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THE EILENBERG-BORSUK DUALITY THEOREM FOR METRIC SPACES

BY TAKEO AKASAKI

1. In [3], S. Eilenberg proved the following useful 'théorème de dualité' on extensions of maps (that is, continuous functions).

THEOREM. *Suppose A is a closed set of a compact metric (hence separable) space X with $\dim (X \setminus A) \leq n$. Then for each map $f: A \rightarrow S^k$ from A to the k -dimensional sphere, there exists a set $E \subset X \setminus A$, E closed in X with $\dim (E) < n - k$ such that f can be extended over $X \setminus E$. ($X \setminus A$ denotes the set-theoretic difference.)*

K. Borsuk showed [1] that the theorem of Eilenberg remained valid under the weaker assumption that the domain be a separable metric space and the range be a separable metric space which is $(k - 1)$ -connected and locally $(n - 1)$ -connected.

By an *absolute neighborhood retract* (ANR) we mean a space Y in the class M of all metrizable spaces such that every homeomorphic image of Y as a closed subspace of a space Z in M is necessarily a neighborhood retract of Z . Recent investigations have prompted the choice of our preferred class M for the definition of ANR [6]. On the one hand, ANR's for the class M can be ANR's for larger classes of spaces if suitable condition of completeness, separability, or compactness is satisfied. On the other hand, ANR's for the class of compact metrizable spaces and the class of separable metrizable spaces are merely special cases of ANR's for the class M . No other weakly hereditary topological class of spaces has such a favorable position.

The objective of this paper is to prove the above-mentioned theorem without the separability condition if the covering definition of dimension is used. We first prove the theorem for maps from a metric space to the k -dimensional sphere S^k by arguments as in [1]. Then we are able to prove the theorem for maps from a metric space into a metric ANR by means of S. T. Hu's bridge theory technique [5]. This method of proof enables us to generalize for metric ANR's some classical theorems which were proved for separable metric ANR's by Hu [5].

2. A metric space X is said to be of dimension $\leq n$, in symbol, $\dim X \leq n$ if and only if every open covering α of X has a locally finite open refinement β such that its nerve K_β contains no simplex of dimension greater than n .

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THEOREM 2.1. *Suppose A is a closed set of a metric space X with $\dim(X \setminus A) \leq n$. Then for each map $f: A \rightarrow S^k$ from A to the k -dimensional sphere, there exists a set $E \subset X \setminus A$, E closed in X with $\dim E < n - k$ such that f can be extended over $X \setminus E$.*

Before proceeding with the proof of (2.1), one should note that since K. Morita's sum theorems [9] in dimension theory will be used, this generalization, in view of P. Roy's result [10], is valid only in the sense ' $d_2(X)$, $d_3(X)$ ' of [8; 153] and $\dim X$ as defined above. It is known that these three definitions of dimension agree for all metric spaces.

The remainder of the present section is devoted to the proof of (2.1). Thus X is assumed to be an arbitrary metric space.

If A_1 , A_2 and B are mutually disjoint subsets of a space X , we say that A_1 and A_2 are *separated in X by B* if $X \setminus B$ can be expressed as the disjoint union of two sets, closed in $X \setminus B$ and containing A_1 and A_2 respectively, i.e. if there exist disjoint sets X_1 and X_2 for which $X \setminus B = X_1 \cup X_2$, $A_1 \subset X_1$, $A_2 \subset X_2$, with X_1 and X_2 both closed in $X \setminus B$. Note that X_1 and X_2 are also open in $X \setminus B$.

If A_1 and A_2 are separated by the empty set, we say simply that A_1 and A_2 are *separated in X* .

PROPOSITION 2.2. *If a space X has dimension ≤ 0 , then any two disjoint closed subsets can be separated in X .*

Proof. Let A_1 and A_2 be two disjoint closed sets in X and assume $\dim X = 0$. Since X is normal, there exist open sets L and M of X such that $A_1 \subset L$, $\bar{L} \subset M$, $\bar{M} \subset X \setminus A_2$. By the definition of dimension ($d_2(X)$), there is an open set V of X satisfying $\bar{L} \subset V \subset M$, $\dim(\partial V) = -1$. V is also closed in X since the intersection $X \cap \partial V$ is empty. Hence, $A_1 \cup V$ and $A_2 \cup (X \setminus V)$ are disjoint closed sets in X . Thus A_1 and A_2 are separated in X .

