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What Did Weierstrass Really Define? The Cognitive Structure of Natural and ϵ - δ Continuity

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The cognitive science of mathematics is the study of mathematical ideas from the perspective of research on our largely unconscious everyday conceptual systems as they are embodied in the human brain. A major result is that most everyday abstract ideas are metaphorical in nature—that is, they involve inference-preserving mappings from one conceptual domain to another. Many mathematical ideas are metaphorical in this respect, as when we conceptualise numbers metaphorically as points on a line, or when we conceptualise lines metaphorically as sets of points.

The concept of ϵ - δ continuity is metaphorical as well. In everyday thought, (natural) continuity is understood in terms of a trajectory of motion, as it was in mathematics until the late nineteenth century. From a cognitive perspective, what Dedekind and Weierstrass really did was to introduce new metaphors for natural continuity. That is, they conceptualised continuity for lines and for functions in terms of two new and radically different concepts: *gaplessness* for lines, and preservation of closeness for functions. But the mathematical community has incorrectly seen Weierstrass as having done something different: defining the essence of the concept of continuity. This mistake has confused generations of mathematics students and has led to a misleading, counterintuitive, and cognitively untenable view of what continuity is.

Continuity is one of the most important ideas in twentieth-century mathematics. It has been fundamental to topology and analysis and has been central to core questions in set theory and metamathematics involving the elusive concept of the *continuum*. The concept of continuity, which in everyday life seems so fundamentally intuitive, immediate, and transparent, has a mathematical counterpart—the so-called ϵ - δ definition—that is elusive and counterintuitive. Teachers of mathematics have long bewailed the fact that it is simply hard to teach the concept of continuity and limits of functions as defined by the ϵ - δ definition (Keisler, 1976; P. Kitcher, personal communication, 1997; Núñez, 1993; Robert, 1982; Tall & Vinner, 1981).

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Our everyday concept, the one that arises naturally, is the so-called “intuitive” and “non-rigorous” concept so central to seventeenth-century mathematics. The second is what is referred to as the “rigorous” one— ϵ - δ definition—adopted by Weierstrass two centuries later, and which has become the standard of modern analysis ever since. Are these two characterisations of “continuity”—the natural, everyday notion and the ϵ - δ definition—two different faces of the same concept? Or are they radically different ideas with independent cognitive structures? These questions are not mathematical in nature, nor are they philosophical. They are empirical questions that must be addressed from the perspective of contemporary cognitive science. In this paper, we analyse the cognitive structure of these ideas building mainly on recent advances in cognitive linguistics. We do not address here the details of educational implications.

We argue that natural, everyday continuity and ϵ - δ continuity are simply different concepts. To conceptualise the first in terms of the second, as Weierstrass did, is to use a conceptual metaphor. This has an important consequence for understanding the nature of mathematical ideas and for teaching: the ϵ - δ definition of continuity, although extremely useful for doing interesting mathematics, is *not* cognitively about the concept of natural, everyday continuity at all. It is *not*, as is sometimes claimed, a distillation of the essence of the continuity idea. It is, rather, a different—and metaphorical—idea. This does not mean that there is anything wrong with it. Conceptual metaphors are inherent in most of mathematics (Lakoff & Núñez, 1997, 1999). But recognising this dispels the idea that the ϵ - δ metaphor is “more rigorous” and therefore superior to the natural everyday idea of continuity. The two ideas are *just different*, and being different, yield different results. Each is perfectly valid in its own sphere.

THE EMBODIED MIND

To discuss mathematical ideas at all from a scientific perspective, one must turn to cognitive science, which over the past two decades has shown that ideas are not purely abstract, free-floating, disembodied, transcendent entities, but rather that *ideas arise from and are shaped by the structure of human bodies and brains and the nature of everyday human experience* (Damasio, 1994; Dehaene, 1997; Edelman, 1992; Johnson, 1987; Lakoff, 1987; Lakoff & Johnson, 1980, 1998; Lakoff & Núñez, 1997, 1999; Núñez, 1995, 1997; Thelen, 1995; Varela, Thompson, & Rosch, 1991). For this reason, the very undertaking of the scientific study of mathematical ideas puts one at odds with the a priorist, human-independent philosophy of mathematics in which Weierstrass’ arithmetisation programme was immersed, and in which much of contemporary mathematics still is. For this reason, our conclusions will not be in harmony with what most mathematicians have been brought up to believe.

