Title
Levelwise Modules and Localization in Derivators

Author
Lagkas Nikolos, Ioannis

Publication Date
2018

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA

Los Angeles

Levelwise Modules and Localization in Derivators

A dissertation submitted in partial satisfaction

of the requirements for the degree

Doctor of Philosophy in Mathematics

by

Ioannis Lagkas Nikolos

2018

ABSTRACT OF THE DISSERTATION

Levelwise Modules and

Localization in Derivators

by

Ioannis Lagkas Nikolos

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2018

Professor Paul Balmer, Chair

We prove that the idempotent completion of a strong additive derivator $\mathbb{D}$ is also a strong additive derivator which is moreover stable if $\mathbb{D}$ is. This recovers the main results in [BS01] and [LS14] assuming that the (pre)triangulated categories we are working with are the base of a strong additive (stable) derivator (which happens in all examples one encounters in practice). We also show that given a separable cocontinuous monad on a triangulated derivator, the levelwise Eilenberg-Moore categories of modules glue together to a triangulated derivator. This allows us to give examples of derivators that are stable but not strong. Furthermore, we study localization of right derivators under classes that are closed under colimits and, as a special case, recover Franke's result [Fra96] that we can form Verdier quotients of small triangulated derivators. Finally, we study compact objects in big triangulated derivators. We show that any compact object can be written as a retract of a finite direct colimit of a coherent diagram that is pointwise in a chosen generating set. We also show that levelwise and pointwise compact objects over small diagrams coincide. As an application, we extend the equivalence of [Nee92] to an equivalence of $\mathbf{Dir}_{\mathrm{f}}$-derivators.

The dissertation of Ioannis Lagkas Nikolos is approved.

Michael Gutperle

Raphaël Alexis Rouquier

Alexander Sergee Merkurjev

Paul Balmer, Committee Chair

University of California, Los Angeles

2018

To my family…

for their unconditional love and support.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

First of all, I would like to thank my advisor, Paul Balmer, for his tremendous support over the past four years. I am sincerely grateful for all the time he has taken to guide me and for being really patient with me. Without him, this dissertation would certainly not have been possible.

I would also like to thank my other committee members: Raphaël Alexis Rouquier, Alexander Sergee Merkurjev and Michael Gutperle for their consideration and input.

I would also like to thank the UCLA Derivator seminar for the great research discussions we've had and the input I've received from them over the years. Special thanks to Ian Coley for his invaluable help in editing some parts of this thesis.

Last but not least, I would like to thank my parents and my brother for supporting me when I needed it the most and always believing I could make it; Nikos Angelopoulos, Dimitris Ntalampekos, Yiannis Kospentaris and George Derpanopoulos for being great friends to me throughout graduate school and Elena Ramirez for making my last year at graduate school special. I don't have enough words for how much I appreciate you all.

VITA

2007	B.Sc. (Mathematics), University of Ioannina, Greece.
2015	M.A. (Mathematics), UCLA.
2012-2018	Teaching Assistant, Department of Mathematics, UCLA.
2014-2018	Graduate Research, Department of Mathematics, UCLA.
2016	Graduate Student Instructor, Department of Mathematics, UCLA.

PUBLICATIONS

“Levelwise modules over separable monads on stable derivators". *Journal of Pure and Applied Algebra*. **222**(7):1704-1726. 2018

CHAPTER 0

Introduction

From a naïve point of view, homotopy theory can be thought of as the study of the localization of some category $\mathcal{M}$ by a class of morphisms $\mathcal{W}$ called weak equivalences. Without any further information however, this approach doesn't take us very far. For one, the resulting category could be too large, i.e. may not be locally small even if the category we start with is. Most importantly however, we cannot make useful constructions on the homotopy category, for instance because it will lack enough (co)limits. The usual remedy is to introduce some extra structure on $(\mathcal{M}, \mathcal{W})$ for example consider only model categories as introduced by Quillen [Qui67], or quasicategories as introduced by Boardman and Vogt [BV73] and studied further by Joyal [Joy02] and Lurie [Lur09]. The homotopy category will still lack (co)limits of course, but the extra structure allows us to consider a special type of (co)limits in $\mathcal{M}$ that are invariant under weak equivalences, the so-called homotopy (co)limits.

In the context of *stable* homotopy theory a model-free approach is given by triangulated categories, introduced by Verdier in his landmark thesis [Ver96]. Initially used in the context of homological algebra to study derived categories of Grothendieck abelian categories and in topology to study the stable homotopy category, triangulated categories have been used extensively in algebraic geometry and modular representation theory among other fields. For all their omnipresence however, triangulated categories suffer from some well-known defects, the most notable of which is the lack of functoriality of the cone construction, which in turn implies that there is no good notion of homotopy colimits in triangulated categories. Although we can still define homotopy colimits in some special cases, such as pushouts and sequenctial colimits, the objects thus defined will be unique up to *non-unique* isomorphism.

(Pre)derivators, introduced independently by Grothendieck [Gro90], Heller [Hel88] and Franke

[Fra96], can be thought of as an intermediate stage between a model and its associated homotopy category. This new approach to homotopy theory gives, as it turns out, a convenient framework for formalizing and proving universal properties up to homotopy (for instance Cisinski [Cis08] has given a universal characterization of the derivator of spaces as the free cocompletion of a point under homotopy colimits). The basic idea is that one should not only study the homotopy category of some model $\mathcal{M}$ but also the homotopy category[1] of functors from any small category to $\mathcal{M}$, together with the induced restrictions between them. We refer to the homotopy category of $\mathcal{M}$ as the base of the derivator, to homotopy categories of diagrams in $\mathcal{M}$ as *coherent diagrams* and to diagrams in the homotopy category of $\mathcal{M}$ as *incoherent diagrams*. A homotopy (co)limit is then simply a left (right) adjoint to the functor that sends any object to a *coherent* constant diagram. The cone construction also becomes functorial provided we interpret its domain as *coherent* morphisms[2]. Moreover, being triangulated is then a *property* of the derivator and *not* an extra structure that one needs to impose [Gro16].

Derivators as it turns out are a lot closer to a model than its homotopy category. For instance, Muro and Raptis introduced a certain simplicial enrichment of the category of prederivators [MR17] providing an upgraded notion of equivalence between derivators that encodes higher coherent data. The authors then define a notion of Waldhausen K-theory[3] of derivators that satisfies agreement and is invariant under this stronger notion of equivalences, while agreement fails [MR11] for the "naïve" derivator K-theory introduced by Maltsiniotis [Mal]. Whether this new K-theory satisfies additivity or localization remains to be seen.

More recently, Carlson [Car16] proved that small quasicategories can be simplicially and 2-categorically embedded into prederivators, extending in some sense an older result of Renaudin [Ren09] showing that a certain bilocalization of nice Quillen model categories up to Quillen equivalence embeds into derivators. In a somewhat different direction, Balmer and Zhang [BZ17] explain how coherent diagrams in a derivator encode a lot of information by showing how, given

[1] With respect to the pointwise weak equivalences

[2] Of course the cone on incoherent morphisms is still not functorial.

[3] For triangulated categories Schlichting [Sch02] showed that K-theory of exact categories cannot factor through triangulated categories so that localization is satisfied.

the derivator $\mathbb{D}_X$ associated to a quasi-compact and quasi-separated scheme X, one can recover the derivator of the affine space over X by considering an appropriate shift of $\mathbb{D}_X$.

Thus, derivators present a serious alternative to traditional ways of doing homotopy theory, and triangulated derivators provide a non-trivial yet minimal enhancement of triangulated categories. With the stable case in mind, we are thus led to consider which traditional constructions that are important to triangulated categories can also be made for derivators.

Our first consideration in this spirit is idempotent completion. We remind the reader that the idempotent completion of a triangulated category is still triangulated [BS01]. From the point of view of tensor triangular geometry, it is a harmless construction in the sense that it doesn't affect the spectrum [Bal05, Proposition 3.13]. However, in the landmark paper of Thomason and Trobaugh [TT90] it is shown that the compact objects in the Verdier quotient of the derived category of a nice scheme X by the derived category of complexes supported in the complement $Z := X - U$ of a nice open subscheme U, is the idempotent completion of the corresponding Verdier quotient of perfect complexes. This result was later generalized by Neeman [Nee92] to localizations of compactly generated triangulated categories by the localizing subcategory generated by a set of compact objects. A further generalization was provided by Balmer [Bal16] in the context of separable extensions of compactly generated triangulated categories. It is thus clear that if we use triangulated derivators as extensions of triangulated categories, it is important to know that the notion of triangulated derivators is invariant under idempotent completion.

The next construction we consider is levelwise modules over monads intrinsic to the 2-category of (pre)derivators. The discussion can be motivated as follows: given a monoid object A in a tensor triangulated category, we can consider the category of modules over it. This category is rarely useful from a homotopical perspective since it often fails to inherit a natural triangulated structure (see Example 3.6.12). A notable exception is when the monoid A is separable (see [Bal11]). Such monoids appear frequently in practice: commutative étale algebras in commutative algebra [Bal11, Corollary 6.6], étale extensions in algebraic geometry [Bal16, Theorem 3.5 and Remark 3.8], $k(G/H)$ for subgroups $H < G$ of finite index in representation theory [Bal15, Theorem 1.2], $\Sigma^\infty(G/H)_+$ for H a closed subgroup of finite index of a compact Lie group in equivariant stable homotopy theory [BDS15, Theorem 1.1], and in other equivariant settings [BDS15, Theorems

1.2, 1.3]. More generally, in [Bal11] the author proves that given a separable exact monad on a triangulated category $\mathscr{C}$, the category of modules over that monad inherits a triangulation from that of $\mathscr{C}$ in a compatible way (this includes Bousfield localization as a special case). Our purpose is to prove a similar statement for triangulated derivators:

Theorem (3.4.4)**.** *Let $M : \mathbb{D} \to \mathbb{D}$ a cocontinuous separable monad on a triangulated idempotent-complete derivator $\mathbb{D}$. Then the levelwise Eilenberg-Moore categories of M define a triangulated derivator M-$\mathbf{Mod}_{\mathbb{D}}$.*

In particular, our theorem gives a new proof that modules over a separable monoid in a tensor triangulated category $\mathscr{C}$ inherit a natural triangulation from that of $\mathscr{C}$ if $\mathscr{C}$ is the base of a triangulated monoidal derivator $\mathbb{D}$ (see Example 3.6.10). It is noteworthy that if we drop the separability assumption on M, we can still show that M-$\mathbf{Mod}_{\mathbb{D}}$ is a stable derivator (see Proposition 3.5.1) but it might not be strong[4]. This allows us to construct examples of derivators that are stable but not strong (see Corollary 3.6.11). Moreover, it explains why the base of M-$\mathbf{Mod}_{\mathbb{D}}$ might fail to be triangulated: our (potential) inability to turn an incoherent M-linear morphism f to a coherent one is exactly the obstruction to constructing an action of M on the cone of f in a way that is compatible with the triangulation.

This naturally leads to the question of localization for derivators, as reflective localizations is a special case of modules over idempotent monads. Reflective localizations have been studied in the literature, initially by Heller [Hel88] and more recently by Loregian [Lor18] where a connection is made with weak factorization systems. However, even in the well-behaved stable case, one cannot always expect localization to admit a right adjoint. While this will typically be true of "large" triangulated categories, it will consistently fail for small ones for instance the compact objects in a big triangulated category.

It thus makes sense to study localizations of derivators which do not admit an adjoint. In general, this type of question amounts to the study of derived functors, see Remark 4.2.19. An exception is when the class we are trying to invert is closed under colimits (cf. Definition 4.2.12):

[4]In the literature, the axioms for a stable derivator include strongness; we depart from this by calling a derivator stable if it satisfies all axioms in the literature *except* strongness (see Definitions 1.7.1 and 1.2.8) and reserving the term "triangulated" for derivators that are both (see Definition 1.7.2).

Theorem (4.2.17)**.** *Let $\mathbb{D}$ be a right derivator and $\mathcal{S}$ a class of morphisms in $\mathbb{D}(e)$ such that $\mathcal{S}$ is saturated and closed under colimits. Moreover, assume that either $\mathbb{D}$ satisfies $(Der5)_{\ulcorner}$ or that $(\mathbb{D}(J), \mathcal{S}_J)$ admits a left calculus of fractions for each category $J \in$ **Dia**. Then $\mathbb{D}[\mathcal{S}^{-1}]$ is a right derivator that satisfies $(Der5)_{\ulcorner}$ if $\mathbb{D}$ does, and the localization morphism $\gamma : \mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$ is cocontinuous.*

Here a right derivator refers to a derivator without limits (cf. Definition 1.2.18). When we say that $\mathbb{D}$ satisfies $(Der5)_{\ulcorner}$, we mean it satisfies "strongness for upper left corners" (cf. Definition 4.2.6). We also remind the reader that a class $\mathcal{S}$ of morphisms in some category $\mathcal{C}$ is saturated if it coincides with the inverse image of isomorphisms under the canonical functor $\gamma : \mathscr{C} \to \mathcal{C}[\mathcal{S}^{-1}]$. Theorem 4.2.17 and its dual imply as a special case Franke's result that Verdier quotients of triangulated derivators are still triangulated derivators[5].

Let us now give a quick overview of the organization of this manuscript:

Chapter 1 starts with an overview of the basics of the theory of derivators and establishes terminology and notation. Using only "diagramatic methods", we establish the following new result[6]:

Theorem (1.4.8)**.** *Let $\mathbb{D}$ be a right $\mathbf{Dir}_f$-derivator. Then:*

(i) A full replete subcategory of $\mathbb{D}(e)$ that is closed under pushouts and contains the initial object of $\mathbb{D}(e)$ is closed under all finite direct colimits.

(ii) A right exact morphism of $\mathbf{Dir}_f$-derivators: $F : \mathbb{D} \to \mathbb{E}$ is cocontinuous.

This is reminiscent of the usual categorical fact that finite colimits can be built out of coproducts and coequalizers; in fact it is an exact analogue provided we interpret homotopy colimits as being

[5] At least when the triangulated derivator in question is defined over all finite direct categories, see Remark 4.3.3.

[6] This is already known to be true for derivators of domain **Cat** [PS16, Theorem 7.1]. However, we shouldn't need existence of all homotopy colimits to prove something that concerns only finite direct ones. The proof in [PS16] doesn't immediately generalize to $\mathbf{Dir}_f$ though, because it relies crucially on the universal property of the homotopy derivator of spaces [Cis08], namely that in some sense, the homotopy derivator of spaces is the free cocompletion of the terminal derivator under all homotopy colimits. We do not know if a similar result for finite CW-complexes in the context of $\mathbf{Dir}_f$-derivators is true.

a relative notion of the subcategory in question relative to the ambient (right) derivator. We then review pointed, additive and stable derivators, and we prove that in an additive derivator we can lift an idempotent of an incoherent morphism to an idempotent of a coherent morphism (cf. Corollary 1.6.9). This relies on the fact that in an additive derivator, if we have two composable morphisms of coherent diagrams both of which are pointwise zero, then the composite of the coherent morphisms must be zero (cf. Theorem 1.6.6). This last theorem is a generalization of the relevant fact for triangulated derivators [Gro11, Part 3, Theorem 3.8].

In Chapter 2 we study idempotent completion of derivators. We begin the section by a general overview of idempotent completion in categories, and explain why idempotent completion of a general derivator will admit homotopy (co)limits in a compatible way but may not be a derivator because we don't know if isomorphisms of coherent diagrams can be tested pointwise. The situation is remedied by passing to additive derivators and, moreover, we show that the idempotent completion of a triangulated derivator is again triangulated. Thus, this recovers the result of [BS01] and [LS14] in the case that our (pre)triangulated category is the base of a strong additive (stable) derivator.

In Chapter 3 we study levelwise modules over monads in the 2-category of derivators, or more generally prederivators. We begin by reviewing the theory of 2-monads and their modules, which we use extensively in the next sections to establish functoriality of pullbacks and Kan extensions in the (pre)derivator of levelwise modules. As expected, left Kan extensions are not for free in general, and come at the price of requiring the monad to be cocontinuous. In the last section, we prove the promised Theorem 3.4.4 and conclude the chapter by giving examples of stable non-strong derivators.

In Chapter 4, we study localization theory of derivators. We have already remarked that reflective localizations have been studied in [Hel88] and [Lor18]. A possible definition of localization is Cisinski's left Bousfield localization (cf. [CT11, Definition A.4] or [Tab08, Definition 4.2]) as an initial cocontinuous morphism from a given derivator to another, that inverts a class of morphisms on the base. For a localization via cocontinuous and idempotent monads (see Lemma 4.4.2) or for the derivator associated to a combinatorial model category (see Example 4.4.3), the two approaches are equivalent. We take an alternate approach instead. We do not define a localization of derivator as universal among derivators, but as a universal prederivator inverting at each level morphisms that are pointwise in a chosen class $\mathcal{S}$ which we think of as weak equivalences. We then proceed to prove the main Theorem 4.2.17 and to give examples in the context of nice Waldhausen categories or more generally right derivable categories. We also make a connection between reflective localizations and monads as developed in Chapter 3. For a more detailed treatment of reflective localizations and the connection with weak factorization systems, the reader is referred to [Lor18].

In the last chapter, we study compact objects in big triangulated derivators. We begin by showing that when our derivator can be defined over all of **Cat** then levelwise compact objects over small diagrams coincide with pointwise compact objects. We then show that any compact object in the base of such a derivator can be obtained as a retract of a finite direct colimit of a coherent diagram that is pointwise in a chosen generating set. As a last application, we study localizations of big compactly generated triangulated derivators and we prove a derivator version of [Nee92]:

Theorem (5.3.3)**.** *Let $\mathbb{T}$ be a big* **Cat***-triangulated derivator and $\mathcal{E}$ a set of compact objects in $\mathbb{T}(e)$ closed under arbitrary suspensions. Let $\mathcal{L}$ be the localizing subcategory generated by $\mathcal{E}$ and $\mathbb{L}$ the associated triangulated derivator (cf. Proposition 5.3.1) and $\mathbb{S} := \mathbb{T}/\mathbb{L}$ the associated Verdier localization 5.3.2. Then there is an equivalence of* $\mathbf{Dir}_f$*-derivators:*

$$(\mathbb{T}^c/\mathbb{L}^c)^\natural \to \mathbb{S}^c$$

where $(-)^c$ refers to taking compact objects (cf. Proposition 5.1.4) and $(-)^\natural$ refers to idempotent completion (cf. Construction 2.2.2 and Theorem 2.3.10).

CHAPTER 1

Preliminary results

1.1 Foundations

In order to set the foundations for category theory, we first recall the following standard definition, originally in [SGA72]:

Definition 1.1.1. A set $\mathcal{U}$ is called a **(Grothendieck) universe** if it satisfies the following axioms:

(U.I) The set $\mathcal{U}$ is transitive: if $x \in \mathcal{U}$ and $y \in x$ then $y \in \mathcal{U}$.

(U.II) If $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$.

(U.III) If $x \in \mathcal{U}$ then also $\mathcal{P}(x) \in \mathcal{U}$ where $\mathcal{P}(x)$ denotes the power-set of x (the set of all subsets of x).

(U.IV) If $I \in \mathcal{U}$ and if $\{x_a\}_{a \in I}$ is a family of elements of $\mathcal{U}$ then also $\bigcup_{a \in I} x_a \in \mathcal{U}$.

Given a Grothendieck universe $\mathcal{U}$, a **$\mathcal{U}$-small set** (or simply a **$\mathcal{U}$-set**) is just an element of $\mathcal{U}$, while a **$\mathcal{U}$-large set** is any set that is not a $\mathcal{U}$-set. Note that while elements of $\mathcal{U}$ are $\mathcal{U}$-small, sets of elements of $\mathcal{U}$ (i.e. elements of the power-set $\mathcal{P}(\mathcal{U})$ of $\mathcal{U}$) are generally $\mathcal{U}$-large. The connection with categories is given in the following standard definition (cf. [DHK04] for instance):

Definition 1.1.2. Given a Grothendieck universe $\mathcal{U}$, a category $\mathscr{C}$ will be called a **$\mathcal{U}$-locally small category** (or simply a **$\mathcal{U}$-category**) if all its Hom-sets are $\mathcal{U}$-sets and its set of objects is a subset of $\mathcal{U}$. A **$\mathcal{U}$-small category** is a $\mathcal{U}$-category such that its set of objects is actually a $\mathcal{U}$-set.

We will assume the reader is familiar with some basic 2-category theory; if not, he or she is referred to [KS74] or [Bor94a]. All 2-categories and 2-functors under consideration will be strict.

Usual composition of morphisms in a category or 2-category will be denoted by $\circ$ or simply by concatenation. The same convention will apply to vertical composition of 2-cells in a 2-category. For horizontal composition, we will use $*$ instead. Identity 1- or 2- cells will generally be denoted by id or 1, possibly with an appropriate subscript. Given a 2-category $\mathcal{K}$, 1-cells $f : x \to y$ and $g : z \to w$ and a 2-cell $\alpha : h \Rightarrow h' : y \to z$, we will write $\alpha * f$ or simply αf for the horizontal composite $\alpha * 1_{\mathrm{f}}$ and $g * \alpha$ or simply $g\alpha$ for the horizontal composite $1_{\mathrm{g}} * \alpha$. Finally, we will use the notation $\underline{\mathsf{Hom}}(-,-)$ for the category of functors between two categories (to be distinguished from the class of functors).

Definition 1.1.3. Given a Grothendieck universe $\mathcal{U}$, a 2-category $\mathcal{K}$ will be called a **$\mathcal{U}$-locally small 2-category** (or simply a **$\mathcal{U}$-2-category**) if all of its Hom-categories are small $\mathcal{U}$-categories and its set of objects is a subset of $\mathcal{U}$. A **$\mathcal{U}$-small 2-category** is a $\mathcal{U}$-2-category such that its set of objects is actually a $\mathcal{U}$-set.

Following the approach in [DHK04], in what follows we fix a Grothendieck universe $\mathcal{U}$ with cardinality bigger than $\aleph_0$. We will also assume that there is a unique successor Grothendieck universe $\mathcal{U}^+$ of our chosen Grothendieck universe $\mathcal{U}$, i.e. a unique smallest Grothendieck universe $\mathcal{U}^+$ such that the set $\mathcal{U}$ is a $\mathcal{U}^+$-set. The following terminology/conventions will be used:

- We will refer to $\mathcal{U}^+$-sets as **classes**, to $\mathcal{U}^+$-large sets as **large classes** and to $\mathcal{U}$-sets as just **sets**.
- Unless otherwise specified by category we will mean a $\mathcal{U}^+$-small category. We will refer to categories that are not $\mathcal{U}^+$-small as **large categories** and to $\mathcal{U}$-small categories as **small categories**. We will also use the term **locally small category** to refer to a $\mathcal{U}$-category. The same terminology will be used for 2-categories.

As usual, we will denote by **CAT** the large[1] category of categories, functors and natural transformations, and by **2-CAT** the large[1] 2-category of 2-categories, 2-functors and 2-natural transformations. By restricting the objects to locally small categories and locally small 2-categories

[1]Elements of **CAT** are $\mathcal{U}^+$-small categories. One can show that the category of functors between $\mathcal{U}^+$-small categories is again $\mathcal{U}^+$-small. Hence **CAT** is actually a $\mathcal{U}^+$-locally small category (but not $\mathcal{U}^+$-small). Similarly, **2-CAT** is $\mathcal{U}^+$-locally small but not $\mathcal{U}^+$-small.

respectively (but keeping all the morphisms and 2-cells) we get the locally small 2-categories **Cat** and **2-Cat** respecively.

1.2 Definition and examples

A prederivator is best thought of as a **CAT**-valued presheaf on **Cat**, which we think of as a category of diagrams. However, sometimes we need to restrict to smaller categories of diagrams, which should satisfy some further properties. To define that a category of diagram is, let us recall first, that given two functors $u : I \to K$ and $v : J \to K$, the comma category $(u \downarrow v)$ is defined as follows:

- objects are pairs (i, j, a), where i, j are objects of the categories I, J respectively and
$$a : u(i) \to v(j)$$
is a morphism in K and
- morphisms $(i, j, a) \to (i', j', a')$ are pairs (f, g) of morphisms $f : i \to i'$ and $g : j \to j'$ in I and J respectively such that the following diagram in K:
$$\begin{tikzcd} u(i) \arrow[r, "u(f)"] \arrow[d, "a"'] & u(i') \arrow[d, "a'"] \\ v(j) \arrow[r, "v(g)"'] & v(j') \end{tikzcd}$$
commutes.

Any such comma category is equipped with functors $\mathrm{pr}_I : (u \downarrow v) \to I$ and $\mathrm{pr}_J : (u \downarrow v) \to J$ mapping (i, j, a) to i and j respectively. There is also a natural transformation as in the following square:

(1.2.1)
$$\begin{tikzcd} (u \downarrow v) \arrow[r, "\mathrm{pr}_I"] \arrow[d, "\mathrm{pr}_J"'] \arrow[dr, phantom, "\overset{\alpha}{\swArrow}"] & I \arrow[d, "u"] \\ J \arrow[r, "v"'] & K \end{tikzcd}$$

whose value at an object (i, j, a) of $(u \downarrow v)$ is just the arrow a. We will refer to the square (1.2.1) as the **comma square associated to $(u \downarrow v)$**.

Of particular interest is the case when either I or J is the terminal category. The terminal category, i.e. the category with only one object and one arrow, will henceforth be denote by e. For a category J, we will denote by $\pi_J : J \to e$ the unique functor to the terminal category.

Given a category K and any object k of K, we will identify it with the functor $e \to K$ that maps the unique object of e to k. The comma categories $(u \downarrow k)$ and $(k \downarrow u)$ for a functor $u : J \to K$ and an object $k \in K$ are sometimes called the (op)lax fibers of u, and they have associated comma squares as in (1.2.1). We remark that objects in those categories can just be described as pairs instead of triples, and morphisms as just arrows in J instead of pairs of arrows[2]. The *strict fiber* of u over an object $k \in K$ is the category $u^{-1}(k)$ with objects those objects $j \in J$ such that $u(j) = k$ and morphisms $j \to j'$ those morphisms of J mapping to the identity on k. Finally, we will write $(K \downarrow k)$ instead of $(\mathrm{id}_K \downarrow k)$ and $(k \downarrow K)$ instead of $(k \downarrow \mathrm{id}_K)$.

Definition 1.2.2. A full[3] 2-subcategory **Dia** of **Cat** is a **diagram category** if:

(i) **Dia** contains all finite posets.

(ii) **Dia** is closed under pullbacks and finite coproducts.

(iii) For any category $J \in$ **Dia** and any object $j \in J$ the slice constructions $(J \downarrow j)$ and $(j \downarrow J)$ also belong to **Dia**.

(iv) If $J \in$ **Dia** then also $J^{\mathrm{op}} \in$ **Dia**.

(v) If $u : J \to K$ is a Grothendieck fibration (see [Bor94b, Chapter 8]) such that $K \in$ **Dia** and for any object $k \in K$ the fiber $u^{-1}(k) \in$ **Dia**, then also $J \in$ **Dia**.

A diagram subcategory of **Dia** is a full[3] 2-subcategory that is also a diagram category.

Example 1.2.3. Examples of diagram categories include **Cat**, the 2-category $\mathbf{Pos}_{\mathrm{f}}$ of finite posets and the 2-category $\mathbf{Dir}_{\mathrm{f}}$ of finite direct categories[4].

[2] Namely, we omit the unique object and unique arrow in e from the notations.

[3] Meaning we just restrict the objects.

[4] A finite direct category is one whose nerve has only finitely many non-degenerate simplices. Equivalently, it is finite and skeletal and has no non-trivial endomorphisms.

The following lemma gathers some properties of diagram categories that are straightforward consequences of the axioms, hence the proof is omitted:

Lemma 1.2.4. *Let* **Dia** *be a diagram category. Then:*

(i) If $u : J \to K$ is a Grothendieck opfibration such that $K \in$ **Dia** *and for any object $k \in K$ the fiber $u^{-1}(k) \in$* **Dia***, then also $J \in$* **Dia**.

(ii) For any functor $u : J \to K$ in **Dia** *and any object $k \in K$, the slice categories $(u \downarrow k)$ and $(k \downarrow u)$ belong to* **Dia**.

(iii) For any functors $u : I \to K$ and $v : J \to K$ in **Dia***, the slice category $(u \downarrow v)$ also belongs to* **Dia**.

Definition 1.2.5. Let **Dia** a diagram category. A **Dia**-prederivator is a (strict) 2-functor

$$\mathbb{D} : \mathbf{Dia}^{1\text{-op}} \to \mathbf{CAT}$$

Here "1-op" refers to reversing the direction of 1-cells in **Dia**. For the rest of this paper, **Dia** will refer to a fixed diagram category, and all prederivators will be **Dia**-prederivators unless specified otherwise. The terminology of the previous section will apply to $\mathbb{D}$ referring to its values. Thus, a locally small prederivator is a prederivator whose values are locally small categories etc.

Example 1.2.6.

(i) Given a category $\mathscr{C}$, the assignment $J \mapsto \mathscr{C}^J$ defines a **Cat**-prederivator $y\mathscr{C}$, called the **prederivator represented by** $\mathscr{C}$.

(ii) A **relative category** is a pair $(\mathscr{C}, \mathcal{W})$ consisting of a (locally small) category $\mathscr{C}$ and a class of arrows $\mathcal{W}$ in $\mathscr{C}$ called weak equivalences. Given a relative category $(\mathscr{C}, \mathcal{W})$ and a small category J, denote by $\mathcal{W}_J$ the class of morphisms in $\mathscr{C}^J$ that are pointwise in $\mathcal{W}$. The assignment $J \mapsto \mathscr{C}^J[\mathcal{W}_J^{-1}]$ defines a **Cat**-prederivator. The values of this prederivator will generally not be locally small [5]. A notable exception is when our relative category is associated to a model category (see example 1.2.26(iii)).

[5] But they will still be $\mathcal{U}^+$-small.

(iii) Let $\mathbb{D}$ be a **Dia**-prederivator. Given any category J in **Dia**, the assignment $K \mapsto \mathbb{D}(J \times K)$ defines a **Dia**-prederivator $\mathbb{D}^J$. A prederivator obtained in this manner is called a **shift** of $\mathbb{D}$.

(iv) Given a prederivator $\mathbb{D}$, the **opposite prederivator** $\mathbb{D}^{\mathrm{op}}$ is defined by $J \mapsto \mathbb{D}(J^{\mathrm{op}})^{\mathrm{op}}$ (where the last two "op" refer to the formation of the opposite of a category).

Remark 1.2.7. Let $\mathbb{D}$ be a prederivator. We call $\mathbb{D}(e)$ the **underlying category or base of** $\mathbb{D}$. Given a functor $u : J \to K$ in **Dia**, we will write u^* for the image of u under $\mathbb{D}$. The same notation applies to a natural transformation $\alpha : u \Rightarrow v$.

Given any category $J \in$ **Dia** and any object $j \in J$, the **evaluation functor** $j^* : \mathbb{D}(J) \to \mathbb{D}(e)$ is defined as the $\mathbb{D}$-pullback along the functor $j : e \to J$. We will often denote j^*X by X_j for any object X in $\mathbb{D}(e)$ and similarly for morphisms. Putting all evaluation functors together, we obtain a functor $\mathrm{dia}_J : \mathbb{D}(J) \to \mathbb{D}(e)^J$ which we call the **underlying diagram functor**. We think of $\mathbb{D}(J)$ as coherent J-shaped diagrams in $\mathbb{D}$, and of $\mathbb{D}(e)^J$ as incoherent J-shaped diagrams in $\mathbb{D}$.

Similarly, given two categories J, K in **Dia**, we can define a **partial underlying diagram functor** $\mathrm{dia}_{J,K} : \mathbb{D}(K \times J) \to \mathbb{D}(J)^K$. We think of $\mathbb{D}(J)^K$ as $(J \times K)$-shaped diagrams in $\mathbb{D}$ that are coherent in the J-direction, and incoherent in the K-direction. See also [Gro13, p. 323].

Definition 1.2.8. A prederivator $\mathbb{D}$ is **strong** if it satisfies the following axiom:

- (*Der*5) For any category J in **Dia**, the partial underlying diagram functor

$$\mathrm{dia}_{J,[1]} : \mathbb{D}([1] \times J) \to \mathbb{D}(J)^{[1]}$$

  is full and essentially surjective.

We remark that there are competing definitions (cf. [Hel88]) who require the above axiom for all finite free categories instead of just the category [1]. Definition 1.2.8 gives a minimal framework to study triangulated derivators and is the one that will be used throughout this thesis. We do not know if the two definitions are equivalent.

We now introduce the useful notion of a semiderivator which is a "derivator without colimits".

Definition 1.2.9. A prederivator $\mathbb{D}$ is a **semiderivator** if it satisfies the following axioms:

- ($Der1$) For any finite family of categories $\{J_i\}$ in **Dia**, the canonical functor

$$\mathbb{D}\Big(\coprod J_i\Big) \to \prod \mathbb{D}(J_i)$$

  induced by all the pullbacks along the canonical inclusions $J_i \hookrightarrow \coprod J_i$, is an equivalence of categories. In particular, if $\emptyset$ is the empty category then $\mathbb{D}(\emptyset) \cong e$.

- ($Der2$) For any category $J \in$ **Dia** the underlying diagram functor $\mathrm{dia}_J : \mathbb{D}(J) \to \mathbb{D}(e)^J$ is conservative. That is, a morphism $f : X \to Y$ in $\mathbb{D}(J)$ is an isomorphism if and only if f_j is an isomorphism in $\mathbb{D}(e)$ for all objects $j \in J$.

Example 1.2.10.

(i) Given a category $\mathscr{C}$, the prederivator represented by $\mathscr{C}$ (cf. Example 1.2.6(i)) is a semiderivator.

(ii) A prederivator $\mathbb{D}$ is a semiderivator if and only if $\mathbb{D}^{\mathrm{op}}$ is.

(iii) A prederivator $\mathbb{D}$ is a semiderivator if and only if $\mathbb{D}^J$ is for any category $J \in$ **Dia**.

Some authors require an infinite version of axiom $Der1$. The following definition is taken from [Hor15]:

Definition 1.2.11. A semiderivator of domain **Dia** is **big** if **Dia** is closed under infinite coproducts and $Der1$ holds for arbitrary families (instead of just finite ones).

Definition 1.2.12. We say that a prederivator $\mathbb{D}$ **admits (homotopy) left Kan extensions** if for any functor $u : J \to K$ in **Dia**, the induced functor $u^* : \mathbb{D}(K) \to \mathbb{D}(J)$ has a left adjoint $u_!$. Dually, if for any such u the functor u^* has a right adjoint u_*, then we say $\mathbb{D}$ **admits (homotopy) right Kan extensions**.

Now consider a square in **Dia**:

(1.2.13)
$$\begin{tikzcd} J \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\swArrow\alpha"] & J' \arrow[d, "v"] \\ K \arrow[r, "g"'] & K' \end{tikzcd}$$

i.e. we have a natural transformation $\alpha : vf \Rightarrow gu$. Given any prederivator $\mathbb{D}$, applying it to this square gives a new one as follows:

(1.2.14)
$$\begin{tikzcd}
\mathbb{D}(J) \arrow[d, phantom, ""] & \mathbb{D}(J') \arrow[l, "f^*"'] \\
\mathbb{D}(K) \arrow[u, "u^*"] \arrow[ur, phantom, "\alpha^* \swarrow"] & \mathbb{D}(K') \arrow[l, "g^*"] \arrow[u, "v^*"']
\end{tikzcd}$$

Assuming that $\mathbb{D}$ admits left Kan extensions, the functors u^*, v^* have left adjoints $u_!$ and $v_!$ respectively. This allows us to define the Beck-Chevalley transform $\alpha_! : u_! f^* \Rightarrow g^* v_!$ of the natural transformation populating the square (1.2.14). We will refer to this transform as the **mate** of the square (1.2.13) (or even just the mate of α). More precisely, this is given by the pasting:

(1.2.15)
$$\begin{tikzcd}
\mathbb{D}(K) & \mathbb{D}(J) \arrow[l, "u_!"'] & \mathbb{D}(J') \arrow[l, "f^*"'] & \\
 & \mathbb{D}(K) \arrow[ul, bend left, "\mathrm{id}_{\mathbb{D}(K)}"] \arrow[u, "u^*"] \arrow[ul, phantom, "\epsilon \swarrow"] & \mathbb{D}(K') \arrow[l, "g^*"] \arrow[u, "v^*"'] \arrow[ul, phantom, "\alpha^* \swarrow"] & \mathbb{D}(J') \arrow[l, "v_!"] \arrow[ul, bend right, "\mathrm{id}_{\mathbb{D}(J')}"'] \arrow[ul, phantom, "\eta \swarrow"]
\end{tikzcd}$$

where ϵ is the counit of the adjunction $u_! \dashv u^*$, and η is the unit of the adjunction $v_! \dashv v^*$. That is, the mate $\alpha_! : u_! f \Rightarrow g^* v_!$ is given as the top arrow in the following defining commutative square:

(1.2.16)
$$\begin{tikzcd}
u_! f^* \arrow[r, "\alpha_!"] \arrow[d, "u_! f^* \eta"'] & g^* v_! \\
u_! f^* v^* v_! \arrow[r, "u_! \alpha^* v_!"'] & u_! u^* g^* v_! \arrow[u, "\epsilon g^* v_!"']
\end{tikzcd}$$

Dually, if $\mathbb{D}$ admits right Kan extensions, then the Beck-Chevalley transform of the square (1.2.14) gives us a natural transformation $\alpha_* : v^* g_* \Rightarrow f_* u^*$ which we will also refer to as the mate of (1.2.13) (or the mate of α). We refer the reader to [Gro13, §1.2] for more details and properties of these constructions.