PROPOSITION 2.3. *If A_1 and A_2 are disjoint closed sets in a space X and A is a subset of X with $\dim A \leq n$, then there exists a closed set B separating A_1 and A_2 with $\dim(A \cap B) \leq n - 1$.*

Proof. Suppose $n = 0$ and assume $\dim A = -1$. Since X is normal, there exist two disjoint open sets U_1 , U_2 in X containing A_1 and A_2 respectively. Then the proposition follows by setting $B = X \setminus (U_1 \cup U_2)$. On the other hand, if $\dim A = 0$, we use the proof of [8; 16, 2F] in which the argument is based on two facts: (i) a statement exactly like (2.2), (ii) X is completely normal.

Suppose then that $n > 0$. By a theorem of K. Morita [9; 354], we have $A = D \cup E$ with $\dim D \leq n - 1$ and $\dim E \leq 0$. By applying this proposition for $n = 0$, we obtain a separation of A_1 and A_2 by a set B not meeting E . Hence $A \cap B \subset D$. But $\dim D \leq n - 1$ and since the notation of dimension is hereditary [9; 355], we have $\dim(A \cap B) \leq n - 1$.

PROPOSITION 2.4. *If A_1 and A_2 are two closed sets in X such that $\dim X \setminus (A_1 \cup A_2) \leq n$, then there exist two closed sets X_1 and X_2 in X satisfying the following conditions:*

- (i) $A_1 \subset X_1, A_2 \subset X_2$
- (ii) $X = X_1 \cup X_2$
- (iii) $\dim [(X_1 \cap X_2) \setminus (A_1 \cap A_2)] \leq n - 1$.

Proof. Let d be a distance function which defines the topology of X . Set $G_1 = \{x \in X \mid d(x, A_1) < \frac{1}{2}d(x, A_2)\}$ and $G_2 = \{x \in X \mid d(x, A_2) < \frac{1}{2}d(x, A_1)\}$. The open sets G_1 and G_2 of X contain $A_1 \setminus (A_1 \cap A_2)$ and $A_2 \setminus (A_1 \cap A_2)$ respectively. One can easily show that the sets $(\bar{G}_1 \setminus A_1)$ and $(\bar{G}_2 \setminus A_2)$ are disjoint. Let $A = A_1 \cup A_2$. The sets $(\bar{G}_1 \setminus A_1)$ and $(\bar{G}_2 \setminus A_2)$ are closed in $X \setminus A$. By a special case of (2.3), $G_1 \setminus A_1$ and $G_2 \setminus A_2$ can be separated in $X \setminus A$ by a closed set of dimension $\leq n - 1$. That is to say, there exists a closed set C of $X \setminus A$ with $\dim C \leq n - 1$ and there are two disjoint closed sets K_1 and K_2 of $(X \setminus A) \setminus C$ such that $G_1 \setminus A_1 \subset K_1, G_2 \setminus A_2 \subset K_2$ and $(X \setminus A) \setminus C = K_1 \cup K_2$. Set $B_1 = K_1 \cup C, B_2 = K_2 \cup C$ and observe that both $B_1 \cap G_2$ and $B_2 \cap G_1$ are empty. Finally, let $X_1 = B_1 \cup A$ and $X_2 = B_2 \cup A$. Clearly (i) and (ii) are satisfied. (iii) follows from the fact that $\dim [(X_1 \cap X_2) \setminus (A_1 \cap A_2)] = \dim (B_1 \cap B_2)$ which is $\leq n - 1$.

COROLLARY 2.5. *If in (2.4) the given sets A_1 and A_2 are disjoint, then the two closed sets X_1 and X_2 in X can be chosen so that the sets $X_1 \cap X_2$ and $A_1 \cup A_2$ are disjoint.*

Proof. The disjointness of A_1 and A_2 implies that the sets $A_1 \cap X_2$ and $A_2 \cap X_1$ in the proof of (2.4) are empty. Thus (2.5) is proved.