Embodied cognition has a special interest in empirically studying domains such as everyday cognition, common-sense understanding, natural language, spontaneous gestures, real-world actions, and so on; that is, in studying natural mental phenomena that in general are manifested below the level of conscious experience. The basic finding is that most thought is unconscious and that concepts are systematically organised. Findings in various scientific disciplines such as evolutionary biology, neuroscience, cognitive linguistics, developmental psychology, and cognitive anthropology have all contributed to the development of embodied cognition as a theoretical approach.

Results about the structure of conceptual systems have come mainly from cognitive linguistics, especially the theory of conceptual metaphor (Lakoff, 1993; Lakoff & Johnson, 1980). Most abstract concepts are metaphorical in nature, drawing upon the inferential structure of everyday bodily experience to reason about abstractions. Time, for example, is primarily conceptualised in terms of motion, either the motions of future times toward an observer (“Christmas is *approaching*”) or the motion of an observer over a time landscape (“We’re *approaching* Christmas”).

Conceptual metaphors are cross-domain “mappings” that project the inferential structure of a *source domain* onto a *target domain*, allowing the use of body-based inference to structure abstract inference. Such “projections” or “mappings” are not arbitrary and can be studied empirically and stated precisely. They are not arbitrary, because they are motivated by our everyday experience—especially bodily experience. For example, “affection” is conceptualised as “warmth (as in “She greeted me *warmly*”), since we experience a correlation between affection and warmth from birth onwards. Research in contemporary conceptual metaphor theory indicates that there is an extensive conventional system of conceptual metaphors in every human conceptual system. Unlike traditional studies of metaphor, contemporary embodied views do not see conceptual metaphors as residing in words, but in thought. Metaphorical linguistic expressions thus are only surface manifestations of metaphorical thought.

It should be clear that these theoretical claims are not based on mere introspection or on anecdotal personal accounts. Rather, they are based on empirical evidence from a variety of sources, including psycholinguistic experiments (Gibbs, 1994), generalisations over inference patterns (Lakoff, 1987, Case Study 1), generalisations over conventional and novel language (Lakoff, 1993; Lakoff & Turner, 1989), and the study of historical semantic change (Sweetser, 1990), of language acquisition (C. Johnson, 1997), of spontaneous gestures (McNeill, 1992), of American sign language (Taub, 1997), and of coherence in discourse (Narayanan, 1997). For surveys of conceptual metaphors, their properties, research methodologies, and theory required to understand how they function, see Lakoff (1993), Lakoff and Johnson (1980, 1998).

Much of mathematical understanding is based on spatial relations concepts, which are expressed in English by prepositions like *in*, *on*, *through*, and so on. These concepts are decomposable into primitive spatial relations concepts called *image schemas*. Image schemas are basic dynamic topological and orientation structures that characterise spatial inferences and link language to visual experience (see Johnson, 1987; Lakoff & Johnson, 1998, Chapter 3; Regier, 1996). Their inferential structure is preserved under metaphorical mappings, which characterise the meanings of abstract uses of prepositions (as in, “He is *in* a depression”). It is claimed that image schemas are realised through such neural structures as topographic maps of the visual field, orientation-sensitive cells, and so on (for a computer simulation, see Regier, 1996). As such, they are dynamic recurrent regular patterns of ongoing perceptions and actions. These neural structures emerge as meaningful for us mainly through the bodily experience of movement in space, manipulation of objects, and perceptual interactions. They are not static, not symbol-like, and not part of some putative objective, disembodied structure in the universe outside of human beings.

Some examples are the container schema (underlying concepts like *IN* and *OUT*), source–path–goal schema (*TO* and *FROM*), the contact schema, and verticality schema. The notion of a path in the source–path–goal schema is central to our everyday notion of a continuous line or curve. Many basic concepts are built on combinations of these schemas. The concept *ON*, as in “the book is *on* the desk”, uses three basic schemas: verticality, contact, and support. Each image-schema has its own inferential structure that is preserved by metaphorical mappings onto abstract domains. In mathematics, for example, the classical notion of a set is based on container schemas. Boolean logic is the logic of container schemas mapped onto categories of any sort via the metaphor that “Categories Are Containers” (Lakoff, 1987; Lakoff & Núñez, 1999).