Definition 1.2.17. Let $\mathbb{D}$ be a prederivator that admits left (respectively right) Kan extensions. A square (1.2.13) in **Dia** is called **$\mathbb{D}$-exact** if the mate $\alpha_!$ (respectively α_*) is an isomorphism.

Note that by [Gro13, Lemma 1.14], if $\mathbb{D}$ admits both left and right homotopy Kan extensions, then the two (seemingly different) definitions above agree.

Definition 1.2.18. A semiderivator $\mathbb{D}$ is a **right derivator** if it satisfies the following two axioms:

- ($Der3R$) $\mathbb{D}$ admits left Kan extensions.
- ($Der4R$) For any functor $u : J \to K$ in **Dia** and any object k of K, the comma square (1.2.1) associated to $(u \downarrow k)$ is $\mathbb{D}$-exact.

Remark 1.2.19. In the literature there is no overall concensus of the proper naming of semiderivators that satisfy axioms ($Der3R$) and ($Der4R$). Some authors choose to call such semiderivators left derivators (and relabel the axioms accordingly). Our terminology is rather inspired by the terminology for abelian categories, where a *left* adjoint is *right* exact. This terminology has persisted even among derivators (cf. Definition 1.3.10). We thus choose this convention to avoid the awkwardness of speaking of right exact morphisms of left derivators.

As in [Gro13, Proposition 1.7], we can show that all values of a right derivator admit finite coproducts (in particular initial objects, henceforth denoted by $\emptyset$) and all values of a big right derivator admit all coproducts. To establish more terminology, consider the category $\square$:

(1.2.20)
$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (1,0) \arrow[d] \\ (0,1) \arrow[r] & (1,1) \end{tikzcd}$$

We will use the notation $\ulcorner$ for the top left corner, i.e. the full subcategory missing $(1,1)$. The corresponding inclusion will be denoted by $i_\ulcorner$. An object $X \in \mathbb{D}(\square)$ will be called a **(coherent) square**.

Definition 1.2.21. Let $\mathbb{D}$ be a right derivator. A coherent square $X \in \mathbb{D}(\square)$ is called **cocartesian** if and only if it lies in the essential image of the functor $(i_\ulcorner)_! : \mathbb{D}(\ulcorner) \to \mathbb{D}(\square)$.

Note that $i_\ulcorner$ is fully faithful, and because homotopy left Kan extensions along fully faithful functors are also fully faithful (see [Gro13, Proposition 1.20]), we see that $X \in \mathbb{D}(\square)$ is cocartesian if and only if the counit $\epsilon_X : (i_\ulcorner)_! i_\ulcorner^* X \to X$ is an isomorphism.

Another important class of Kan extensions consists of (co)sieves. We recall the definition below:

Definition 1.2.22. A fully faithful inclusion $u : J \to K$ is called a **cosieve**, if whenever we have a morphism $u(j) \to k$ in K then k lies in the image of u.

The importance of cosieves comes from the following proposition:

Proposition 1.2.23. *Let $\mathbb{D}$ be a prederivator satisfying Der1, Der3R and Der4R. Consider a cosieve $u : J \to K$ in* **Dia**. *If an object $X \in \mathbb{D}(K)$ lies in the essential image of $u_!$, then we have $X_k \cong \emptyset$ for any object $k \in K - u(J)$, where $\emptyset$ is the initial object of $\mathbb{D}(e)$. The converse is true if $\mathbb{D}$ is a right derivator.*

Proof. By definition, there is an object $Y \in \mathbb{D}(J)$ such that $X \cong u_! Y$. Given an object $k \in K - u(J)$, by *Der*4*R* we have $X_k \cong (\pi_{(u\downarrow k)})_! \mathrm{pr}_J^* Y$, where $\mathrm{pr}_J : (u \downarrow k) \to J$ is the canonical projection. Since $u : J \to K$ is a cosieve and $k \notin u(J)$, the category $(u \downarrow k)$ is empty, from which the result follows using *Der*1. For the converse, cf. [Gro13, Lemma 1.21]. Note that the proof of the first part given over there cannot be applied here, since without *Der*2 we cannot use [Gro13, Proposition 1.20]. □

Definition 1.2.24. A semiderivator $\mathbb{D}$ is a **left derivator** if it satisfies the following two axioms:

- (*Der*3*L*) $\mathbb{D}$ admits right Kan extensions.
- (*Der*4*L*) For any functor $u : J \to K$ in **Dia** and any object $k \in K$, the comma square (1.2.1) associated to $(k \downarrow u)$ is $\mathbb{D}$-exact.

Dually to our remarks above, a left derivator admits finite products (in particular a terminal object, henceforth denoted by $*$) and a big left derivator admits all products. The full subcategory of (1.2.20) on all objects except $(0, 0)$ will be denoted by $\lrcorner$ and the corresponding fully faithful inclusion by $i_\lrcorner$. The dual of a cocartesian square is a cartesian square, the dual of a cosieve is a sieve, and there is a dual version of Proposition 1.2.23, which we won't make explicit here.

Definition 1.2.25. A **derivator** is a prederivator that is both a left and a right derivator.

Example 1.2.26.

(i) If $\mathscr{C}$ is a category, then the prederivator represented by $\mathscr{C}$ (see Example 1.2.6(i)) is a **Cat**-derivator if and only if $\mathscr{C}$ is bicomplete.

(ii) A prederivator $\mathbb{D}$ is a derivator if and only if its opposite $\mathbb{D}^{\mathrm{op}}$ is.

(iii) Given a combinatorial model category M with $\mathcal{W}$ as weak equivalences, the **Cat**-prederivator $\mathbb{D}_M$ associated to the relative category $(M, \mathcal{W})$ (see Example 1.2.6(ii)) is a derivator (see for instance [Gro13, Proposition 1.30]). More generally this is true for any model category (see [Cis03]).

(iv) Given an exact category $\mathcal{E}$, the assignment $J \mapsto D^b(\mathcal{E}^J)$ defines a $\mathbf{Dir}_{\mathrm{f}}$-derivator (see [Kel07]). Here D^b denotes the bounded derived category of an exact category.

1.3 Morphisms of prederivators

Definition 1.3.1. Let $\mathbb{D}, \mathbb{E}$ two prederivators with domain **Dia**. A **morphism of prederivators** from $\mathbb{D}$ to $\mathbb{E}$ is a pseudonatural transformation of 2-functors $\mathbf{Dia}^{1\text{-op}} \to \mathbf{CAT}$.

More precisely, a morphism of prederivators $F : \mathbb{D} \to \mathbb{E}$ consists of the following data:

- For any category $J \in \mathbf{Dia}$, a functor $F_J : \mathbb{D}(J) \to \mathbb{E}(J)$.
- For any functor $u : J \to K$ in **Dia**, a natural isomorphism $\gamma_u^F : u^* F_K \overset{\cong}{\Rightarrow} F_J u^*$ as indicated in the diagram:

$$\begin{tikzcd}
\mathbb{E}(J) & \mathbb{E}(K) \arrow[l, "u^*"'] \\
\mathbb{D}(J) \arrow[u, "F_J"] & \mathbb{D}(K) \arrow[l, "u^*"] \arrow[u, "F_K"'] \arrow[ul, phantom, "\gamma_u^F \Swarrow"]
\end{tikzcd}$$

subject to the following coherence conditions:

(i) For any category $J \in \mathbf{Dia}$ we have $\gamma_{\mathrm{id}_J}^F = \mathrm{id}_{F_J}$.

(ii) Given a pair of composable functors $J \xrightarrow{u} K \xrightarrow{v} L$ in **Dia**, the following pastings are equal:

$$\begin{tikzcd}
\mathbb{E}(J) & \mathbb{E}(K) \arrow[l, "u^*"'] & \mathbb{E}(L) \arrow[l, "v^*"'] \\
\mathbb{D}(J) \arrow[u, "F_J"] & \mathbb{D}(K) \arrow[l, "u^*"] \arrow[u, "F_K"] \arrow[ul, phantom, "\gamma_u^F \Swarrow"] & \mathbb{D}(L) \arrow[l, "v^*"] \arrow[u, "F_L"] \arrow[ul, phantom, "\gamma_v^F \Swarrow"]
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{E}(J) & \mathbb{E}(L) \arrow[l, "(vu)^*"'] \\
\mathbb{D}(J) \arrow[u, "F_J"] & \mathbb{D}(L) \arrow[l, "(vu)^*"] \arrow[u, "F_L"'] \arrow[ul, phantom, "\gamma_{vu}^F \Swarrow"]
\end{tikzcd}$$

In other words we have a commutative diagram:

(1.3.2)
$$\begin{tikzcd} u^*v^*F_L \arrow[r, "u^*\gamma_v^F"] \arrow[dr, "\gamma_{vu}^F"'] & u^*F_K v^* \arrow[d, "\gamma_u^F v^*"] \\ & F_J u^* v^* \end{tikzcd}$$

(iii) Given a natural transformation $\begin{tikzcd} J \arrow[r, bend left, "u"{name=U}] \arrow[r, bend right, "v"'{name=V}] & K \arrow[from=U, to=V, Rightarrow, shorten=2pt, "\alpha"] \end{tikzcd}$ in **Dia**, we have a commutative diagram:

(1.3.3)
$$\begin{tikzcd} u^*F_K \arrow[r, "\alpha^*F_K"] \arrow[d, "\gamma_u^F"'] & v^*F_K \arrow[d, "\gamma_v^F"] \\ F_J u^* \arrow[r, "F_J\alpha^*"'] & F_J v^* \end{tikzcd}$$

Such a morphism F is called **strict** if all its coherence isomorphisms γ_u^F are identities.

Given two morphisms F, G from a prederivator $\mathbb{D}$ to a prederivator $\mathbb{E}$ (both of the same domain **Dia**), a **modification** ρ from F to G consists of a natural transformation $\rho_J : F_J \Rightarrow G_J$ for each category $J \in$ **Dia** compatible with the coherence isomorphisms of F and G. More explicitly, for any functor $u : J \to K$ in **Dia**, we have a commutative diagram:

(1.3.4)
$$\begin{tikzcd} u^*F_K \arrow[r, "u^*\rho_K"] \arrow[d, "\gamma_u^F"'] & u^*G_K \arrow[d, "\gamma_u^G"] \\ F_J u^* \arrow[r, "\rho_J u^*"'] & G_J u^* \end{tikzcd}$$

We will denote by **PDer** the 2-category[6] of prederivators with 1-cells pseudonatural transformations, and 2-cells modifications (see [Gro13, Section 2]), and by **Der** its full 2-subcategory[6] spanned by derivators. We will write $\underline{\mathsf{Hom}}(\mathbb{D}, \mathbb{E})$ for the category of morphisms of (pre)derivators from $\mathbb{D}$ to $\mathbb{E}$.

Example 1.3.5.

(i) Any morphism of prederivators $F : \mathbb{D} \to \mathbb{E}$ gives rise to a morphism of shifted prederivators $F^J : \mathbb{D}^J \to \mathbb{E}^J$ for any category $J \in$ **Dia**.

[6] Since **CAT** is $\mathcal{U}^+$-locally small with our conventions, it follows that **PDer** and **Der** are $\mathcal{U}^+$-locally small.

(ii) Any functor $F : \mathscr{C} \to \mathscr{D}$ gives rise to a strict morphism of the associated represented **Cat**-prederivators $yF : y\mathscr{C} \to y\mathscr{D}$ (cf. Examples 1.2.6(i).

(iii) For a relative (locally small) category $(\mathscr{C}, \mathcal{W})$ with associated prederivator $\mathbb{D}_{(\mathscr{C},\mathcal{W})}$ (cf. Example 1.2.6(ii)), the localization functors $\mathscr{C}^J \to \mathscr{C}^J[\mathcal{W}_J^{-1}]$ assemble to a morphism of prederivators $y\mathscr{C} \to \mathbb{D}_{(\mathscr{C},\mathcal{W})}$. This is true in particular for the relative category associated to a model category (cf. Example 1.2.26(iii)).

(iv) Let $\mathbb{D}$ be a **Dia**-prederivator. Any functor $u : J \to K$ in **Dia** gives rise to a strict morphism of shifted prederivators $u^* : \mathbb{D}^K \to \mathbb{D}^J$ (cf. Example 1.2.6(iii)). Furthermore, any natural transformation $a : u \Rightarrow v$ of functors $J \to K$ in **Dia** gives rise to a modification a^* from u^* to v^*.

Let $F : \mathbb{D} \to \mathbb{E}$ be a morphism of **Dia**-prederivators and let $u : J \to K$ be a functor in **Dia**. If $\mathbb{D}, \mathbb{E}$ are right derivators, then there is a canonical mate:

$$\gamma_{u,!}^F : u_! F_J \Rightarrow F_K u_!$$

given as the pasting

$$\begin{tikzcd}
\mathbb{E}(K) & \mathbb{E}(J) \arrow[l, "u_!"'] \arrow[dr, phantom, "\Swarrow (\gamma_u^F)^{-1}"] & \mathbb{D}(J) \arrow[l, "F_J"'] \arrow[dr, phantom, "\Swarrow \eta"] & \\
 & \mathbb{E}(K) \arrow[ul, bend left, "\mathrm{id}_{\mathbb{E}(K)}"] \arrow[u, "u^*"'] \arrow[ul, phantom, "\Swarrow \epsilon"] & \mathbb{D}(K) \arrow[l, "F_K"] \arrow[u, "u^*"'] & \mathbb{D}(J) \arrow[l, "u_!"] \arrow[ul, bend right, "\mathrm{id}_{\mathbb{D}(J)}"']
\end{tikzcd} \tag{1.3.6}$$

where ϵ, η are the counit and unit respectively of the adjunctions $u_!^{\mathbb{E}} \dashv u_{\mathbb{E}}^*$ and $u_!^{\mathbb{D}} \dashv u_{\mathbb{D}}^*$.

Dually, if $\mathbb{D}, \mathbb{E}$ are left derivators, there is a canonical mate:

$$\gamma_{u,*}^F : F_K u_* \Rightarrow u_* F_J$$

given by a dual pasting to (1.3.6) above.

Definition 1.3.7. Let $F : \mathbb{D} \to \mathbb{E}$ be a morphism of right (respectively left) derivators of domain **Dia**. If $u : J \to K$ is a functor in **Dia** such that the mate $\gamma_{u,!}^F$ (respectively $\gamma_{u,*}^F$) is an isomorphism, then we say F **commutes with left (respectively right) Kan extensions along** u. If this happens for all functors $u : J \to K$ in **Dia** then we say F is **cocontinuous** (respectively **continuous**).

Given two left (respectively right) **Dia**-derivators $\mathbb{D}, \mathbb{E}$, we will write $\underline{\mathsf{Hom}}_!(\mathbb{D}, \mathbb{E})$ (respectively $\underline{\mathsf{Hom}}_*(\mathbb{D}, \mathbb{E})$) for the category of cocontinuous (respectively continuous) morphisms from $\mathbb{D}$ to $\mathbb{E}$. We remark that a morphism of right (respectively left) derivators F preserves left (respectively right) Kan extensions along a functor u in **Dia** if and only if so does the shifted morphism F^J for any category $J \in$ **Dia** (cf. [Gro13, Corollary 2.7]).

As with any 2-category there is an internal notion of adjunctions in **PDer**. Specifically, an adjunction $F \dashv G$ in **PDer** consists of two morphisms of prederivators $F : \mathbb{D} \to \mathbb{E}$ and $G : \mathbb{E} \to \mathbb{D}$, together with two modifications $\epsilon : FG \to \mathrm{id}_{\mathbb{E}}$ and $\eta : GF \to \mathrm{id}_{\mathbb{D}}$ that satisfy the triangular identities. Similarly, there is an internal notion of equivalence: a morphism $F : \mathbb{D} \to \mathbb{E}$ is an equivalence if there is a morphism $G : \mathbb{E} \to \mathbb{D}$ and isomodifications $FG \cong \mathrm{id}_{\mathbb{E}}$ and $GF \cong \mathrm{id}_{\mathbb{D}}$.

Proposition 1.3.8. *A morphism of right (respectively left) derivators is a left (respectively right) adjoint if and only if it admits a right (respectively left) adjoint levelwise and is cocontinuous (respectively continuous).*

Proof. See [Gro13, Lemma 2.8 and Proposition 2.9]. Alternatively, one can use the machinery of Section 3.1 to prove this more abstractly. □

Example 1.3.9.

(i) Let $\mathbb{D}$ be a derivator and $u : J \to K$ a functor in **Dia**. Then the strict morphism of derivators $u^* : \mathbb{D}^K \to \mathbb{D}^J$ is both continuous and cocontinuous (cf. [Gro13, Proposition 2.5]). In particular, we get adjunctions of derivators $u_! : \mathbb{D}^J \leftrightarrows \mathbb{D}^K : u^*$ and $u^* : \mathbb{D}^K \leftrightarrows \mathbb{D}^J : u_*$.

(ii) Let $F : M \leftrightarrows N : G$ be a Quillen adjunction between model categories. Then forming levelwise derived functors gives us an adjunction of derivators $\mathbb{L}F : \mathbb{D}_M \leftrightarrows \mathbb{D}_N : \mathbb{R}G$ (cf. [Cis08, Proposition 6.12]).

Finally, we recall the notion of exact morphisms:

Definition 1.3.10. A morphism $F : \mathbb{D} \to \mathbb{E}$ of right (respectively left) derivators is **right exact** (respectively **left exact**) if and only if F preserves initial objects and cocartesian squares (respectively final objects and cartesian squares). A morphism of derivators is **exact** if and only if it is both left and right exact.

Example 1.3.11.

(i) Any cocontinuous morphism of right derivators is right exact. Dually a continuous morphism of left derivators is left exact.

(ii) For a right (respectively left) derivator $\mathbb{D}$ and a functor $u : J \to J$ in **Dia** the induced morphism $u^* : \mathbb{D}^K \to \mathbb{D}^J$ is right exact (respectively left exact) (cf. Example 1.3.9(i)).

(iii) For a Quillen adjunction $F : M \leftrightarrows N : G$ the morphism of derivators $\mathbb{L}F : \mathbb{D}_M \to \mathbb{D}_N$ is right exact and the morphism $\mathbb{R}G : \mathbb{D}_N \to \mathbb{D}_M$ is left exact.

1.4 Homotopy finite categories

It is a well-known fact of ordinary category theory that we can construct finite colimits out of pushouts and finite coproducts. The goal of this section is to prove an analogous result for derivators. In the homotopical case, “finite categories" should be replaced by “categories whose nerve is finite". Those are precisely finite direct categories, or “strictly homotopy finite categories" in the terminology of [PS16].

To present our argument, we first need to recall the notion of homotopy final functors.

Definition 1.4.1. Let $\mathbb{D}$ be a right derivator. A functor $u : J \to K$ is called **$\mathbb{D}$-homotopy final** if the following square in **Dia**:

$$\begin{tikzcd} J \arrow[r, "u"] \arrow[d, "\pi_J"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\swArrow}"] & K \arrow[d, "\pi_K"] \\ e \arrow[r, "="'] & e \end{tikzcd}$$

is $\mathbb{D}$-exact.

Thus, a functor is $\mathbb{D}$-homotopy final if restriction along it preserves colimits. To give a good criterion for a functor to be homotopy final, we need to also recall the following definition:

Definition 1.4.2. Let $\mathbb{D}$ be a right derivator and $I \in$ **Dia**. If the functor $\pi_I : I \to e$ is homotopy final, then we say I is **$\mathbb{D}$-contractible**.

Equivalently, the counit morphism $(\pi_I)_!\pi_I^* \to \mathrm{id}$ is an isomorphism, i.e. the functor π_I^* is fully faithful. Important examples of homotopy final functors are right adjoints:

Proposition 1.4.3. *Let $\mathbb{D}$ be a right derivator. If a functor $u : J \to K$ is a right adjoint, then it is $\mathbb{D}$-homotopy final.*

Proof. The proof can be found in [Gro13, Proposition 1.18]. Note that the statement there is for derivators, but the proof only uses left Kan extensions. □

Recall that there is an equivalence relation on **Cat** generated by adjunctions. More precisely, two small categories A, B are **strongly homotopic** if there is a zig-zag of adjunctions connecting A and B. Using Proposition 1.4.3, the following corollary is immediate:

Corollary 1.4.4. *Let $\mathbb{D}$ be a right derivator. If two categories I, J in* **Dia** *are strongly homotopic, then I is $\mathbb{D}$-homotopy contractible if and only if J is.*

Another consequence of Proposition 1.4.3 allows us to replace *Der*4*R* in the definition of a right derivator by a seemingly stronger axiom:

Proposition 1.4.5. *Let $\mathbb{D}$ be a semiderivator that admits left Kan extensions. Then $\mathbb{D}$ is a right derivator if and only if any comma square* (1.2.1) *in* **Dia** *is $\mathbb{D}$-exact. Moreover, any pullback square in* **Dia***:*

(1.4.6)
$$\begin{tikzcd} P \arrow[r, "v'"] \arrow[d, "u'"'] \arrow[dr, phantom, "id \swarrow"] & I \arrow[d, "u"] \\ J \arrow[r, "v"'] & K \end{tikzcd}$$

where u is a Grothendieck opfibration is $\mathbb{D}$-exact.

Proof. For the first part of the proof: for the forward direction see the proof that (1) $\Rightarrow$ (3) in [Gro13, Proposition 1.26] noting that it doesn't use right Kan extensions at all, while the converse direction is trivial. For the second part, see the dual of the proof in [Gro13, Proposition 1.24]. □

By duality, the first part of the proposition remains true if $\mathbb{D}$ is a semiderivator that admits right Kan extensions, and the second part is also true if we require v to be a Grothendieck fibration instead of u being an opfibration.

Moreover, in the presence of both left and right Kan extensions, if any square (1.4.6) with u an opfibration or v a fibration is $\mathbb{D}$-exact, then $\mathbb{D}$ is a derivator. We do not know how to show this in the one-sided case, but we won't need it anyway.

Proposition 1.4.7. *Let $\mathbb{D}$ be a right derivator. A functor $u : J \to K$ in* **Dia** *is $\mathbb{D}$-homotopy final if for any object $k \in K$ the category $(k \downarrow u)$ is $\mathbb{D}$-homotopy contractible.*

Proof. See [GPS14b, Corollary 3.13 and Theorem 3.8], noting that only left Kan extensions are used throughout the proof. □

Let us recall from [PS16], that given a right derivator $\mathbb{D}$ and a category $J \in$ **Dia**, a full subcategory $\mathcal{E} \subset \mathbb{D}(e)$ is said to be **closed under J-colimits** if for any object $X \in \mathbb{D}(I)$ that is pointwise in $\mathcal{E}$ (meaning that $X_j \in \mathcal{E}$ for all objects $j \in J$) we also have $(\pi_J)_! X \in \mathcal{E}$. When we say that $\mathcal{E}$ is **closed under pushouts**, we will mean that $\mathcal{E}$ is closed under $\ulcorner$-colimits. We also recall from [Cis10] that the length of a finite direct category I is the dimension of the longest non-degenerate simplex in the nerve NI. We are now ready to prove the main theorem in this section:

Theorem 1.4.8. *Let $\mathbb{D}$ be a right* $\mathbf{Dir}_f$*-derivator. Then:*

(i) A full replete[7] subcategory of $\mathbb{D}(e)$ that is closed under pushouts and contains the initial object of $\mathbb{D}(e)$ is closed under all finite direct colimits.

(ii) A right exact morphism of $\mathbf{Dir}_f$*-derivators: $F : \mathbb{D} \to \mathbb{E}$ is cocontinuous.*

Proof. We'll just prove (i) since (ii) is formally very similar. Let $\mathcal{E}$ be a full subcategory of $\mathbb{D}(e)$ that is closed under homotopy pushouts and contains the initial object of $\mathbb{D}(e)$. We will show that for any finite direct category I, the category $\mathcal{E}$ is closed under I colimits. The proof will be done by induction on the length n of I. For $n = 0$, this is the content of [GPS14b, Corollary 4.11]. Assume

[7] *Meaning closed under isomorphisms in the ambient category.*

now that $n > 0$ and $\mathcal{E}$ is closed under colimits over finite direct categories of smaller length. Let I be a finite direct category of length n.

Using Corollaries 1.4.4 and 1.4.5 as well as [GPS14b, Corollary 4.11], we can reduce to the case $I = (\Delta'_n)^{\mathrm{op}}$ exactly as in the proof of [PS16, Theorem 7.1]. Here, the category Δ'_n is the subcategory of the simplex category Δ containing only the objects $[0], \cdots, [n]$ and only injective order-preserving maps. Now consider any object $X \in \mathbb{D}((\Delta'_n)^{\mathrm{op}})$ that is pointwise in $\mathcal{E}$, and we'll show that $(\pi_{(\Delta'_n)^{\mathrm{op}}})_! X \in \mathcal{E}$.

Consider the full category $B \subset ((\Delta'_n)^{\mathrm{op}})^{[1]}$ spanned by all non-identity arrows whose domain is $[n]$ as well as the identity arrows on $[0], \cdots, [n-1]$. Note that as a full subcategory of a finite direct category, B is still finite direct. Let A be the category that occurs by freely attaching a terminal object to all arrows with domain $[n]$. In other words, A contains B as a full subcategory and has a new object ∞ together with a unique arrow $b \to \infty$ for any object $b \in B$ that corresponds to an arrow $[n] \to [k]$ in $(\Delta'_n)^{\mathrm{op}}$ with $k \neq n$. Note that A is also finite direct. We extend the source functor $B \to (\Delta'_n)^{\mathrm{op}}$ mapping an arrow to its source, to a functor $s : A \to (\Delta'_n)^{\mathrm{op}}$ by mapping ∞ to $[n]$. We claim that s is homotopy final. By Proposition 1.4.7, it is enough to show that $([k] \downarrow s)$ is $\mathbb{D}$-homotopy contractible for $k = 0, \cdots, n$. Recall that objects in this category are pairs (a, f) where $a \in A$ is an object and $f : [k] \to s(a)$ is a morphism in $(\Delta'_n)^{\mathrm{op}}$.

First, consider the case $k \in \{0, \cdots, n-1\}$. Since there are no maps $[k] \to [n]$ in $(\Delta'_n)^{\mathrm{op}}$ all objects in $([k] \downarrow s)$ have the form $(\mathrm{id}_{[l]}, f)$ where $f : [k] \to [l]$ is an arrow in $(\Delta'_n)^{\mathrm{op}}$. If we identify such an object with f, then a morphism $f \to f'$ in $([k] \downarrow s)$ is just a commutative triangle in Δ'^{op}_n:

$$\begin{tikzcd} {[k]} \arrow[r, "f"] \arrow[dr, "f'"'] & {[l]} \arrow[d] \\ & {[l']} \end{tikzcd}$$

Thus, the category $([k] \downarrow s)$ is isomorphic to the category $([k] \downarrow (\Delta'_n)^{\mathrm{op}})$, which has $([k], \mathrm{id}_{[k]})$ as an initial object and is, hence, contractible.

To prove our claim, it remains to show that the category $([n] \downarrow s)$ is homotopy contractible. Objects in $([n] \downarrow s)$ look like $(\infty, \mathrm{id}_{[n]})$ or $(a, \mathrm{id}_{[n]})$ where $a : [n] \to [l]$ is a non identity arrow in $(\Delta'_n)^{\mathrm{op}}$, i.e. as commutative diagrams:

$$\begin{tikzcd} {[n]} \arrow[r, "\mathrm{id}"] \arrow[dr, "a"'] & {[n]} \arrow[d, "a"] \\ & {[l]} \end{tikzcd} \tag{1.4.9}$$

or as $(\mathrm{id}_{[l]}, a)$ where $a : [n] \to [l]$ is a non identity arrow in $(\Delta'_n)^{\mathrm{op}}$, i.e. as commutative diagrams:

$$\begin{tikzcd} {[n]} \arrow[r, "a"] \arrow[dr, "a"'] & {[l]} \arrow[d, "="] \\ & {[l]} \end{tikzcd} \tag{1.4.10}$$

Note that there are no maps from objects of type (1.4.10) to object of type (1.4.9).

Consider the full subcategory C of $([n] \downarrow s)$ generated by all objects of type (1.4.9) and the object $(\infty, \mathrm{id}_{[n]})$. This subcategory is reflective, the right adjoint being given by the identity on objects already in C and by mapping an object:

$$\begin{tikzcd} {[n]} \arrow[r, "a"] \arrow[dr, "a"'] & {[l]} \arrow[d, "\mathrm{id}"] \\ & {[l]} \end{tikzcd}$$

of type (1.4.10) to the object:

$$\begin{tikzcd} {[n]} \arrow[r, "\mathrm{id}"] \arrow[dr, "a"'] & {[n]} \arrow[d, "a"] \\ & {[l]} \end{tikzcd}$$

of type (1.4.9). But C has a terminal object, namely $(\infty, \mathrm{id}_{[n]})$. By Corollary1.4.4, we deduce that $([n] \downarrow s)$ is homotopy contractible, hence the functor $s : A \to (\Delta'_n)^{\mathrm{op}}$ is homotopy final as claimed.

Thus, we have $(\pi_{(\Delta'_n)^{\mathrm{op}}})_! X \cong (\pi_A)_! s^* X$ and $s^* X$ is still pointwise in $\mathcal{E}$. Next, consider the functor $q : A \to \ulcorner$ that maps the object $(\infty, \mathrm{id}_{[n]})$ to $(1, 0)$, objects of type (1.4.9) to $(0, 0)$ and objects of type (1.4.10) to $(0, 1)$. Since $(\pi_{\ulcorner})_! q_! s^* X \cong (\pi_A)_! s^* X$, to finish the proof, it is enough to show that $q_! s^* X$ is pointwise in $\mathcal{E}$. By *Der*4*R*, it is enough to show that $\mathcal{E}$ is closed under colimits of shape $(q \downarrow (0, 0)), (q \downarrow (1, 0))$ and $(q \downarrow (0, 1))$.

For the first one: note that $(q \downarrow (0, 0)) \cong \partial([n] \downarrow (\Delta'_n)^{\mathrm{op}})$. Here $\partial([n] \downarrow (\Delta'_n)^{\mathrm{op}})$ is the full subcategory of $([n] \downarrow (\Delta'_n)^{\mathrm{op}})$ spanned by non-identity arrows. This has clearly length smaller than n. Hence $\mathcal{E}$ is closed under $(q \downarrow (0, 0))$-colimits by induction.

For the second one: note that $(q \downarrow (1,0))$ is isomorphic to the full subcategory of A on all objects except $\mathrm{id}_{[k]}$ for $k = 0, \cdots, n-1$, which has a final object. Thus, $(q \downarrow (1,0))$-colimits are just evaluation at the final object (cf. [Gro13, Lemma 1.19]), so that clearly $\mathcal{E}$ is closed under colimits of type $(q \downarrow (1,0))$.

Finally, note that $(q \downarrow (0,1))$ is isomorphic to B. The category $(\Delta'_{n-1})^{\mathrm{op}}$ admits a fully faithful inclusion to B given by $[k] \mapsto \mathrm{id}_{[k]}$ on objects. This inclusion admits a left adjoint given by mapping an object of B, which is really an arrow in $(\Delta'_n)^{\mathrm{op}}$, to its codomain[8]. Since right adjoint functors are homotopy final, we conclude by Proposition 1.4.3 and induction. □

As an application, we now study maximal subprederivators.

Lemma 1.4.11. *Let $\mathbb{D}$ be a prederivator and $\mathcal{E} \subset \mathbb{D}(e)$ a full subcategory. For any category $J \in$ **Dia** let $\mathbb{E}(J)$ be the full subcategory of $\mathbb{D}(J)$ spanned by all coherent diagrams $X \in \mathbb{D}(J)$ that are pointwise in $\mathcal{E}$. Then the assignment:*

$$\mathbf{Dia} \ni J \mapsto \mathbb{E}(J) \in \mathbf{CAT}$$

defines a prederivator $\mathbb{E} : \mathbf{Dia}^{1\text{-}op} \to \mathbf{CAT}$ and the levelwise inclusions $\mathbb{E}(J) \hookrightarrow \mathbb{D}(J)$ assemble to a strict morphism of prederivators $\mathbb{E} \hookrightarrow \mathbb{D}$.

Proof. The proof is immediate by the observation that for any functor $u : J \to K$ in **Dia** and any object $X \in \mathbb{E}(K)$ we have $u^*X \in \mathbb{E}(J)$. □

Definition 1.4.12. Let $\mathbb{D}$ be a prederivator and $\mathcal{E} \subset \mathbb{D}(e)$ a full subcategory. The prederivator $\mathbb{E}$ of Lemma 1.4.11 will be called the **maximal subprederivator** of $\mathbb{D}$ associated to (or spanned by) $\mathcal{E}$.

Proposition 1.4.13. *Let $\mathbb{D}$ be a prederivator and $\mathbb{E}$ the maximal subprederivator of $\mathbb{D}$ associated to a full replete subcategory $\mathcal{E} \subset \mathbb{D}(e)$. Then:*

(i) If $\mathbb{D}$ is a semiderivator, then so is $\mathbb{E}$.

(ii) If $\mathbb{D}$ is strong, then so is $\mathbb{E}$.

---

[8] This is well-defined because arrows in $(\Delta'_n)^{\mathrm{op}}$ that are objects of B cannot have $[n]$ as a codomain.

(iii) If $\mathbb{D}$ is a right derivator and $\mathcal{E}$ is closed under colimits, then $\mathbb{E}$ is also a right derivator and the inclusion morphism $i : \mathbb{E} \hookrightarrow \mathbb{D}$ is cocontinuous.

Proof. The first two assertions are immediate from the definitions and left to the reader. For the last assertion, note that for any functor $u : J \to K$ in **Dia**, any object $X \in \mathbb{D}(J)$ and any $k \in K$ we have by ($Der4R$) for $\mathbb{D}$ that $(u_!X)_k \cong (\pi_{(u\downarrow k)})_! X|_{(u\downarrow k)} \in \mathcal{E}$. By repleteness, it thus follows that $u_!X$ is pointwise in $\mathcal{E}$, i.e. $u_!X \in \mathbb{E}(K)$. We thus immediately verify that $\mathbb{E}$ satisfies ($Der3R$) and ($Der4R$) and i is cocontinuous. □

Corollary 1.4.14. *Let $\mathbb{D}$ be a right $\mathbf{Dir}_f$-derivator and $\mathbb{E}$ the maximal subprederivator associated to a full subcategory $\mathcal{E} \subset \mathbb{D}(e)$. If $\mathcal{E}$ contains the initial object of $\mathbb{E}(e)$ and is closed under pushouts, then $\mathbb{E}$ is also a right derivator.*

Proof. Follows immediately from Proposition 1.4.13 and Theorem 1.4.8. □

1.5 Pointed derivators

Recall that a pointed category $\mathscr{C}$ is just a category with a zero object[9], and a pointed functor is a functor of pointed categories preserving the zero object.