The proof of (2.1) can now be easily obtained by means of (2.5), (2.4) and arguments as in [1; 164]. We omit the complete proof.

3. Let X denote a paracompact Hausdorff space and A a closed subspace of X .

Consider an arbitrary open covering α of X and its nerve K_α geometrically realized as a simplicial polytope with the Whitehead topology. The non-void sets of $\{A \cap U \mid U \in \alpha\}$ form an open covering of the subspace A whose nerve L_α is a subpolytope of K_α .

Assume the open covering α of X to be locally finite. A map $\varphi_\alpha: X \rightarrow K_\alpha$ is called a *canonical map* of α if and only if, for each point $x \in X$, $\varphi_\alpha(x)$ is contained in the closed simplex of K_α whose vertices correspond to the members of α containing x . The existence of a canonical map is proved by Dowker [2; 202]. Clearly every canonical map φ_α sends A into L_α .

Let $f: A \rightarrow Y$ be an arbitrary map defined on A into any space Y . A map $\psi_\alpha: L_\alpha \rightarrow Y$ is called a *bridge map* for the given map f , if the partial map $\psi_\alpha \circ (\varphi_\alpha|_A)$ is homotopic with f for every canonical map $\varphi_\alpha: X \rightarrow K_\alpha$ of the covering α .

If a bridge map $\psi_\alpha: L_\alpha \rightarrow Y$ exists for f , then α is said to be a *bridge* for f .

THEOREM 3.1. *If Y is an ANR, then, for an arbitrary map $f: A \rightarrow Y$ defined on a closed subspace A of a paracompact Hausdorff space X , there is a locally finite open covering α of X which is a bridge for f .*

Proof. Since Y is a metric space, we may consider Y as a closed subset of a convex set Z in the Banach space $L = C(Y)$ of all bounded continuous real functions defined on Y [6, III, 2.1]. The hypothesis that Y is an ANR implies the existence of an open neighborhood V of Y in Z together with a retraction $r: V \supset Y$. Since V is an open set of a convex set Z in a locally convex space L , there exists an open covering $\{V_\mu \mid \mu \in M\}$ of V such that V_μ is convex for every $\mu \in M$. For each $\mu \in M$, let $W_\mu = V_\mu \cap Y$. Then $\{W_\mu \mid \mu \in M\}$ is an open covering of Y .

Now let $f: A \rightarrow Y$ be a given map. The inverse images $f^{-1}(W_\mu)$, $\mu \in M$, cover A . The paracompactness of A implies the existence of a locally finite open covering of A , $\alpha^* = \{a_\lambda^* \mid \lambda \in \Lambda\}$ with the property that for each $\lambda \in \Lambda$, $f(a_\lambda^*) \subset W_\mu \subset V_\mu$ for some $\mu \in M$. Since X is fully normal and A is closed in X , it follows by [4; 324] that there exists a locally finite open covering $\alpha = \{a_\lambda \mid \lambda \in \Lambda\}$ of X such that $a_\lambda^* = a_\lambda \cap A$ for every $\lambda \in \Lambda$.

It remains to show that α is a bridge for f . We omit this part of the proof since obvious modifications of the proof in [5; 341] yield the desired result. Thus the theorem is proved.

Throughout the remainder of this paper, X will denote a given metric space, A a closed subspace of X and Y a connected ANR (and hence pathwise connected). We shall denote by K^n the n -dimensional skeleton of a given complex K , that is, the totality of the closed simplexes of K with dimensions not exceeding n .

A map $f: A \rightarrow Y$ defined on A into Y is said to be n -*extensible over X* if f has a bridge α and a bridge map $\psi_\alpha: L_\alpha \rightarrow Y$ which has an extension $\psi'_\alpha: L_\alpha \cup K_\alpha^n \rightarrow Y$. Since Y is pathwise connected, the following proposition is evident.

PROPOSITION 3.1. *Every map $f: A \rightarrow Y$ is 1-extensible over X .*

For a given map $f: A \rightarrow Y$, the least upper bound of the set of integers n such that f is n -extensible over X is called the *extension index* of the map f . The following proposition is obvious from the very definition.