NATURAL CONTINUITY

For centuries, the characterisation of continuity was based on the idea of motion—the motion of a physical object with definite direction and speed. Such motion, based on the source–path–goal schema, proceeds without gaps, interruptions or “discontinuities”. Great mathematicians such as Kepler, Leibniz, Newton, and Euler based their mathematical work involving continuity on this intuitive notion. Euler referred to a continuous curve as “a curve described by freely leading the hand” (cited in Stewart, 1995, p. 237). It is the same intuitive idea that earlier allowed Kepler to measure “an area swept out by the motion of a (celestial) point on a physical ‘continuous curve’ ” (Kramer, 1970, p. 528). This idea, although simple, proved to be extremely rich and powerful in generating one of the most beautiful and productive branches of all mathematics: seventeenth-century calculus. We call this everyday idea of continuity, *natural continuity*.

Everyday, natural continuity—continuity as we normally conceptualise it outside of mathematics—has the following essential features.

1. It is continuity traced by motion, which takes place over time.
2. The trace of the motion is a static holistic line with no “jumps”.

The motion and the line tracing the motion are related by a well-known cognitive mechanism, *fictive motion* (Talmy, 1996). It is through this everyday cognitive mechanism that we can conceptualise a (static) curve in terms of the motion tracing that curve. In terms of conceptual metaphor, this can be stated as follows (Fictive Motion Metaphor):

- A Line Is the Motion of an Object Tracing That Line.

Examples of this fundamental metaphor are abundant in everyday language: “Highway 101 *goes to* Los Angeles.” “After *crossing* the bay, Highway 80 *reaches* San Francisco.” “Just before *reaching* the border, that highway *goes through* several tunnels.” In these everyday expressions, a highway (one-dimensional line), which is a static object, is conceptualised in terms of a traveller (object) moving along the route of the highway. It is because of this cognitive mechanism that in mathematics we can speak of a function as *growing, moving, oscillating, approaching* values, and *reaching* limits. Expressions of this kind are not limited to students. They are manifested in professional mathematicians as well. This is not surprising, since mathematical ideas are systematic extensions of everyday common forms of understanding such as fictive motion. What is extremely important to keep in mind is that, formally speaking, the mathematical function does not move, but cognitively speaking—which is what we really care about here—under this conceptual metaphor, the function *does move, does approach* limits.

But how did mathematicians of the seventeenth century conceptualise a function as a naturally continuous line? They used conceptual metaphor. A function from the real numbers to the real numbers was conceptualised geometrically, using analytic geometry. At the heart of analytic geometry is a central conceptual metaphor: Real Numbers Are Points on a Line (Lakoff & Núñez, 1997, 1999). The structure of that metaphorical mapping is shown in Table 1.

This mapping projects aspects of the inferential structure of the line onto the real numbers, with properties of points on a line projected onto properties of numbers. For example, the denseness of the line (between every two points there is a point) becomes denseness for numbers (between every two numbers there is a number). In addition, the continuity of the line (there are no “holes” in the line) is projected onto continuity for the real numbers (there are no “gaps” in the real numbers). Similarly, the fact that *distance* maps onto *difference* has the consequence that distinctness for points (in order for two points to be distinct, there

TABLE 1
Real Numbers Are Points on a Line

<i>Source Domain: Points on a Line</i>		<i>Target Domain: Real Numbers</i>
The points P_i on a line L	→	The real numbers R_i
A distance $D_i(P_j, P_k)$ between each pair of points	→	A difference relation $Diff_i(R_j, R_k) = R_j - R_k $ between each pair of real numbers
A spatial ordered-before relation holding between all pairs of points	→	A less-than relation holding between all pairs of real numbers
Designated points a and b , with a spatially ordered-before b	→	Designated real numbers 0 and 1, with 0 less than 1

must be a nonzero *distance* between them) maps onto difference for numbers (in order for two numbers to be different, there must be a nonzero *difference* between them).

This metaphorical mapping from points on a line to numbers allows for the formation of a new conceptual structure, in which the source and target domains of the metaphor are combined into what is called in cognitive science a “conceptual blend” (Fauconnier & Turner, 1998). What results in this case is *the Number-Line Blend*, which defines the familiar “number line” in which points and numbers are blended to form number-points (for further discussion, see Lakoff & Núñez, 1999). It is the Number-Line Blend that is the basis of the analytic geometry used in the development of calculus in the seventeenth century through the late nineteenth century.

