\hat{\beta}_1 - \hat{\beta}_0) + (e_1 + ve_2)' \hat{\gamma}_{\hat{U}}(v, 1) - (e_1 - (1-v)e_2)' \hat{\gamma}_{\hat{U}}(v, 0).$$

First, we are going to show

$$\hat{\beta}_d - \beta_d^* = m_X(d, \eta^*)^{-1} \mathbb{E}_n[\mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)] + o_p(n^{-1/2}). \quad (3.10)$$

Although this statement follows from Andrews (1994), we provide its proof for completeness.

We have

$$\begin{aligned} & \hat{\beta}_d - \beta_d^* \\ &= \mathbb{E}_n[\mathcal{G}_X(d, \hat{\eta})]^{-1} \mathbb{E}_n[\mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)] + \mathbb{E}_n[\mathcal{G}_X(d, \hat{\eta})]^{-1} (\mathbb{E}_n[\mathcal{G}_{X,Y}(d, \beta_d^*, \hat{\eta}) - \mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)]). \end{aligned}$$

By Assumption 3.4.6, Lemma 3.4.2, 3.4.4, and 3.4.5, we have (3.10).

Second, we are going to show

$$\begin{aligned} e'_1 \hat{\gamma}_{\hat{U}}(v, d) - e'_1 \gamma_U(v, d) &= e'_1 m_P(v, d, \vartheta^*)^{-1} \mathbb{E}_n[\mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \vartheta^*)] \\ &\quad + e'_1 \gamma_{X'}(v, d) m_X(d, \eta^*)^{-1} \mathbb{E}_n[\mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)] \\ &\quad + e'_1 m_P(v, d, \vartheta^*)^{-1} \frac{\partial}{\partial \vartheta'} m_{P,U}(v, d, \gamma_U(v, d), \vartheta) |_{\vartheta=\vartheta^*} \mathbb{E}_n[\varphi] \\ &\quad + o_p(n^{-1/2}) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} e'_2 \hat{\gamma}_{\hat{U}}(v, d) - e'_2 \gamma_U(v, d) &= \frac{1}{h} e'_2 m_P(v, d, \vartheta^*)^{-1} \mathbb{E}_n[\mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \vartheta^*)] \\ &\quad + e'_2 \gamma_{X'}(v, d) m_X(d, \eta^*)^{-1} \mathbb{E}_n[\mathcal{G}_{X,Y}(d, \beta_d^*, \eta^*)] \\ &\quad + \frac{1}{h} e'_2 m_P(v, d, \vartheta^*)^{-1} \frac{\partial}{\partial \vartheta'} m_{P,U}(v, d, \gamma_U(v, d), \vartheta) |_{\vartheta=\vartheta^*} \mathbb{E}_n[\varphi] \\ &\quad + o_p(n^{-1/2}). \end{aligned} \tag{3.12}$$

Our proof is based on the one of Mammen et al. (2012). We have

$$\begin{aligned} & \hat{\gamma}_{\hat{U}}(v, d) - \gamma_U(v, d) \\ &= \text{diag}(Q(h))^{-1} \mathbb{E}_n[\mathcal{G}_P(v, d, \hat{\vartheta})]^{-1} \mathbb{E}_n[\mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \vartheta^*)] - \hat{\gamma}_{X'}(v, d) (\hat{\beta}_d - \beta_d^*) \\ &\quad + \text{diag}(Q(h))^{-1} \mathbb{E}_n[\mathcal{G}_P(v, d, \hat{\vartheta})]^{-1} (\mathbb{E}_n[\mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \hat{\vartheta}) - \mathcal{G}_{P,U}(v, d, \gamma_U(v, d), \vartheta^*)]) \end{aligned}$$

and

$$\begin{aligned}
& \hat{\gamma}_{X'}(v, d) - \gamma_{X'}(v, d) \\
&= \text{diag}(Q(h))^{-1} \mathbb{E}_n[\mathcal{G}_P(v, d, \hat{\vartheta})]^{-1} \mathbb{E}_n[\mathcal{G}_{P, X'}(v, d, \gamma_{X'}(v, d), \vartheta^*)] \\
&\quad + \text{diag}(Q(h))^{-1} \mathbb{E}_n[\mathcal{G}_P(v, d, \hat{\vartheta})]^{-1} (\mathbb{E}_n[\mathcal{G}_{P, X'}(v, d, \gamma_{X'}(v, d), \hat{\vartheta}) - \mathcal{G}_{P, X'}(v, d, \gamma_{X'}(v, d), \vartheta^*)]).
\end{aligned}$$