PROPOSITION 3.2. *Homotopic maps have the same extension index.*

THEOREM 3.3. *If $\dim(X \setminus A) \leq m$ and a given map $f: A \rightarrow Y$ is n -extensible over X , then there exists a closed set E of X contained in $X \setminus A$ such that $\dim E < m - n$ and f has an extension $f': X \setminus E \rightarrow Y$.*

Proof. By the definition of n -extensibility, there exists a bridge α and a bridge map $\psi_\alpha: L_\alpha \rightarrow Y$ for f , which has an extension $\psi'_\alpha: L_\alpha \cup K_\alpha^n \rightarrow Y$. Let $\varphi_\alpha: X \rightarrow K_\alpha$ be an arbitrary canonical map. We shall prove an assertion that *there exists a closed set E of X contained in $X \setminus A$ with $\dim E < m - n$ and*

a homotopy $h_t : X \setminus E \rightarrow K_\alpha$ ($0 \leq t \leq 1$) such that $h_0 = \varphi_\alpha \mid X \setminus E$, $h_1(X \setminus E) \subset L_\alpha \cup K_\alpha^n$, and $h_t(x) = \varphi_\alpha(x)$ for each $x \in \varphi_\alpha^{-1}(L_\alpha \cup K_\alpha^n)$ and each $0 \leq t \leq 1$.

Our assertion is trivial if $\dim L_\alpha \leq n$. Assume the assertion for $\dim L_\alpha \leq p-1$, and now suppose $\dim L_\alpha = p > n$. Let $\{\sigma_\lambda \mid \lambda \in \Lambda\}$ denote the collection of all p -dimensional closed simplexes of $\overline{K_\alpha \setminus L_\alpha}$. For each $\lambda \in \Lambda$, let $F_\lambda = \varphi_\alpha^{-1}(\sigma_\lambda)$, $B_\lambda = \varphi_\alpha^{-1}(\partial\sigma_\lambda)$. Then F_λ and B_λ are closed sets of X and $F_\lambda \setminus B_\lambda \subset X \setminus A$. Hence we have $\dim(F_\lambda \setminus B_\lambda) \leq m$ [9; 355, 5.1]. It follows from (2.1) that there is a closed subset H_λ of F_λ contained in $F_\lambda \setminus B_\lambda$ with $\dim H_\lambda \leq m - (p-1) - 1 < m - n$ such that the partial map $\varphi_\alpha \mid B_\lambda$ has an extension $\theta_\lambda : F_\lambda \setminus H_\lambda \rightarrow \partial\sigma_\lambda$. Let $H = \bigcup_{\lambda \in \Lambda} H_\lambda \subset X \setminus A$. We now show that the set H is closed in X . In order to prove this assertion we first show that the collection $\{H_\lambda \mid \lambda \in \Lambda\}$ is locally finite in X . For this purpose let x be an arbitrary point of X and let $U_0, U_1, \dots, U_k$ denote the members of α containing x . Let $\sigma_1, \sigma_2, \dots, \sigma_k$ be the p -dimensional closed simplexes of $\overline{K_\alpha \setminus L_\alpha}$ with vertices from $U_0, U_1, \dots, U_k$. As before, set $F_i = \varphi_\alpha^{-1}(\sigma_i)$ ($i = 1, 2, \dots, k$) and let $H_1, H_2, \dots, H_k$ be the closed subsets of $F_1, F_2, \dots, F_k$ respectively. It suffices to show that if x meets some H_λ from the collection $\{H_\lambda \mid \lambda \in \Lambda\}$, then H_λ is one of the sets $H_1, H_2, \dots, H_k$. Thus assume $x \in H_\lambda$ and denote the vertices of the p -dimensional simplex σ_λ by $U_{\lambda_0}, U_{\lambda_1}, \dots, U_{\lambda_p}$. Since $H_\lambda \subset F_\lambda \setminus B_\lambda$, $\varphi_\alpha(x)$ is an interior point of the closed simplex σ_λ . By definition of φ_α , this means that x belongs to the intersection $U_{\lambda_0} \cap U_{\lambda_1} \cap \dots \cap U_{\lambda_p}$ and therefore σ_λ is one of the simplexes $\sigma_1, \sigma_2, \dots, \sigma_k$. Hence H_λ is one of the sets $H_1, H_2, \dots, H_k$. This shows that $\{H_\lambda \mid \lambda \in \Lambda\}$ is a locally finite collection of X . It follows immediately from the definition of local finiteness that a point is a limit point of H if and only if it is a limit point of some H_λ and hence the closure of the union H is the union of the closures. Therefore, since H_λ is closed in X for every $\lambda \in \Lambda$, we have $\bar{H} = (\bigcup_{\lambda \in \Lambda} H_\lambda) = \bigcup_{\lambda \in \Lambda} \bar{H}_\lambda = \bigcup_{\lambda \in \Lambda} H_\lambda = H$ and hence H is closed in X .