Definition 1.5.1. A prederivator $\mathbb{D}$ is **pointed** if all its values are pointed categories, and $\mathbb{D}$-pullbacks are pointed functors.

Remark 1.5.2. A derivator $\mathbb{D}$ is pointed if and only if it is pointed as a prederivator. This happens if and only if the category $\mathbb{D}(e)$ is pointed (cf. [Gro13, Proposition 3.2]).

Example 1.5.3.

(i) A derivator $\mathbb{D}$ is pointed if and only if the shifted derivator $\mathbb{D}^J$ is pointed for all categories $J \in$ **Dia** (cf. [Gro13, Proposition 3.2]).

(ii) A derivator $\mathbb{D}$ is pointed if and only if its opposite $\mathbb{D}^{\mathrm{op}}$ is pointed.

[9] That is, $\mathscr{C}$ admits both an initial object $\emptyset$ and a final object $*$ and the unique morphism $\emptyset \to *$ is an isomorphism.

(iii) The derivator represented by a bicomplete category $\mathscr{C}$ is pointed if and only if $\mathscr{C}$ is pointed.

(iv) The derivator associated to a model category M is pointed if and only if M is pointed.

In the pointed case, Proposition 1.2.23 takes the following form:

Proposition 1.5.4. *Let $\mathbb{D}$ be a pointed prederivator satisfying Der1, Der3R and Der4R. Consider a cosieve $u : J \to K$ in* **Dia**. *If an object $X \in \mathbb{D}(K)$ lies in the essential image of $u_!$, then we have $X_k \cong 0$ for any object $k \in K - u(J)$. The converse is true if $\mathbb{D}$ is a pointed right derivator.*

To finish this section, we recall the construction of suspension, loop, cone and fiber functors. In what follows, we write $[n]$ for the poset $\{0 < 1 < ... < n\}$ viewed as a category.

Definition 1.5.5. Let $\mathbb{D}$ be a pointed derivator.

(i) The **suspension functor** $\Sigma : \mathbb{D}(e) \to \mathbb{D}(e)$ is defined as the composite:

$$\mathbb{D}(e) \xrightarrow{(0,0)_*} \mathbb{D}(\ulcorner) \xrightarrow{(i_\ulcorner)_!} \mathbb{D}(\square) \xrightarrow{(1,1)^*} \mathbb{D}(e)$$

where $(0,0), (1,1)$ are the functors $e \to \square$ classifying the objects $(0,0)$ and $(1,1)$ of $\square$ respectively (cf. (1.2.20)).

(ii) The **cone functor Cone** : $\mathbb{D}([1]) \to \mathbb{D}([1])$ is defined as the composite:

$$\mathbb{D}([1]) \xrightarrow{i_*} \mathbb{D}(\ulcorner) \xrightarrow{(i_\ulcorner)_!} \mathbb{D}(\square) \xrightarrow{j^*} \mathbb{D}([1])$$

where $i : [1] \to \ulcorner$ is the functor classifying the top arrow and $j : [1] \to \square$ is the functor classifying the right arrow.

Using properties of Kan extensions along (co)sieves (cf. Proposition 1.5.4 and its dual), we can show that the suspension functor $\Sigma : \mathbb{D}(e) \to \mathbb{D}(e)$ takes an object $X \in \mathbb{D}(e)$ to the bottom right corner of a cocartesian square with underlying diagram:

(1.5.6)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

Similarly, if we have a coherent arrow $F \in \mathbb{D}([1])$ with underlying diagram $f : X \to Y$, we can form a cocartesian square with underlying diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \tag{1.5.7}$$

and $\mathbf{Cone}(F)$ is the coherent arrow $Y \to Z$. It is thus clear that $\Sigma \cong (1,1)^*\mathbf{Cone} \circ 0_*$. These functors actually extend to morphisms of derivators:

$$\Sigma : \mathbb{D} \to \mathbb{D} \text{ and } \mathbf{Cone} : \mathbb{D}^{[1]} \to \mathbb{D}^{[1]}$$

We can define the **loop** and **fiber** $\Omega : \mathbb{D} \to \mathbb{D}$ and $\mathbf{F} : \mathbb{D}^{[1]} \to \mathbb{D}^{[1]}$ dually to Σ and **Cone** respectively. Moreover we have adjunctions of derivators:

$$\Sigma : \mathbb{D} \leftrightarrows \mathbb{D} : \Omega \text{ and } \mathbf{Cone} : \mathbb{D}^{[1]} \leftrightarrows \mathbb{D}^{[1]} : \mathbf{F}$$

For proofs and more in depth explanation, we refer to [Gro13].

1.6 Additive Derivators

Definition 1.6.1. A prederivator $\mathbb{D}$ is **additive** if all its values and precomposition functors are additive.

Remark 1.6.2. A derivator is additive if and only if its underlying category is additive. That forces all left and right Kan extensions to be additive as well. See [Gro12, Section 3.1] for more details.

The goal of this section is to show that in additive strong derivators we can lift idempotents. We will make this more precise shortly, but for now we remark that the main ingredient is [Gro11, Theorem 3.8 in Part 3]. That theorem as stated works only for triangulated derivators that are linear over a field. The important (for us) part of that theorem, namely that the kernel is a square zero ideal, works for any strong additive derivator using partial triangulations. In fact, the proof is essentially identical, but we reproduce it here for convenience. We first recall the definition of a right triangulated category [Gro12, Definition 3.4] along with some relevant facts. The two original references are [KV87], where right triangulated categories are called suspended categories, and [BM92]:

Definition 1.6.3. Let $\mathscr{A}$ be an additive category with an additive endofunctor $\Sigma : \mathscr{A} \to \mathscr{A}$, and a class of so-called distinguished right triangles $X \to Y \to Z \to \Sigma X$. We say that Σ together with the class of distinguished right triangles defines a **right triangulated structure** on $\mathscr{A}$ if the following axioms are satisfied:

(RT.1) For every object $X \in \mathscr{A}$ the right triangle $0 \to X \xrightarrow{1} X \to 0$ is distinguished. The class of distinguished right triangles is replete and every morphism in $\mathscr{A}$ occurs as the first morphism in a distinguished right triangle.

(RT.2) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is a distinguished right triangle, then so is the rotated right triangle $Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$.

(RT.3) Given a commutative solid arrow diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, dashed, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the top and bottom rows are right distinguished triangles, there is a dashed arrow $w : Z \to Z'$ as indicated that makes the extended diagram commute.

(RT.4) For any two composable morphisms $X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$, there is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, equal] & Y \arrow[r, "g_1"] \arrow[d, "f_2"] & C_1 \arrow[r, "h_1"] \arrow[d] & \Sigma X \arrow[d, equal] \\ X \arrow[r, "f_2 f_1"'] & Z \arrow[r, "g_3"'] \arrow[d, "g_2"'] & C_3 \arrow[r, "h_3"'] \arrow[d] & \Sigma X \arrow[d, "\Sigma f_1"] \\ & C_2 \arrow[r, equal] \arrow[d, "h_2"'] & C_2 \arrow[r, "h_2"'] \arrow[d, "\Sigma g_1 \circ h_2"] & \Sigma Y \\ & \Sigma Y \arrow[r, "\Sigma g_1"'] & \Sigma C_1 & \end{tikzcd}$$

where the first two rows and middle columns are distinguished right triangles.

If those axioms are satisfied, we refer to the triple consisting of the additive category $\mathscr{A}$, the additive endofunctor Σ and the class of distinguished right triangles as a **right triangulated category**.

A **right exact functor** between right triangulated categories is a functor that commutes with suspension up to a natural isomorphism, such that the two together map distinguished right triangles to distinguished right triangles.

As with usual triangulated categories, if we have a distinguished right triangle as in ($RT.2$) then $gf = 0$. Following (the dual of) the terminology in [BM92], we will call an additive functor $H : \mathscr{A} \to \mathscr{B}$ from a right triangulated category to an abelian category **right homological** if for any distinguished right triangle $X \to Y \to Z \to \Sigma X$, the sequence: $HX \to HY \to HZ$ in $\mathscr{B}$ is exact. In particular, we get a long exact sequence $HX \to HY \to HZ \to H\Sigma X \to \cdots$. Note that this sequence does not extend to the left as in the case of usual triangulated categories! The following lemma is proved just as in the usual triangulated case:

Lemma 1.6.4. *Let $\mathscr{A}$ be a right triangulated category. For any object $X \in \mathscr{A}$, the contravariant functor* $\mathsf{Hom}(-, X) : \mathscr{A} \to \mathcal{A}b$ *is right cohomological.*

Note however that $\mathsf{Hom}(X, -)$ is **not** right homological in general. Nevertheless, the previous lemma allows us to conclude more or less as in the triangulated case that if we have a commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"] & Z \arrow[r] \arrow[d, "w"] & \Sigma X \arrow[d, "\Sigma u"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

whose rows are distinguished right triangles and such that u and v are isomorphisms, then so is w. In turn, this and the same proof as in the triangulated case imply that the sum of two distinguished right triangles is a distinguished right triangle. In particular, we have a distinguished right triangle:

$$X \xrightarrow{\binom{1}{0}} X \oplus Y \xrightarrow{(0\ 1)} Y \xrightarrow{0} \Sigma X$$

for any two objects $X, Y \in \mathscr{A}$.

Lemma 1.6.5. *Let $\mathscr{A}$ be a right triangulated category and $f : X \to Y$ a morphism. If:*

$$X \oplus Y \xrightarrow{(f,1)} Y \to Z \to \Sigma(X \oplus Y)$$

is a distinguished right triangle, then $Z \cong \Sigma X$.

Proof. Applying $(RT.4)$ to the composite $X \xrightarrow{\binom{1}{0}} X \oplus Y \xrightarrow{(f\ 1)} Y$, we get a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\binom{1}{0}"] \arrow[d, equal] & X \oplus Y \arrow[r, "(0\ 1)"] \arrow[d, "(f\ 1)"] & Y \arrow[r] \arrow[d, "h"] & \Sigma X \arrow[d, equal] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] \arrow[d] & C \arrow[r] \arrow[d] & \Sigma X \arrow[d] \\
& Z \arrow[r, equal] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma(X \oplus Y) \\
& \Sigma(X \oplus Y) \arrow[r] & \Sigma Y &
\end{tikzcd}$$

where the top two rows and middle two columns are distinguished right triangles. Commutativity of the middle square in the top two rows implies that $g = h$. By $(RT.2)$ applied to the second row, we get a distinguished right triangle:

$$Y \xrightarrow{g} C \to \Sigma X \to \Sigma Y$$

Comparing with the third column we deduce that $Z \cong \Sigma X$.

□

There is a dual notion of a left triangulated category, which the reader can deduce from Definition 1.6.3 or just look up in [BM92]. The important part for us is Theorem [Gro12, Theorem 3.6], which states that for a strong additive derivator $\mathbb{D}$, its values equipped with suspension (cf. Section 1.5) can be canonically turned into a right triangulated category and moreover the $\mathbb{D}$-pullback along any functor in **Dia** is naturally a right exact functor (cf. [Gro12, Corollary 3.10]). Dually, the values of $\mathbb{D}$ with loop can be canonically turned into a left triangulated category and the $\mathbb{D}$-pullback along any functor in **Dia** is naturally a left exact functor. In fact, [Gro12, Theorem 3.8] there is a compatibility between the two structures giving rise to what is called a pretriangulated category. We are now ready to go through the argument in [Gro11, Theorem 3.8 in Part 3] ensuring that the relevant part works in the additive case:

Theorem 1.6.6. *Let $\mathbb{D}$ be a strong additive derivator. Let K be the kernel of the underlying diagram functor:*

$$dia_{[1]} : \mathbb{D}([1]) \to \mathbb{D}(e)^{[1]}$$

meaning that K is the wide subcategory of $\mathbb{D}([1])$ on all morphisms that are the zero morphism pointwise. Then any two morphisms in K compose to the zero morphism.

Proof. We will follow the notation in [Gro11]. Consider morphisms $f : X \to Y$ and $g : Y \to Z$ in K that are zero pointwise, i.e. all the maps f_0, f_1, g_0, g_1 are zero. We will show that $gf = 0$.

First, we claim that f factors as $X \xrightarrow{f'} 1_!\Sigma X_0 \to Y$ where the first map f' is still in K, i.e. is zero pointwise. Let $X_0 \xrightarrow{u} X_1$ and $Y_0 \xrightarrow{v} Y_1$ be the underlying diagrams of X and Y respectively. The underlying diagram of f is:

(1.6.7)
$$\begin{tikzcd} X_0 \arrow[r, "0"] \arrow[d, "u"'] & Y_0 \arrow[d, "v"] \\ X_1 \arrow[r, "0"'] & Y_1 \end{tikzcd}$$

Using [Gro13, Lemma 1.17] the underlying diagram of $0_!0^*X$ looks like $X_0 \xrightarrow{1} X_0$, and using Proposition 1.5.4 the object $1_!1^*X \in \mathbb{D}([1])$ has underlying diagram $0 \to X_1$. By adding the adjunction counits, we get a morphism:

$$r = (\epsilon_0, \epsilon_1) : RX = 0_!0^*X \oplus 1_!1^*X \to X$$

Pick a distinguished right triangle $RX \xrightarrow{r} X \to C \to \Sigma RX$ in $\mathbb{D}([1])$. Using Lemma 1.6.5 and [Gro12, Corollary 3.10] this distinguished right triangle has, up to isomorphism, underlying diagram:

(1.6.8)
$$\begin{tikzcd} X_0 \arrow[r, "1"] \arrow[d, "\binom{1}{0}"'] & X_0 \arrow[r] \arrow[d, "u"'] & 0 \arrow[r] \arrow[d] & \Sigma X_0 \arrow[d] \\ X_0 \oplus X_1 \arrow[r, "(u\ 1)"'] & X_1 \arrow[r, "h"'] & \Sigma X_0 \arrow[r] & \Sigma(X_0 \oplus X_1) \end{tikzcd}$$

where both rows are distinguished right triangles. Since the composite $X_0 \oplus X_1 \xrightarrow{(u\ 1)} X_1 \xrightarrow{h} \Sigma X_0$ is zero (cf. the dual of [BM92, Lemma 2.1]), we conclude that $h = 0$. Using the converse in Proposition 1.5.4 we conclude that $C \cong 1_!\Sigma X_0$. We thus have a distinguished right triangle in $\mathbb{D}([1])$:

$$RX \xrightarrow{r} X \xrightarrow{f'} 1_!\Sigma X_0 \to \Sigma RX$$

where the middle morphism f' is pointwise zero. Using Lemma 1.6.4 applied to this distinguished right triangle, we deduce that the sequence:

$$\mathsf{Hom}_{\mathbb{D}([1])}(1_!\Sigma X_0, Y) \xrightarrow{(f')^*} \mathsf{Hom}_{\mathbb{D}([1])}(X, Y) \xrightarrow{r^*} \mathsf{Hom}_{\mathbb{D}([1])}(RX, Y)$$

is exact. Under the identification:

$$\begin{aligned}\mathsf{Hom}_{\mathbb{D}([1])}(RX, Y) &\cong \mathsf{Hom}_{\mathbb{D}([1])}(0_! X_0 \oplus 1_! X_1, Y)\\ &\cong \mathsf{Hom}_{\mathbb{D}([1])}(0_! X_0, Y) \oplus \mathsf{Hom}_{\mathbb{D}([1])}(1_! X_1, Y)\\ &\cong \mathsf{Hom}_{\mathbb{D}(e)}(X_0, Y_0) \oplus \mathsf{Hom}_{\mathbb{D}(e)}(X_1, Y_1)\end{aligned}$$

the map r^* sends f to $(f_0, f_1) = (0, 0)$. We thus conclude that $f \in \ker(r^*)$ and so f factors through f' proving our claim. Dually, any pointwise zero morphism $X' \to Y'$ in $\mathbb{D}([1])$ has a factorization of the form $X' \to 0_*\Omega Y'_1 \to Y'$ where the second map is pointwise zero. Applying this dual claim to f', we deduce a further factorization of f as $X \to 0_*\Omega\Sigma X_0 \xrightarrow{f''} 1_!\Sigma X_0 \to Y$, where the map f'' is zero pointwise.

It follows that the composite gf has a factorization as:

$$X \to 0_*\Omega\Sigma X_0 \to 1_!\Sigma X_0 \to Y \to 0_*\Omega\Sigma Y_0 \to 1_!\Sigma Y_0 \to Z$$

Now note that the composite $1_!\Sigma X_0 \to Y \to 0_*\Omega\Sigma Y_0$ is zero since for any objects $Z, W \in \mathbb{D}(e)$ we have:

$$\mathsf{Hom}_{\mathbb{D}([1])}(1_! Z, 0_* W) \cong \mathsf{Hom}_{\mathbb{D}(e)}(Z, 1^* 0_* W) = 0$$

by the dual of Proposition 1.5.4. □

The previous theorem allows us to lift idempotents coherently using the usual trick of "lifting modulo nilpotence":

Corollary 1.6.9. *Let $\mathbb{D}$ be a strong additive derivator. Let $f : X \to Y$ a morphism in $\mathbb{D}(e)$ and $F \in \mathbb{D}([1])$ a lift of f i.e. $\mathrm{dia}_{[1]}(F) = f$. Assume furthermore that $(a, b) : f \to f$ is an idempotent in $\mathbb{D}(e)^{[1]}$. In other words, we have a commutative diagram:*

(1.6.10)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in $\mathbb{D}(e)$ such that $a^2 = a$ and $b^2 = b$. Then there is an idempotent $e : F \to F$ whose underlying diagram is (1.6.10).

Proof. Since the derivator $\mathbb{D}$ is strong, there is some $r : F \to F$ whose underlying diagram is 1.6.10. Then $s = r^2 - r : F \to F$ is pointwise zero. By Theorem 1.6.6, we have $s^2 = 0$. Then $e = 3r^2 - 2r^3 : F \to F$ is an idempotent by direct computation, whose underlying diagram is (1.6.10). $\square$

1.7 Triangulated derivators

In this section we recall the definition and some basic properties of triangulated derivators, with a focus on triangulated derivators over small diagrams.

Definition 1.7.1. A derivator $\mathbb{D}$ is **stable** if it is pointed, and each square in $\mathbb{D}(\square)$ is cocartesian if and only if it is cartesian.

We remark that this definition is different than the one appearing in [Gro13]. Indeed, in the literature strongness is included in the definition of a stable derivator; however, we will call those derivators triangulated instead.

Definition 1.7.2. A derivator $\mathbb{D}$ is **triangulated** if it is both stable and strong.

If $\mathbb{D}$ is a triangulated derivator, then each of its values is an additive category, and moreover all $\mathbb{D}$-pullbacks and Kan extensions are additive functors (cf. [Gro13, Corollary 4.14]). More than that though, we can endow each value of $\mathbb{D}(J)$ with a canonical triangulated structure. It works on $\mathbb{D}(e)$ as follows: There is a functor $\mathbf{Cof} : \mathbb{D}([1]) \to \mathbb{D}([2] \times [1])$ the value of which at a coherent arrow $F \in \mathbb{D}([1])$ with underlying diagram $f : X \to Y$ has underlying diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & \Sigma X \end{tikzcd}$$

where all squares are cocartesian. Restricting to an appropriate subcategory of $[2] \times [1]$ and taking underlying diagram we arrive at a triangle:

$$X \xrightarrow{f} Y \to Z \to \Sigma X$$

We then declare a triangle in $\mathbb{D}(e)$ to be distinguished, if it is isomorphic to a triangle obtained via the above process. This is referred to as **the canonical triangulation on** $\mathbb{D}(e)$, and the canonical triangulation on $\mathbb{D}(J)$ is exactly the canonical triangulation on $\mathbb{D}^J(e)$ for any category $J \in$ **Dia**.

Theorem 1.7.3. *Let $\mathbb{D}$ be a triangulated derivator of domain* **Dia** *and let* **TCAT** *be the 2-category of triangulated categories, exact functors and exact natural tranformations. Then $\mathbb{D}$ admits a lift against the forgetful functor* **TCAT** $\to$ **CAT***:*

$$\begin{tikzcd} & \mathbf{TCAT} \arrow[d] \\ \mathbf{Dia}^{op} \arrow[r, "\mathbb{D}"'] \arrow[ur, dotted] & \mathbf{CAT} \end{tikzcd}$$

given by endowing $\mathbb{D}(J)$, $J \in$ **Dia** *with the canonical triangulations.*

Proof. See [Gro16, Theorem 10.14]. □

The following lemma relies on [GPS14b, Theorems 6.1 and 7.1]. Note the authors there give a proof for **Cat**-derivators, but all the diagramatic constructions used involve posets.

Lemma 1.7.4. *Let $\mathbb{D}$ be a triangulated* $\mathbf{Pos}_f$*-derivator, and let $\mathscr{E}$ be a full replete subcategory of $\mathbb{D}(e)$. The following are equivalent:*

(i) $\mathscr{E}$ is a triangulated subcategory of $\mathbb{D}(e)$.

(ii) $\mathscr{E}$ contains the zero object and is closed under homotopy pushouts.

(iii) $\mathscr{E}$ contains the zero object and is closed under homotopy pullbacks.

(iv) $\mathscr{E}$ contains the zero object and is closed under both homotopy pushouts and homotopy pullbacks.

Proof. This is immediate using the fact the definition of stability and [GPS14b, Theorems 6.1 and 7.1]. □

Using Theorem 1.4.8 and Corollary 1.4.14 the following corollary is immediate:

Corollary 1.7.5. *Let $\mathbb{D}$ be a triangulated $\mathbf{Dir}_f$-derivator and let $\mathscr{E}$ be a full replete subcategory of $\mathbb{D}(e)$. Then the following are equivalent:*

(i) *$\mathscr{E}$ is a triangulated subcategory of $\mathbb{D}(e)$.*

(ii) *$\mathscr{E}$ contains the zero object and is closed under homotopy pushouts.*

(iii) *$\mathscr{E}$ contains the zero object and is closed under colimits.*

(iv) *$\mathscr{E}$ contains the zero object and is closed under homotopy pullbacks.*

(v) *$\mathscr{E}$ contains the zero object and is closed under limits.*

(vi) *$\mathscr{E}$ contains the zero object and is closed under both homotopy pushouts and homotopy pullbacks.*

(vii) *$\mathscr{E}$ contains the zero object and is closed under both colimits and limits.*

Under the previous conditions, if $\mathbb{E}$ is the maximal subprederivator (cf. Definition 1.4.12) of $\mathbb{D}$ associated to $\mathcal{E}$, then $\mathbb{E}$ is also a triangulated $\mathbf{Dir}_f$-derivator and the inclusion $i : \mathbb{E} \hookrightarrow \mathbb{D}$ is exact. If $\mathcal{E}$ is thick, then so is $\mathbb{E}(J)$ for any category $J \in \mathbf{Dia}$.

We will study this setting in more detail when we talk about Verdier localizations of small triangulated derivators (cf. Section 4.3).

CHAPTER 2

Idempotent Completion

2.1 Idempotent Completion of Categories

In this section we examine some basic properties of idempotent completion (or Karoubi envelope) of categories, and in particular show that idempotent completion is a 2-functor. The result seems like it should be well-known but the author was unable to find a reference, hence a detailed proof is given. Readers comfortable with idempotent completion may want to read only Constructions 2.1.6, 2.1.10 and 2.1.11. We begin with the following definition:

Definition 2.1.1. Let $\mathscr{C}$ be a category. An **idempotent** in $\mathscr{C}$ is an endomorphism $e : X \to X$ of $\mathscr{C}$ such that $e^2 = e$. We say that the idempotent e **splits**, if there exist morphisms $i : X \to Y$ and $r : Y \to X$ with $ri = e$ and $ir = 1_Y$, and we call such a factorization a **splitting** of e. Finally, we say $\mathscr{C}$ is **idempotent complete** if every idempotent in $\mathscr{C}$ splits.

Example 2.1.2. Let $\mathscr{C}$ be a category and $e : X \to X$ an idempotent. One can show that a colimit (or limit) of the diagram $X \overset{1}{\underset{e}{\rightrightarrows}} X$ is precisely a splitting of e. Thus, any category with finite colimits or limits is idempotent complete. In particular, splittings of idempotents are unique up to isomorphism.

Remark 2.1.3. An additive category $\mathscr{C}$ is idempotent-complete if and only if for any idempotent $e : X \to X$ in $\mathscr{C}$ there is a splitting $X \cong A \oplus B$ under which e corresponds to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) : A \oplus B \to A \oplus B$. Equivalently, every idempotent $e : X \to X$ gives rise to a splitting $X \cong \mathrm{im}(e) \oplus \ker(e)$.

Definition 2.1.4. Let $\mathscr{C}$ be a category. We say that a functor $i : \mathscr{C} \to \mathscr{D}$ **exhibits $\mathscr{D}$ as the idempotent completion of** $\mathscr{C}$ if, for any idempotent complete category $\mathscr{E}$, precomposition with i

induces an equivalence of categories:

$$\underline{\mathsf{Hom}}(\mathscr{D}, \mathscr{E}) \to \underline{\mathsf{Hom}}(\mathscr{C}, \mathscr{E})$$

We leave the proof of the following proposition to the reader:

Proposition 2.1.5. *Let $i : \mathscr{C} \to \mathscr{D}$ be a fully faithful functor. The following conditions are equivalent:*

(i) The category $\mathscr{D}$ is idempotent complete and every object of $\mathscr{D}$ is a retract of an object of $\mathscr{C}$ under the functor i.

(ii) For every idempotent e of $\mathscr{C}$ the idempotent $i(e)$ of $\mathscr{D}$ splits and every object of $\mathscr{D}$ is a retract of an object of $\mathscr{C}$ under the functor i.

In this case, the functor i exhibits $\mathscr{D}$ as an idempotent completion of $\mathscr{C}$.

The rest of this section is devoted to explicitly constructing the idempotent completion of an arbitrary category and showing it is 2-functorial.

Construction 2.1.6. Let $\mathscr{C}$ be a category. The **Karoubi envelope $\mathscr{C}^\natural$ of $\mathscr{C}$** is the category which has as objects ordered pairs (X, e) where X is an object of $\mathscr{C}$ and $e : X \to X$ is an idempotent, and as morphisms $(X, e) \to (X', e')$ precisely those arrows $f : X \to X'$ in $\mathscr{C}$ with $fe = f = e'f$. Composition of arrows in $\mathscr{C}^\natural$ is given by composition in $\mathscr{C}$ and the identity arrow on an object (X, e) of $\mathscr{C}^\natural$ is precisely the morphism e.

Proposition 2.1.7. *Let $\mathscr{C}$ be a category.*

(i) The Karoubi envelope $\mathscr{C}^\natural$ of $\mathscr{C}$ (cf. Construction 2.1.6) is idempotent complete.

(ii) The functor $\eta_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\natural$ given on objects by $X \mapsto (X, 1)$ and on morphisms by $f \mapsto f$ is fully faithful and exhibits $\mathscr{C}^\natural$ as an idempotent completion of $\mathscr{C}$.

Proof. (i) A splitting of the idempotent $f : (X, e) \to (X, e)$ of $\mathscr{C}^\natural$ is given by the factorization

$$(X, e) \xrightarrow{f} (X, f) \xrightarrow{f} (X, e)$$

This shows $\mathscr{C}^\natural$ is idempotent-complete.

(ii) It is immediate that the functor $\eta_{\mathscr{C}}$ is fully faithful. Note that for any object (X, e) in $\mathscr{C}^\natural$ we have a factorization $(X, e) \xrightarrow{e} (X, 1) \xrightarrow{e} (X, e)$ of the identity morphism on (X, e). This finishes the proof by part (i) of Proposition 2.1.5. $\square$

Remark 2.1.8. The above proposition, implies that if a functor $\mathscr{C} \to \mathscr{D}$ is an idempotent completion, then it has to be fully faithful. Moreover, it also implies the converse in Proposition 2.1.5. In the literature, the terms idempotent completion and Karoubi envelope are interchangeable. However, we will use the latter term to refer to the specific construction of the idempotent completion given above.

Remark 2.1.9. By Remark 2.1.3, given a fully faithful functor $i : \mathscr{C} \to \mathscr{D}$ between additive categories with $\mathscr{D}$ idempotent-complete, a sufficient condition for the functor i to exhibit $\mathscr{D}$ as an idempotent completion of $\mathscr{C}$ is the following: for every object X of $\mathscr{D}$ there is an object Y of $\mathscr{D}$ such that $X \oplus Y$ is isomorphic to some object of $\mathscr{C}$.

Construction 2.1.10. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. The **Karoubi envelope $F^\natural$ of F** is the functor $F^\natural : \mathscr{C}^\natural \to \mathscr{D}^\natural$ given on objects by $(X, e) \mapsto (FX, Fe)$ and on arrows by $f \mapsto Ff$.

Construction 2.1.11. Let $\alpha : F \Rightarrow G : \mathscr{C} \to \mathscr{D}$ be a natural transformation. Let (X, e) be an object of $\mathscr{C}^\natural$, i.e. $e : X \to X$ is an idempotent of $\mathscr{C}$. By naturality of α the following diagram in $\mathscr{C}$ commutes:

$$\begin{tikzcd}
FX \arrow[r, "Fe"] \arrow[d, "\alpha_X"'] & FX \arrow[d, "\alpha_X"] \\
GX \arrow[r, "Ge"'] & GX
\end{tikzcd}$$

Consider the morphism $\alpha^\natural_{(X,e)} = \alpha_X F(e) = G(e)\alpha_X : FX \to GX$. Using that e is an idempotent we have:

$$\alpha^\natural_{(X,e)} F(e) = \alpha_X F(e) F(e) = \alpha_X F(e) = \alpha^\natural_{(X,e)}$$

and

$$G(e)\alpha^\natural_{(X,e)} = G(e)G(e)\alpha_X = G(e)\alpha_X = \alpha^\natural_{(X,e)}$$

In other words, $\alpha^\natural_{(X,e)}$ can be viewed as a morphism $F^\natural(X, e) \to G^\natural(X, e)$ in $\mathscr{D}^\natural$. One checks that it is natural in (X, e). The **Karoubi envelope $\alpha^\natural$ of α** is defined as the natural transformation $F^\natural \Rightarrow G^\natural$ whose component at an object (X, e) of $\mathscr{C}^\natural$ is the morphism $\alpha^\natural_{(X,e)}$.

Proposition 2.1.12. *The assignment $(-)^\natural : \mathbf{CAT} \to \mathbf{CAT}$ given on 0-cells by Construction 2.1.6, on 1-cells by Construction 2.1.10 and on 2-cells by Construction 2.1.11 is 2-functiorial. Moreover, there is a 2-natural tranformation $\eta : id_{\mathbf{CAT}} \Rightarrow (-)^\natural$ such that the functor $\eta_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\natural$ exhibits $\mathscr{C}^\natural$ as an idempotent completion of $\mathscr{C}$ for any category $\mathscr{C}$.*

Proof. We will only show that $(-)^\natural$ respects horizontal composition of natural transformations, and leave the rest of the verification of 2-functoriality to the reader. Consider natural transformations $\alpha : F \Rightarrow F' : \mathscr{C} \to \mathscr{D}$ and $\beta : G \Rightarrow G' : \mathscr{D} \to \mathscr{E}$. Recall that their horizontal composition is the natural transformation $\beta * \alpha : GF \Rightarrow G'F' : \mathscr{C} \to \mathscr{E}$ whose component at an object X in $\mathscr{C}$ is given by the dashed arrow in the following commutative square:

$$\begin{tikzcd}
G(F(X)) \arrow[r, "G(\alpha_X)"] \arrow[d, "\beta_{F(X)}"'] \arrow[dr, dashed] & G(F'(X)) \arrow[d, "\beta_{F'(X)}"] \\
G'(F(X)) \arrow[r, "G'(\alpha_X)"'] & G'(F'(X))
\end{tikzcd}$$

For any object (X, e) of $\mathscr{C}^\natural$ we have:

$$\begin{aligned}
(\beta * \alpha)^\natural_{(X,e)} &= (\beta * \alpha)_X G(F(e)) = \beta_{F'(X)} G(\alpha_X) G(F(e)) = \beta_{F'(X)} G(\alpha_X) G(F(e)) G(F(e)) \\
&= \beta_{F'(X)} G(\alpha_X F(e)) G(F(e)) = \beta_{F'(X)} G(F'(e)\alpha_X) G(F(e)) \\
&= \beta_{F'(X)} G(F'(e)) G(\alpha_X) G(F(e) = \beta_{F'(X)} G(F'(e)) G(\alpha_X F(e)) = \\
&= \beta^\natural_{F'^\natural(X,e)} G^\natural(\alpha^\natural_{(X,e)}) = (\beta^\natural * \alpha^\natural)_{(X,e)}
\end{aligned}$$

which shows that $(\beta * \alpha)^\natural = \beta^\natural * \alpha^\natural$ as wanted.

For the second part, for any category $\mathscr{C}$, we define the functor $\eta_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\natural$ by $X \mapsto (X, 1_X)$ on objects and by $f \mapsto f$ on morphisms. By Proposition 2.1.7, we just have to verify that η is 2-natural. So, consider two categories $\mathscr{C}, \mathscr{D}$. By [Bor94a, Definition 7.2.2], it is enough to show that the following diagram of categories commutes:

$$\begin{tikzcd}
 & \mathsf{Hom}(\mathscr{C}, \mathscr{D}) \arrow[dl, "(-)^\natural"'] \arrow[dr, "\eta_{\mathscr{D}} \circ -"] & \\
\mathsf{Hom}(\mathscr{C}^\natural, \mathscr{C}^\natural) \arrow[rr, "- \circ \eta_{\mathscr{C}}"'] & & \mathsf{Hom}(\mathscr{C}, \mathscr{C}^\natural)
\end{tikzcd}$$

We leave the details to the reader. $\square$

The previous proposition allows us to identify $\mathscr{C}$ with a full subcategory of $\mathscr{C}^\natural$ via $\eta_{\mathscr{C}}$. We will be particularly interested in the additive case. The following two propositions are well-known in the literature, cf. for instance [B10, Section 6]:

Proposition 2.1.13. *The Karoubi envelope $\mathscr{C}^\natural$ of an additive category $\mathscr{C}$ is additive and the fully faithful functor $\eta_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}^\natural$ of Proposition 2.1.12 is additive.*

Proposition 2.1.14. *The Karoubi envelope of an additive functor is additive.*

2.2 Idempotent Completion of prederivators

Let **Dia** be a fixed diagram category. All (pre)derivators in the rest of this chapter will be of fixed domain **Dia**.

Definition 2.2.1. A prederivator $\mathbb{D}$ is called **idempotent complete** if all its values are idempotent complete categories. A morphism of prederivators $i : \mathbb{D} \to \mathbb{E}$ **exhibits $\mathbb{E}$ as an idempotent completion of $\mathbb{D}$** if for any idempotent complete prederivator $\mathbb{E}'$, precomposition with i induces an equivalence of categories:

$$\underline{\mathsf{Hom}}(\mathbb{E}, \mathbb{E}') \to \underline{\mathsf{Hom}}(\mathbb{D}, \mathbb{E}')$$

Construction 2.2.2. Let $\mathbb{D}$ be a prederivator. The **Karoubi envelope $\mathbb{D}^\natural$ of $\mathbb{D}$** is the prederivator $\mathbb{D}^\natural = (-)^\natural \circ \mathbb{D}$ where $(-)^\natural$ is the 2-endofunctor of **CAT** defined on Proposition 2.1.12.

Proposition 2.2.3. *Let $\mathbb{D}$ be a prederivator. Then:*

(i) The levelwise functors $\eta_{\mathbb{D}(J)} : \mathbb{D}(J) \to \mathbb{D}(J)^\natural$ of Proposition 2.1.12 assemble to a strict morphism of prederivators $\eta_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}^\natural$ exhibiting $\mathbb{D}^\natural$ as an idempotent completion of $\mathbb{D}$.

(ii) If $\mathbb{D}$ satisfies Der1, then so does $\mathbb{D}^\natural$.

(iii) If the prederivator $\mathbb{D}$ satisfies Der3R, Der4R (respectively Der3L, Der4L) then so does the prederivator $\mathbb{D}^\natural$. In that case, the morphism of prederivators $\eta_{\mathbb{D}}$ is cocontinuous.