Given the Number-Line Blend, pairs of lines (coordinates) generate the Cartesian plane in which a naturally continuous function from the reals into the reals can be characterised. Then the function can be visualised as a curve in the Cartesian plane, with each point on the curve defined by an ordered pair of points $(x, f(x))$. Here the curve, as a conceptual curve, continues to have the properties of a natural everyday curve. As James Pierpont, a noted analyst at Yale, observed a century ago (Pierpont, 1899), a curve taken as representing a function was seen as having the following properties:

1. It can be generated by the motion of a point.
2. It is continuous.
3. It has a tangent.
4. It has a length.
5. When closed, it forms a complete boundary of a region.
6. This region has an area.
7. A curve is not a surface.
8. It is formed by the intersection of two surfaces.

These properties were presented as “intuitive” and “more or less undisputed”.

SOME HISTORICAL ORIGINS OF ϵ - δ CONTINUITY

But this intuitive idea of a function as a naturally continuous curve changed dramatically towards the end of the nineteenth century, when mathematicians created new, rich, and complex mathematical objects that demanded a revision of major basic concepts. Here are some of the famous examples:

1. $f(x) = \sin(1/x)$, for $x \neq 0$, and $f(x) = 0$, for $x = 0$. As this function approaches 0, the number of periods increases to infinity. At 0, it has no tangent; that is, there is no specified direction from which the "curve" "approaches zero". It cannot, therefore, be characterised by the motion of a point.
2. $f(x) = 1$, if x is irrational, and $f(x) = 0$, if x is rational. This cannot be thought of as a curve at any point. It violates all of the eight conditions.
3. Space-filling curves, like the Peano curve or the Hilbert curve, map from the interval $[0,1]$ onto the unit square, going through every point in the unit square. This "curve" "fills" the square and therefore has an area and does not form a boundary of a region.
4. The Cantor odd-even function, which maps the interval $[0,1]$ onto the unit square, by mapping each infinite decimal, r , onto the pair of infinite decimals (r_1, r_2) , where r_1 consists of the sequences of odd places in r , and r_2 consists of the sequence of even places in r . There is no way even to begin to think about this function as a curve. It violates all the properties given above.
5. The so-called "Cantor function" from the interval $[0,1]$ to the Cantor set: The function can be characterised algorithmically by successively removing the middle third of each continuous portion of the interval from 0 to 1; for example, first removing $[1/3, 2/3]$, then removing $[1/9, 2/9]$ and $[7/9, 8/9]$, and so on till infinity. The resulting function also violates the properties given by Pierpont and cannot be conceptualised in natural geometrical terms.

For these cases and many more, it became impossible to give a uniform definition of a function in geometric terms as a curve. This became a crisis, rather than a discussion of mere curiosities, because of the cultural values of the mathematical community of the times, which saw a need for "secure and rigorous foundations". As a consequence, the very notion of rigour was redefined in such a way as to banish or tame the monsters. This contributed to what we will call:

The Formal Foundations Principle

- Every basic concept has to be defined in symbolic notation using symbolic logic by a set of necessary and sufficient conditions that characterise the essence of the concept.

- The definition must be in terms of the ontology taken as characterising the foundations of the discipline (e.g. sets and formal logic; sets, numbers, and formal logic; or whatever ontology is to be taken as “foundational”).

Many contemporary mathematicians take this principle as being so obviously necessary that it does not have to be stated, much less justified, and indeed no mathematics text that we have ever seen has ever stated or tried to justify it. Yet mathematics proceeded quite well up until the late nineteenth century without it. This principle is part of the cultural history of mathematics and suited the tenor of the times. At that period, mathematicians tended to view mathematics in the following way:

1. Mathematics is transcendentally true. Its truths are universal, unique, disembodied, and not dependent on anything physical.
2. Mathematics is about absolute certainty.
3. All truths could be proved rigorously and without any doubt from a finite number of axioms and definitions.
4. Therefore, mathematics had to be completely symbolisable in a rigorous way.
5. Nothing could be left to mere intuition.
6. Definitions must be formulated in terms of some ontology, language, and form of reason. That ontology had to be absolutely clear and unambiguous. The language had to be universal, and the form of reason had to be universally valid.
7. Concepts are defined by essences—necessary and sufficient conditions—and therefore each formal definition must specify such conditions.
8. Concepts are assumed to be literal. No metaphor could possibly enter the definition of a concept.

Findings in contemporary cognitive science are at odds with such a view of mathematics, if mathematics is viewed as a human conceptual system (Dehaene, 1997; Lakoff & Núñez, 1999). Human concepts, on the whole, are not characterised in this way. They are embodied, not defined by essences, not by any means entirely literal, and so on. Yet, to the mind of the late nineteenth-century mathematician, these eight properties of mathematics seemed obviously true and beyond dispute, as did the Formal Foundations Principle that arose from them. For these reasons, the problems with the geometric definition of a real-valued function as a curve in the Cartesian plane created a crisis in late nineteenth-century mathematics. The crisis could only be resolved using the Formal Foundations Principle.