By (3.10) and Lemma 3.4.1, 3.4.3, 3.4.6, 3.4.7, 3.4.8, we have (3.11) and (3.12). The first statement of this theorem follows from (3.10)-(3.12). Moreover, the first statement of this theorem and Theorem 3.4.1 imply the second statement. $\square$

Now define

$$\hat{\omega}(x, v) = \hat{\omega}_1(x, v) + \hat{\omega}_2(x, v) + \hat{\omega}_3(x, v), \quad (3.13)$$

with

$$\begin{aligned}
\hat{\omega}_1 = & \frac{1}{h} e'_2 \left(v \mathbb{E}_n[\mathcal{G}_P(v, 1, \hat{\vartheta})]^{-1} \mathcal{G}_{P,U}(v, 1, \hat{\gamma}_{\hat{U}}(v, 1), \hat{\vartheta}) \right. \\
& + (1 - v) \mathbb{E}_n[\mathcal{G}_P(v, 0, \hat{\vartheta})]^{-1} \mathcal{G}_{P,U}(v, 0, \hat{\gamma}_{\hat{U}}(v, 0), \hat{\vartheta}) \Big) \\
& + e'_2 \left(v \hat{\gamma}_{X'}(v, 1) \mathbb{E}_n[\mathcal{G}_X(1, \hat{\eta})]^{-1} \mathcal{G}_{X,Y}(1, \hat{\beta}_1, \hat{\eta}) \right. \\
& + (1 - v) \hat{\gamma}_{X'}(v, 0) \mathbb{E}_n[\mathcal{G}_X(0, \hat{\eta})]^{-1} \mathcal{G}_{X,Y}(0, \hat{\beta}_0, \hat{\eta}) \Big) \\
& + \frac{1}{h} e'_2 \left(v \mathbb{E}_n[\mathcal{G}_P(v, 1, \hat{\vartheta})]^{-1} \mathbb{E}_n \left[\frac{\partial}{\partial \vartheta'} \mathcal{G}_{P,U}(v, 1, \hat{\gamma}_{\hat{U}}(v, 1), \vartheta) \Big|_{\vartheta=\hat{\vartheta}} \right] \hat{\varphi} \right. \\
& + (1 - v) \mathbb{E}_n[\mathcal{G}_P(v, 0, \hat{\vartheta})]^{-1} \mathbb{E}_n \left[\frac{\partial}{\partial \vartheta'} \mathcal{G}_{P,U}(v, 0, \hat{\gamma}_{\hat{U}}(v, 0), \vartheta) \Big|_{\vartheta=\hat{\vartheta}} \right] \hat{\varphi} \Big), \\
\hat{\omega}_2 = & e'_1 \left(\mathbb{E}_n[\mathcal{G}_P(v, 1, \hat{\vartheta})]^{-1} \mathcal{G}_{P,U}(v, 1, \hat{\gamma}_{\hat{U}}(v, 1), \hat{\vartheta}) - \mathbb{E}_n[\mathcal{G}_P(v, 0, \hat{\vartheta})]^{-1} \mathcal{G}_{P,U}(v, 0, \hat{\gamma}_{\hat{U}}(v, 0), \hat{\vartheta}) \right) \\
& + e'_1 \left(\hat{\gamma}_{X'}(v, 1) \mathbb{E}_n[\mathcal{G}_X(1, \hat{\eta})]^{-1} \mathcal{G}_{X,Y}(1, \hat{\beta}_1, \hat{\eta}) - \hat{\gamma}_{X'}(v, 0) \mathbb{E}_n[\mathcal{G}_X(0, \hat{\eta})]^{-1} \mathcal{G}_{X,Y}(0, \hat{\beta}_0, \hat{\eta}) \right) \\
& + e'_1 \left(\mathbb{E}_n[\mathcal{G}_P(v, 1, \hat{\vartheta})]^{-1} \mathbb{E}_n \left[\frac{\partial}{\partial \vartheta'} \mathcal{G}_{P,U}(v, 1, \hat{\gamma}_{\hat{U}}(v, 1), \vartheta) \Big|_{\vartheta=\hat{\vartheta}} \right] \hat{\varphi} \right. \\
& \quad \left. - \mathbb{E}_n[\mathcal{G}_P(v, 0, \hat{\vartheta})]^{-1} \mathbb{E}_n \left[\frac{\partial}{\partial \vartheta'} \mathcal{G}_{P,U}(v, 0, \hat{\gamma}_{\hat{U}}(v, 0), \vartheta) \Big|_{\vartheta=\hat{\vartheta}} \right] \hat{\varphi} \right), \\
\hat{\omega}_3 = & x' \left(\mathbb{E}_n[\mathcal{G}_X(1, \hat{\eta})]^{-1} \mathcal{G}_{X,Y}(1, \hat{\beta}_1, \hat{\eta}) - \mathbb{E}_n[\mathcal{G}_X(0, \hat{\eta})]^{-1} \mathcal{G}_{X,Y}(0, \hat{\beta}_0, \hat{\eta}) \right).
\end{aligned}$$