Proof. (i) The fact that $\eta_{\mathbb{D}}$ is a strict morphism of prederivators follows immediately by 2-naturality of $\eta : \mathrm{id}_{\mathbf{CAT}} \to (-)^\natural$. It remains to show that for any idempotent-complete prederivator $\mathbb{E}$, the functor:

$$- \circ \eta_{\mathbb{D}} : \underline{\mathsf{Hom}}(\mathbb{D}^\natural, \mathbb{E}) \to \underline{\mathsf{Hom}}(\mathbb{D}, \mathbb{E})$$

is an equivalence of categories. It is straightforward to verify it is fully faithful using the corresponding level-wise property of η together with 2-functoriality of η. To show essential surjectivity, consider any morphism of prederivators $F : \mathbb{D} \to \mathbb{E}$. For each category $J \in \mathbf{Dia}$, essential surjectivity of precomposing with $\eta_{\mathbb{D}(J)}$ furnishes a functor $\hat{F}_J : \mathbb{D}(J)^\natural \to \mathbb{E}(J)$ together with a natural isomorphism $\tau_J : \hat{F}_J \eta_{\mathbb{D}(J)} \Rightarrow F_J$. Given a functor $u : J \to K$ in **Dia** we have a series of natural isomorphisms:

$$u^* \hat{F}_K \eta_{\mathbb{D}(K)} \cong u^* F_K \cong F_J u^* \cong \hat{F}_J \eta_{\mathbb{D}(J)} u^* \cong \hat{F}_J u^* \eta_{\mathbb{D}(K)} \tag{2.2.4}$$

Thus, since precomposition with $\eta_{\mathbb{D}(K)}$ is an equivalence of categories, there is a unique natural isomorphism $\gamma_u^{\hat{F}} : u^* \hat{F}_K \cong \hat{F}_J u^*$ such that $\gamma_u^{\hat{F}} * \eta_{\mathbb{D}(K)}$ is exactly the morphism (2.2.4). By repeatedly using the level-wise universal property of idempotent completion, it is a tedious but straightforward verification to check that $\hat{F}$ together with the various $\gamma_u^{\hat{F}}$ assemble to a morphism of prederivators $\hat{F} : \mathbb{D}^\natural \to \mathbb{E}$ and that the various τ_J assemble to an isomodification $\tau : \hat{F}\eta_{\mathbb{D}} \to F$. This finishes the proof.

(ii) This follows immediately by 2-functoriality of $(-)^\natural$ and the observation that $(-)^\natural$ commutes with finite products.

(iii) The first part is immediate by 2-functoriality of η. For the second part, consider a functor $u : J \to K$ in **Dia**. We will use the notation $u^*, u_!$ to refer to $\mathbb{D}$-pullbacks and left Kan extensions respectively, while the corresponding ones for $\mathbb{D}^\natural$ will be explicitly written as $(u^*)^\natural$ and $(u_!)^\natural$ respectively. With this in mind, we have to show that the top arrow in the following commutative diagram:

$$\begin{tikzcd}
(u_!)^\natural \eta_{\mathbb{D}(J)} \arrow[r] \arrow[d] & \eta_{\mathbb{D}(K)} u_! \\
(u_!)^\natural \eta_{\mathbb{D}(J)} u^* u_! \arrow[r, equal] & (u_!)^\natural (u^*)^\natural \eta_{\mathbb{D}(K)} u_! \arrow[u]
\end{tikzcd}$$

is an isomorphism. The left vertical arrow is the unit θ of the adjunction $u_! \dashv u^*$, the bottom arrow follows from strictness of $\eta_{\mathbb{D}}$ and the right arrow is the counit of the adjunction $(u_!)^\natural \dashv$

$(u^*)^\natural$, i.e. $(-)^\natural$ applied to the counit ϵ of the adjunction $u_! \dashv u^*$. Consider an object X of $\mathbb{D}(J)$. The above commutative diagram at X becomes:

$$\begin{tikzcd}
(u_!)^\natural(X,1) \arrow[r] \arrow[d] & (u_!X,1) \\
(u_!)^\natural(u^*u_!X,1) \arrow[r, equal] & (u_!)^\natural(u^*)^\natural(u_!X,1) \arrow[u]
\end{tikzcd}$$

Unraveling the definitions, the left arrow is given by $u_!(\theta_X)$ while the right arrow is given by $\epsilon_{u_!X}$. Thus, the top arrow is an identity by the triangular identities.

□

Since η is fully faithful, we will identify $\mathbb{D}$ with a full subprederivator of $\mathbb{D}^\natural$, i.e. identify each level of $\mathbb{D}$ as a full subcategory of the corresponding level of $\mathbb{D}^\natural$.

Remark 2.2.5. We remark that for a prederivator $\mathbb{D}$, the underlying diagram functor $\mathrm{dia}^\natural$ associated to the Karoubi envelope $\mathbb{D}^\natural$ is not the same as the Karoubi envelope of the underlying diagram functor dia associated to $\mathbb{D}$. To make this more concrete, consider a coherent morphism $(X, e) \in \mathbb{D}^\natural([1])$. Let $f := \mathrm{dia}(X) : X_0 \to X_1$ be the underlying diagram of $X \in \mathbb{D}(e)$. By Construction 2.1.11 the underlying diagram of (X, e) is actually $fe_0 = e_1 f : (X_0, e_0) \to (X_1, e_1)$ and **not just** f. In fact, there is not a good reason a priori why f should even define a morphism $(X_0, e_0) \to (X_1, e_1)$, namely we may have $fe_0 = e_1 f \neq f$.

2.3 Idempotent completion of additive derivators

By the results of the previous section, we know that the idempotent completion of a derivator exists as a prederivator, and is almost a derivator (satisfies all axioms except *Der*2). The proof of *Der*2 turns out to be tricky and the author did not succeed in proving it in full generality. However, the case turns out to be considerably simpler for additive derivators.

Remark 2.3.1. Let $\mathbb{D}$ be an additive derivator. Propositions 2.1.13, 2.1.14 and 2.2.3 allow us to identify $\mathbb{D}$ as a full additive subprederivator of $\mathbb{D}^\natural$, i.e. each level of $\mathbb{D}$ is a full additive subcategory of the corresponding level of $\mathbb{D}^\natural$ via the morphism $\eta_{\mathbb{D}}$.

The following lemma is a first step towards proving *Der*2:

Lemma 2.3.2. *Let $\mathbb{D}$ be a strong additive derivator. Let $Z \in \mathbb{D}([1])^\natural$ be a coherent morphism such that Z_0, Z_1 are zero objects of $\mathbb{D}(e)^\natural$. Then Z is a zero object of $\mathbb{D}([1])^\natural$.*

Proof. We can represent Z as a pair $Z = (X, e)$, where $X \in \mathbb{D}([1])$ and $e : X \to X$ is an idempotent. By assumption, $Z_0 = (X_0, e_0)$ and $Z_1 = (X_1, e_1)$ are zero objects of $\mathbb{D}(e)^\natural$. This implies that the two morphisms $e_0 : X_0 \to X_0$ and $e_1 : X_1 \to X_1$ are actually zero. Thus, the idempotent morphism $e : X \to X$ in $\mathbb{D}([1])$ is in the kernel of the underlying diagram functor $\mathbb{D}([1]) \to \mathbb{D}(e)^{[1]}$. By Theorem 1.6.6, we get that $e = e^2 = 0$, which proves our assertion. □

Lemma 2.3.3. *Let $\mathbb{D}$ be an additive strong derivator. Then the prederivator $\mathbb{D}^\natural$ is also strong.*

Proof. By shifting, it is enough to show that the underlying diagram functor:

$$\widehat{\mathrm{dia}_{[1]}} : \mathbb{D}([1])^\natural \to (\mathbb{D}(e)^\natural)^{[1]}$$

is full and essentially surjective (cf. Remark 2.2.5 for why we don't use the notation $\mathrm{dia}^\natural_{[1]}$ instead). We will reserve the notation $\mathrm{dia}_{[1]}$ for the underlying diagram functor:

$$\mathrm{dia}_{[1]} : \mathbb{D}([1]) \to \mathbb{D}(e)^{[1]}$$

First, we prove essential surjectivity of $\widehat{\mathrm{dia}_{[1]}}$. Consider a morphism $f : (X_0, e_0) \to (X_1, e_1)$ in $\mathbb{D}(e)^\natural$. In other words, we have an incoherent morphism $f : X_0 \to X_1$ in $\mathbb{D}(e)$ such that the following diagram in $\mathbb{D}(e)$:

(2.3.4)
$$\begin{tikzcd} X_0 \arrow[r, "e_0"] \arrow[d, "f"'] \arrow[dr, "f"] & X_0 \arrow[d, "f"] \\ X_1 \arrow[r, "e_1"'] & X_1 \end{tikzcd}$$

commutes. Pick a coherent morphism $X \in \mathbb{D}([1])$ such that $\mathrm{dia}_{[1]}(X) \cong f : X_0 \to X_1$ in $\mathbb{D}(e)$. By Corollary 1.6.9, there is an idempotent $e : X \to X$, whose underlying diagram is isomorphic to:

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "e_0"'] & X_1 \arrow[d, "e_1"] \\ X_0 \arrow[r, "f"'] & X_1 \end{tikzcd}$$

By Remark 2.2.5, the underlying diagram $\widehat{\mathrm{dia}_{[1]}}(X, e)$ is $fe_0 = e_1 f : (X_0, e_0) \to (X_1, e_1)$ up to an isomorphism, which is the same as $f : (X_0, e_0) \to (X_1, e_1)$ using commutativity of the diagram 2.3.4.

For fullness: consider two objects (X, e) and (X', e') in $\mathbb{D}^\natural([1])$ for two coherent morphisms $X, X' \in \mathbb{D}([1])$. Let $\alpha : \widehat{\mathrm{dia}_{[1]}}(X, e) \to \widehat{\mathrm{dia}_{[1]}}(X', e')$ be a morphism of underlying diagrams in $\mathbb{D}([1])$. This induces a morphism

$$\alpha' = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right) : \widehat{\mathrm{dia}_{[1]}}(X, e) \oplus \widehat{\mathrm{dia}_{[1]}}(X, 1-e) \to \widehat{\mathrm{dia}_{[1]}}(X', e') \oplus \widehat{\mathrm{dia}_{[1]}}(X', 1-e')$$

Since up to an isomorphism this is a morphism in $\mathbb{D}(e)^{[1]}$ there is a morphism:

$$A' : (X, e) \oplus (X', 1-e') \to (X', e') \oplus (X', 1-e')$$

whose underlying diagram in $\mathbb{D}^\natural(e)$ is exactly α'. This finishes the proof. $\square$

Lemma 2.3.5. *Let $\mathbb{D}$ be a strong additive derivator. If $X \in \mathbb{D}([1])^\natural$ is a coherent morphism such that X_j is isomorphic to an object of $\mathbb{D}(e)$ for $j = 0, 1$, then X is isomorphic to an object of $\mathbb{D}([1])$.*

Proof. As in the proof of Lemma 2.3.3, we will write:

$$\widehat{\mathrm{dia}_{[1]}} : \mathbb{D}([1])^\natural \to (\mathbb{D}(e)^\natural)^{[1]}$$

for the underlying diagram functor associated with $\mathbb{D}^\natural$ and $\mathrm{dia}_{[1]}$ for the underlying diagram functor associated with $\mathbb{D}$. Set $f = \widehat{\mathrm{dia}_{[1]}}(X, e) : (X_0, e_0) \to (X_1, e_1)$. By assumption, we can find isomorphisms $\alpha_i : (X_i, e_i) \xrightarrow{\cong} (Y_i, 1_{Y_i})$ in $\mathbb{D}(e)^\natural$ for $i = 0, 1$. Set

$$g := \alpha_1 \circ f \circ \alpha_0^{-1} : (Y_0, 1_{Y_0}) \to (Y_1, 1_{Y_1})$$

By strongness for $\mathbb{D}$, we can find an object $Y \in \mathbb{D}([1])$ such that $\mathrm{dia}_{[1]}(Y) \cong g : Y_0 \to Y_1$. Then, clearly

$$\widehat{\mathrm{dia}_{[1]}}(Y, 1_Y) \cong g : (Y_0, 1) \to (Y_1, 1) \tag{2.3.6}$$

By Lemma 2.3.3, we can find a morphism $A : (X, e) \to (Y, 1_Y)$ with underlying diagram (up to the isomorphism (2.3.6)):

$$\begin{array}{ccc} (X_0, e_0) & \xrightarrow{f} & (X_1, e_1) \\ {\scriptstyle\alpha_0}\downarrow & & \downarrow{\scriptstyle\alpha_1} \\ (Y_0, 1_{Y_0}) & \xrightarrow[g]{} & (Y_1, 1_{Y_1}) \end{array} \tag{2.3.7}$$

and a morphism $B : (Y, 1_Y) \to (X, e)$ with underlying diagram (again up to the isomorphism (2.3.6)):

$$\begin{tikzcd}
(Y_0, 1_{Y_0}) \arrow[r, "g"] \arrow[d, "\alpha_0^{-1}"'] & (Y_1, 1_{Y_1}) \arrow[d, "\alpha_1^{-1}"] \\
(X_0, e_0) \arrow[r, "f"'] & (X_1, e_1)
\end{tikzcd} \tag{2.3.8}$$

The morphism $\rho = AB : (Y, 1_Y) \to (Y, 1_Y)$ is pointwise an identity, hence an isomorphism (by *Der*2 for $\mathbb{D}$). Let $A' = \rho^{-1}A$. Then, the underlying diagram of A' is still (isomorphic to) (2.3.7) and we have $A'B = 1_{(Y,1_Y)}$. It follows that $r = BA'$ is an idempotent on (X, e) with image $(Y, 1_Y)$ and whose underlying diagram is exactly:

$$\begin{tikzcd}
(X_0, e_0) \arrow[r, "f"] \arrow[d, "="'] & (X_1, e_1) \arrow[d, "="] \\
(X_0, e_0) \arrow[r, "f"'] & (X_1, e_1)
\end{tikzcd}$$

In particular, we have a splitting $(X, e) \cong (Y, 1_Y) \oplus (Z, e')$, where (Z, e') is a kernel for r. Since $\widehat{\mathrm{dia}_{[1]}}$ preserves splitting of idempotents and those are unique up to isomorphism, it follows that (Z, e') is pointwise a zero object of $\mathbb{D}^\natural(e)$. Hence, by 2.3.2, (Z, e') is also a zero object for $\mathbb{D}^\natural([1])$, which finishes the proof. □

Given an additive derivator $\mathbb{D}$, we have already seen that we can identify it as a full additive subprederivator of its Karoubi envelope $\mathbb{D}^\natural$. We already know that $\mathbb{D}^\natural$ is almost a derivator (satisfies all axioms except possibly for *Der*2, i.e. we cannot test isomorphisms pointwise, not yet at least). We summarize the facts/constructions of interest from usual derivators that we can use on $\mathbb{D}^\natural$ so far:

- Right Kan extensions along sieves (dually left Kan extensions along cosieves) are extensions by zero (cf. Proposition 1.5.4). More precisely, suppose $u : J \to K$ is a sieve and $X \in \mathbb{D}(J)$. Proposition 1.5.4 ensures that $(u_*X)_k \cong 0$ for $k \in K - u(J)$ and an easy application of *Der*4*L* shows that $(u_*X)_{u(j)} \cong X_j$. Thus, the restriction of u_*X to the image of u is pointwise isomorphic to X, but we emphasize that without *Der*2 we cannot conclude that the same is true globally!

- The homotopy pushout of an isomorphism along any morphism is also an isomorphism. This is the content of [Gro13, Proposition 3.12]. Again, *Der*2 is only used to prove a converse which we don't care about here.

- We can define the suspension $\Sigma_J : \mathbb{D}^\natural(J) \to \mathbb{D}^\natural(J)$ for any category $J \in \mathbf{Dia}$ just like for usual derivators. The first bullet-point above ensures that we do indeed have a cocartesian square as in (1.5.6). However, note that without *Der*2 we do not know this defines a morphism of derivators (!) but we won't need this to prove *Der*2.

Lemma 2.3.9. *Let $\mathbb{D}$ be an additive strong derivator and $\Sigma : \mathbb{D} \to \mathbb{D}$ the suspension (cf. Section 1.5). For any category $J \in$ **Dia** the canonical functor $id_{\mathbb{D}^\natural(J)} \oplus \Sigma_J : \mathbb{D}^\natural(J) \to \mathbb{D}^\natural(J)$ takes values in $\mathbb{D}(J)$ up to an isomorphism (possibly non-functorial even level-wise).*

Proof. Since $\mathbb{D}$ is a full subprederivator of $\mathbb{D}^\natural$, it suffices to show that $x \oplus \Sigma x$ is isomorphic to an object of $\mathbb{D}(J)$ for every category $J \in$ **Dia** and object $x \in \mathbb{D}(J)^\natural$. By shifting, we may also reduce to the case $J = e$. To conclude, write $i : [1] \to \ulcorner$ for the inclusion of the top arrow. Consider any object $x \in \mathbb{D}(e)^\natural$. By Remark 2.1.9 there is some $y \in \mathbb{D}(e)^\natural$ such that $x \oplus y$ is isomorphic to some object of $\mathbb{D}(e)$. Since i is a sieve, the underlying diagram of $i_*\pi_{[1]}^* y$ looks like:

$$\begin{tikzcd} y \arrow[r, "1"] \arrow[d] & y \\ 0 & \end{tikzcd}$$

Thus, we get a cocartesian square $Y = (i_\ulcorner)_! i_* \pi_{[1]}^* y$ in $\mathbb{D}^\natural(\square)$ with underlying diagram (cf. the remarks before the statement of the lemma):

$$\begin{tikzcd} y \arrow[r, "1"] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & 0 \end{tikzcd}$$

Similarly, we have a cocartesian square $X = (i_\ulcorner)_! i_* 1_! x$ with underlying diagram:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & x \arrow[d, "1"] \\ 0 \arrow[r] & x \end{tikzcd}$$

Finally, the very definition of Σ implies the existence of a cocartesian square Z with underlying diagram:

$$\begin{tikzcd} x \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma x \end{tikzcd}$$

Note that since left Kan extensions are additive functors, we have a cocartesian square $X \oplus Y \oplus Z$ with underlying diagram:

$$\begin{tikzcd} x \oplus y \arrow[r] \arrow[d] & x \oplus y \arrow[d] \\ 0 \arrow[r] & x \oplus \Sigma x \end{tikzcd}$$

Let $Q \in \mathbb{D}([1])^\natural$ be the top (coherent) arrow of this coherent square. By Lemma 2.3.5, Q is isomorphic to some object of $\mathbb{D}([1])$. Since the inclusion of $\mathbb{D}$ to $\mathbb{D}^\natural$ commutes with left and right Kan extensions (cf. Proposition 2.2.3), the colimit of $i_* Q$ is isomorphic to some object of $\mathbb{D}(e)$. This finishes the proof. □

We are now ready for the main theorem of this section:

Theorem 2.3.10. *Let $\mathbb{D}$ be a strong additive derivator. Then $\mathbb{D}^\natural$ is also a strong additive derivator. Moreover, if $\mathbb{D}$ is stable, then so is $\mathbb{D}^\natural$.*

Proof. For the first part, by Lemma 2.3.3 and Remark 2.3.1 it is enough to show that $\mathbb{D}^\natural$ satisfies *Der*2. Let J be a category in **Dia** and f an arrow in $\mathbb{D}^\natural(J)$ such that f_j is an isomorphism in $\mathbb{D}^\natural(e)$ for any object $j \in J$. By Lemma 2.3.9, we know that $f \oplus \Sigma f$ is isomorphic to a morphism g in $\mathbb{D}(J)$ that is a pointwise isomorphism since f is. By *Der*2 for $\mathbb{D}$ we conclude that g is an isomorphism. Hence $f \oplus \Sigma f$ is an isomorphism which implies that f is.

For the second part, let us write $\Sigma, \Sigma^\natural$ for the suspension morphisms associated with $\mathbb{D}$ and $\mathbb{D}^\natural$ respectively. It is straightforward to see that $\Sigma^\natural$ is actually equal to the image of Σ by $(-)^\natural$. Since 2-functors on **CAT** preserve equivalences of categories, it follows that $\Sigma^\natural$ is a (levelwise) equivalence of derivators. The assertion then follows by the main result in [GPS14b]. □

Example 2.3.11. As advertised in the introduction of this section, Theorem 2.3.10 recovers the main result in [LS14] or [BS01] if the (pre)triangulated category in question is the base of a strong stable/additive derivator, which is the case in all examples one encounters in practice.

CHAPTER 3

Levelwise modules over separable monads on stable derivators

3.1 2-Monads and their modules

In this section, we introduce some results on the theory of 2-monads and their modules that will be used later, and in particular in Section 3.2. We begin with the general definition of a monad internal to a 2-category:

Definition 3.1.1. Let $\mathcal{K}$ be a 2-category, and let x be an object of $\mathcal{K}$. A **monad** on x consists of a triple $(M, \mu, \mathbb{S})$, where $M : x \to x$ is a 1-cell in $\mathcal{K}$, and $\mu : M^2 \to M$ (the multiplication) and $\mathbb{S} : 1_x \to M$ (the unit) are 2-cells such that the two diagrams below commute:

(3.1.2)
$$\begin{tikzcd} M^3 \arrow[r, "\mu M"] \arrow[d, "M\mu"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"'] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\mathbb{S}M"] \arrow[dr, "1_M"'] & M^2 \arrow[d, "\mu"] & M \arrow[l, "M\mathbb{S}"'] \arrow[dl, "1_M"] \\ & M & \end{tikzcd}$$

We will often denote a monad by M, leaving the multiplication and unit out of the notation.

Note that when $\mathcal{K} = \mathbf{CAT}$ the above definition specializes to the usual definition of a monad in ordinary category theory (see, for instance [Mac98, Chapter VI]). In this context, M is an endofunctor of some category, and $\mu, \mathbb{S}$ are natural transformations as in Definition 3.1.1 making the diagrams (3.1.2) commute. When appropriate we will refer to such a monad as a **classical monad** to distinguish from the more general notion of a 2-monad given below. To avoid confusion, we will also denote a 2-monad by T and reserve M for a classical monad or (later) for a monad of (pre)derivators.

Definition 3.1.3. A **2-monad on** a 2-category $\mathcal{K}$ is a monad on $\mathcal{K}$, where $\mathcal{K}$ is viewed as an object of the 2-category **2-CAT**.

Thus, a 2-monad consists of a 2-endofunctor $T : \mathcal{K} \to \mathcal{K}$, with 2-natural transformations for its multiplication and unit. Note that T has an underlying classical monad on the underlying 1-category of $\mathcal{K}$. We remark that this definition can be weakened in various ways. For instance, we could require the multiplication or the unit (or both) to be lax natural transformations instead of 2-natural, even if we still require the 2-functor T to be strict. For our purposes though, the above definition will be sufficient.

Definition 3.1.4. Let T be a 2-monad on a 2-category $\mathcal{K}$. A ***T*-module** (or ***T*-algebra**) is a pair (x, λ), where x is an object of $\mathcal{K}$ and $\lambda : Tx \to x$ is a morphism, such that the following diagrams commute:

(3.1.5)
$$\begin{tikzcd} T^2x \arrow[r, "T\lambda"] \arrow[d, "\mu_x"'] & Tx \arrow[d, "\lambda"] \\ Tx \arrow[r, "\lambda"'] & x \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\mathbb{S}_x"] \arrow[dr, "1_x"'] & Tx \arrow[d, "\lambda"] \\ & x \end{tikzcd}$$

Again, this is only the strict version of the definition, but we will omit the adjective since we will have no use for the weakened definitions. Note that this definition does not in any way use the 2-categorical structure on $\mathcal{K}$ or T; in fact, a module over the 2-monad T in our sense is the same as a module over the underlying classical monad of T (see Example 3.1.12(iii)). Where the 2-structure appears for us is in the definition of a morphism of modules, given below:

Definition 3.1.6. Let T be a 2-monad on $\mathcal{K}$, and let $(x_1, \lambda_1), (x_2, \lambda_2)$ be T-modules. A **lax *T*-morphism** from (x_1, λ_1) to (x_2, λ_2) consists of a pair $\left(f, \overline{f}\right)$, where $f : x_1 \to x_2$ is a 1-cell, and $\overline{f}$ is a 2-cell (called the lax enrichment of f) as in the following diagram:

(3.1.7)
$$\begin{tikzcd} Tx_1 \arrow[r, "\lambda_1"] \arrow[d, "Tf"'] & x_1 \arrow[d, "f"] \\ Tx_2 \arrow[r, "\lambda_2"'] \arrow[ur, phantom, "\overline{f}\,\Nearrow"] & x_2 \end{tikzcd}$$

such that we have an equality of pastings:

(3.1.8)
$$\begin{tikzcd} T^2x_1 \arrow[r, "\mu_{x_1}"] \arrow[d, "T^2f"'] & Tx_1 \arrow[r, "\lambda_1"] \arrow[d, "Tf"] & x_1 \arrow[d, "f"] \\ T^2x_2 \arrow[r, "\mu_{x_2}"'] & Tx_2 \arrow[r, "\lambda_2"'] \arrow[ur, phantom, "\overline{f}\,\Nearrow"] & x_2 \end{tikzcd} = \begin{tikzcd} T^2x_1 \arrow[r, "T\lambda_1"] \arrow[d, "T^2f"'] & Tx_1 \arrow[r, "\lambda_1"] \arrow[d, "Tf"] & x_1 \arrow[d, "f"] \\ T^2x_2 \arrow[r, "T\lambda_2"'] \arrow[ur, phantom, "T\overline{f}\,\Nearrow"] & Tx_2 \arrow[r, "\lambda_2"'] \arrow[ur, phantom, "\overline{f}\,\Nearrow"] & x_2 \end{tikzcd}$$

and the following pasting is the identity:

$$
\begin{tikzcd}
x_1 \arrow[r, "\mathbb{S}_{x_1}"] \arrow[d, "f"'] & Tx_1 \arrow[r, "\lambda_1"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overline{f} \nearrow"] & x_1 \arrow[d, "f"] \\
x_2 \arrow[r, "\mathbb{S}_{x_2}"'] & Tx_2 \arrow[r, "\lambda_2"'] & x_2
\end{tikzcd}
\tag{3.1.9}
$$

A **strong T-morphism** is a lax T-morphism $(f, \overline{f})$ such that $\overline{f}$ is invertible. We will often refer to a lax T-morphism $(f, \overline{f})$ by its underlying morphism f, omitting the lax enrichment $\overline{f}$ from our notation.

To organize modules into a 2-category, we need to define what 2-cells are:

Definition 3.1.10. Let T be a 2-monad on $\mathcal{K}$, and let $(f, \overline{f}), (g, \overline{g}) : (x_1, \lambda_1) \to (x_2, \lambda_2)$ be lax T-morphisms. A T**-2-cell** from $\left(f, \overline{f}\right)$ to $(g, \overline{g})$ is a 2-cell $\alpha : f \Rightarrow g$ in $\mathcal{K}$ such that the following two pastings are equal:

$$
\begin{tikzcd}
Tx_1 \arrow[r, "\lambda_1"] \arrow[d, "Tg"] \arrow[d, bend right=60, "Tf"'] \arrow[d, phantom, shift right=4, "\overset{T\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\overline{g} \nearrow"] & x_1 \arrow[d, "g"] \\
Tx_2 \arrow[r, "\lambda_2"'] & x_2
\end{tikzcd}
=
\begin{tikzcd}
Tx_1 \arrow[r, "\lambda_1"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overline{f} \nearrow"] & x_1 \arrow[d, "f"'] \arrow[d, bend left=60, "g"] \arrow[d, phantom, shift left=4, "\overset{\alpha}{\Rightarrow}"] \\
Tx_2 \arrow[r, "\lambda_2"'] & x_2
\end{tikzcd}
\tag{3.1.11}
$$

There are 2-categories $T\text{-}\mathbf{Mod}^{\mathrm{lax}}_{\mathcal{K}}$ (respectively $T\text{-}\mathbf{Mod}^{\mathrm{str}}_{\mathcal{K}}$) which have T-modules as 0-cells, (respectively strong) T-morphisms as 1-cells, and T-2-cells as 2-cells.

Example 3.1.12.

(i) The algebraic simplex category or augmented simplex category Δ_+ has objects finite ordinal numbers $\underline{n} = \{0, 1, ..., n-1\}$ (where $\underline{0} = \varnothing$), and morphisms order-preserving maps. This is a strict monoidal category under the usual ordinal addition (see [Mac98, VII.5]). The multiplication and unit for this monoidal structure induce the structure of a 2-monad on the 2-functor $T := \Delta_+ \times - : \mathbf{CAT} \to \mathbf{CAT}$. We will describe $T\text{-}\mathbf{Mod}^{\mathrm{lax}}_{\mathbf{CAT}}$ and $T\text{-}\mathbf{Mod}^{\mathrm{str}}_{\mathbf{CAT}}$ in concrete terms in Section 3.2 (see Definition 3.2.1 and Proposition 3.2.4).

(ii) Consider a small 2-category $\mathcal{L}$, and write X for the set of objects of $\mathcal{L}$. Consider the 2-category $\mathcal{K} = \mathbf{CAT}^X$ of X-indexed collections of categories, with 1 and 2-cells being taken componentwise. There is a 2-monad T on $\mathcal{K}$ such that:

- T-modules are (strict) 2-functors $\mathcal{L} \to \mathbf{CAT}$
- Lax (respectively strong) T-morphisms are lax (respectively pseudo-natural) transformations of 2-functors $\mathcal{L} \to \mathbf{CAT}$.
- T-2-cells are modifications.

We refer the reader to [BKP89, 6.6] for details. In fact, Section 6 of [BKP89] provides a plethora of other examples such as monoidal categories (possibly symmetric and/or strict), categories with finite coproducts, and more.

(iii) Although we have already alluded to what modules are in ordinary category theory (see the paragraph following Definition 3.1.4), we examine them in more detail here since they are central to this chapter. Let M be a (classical) monad on a category $\mathscr{C}$. We can view $\mathscr{C}$ as a 2-category $\mathcal{K}$ where 2-cells are identities, and then the classical monad M automatically upgrades to a 2-monad T, whose underlying classical monad is M. By an M-module we mean a T-module in the sense of Definition 3.1.4. Notice that strong T-morphisms coincide with lax T-morphisms (since all 2-cells are identities) and such a morphism is simply called a morphism of M-modules (or M-linear map). Of course, this can all be defined without any reference to 2-categories (see, for instance [Mac98, Chapter VI]).

Let $M\text{-}\mathbf{Mod}_{\mathscr{C}}$ be the **Eilenberg-Moore category** with M-modules as objects and M-linear maps as morphisms. Then, there is an adjunction $F_M : \mathscr{C} \leftrightarrows M\text{-}\mathbf{Mod}_{\mathscr{C}} : U_M$, where the functor $F_M : \mathscr{C} \to M\text{-}\mathbf{Mod}_{\mathscr{C}}$ defined by $x \mapsto (Mx, \mu_x)$ on objects and $f \mapsto Mf$ on morphisms is called the **free module functor**, and the functor $U_M : M\text{-}\mathbf{Mod}_{\mathscr{C}} \to \mathscr{C}$ defined by forgetting the action of M, is called the **forgetful functor**. This is described in more detail in [Mac98, Chapter VI]. We will refer to the adjunction $F_M \dashv U_M$ as the **Eilenberg-Moore adjunction**.

The following two results are proved in [Kel74] for the 2-category **CAT** but are true for any 2-category:

Theorem 3.1.13. *[Kel74, Theorem 1.4] Let T be a 2-monad on a 2-category $\mathcal{K}$. Let $U = (g, \overline{g}) : (x_2, \lambda_2) \to (x_1, \lambda_1)$ be a lax T-morphism. Then U has a left adjoint F in $T\text{-}\mathbf{Mod}_{\mathcal{K}}^{lax}$ if and only if g*

has a left adjoint f in $\mathcal{K}$ and the mate $\overline{f}'$ of $\overline{g}$ in the following pasting is an isomorphism:

(3.1.14)
$$\begin{tikzcd}
& Tx_1 \arrow[r, "\lambda_1"] \arrow[dr, phantom, "\Downarrow \overline{g}"] & x_1 \arrow[r, "f"] \arrow[dr, phantom, "\Downarrow \epsilon"] & x_2 \\
Tx_1 \arrow[ur, bend left, "1_{Tx_1}"] \arrow[r, "Tf"'] \arrow[ur, phantom, "\Downarrow T\eta"] & Tx_2 \arrow[u, "Tg"] \arrow[r, "\lambda_2"'] & x_2 \arrow[u, "g"] \arrow[ur, bend right, "1_{x_2}"'] &
\end{tikzcd}$$

Here ϵ, η are the counit and unit respectively of the adjunction $f \dashv g$. In that case, the left adjoint $F := \left(f, \overline{f}\right)$ of G is necessarily strong, and $\overline{f} = (\overline{f}')^{-1}$.

Theorem 3.1.15. *[Kel74, Theorem 1.5] Let T be a 2-monad on a 2-category $\mathcal{K}$. Let $F = \left(f, \overline{f}\right)$: $(x_1, \lambda_1) \to (x_2, \lambda_2)$ be a lax T-morphism. Then F admits a right adjoint in T-$\mathbf{Mod}^{lax}_{\mathcal{K}}$ if and only if f admits a right adjoint g in $\mathcal{K}$ and F is strong. In that case, the lax enrichment $\overline{g}$ of g is given by the mate of $\left(\overline{f}\right)^{-1}$, namely as the pasting:*

(3.1.16)
$$\begin{tikzcd}
Tx_2 \arrow[r, "Tg"] \arrow[dr, bend right, "1_{Tx_2}"'] \arrow[dr, phantom, "T\epsilon \Swarrow"] & Tx_1 \arrow[r, "\lambda_1"] \arrow[d, "Tf"'] \arrow[dr, phantom, "(\overline{f})^{-1} \Swarrow"] & x_1 \arrow[d, "f"] \arrow[dr, bend left, "1_{x_1}"] \arrow[dr, phantom, "\eta \Swarrow"] & \\
& Tx_2 \arrow[r, "\lambda_2"'] & x_2 \arrow[r, "g"'] & x_1
\end{tikzcd}$$

where ϵ, η are the counit and unit respectively of the adjunction $f \dashv g$.

3.2 2-functoriality of levelwise modules

The goal of this section is to show that given a monad M on some prederivator $\mathbb{D}$, its levelwise modules assemble to a prederivator. This result as well as various other proofs later in this chapter boil down to the following situation: we have an adjunction $F \dashv G$ between two categories, each category has a specified (classical) monad on it and we want to get an induced adjunction between the corresponding categories of modules. To tackle this, we consider a suitable 2-category $\mathbf{Mnd}^{\mathrm{lax}}$ of categories with a monad action (see Definition 3.2.1), and reduce to upgrading our original adjunction to an adjunction in $\mathbf{Mnd}^{\mathrm{lax}}$ (see Proposition 3.2.7). Finally, we show that $\mathbf{Mnd}^{\mathrm{lax}}$ can be described as modules over a 2-monad, allowing us to conclude using Theorems 3.1.13 and 3.1.15.

Definition 3.2.1. Consider two (classical) monads M, M' on categories $\mathscr{C}, \mathscr{C}'$ respectively. A **lax monad functor** from M to M' is a pair (F, ϕ), where $F : \mathscr{C} \to \mathscr{C}'$ is a functor and $\phi : M'F \Rightarrow FM$

is a natural transformation such that the following diagrams commute:

(3.2.2)
$$\begin{tikzcd} M'^2F \arrow[r, "M'\phi"] \arrow[d, "\mu' F"'] & M'FM \arrow[r, "\phi M"] & FM^2 \arrow[d, "F\mu"] \\ M'F \arrow[rr, "\phi"'] & & FM \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\mathbb{S}'F"] \arrow[dr, "F\mathbb{S}"'] & M'F \arrow[d, "\phi"] \\ & FM \end{tikzcd}$$

A **strong monad functor** is a lax monad functor (F, ϕ) such that ϕ is an isomorphism. Given two lax monad functors $(F_1, \phi_1), (F_2, \phi_2)$ from M to M', a **monad functor transformation** is a natural transformation $\alpha : F_1 \Rightarrow F_2$ such that the diagram below commutes:

(3.2.3)
$$\begin{tikzcd} M'F_1 \arrow[r, "M'\alpha"] \arrow[d, "\phi_1"'] & M'F_2 \arrow[d, "\phi_2"] \\ F_1M \arrow[r, "\alpha M"'] & F_2M \end{tikzcd}$$

We write $\mathbf{Mnd}^{\mathrm{lax}}$ (respectively $\mathbf{Mnd}^{\mathrm{str}}$) for the 2-category with monads as 0-cells, lax (respectively strong) monad functors as 1-cells, and monad functor transformations as 2-cells. For notational convenience, we will write the 0-cells in $\mathbf{Mnd}^{\mathrm{lax}}$ as $(\mathscr{C}, M)$ where $\mathscr{C}$ is the underlying category of the monad M.