Since $\{H_\lambda \mid \lambda \in \Lambda\}$ is locally finite and $\dim H_\lambda < m - n$ for every $\lambda \in \Lambda$, it follows by [9; 355, 5.2] that $\dim H < m - n$.

Define a function

$$\theta : X \setminus H \rightarrow L_\alpha \cup K_\alpha^{p-1}$$

by taking

$$\theta(x) = \begin{cases} \theta_\lambda(x) & \text{if } x \in F_\lambda \setminus H_\lambda, \\ \varphi_\alpha(x) & \text{if } x \in X \setminus \bigcup_{\lambda \in \Lambda} F_\lambda, \end{cases}$$

for every $x \in X \setminus H$. In order to verify that θ is continuous let us first observe that if $\lambda \neq \lambda'$, then $\theta_\lambda(x) = \theta_{\lambda'}(x)$ for each $x \in (F_\lambda \setminus H_\lambda) \cap (F_{\lambda'} \setminus H_{\lambda'})$. In fact, let x be an arbitrary point in the intersection

$$(F_\lambda \setminus H_\lambda) \cap (F_{\lambda'} \setminus H_{\lambda'}) = [\varphi_\alpha^{-1}(\sigma_\lambda) \setminus H_\lambda] \cap [\varphi_\alpha^{-1}(\sigma_{\lambda'}) \setminus H_{\lambda'}].$$

Then $\varphi_\alpha(x) \in \sigma_\lambda \cap \sigma_{\lambda'}$. Since σ_λ and $\sigma_{\lambda'}$ are different simplexes, $\varphi_\alpha(x) \in \partial\sigma_\lambda \cap \partial\sigma_{\lambda'}$.

and therefore $x \in B_\lambda \cap B_{\lambda'}$. Hence $\theta_\lambda(x) = \varphi_\alpha(x) = \theta_{\lambda'}(x)$. Second, consider an arbitrary boundary point x of $X \setminus \bigcup_{\lambda \in \Lambda} F_\lambda$ in $X \setminus H$. Then x is a boundary point of $F_\lambda \setminus H \subset F_\lambda \setminus H_\lambda$ for some $\lambda \in \Lambda$ in $X \setminus H$. But $F_\lambda \setminus B_\lambda$ is closed in $X \setminus H$ and thus x is a point of $F_\lambda \setminus H_\lambda$. Since θ_λ is an extension of φ_α from B_λ to $F_\lambda \setminus H_\lambda$, this shows that θ agrees on the intersection of the boundary $X \setminus \bigcup_{\lambda \in \Lambda} F_\lambda$ in $X \setminus H$ with $\bigcup_{\lambda \in \Lambda} (F_\lambda \setminus B_\lambda)$. Finally let x be an arbitrary point of $X \setminus H$ and let V be any open set of $L_\alpha \cup K_\alpha^{p-1}$ containing the point $\theta(x)$. Let τ be any closed simplex of $L_\alpha \cup K_\alpha^{p-1}$ containing $\theta(x)$. By the Whitehead topology, $V \cap \tau$ is an open set of $L_\alpha \cup K_\alpha^{p-1}$ and contains $\theta(x)$. Hence there is an open set W of $X \setminus H$ containing x such that $\theta(W) \subset V \cap \tau \subset V$. This shows that the function θ is continuous.