Based on Theorem 3.4.2 and (3.13), we propose the following variance estimator

$$\widehat{Var}(\widehat{MTE}(x, v)) = \frac{1}{n^2} \sum_{i=1}^n \hat{\omega}_i(x, v)^2. \quad (3.14)$$

3.5 Monte Carlo Simulations

In this section, we evaluate the finite-sample coverage of our proposed variance estimator for MTE, and compare it with the coverage of the variance estimator suggested by Carneiro and Lee (2009). We consider sample sizes $N = 500$ and $N = 1000$, and the results are based on 5000 Monte Carlo replications. The MTE is evaluated at $x = (2, 4, 8, 16)'$ and multiple

values of v with $v = (0.25, 0.5, 0.75)$. We generate (X, Z) jointly from a multivariate normal distribution, $(X_1, Z_1, Z_2, Z_3) \sim \mathcal{N}(\mu_{XZ}, \Sigma_{XZ})$ with

$$\mu_{XZ} = (0, 0, 0, 0)'$$

and

$$\Sigma_{XZ} = \begin{pmatrix} 1 & 0.3 & 0.3 & 0.3 \\ 0.3 & 1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 1 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{pmatrix}.$$

The first component in (X_1, Z_1, Z_2, Z_3) is used to construct regressors $X = (X_1, X_1^2, X_1^3, X_1^4)'$ and $Z = (Z_1, Z_2, Z_3)'$ are the instruments. Potential outcomes are generated as

$$Y_1 = X'\beta_1 + U_1, \quad Y_0 = X'\beta_0 + U_0,$$

where

$$\beta_1 = (1, 2, 2, 1)', \quad \beta_0 = (-4, 2, 1, 1)'.$$

Treatment follows from

$$D = \mathbf{1}\{U_S < Z'\theta\}, \quad \theta = (0.1, 0.4, 0.2)',$$

and the observed outcome is

$$Y = DY_1 + (1 - D)Y_0.$$

The unobservables (U_S, U_1, U_0) are jointly normal with mean zero and covariance matrix

$$\Sigma_U = \begin{pmatrix} 1 & 0.3 & 0.3 \\ 0.3 & 1 & 0.3 \\ 0.3 & 0.3 & 1 \end{pmatrix}.$$

Then our variance estimation follows from (3.13) and (3.14). In each replication, we estimate θ and the propensity score using a probit model. For the local polynomial regression, we choose a quadratic basis and the Epanechnikov kernel. For bandwidth selection, we choose $h = 0.4$ for $n = 500$ and $h = 0.35$ for $n = 1000$ (undersmoothing).

For the variance of the MTE estimator, Carneiro and Lee (2009) focuses on the variance estimator for the asymptotic variance, which is only driven by the estimators of $\frac{\partial \mathbb{E}[U_1|P=v, D=1]}{\partial v}$ and $\frac{\partial \mathbb{E}[U_0|P=v, D=0]}{\partial v}$. Here we use a plug-in variance estimator for the asymptotic variance in Theorem 4 and 5 from Carneiro and Lee (2009), following Fan and Gijbels (1996).