The definition above is taken from [Str72], where the author simply refers to lax monad functors as monad functors. For the following theorem, recall from Example 3.1.12(i) that we have a monad $T = \Delta_+ \times -$ on **CAT** where Δ_+ has objects finite ordinals with order preserving maps.

Proposition 3.2.4. *Let $T = \Delta_+ \times -$ be the 2-monad on* **CAT** *of Example 3.1.12(i). There is an isomorphism of 2-categories between T-$\mathbf{Mod}^{lax}_{\mathbf{CAT}}$ and $\mathbf{Mnd}^{lax}$, under which strong T-morphisms correspond to strong monad functors.*

Proof. A 0-cell in T-$\mathbf{Mod}^{\mathrm{lax}}_{\mathbf{CAT}}$ is, by definition, a category $\mathscr{C}$, equipped with a functor

$$\Delta_+ \times \mathscr{C} \to \mathscr{C}$$

making the diagrams (3.1.5) commute. By adjunction, such a functor is equivalent to a functor $\Delta_+ \to \mathrm{End}(\mathscr{C})$; commutativity of the diagrams (3.1.5) means that the latter functor is strictly monoidal. By [Mac98, VII.5], this is the same as a monoid object in the endomorphism category of $\mathscr{C}$, i.e. a monad M on $\mathscr{C}$. More specifically, M is the action of $\underline{1}$ on $\mathscr{C}$, its unit is induced by the unique morphism $\underline{0} \to \underline{1}$, and its multiplication is induced by the unique morphism $\underline{2} \to \underline{1}$ in Δ_+. Hence T-modules are categories $\mathscr{C}$ equipped with a monad M.

Consider now categories $\mathscr{C}, \mathscr{C}'$ equipped with T-actions λ, λ' turning them into T-modules. Let M, M' be the associated monads. Let $(F, \overline{F})$ be a lax T-morphism between them. Thus, F is a functor $\mathscr{C} \to \mathscr{C}'$ and $\overline{F}$ is a natural transformation populating the square:

(3.2.5)
$$\begin{tikzcd} \Delta_+ \times \mathscr{C} \arrow[r, "\lambda"] \arrow[d, "\mathrm{id}\times F"'] \arrow[dr, phantom, "\overline{F}\nearrow"] & \mathscr{C} \arrow[d, "F"] \\ \Delta_+ \times \mathscr{C}' \arrow[r, "\lambda'"'] & \mathscr{C}' \end{tikzcd}$$

Restricting $\overline{F}$ to $\{\underline{1}\} \times \mathscr{C}$ yields a natural transformation $\phi : M'F \Rightarrow FM$. Using commutativity of the diagrams (3.1.8) and (3.1.9) we deduce that the diagrams (3.2.2) commute, i.e. the pair (F, ϕ) defines a lax monad functor. Conversely, given a lax monad functor (F, ϕ), we define $\overline{F}$ on $\{\underline{n}\} \times \mathscr{C}$ as the horizontal pasting of n squares

(3.2.6)
$$\begin{tikzcd} \mathscr{C} \arrow[r, "M"] \arrow[d, "F"'] \arrow[dr, phantom, "\overline{F}\nearrow"] & \mathscr{C} \arrow[d, "F"] \\ \mathscr{C}' \arrow[r, "M'"'] & \mathscr{C}' \end{tikzcd}$$

It is then straightforward to see that this defines a natural transformation as in the square (3.2.5) above and that $(F, \overline{F})$ is a lax T-morphism. From the construction it is clear also that $\overline{F}$ is an isomorphism if and only if ϕ is. The correspondence between 2-cells in the two categories is similar and left to the reader. The functoriality is immediate since the mapping from T-$\mathbf{Mod}^{\mathrm{lax}}_{\mathbf{CAT}}$ to $\mathbf{Mnd}^{\mathrm{lax}}$ is just restriction. This completes the proof. □

Proposition 3.2.7. *There is a* 2*-functor* $\mathbf{Mod} : \mathbf{Mnd}^{lax} \to \mathbf{CAT}$ *given on objects by assigning to a (classical) monad* M *on a category* $\mathscr{C}$ *its Eilenberg-Moore category* M-$\mathbf{Mod}_{\mathscr{C}}$.

Proof. First, we define $\mathbf{Mod}$ on 1-cells: Given a lax monad functor

$$(F, \phi) : (\mathscr{C}, M) \to (\mathscr{C}', M')$$

we define a functor

$$\mathbf{Mod}(F, \phi) : \ M\text{-}\mathbf{Mod}_{\mathscr{C}} \ \to \ M'\text{-}\mathbf{Mod}_{\mathscr{C}'}$$

given on objects by $(X, \lambda) \mapsto (FX, F\lambda \circ \phi_X)$ and on morphisms as the identity. Commutativity of the diagrams below shows that $\mathbf{Mod}(F, \phi)$ is well-defined on objects:

$$
\begin{tikzcd}
M'^2FX \arrow[r, "M'\phi_X"] \arrow[dd, "\mu'_{FX}"'] & M'FMX \arrow[r, "M'F\lambda"] \arrow[d, "\phi_{MX}"] & M'FX \arrow[d, "\phi_X"] \\
 & FM^2X \arrow[r, "FM\lambda"] \arrow[d, "F\mu_X"] & FMX \arrow[d, "F\lambda"] \\
M'FX \arrow[r, "\phi_X"] & FMX \arrow[r, "F\lambda"] & FX
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "\mathbb{S}'_{FX}"] \arrow[dr, "F\mathbb{S}_X"'] \arrow[ddr, bend right, "1_{FX}"'] & M'FX \arrow[d, "\phi_X"] \\
 & FMX \arrow[d, "F\lambda"] \\
 & FX
\end{tikzcd}
$$

and one checks similarly that it is well-defined on morphisms, while functoriality is straightforward.

We now define $\mathbf{Mod}$ on 2-cells: Let $\alpha : (F_1, \phi_1) \Rightarrow (F_2, \phi_2)$ a monad functor transformation between lax monad functors $(\mathscr{C}, M) \to (\mathscr{C}', M')$. It is then straightforward from (3.2.3) and naturality of α that for any M-module (X, λ), α_X is an M'-linear map

$$(F_1X, F_1\lambda \circ (\phi_1)_X) \to (F_2X, F_2\lambda \circ (\phi_2)_X)$$

and hence $\beta_{(X,\lambda)} = \alpha_X$ defines a natural transformation $\mathbf{Mod}(F_1, \phi_1) \to \mathbf{Mod}(F_2, \phi_2)$.

Verification of 2-functoriality is straightforward and omitted. $\square$

Remark 3.2.8. Let $\mathscr{C}, \mathscr{C}'$ be two monoidal categories and let $F : \mathscr{C} \to \mathscr{C}'$ be a lax monoidal functor. If A is a monoid in $\mathscr{C}$, then $A' = FA$ is canonically a monoid in $\mathscr{C}'$ and F extends to a functor $A\text{-}\mathbf{Mod}_{\mathscr{C}} \to A'\text{-}\mathbf{Mod}_{\mathscr{C}'}$.

Corollary 3.2.9. *Let $(F, \phi) : (\mathscr{C}, M) \to (\mathscr{C}', M')$ be a strong monad functor. If F admits a right adjoint, then so does $\mathbf{Mod}(F) : M\text{-}\mathbf{Mod}_{\mathscr{C}} \to M'\text{-}\mathbf{Mod}_{\mathscr{C}'}$.*

Proof. Since a 2-functor sends adjunctions to adjunctions, by Proposition 3.2.7, it is enough to show that (F, ϕ) admits a right adjoint in $\mathbf{Mnd}^{\mathrm{lax}}$. By Proposition 3.2.4, this 2-category is isomorphic to $T\text{-}\mathbf{Mod}^{\mathrm{lax}}_{\mathbf{CAT}}$ for a certain 2-monad T on $\mathbf{CAT}$ and this isomorphism maps (F, ϕ) to a strong T-functor. Hence, the result follows by Theorem 3.1.15. $\square$

Corollary 3.2.10. *Let $(G, \psi) : (\mathscr{C}', M') \to (\mathscr{C}, M)$ be a lax monad functor. Assume the functor G*

has a left adjoint F, and that the mate of ψ:

(3.2.11)
$$\begin{tikzcd} & \mathscr{C} \arrow[r, "M"] & \mathscr{C} \arrow[r, "F"] & \mathscr{C}' \\ \mathscr{C} \arrow[ur, bend left, "id_{\mathscr{C}}"] \arrow[r, "F"'] \arrow[ur, phantom, "\Downarrow\eta"] & \mathscr{C}' \arrow[u, "G"'] \arrow[r, "M'"'] \arrow[ur, phantom, "\Downarrow\psi"] & \mathscr{C}' \arrow[u, "G"'] \arrow[ur, bend right, "id_{\mathscr{C}'}"'] \arrow[ur, phantom, "\Downarrow\epsilon"] & \end{tikzcd}$$

is invertible (where ϵ, η are the counit and unit respectively of the adjunction $F \dashv G$). Then the functor $\mathbf{Mod}(G) : M'\text{-}\mathbf{Mod}_{\mathscr{C}} \to M\text{-}\mathbf{Mod}_{\mathscr{C}}$ *has a left adjoint.*

Proof. By Proposition 3.2.4 we know that (G, ψ) corresponds to a lax T-morphism (G, ψ') for the monad $T = \Delta_+ \times -$ on **CAT**. Using Theorem 3.1.15, the proof is similar to the one in Corollary 3.2.9 above, only we need to remark that the mate of ψ' is an isomorphism if and only if the mate of ψ is. But from the proof of Proposition 3.2.4 ψ is a restriction of ψ', and ψ' is given on each "piece" $\{\underline{n}\} \times \mathscr{C}$ of $\Delta_+ \times \mathscr{C}$ as a horizontal pasting of ψ with itself n times. Hence the assertion is immediate, using the functoriality of mates with respect to pasting (see [Gro13, Lemma 1.14]). □

We now return our attention to prederivators. Recall that **Dia** is a fixed diagram category and that **PDer** is the 2-category of (**Dia**-)prederivators with pseudonatural transformations as 1-cells, and modifications as 2-cells.

Definition 3.2.12. A **monad on a prederivator** $\mathbb{D}$ is a monad on $\mathbb{D}$ in the 2-category **PDer** (see Definition 3.1.1).

Thus, a monad on a prederivator $\mathbb{D}$ will consist of an endomorphism M on $\mathbb{D}$, that is, a pseudonatural transformation $M : \mathbb{D} \to \mathbb{D}$, together with a modification $\mu : M^2 \to M$ and a modification $\mathbb{S} : \mathrm{id}_{\mathbb{D}} \to M$ such that the diagrams (3.1.2) commute. In particular, for each category $J \in$ **Dia**, the triple $(M_J, \mu_J, \mathbb{S}_J)$ defines a (classical) monad on $\mathbb{D}(J)$.

Lemma 3.2.13. *Let M be a monad on a prederivator $\mathbb{D}$. Then the assignment $J \mapsto (\mathbb{D}(J),\ M_J)$ extends to a 2-functor $\widetilde{\mathbb{D}} : \mathbf{Dia}^{1\text{-}op} \to \mathbf{Mnd}^{lax}$ lifting $\mathbb{D}$ against the forgetful 2-functor $U : \mathbf{Mnd}^{lax} \to$* **CAT**, *i.e. we have a factorization:*

$$\begin{tikzcd} & \mathbf{Mnd}^{\mathrm{lax}} \arrow[d, "\mathrm{U}"] \\ \mathbf{Dia}^{1\text{-op}} \arrow[ur, dotted, "\widetilde{\mathbb{D}}"] \arrow[r, "\mathbb{D}"'] & \mathbf{CAT} \end{tikzcd}$$

Proof. Define $\widetilde{\mathbb{D}}$ on 1-cells by $u \mapsto (u^*, (\gamma_u^M)^{-1})$ where $\gamma_u^M : u^*M \Rightarrow Mu^*$ is the coherence isomorphism of the pseudonatural transformation M, and as the identity on 2-cells.

Given a functor $u : J \to K$ in **Dia**, it follows using (1.3.4) for μ and $\mathbb{S}$, that $(u^*, (\gamma_u^M)^{-1})$ is a lax (even strong) monad functor; and given a natural transformation

$$\begin{tikzcd} J \arrow[r, bend left, "u"{name=U}] \arrow[r, bend right, "v"'{name=V}] \arrow[from=U, to=V, phantom, "\Downarrow\alpha"] & K \end{tikzcd}$$

it is clear by (1.3.3) for M, that α^* is a monad functor transformation from $(u^*, (\gamma_u^M)^{-1})$ to $(v^*, (\gamma_v^M)^{-1})$. Hence $\widetilde{\mathbb{D}}$ is well-defined, and its 2-functoriality is immediately verified. □

Proposition 3.2.14. *Let M be a monad on a prederivator $\mathbb{D}$. The assignment:*

$$J \mapsto M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)}$$

extends to a prederivator M-$\mathbf{Mod}_{\mathbb{D}} : \mathbf{Dia}^{1\text{-}op} \to \mathbf{CAT}$. Furthermore, the free module functors $F_{M,J} : \mathbb{D}(J) \to M_J$-$\mathbf{Mod}_{\mathbb{D}(J)}$ define a morphism of prederivators $F_M : \mathbb{D} \to M$-$\mathbf{Mod}_{\mathbb{D}}$ and the forgetful functors $U_{M,J} : M_J$-$\mathbf{Mod}_{\mathbb{D}(J)} \to \mathbb{D}(J)$ define a strict morphism of prederivators $U_M : M$-$\mathbf{Mod}_{\mathbb{D}} \to \mathbb{D}$ such that F_M is left adjoint to U_M in the 2-category **PDer** *and $M = U_M F_M$.*

Proof. For the first part of the proof, note that by Lemma 3.2.13, we can factor $\mathbb{D}$ as

$$\begin{tikzcd} & \mathbf{Mnd}^{\text{lax}} \arrow[d, "U"] \\ \mathbf{Dia}^{\text{1-op}} \arrow[ur, dotted, "\widetilde{\mathbb{D}}"] \arrow[r, "\mathbb{D}"'] & \mathbf{CAT} \end{tikzcd}$$

and the result follows by postcomposing $\widetilde{\mathbb{D}}$ with the 2-functor $\mathbf{Mod} : \mathbf{Mnd}^{\text{lax}} \to \mathbf{CAT}$ of Proposition 3.2.7.

For the second part of the proof, write $\overline{u^*} := \mathbf{Mod}(u^*, (\gamma_u^M)^{-1})$ for a functor $u : J \to K$ in **Dia**. Given such u, the diagram:

$$\begin{tikzcd} M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)} \arrow[d, "U_{M,J}"'] & M_K\text{-}\mathbf{Mod}_{\mathbb{D}(K)} \arrow[l, "\overline{u^*}"'] \arrow[d, "U_{M,K}"] \\ \mathbb{D}(J) & \mathbb{D}(K) \arrow[l, "u^*"] \end{tikzcd}$$

commutes, which shows that U_M is a 2-natural transformation (instead of just pseudonatural). We now want to define a natural isomorphism $\gamma_u^{F_M}$ populating the square:

$$\begin{tikzcd}
M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)} & M_K\text{-}\mathbf{Mod}_{\mathbb{D}(K)} \arrow[l, "\overline{u^*}"'] \\
\mathbb{D}(J) \arrow[u, "F_{M,J}"] \arrow[ur, phantom, "\gamma_u^{F_M}\swArrow"] & \mathbb{D}(K) \arrow[l, "u^*"] \arrow[u, "F_{M,K}"']
\end{tikzcd}$$

Given $X \in \mathbb{D}(K)$, we note that the M_J-module $\overline{u^*}F_{M,K}X$ has underlying object u^*M_KX with action

$$\lambda : M_Ju^*M_KX \xrightarrow{(\gamma_u^M)^{-1}_{M_KX}} u^*M_K^2X \xrightarrow{u^*(\mu_K)_X} u^*M_KX$$

while $F_{M,J}u^*X$ has underlying object M_Ju^*X with action

$$(\mu_J)_{u^*X} : M_J^2u^*X \to M_Ju^*X$$

Thus, taking $\gamma_u^{F_M} := \gamma_u^M$, we need to show that $(\gamma_u^M)_X$ is M_J-linear. That is, we need to show that the following diagram commutes:

$$\begin{tikzcd}
M_Ju^*M_KX \arrow[d, "M_J(\gamma_u^M)_X"'] \arrow[rr, bend left, "\lambda"] & u^*M_K^2X \arrow[l, "(\gamma_u^M)_{M_KX}"', "\cong"] \arrow[r, "u^*(\mu_K)_X"] \arrow[dl, "(\gamma_u^{M^2})_X"] & u^*M_KX \arrow[d, "(\gamma_u^M)_X"] \\
M_J^2u^*X \arrow[rr, "(\mu_J)_{u^*X}"'] & & M_Ju^*X
\end{tikzcd}$$

But the triangle on the left commutes by the very definition of horizontal composite of pseudonatural transformations while the right square commutes because of diagram (1.3.4) for μ. The coherence axioms now follow directly from the fact that γ_u^M satisfies them, so F_M becomes a morphism of prederivators with structure maps those of M.

From the definitions, it is clear that $M = U_MF_M$. Hence, to finish, we need to show that the levelwise counits and units of the Eilenberg-Moore adjunctions assemble to modifications between the corresponding prederivators. This can be directly checked by hand, or as follows: by Example 3.1.12(ii), there is a 2-monad such that T-modules are **Dia**-prederivators, lax (respectively strong) T-morphisms are lax (respectively pseudo) natural transformations, and T-2-cells are modifications. It follows by Theorem 3.1.13 that F_M is left adjoint to U_M in the 2-category of prederivators, *lax* natural transformations and modifications. Since **PDer** is a 2-subcategory of the latter (with the same 2-cells) containing the pseudonatural transformations F_M and U_M, the result is immediate. □

3.3 The left derivator of levelwise modules

Throughout this section M will be a fixed monad on a fixed prederivator $\mathbb{D}$.

Remark 3.3.1. As in the proof of Proposition 3.2.14, an undecorated pullback (or left/right Kan extension when they exist) notation will refer to $\mathbb{D}$, while a bar above the corresponding notation will refer to M-$\mathbf{Mod}_{\mathbb{D}}$.

Proposition 3.3.2. *Assume $\mathbb{D}$ satisfies Der1 and Der2 (see Definition 1.2.24). Then so does M-$\mathbf{Mod}_{\mathbb{D}}$.*

Proof. For *Der*1: Consider a finite family $\{J_i\}_{i\in I}$ of categories in **Dia**. Let $J = \coprod J_i$ and for each i, let $j_i : J_i \to J$ the canonical inclusions, and $\pi_i : \prod \mathbb{D}(J_i) \to \mathbb{D}(J_i)$ the canonical projections. For each index i, we have a strong monad functor

$$(j_i^*, \gamma_{j_i}^M) : (\mathbb{D}(J), M_J) \to (\mathbb{D}(J_i), M_{J_i})$$

Note that $\prod \mathbb{D}(J_i)$ is a 2-product, meaning that for any category $\mathcal{K}$, the induced functor

$$\mathbf{CAT}\left(\mathcal{K}, \prod \mathbb{D}(J_i)\right) \to \prod \mathbf{CAT}(\mathcal{K}, \mathbb{D}(J_i))$$

(which is component-wise post-composition with π_i) is an isomorphism of categories. Therefore, the j_i^* induce a unique functor $j^* : \mathbb{D}(J) \to \prod \mathbb{D}(J_i)$ (by abuse of the * notation) such that $\pi_i j^* = j_i^*$. It is then easy to see that the corresponding induced functor of $\{M_{J_i} j_i^*\}$ on the product is $(\prod M_{J_i})j^*$ and the corresponding induced functor of $\{j_i^* M_J\}$ is $j^* M_J$. Therefore by the 2-universal property above, the $\gamma_{J_i}^M$ induce a natural isomorphism $\phi : j^* M_J \Rightarrow (\prod M_{J_i})j^*$. Since the $\gamma_{J_i}^M$ make the diagrams (3.2.2) commute (with M_{J_i} for M), by uniqueness of the induced natural isomorphism, ϕ also has to make these diagrams commute; that is, (j^*, ϕ) is a strong monad functor.

By *Der*1 for $\mathbb{D}$, the functor j^* is an equivalence. Since (j^*, ϕ) is a strong monad functor, Theorem 3.1.13 guarantees this is an equivalence $(\mathbb{D}(J), M_J) \to (\prod \mathbb{D}(J_i), \prod M_{J_i})$ in $\mathbf{Mnd}^{\mathrm{lax}}$.

Since 2-functors preserve equivalences, by Proposition 3.2.7 this induces a further equivalence $\overline{j^*}$: $M\text{-}\mathbf{Mod}_{\mathbb{D}}(J) \to \prod M\text{-}\mathbf{Mod}_{\mathbb{D}}(J_i)$. Finally, $\overline{j^*}$ is the functor induced by $\left\{\overline{j_i^*}\right\}$ so $M\text{-}\mathbf{Mod}_{\mathbb{D}}$ satisfies $Der1$.

For $Der2$: Let J be any category in **Dia**, and $f : X \to Y$ a morphism in $M\text{-}\mathbf{Mod}_{\mathbb{D}}(J) = M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)}$. Since the forgetful functor $U_{M,J} : M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)} \to \mathbb{D}(J)$ is conservative, f is an isomorphism if and only if $U_{M,J}f$ is. By $Der2$ for $\mathbb{D}$, the latter morphism is an isomorphism if and only if $m^*U_{M,J}f$ is an isomorphism for all objects $m \in J$. But U_M is a morphism of prederivators (see Proposition 3.2.14), and hence, we have $m^*U_{M,J}f \cong U_{M,e}\overline{m^*}f$ for all objects $m \in J$. Using again conservativity of the forgetful functor, we deduce this holds if and only if $\overline{m^*}f$ is an isomorphism for all objects $m \in J$. □

Assume $\mathbb{D}$ is a left derivator (see Definition 1.2.24). Given any functor $u : J \to K$ in **Dia** we have a natural transformation $\gamma_u^M : u^*M_K \Rightarrow M_Ju^*$. Its mate is a natural transformation (cf. the dual of the diagram (1.3.6)):

$$\gamma_{u,*}^M : M_Ku_* \Rightarrow u_*M_J$$

Proposition 3.3.3. *If $\mathbb{D}$ is a left derivator, then the prederivator M-$\mathbf{Mod}_{\mathbb{D}}$ is also a left derivator.*

Proof. Let us fix a functor $u : J \to K$ in **Dia**. By (1.3.4) for μ and $\mathbb{S}$, (u^*, γ_u^M) is a strong monad functor. Since u^* has a right adjoint u_*, by Corollary 3.2.9 we get an induced adjunction

$$\overline{u^*} : M\text{-}\mathbf{Mod}_{\mathbb{D}}(K) \leftrightarrows M\text{-}\mathbf{Mod}_{\mathbb{D}}(J) : \overline{u_*}$$

proving $Der3L$ for M-$\mathbf{Mod}_{\mathbb{D}}$.

For $Der4L$: For any functor $u : J \to K$ in **Dia**, and any object $k \in K$, we need to show that the following comma square:

$$\begin{tikzcd} (k \downarrow u) \arrow[r, "\mathrm{pr}_J"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & J \arrow[d, "u"] \\ e \arrow[r, "k"'] & K \end{tikzcd}$$

where $\pi = \pi_{(k\downarrow u)}$, is M-$\mathbf{Mod}_{\mathbb{D}}$-exact. By assumption the mate $\alpha_* : k^*u_* \Rightarrow \pi_*\mathrm{pr}_J^*$ of α^* given by the following pasting is an isomorphism:

$$
\begin{tikzcd}
\mathbb{D}(e) & \mathbb{D}(J_{k/}) \arrow[l, "\pi_*"'] & \mathbb{D}(J) \arrow[l, "\mathrm{pr}_J^*"'] & \\
& \mathbb{D}(e) \arrow[ul, bend left, "\mathrm{id}_{\mathbb{D}(e)}"] \arrow[ul, phantom, "\eta \Nearrow"] \arrow[u, "\pi^*"'] \arrow[ur, phantom, "\alpha^* \Nearrow"] & \mathbb{D}(K) \arrow[l, "k^*"] \arrow[u, "u^*"] \arrow[ur, phantom, "\epsilon \Nearrow"] & \mathbb{D}(J) \arrow[l, "u_*"] \arrow[ul, bend right, "\mathrm{id}_{\mathbb{D}(J)}"']
\end{tikzcd}
$$

Here ϵ is the counit of the adjunction $u^* \dashv u_*$ and η is the unit of the adjunction $\pi^* \dashv \pi_*$. Given an object $(X, \lambda) \in M\text{-}\mathbf{Mod}_{\mathbb{D}}(J)$ we clearly have $U_{M,e}((\overline{\alpha_*})_{(X,\lambda)}) = (\alpha_*)_X$. Since the forgetful functor detects isomorphisms, we are done. $\square$

Remark 3.3.4. Dually, given a comonad M on a right derivator $\mathbb{D}$, the levelwise comodules over M form a right derivator.

3.4 Left Kan extensions

Throughout this section, let M be a fixed monad on a prederivator $\mathbb{D}$.

Proposition 3.4.1. *Let $u : J \to K$ be a functor in* **Dia** *such that $\mathbb{D}$ admits left Kan extensions along u. If M commutes with left Kan extensions along u then M-$\mathbf{Mod}_{\mathbb{D}}$ admits left Kan extensions along u.*

Proof. We have a lax monad functor $(u^*, (\gamma_u^M)^{-1}) : (\mathbb{D}(K), M_K) \to (\mathbb{D}(J), M_J)$. By Corollary 3.2.10, the monad M commutes with left Kan extensions along u if and only if $(u^*, (\gamma_u^M)^{-1})$ has a lax left adjoint, which is then necessarily strong. Applying the 2-functor of Proposition 3.2.7 the result follows immediately. $\square$

Corollary 3.4.2. *Let $\mathbb{D}$ be a right derivator. If M is cocontinuous, then M-$\mathbf{Mod}_{\mathbb{D}}$ is also a right derivator and the right adjoint $U_M : M\text{-}\mathbf{Mod}_{\mathbb{D}} \to \mathbb{D}$ of the Eilenberg-Moore adjunction (see Proposition 3.2.14) is cocontinuous.*

Proof. The last proposition guarantees that M-$\mathbf{Mod}_{\mathbb{D}}$ satisfies *Der*3*R*, while Proposition 3.3.2 guarantees it satisfies *Der*1 and *Der*2. The verification of *Der*4*R* follows exactly as in Proposition 3.3.3.

To finish the proof it remains to show that U_M is cocontinuous: let $u : J \to K$ a functor in **Dia**. Let us write ϵ for the counit of the adjunction $u_! \dashv u^*$ and $\overline{\eta}$ for the unit of the adjunction $\overline{u_!} \dashv \overline{u^*}$. We have to show that the top arrow in the following commutative diagram is an isomorphism:

$$\begin{tikzcd} u_! U_{M,J} \arrow[r] \arrow[d, "u_! U_{M,J}\overline{\eta}"'] & U_{M,K}\overline{u_!} \\ u_! U_{M,J}\overline{u^*}\,\overline{u_!} \arrow[r, "u_!\gamma_u^{U_M}\overline{u_!}"'] & u_! u^* U_{M,K}\overline{u_!} \arrow[u, "\epsilon U_{M,K}\overline{u_!}"'] \end{tikzcd}$$

Evaluated at an object $(X, \lambda) \in M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)}$ this diagram becomes

$$\begin{tikzcd} u_! X \arrow[r] \arrow[d, "u_!\eta"'] & u_! X \\ u_! u^* u_! X \arrow[r] & u_! u^* u_! X \arrow[u, "\epsilon u_!"] \end{tikzcd}$$

where the bottom arrow is the identity because U is strict. By the triangle identities, it follows the top arrow is an identity as well, finishing the proof. ☐

Remark 3.4.3. Proposition 3.3.3 says that for the 2-full subcategory $\mathcal{K}$ of **PDer** spanned by left derivators, the prederivator $M\text{-}\mathbf{Mod}_{\mathbb{D}}$ is actually an object of $\mathcal{K}$ and furthermore the adjunction $F_M \dashv U_M$ is internal to $\mathcal{K}$. In general, this will fail for the 2-full subcategory of **PDer** spanned by right derivators, unless we further restrict the 1-cells to **cocontinuous** morphisms. It is worth noting that when the monad in question is idempotent (i.e. a localization) we do not need M to be cocontinuous in order for $M\text{-}\mathbf{Mod}_{\mathbb{D}}$ to be a right derivator (see [Cis08, Lemma 4.2]).

Combining Proposition 3.3.3 and Corollary 3.4.2, we arrive at the main result of this section:

Theorem 3.4.4. *Assume that $\mathbb{D}$ is a derivator and M a cocontinuous monad on $\mathbb{D}$. Then the prederivator M-$\mathbf{Mod}_{\mathbb{D}}$ is also a derivator and the forgetful functor $U_M : M\text{-}\mathbf{Mod}_{\mathbb{D}} \to \mathbb{D}$ of Proposition 3.2.14 is cocontinuous.*

3.5 Stability

For this section, assume $\mathbb{D}$ is a derivator and M is cocontinuous, so that by Theorem 3.4.4 above $M\text{-}\mathbf{Mod}_{\mathbb{D}}$ is a derivator.

Proposition 3.5.1. *If $\mathbb{D}$ is a stable derivator and M is a cocontinuous monad on $\mathbb{D}$, then M-$\mathbf{Mod}_{\mathbb{D}}$ is also stable.*

Proof. Since M is cocontinuous, M-$\mathbf{Mod}_{\mathbb{D}}$ is a derivator. Furthermore, M is in particular pointed and hence, M-$\mathbf{Mod}_{\mathbb{D}}$ is also pointed. Let us write ϵ for the counit of the adjunction $(i_{\ulcorner})_! \dashv i_{\ulcorner}^*$, and $\overline{\epsilon}$ for that of $\overline{(i_{\ulcorner})_!} \dashv \overline{i_{\ulcorner}^*}$. Let $(X, \lambda) \in M$-$\mathbf{Mod}_{\mathbb{D}}(\square)$. Note that $U_M(\overline{\epsilon}_{(X,\lambda)}) = \epsilon_X$ so (X, λ) is cocartesian if and only if $X \in \mathbb{D}(\square)$ is. Similarly (X, λ) is cartesian if and only if X is. The result follows immediately. $\square$

3.6 Separable monads and strongness

In this section, we investigate under which conditions on the monad M, strongness of $\mathbb{D}$ implies strongness of the prederivator M-$\mathbf{Mod}_{\mathbb{D}}$ of Proposition 3.2.14. It turns out that a sufficient condition is the following, which is a straightforward generalization of the condition given in [Bal11]:

Definition 3.6.1. A monad $M : \mathbb{D} \to \mathbb{D}$ in the 2-category **PDer** is called **separable** if $\mu : M^2 \to M$ admits a section $\sigma : M \to M^2$ such that $M\mu \circ \sigma M = \sigma \circ \mu = \mu M \circ M\sigma$, i.e. if the following diagram commutes:

(3.6.2)
$$
\begin{tikzcd}
 & M^2 \arrow[dl, "M\sigma"'] \arrow[d, "\mu"] \arrow[dr, "\sigma M"] & \\
M^3 \arrow[dr, "\mu M"'] & M \arrow[d, "\sigma"] & M^3 \arrow[dl, "M\mu"] \\
 & M^2 &
\end{tikzcd}
$$

Lemma 3.6.3. *Let $M : \mathbb{D} \to \mathbb{D}$ a monad on a prederivator $\mathbb{D}$. Then M is separable if and only if the counit of the Eilenberg-Moore adjunction admits a section.*

Proof. Let $\xi : \mathrm{id}_{M\text{-}\mathbf{Mod}_{\mathbb{D}}} \to F_M U_M$ be a section of the counit $\epsilon : F_M U_M \to \mathrm{id}_{M\text{-}\mathbf{Mod}_{\mathbb{D}}}$. Define $\sigma : M \to M^2$ as the composite

$$M = U_M F_M \xrightarrow{U_M \xi F_M} U_M F_M U_M F_M = M^2$$

which is necessarily a modification. Then by [BV07, Proposition 6.3], M is separable.

Conversely, assume M is separable and let $\sigma : M \to M^2$ a section of μ making diagram (3.6.2) commute. For any category J in **Dia**, and any $(X, \lambda) \in M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)}$, define

$$(\xi_J)_{(X,\lambda)} : X \xrightarrow{\mathbb{S}_{J,X}} M_J X \xrightarrow{(\sigma_J)_X} M_J^2 X \xrightarrow{M_J \lambda} M_J X$$

where $\mathbb{S}$ is the unit of the monad M, which is also the unit of the Eilenberg-Moore adjunction. It is proved in [BV07, Proposition 6.3] that this defines a natural transformation

$$\xi_J : \mathrm{id}_{M\text{-}\mathbf{Mod}_{\mathbb{D}}(J)} \to F_{M,J} U_{M,J}$$

which is a section of the counit ϵ_J. All that is left for us is to show that this is a modification. So let $u : J \to K$ a functor in **Dia**. We need to show that the diagram:

$$\begin{tikzcd} u^* \arrow[r, "u^*\xi_K"] \arrow[dr, "\xi_J u^*"'] & u^* F_{M,K} U_{M,K} \arrow[d, "\gamma_u^M"] \\ & F_{M,J} U_{M,J} u^* \end{tikzcd}$$

commutes. So, let $(X, \lambda) \in M\text{-}\mathbf{Mod}_{\mathbb{D}}(K) = M_K\text{-}\mathbf{Mod}_{\mathbb{D}(K)}$. Then $u^*((\xi_K)_{(X,\lambda)})$ is the composite:

$$u^* X \xrightarrow{u^*((\mathbb{S}_K)_X)} u^* M_K X \xrightarrow{u^*((\sigma_K)_X)} u^* M_K^2 X \xrightarrow{u^* M_K \lambda} u^* M_K X$$

and $(\xi_J)_{u^* X}$ is the composite:

$$u^* X \xrightarrow{(\mathbb{S}_J)_{u^* X}} M_J u^* X \xrightarrow{(\sigma_J)_{u^* X}} M_J^2 u^* X \xrightarrow{M_J (\gamma_u^M)_X^{-1}} M_J u^* M_K X \xrightarrow{M_J u^* \lambda} M_J u^* X$$

Now it suffices to note that the diagram

$$\begin{tikzcd} u^* X \arrow[r, "u^*(\mathbb{S}_K)_X"] \arrow[d, "(\mathbb{S}_J)_{u^* X}"'] & u^* M_K X \arrow[r, "u^*(\sigma_K)_X"] \arrow[dl, "(\gamma_u^M)_X"'] & u^* M_K^2 X \arrow[r, "u^* M_K \lambda"] \arrow[dl, "(\gamma_u^{M^2})_X"'] \arrow[d, "(\gamma_u^M)_{M_K X}"] & u^* M_K X \arrow[d, "(\gamma_u^M)_X"] \\ M_J u^* X \arrow[r, "(\sigma_J)_{u^* X}"'] & M_J^2 u^* X \arrow[r, "M_J (\gamma_u^M)_X^{-1}"'] & M_J u^* M_K X \arrow[r, "M_J u^* \lambda"'] & M_J u^* X \end{tikzcd}$$

commutes: the left triangle because $\mathbb{S}$ is a modification (see (1.3.4)), the other triangle by definition of the composition of pseudonatural transformations, the skew parallelogram because σ is a modification, and the right square by naturality of γ_u^M. □

The following is the main ingredient in the proof of the main theorem and is analogous to [Bal11, Theorem 4.1]:

Proposition 3.6.4. *Let $F : \mathbb{D} \leftrightarrows \mathbb{E} : G$ be an adjunction of additive derivators such that $\mathbb{D}$ is strong and $\mathbb{E}$ is idempotent complete. Furthermore, assume that the counit $\epsilon : FG \to id_{\mathbb{E}}$ admits a section $\xi : id_{\mathbb{E}} \to FG$. Then $\mathbb{E}$ is strong.*

Proof. Note that the right adjoint G is levelwise faithful (by Yoneda and the fact that at each level, the components of the counit are epimorphisms since they admit sections). The assumptions are preserved under shifting (see Example 1.2.6(iii)), so it is enough to show that the underlying diagram functor $\mathrm{dia}^{\mathbb{E}}_{[1]} : \mathbb{E}([1]) \to \mathbb{E}(e)^{[1]}$ is full and essentially surjective.