THE COGNITIVE STRUCTURE OF THE ARITHMETISATION OF CALCULUS

Applying the Formal Foundations Principle

If the concept of a function—as basic a concept as there is in all of mathematics—could not be defined for *all* cases in accord with the Formal Foundations Principle, then it was thought that the whole field of mathematics as the epitome of rigorous thought would be called into question. If the Formal Foundations Principle was to be met, the geometric definition of functions as curves would have to be replaced. Geometry would have to be rooted out of calculus altogether, and functions from numbers to numbers would have to be conceptualised in purely arithmetic terms, as would notions like derivative and integral. Accordingly, the central notions of calculus would have to be redefined in non-geometric terms, especially the notion of “approaching a limit”, which, of course, had been thought of in terms of geometry and motion. Similarly, the notion of continuity itself, which had only been thought of in geometric terms, would have to be reconceptualised.

How do you get rid of geometry and motion in characterising calculus, when the whole idea of calculus was thought of, and formulated in terms of, geometry and motion? All of calculus had been built on the conceptual metaphor that Real Numbers Are Points on a Line. The notion of continuity used in calculus was everyday, natural continuity defined in terms of the intuitively clear notion of a continuous curve, “described”, as Euler has said, “by freely leading the hand”. The answer, from a cognitive perspective, was to re-metaphorise calculus, to find new non-geometric, static conceptual metaphors to replace the dynamic geometric metaphors. This re-metaphorisation of calculus involved three new conceptual metaphors:

- *A Line Is a Set of Points.*
- *Continuity Is Gaplessness.*
- *Approaching a Limit Is Preservation of Closeness Near a Point.*

1. “A Line Is a Set of Points”

How do we get from a dynamic geometrised calculus to a static arithmetised calculus? Start with the Number-Line Blend and gradually eliminate the geometry and the motion. The first step is to reconceptualise the line metaphorically as a set of points.

Traditionally, a line was seen as holistic—traced by continuous motion and not made up of discrete elements. Points were seen as *locations on* the line. The line as a whole was taken to be ontologically independent of any locations one might

happen to pick on it, just as a bird resting on a telephone line is ontologically independent of any location on the phone line.

According to the new metaphor for a line, *A Line Is a Set of Points*, a line is constituted by its points and does not exist independent of its points. It is a *composite* entity, rather than a holistic one—like a flock of birds. Just as there is no flock without the birds, so there is no line without the points.

It is important to understand how deep the traditional holistic conception of a line is. Consider how we think and talk about “a gap in a line”. To conceptualise this at all, one must be thinking of a traditional holistic line. Notice that it is “a gap *in* a line”; that is, the line is being conceptualised as a container relative to the “gap” and the “gap” is the empty interior of that container. This is especially clear when we think of “filling in” the gap, where the line as whole without the gap defines what counts as “filling” the gap. Moreover, it is “a gap *in a* line”. That is, there is only one line, which includes the space where the gap is. Consider the real line with a gap at $\sqrt{2}$. Notice that we do not speak of two lines with a gap between them, but one line with a gap “at” a location on that line. “The” line here is the holistic line.

Compare this with the composite line, which is just a set of points. If we literally conceptualise the number line in this way, it would make no sense to speak of a “gap in the” real line. The real line minus the point at $\sqrt{2}$ would be just the set of real numbers without $\sqrt{2}$. This set, as a set (not a holistic line), has no “gap”; it is complete in itself; it is the set it is. It is only relative to the holistic line that the notion “gap” makes sense here.

When mathematicians speak of the irrationals as “filling in the gaps in the rationals”, they again are taking the holistic line as a background, as defining what “filling in” means. In short, even though most mathematicians consciously believe the metaphor that *A Line Is a Set of Points*, they unconsciously use the traditional concept of the holistic line.

When mathematicians think of the line as a set of points, they usually think unconsciously in terms of a conceptual blend of the source and target domains of the *Line Is a Set of Points* metaphor. That is, they construct a conceptual blend of both the holistic line and the composite line—what we will call “The Pointset-Line Blend”—in which the line is *both* a set of points and a holistic line with points as locations. It is relative to this blended conception of the line that a line segment can both have length yet be composed of points of zero length, which add up to no length at all. From a cognitive perspective, the length of a line segment is a property of the holistic part of the Pointset-Line Blend, not of the composite part.