Define a homotopy $1_t : X \setminus H \rightarrow K_\alpha$ as follows. If x belongs to some $F_\lambda \setminus H_\lambda$, then both $\varphi_\alpha(x)$ and $\theta(x)$ are contained in the closed simplex σ_λ ; let $1_t(x)$ be the point which divides the line segment joining $\varphi_\alpha(x)$ to $\theta(x)$ in the ratio $t : (1 - t)$. Otherwise, let $1_t(x) = \varphi_\alpha(x)$. Then we have $1_0 = \varphi_\alpha \mid X \setminus H$, $1_1 = \theta$ and $1_t(x) = \varphi_\alpha(x)$ for each $x \in \varphi_\alpha^{-1}(L_\alpha \cup K_\alpha^{p-1})$ and each $0 \leq t \leq 1$.

By the induction hypothesis, it follows that there exists a closed subset J of $X \setminus H$ contained in $X \setminus (H \cup A)$ with $\dim J < m - n$ and a homotopy

$$g_t : x \setminus (H \cup J) \rightarrow L_\alpha \cup K_\alpha^{p-1} \quad 0 \leq t \leq 1$$

such that $g_0 = \theta \mid X \setminus (H \cup J)$, $g_1[X \setminus (H \cup J)] \subset L_\alpha \cup K_\alpha^n$, and that $g_t(x) = \theta(x)$ for each $x \in \theta^{-1}(L_\alpha \cup K_\alpha^n)$ and each $0 \leq t \leq 1$. Let $E = H \cup J$. Since J is closed in $X \setminus H$ and H in X , E is a closed set of X contained in $X \setminus A$. Since H is closed, it follows by means of [9; 355, 5.2] that $\dim E < m - n$.

Define a homotopy

$$h_t : X \setminus E \rightarrow K_\alpha$$

by taking

$$h_t(x) = \begin{cases} 1_{2t}(x) & 0 \leq t \leq 1/2 \\ g_{2t-1}(x) & 1/2 \leq t \leq 1 \end{cases}$$

for every $x \in X \setminus E$; then $h_0 = \varphi_\alpha \mid X \setminus E$, $h_1(X \setminus E) \subset L_\alpha \cup K_\alpha^n$, and $h_t(x) = \varphi_\alpha(x)$ for each $0 \leq t \leq 1$ and each $x \in \varphi_\alpha^{-1}(L_\alpha \cup K_\alpha^n)$. Hence our assertion is proved.

Finally, since $\psi'_\alpha \circ h_1 : X \setminus E \rightarrow Y$ is an extension of $\psi_\alpha \circ (\varphi_\alpha \mid A)$, and since A is closed in $X \setminus E$ and has the homotopy extension property in $X \setminus E$ with respect to Y , it follows that f has an extension $f' : X \setminus E \rightarrow Y$. This completes our proof of the theorem.

4. In the final section of this paper we give two applications of (3.3) for metric ANR's. The proofs of these statements involve the method of obstruction with Čech cohomology based on all open coverings. We omit the proofs since the arguments used in Hu's work [5] can clearly be modified for the present situation. These results were proved by Hu for separable metric ANR's.

THEOREM 4.1. Suppose A is a closed connected ANR in a metric space X with $\dim (X \setminus A) \leq m$. If A is r -simple and $H^r[X, A; \pi_r(A)] = 0$ for each $0 < r < n$, then there exists a closed set E of X contained in $X \setminus A$ with $\dim E < m - n$ and a retraction $\theta: X \setminus E \rightarrow A$.

THEOREM 4.2. Suppose Y is a connected metric ANR, $\dim Y \leq m$ and Y_0 is a connected closed ANR subspace of Y . Then Y_0 is a strong deformation retract of Y , if the following conditions are satisfied.

- (i) Y_0 is r -simple and $H^r[Y, Y_0; \pi_r(Y_0)] = 0$ for each $0 < r < m$.
- (ii) Y is r -simple and $H^r[Y; \pi_r(Y)] = 0$ for each $0 < r \leq m$.

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