For essential surjectivity: Let $f : X_0 \to X_1$ be a morphism in $\mathbb{E}(e)$. By strongness for $\mathbb{D}$, there is some $X \in \mathbb{D}([1])$ such that

$$\mathrm{dia}_{[1]}(X) \cong G_e f : G_e X_0 \to G_e X_1$$

Set $Y = F_{[1]}X$. We then have

$$\mathrm{dia}^{\mathbb{E}}_{[1]}\, Y \cong F_e G_e f \tag{3.6.5}$$

hence also

$$\mathrm{dia}^{\mathbb{D}}_{[1]}\, G_{[1]}Y \cong G_e\, \mathrm{dia}^{\mathbb{E}}_{[1]}\, Y \cong G_e F_e G_e f \tag{3.6.6}$$

Set $r = \xi_e \epsilon_e$. Note that (r_{X_0}, r_{X_1}) is an idempotent of $F_e G_e f$. Then $(G_e r_{X_0}, G_e r_{X_1})$ is an idempotent of $G_e F_e G_e f$. By Corollary 1.6.9 for $\mathbb{D}$, it follows that there is an idempotent r' of $G_{[1]}Y$ with $\mathrm{dia}^{\mathbb{D}}_{[1]}\, r' \cong (G_e r_{X_0}, G_e r_{X_1})$ under the isomorphism (3.6.6). Set $\overline{r} = \epsilon_X \circ F_{[1]}r' \circ \xi_X$, which is an endomorphism of X, lifting (r_{X_0}, r_{X_1}). Since the last is an idempotent of $F_e G_e f$, it follows that $a = \overline{r}^2 - \overline{r}$ lifts the 0 endomorphism of $F_e G_e f$. Because G_e is faithful, Corollary 1.6.9 again for $\mathbb{D}$ implies that $a^2 = 0$. Finally, $b = 3\overline{r}^2 - 2\overline{r}^3$ is an idempotent of Y by direct computation, which lifts the idempotent (r_{X_0}, r_{X_1}) of $F_e G_e f$.

So far, we showed that the idempotent (r_{X_0}, r_{X_1}) of $F_e G_e f$ lifts to an idempotent $\overline{r}$ of the lift Y of $F_e G_e f$. Since by assumption $\mathbb{E}$ is idempotent complete, there is a splitting $Y \cong K \oplus H$ such that $\overline{r}$ corresponds to $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Now using additivity of pullbacks in any additive derivator, it follows that the underlying diagram of $\overline{r}$, which under (3.6.5) is isomorphic to:

$$\begin{tikzcd}
F_eG_eX_0 \arrow[r, "F_eG_ef"] \arrow[d, "r_{X_0}"'] & F_eG_eX_1 \arrow[d, "r_{X_1}"] \\
F_eG_eX_0 \arrow[r, "F_eG_ef"'] & F_eG_eX_1
\end{tikzcd}$$

is isomorphic to a diagram of the form:

$$\begin{tikzcd}
K_0 \oplus H_0 \arrow[r, "\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"'] & K_1 \oplus H_1 \arrow[d, "\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \\
K_0 \oplus H_0 \arrow[r, "\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)"'] & K_1 \oplus H_1
\end{tikzcd}$$

It then follows easily from uniqueness (up to isomorphism) of splitting of idempotents, that $f \cong b$ and hence $\mathrm{dia}^{\mathbb{E}}_{[1]} H \cong f$. This proves essential surjectivity.

For fullness, assume that we have a commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, "u_0"'] & X_1 \arrow[d, "u_1"] \\
Y_0 \arrow[r, "g"'] & Y_1
\end{tikzcd}$$

in $\mathbb{D}(e)$ and chosen lifts X, Y of f, g respectively in $\mathbb{E}([1])$ (here, we assume X, Y are strict lifts, in the sense that $\mathrm{dia}^{\mathbb{E}}_{[1]}(X) = f$ and $\mathrm{dia}^{\mathbb{E}}_{[1]}(Y) = g$). Then, applying strongness of $\mathbb{D}$ we deduce that there is a lift $u' : G_{[1]}X \to G_{[1]}Y$ lifting (G_eu_0, G_eu_1). Hence $F_{[1]}u'$ lifts $(F_eG_eu_0, F_eG_eu_1)$.

Set $\overline{u} = \epsilon_Y \circ Fu' \circ \xi_X : X \to Y$. By the fact that ξ, ϵ are modifications, and functoriality of the underlying diagram functors, we deduce that the underlying diagram of $\overline{u}$ is isomorphic to the outer square in the diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, "\xi_{X_0}"'] & X_1 \arrow[d, "\xi_{X_1}"] \\
F_eG_eX_0 \arrow[r, "F_eG_ef"] \arrow[d, "F_eG_eu_0"'] & F_eG_eX_1 \arrow[d, "F_eG_eu_1"] \\
F_eG_eY_0 \arrow[r, "F_eG_eg"] \arrow[d, "\epsilon_C"'] & F_eG_eY_1 \arrow[d, "\epsilon_D"] \\
Y_0 \arrow[r, "g"'] & Y_1
\end{tikzcd}$$

which by naturality of ϵ_e is in turn isomorphic to

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, "\xi_{X_0}"'] & X_1 \arrow[d, "\xi_{X_1}"] \\
F_eG_eX_0 \arrow[r, "F_eG_ef"] \arrow[d, "\epsilon_{X_0}"'] & F_eG_eX_1 \arrow[d, "\epsilon_{X_1}"] \\
X_0 \arrow[r, "f"] \arrow[d, "u_0"'] & X_1 \arrow[d, "u_1"] \\
Y_0 \arrow[r, "g"'] & Y_1
\end{tikzcd}$$

which is the original diagram (since $\epsilon_e \circ \xi_e = \mathrm{id}_{\mathrm{id}_{E(e)}}$). □

We are now ready to prove the main theorem in this chapter. For the notion of a (right or left) exact morphism of derivators (cf. Definition 1.3.10):

Theorem 3.6.7. *Let $M : \mathbb{D} \to \mathbb{D}$ a cocontinuous separable monad on a triangulated idempotent-complete derivator $\mathbb{D}$. Then the prederivator M-$\mathbf{Mod}_{\mathbb{D}}$ of Proposition 3.2.14 is a triangulated derivator. Moreover, the morphisms F_M, U_M of Proposition 3.2.14 are exact.*

Proof. By Proposition 3.5.1, M-$\mathbf{Mod}_{\mathbb{D}}$ is a stable derivator. The monad M is realized by the adjunction $F_M : \mathbb{D} \leftrightarrows M\text{-}\mathbf{Mod}_{\mathbb{D}} : U_M$ of Proposition 3.2.14. The separability assumption on M guarantees the existence of a section of the counit by Lemma 3.6.3. We know that each M_J-$\mathbf{Mod}_{\mathbb{D}(J)}$ is additive and idempotent-complete since each $\mathbb{D}(J)$ is and M_J is additive. Thus the Eilenberg-Moore adjunction satisfies all the conditions in the previous proposition and, hence, $\mathbb{D}$ is strong. For the last part, note that F_M is cocontinuous as with all left adjoint morphisms of derivators (see [Gro13, Proposition 2.9]). Hence, in particular, F_M is right exact. Dually, U_M is left exact. Since these are morphisms between triangulated derivators, by [Gro13, Corollary 4.17] it follows that they are exact. □

Remark 3.6.8. If the derivator $\mathbb{D}$ is big (see Definition 1.2.11), then the assumption of idempotent completeness is redundant (cf. [Nee01, Proposition 1.6.8]).

Remark 3.6.9. Assume $\mathbb{D}$ is a triangulated derivator of domain $\mathbf{Dir}_f$, and M is a separable exact monad on $\mathbb{D}$. Then Theorem 3.6.7 together with Theorem 1.4.8 imply that the derivator M-$\mathbf{Mod}_{\mathbb{D}}$

is triangulated, and that F_M, U_M (see Proposition 3.2.14) are exact morphisms of derivators. This reproves [Bal11, Corollary 4.3] in the case the triangulated category $\mathscr{C}$ is the base of a triangulated derivator $\mathbb{D}$ of domain $\mathbf{Dir}_\mathrm{f}$ and M extends to a separable exact monad on $\mathbb{D}$.

Example 3.6.10. Let $\mathbb{D}$ be a triangulated monoidal derivator. Then its underlying category $\mathscr{C} = \mathbb{D}(e)$ is a tensor triangulated category (see [GPS14a]). Write $\boxtimes$ for the external tensor product associated to the monoidal structure on $\mathbb{D}$, and let A be a monoid object in $\mathscr{C}$. It is an easy but tedious verification to see that $M = A \boxtimes - : \mathbb{D} \to \mathbb{D}$ is a monad on $\mathbb{D}$, with multiplication and unit those of A. This monad is cocontinuous because the tensor product is assumed to be cocontinuous in each variable separately. Thus, by Theorem 3.4.4, we get a derivator $A\text{-}\mathbf{Mod}_\mathbb{D} := M\text{-}\mathbf{Mod}_\mathbb{D}$ whose underlying category is exactly $A\text{-}\mathbf{Mod}_\mathscr{C}$. If A is separable then so is M and $A\text{-}\mathbf{Mod}_\mathbb{D}$ is again a stable strong derivator. In particular $A\text{-}\mathbf{Mod}_\mathscr{C}$ is a triangulated category. For examples of separable monoids in tensor triangulated categories we refer to the introduction.

Corollary 3.6.11. *Let $\mathbb{D}$ be a triangulated monoidal derivator and let A be a monoid object in $\mathbb{D}(e)$ such that there is no triangulation on $A\text{-}\mathbf{Mod}_{\mathbb{D}(e)}$ compatible with that on $\mathbb{D}(e)$. Then $A\text{-}\mathbf{Mod}_\mathbb{D}$ is a stable non-strong derivator.*

Proof. By Example 3.6.10 $A\text{-}\mathbf{Mod}_\mathbb{D}$ is a stable derivator. If it is also strong, then its underlying category $A\text{-}\mathbf{Mod}_\mathbb{D}(e) = A\text{-}\mathbf{Mod}_{\mathbb{D}(e)}$ has to be triangulated. Furthermore, as in the proof of Theorem 3.6.7, the morphism of derivators $U : A\text{-}\mathbf{Mod}_\mathbb{D} \to \mathbb{D}$ has to be exact. But then [Gro13, Proposition 4.18] guarantees that the forgetful functor $U_e : A\text{-}\mathbf{Mod}_{\mathbb{D}(e)} \to \mathbb{D}(e)$ can be (canonically) endowed with the structure of an exact functor of triangulated categories, a contradiction. $\square$

The following example together with Corollary 3.6.11 above gives an explicit example of a stable non-strong derivator.

Example 3.6.12. Let k be a field, and let A be any k-algebra which admits non-projective modules. Considering A as a complex concentrated in degree 0 defines a monoid in the derived category $D(k)$. Assume that the category $\mathcal{T} := A\text{-}\mathbf{Mod}_{D(k)}$ admits a triangulation such that the forgetful functor $\mathcal{T} \to D(k)$ is exact. Then for any object $X \in \mathcal{T}$, the counit ϵ_X admits a section (see the proof of [Bal11, Proposition 2.10]). Since A-modules are equivalent to graded k-vector spaces with a compatible action of A, this implies that any A-module is projective, a contradiction.

CHAPTER 4

Localization theory

4.1 Preliminaries

Recall from Example 1.2.6(ii) that a relative category is a pair $(\mathscr{C}, \mathcal{S})$ where $\mathscr{C}$ is a category and $\mathcal{S}$ is a class of arrows in $\mathscr{C}$. Given another category $\mathscr{D}$, we will denote by $\underline{\mathsf{Hom}}_{\mathcal{S}}(\mathscr{C}, \mathscr{D})$ the full subcategory of $\underline{\mathsf{Hom}}(\mathscr{C}, \mathscr{D})$ spanned by functors that invert $\mathcal{S}$. Henceforth, for convenience reasons, the class $\mathcal{S}$ will always be assumed to contain all isomorphisms and be closed under composition (i.e. $\mathcal{S}$ is assumed to be a replete subcategory of $\mathscr{C}$). This is a very mild assumption and doesn't affect localization:

Definition 4.1.1. Given a relative category $(\mathscr{C}, \mathcal{S})$, a localization of $\mathscr{C}$ by $\mathcal{S}$ is given by a category $\mathscr{C}[\mathcal{S}^{-1}]$ together with a functor $\gamma : \mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]$ with the following universal property: for any category $\mathscr{D}$ and any functor $F : \mathscr{C} \to \mathscr{D}$ that inverts $\mathcal{S}$ (i.e. maps all arrows in $\mathcal{S}$ to isomorphisms), there is a unique functor $\overline{F} : \mathscr{C}[\mathcal{S}^{-1}] \to \mathscr{D}$ such that $F = \overline{F}\gamma$.

It is well-known that a localization in the above sense is 2-universal meaning that for any category $\mathscr{D}$, precomposition with γ:

$$\gamma^* = - \circ \gamma : \underline{\mathsf{Hom}}(\mathscr{C}[\mathcal{S}^{-1}], \mathscr{D}) \to \underline{\mathsf{Hom}}_{\mathcal{S}}(\mathscr{C}, \mathscr{D})$$

is an equivalence of categories. Let us now fix a relative category $(\mathscr{C}, \mathcal{S})$. The localized category $\mathscr{C}[\mathcal{S}^{-1}]$ is constructed by taking as objects the same objects with $\mathscr{C}$ and as morphisms arbitrarily long zig-zags of morphisms in $\mathscr{C}$ where the backward arrows are in $\mathcal{S}$, subject to some equivalence relations (cf. [GZ67]). The functor γ is the identity on objects and takes a morphism in $\mathscr{C}$ to the class of the corresponding zig-zag of length 1 in $\mathscr{C}[\mathcal{S}^{-1}]$. The set theoretic issues usually encountered in this construction are circumvented by an appropriate choice of foundations (cf.

Section 1.1). In particular, the localization of a locally small category might not be locally small, but it will still be a category in our sense. We will write arrows in $\mathscr{C}[\mathcal{S}^{-1}]$ with $\rightsquigarrow$ and use normal arrows in a zig-zag representative.

We say that a relative category $(\mathscr{C}, \mathcal{S})$ satisfies the **weakly universal left Ore condition** if every diagram in $\mathscr{C}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"'] & Y \\ Z & \end{tikzcd} \tag{4.1.2}$$

where $\sigma \in \mathcal{S}$ admits a weak pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"'] & Y \arrow[d, "\sigma'"] \\ Z \arrow[r, "f'"'] & W \end{tikzcd} \tag{4.1.3}$$

such that $\sigma' \in \mathcal{S}$; moreover, if $f \in \mathcal{S}$ then (4.1.3) can be chosen so that $f' \in \mathcal{S}$ as well. The rest of this section is devoted to showing that in this case morphisms in $\mathscr{C}[\mathcal{S}^{-1}]$ can be described by an equivalence relation on left fractions. This seems very natural since the condition above is almost the same as requiring that $\mathcal{S}$ admits a left calculus of fractions, but we were unable to prove that this condition implies left cancelability in general[1].

In what follows, fix a relative category $(\mathscr{C}, \mathcal{S})$ that satisfies the weak universal left Ore condition. Assume also that $\mathcal{S}$ satisfies 2-out-of-3. For given objects $X, Y \in \mathscr{C}$, define $\alpha(X, Y)$ as the class of triples (f, W, σ) where $W \in \mathscr{C}$ is an object, and

$$X \xrightarrow{f} W \xleftarrow{\sigma} Z$$

are morphisms in $\mathscr{C}$ such that $\sigma \in \mathcal{S}$. Given two such triples (f_1, W_1, σ_1) and (f_2, W_2, σ_2), we will write $(f_1, W_1, \sigma_1) \sim (f_2, W_2, \sigma_2)$ if and only if there exists a commutative diagram in $\mathscr{C}$ of the form:

$$\begin{tikzcd} & W_1 \arrow[d, "\sim"'] & \\ X \arrow[ur, "f_1"] \arrow[r] \arrow[dr, "f_2"'] & W_3 & Y \arrow[ul, "\sigma_1"'] \arrow[l, "\sim"'] \arrow[dl, "\sigma_2"] \\ & W_2 \arrow[u, "\sim"] & \end{tikzcd}$$

[1] In the additive case we can show the two are equivalent.

where all arrows decorated with $\sim$ are in $\mathcal{S}$. Note that the middle arrows are forced by commutativity so we may omit them from the notation.

Lemma 4.1.4. *This is an equivalence relation on $\alpha(X, Y)$.*

Proof. The only non-trivial part is transitivity. Thus, assume that we have relations in $\alpha(X, Y)$ as follows: $(f_1, Z_1, \sigma_1) \sim (f_2, W_2, \sigma_2)$ and $(f_2, W_2, \sigma_2) \sim (f_3, W_3, \sigma_3)$. Then there are commutative diagrams in $\mathscr{C}$:

$$\begin{tikzcd}
 & W_1 \arrow[d, "s_1"'] & \\
X \arrow[ur, "f_1"] \arrow[r, "g"] \arrow[dr, "f_2"'] & U & Y \arrow[ul, "\sigma_1"'] \arrow[l, "s"'] \arrow[dl, "\sigma_2"] \\
 & W_2 \arrow[u, "s_2"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & W_2 \arrow[d, "\tau_2"'] & \\
X \arrow[ur, "f_2"] \arrow[r, "h"] \arrow[dr, "f_3"'] & V & Y \arrow[ul, "\sigma_2"'] \arrow[l, "\tau"'] \arrow[dl, "\sigma_3"] \\
 & W_3 \arrow[u, "\tau_3"] &
\end{tikzcd}$$

where $s_1, s_2, s, \tau_2, \tau_3, \tau \in \mathcal{S}$. Now pick a weak pushout diagram:

$$\begin{tikzcd}
W_2 \arrow[r, "s_2"] \arrow[d, "\tau_2"'] & U \arrow[d, "\tau'"] \\
V \arrow[r, "s'"'] & W
\end{tikzcd}$$

where $s', \tau' \in \mathcal{S}$. We have a commutative diagram:

$$\begin{tikzcd}
 & W_1 \arrow[d, "s''"'] & \\
X \arrow[ur, "f_1"] \arrow[r, "g''"] \arrow[dr, "f_3"'] & W & Y \arrow[ul, "\sigma_1"'] \arrow[l, "\sigma''"'] \arrow[dl, "\sigma_3"] \\
 & W_3 \arrow[u, "\tau''"] &
\end{tikzcd} \tag{4.1.5}$$

where $s'' = \tau' s_1, g'' = \tau' g, \sigma'' = \tau' s$ and $\tau'' = s' \tau_3$. Indeed we have:

$$g'' = \tau' g = \tau' s_1 f_1 = s'' f_1$$

and:

$$g'' = \tau' g = \tau' s_2 f_2 = s' \tau_2 f_2 = s' \tau_3 f_3 = \tau'' f_3$$

Thus the left part of the diagram (4.1.5) commutes. We also have:

$$\sigma'' = \tau' s = \tau' s_1 \sigma_1 = s'' \sigma_1$$

and:

$$\sigma'' = \tau' s = \tau' s_2 \sigma_2 = s' \tau_2 \sigma_2 = s' \tau_3 \sigma_3 = \tau'' \sigma_3$$

Thus the diagram (4.1.5) commutes showing that $(f_1, W_1, \sigma_1) \sim (f_3, W_3, \sigma_3)$. $\square$

We let "Hom"$(X, Y) = \alpha(X, Y)/\sim$. The class of $(f, W, \sigma) \in \alpha(X, Y)$ will be denoted by $[(f, W, \sigma)]$.

Lemma 4.1.6. *Let $(f_1, W_1, \sigma_1) \in \alpha(X, Y)$ and $(f_2, W_2, \sigma_2) \in \alpha(Y, Z)$. Given commutative diagrams:*

$$\begin{tikzcd}
 & & Z \arrow[d, "\sigma_2"] \\
 & Y \arrow[r, "f_2"] \arrow[d, "\sigma_1"'] & W_2 \arrow[d, "\sigma"] \\
X \arrow[r, "f_1"'] & W_1 \arrow[r, "f"'] & W
\end{tikzcd}$$

and:

$$\begin{tikzcd}
 & & Z \arrow[d, "\sigma_2"] \\
 & Y \arrow[r, "f_2"] \arrow[d, "\sigma_1"'] & W_2 \arrow[d, "\sigma'"] \\
X \arrow[r, "f_1"'] & W_1 \arrow[r, "f'"'] & W'
\end{tikzcd}$$

where $\sigma, \sigma' \in \mathcal{S}$, then $(f f_1, W, \sigma\sigma_2) \sim (f' f_1, W', \sigma'\sigma_2)$ in "Hom"(X, Z).

Proof. Note that the given commutative squares are arbitrary and *not necessarily* weak pushout squares. By the weakly universal left Ore condition, there is a weak pushout square:

$$\begin{tikzcd}
Y \arrow[r, "f_2"] \arrow[d, "\sigma_1"'] & W_2 \arrow[d, "\sigma''"] \\
W_1 \arrow[r, "f''"'] & W''
\end{tikzcd}$$

where $\sigma'' \in \mathcal{S}$. Then there is a morphism $w : W'' \to W$ (necessarily in $\mathcal{S}$ by 2-out-of-3) such that $w\sigma'' = \sigma$ and $wf'' = f$. Then, the diagram:

$$\begin{tikzcd} & W \arrow[d, "1"'] & \\ X \arrow[ur, "ff_1"] \arrow[dr, "f''f_1"'] & W & Z \arrow[ul, "\sigma\sigma_2"'] \arrow[dl, "\sigma''\sigma_2"] \\ & W'' \arrow[u, "w"] & \end{tikzcd}$$

shows that $(ff_1, W, \sigma\sigma_2) \sim (f''f_1, W'', \sigma''\sigma_2)$. Similarly $(f''f_1, W'', \sigma''\sigma_2) \sim (f'f_1, W', \sigma'\sigma_2)$. $\square$

It follows that we get a well-defined map:

$$\alpha(Y,Z) \times \alpha(X,Y) \to \text{“}\,\mathsf{Hom}\,\text{”}(X,Z) \tag{4.1.7}$$

by sending the 2-tuple (G, F) where $F = (f_1, W_1, \sigma_1)$ and $G = (f_2, W_2, \sigma_2)$ to the class $[(ff_1, W, \sigma\sigma_2)]$ in “ Hom ”(X,Z) where:

$$\begin{tikzcd} Y \arrow[r, "\sigma_2"] \arrow[d, "\sigma_1"'] & W_2 \arrow[d, "\sigma"] \\ W_1 \arrow[r, "f"'] & W \end{tikzcd}$$

is any commutative square with $\sigma \in \mathcal{S}$.

Lemma 4.1.8. *The map* (4.1.7) *is compatible with the equivalence relations of Lemma 4.1.4 on $\alpha(X,Y)$ and $\alpha(Y,Z)$.*

Proof. We show that 4.1.7 respects the equivalence relation on $\alpha(X,Y)$ the proof for $\alpha(Y,Z)$ being similar. So let $F_1 := (f_1, W_1, \sigma_1) \sim (f_2, W_2, \sigma_2) =: F_2 \in \alpha(X,Y)$ and $G := (g, V, \tau)$ in $\alpha(Y,Z)$. Our goal is to show that the 2-tuples $(G, F_i), i = 1, 2$ have the same image under (4.1.7). Since $F_1 \sim F_2$, there is by definition a commutative diagram of the form:

$$\begin{tikzcd} & W_1 \arrow[d, "s_1"'] & \\ X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & W_3 & Y \arrow[ul, "\sigma_1"'] \arrow[l, "\sigma_3"'] \arrow[dl, "\sigma_2"] \\ & W_2 \arrow[u, "s_2"] & \end{tikzcd} \tag{4.1.9}$$

where $s_1, s_2, \sigma_3 \in \mathcal{S}$. Pick a commutative square:

(4.1.10)
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "\sigma_1"'] & V \arrow[d, "\sigma'_1"] \\ W_1 \arrow[r, "g'_1"'] & V'_1 \end{tikzcd}$$

where $\sigma'_1 \in \mathcal{S}$. Then the map (4.1.7) maps the 2-tuple (G, F_1) to the class of $(g'_1 f_1, V'_1, \sigma'_1 \tau)$ in "Hom"(X, Z). Note that the particular choice of the commutative square (4.1.10) above doesn't matter by Lemma 4.1.6. Pick a further commutative square:

(4.1.11)
$$\begin{tikzcd} W_1 \arrow[r, "g'_1"] \arrow[d, "s_1"'] & V'_1 \arrow[d, "\sigma''_1"] \\ W_3 \arrow[r, "g''_1"'] & V''_1 \end{tikzcd}$$

where $\sigma''_1 \in \mathcal{S}$. Putting (4.1.10) and (4.1.11) together, we get a commutative square:

(4.1.12)
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "s_1\sigma_1"'] & V \arrow[d, "\sigma''_1\sigma'_1"] \\ W_3 \arrow[r, "g''_1"'] & V''_1 \end{tikzcd}$$

where both vertical arrows are in $\mathcal{S}$. Thus, Lemma 4.1.6 shows that the map (4.1.7) maps the 2-tuple (G, F_3), where $F_3 = (f_3, W_3, \sigma_3) \in \alpha(X, Y)$ to the class of $(g''_1 f_3, V''_1, \sigma''_1\sigma'_1\tau)$ in "Hom"(X, Z).

One checks immediately that the following diagram commutes:

$$\begin{tikzcd} & V'_1 \arrow[d, "\sigma''_1"'] & \\ X \arrow[ur, "g'_1 f_1"] \arrow[r] \arrow[dr, "g''_1 f_3"'] & V''_1 & Z \arrow[ul, "\sigma'_1\tau"'] \arrow[l, "\sigma_3"'] \arrow[dl, "\sigma''_1\sigma'_1\tau"] \\ & V''_1 \arrow[u, "1"] & \end{tikzcd}$$

showing that under (4.1.7) the 2-tuples (G, F_1) and (G, F_3) have the same image. The same argument shows that the last 2-tuple has the same image under (4.1.7) with the 2-tuple (G, F_2) completing the proof. □

Thus, the map (4.1.7) induces a well-defined map:

(4.1.13)
$$\text{“}\mathsf{Hom}\text{”}(Y, Z) \times \text{“}\mathsf{Hom}\text{”}(X, Y) \to \text{“}\mathsf{Hom}\text{”}(X, Z)$$

Proposition 4.1.14. *There is a category $\mathcal{S}^{-1}\mathscr{C}$ with objects $Ob(\mathcal{S}^{-1}\mathscr{C}) = Ob(\mathscr{C})$ and morphisms*

$$\mathsf{Hom}_{\mathcal{S}^{-1}\mathscr{C}}(X,Y) = \text{``}\,\mathsf{Hom}\,\text{''}(X,Y)$$

for all $X, Y \in Ob(\mathscr{C})$. Composition is given by (4.1.13) *and the identity on X is* $[(1, X, 1)]$. *Moreover, there is a functor $\Gamma : \mathscr{C} \to \mathcal{S}^{-1}\mathscr{C}$ that is the identity on objects and sends a morphism $f : X \to Y$ in $\mathscr{C}$ to* $[(f, Y, 1)]$.

Proof. The fact that $[(1, X, 1)]$ is an identity for composition is readily verified using Lemma 4.1.6. For the first assertion, consider morphisms $G_i = (f_i, W_i, \sigma_i) \in \mathsf{Hom}_{\mathcal{S}^{-1}\mathscr{C}}(X_i, X_{i+1})$ for $i = 1, 2, 3$. We can find a commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd}
 & & & X_4 \arrow[d, "\sigma_3"] \\
 & & X_3 \arrow[r, "f_3"] \arrow[d, "\sigma_2"] & W_3 \arrow[d, "\sim"] \\
 & X_2 \arrow[r, "f_2"'] \arrow[d, "\sigma_1"'] & W_2 \arrow[r, "g_2"'] \arrow[d, "\sim"] & U \arrow[d, "\sim"] \\
X_1 \arrow[r, "f_1"'] & W_1 \arrow[r, "g_1"'] & V \arrow[r, "g"'] & W
\end{tikzcd}$$

where all arrows decorated by $\sim$ are in $\mathcal{S}$. The diagram on the "left" of the column of U is a representative for the composite G_2G_1, thus the whole diagram computes $G_3(G_2G_1)$ (again using Lemma 4.1.6). The diagram above the row of V is a representative for G_3G_2 hence the whole diagram also computes $(G_3G_2)G_1$.

For the last part of the proposition, it is clear that Γ respects identities, and the fact that it preserves compositions is Lemma 4.1.6 together with the observation that for $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{C}$ we have a commutative diagram:

$$\begin{tikzcd}
 & & Z \arrow[d, "1"] \\
 & Y \arrow[r, "g"] \arrow[d, "1"'] & Z \arrow[d, "1"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

□

Lemma 4.1.15. *The functor* $\Gamma : \mathscr{C} \to \mathcal{S}^{-1}\mathscr{C}$ *of Proposition 4.1.14 inverts* $\mathcal{S}$*.*

Proof. Let $\sigma : X \to Y$ be a morphism in $\mathcal{S}$ and set $F = [(\sigma, Y, 1)]$ and $G = [(1, Y, \sigma)]$. The diagram:

$$\begin{tikzcd}
& & Y \arrow[d, "1"] \\
& X \arrow[r, "\sigma"] \arrow[d, "\sigma"'] & Y \arrow[d, "1"] \\
Y \arrow[r, "1"'] & Y \arrow[r, "1"'] & Y
\end{tikzcd}$$

shows that FG is the identity on Y in $\mathcal{S}^{-1}\mathscr{C}$, and the diagram:

$$\begin{tikzcd}
& & X \arrow[d, "\sigma"] \\
& Y \arrow[r, "1"] \arrow[d, "1"'] & Y \arrow[d, "1"] \\
X \arrow[r, "\sigma"'] & Y \arrow[r, "1"'] & Y
\end{tikzcd}$$

shows that $GF = [(\sigma, Y, \sigma)]$ which is equal to $[(1, X, 1)]$ by virtue of the following commutative diagram:

$$\begin{tikzcd}
& Y \arrow[d, "1"] & \\
X \arrow[ur, "\sigma"] \arrow[r, "\sigma"] \arrow[dr, "1"'] & Y & X \arrow[ul, "\sigma"'] \arrow[l, "\sigma"'] \arrow[dl, "1"] \\
& X \arrow[u, "\sigma"'] &
\end{tikzcd}$$

□

Lemma 4.1.16. *For any morphism* $[(f, W, \sigma)] \in \mathsf{Hom}_{\mathcal{S}^{-1}\mathscr{C}}(X, Y)$ *we have*

$$[(f, W, \sigma)] = \Gamma(\sigma)^{-1}\Gamma(f)$$

Proof. This is immediate from Lemma 4.1.6, the following commutative diagram:

$$\begin{tikzcd}
& & Y \arrow[d, "\sigma"] \\
& W \arrow[r, "1"] \arrow[d, "1"'] & W \arrow[d, "1"] \\
X \arrow[r, "f"'] & W \arrow[r, "1"'] & W
\end{tikzcd}$$

and the fact that $\Gamma(\sigma)^{-1} = [(1, W, \sigma)]$ by the proof of Lemma 4.1.15. □

Lemma 4.1.17. *Let $F : \mathscr{C} \to \mathscr{D}$ be any functor that inverts $\mathcal{S}$. If*

$$[(f_1, W_1, \sigma_1)] = [(f_2, W_2, \sigma_2)] \in \mathsf{Hom}_{\mathcal{S}^{-1}\mathscr{C}}(X, Y)$$

then $F(\sigma_1)^{-1}F(f_1) = F(\sigma_2)^{-1}F(f_2) \in \mathsf{Hom}_{\mathscr{D}}(FX, FY)$.

Proof. The assumption implies that there is a commutative diagram in $\mathscr{C}$:

$$\begin{tikzcd}
 & W_1 \arrow[d, "s_1"] & \\
X \arrow[ur, "f_1"] \arrow[r, "f_3"] \arrow[dr, "f_2"'] & W_3 & Y \arrow[ul, "\sigma_1"'] \arrow[l, "\sigma_3"'] \arrow[dl, "\sigma_2"] \\
 & W_2 \arrow[u, "s_2"'] &
\end{tikzcd}$$

where $s_1, s_2, \sigma_3 \in \mathcal{S}$. The proof follows immediately by applying F to this diagram. $\square$

Proposition 4.1.18. *We have an isomorphism of categories $\mathcal{S}^{-1}\mathscr{C} \cong \mathscr{C}[\mathcal{S}^{-1}]$.*

Proof. It suffices to verify that the pair $(\mathcal{S}^{-1}\mathscr{C}, \Gamma)$ is a localization of the relative category $(\mathscr{C}, \mathcal{S})$. Thus, we consider any functor $F : \mathscr{C} \to \mathscr{D}$ that inverts $\mathcal{S}$. We then define

$$\overline{F} : \mathcal{S}^{-1}\mathscr{C} \to \mathscr{D}$$

given by the identity on objects and by

$$\mathsf{Hom}_{\mathcal{S}^{-1}\mathscr{C}}(X, Y) \ni (f, Z, \sigma) \mapsto F(\sigma)^{-1}F(f) \in \mathsf{Hom}_{\mathscr{D}}(FX, FY)$$

on morphisms. Note that the map on morphisms in well-defined by Lemma 4.1.17. If we verify that $\overline{F}$ is a functor then we're done since it will clearly satisfy $\overline{F}\Gamma = F$ and by Lemma 4.1.16 this is the only possible way to define $\overline{F}$ so that $\overline{F}\Gamma = F$. It is immediate that $\overline{F}$ preserves identities. To see that it is compatible with composition, consider two morphisms $A = [(f_1, W_1, \sigma_1)] \in \mathsf{Hom}_{\mathcal{S}^{-1}\mathscr{C}}(X, Y)$ and $B = [(f_2, W_2, \sigma_2)] \in \mathsf{Hom}_{\mathcal{S}^{-1}\mathscr{C}}(Y, Z)$. Then there is a commutative diagram:

(4.1.19)
$$\begin{tikzcd}
 & & Z \arrow[d, "\sigma_2"] \\
 & Y \arrow[r, "f_2"] \arrow[d, "\sigma_1"'] & W_2 \arrow[d, "\sigma"] \\
X \arrow[r, "f_1"'] & W_1 \arrow[r, "f"'] & W
\end{tikzcd}$$

where $\sigma \in \mathcal{S}$. Thus $BA = [(ff_1, W, \sigma\sigma_2)]$ and hence $\overline{F}(BA) = F(\sigma\sigma_2)^{-1}F(ff_1)$. On the other hand

$$\overline{F}(B)\overline{F}(A) = F(\sigma_2)^{-1}F(f_2)F(\sigma_1)^{-1}F(f_1) = F(\sigma_2)^{-1}F(\sigma)^{-1}F(f)F(f_1)$$

which finishes the proof by functoriality of F. □

We can now forgo the notation $(\mathcal{S}^{-1}\mathscr{C}, \Gamma)$ and keep the original one $(\mathscr{C}[\mathcal{S}^{-1}], \gamma)$. We summarize the important bit as far as derivators will be concerned in the following Corollary:

Corollary 4.1.20. *Let $(\mathscr{C}, \mathcal{S})$ be a relative category that satisfies the weakly universal left Ore condition. Assume also that $\mathcal{S}$ satisfies 2-out-of-3. Two parallel arrows $f, g : X \to Y$ in $\mathscr{C}$ have the same image under the localization functor $\gamma : \mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]$ if and only if there is a commutative diagram:*

$$\begin{tikzcd} & Y \arrow[d, "s"] \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & Z \\ & Y \arrow[u, "s"'] \end{tikzcd}$$

with $s \in \mathcal{S}$.