Thus, in the process of getting rid of the geometric portion of the number line, the holistic line is not cognitively eliminated but only further metaphorised as a set of points.

2. Natural Continuity Is Gaplessness

Dedekind considered “a one-dimensional or *linear continuum* to be, like a line segment, a *dense* aggregate with *no gaps*” (Kramer, 1970, p. 38). Notice that, in using the term “aggregate”, Dedekind is viewing the line as a set of points—a composite line. Notice also that, in seeing that “aggregate” as having “no gaps”, he is using the concept of a gap, which makes sense only relative to the holistic line. In short, Dedekind is conceptualising the line using both concepts at once. That is, he is thinking in terms of the Pointset-Line Blend.

Dedekind is, of course, celebrated for defining the real numbers in terms of what has come to be called “the Dedekind cut”. Dedekind conceived of the irrational numbers as filling the “gaps” in the rationals. That is, he thought about irrationals in terms of the Number-Line Blend, with the rationals spread out along the line. From this geometric perspective, an irrational number I was a point on the line, dividing or “cutting” the line into two parts, A and B , with A containing all the rationals less than I , and B containing all the rationals greater than I . Dedekind then got rid of the geometry by creating a new metaphor using only numbers and sets. In that metaphor, the set of rational numbers is divided into two subsets, A and B . We then form the ordered pair (A, B) . If A has a largest rational, or B has a smallest rational, then that rational is conceptualised metaphorically as the pair (A, B) . But if A has no largest rational and B has no smallest rational, then the pair (A, B) is conceptualised metaphorically as *being* the irrational number I . This is the same I that, in the number-line blend, would “cut” the rationals into sets A and B . The “cut”, of course, is not performed on sets but is (unconsciously) performed on the holistic part of the Pointset-Line Blend. In short, the Dedekind Cut Metaphor states: *A real number is an ordered pair of sets (A, B) of rational numbers, with all the rationals in A being less than all the rationals in B . Through this conceptual metaphor we have the domain of real numbers (target domain) being conceptualised in terms of an ontologically different domain, the domain of ordered pairs of sets of rational numbers (source domain). (For details, see Lakoff & Núñez, 1999.)*

The Dedekind Cut Metaphor is stated in such a way as to guarantee that there will be no “gaps” in the real numbers. The reason is that *all* of the pairs (A, B) of rationals (with all the rationals in A being less than all the rationals in B) are metaphorically defined as *being* the real numbers. There are no more left. Hence, there can be no gaps between the real numbers. “Continuity” can then be metaphorically defined for the real numbers so defined, which is an aggregate of discrete entities: Continuity Is Gaplessness. The result is a metaphorical “continuum” of discrete entities: the “real number continuum”—the “gapless” sequence of all real numbers defined metaphorically as “cuts” without any geometry.

This kind of metaphorical “continuity” is not our everyday notion of what continuity is about. It is not the *natural continuity* of the holistic line; that is, it is not the continuity of our image-schematic concept of a path in the source–path–goal schema. Natural continuity is a product of our everyday conceptual systems. And it is this concept of continuity that students have when they come into a mathematics class. Natural continuity for a holistic line is fundamentally different to Dedekind’s metaphorical notion of “continuity” as “gaplessness” in the set of discrete real numbers defined metaphorically as “cuts”.

Since textbooks try to give intuitive motivation for the gaplessness notion of continuity using the natural notion of continuity, students are faced with two fundamentally different ideas, both given the same name. Moreover, they are told—incorrectly—that the gaplessness version defines the essence of the natural version, which, as we have seen, is simply not true.

3. *Approaching a Limit Is Preserving Closeness Near a Point (Metaphorical Continuity for Functions)*

It is to Weierstrass that we owe the contemporary metaphor for continuity of functions in static, purely arithmetic terms. Weierstrass’ classic “ ϵ - δ definition of continuity”, as it is normally introduced in textbooks, goes as follows:

A function f is continuous at a number a if the following three conditions are satisfied:

1. f is defined on an open interval containing a ,
2. $\lim_{x \rightarrow a} f(x)$ exists, and
3. $\lim_{x \rightarrow a} f(x) = f(a)$

where $\lim_{x \rightarrow a} f(x)$ (the limit of the function f at a) is defined as follows:

Let a function f be defined on an open interval containing a , except possibly at a itself, and let L be a real number. The statement

$$\lim_{x \rightarrow a} f(x) = L$$

means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - L| < \epsilon$$

This definition of continuity is considered a very important technical and theoretical achievement, for the reasons given above. First, it eliminated geometry so that it could be applied straightforwardly to functions that did not fit the definition of a function as curve. Second, it fit the requirements of the Formal Foundations Principle.