Using the local polynomial estimator $\hat{\gamma}_{\hat{U}}(v, d)$ defined above, let

$$\hat{\Sigma}_{\hat{U}}(v, d) = \hat{\Gamma}(v, d)^{-1} \hat{\Omega}(v, d) \hat{\Gamma}(v, d)^{-1} \hat{\sigma}_d^2(v),$$

where

$$\hat{\Gamma}(v, d) = \sum_{i=1}^n 1\{D_i = d\} K_h(\rho(Z_i, \hat{\vartheta}) - v) Q\left(\frac{\rho(Z_i, \hat{\vartheta}) - v}{h}\right) Q\left(\frac{\rho(Z_i, \hat{\vartheta}) - v}{h}\right)',$$

$$\hat{\Omega}(v, d) = \sum_{i=1}^n 1\{D_i = d\} K_h(\rho(Z_i, \hat{\vartheta}) - v)^2 Q\left(\frac{\rho(Z_i, \hat{\vartheta}) - v}{h}\right) Q\left(\frac{\rho(Z_i, \hat{\vartheta}) - v}{h}\right)'.$$

Let

$$\hat{U}_{d,i} = Y_i - X_i' \hat{\beta}_d,$$

and define

$$\hat{\varepsilon}_{d,i}(v) = \hat{U}_{d,i} - Q\left(\frac{\rho(Z_i, \hat{\vartheta}) - v}{h}\right)' \hat{\gamma}_{\hat{U}}(v, d).$$

We estimate the local error variance by

$$\hat{\sigma}_d^2(v) = \frac{\sum_{i=1}^n 1\{D_i = d\} K_h(\rho(Z_i, \hat{\vartheta}) - v) \hat{\varepsilon}_{d,i}(v)^2}{\sum_{i=1}^n 1\{D_i = d\} K_h(\rho(Z_i, \hat{\vartheta}) - v) - \text{tr}(\hat{\Omega}(v, d) \hat{\Gamma}(v, d)^{-1})}.$$

Since the derivative component is given by $e_2' \hat{\gamma}_{\hat{U}}(v, d)/h$, we define

$$\widehat{\text{Var}}_d'(v) = \frac{1}{h^2} e_2' \hat{\Sigma}_{\hat{U}}(v, d) e_2.$$

Finally, we define the derivative-based plug-in variance estimator

$$\widetilde{\text{Var}}(\widehat{MTE}(v)) = v^2 \widehat{\text{Var}}_1'(v) + (1 - v)^2 \widehat{\text{Var}}_0'(v).$$

Given the DGP constructed above, now we provide the coverage rates following our variance

Table 3.1: Coverage of 95% Confidence Intervals ($N = 500$)

v	$\widehat{\text{Var}}(\widehat{MTE}(x, v))$	$\widetilde{\text{Var}}_1(\widehat{MTE}(x, v))$	$\widetilde{\text{Var}}(\widehat{MTE}(v))$
0.25	0.900	0.879	0.870
0.50	0.924	0.655	0.600
0.75	0.965	0.883	0.874

Table 3.2: Coverage of 95% Confidence Intervals ($N = 1000$)

v	$\widehat{\text{Var}}(\widehat{MTE}(x, v))$	$\widetilde{\text{Var}}_1(\widehat{MTE}(x, v))$	$\widetilde{\text{Var}}(\widehat{MTE}(v))$
0.25	0.862	0.833	0.824
0.50	0.871	0.600	0.559
0.75	0.957	0.883	0.877

estimator and $\widetilde{\text{Var}}(\widehat{MTE}(v))$. The coverage result following $\hat{\omega}_1$ alone is also included for comparison, it is the influence function component that accounts for the derivative part in

the local polynomial regression, let $\widehat{Var}_1(\widehat{MTE}(x, v)) = \frac{1}{n^2} \sum_{i=1}^n \hat{\omega}_{1i}(x, v)^2$. From Table 3.1 and 3.2 we find that our variance estimator achieves better finite-sample coverage at different values of v . Though at some points there is undercoverage, our method is generally much closer to the correct coverage or achieves the correct coverage, compared to the other two methods.

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