In other words, under the above conditions, two parallel arrows f, g in $\mathscr{C}$ become equal in $\mathscr{C}[\mathcal{S}^{-1}]$ if and only if some arrow in $\mathcal{S}$ coequalizes f and g. Note that the above corollary admits a dual formulation which we won't make explicit here.

4.2 Localization of right derivators

A **relative prederivator** is a pair $(\mathbb{D}, \mathcal{S})$, where $\mathbb{D}$ is a prederivator and $\mathcal{S}$ a class of morphisms in $\mathbb{D}(e)$. Given a category $J \in \mathbf{Dia}$, we will write $\mathcal{S}_J$ for the class of morphisms in $\mathbb{D}(J)$ that is pointwise in $\mathbb{D}(e)$. Given another prederivator $\mathbb{E}$, we will write $\underline{\mathsf{Hom}}_{\mathcal{S}}(\mathbb{D}, \mathbb{E})$ for the full subcategory of $\underline{\mathsf{Hom}}(\mathbb{D}, \mathbb{E})$ spanned by morphisms $F : \mathbb{D} \to \mathbb{E}$ such that the functor F_J inverts $\mathcal{S}_J$ for all categories $J \in \mathbf{Dia}$. As before we will assume $\mathcal{S}$ is replete and closed under compositions throughout, and will not state it explicitly every time. Note that this implies that $\mathcal{S}_J$ is also replete and closed under compositions for all categories $J \in \mathbf{Dia}$.

Definition 4.2.1. Let $(\mathbb{D}, \mathcal{S})$ be a relative prederivator. A localization of $\mathbb{D}$ by $\mathcal{S}$ is given by a prederivator $\mathbb{D}[\mathcal{S}^{-1}]$ together with a morphism of prederivators

$$\gamma : \mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$$

such that precomposition with γ induces an equivalence of categories:

$$\gamma^* = - \circ \gamma : \underline{\mathsf{Hom}}(\mathbb{D}[\mathcal{S}^{-1}], \mathbb{D}') \to \underline{\mathsf{Hom}}_{\mathcal{S}}(\mathbb{D}, \mathbb{D}')$$

for all prederivators $\mathbb{D}'$.

Remark 4.2.2. It would seem more natural to define a localization of $\mathbb{D}$ by a **subprederivator**, i.e. a prederivator $\mathbb{E}$ whose values are levelwise subcategories of the corresponding values of $\mathbb{D}$ and such that the inclusion functors assemble to a strict morphism of prederivators $\mathbb{E} \hookrightarrow \mathbb{D}$. However, note that for semiderivators $\mathbb{D}$ and $\mathbb{E}$ we have $F \in \underline{\mathsf{Hom}}_{\mathcal{S}}(\mathbb{D}, \mathbb{E})$ if and only if the functor F_e inverts $\mathcal{S}$. So Definition 4.2.1 is given with the goal of applying it to (right/left) derivators in mind.

Proposition 4.2.3. *Let $(\mathbb{D}, \mathcal{S})$ be a relative prederivator. The assignment*

$$J \mapsto \mathbb{D}(J)[\mathcal{S}_J^{-1}]$$

defines a prederivator $\mathbb{D}[\mathcal{S}^{-1}]$ and the localization morphisms $\gamma_J : \mathbb{D}(J) \to \mathbb{D}(J)[\mathcal{S}_J^{-1}]$ assemble to a strict morphism of prederivators $\gamma : \mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$ such that the pair $(\mathbb{D}[\mathcal{S}^{-1}], \gamma)$ defines a localization of $\mathbb{D}$ by $\mathcal{S}$. Moreover, if $\mathbb{D}$ and $\mathbb{D}[\mathcal{S}^{-1}]$ are actually right derivators and γ is cocontinuous, then precomposition with γ induces an equivalence of categories:

$$\gamma^* = - \circ \gamma : \underline{\mathsf{Hom}}_!(\mathbb{D}[\mathcal{S}^{-1}], \mathbb{D}') \to \underline{\mathsf{Hom}}_{!,\mathcal{S}}(\mathbb{D}, \mathbb{D}')$$

for all right derivators $\mathbb{D}'$.

Proof. The assertions that $\mathbb{D}[\mathcal{S}^{-1}]$ is a prederivator and that γ is a strict morphism of prederivators follow directly from the universal property of localization of categories.

It remains to verify the universal property of localization of prederivators. Consider any prederivator $\mathbb{D}'$. It is well-known that localization of categories is actually a 2-localization, implying that for any category $J \in \mathbf{Dia}$, the functor:

$$\gamma_J^* = - \circ \gamma_J : \underline{\mathsf{Hom}}(\mathbb{D}(J)[\mathcal{S}_J^{-1}], \mathbb{D}'(J)) \to \underline{\mathsf{Hom}}_{\mathcal{S}_J}(\mathbb{D}(J), \mathbb{D}'(J))$$

is actually an equivalence of categories (in fact, an isomorphism since we consider localizations to be strict). Thus, the functor γ^* of the statement is seen immediately to be faithful.

To see that γ^* is full, consider two morphisms of prederivators $F, G : \mathbb{D}[\mathcal{S}^{-1}] \to \mathbb{D}'$, and let $\beta : F\gamma \to G\gamma$ be a modification. For each $J \in \mathbf{Dia}$, since γ_J^* is fully faithful, we can find a unique natural transformation $\alpha_J : F_J \Rightarrow G_J$ such that $\alpha_J * \gamma_J = \beta_J$. It is enough to show that the α is a modification: Let $u : J \to K$ any functor in $\mathbf{Dia}$. The square:

$$\begin{tikzcd} u^*F_K \arrow[r, "u^*\alpha_K"] \arrow[d, "\gamma_u^F"'] & u^*G_K \arrow[d, "\gamma_u^G"] \\ F_Ju^* \arrow[r, "\alpha_J u^*"'] & G_Ju^* \end{tikzcd}$$

commutes because applying the faithful functor γ_K^* yields the following square:

$$\begin{tikzcd} u^*F_K\gamma_K \arrow[r, "u^*\beta_K"] \arrow[d, "\gamma_u^F\gamma_K"'] & u^*G_K\gamma_K \arrow[d, "\gamma_u^G\gamma_K"] \\ F_J\gamma_Ju^* \arrow[r, "\beta_J u^*"'] & G_J\gamma_Ju^* \end{tikzcd}$$

which commutes since β is a modification.

To finish the first part of this proposition, it remains to show that γ^* is essentially surjective: Let $H \in \underline{\mathsf{Hom}}_{\mathcal{S}}(\mathbb{D}, \mathbb{D}')$. By the universal property of localization of categories, there is, for each category $J \in \mathbf{Dia}$, a unique functor $\overline{H}_J : \mathbb{D}(J)[\mathcal{S}_J^{-1}] \to \mathbb{D}'(J)$ such that $\overline{H}_J \circ \gamma_J = H_J$. For a functor $u : J \to K$ in $\mathbf{Dia}$, the composite

$$\delta_u : u^*\overline{H}_K\gamma_K = u^*H_K \xrightarrow{\gamma_u^H} H_Ju^* = \overline{H}_J\gamma_Ju^* = \overline{H}_Ju^*\gamma_K$$

is a natural isomorphism. Because precomposition with γ_K is fully faithful, there is a unique natural isomorphism $\gamma_u^{\overline{H}} : u^*\overline{H}_K \to \overline{H}_Ju^*$ such that $\gamma_u^{\overline{H}}\gamma_K = \delta_u$. It remains to check that $\overline{H}$ together with the various $\gamma_u^{\overline{H}}$ defines a morphism of prederivators, which follows as before by applying γ^* to the necessary diagrams and using that H is a morphism of prederivators.

For the last part of the proof, assume that both $\mathbb{D}$ and $\mathbb{D}[\mathcal{S}^{-1}]$ are right derivators and that the localization morphism γ is cocontinuous. Let $\mathbb{D}'$ be another right derivator and let $H \in \underline{\mathsf{Hom}}_{!,\mathcal{S}}(\mathbb{D}, \mathbb{D}')$. Consider $\overline{H}, \delta_u$ as above. It suffices to show that $\overline{H}$ is cocontinuous. The mate of δ_u is given as the composite:

$$u_!\overline{H}_JF_J \xrightarrow{\gamma_{u,!}^{\overline{H}} * F_J} \overline{H}_Ku_!F_J \xrightarrow{H_K * \gamma_{u,!}^F} \overline{H}_KF_Ku_!$$

Since $\overline{H}F = H$ as we just showed, and H is cocontinuous, this composite is an isomorphism. The second map in the composite is an isomorphism because γ is cocontinuous. It follows that the natural transformation $\gamma^{\overline{H}}_{u,!} * \gamma_J$ is an isomorphism, hence so is the natural transformation

$$\gamma^{\overline{H}}_{u,!} : u_! \overline{H}_J \Rightarrow \overline{H}_K u_!$$

because precomposition with γ_J is fully faithful. □

Remark 4.2.4. Let $\mathbb{D}$ be a derivator and $\mathcal{S}$ a class of morphisms in $\mathbb{D}(e)$. A **left Bousfield localization** of $\mathbb{D}$ by $\mathcal{S}$ is a cocontinuous morphism of derivators $\gamma : \mathbb{D} \to L_{\mathcal{S}}\mathbb{D}$ such that precomposition with γ induces an equivalence of categories:

$$\gamma^* = - \circ \gamma : \underline{\mathsf{Hom}}_!(\mathbb{L}_S\mathbb{D}, \mathbb{D}') \to \underline{\mathsf{Hom}}_{!,\mathcal{S}}(\mathbb{D}, \mathbb{D}')$$

for all derivators $\mathbb{D}'$ [CT11, Definition A.4]. Proposition 4.2.3 implies that if $\mathbb{D}[\mathcal{S}^{-1}]$ is a derivator and the localization morphism $\gamma : \mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$ is cocontinuous, then γ is also a left Bousfield localization in the sense of Cisinski. We do not know if the converse holds in general, except when the localization morphism admits a cocontinuous right adjoint (cf. Lemma 4.4.2).

Recall, that given a relative category $(\mathscr{C}, \mathcal{S})$ the class $\mathcal{S}$ is **saturated** if it is exactly the class of morphisms that the localization functor $\mathscr{C} \to \mathscr{C}[\mathcal{S}^{-1}]$ inverts. The following corollary shows that we may assume without loss of generality that $\mathcal{S}$ is saturated:

Corollary 4.2.5. *Let $(\mathbb{D}, \mathcal{S})$ be a relative prederivator. If $\mathcal{S}$ is saturated, then so is $\mathcal{S}_J$ for any category $J \in$ **Dia**.*

Proof. Let $J \in$ **Dia** and $f : X \to Y$ be a morphism in $\mathbb{D}(J)$ such that $\gamma_J(f)$ is an isomorphism. Hence so is $\gamma_J(f)_j$ for any object $j \in J$. Since γ is strict, we conclude that $\gamma_e(f_j) = \gamma_J(f)_j$ is an isomorphism for any object $j \in J$. Since $\mathcal{S}$ is saturated, this implies that f is pointwise in $\mathcal{S}$ and, hence, $f \in \mathcal{S}_J$. □

In order to study localization in general we need a stronger form of (*Der*5):

Definition 4.2.6. Let $\mathbb{D}$ be a prederivator and $K \in \mathbf{Dia}$. We will say that $\mathbb{D}$ satisfies $(Der5)_K$ if for all categories $J \in \mathbf{Dia}$ the partial underlying diagram functor:

$$\mathbb{D}(J \times K) \to \mathbb{D}(J)^K$$

is full and essentially surjective.

Thus, the usual $(Der5)$ is just $(Der5)_{[1]}$. In the literature some authors require that $\mathbb{D}$ satisfies $(Der5)_K$ for all finite free categories K (which is true in all known examples of strong right derivators cf. for instance [Cis10]), but for our purposes we will restrict attention to $K = \ulcorner$. We first show there is $(Der5)_{\ulcorner}$ implies our previous $(Der5)_{[1]}$.

Lemma 4.2.7. *Let $\mathbb{D}$ be a prederivator that satisfies $(Der5)_{\ulcorner}$. Then $\mathbb{D}$ is strong.*

Proof. The inclusion $i : [1] \to \ulcorner$ of the top arrow has a retraction $p : \ulcorner \to [1]$ yielding a commutative diagram:

$$\begin{tikzcd}
\mathbb{D}([1]) \arrow[r, "\mathrm{dia}_{[1]}"] \arrow[d, "p^*"'] \arrow[dd, bend right=60, "\mathrm{id}"'] & \mathbb{D}(e)^{[1]} \arrow[d, "-\circ p"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
\mathbb{D}(\ulcorner) \arrow[r, "\mathrm{dia}_{\ulcorner}"] \arrow[d, "i^*"'] & \mathbb{D}(e)^{\ulcorner} \arrow[d, "-\circ i"] \\
\mathbb{D}([1]) \arrow[r, "\mathrm{dia}_{[1]}"'] & \mathbb{D}(e)^{[1]}
\end{tikzcd}$$

from which it follows immediately that the functor $\mathrm{dia}_{[1]}$ is full and essentially surjective. □

Proposition 4.2.8. *Let $(\mathbb{D}, \mathcal{S})$ be a relative semiderivator such that $\mathcal{S}$ is saturated and that for each category $J \in \mathbf{Dia}$, the relative category $(\mathbb{D}(J), \mathcal{S}_J)$ either satisfies the weakly universal left Ore condition or admits a calculus of left fractions. Then the localization $\mathbb{D}[\mathcal{S}^{-1}]$ is a semiderivator. Moreover, if $\mathbb{D}$ satisfies $(Der5)_K$ for $K = [1]$ or $K = \ulcorner$, then so does $\mathbb{D}[\mathcal{S}^{-1}]$.*

Proof. The axiom $Der1$ is immediately verified for the prederivator $\mathbb{D}[\mathcal{S}^{-1}]$ using the universal property of localizations of categories. To show $Der2$, fix a category $J \in \mathbf{Dia}$. We need to show that the functor

$$\mathbb{D}(J)[\mathcal{S}_J^{-1}] \to \mathbb{D}(e)[\mathcal{S}^{-1}]^J$$

is conservative. Consider a morphism $h : X \rightsquigarrow Y$ in $\mathbb{D}(J)[\mathcal{S}_J^{-1}]$ such that h_j is an isomorphism for any object $j \in J$. The assumption on $(\mathbb{D}(J), \mathcal{S}_J)$ imply that $h = \gamma_J(\sigma)^{-1}\gamma_J(f)$ for some morphisms $f \in \mathbb{D}(e)^{[1]}$ and $\sigma \in \mathcal{S}$. Thus, we can write $\gamma_J(f) = \gamma_J(\sigma)h$ which implies that $\gamma_J(f)$ is pointwise an isomorphism. Since γ is strict, we have $\gamma_J(f)_j = \gamma_e(f_j)$ for any object $j \in J$. Since $\mathcal{S}$ is saturated, we deduce that $f_j \in \mathcal{S}$ for any object $j \in J$, and so $f \in \mathcal{S}$. It follows that $h = \gamma_J(\sigma)^{-1}\gamma_J(f)$ is invertible.

Now assume $\mathbb{D}$ satisfies $(Der5)_\ulcorner$ (the case of $(Der5)_{[1]}$ works exactly the same way just pretend $\ulcorner$ is [1] and ignore morphisms with subscript 2 throughout). The assumptions are stable under shifting by Corollary 4.2.5. Thus, it is enough to show that the underlying diagram functor

$$\widehat{\mathrm{dia}}_\ulcorner : \mathbb{D}(\ulcorner)[\mathcal{S}_\ulcorner^{-1}] \to \mathbb{D}(e)[\mathcal{S}^{-1}]^\ulcorner$$

is full and essentially surjective. The corresponding underlying diagram functor for $\mathbb{D}$ will be denoted as usual by $\mathrm{dia}_\ulcorner : \mathbb{D}(\ulcorner) \to \mathbb{D}(e)^\ulcorner$. We will not use the notation introduced in (1.2.20); instead we will label $\ulcorner$ as follows:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \\ 2 \end{tikzcd}$$

Consider a diagram $h : \cdot \leftsquigarrow \cdot \rightsquigarrow \cdot \in \mathbb{D}(e)[\mathcal{S}^{-1}]^\ulcorner$. Both arrows in the diagram can be represented in the form $\cdot \to \cdot \xleftarrow{\sim} \cdot$. Thus, the whole diagram can be represented in the form:

$$\begin{tikzcd} X_0 \arrow[r, "\gamma f_1"] \arrow[d, "\gamma f_2"'] & X_1 & Z_1 \arrow[l, "\gamma s_1"'] \\ X_2 \\ Z_2 \arrow[u, "\gamma s_2"] \end{tikzcd}$$

where $s_1, s_2 \in \mathcal{S}$. Now let $f \in \mathbb{D}(e)^\ulcorner$ be the diagram:

$$\begin{tikzcd} X_0 \arrow[r, "f_1"] \arrow[d, "f_2"'] & X_1 \\ X_2 \end{tikzcd}$$

By $(Der5)_\ulcorner$ for $\mathbb{D}$ there is some $X \in \mathbb{D}(\ulcorner)$ with $\mathrm{dia}_\ulcorner X \cong f$. Thus we have

$$h \cong \gamma_\ulcorner f \cong \gamma_\ulcorner(\mathrm{dia}_\ulcorner(X)) = \widehat{\mathrm{dia}}_\ulcorner(\gamma_\ulcorner f)$$

which proves essential surjectivity.

For fullness, let X, Y be coherent diagrams in $\mathbb{D}(\ulcorner)[\mathcal{S}_{\ulcorner}^{-1}]$, and $h : \widehat{\mathrm{dia}}_{\ulcorner}(X) \rightsquigarrow \widehat{\mathrm{dia}}_{\ulcorner}(Y)$ be a morphism between their underlying diagrams. Since γ is surjective on objects and strict, we can[2] actually assume $X, Y \in \mathbb{D}(\ulcorner)$ and $h : \gamma\mathrm{dia}_{\ulcorner}(X) \rightsquigarrow \gamma\mathrm{dia}_{\ulcorner}(Y)$. Let us denote the underlying diagrams of X, Y respectively by $X_2 \xleftarrow{f_2} X_0 \xrightarrow{f_1} X_1$ and $Y_2 \xleftarrow{g_2} Y_0 \xrightarrow{g_1} X_1$. We will break down the proof in two steps.

Assume first that h is pointwise in the image of γ. We can rephrase this by requiring that there are morphisms $h_i : X_i \to Y_i$ for $i = 0, 1, 2$ in $\mathbb{D}(e)$ such that $\gamma(h_1 f_1) = \gamma(g_1 h_0)$ and $\gamma(h_2 f_2) = \gamma(g_2 h_0)$. By Corollary 4.1.20 there is some commutative diagram in $\mathbb{D}(e)$ of the form[3]:

$$\begin{tikzcd}
Y_2 \arrow[d, "s_2"'] & X_0 \arrow[l, "h_2 f_2"'] \arrow[r, "h_1 f_1"] \arrow[d, "1"'] & Y_1 \arrow[d, "s_1"] \\
Z_2 & X_0 \arrow[l, "u_2"'] \arrow[r, "u_1"] & Z_1 \\
Y_2 \arrow[u, "s_2"] & X_0 \arrow[l, "g_2 h_0"] \arrow[r, "g_1 h_0"'] \arrow[u, "1"'] & Y_1 \arrow[u, "s_1"']
\end{tikzcd}$$

where $s_1, s_2 \in \mathcal{S}$ and after applying γ the composites of the vertical arrows on both sides are identities. Thus, we get a further commutative diagram in $\mathbb{D}(e)$:

$$\begin{tikzcd}
X_2 \arrow[d, "h_2"'] & X_0 \arrow[l, "f_2"'] \arrow[r, "f_1"] \arrow[d, "1"'] & X_1 \arrow[d, "h_1"] \\
Y_2 \arrow[d, "s_2"'] & X_0 \arrow[l, "h_2 f_2"'] \arrow[r, "h_1 f_1"] \arrow[d, "1"'] & Y_1 \arrow[d, "s_1"] \\
Z_2 & X_0 \arrow[l, "u_2"'] \arrow[r, "u_1"] & Z_1 \\
Y_2 \arrow[u, "s_2"] \arrow[d, "1"'] & X_0 \arrow[l, "g_2 h_0"'] \arrow[r, "g_1 h_0"] \arrow[u, "1"'] \arrow[d, "h_0"'] & Y_1 \arrow[u, "s_1"'] \arrow[d, "1"] \\
Y_2 & Y_0 \arrow[l, "g_2"] \arrow[r, "g_1"'] & Y_1
\end{tikzcd} \tag{4.2.9}$$

[2]Here we are being a bit sloppy with notation and writing γ for any of the following: the morphism of semiderivators $\mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$, the underlying functor $\gamma_e : \mathbb{D}(e) \to \mathbb{D}(e)[\mathcal{S}^{-1}]$ and the functor γ_e induces on diagram categories $\mathbb{D}(e)^{[1]} \to \mathbb{D}(e)[\mathcal{S}^{-1}]^{[1]}$.

[3]The key is that arrows in the left and right part have the same source; this wouldn't work for instance for $\lrcorner$; for that we'd need to be able to present the localization by right fractions instead.

where, after applying γ, the left, middle and right columns compose to $\gamma h_2, \gamma h_0$ and γh_1 respectively. Using $(Der5)_\ulcorner$ for $\mathbb{D}$, pick $X', Y', Z' \in \mathbb{D}(\ulcorner)$ whose underlying diagrams are isomorphic (in $\mathbb{D}(e)^\ulcorner$) to the second, third and fourth rows in (4.2.9). Without loss of generality, we can assume these lifts are strict (in particular, this can be done without changing the composites of the vertical arrows after applying γ). Thus, by $(Der5)_\ulcorner$ for $\mathbb{D}$ there is a zig-zag in $\mathbb{D}(\ulcorner)$:

$$X \to X' \to Z' \xleftarrow{\sim} Y' \to Y$$

whose underlying diagram is precisely (4.2.9). Since γ is a strict morphism of prederivators, it follows that $\gamma_\ulcorner$ applied to the above zig-zag is a lift of h.

Now we treat the general case: pick representations of h_0, h_1 and h_2 of the form $\cdot \to \cdot \xleftarrow{\sim} \cdot$. That is, we have a diagram in $\mathbb{D}(e)$ which becomes commutative after applying γ:

(4.2.10)
$$\begin{tikzcd} X_2 \arrow[d, "u_2"'] & X_0 \arrow[l, "f_2"'] \arrow[r, "f_1"] \arrow[d, "u_0"'] & X_1 \arrow[d, "u_1"] \\ Z_2 & Z_0 & Z_1 \\ Y_2 \arrow[u, "s_2"] & Y_0 \arrow[l, "g_2"] \arrow[r, "g_1"'] \arrow[u, "s_0"] & Y_1 \arrow[u, "s_1"'] \end{tikzcd}$$

(where $s_0, s_1, s_2 \in \mathcal{S}$) and such that $h_i = \gamma(s_i)^{-1}\gamma(u_i)$ for $i = 0, 1, 2$. Now consider commutative squares for $i = 1, 2$:

$$\begin{tikzcd} Y_0 \arrow[r, "s_i g_i"] \arrow[d, "s_0"'] & Z_i \arrow[d, "\sigma_i"] \\ Z_0 \arrow[r, "g'_i"'] & W_i \end{tikzcd}$$

where $\sigma_1, \sigma_2 \in \mathcal{S}$. It follows that we have a diagram in $\mathbb{D}(e)$

$$\begin{tikzcd} X_2 \arrow[d, "\sigma_2 u_2"'] & X_0 \arrow[l, "f_2"'] \arrow[r, "f_1"] \arrow[d, "u_0"'] & X_1 \arrow[d, "\sigma_1 u_1"] \\ W_2 & Z_0 \arrow[l, "g'_2"'] \arrow[r, "g'_1"] & W_1 \\ Y_2 \arrow[u, "\sigma_2 s_2"] & Y_0 \arrow[l, "g_2"] \arrow[r, "g_1"'] \arrow[u, "s_0"] & Y_1 \arrow[u, "\sigma_1 s_1"'] \end{tikzcd}$$

where the bottom squares commute in $\mathbb{D}(e)$, the top ones commute after applying γ and such that after applying γ the vertical arrows on the left, middle and right columns compose to h_1, h_0

and h_2 respectively. By using $(Der5)_\ulcorner$ for $\mathbb{D}$ and replacing the middle row up to isomorphism in $\mathbb{D}(e)^\ulcorner$, we can assume that row admits a strict lift $Z \in \mathbb{D}(\ulcorner)$. Using $(Der5)_\ulcorner$ for $\mathbb{D}$ again, pick a morphism $s : Y \to Z$ in $\mathbb{D}(\ulcorner)$ with underlying diagram $(\sigma_2 s_2, s_0, \sigma s_1)$. Note that $s \in \mathcal{S}_\ulcorner$ since this is true pointwise. Moreover, by the previous case, we can find a morphism $u : X \rightsquigarrow Z$ with $\widehat{\mathrm{dia}}_\ulcorner(u) = (\sigma_2 u_2, u_0, \sigma_1 u_1)$. It follows that $\gamma(s)^{-1}u : X \rightsquigarrow Y$ is a lift of h. □

Remark 4.2.11. Without assuming a left calculus of fractions or the universal left Ore condition for each $(\mathbb{D}(J), \mathcal{S}_J)$, the localization of $\mathbb{D}$ by $\mathcal{S}$ will not in general be strong, even if $\mathbb{D}$ is. See [Len17] for a simple example.

Definition 4.2.12. Given a relative semiderivator $(\mathbb{D}, \mathcal{S})$ that admits J-colimits for some category $J \in$ **Dia**, we say that $\mathcal{S}$ is **closed under J-colimits** if for any $f \in \mathcal{S}_J$ we have $(\pi_J)_! f \in \mathcal{S}$. If $\mathbb{D}$ is a right derivator and $\mathcal{S}$ is closed under J-colimits for all $J \in$ **Dia**, then we simply say that $\mathcal{S}$ is **closed under colimits**.

Lemma 4.2.13. *Let $(\mathbb{D}, \mathcal{S})$ be a relative right derivator and J a category in* **Dia**. *Then $\mathcal{S}$ is closed under J-colimits if and only if so is $\mathcal{S}_K$ for any category $K \in$* **Dia**.

Proof. The inverse direction is trivially true so let's show the forward direction. Let K be a category in **Dia** and $f \in \mathcal{S}_{J\times K}$. The claim is then that $(\pi_J \times \mathrm{id}_K)_! f \in S_K$. By $(Der4R)$ we have a $\mathbb{D}$-exact square:

$$\begin{tikzcd} ((\pi_J \times \mathrm{id}_K) \downarrow k) \arrow[r, "\mathrm{pr}_{J\times K}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\swArrow\alpha"] & J \times K \arrow[d, "\pi_J\times\mathrm{id}_K"] \\ e \arrow[r, "k"'] & K \end{tikzcd}$$

Note that $((\pi_J \times \mathrm{id}_K) \downarrow k) \cong J \times (K \downarrow k)$ and that $(K \downarrow k)$ has a final object, hence is contractible. Thus, we get a $\mathbb{D}$-exact square:

$$\begin{tikzcd} J \arrow[r, "\mathrm{id}_J\times k"] \arrow[d, "\pi_J"'] \arrow[dr, phantom, "\swArrow\alpha"] & J \times K \arrow[d, "\pi_J\times\mathrm{id}_K"] \\ e \arrow[r, "k"'] & K \end{tikzcd}$$

which proves our claim since $(\mathrm{id}_J \times k)^*(S_{J\times K}) \subset S_J$. □

We say that a class $\mathcal{S} \subset D(e)^{[1]}$ is **closed under pushouts** if for any cocartesian square with underlying diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\sigma"'] & Y \arrow[d, "\sigma'"] \\
Z \arrow[r, "f'"'] & W
\end{tikzcd}$$

such that $\sigma \in \mathcal{S}$ we have $\sigma' \in \mathcal{S}$.

Lemma 4.2.14. *Let $(\mathbb{D}, \mathcal{S})$ be a strong relative right derivator such that $\mathcal{S}$ satisfies 2-out-of-3. If $\mathcal{S}$ is closed under pushouts then it is closed under $\ulcorner$-colimits. The converse is true if $\mathbb{D}$ satisfies $(Der5)_\ulcorner$.*

Proof. Assume first that $\mathbb{D}$ satisfies $(Der5)_\ulcorner$ and $\mathcal{S}$ is closed under $\ulcorner$-colimits. Consider a coherent diagram $X \in \mathbb{D}(\ulcorner)$ with underlying diagram:

$$\begin{tikzcd}
X_{(0,0)} \arrow[r, "f"] \arrow[d, "\sigma"'] & X_{(1,0)} \\
X_{(0,1)} &
\end{tikzcd}$$

with $\sigma \in \mathcal{S}$. The underlying diagram of $(i_\ulcorner)_! X$ has the form:

$$\begin{tikzcd}
X_{(0,0)} \arrow[r, "f"] \arrow[d, "\sigma"'] & X_{(1,0)} \arrow[d, "\sigma'"] \\
X_{(0,1)} \arrow[r, "f'"'] & W
\end{tikzcd}$$

Pick a lift $X' \in \mathbb{D}(\ulcorner)$ of the diagram:

$$\begin{tikzcd}
X_{(0,0)} \arrow[r, "f"] \arrow[d, "1"'] & X_{(1,0)} \\
X_{(0,0)} &
\end{tikzcd}$$

We then have a morphism of incoherent diagrams:

$$\begin{tikzcd}
X_{(0,0)} \arrow[rr, "f"] \arrow[dr, "1"] \arrow[dd, "1"'] & & X_{(1,0)} \arrow[dd, dashed, "1"] \\
& X_{(0,0)} \arrow[dd] & \\
X_{(0,0)} \arrow[rr, dashed] \arrow[dr, "\sigma"'] & & X_{(1,0)} \\
& X_{(0,1)} &
\end{tikzcd}$$

which admits a lift $s : X' \to X$ by fullness of $\mathrm{dia}_{\ulcorner}$. Note that $s \in \mathcal{S}_{\ulcorner}$ since this is true pointwise. The underlying diagram of $(i_{\ulcorner})_! s$ then looks like:

$$\begin{tikzcd}
X_{(0,0)} \arrow[rr, "f"] \arrow[dr, "1"] \arrow[dd, "1"'] & & X_{(1,0)} \arrow[dr, "1"] \arrow[dd, dashed] & \\
& X_{(0,0)} \arrow[rr] \arrow[dd] & & X_{(1,0)} \arrow[dd, "\sigma'"] \\
X_{(0,0)} \arrow[rr, dashed] \arrow[dr, "\sigma"'] & & X_{(1,0)} \arrow[dr, dashed, "\sigma'"] & \\
& X_{(0,1)} \arrow[rr, "f'"'] & & W
\end{tikzcd}$$

where $\sigma' \in \mathcal{S}$ since $\mathcal{S}$ is closed under pushouts. Thus, $\mathcal{S}$ is closed under $\ulcorner$-colimits.

Converesely, assume that $\mathbb{D}$ is strong and $\mathcal{S}$ is closed under pushouts. Let K be the poset

$$\begin{tikzcd}
& & (0,0,0) \arrow[rr] \arrow[dl] \arrow[dd] & & (0,1,0) \arrow[dl] \arrow[dd, dashed] \arrow[ddrr] & & \\
& (1,0,0) \arrow[rr] \arrow[dd, dashed] \arrow[dddl] & & (1,1,0) \arrow[dd, dashed] \arrow[dddl] \arrow[dddr] \arrow[ddrr] & & & \\
& & (0,0,1) \arrow[rr, dashed] \arrow[dl, dashed] & & (0,1,1) \arrow[rr, dashed] \arrow[dl, dashed] & & (0,2,1) \arrow[dl] \\
& (1,0,1) \arrow[rr, dashed] \arrow[dl, dashed] & & (1,1,1) \arrow[rr, dashed] \arrow[dl, dashed] & & (1,2,1) \arrow[dl] & \\
(2,0,1) \arrow[rr] & & (2,1,1) \arrow[rr] & & (2,2,1) & &
\end{tikzcd}$$

We consider a sequence of poset maps:

$$\ulcorner \times [1] \xrightarrow{i} I_3 \xrightarrow{i_3} I_2 \xrightarrow{i_2} I_1 \xrightarrow{i_1} J_2 \xrightarrow{j_2} J_1 \xrightarrow{j_1} K \tag{4.2.15}$$

as follows: J_1 is the full subcategory of K spanned by all objects except $(1,1,0)$, J_2 is the full subcategory of J_1 spanned by all objects except $(2,2,1)$, I_1 is the full subcategory of J_2 spanned by all objects except $(1,2,1)$ and I_2 is the full subcategory of I_1 spanned by all objects except $(2,1,1)$ and I_3 is the full subcategory of I_2 spanned by all objects except $(1,1,1)$. Finally, $i : \ulcorner \times [1] \to I_3$ is the inclusion of the full subcategory of I_3 spanned by all objects except $(1,0,1)$ and $(0,1,1)$ and the other maps are just inclusions. Let $u : \ulcorner \times [1] \to K$ be the composite of all those maps.

Let $\sigma : X \to Y$ be a morphism in $\mathcal{S}_{\ulcorner}$ with underlying diagram:

$$\begin{tikzcd}
& X_0 \arrow[r, "f_1"] \arrow[dl, "f_2"'] \arrow[dd, "\sigma_0"] & X_1 \arrow[dd, "\sigma_1"] \\
X_2 \arrow[dd, "\sigma_2"'] & & \\
& Y_0 \arrow[r, "g_1"'] \arrow[dl, "g_2"] & Y_1 \\
Y_2 & &
\end{tikzcd}$$

Pick a coherent diagram $Z \in \mathbb{D}(\ulcorner \times [1])$ whose underlying diagram is σ up to isomorphism. Then $i_! Z \in \mathbb{D}(I_3)$ has underlying diagram that looks like:

$$\begin{tikzcd}
& & X_0 \arrow[r, "f_1"] \arrow[dl, "f_2"'] \arrow[dd, "\sigma_0"] & X_1 \arrow[dd, "\sigma_1'"'] \arrow[ddr, "\sigma_1"] & \\
& X_2 \arrow[dd, "\sigma_2'"] \arrow[dddl, "\sigma_2"'] & & & \\
& & Y_0 \arrow[r, "f_1'"'] \arrow[dl, "f_2'"] & P_1 \arrow[r, "\tau_1"'] & Y_1 \\
& P_2 \arrow[dl, "\tau_2"] & & & \\
Y_2 & & & &
\end{tikzcd}$$

We claim that the square:

$$\begin{tikzcd}
X_0 \arrow[r, "f_1"] \arrow[d, "\sigma_0"'] & X_1 \arrow[d, "\sigma_1'"] \\
Y_0 \arrow[r, "f_1'"'] & P_1
\end{tikzcd}$$

is cocartesian. Indeed, consider the inclusion $\square \to I_3$ of the square spanned by all objects of the form $(0, i, j)$ for $i, j = 0, 1$. The induced functor $\ulcorner \to (I_3 - (0,1,1) \downarrow (0,1,1))$ is an isomorphism (in particular has a left adjoint); hence our claim follows by [Gro13, Proposition 3.10]. Similarly, we deduce that the square:

$$\begin{tikzcd}
X_0 \arrow[r, "f_2"] \arrow[d, "\sigma_0"'] & X_2 \arrow[d, "\sigma_2'"] \\
Y_0 \arrow[r, "f_1'"'] & P_2
\end{tikzcd}$$

is also cocartesian. By the assumption on $\mathcal{S}$, we have $\sigma_1', \sigma_2' \in \mathcal{S}$ and, hence, $\tau_1, \tau_2 \in \mathcal{S}$ by 2-out-of-3. By applying left Kan extensions along the maps in (4.2.15) in order and repeated application of [Gro13, Proposition 3.10] at each step as illustrated above, we deduce that $u_! Z$ has underlying diagram:

$$\begin{tikzcd}
 & & X_0 \arrow[rr] \arrow[dl] \arrow[dd] & & X_1 \arrow[dl] \arrow[dd, dashed] \arrow[ddrr] & & \\
 & X_2 \arrow[rr] \arrow[dd, dashed] \arrow[dddl] & & Q \arrow[dd, dashed] \arrow[ddrr] \arrow[dddl] \arrow[dddr] & & & \\
 & & Y_0 \arrow[rr, dashed] \arrow[dl, dashed] & & P_1 \arrow[rr, dashed] \arrow[dl, dashed] & & Y_1 \arrow[dl] \\
 & P_2 \arrow[rr, dashed] \arrow[dl, dashed] & & P_3 \arrow[rr, dashed] \arrow[dl, dashed] & & P_4 \arrow[dl] & \\
Y_2 \arrow[rr] & & P_5 \arrow[rr] & & P & &
\end{tikzcd}$$

where all the bottom squares as well as the front and back faces of the cube in the back left corner are cocartesian. Restricting to the top and front faces of the cube we get a coherent diagram in $[1] \times [2]$ with underlying diagram:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\
X_2 \arrow[r] \arrow[d] & Q \arrow[d] \\
P_2 \arrow[r] & P_3
\end{tikzcd}$$

The top square is cocartesian as already explained, and the composite square is also cocartesian since it is isomorphic to the composite of two cocartesian squares (namely the back and bottom of the cube). We conclude that the coherent square:

$$\begin{tikzcd}
X_2 \arrow[r] \arrow[d] & Q \arrow[d] \\
P_2 \arrow[r] & P_3
\end{tikzcd}$$

is cocartesian (cf. [Gro13, Proposition 3.13]). Since the morphism $X_2 \to P_2$ is in $\mathcal{S}$, so is the morphism $Q \to P_3$ by our assumption on $\mathcal{S}$. We also know that the map $P_2 \to Y_2$ is in $\mathcal{S}$ which implies that so are the maps $P_3 \to P_5$ and $P_4 \to P$ since all bottom squares are cocartesian.