For these reasons, it is widely accepted in mathematics as *the* definition of continuity, as if no other concept of continuity existed. It is seen as *the* ultimate

characterisation of continuity. Moreover, it is seen as a triumph of reason and rigour over the vagueness of intuition: that is, the concept of natural continuity for the holistic line—the normal concept that we use everyday, both outside of mathematics and in our attempt to understand this technical definition.

Today, this definition is taught in regular calculus courses all over the world, implicitly supporting the idea that, with rigour, mathematics is able to define the essence of objects and properties in a way that is completely independent of human intuitions and the peculiarities of the human mind. Generation after generation of students is trained in this mode of thinking, sustaining the widely spread view that mathematics is timeless, unique, absolutely objective, literal, disembodied, and independent of human understanding.

From a cognitive perspective, however, Weierstrass has given us a new metaphor for the continuity of a function in static, non-geometric terms to go with Dedekind's metaphors. Weierstrass' metaphor does the job that it was intended to do. But it is at a cost, since it differs considerably from our ordinary notion of natural continuity for functions conceptualised as holistic curves.

In Weierstrass' ϵ - δ definition of limit, there is no motion, no time, and no "approach". Instead, there are static elements. In this definition, there are no holistic lines and no holistic surfaces in the metaphorical ontology for the Cartesian plane. The "plane" itself is not what we naturally take a "surface" to be, but is a made up of a set of discrete elements, each of which is a pair of real numbers. Each real number is in turn conceptualised in terms of another metaphor—for example, Dedekind's "cut" metaphor. It is worth mentioning that this is not the only metaphor for the real numbers. Other metaphors, which have different entailments, are Real Numbers Are Infinite Nested Intervals and Real Numbers Are Limits of Infinite Sequences of Rational Numbers.

Weierstrass' ϵ - δ definition has more peculiarities. It calls for a *gapless* "open interval" of numbers. The notion of an "open interval" is not the same as the notion of an open interval on the holistic line—namely, a holistic line segment without end points. Rather, it is cognitively produced by a combination of metaphors—The Line Is a Set of Points metaphor and Dedekind's metaphors of Real Numbers Are Cuts and Continuity Is Gaplessness. This "open interval" is thus a set of discrete numbers, not a line segment without endpoints. To comprehend it takes at least three metaphors.

The idea of the function f approaching a limit L as x approaches a is replaced by a different idea in order to arithmetise and to avoid motion. The new idea is also metaphorical, that Approaching a Limit Is the Preservation of Closeness Near a Real Number, namely, that $f(x)$ is arbitrarily close to L when x is sufficiently close to a . The ϵ - δ condition expresses in formal logic exactly what "arbitrarily close" and "sufficiently close" mean.

Weierstrass characterises this new metaphor in two steps: first at a single arbitrary real number in a (gapless) "open interval", and then throughout that interval. His metaphor for continuity uses the same basic idea as his metaphor for

a limit: *preservation of closeness*. Continuity at a real number is conceptualised as preservation of closeness not just near a real number but also *at* it. Continuity of a function throughout an open interval is thus preservation of closeness near and at every real number in the interval. Since Weierstrass, preservation of closeness by a function has become a central idea in modern mathematics. Unfortunately, this idea is not presented as such in textbooks.

DISCUSSION

It should be clear by now that the two definitions of continuity—natural and ϵ - δ continuity—are radically different from a cognitive perspective. They are realised through completely different cognitive mechanisms. This does not mean that one is *essentially* better or worse than the other. Moreover, the terminology is confusing to beginning students, for whom continuity means only natural continuity for holistic curves. Epsilon-delta continuity is a technical concept, and the word “continuity” is a misleading term for it.

Epsilon-delta continuity serves an important function in mathematics, but it is important to see that it is not “superior” to natural continuity. It is simply a different concept with a different purpose. The difference can be seen in what each says about the function $f(x) = x \sin(1/x)$ when $x \neq 0$ and 0 when $x = 0$. This function is not naturally continuous, since it violates the idea that natural continuity is conceptualised in terms of motion, which has direction. As it “hits” zero, that function cannot be said to have any specific direction. Hence, it cannot be characterised in terms of motion and so is not naturally continuous. It is, however, ϵ - δ continuous, since it preserves closeness at every real number, including 0 . These are not contradictory results. They are simply different concepts applying differently to the same case. The result seems strange or counter-intuitive only if you believe that ϵ - δ continuity defines the *essence* of natural continuity. It does not, as this example and the above discussion should make clear; it is just a different concept that serves a different purpose. We are aware that a detailed analysis of the implications of our results for mathematics education go beyond the scope of this article. We must say, however, that in order to teach meaningful mathematics, students should be told how formal definitions arise through a series of conceptual metaphors and other natural cognitive mechanisms. They should be told that, underlying abstract mathematical concepts, there are human cognitive mechanisms that make these concepts possible.

Precision

What makes ϵ - δ continuity precise and formal? Often we are led to believe that it is the ϵ - δ portion of these definitions that constitutes the rigour of the arithmetisation of analysis and that the ϵ - δ portion is part of the arithmetisation. But the ϵ - δ aspect of the definition actually plays a far more limited role. All it

accomplishes is a precise characterisation of the notion “correspondingly”. We could use it perfectly well without arithmetisation in a precise geometric characterisation of “correspondingly”. For example, we could easily define limits and continuity of functions in dynamic terms (i.e. based on motion), by considering the values of $f(x)$ as getting “correspondingly” closer to L as x gets closer to a . For instance, $\lim_{x \rightarrow a} f(x) = L$ could be defined to mean:

for every $\epsilon > 0$, there exists a $\delta > 0$, such that
 as x moves toward a and gets and stays within the distance δ of a ,
 $f(x)$ moves toward L and gets and stays within the distance ϵ of L .

The Preservation of Gaplessness

Another interesting facet of the ϵ - δ definition is its interaction with the idea of “gaplessness”. Weierstrass formulates the definition of continuity with the explicit condition that the function is defined over an *open interval*. It assumes this open interval to be gapless. What the ϵ - δ definition does is to guarantee that (a) when lines are metaphorically conceptualised as sets of real numbers, and (b) when the input of the function is gapless, and (c) when the function preserves closeness in that gapless interval, then (d) the output is also gapless. In short, the ϵ - δ definition functions to preserve gaplessness.

Rigour

As we have seen, the ϵ - δ condition is separate from the arithmetisation of calculus, since it can equally well make precise the notions of continuity and limits for the dynamic geometric notion of a function. It is therefore not the ϵ - δ condition that makes calculus in the Weierstrass tradition “rigorous”. What makes it “rigorous” is a set of metaphors that allows us to replace dynamic geometric calculus with static arithmetic calculus.

So far as we can see, the static arithmetic metaphors are neither more nor less rigorous in any absolute sense than the dynamic geometric metaphors. The static arithmetic metaphors do, however, serve an important *cultural* function in the mathematical community. They allow one to satisfy the Formal Foundations Principle.

Now, we know from results in twentieth-century mathematics that the formal foundations movement collapsed. Even if you believed in it at some past date, there would be no good mathematical reason for believing in it now. Yet the community of mathematicians has, for the most part, preserved the Formal Foundations Principle. It is an anachronism, in a way. But it has come to serve a different function from the one it was intended to serve. It is used to define new forms of mathematics through formal axiomatisations. It is not unusual in cultural processes that a cultural anachronism has come to serve a new function when its original function is lost.

A century after Weierstrass' ϵ - δ definitions of limits and continuity were adopted, they have become a cultural anachronism. Their original function as part of a foundations movement has been lost. They are still taught, but not because anyone believes in the foundations movement. They are taught because another cultural anachronism, the Formal Foundations Principle, has also been retained by the mathematical community long after its function in the foundations movement of the last century ceased to be of any relevance. It is the Formal Foundations Principle that tells us that the Weierstrass ϵ - δ definitions are "rigorous". But that notion of rigour as guaranteeing secure foundations and absolute certainty within mathematics is itself an anachronism.

Mathematics is not what the foundations movement took it to be. It is not transcendental, abstract, disembodied, unique, and independent of anything physical including anything human. Instead, it is a product of the human body, brain, and mind and of human experience in the physical world. The Formal Foundations Principle does not characterise human mathematics. It is, however, part of metamathematics as it has been practised over the last century. As such, it is part of the subject matter for study in the cognitive science of mathematics.

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