Similarly, the maps $P_3 \to P_4$ and $P_5 \to P$ are in $\mathcal{S}$. Since $\mathcal{S}$ is closed under compositions, we deduce that the map $Q \to P$ is in $\mathcal{S}$. But that map is isomorphic to $(\pi_{\ulcorner})_!\sigma$ showing that $\mathcal{S}$ is closed under $\ulcorner$-colimits. $\square$

Corollary 4.2.16. *Let $(\mathbb{D}, \mathcal{S})$ be a relative right derivator that satisfies $(Der5)_{\ulcorner}$. Then $\mathcal{S}$ is closed under pushouts if and only if so is $\mathcal{S}_J$ for any category $J \in$ **Dia**.*

Proof. This is immediate from Lemmas 4.2.13 and 4.2.14. $\square$

Theorem 4.2.17. *Let $(\mathbb{D}, \mathcal{S})$ be a relative right derivator such that $\mathcal{S}$ is saturated and closed under colimits. Moreover, assume that either $\mathbb{D}$ satisfies $(Der5)_{\ulcorner}$ or that $(\mathbb{D}(J), \mathcal{S}_J)$ admits a left calculus of fractions for each category $J \in$ **Dia**. Then $\mathbb{D}[\mathcal{S}^{-1}]$ is a right derivator that satisfies $(Der5)_{\ulcorner}$ if $\mathbb{D}$ does, and the localization morphism $\gamma : \mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$ is cocontinuous.*

Proof. For the first part, it suffices to show that for any category $J \in$ **Dia**, the relative category $(\mathbb{D}(J), \mathcal{S}_J)$ satisfies the weakly universal left Ore condition. But that follows from the fact that $\mathbb{D}$ admits weak pushouts[4] and that $\mathcal{S}$ is closed under pushouts. Hence, by Proposition 4.2.8, $\mathbb{D}$ is a semiderivator that satisfies $(Der5)_{\ulcorner}$.

To show that $\mathbb{D}[\mathcal{S}^{-1}]$ is a right derivator, note that for any functor $u : J \to K$ and any $k \in K$ we have a $\mathbb{D}$-exact square:

$$\begin{tikzcd} (k \downarrow u) \arrow[r, "\mathrm{pr}_J"] \arrow[d, "\pi_{(k\downarrow u)}"'] \arrow[dr, phantom, "\overset{\alpha}{\swarrow}"] & J \arrow[d, "u"] \\ e \arrow[r, "k"'] & K \end{tikzcd}$$

Since $\mathbb{D}$-pullbacks preserve the property of being pointwise in $\mathcal{S}$ and $\mathcal{S}$ is closed under all colimits, we deduce that $k^* u_! s \cong (\pi_{(k\downarrow u)})_! \mathrm{pr}_J^* s \in \mathcal{S}$ for any morphism $s \in \mathcal{S}_J$ and hence we have a relative functor:

$$u_! : (\mathbb{D}(J), \mathcal{S}_J) \to (\mathbb{D}(K), \mathcal{S}_K)$$

It is shown in [Rod14b] that there is a 2-category **RelCAT** whose 0-cells are relative categories and 1-cells are relative functors. The 2-cells are zig-zags of natural transformations such that the

[4] This follows by $(Der5)_{\ulcorner}$ and existence of homotopy pushouts.

ones pointing the wrong way are pointwise weak equivalences. Moreover [Rod14b, Lemma 1.9] shows that the assignment $(\mathscr{C}, \mathcal{W}) \mapsto \mathscr{C}[\mathcal{W}^{-1}]$ is actually a 2-functor

$$\mathrm{Loc} : \mathbf{RelCAT} \to \mathbf{CAT}$$

Note also that a natural transformation between relative functors defines a 2-cell in **RelCAT** (just a zig-zag of length 1). Thus it is clear that $\mathbb{D}[\mathcal{S}^{-1}]$ satisfies (*Der*3*R*) (simply by applying Loc to the adjunction $u_! \dashv u^*$ in $\mathbb{D}$) and (*Der*4*R*) (by applying Loc to the mate in $\mathbb{D}$ associated with a comma square). The fact that γ is cocontinuous follows immediately by the fact that $\mathrm{Loc}(u_!)\gamma_J = \gamma_K u_!$. $\square$

Remark 4.2.18. If both $\mathbb{D}$ and $\mathbb{D}[\mathcal{S}^{-1}]$ are derivators, then the localization $\gamma : \mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$ is also a left Bousfield localization in Cisinski's sense (cf. Remark 4.2.4).

Remark 4.2.19. Let $\mathbb{D}$ be a right derivator and $\mathcal{S}$ a class of morphisms in $\mathbb{D}(e)$. By [Rod14a, Theorem 3.4], the following two questions are equivalent:

- For a given functor $u : J \to K$, does the induced pullback functor

  $$\mathbb{D}(K)[\mathcal{S}_K^{-1}] \to \mathbb{D}(J)[\mathcal{S}_J^{-1}]$$

  admit a left adjoint?

- Does the $\mathbb{D}$-pullback $u^* : \mathbb{D}(K) \to \mathbb{D}(J)$ admit a left derived functor when viewed as a map of relative categories $(\mathbb{D}(K), \mathcal{S}_K) \to (\mathbb{D}(J), \mathcal{S}_J)$?

Thus, it is not at all clear that $\mathbb{D}[\mathcal{S}^{-1}]$ will have left or right Kan extensions. The Theorem above works essentially because we don't have to derive anything, i.e. all possible functors involved are relative. We were unable to find a good axiomatic that both works in known examples and seems natural to axiomatize when a relative right derivator will have a localization that is again a relative right derivator. For example, an easy fix in regards to the issue of left derivability, would be to define relative versions of (*Der*3*R*) and (*Der*4*R*), i.e. require that all pullbacks admit left adjoints in **RelCAT** and that (*Der*4*R*) is satisfied in **RelCAT**. However, this still leaves the issue of why the localized prederivator would satisfy (*Der*2) or (*Der*5). These two are satisfied in the presence of a 2 or 3-arrow calculus of fractions on each level, but we were unable to find a good axiomatic on the base that would guarantee these exist on all levels.

Corollary 4.2.20. *Let* $(\mathbb{D}, \mathcal{S})$ *be a relative right* $\mathbf{Dir}_{\mathrm{f}}$*-derivator that satisfies* $(Der5)_{\ulcorner}$ *and such that* $\mathcal{S}$ *is saturated and closed under pushouts. Then* $\mathbb{D}[\mathcal{S}^{-1}]$ *is a right* $\mathbf{Dir}_{\mathrm{f}}$*-derivator that satisfies* $(Der5)_{\ulcorner}$ *and the localization morphism* $\gamma : \mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$ *is cocontinuous.*

Proof. By Theorem 4.2.17, it is enough to show that $\mathcal{S}$ is closed under colimits. For any category $J \in \mathbf{Dia}$ forming colimits with respect to J is a morphism of right derivators $(\pi_J)_! : \mathbb{D}^J \to \mathbb{D}$, which implies that $(\pi_J)_!(\mathrm{dia}_{J,[1]}X) \cong \mathrm{dia}_{[1]}(\pi_J \times \mathrm{id}_{[1]})_! X$ for any object $X \in \mathbb{D}(J \times [1])$. Hence, if we let $\widehat{\mathcal{S}} \subset \mathbb{D}([1])$ be the full subcategory spanned by all objects $X \in \mathbb{D}([1])$ such that $\mathrm{dia}_{[1]}(X) \in \mathcal{S}$, then strongess (cf. Lemma 4.2.7) shows that $\widehat{\mathcal{S}}$ is closed under J-colimits if and only if so is $\mathcal{S}$. We can thus conclude by Theorem 1.4.8 and Lemma 4.2.14 (and the observation that $\widehat{\mathcal{S}}$ contains the initial object of $\mathbb{D}([1])$ since $\mathcal{S}$ is saturated, hence, replete). □

Example 4.2.21. A general situation where Corollary 4.2.20 applies is when $\mathbb{D} := \mathbb{D}_{\mathcal{A}}$ is the $\mathbf{Dir}_{\mathrm{f}}$ derivator associated to a right derivable category[5] $\mathcal{A} := (\mathcal{M}, \mathrm{cof}, \mathcal{W})$ (cf. [Cis10, Corollary 2.24]). Let $\mathcal{W} \subset \mathcal{W}'$ be another set of weak equivalences such that $\mathcal{A}' := (\mathcal{M}, \mathrm{cof}, \mathcal{W}')$ is still right derivable. If $\mathcal{W}'$ is saturated, then so is $\mathcal{S} := \gamma(\mathcal{W}')$ where $\gamma : \mathcal{M} \to \mathcal{M}[\mathcal{W}^{-1}]$ is the localization functor. Moreover $\mathcal{S}$ is closed under pushouts (exactly because pushouts in $\mathbb{D}_{\mathcal{A}}$ are computed by first doing a cofibrant replacement in $\mathcal{A}$ which has also got to be a cofibrant replacement in $\mathcal{A}'$). Thus, we can form the localization $\mathbb{D}_{\mathcal{A}} \to \mathbb{D}_{\mathcal{A}}[\mathcal{W}'^{-1}]$ and it is not too hard to see that this is equivalent to considering the right exact functor $\mathbb{D}_{\mathcal{A}} \to \mathbb{D}_{\mathcal{A}'}$ induced by the identity functor on $\mathcal{M}$ viewed as a relative functor $\mathcal{A} \to \mathcal{A}'$.

4.3 The stable case

The goal of this section is to study levelwise Verdier localizations of triangulated $\mathbf{Dir}_{\mathrm{f}}$-derivators, and specifically to show that the levelwise Verdier localization of such a derivator $\mathbb{D}$ is again a triangulated derivator. This has been proved directly in [Fra96], but we obtain a new proof as a special case of the results of the previous section.

[5]For example cofibrant objects in any model category or a Waldhausen category satisfying the cylinder and saturation axioms.

Recall that given a triangulated category $\mathcal{T}$ and a triangulated subcategory $\mathscr{E}$, the Verdier localization $\mathcal{T}/\mathscr{E}$ is defined as the localization $\mathcal{T}[\Sigma(\mathscr{E})^{-1}]$, where $\Sigma(\mathscr{E})$ is the class of all morphisms in $\mathcal{T}$ whose cone lies in $\mathscr{E}$. It is also well-known that there is a unique triangulation on the quotient $\mathcal{T}/\mathscr{E}$ such that the canonical functor $\mathcal{T} \to \mathcal{T}/\mathscr{E}$ is exact. Furthermore, the Verdier localization is unchanged (up to isomorphism) if we replace $\mathscr{E}$ by its thick envelope in $\mathcal{T}$, i.e. we may as well assume that $\mathscr{E}$ is thick.

Now fix a small triangulated $\mathbf{Dir}_{\mathrm{f}}$-derivator $\mathbb{D}$ and a maximal subprederivator $\mathbb{E}$ associated to a thick subcategory $\mathcal{E} \subset \mathbb{D}(e)$. Recall from Corollary 1.7.5 that $\mathbb{E}$ is also a triangulated $\mathbf{Dir}_{\mathrm{f}}$-derivator such that the inclusion $\mathbb{E} \hookrightarrow \mathbb{D}$ is exact and $\mathbb{E}(J)$ is a thick subcategory of $\mathbb{D}(J)$ for any category $J \in \mathbf{Dia}$. Write $\mathcal{S} = \Sigma(\mathscr{E})$ for the class of morphisms in $\mathbb{D}(e)$ whose cone lies in $\mathscr{E}$. As before, we let $\mathcal{S}_J$ for the class of morphisms in $\mathbb{D}(J)$ that are pointwise in $\mathcal{S}$.

Lemma 4.3.1. *We have $\mathcal{S}_J = \Sigma(\mathbb{E}(J))$ for any category $J \in$ **Dia**.*

Proof. Let J be any category in **Dia**. Since the cone commutes with pullbacks (non-canonically on incoherent morphisms), we see that $f \in \mathcal{S}_J$ if and only if $f_j \in \mathcal{S}$ for any $j \in J$ if and only if the cone of f_j is in $\mathscr{E}$ for any $j \in J$ if and only if the cone of f is in $\mathbb{E}(J)$, if and only if $f \in \Sigma(\mathbb{E}(J))$. □

Proposition 4.3.2. *Let $\mathbb{D}$ be a small triangulated $\mathbf{Dir}_f$-derivator and $\mathbb{E}$ the maximal subprederivator associated to a thick subcategory $\mathcal{E} \subset \mathbb{D}(e)$. Then the levelwise Verdier localizations assemble to a small triangulated $\mathbf{Dir}_f$-derivator $\mathbb{D}/\mathbb{E}$ and the localization morphisms define a strict exact morphism of triangulated derivators $\gamma : \mathbb{D} \to \mathbb{D}/\mathbb{E}$.*

Proof. By Lemma 4.3.1 it follows that $\mathbb{D}/\mathbb{E}$ is exactly $\mathbb{D}[\mathcal{S}^{-1}]$. By [Ver96, II, Proposition 2.1.8] we know that each $\mathcal{S}_J$ admits a left calculus of fractions hence $\mathbb{D}/\mathbb{E}$ is a strong semiderivator by Proposition 4.2.8. The fact that $\mathbb{E}$ is a derivator implies that $\mathcal{E} = \mathbb{E}(e)$ is closed under both colimits and limits. We claim that so is $\mathcal{S}$. Indeed, let $J \in$ **Dia** and $f : X \to Y$ a morphism in $\mathcal{S}_J$. Pick a distinguished triangle:

$$X \xrightarrow{f} Y \to Z \rightsquigarrow \Sigma X$$

By Lemma 4.3.1 we have $Z \in \mathbb{E}(J)$ hence $u_! Z \in \mathbb{E}(e) = \mathcal{E}$ since $\mathbb{E}$ is closed under colimits. But the functor $(\pi_J)_! : \mathbb{D}(J) \to \mathbb{D}(e)$ is exact so we have a distinguished triangles in $\mathbb{D}(e)$:

$$u_!X \xrightarrow{u_!f} u_!Y \to u_!Z \rightsquigarrow \Sigma u_!X$$

which shows that $u_!f \in \mathcal{S}$. Hence $\mathcal{S}$ is closed under colimits, and dually it is also closed under limits. Theorem 4.2.17 and its dual then imply that $\mathbb{D}/\mathbb{E}$ is a strong small $\mathbf{Dir}_\mathrm{f}$-derivator and the canonical morphism $\gamma : \mathbb{D} \to \mathbb{D}/\mathbb{E}$ is exact[6]. In what follows, we will put an overbar over $\mathbb{D}/\mathbb{E}$-pullbacks and Kan extensions to distinguish them from te corresponding ones in $\mathbb{D}$.

It remains to show that $\mathbb{D}/\mathbb{E}$ is stable. The counit $\overline{\epsilon} : \overline{(i_\ulcorner)}_!\overline{(i_\ulcorner)}^* \Rightarrow \mathrm{id}_{\mathbb{D}/\mathbb{E}(\square)}$ can be chosen so that $\overline{\epsilon} * \gamma_\square = \gamma_\square * \epsilon$, where ϵ is the counit of the adjunction $(i_\ulcorner)_! \dashv (i_\ulcorner)^*$ (cf. [Mal07]). Now consider a cocartesian square in $\mathbb{D}/\mathbb{E}(\square)$ which is the image of some $X \in \mathbb{D}(\square)$ by essential surjectivity of $\gamma_\square$. Since $\gamma_\square X$ is cocartesian, we know that $\overline{\epsilon}_{\gamma_\square X} = \gamma_\square(\epsilon_X)$ is an isomorphism in $\mathbb{D}(\square)/\mathbb{E}(\square)$. In other words, any cocartesian square in $\mathbb{D}/\mathbb{E}$ is the image under γ of a cocartesian square in $\mathbb{D}$. Dually, any cartesian square in $\mathbb{D}(\square)/\mathbb{E}(\square)$ is the image of a cartesian square in $\mathbb{D}(\square)$; thus stability of $\mathbb{D}/\mathbb{E}$ follows directly from that of $\mathbb{D}$. $\square$

Remark 4.3.3. It is further showed in [Fra96, Chapter 1, Theorem 3] that if $\mathbb{D}(e)/\mathcal{E}$ is locally small, then so is every value of $\mathbb{D}/\mathbb{E}$. Note that the result in [Fra96] is slightly more general because it applies also to small triangulated $\mathbf{Pos}_\mathrm{f}$-derivators, while our proof only works when $\mathbf{Dia} = \mathbf{Dir}_\mathrm{f}$. However, this is not a big deal because all examples of small triangulated derivators we are aware of can be defined over all finite direct categories. Moreover, it is shown in [Hor17] that big derivators over posets can be extended to all categories. In lieu of this result, it is likely that small derivators over finite posets can be extended to all finite direct categories.

Remark 4.3.4. Proposition 4.3.2 together with Corollary 1.7.5 show that given a small triangulated $\mathbf{Dir}_\mathrm{f}$-derivator and a thick subcategory $\mathcal{E}$ with associated maximal subprederivator $\mathbb{E}$ (cf. Definition 1.4.12), we have an exact sequence of small triangulated derivators:

$$\mathbb{E} \hookrightarrow \mathbb{D} \twoheadrightarrow \mathbb{D}/\mathbb{E} \tag{4.3.5}$$

in the sense that all functors are exact, the right arrow is a localization and the left arrow is a kernel of that localization. In general, if $\gamma : \mathbb{D} \to \mathbb{D}[\mathcal{S}^{-1}]$ is a localization of (arbitrary) triangulated

[6] It is both continuous and cocontinuous by Theorem 4.2.17 and its dual.

derivators then $\mathbb{D}[\mathcal{S}^{-1}]$ has to be given levelwise as a Verdier quotient of $\mathbb{D}$ by the kernel of γ. Hence every localization of triangulated derivators arises precisely as a sequence of the form (4.3.5).

4.4 Localization via idempotent monads

In this section, we study the connection between *reflective* localizations and modules over idempotent monads. The results in this section are not new and are included only for completeness. A much more detailed treatment of reflective localization and its connection to idempotent monads and derivator factorization systems can be found in [Lor18]. An independent treatment of some aspects of reflective localization of derivators can also be found in [Col18, Section 3] and in [Hel88]. All (pre)derivators are assumed to have domain a fixed diagram category **Dia**.

Lemma 4.4.1. *Let $F : \mathbb{D} \leftrightarrows \mathbb{E} : G$ be an adjunction of prederivators, with G levelwise fully faithful. If $\mathbb{D}$ is a right (respectively left) derivator, then so is $\mathbb{E}$.*

Proof. The original proof is in [Cis08, Lemma 4.2]. For more details, see [Col18, Lemma 3.3] or [Lor18, Proposition 3.13] for more details. □

The following lemma shows that a left adjoint morphism of derivators is a localization if and only if its right adjoint is levelwise fully faithful.

Lemma 4.4.2. *Let $F : \mathbb{D} \leftrightarrows \mathbb{E} : G$ be an adjunction of derivators. Let $\mathcal{S}$ be the class of morphisms in $\mathbb{D}(e)$ that the functor F_e inverts. The following are equivalent:*

(i) The counit $\varepsilon : FG \to 1_{\mathbb{E}}$ is an isomorphism.

(ii) The functor F exhibits $\mathbb{E}$ as the localization of $\mathbb{D}$ by $\mathcal{S}$.

(iii) For any derivator $\mathbb{D}'$, precomposition with F induces an equivalence of categories

$$\underline{\mathsf{Hom}}(\mathbb{E}, \mathbb{D}') \to \underline{\mathsf{Hom}}_{\mathcal{S}}(\mathbb{D}, \mathbb{D}')$$

In that case, precomposition by F also yields an equivalence of categories

$$\underline{\mathsf{Hom}}_!(\mathbb{E}, \mathbb{D}') \to \underline{\mathsf{Hom}}_{!,\mathcal{S}}(\mathbb{D}, \mathbb{D}')$$

Proof. The equivalence of the first three assertions is proved in [Lor18, Proposition 3.8]. We just remark here that once we know the localization $\mathbb{D}[\mathcal{S}^{-1}]$ is actually a derivator, then it doesn't matter if we formulate the universal property in terms of derivators or just prederivators. The last part is in Proposition 4.2.3. $\square$

Example 4.4.3. Let $\mathcal{M}$ be a combinatorial model category and $\mathcal{S}$ a class of morphisms in $\mathcal{M}$ such that the left Bousfield localization $L_{\mathcal{S}}\mathcal{M}$ of $\mathcal{M}$ by $\mathcal{S}$ exists. Then it follows by [Hir03, Theorem 3.3.19] that the derivator associated to $L_{\mathcal{S}}\mathcal{M}$ is a Bousfield localization of that associated to $\mathcal{M}$ by $\mathcal{S}$.

The following corollary shows that every reflective localization of derivators arises precisely as the Eilenberg-Moore adjunction associated to an idempotent monad.

Corollary 4.4.4. *Let $\mathbb{D}$ be a derivator, M an endomorphism of $\mathbb{D}$, and $\eta : 1_{\mathbb{D}} \to M$ a modification. Then the following are equivalent:*

(i) There is an adjunction of derivators $F : \mathbb{D} \leftrightarrows \mathbb{E} : G$ with G levelwise fully faithful such that $M = G \circ F$ and $\eta : 1_{\mathbb{D}} \to G \circ F$ is the adjunction unit.

(ii) $M\eta : M \to M^2$ is invertible and $M\eta = \eta M$.

If moreover $\mathbb{D}$ is triangulated, and M is exact, then $\mathbb{E}$ is triangulated.

Proof. $(i) \Rightarrow (ii)$ Using Lemma 4.4.2, the proof is exactly as the one in [Kra10, Proposition 2.4.1]. $(ii) \Rightarrow (i)$ Set $\mu = (M\eta)^{-1} : M^2 \to M$. The assumptions imply that (M, μ, η) is an idempotent monad on $\mathbb{D}$. Consider the Eilenberg-Moore adjunction $F_M : \mathbb{D} \leftrightarrows M\text{-}\mathbf{Mod}_{\mathbb{D}} : U_M$. Given a category $J \in \mathbf{Dia}$ and $(X, \lambda) \in M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)}$, note that $\lambda \circ \eta_X = 1_X$ simply because X is an M_J-module. By naturality of η the following square commutes:

$$\begin{tikzcd} MX \arrow[r, "\eta_{MX}"] \arrow[d, "\lambda"'] & M^2X \arrow[d, "M\lambda"] \\ X \arrow[r, "\eta_X"'] & MX \end{tikzcd}$$

Hence $\eta_X \circ \lambda = M\lambda \circ \eta_{MX} = M\lambda \circ M\eta_X = M(\lambda \circ \eta_X) = 1_{MX}$ which shows η_X is invertible. Thus it follows that $M_J\text{-}\mathbf{Mod}_{\mathbb{D}(J)}$ coincides with the M_J-local objects (see [Kra10]) and $U_{M,J}$ can be identified with the inclusion of the M_J-local objects into $\mathbb{D}(J)$ and is hence fully faithful. By Proposition 3.3.3 and Lemma 4.4.1, the result follows.

For the last part of the proof, assume $\mathbb{D}$ is triangulated. Because U_M is levelwise fully faithful, by Lemma 4.4.1 and Proposition 3.3.3 M-$\mathbf{Mod}_{\mathbb{D}}$ is actually a derivator. Then, by Lemma 4.4.2, both $\mathbb{E}$ and M-$\mathbf{Mod}_{\mathbb{D}}$ satisfy the same universal property and are hence equivalent. Since an idempotent monad is always separable, it follows from Proposition 3.6.4 that $\mathbb{E}$ is strong. To finish the proof, we just need to show that $\mathbb{E}$ is stable. Let $\mathcal{S}$ the class of morphisms inverted by M (or equivalently by F, since G is fully faithful levelwise). Since M is exact by assumption, $\mathcal{S}$ is stable by loop by [Gro13, Proposition 3.21]. It follows that $\mathbb{E}$ is stable by [Tab08, Lemma 4.3] and Remark 4.2.4. $\square$

CHAPTER 5

Compact objects

5.1 Levelwise vs pointwise compact objects

Recall that given a triangulated category $\mathcal{T}$ with (small) coproducts, an object $X \in \mathcal{T}$ is compact if every map out of X to a (small) coproduct factors through a finite subcoproduct. The full subcategory $\mathcal{T}^c$ spanned by all compact objects is a thick subcategory of $\mathcal{T}$.

For this section, assume that **Dia** = **Cat** unless stated otherwise.

Lemma 5.1.1. *Let $\mathbb{D}$ be a big triangulated derivator. Let $u : J \to K$ be a functor between finite direct categories. Then the exact functors $u_!, u^*, u_*$ all preserve compact objects.*

Proof. The main theorem in [Bec13] implies that the adjunctions $u_! \dashv u^* \dashv u^*$ extends to an infinite chain of adjunctions. In particular, each of the exact functors $u_!, u^*, u_*$ admits a right adjoint that preserves coproducts. The assertion then follows by the well-known fact that a left adjoint exact functor between triangulated categories preserves compact objects if and only if its right adjoint commutes with coproducts [Bal16, Lemma 4.4]. $\square$

This allows us to give the following definition:

Definition 5.1.2. Let $\mathbb{D}$ be a big **Cat**-triangulated derivator. The **derivator of levelwise compact objects** is the $\mathbf{Dir}_\mathrm{f}$-subderivator $\mathbb{D}^{lc}$ of $\mathbb{D}|_{\mathbf{Dir}_\mathrm{f}}$ with $\mathbb{D}^{\mathrm{lc}}(J) = \mathbb{D}(J)^\mathrm{c}$ for any finite direct category J.

It is a subderivator of $\mathbb{D}|_{\mathbf{Dir}_\mathrm{f}}$ in the sense that there is a strict exact[1] morphism $\mathbb{D}^{\mathrm{lc}} \hookrightarrow \mathbb{D}|_{\mathbf{Dir}_\mathrm{f}}$ that is levelwise the inclusion of a full subcategory.

[1]Equivalently bicontinuous by Theorem 1.4.8.

On the other hand, we can also define the maximal subprederivator (cf. Definition 1.4.12) associated to $\mathbb{D}(e)^c$:

Definition 5.1.3. Let $\mathbb{D}$ be a big triangulated derivator. The **prederivator of pointwise compact objects** is the maximal $\mathbf{Dir}_{\mathrm{f}}$ subprederivator $\mathbb{D}^{\mathrm{pc}}$ of $\mathbb{D}|_{\mathbf{Dir}_{\mathrm{f}}}$ associated to the thick subcategory $\mathbb{D}(e)^c$ of $\mathbb{D}(e)$.

In other words, $\mathbb{D}^{\mathrm{pc}}(J)$ is the full subcategory of $\mathbb{D}(J)$ spanned by the objects $X \in \mathbb{D}(J)$ such that $X_j \in \mathbb{D}(e)$ is compact for any $j \in J$. Note that $\mathbb{D}^{\mathrm{pc}}$ is in fact a triangulated $\mathbf{Dir}_{\mathrm{f}}$-derivator and the inclusion $\mathbb{D}^{\mathrm{pc}} \hookrightarrow \mathbb{D}$ is strict and exact; moreover $\mathbb{D}^{\mathrm{pc}}(J)$ is a thick subcategory of $\mathbb{D}(J)$ (cf. Corollary 1.7.5). Moreover, Lemma 5.1.1 implies that any levelwise compact object is pointwise compact. Thus, the inclusion $\mathbb{D}^{\mathbf{lc}} \hookrightarrow \mathbb{D}|_{\mathbf{Dir}_{\mathrm{f}}}$ factors through $\mathbb{D}^{\mathbf{pc}}$. Now [Fra96, Section 1.5, Proposition 1] implies that the morphism $\mathbb{D}^{\mathrm{lc}} \to \mathbb{D}^{\mathrm{pc}}$ is an equivalence. However, since the values of both derivators are actually full and replete subcategories of the corresponding values of $\mathbb{D}$, this implies that they are in fact equal. We have thus proved the following:

Proposition 5.1.4. *Let $\mathbb{D}$ be a big* **Cat***-triangulated derivator. Then for each finite direct category J, the compact objects in $\mathbb{D}(J)$ are precisely those $X \in \mathbb{D}(J)$ such that $X_j \in \mathbb{D}(e)$ is compact for any $j \in J$.*

Henceforth, we will call $\mathbb{D}^{\mathrm{lc}} = \mathbb{D}^{\mathrm{pc}}$ the **subderivator of compact objects** in $\mathbb{D}|_{\mathbf{Dir}_{\mathrm{f}}}$ and denote it by $\mathbb{D}^{\mathrm{c}}$. Note that this is a triangulated derivator since $\mathbb{D}^{\mathrm{pc}}$ is.

Remark 5.1.5. If **Dia** $\neq$ **Cat** we can still show that $\mathbb{D}^{\mathrm{lc}}$ is a right derivator and that every levelwise compact object is pointwise compact. However, in that case we do not have a theorem like the main theorem in [Bec13] guaranteeing that right Kan extensions will preserve compact objects and without it, we cannot show that every pointwise compact object is levelwise compact.

Example 5.1.6. Let X be a quasi-compact and separated scheme. Then the derived category $D^{\mathrm{perf}}(X)$ of perfect complexes is the base of a $\mathbf{Dir}_{\mathrm{f}}$ triangulated derivator $\mathbb{D}_X^{\mathrm{perf}}$. This follows immediately from Proposition 5.1.4 above applied to the big **Cat**-derivator $\mathbb{D}_X$ of quasi-coherent O_X-modules.

5.2 Compact generation

Let $\mathcal{T}$ be a triangulated category and $\mathcal{I}$ a full subcategory. We will write

$$\Sigma^*\mathcal{I} = \{\Sigma^n x | n \in \mathbb{Z}, x \in \mathcal{I}\}$$

By $\langle\mathcal{I}\rangle$ we will mean the smallest triangulated subcategory of $\mathcal{T}$ that contains $\mathcal{I}$. If $\mathcal{J}$ is another full subcategory, we will write[2] $\mathcal{I} * \mathcal{J}$ for the full subcategory of $\mathcal{T}$ spanned by cones of morphisms in $\mathcal{T}$ with source in $\mathcal{I}$ and target in $\mathcal{J}$. We also set $\langle\mathcal{I}\rangle_0 = \{0\}$ and $\langle\mathcal{I}\rangle_{n+1} = \langle\mathcal{I}\rangle_n * \langle\mathcal{I}\rangle_1$.

Lemma 5.2.1. *Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{I}$ be a full subcategory of $\mathcal{T}$. Then we have $\langle\mathcal{I}\rangle = \cup\langle\mathcal{I}\rangle_n$. Moreover, the full subcategory of direct summands of objects of $\langle\mathcal{I}\rangle$ is the smallest full thick subcategory of $\mathcal{T}$ which contains $\mathcal{I}$.*

Proof. An easy application of the octahedron axiom guarantees that $*$ is associative, from which the first assertion follows immediately. The second assertion amounts to showing that all direct summands of objects in $\cup\langle\mathcal{I}\rangle_n$ form a full subcategory. Details can be found in the first Lemma in [Kra07, Section 3.3]. □

Note that each $\langle\mathcal{I}\rangle_n$ is replete, closed under arbitrary shifts and $\langle\mathcal{I}\rangle_n \subset \langle\mathcal{I}\rangle_{n+1}$ for all n.

We will now apply the previous lemma to give an explicit description of the smallest triangulated subcategory that contains $\mathcal{I}$ as one-step colimits of coherent diagrams that are pointwise in the generating class $\mathcal{I}$. Fix a derivator $\mathbb{D}$ of domain **Dia**. We will assume that **Dia** is a diagram category that is closed under all operations described in this section and necessary for the relevant proofs. For example **Dia** = **Dir**$_\mathrm{f}$, **Pos**$_\mathrm{f}$ and **Cat** will all work.

Recall the following construction from [Gro16]: Given a category I, let $I^{\triangleright}$ be I with a free terminal object (denoted ∞) attached to it. Let $t : I \to I^{\triangleright}$ be the inclusion. A **(coherent) cocone** over I is an object $X \in D(I^{\triangleright})$. We say X is a **colimiting cocone** if it is in the essential image of $t_!$. The following lemma generalizes [Gro13, Lemma 3.11]:

[2] In the literature, this notation usually refers to *extensions* of $\mathcal{I}$ by $\mathcal{J}$. The reason we changed the standard notation is because it is easier to write cones in terms of pushouts while for extensions we'd need to rotate and shift.

Lemma 5.2.2. *Let $\mathbb{D}$ be a derivator and consider two functors $f : J \to K$ and $i : I^{\triangleright} \to K$. Assume that the induced functor $R : I \to (K - i(\infty) \downarrow i(\infty))$ has a left adjoint, and that $i(\infty)$ does not lie in the image of f. Then for every $X = f_!(Y) \in \mathbb{D}(K)$ where $Y \in \mathbb{D}(J)$, the induced cocone i^*X is colimiting.*

Proof. By assumption we have a factorization $J \xrightarrow{\overline{f}} K - i(\infty) \xrightarrow{k} K$. Write ϵ for the counit of the adjunction $t_! \dashv t^*$. Pick any $Y \in \mathbb{D}(J)$ and let $X = f_! Y$. By [Gro13, Lemma 1.27], since t is fully faithful, we have to show that the map $(t_! t^* i^* X)_\infty \to (i^*X)_\infty$ induced by ϵ is an isomorphism. This amounts to showing that the base change morphism of the left pasting evaluated at X is an isomorphism:

$$\begin{tikzcd}
I \cong I_{/\infty} \arrow[r] \arrow[d, "\pi_I"'] \arrow[dr, phantom, "\Swarrow"] & I \arrow[r, "t"] \arrow[d, "t"] \arrow[dr, phantom, "\Swarrow"] & I^{\triangleright} \arrow[r, "i"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\Swarrow"] & K \arrow[d, "\mathrm{id}"] \\
e \arrow[r, "\infty"'] & I^{\triangleright} \arrow[r, "\mathrm{id}"'] & I^{\triangleright} \arrow[r, "i"'] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "R"] \arrow[d, "\pi_I"'] \arrow[dr, phantom, "\Swarrow"] & (K - i(\infty) \downarrow i(\infty)) \arrow[r, "\mathrm{pr}"] \arrow[d, "\pi"] \arrow[dr, phantom, "\Swarrow"] & K - i(\infty) \arrow[r, "k"] \arrow[d, "k"] \arrow[dr, phantom, "\Swarrow"] & K \arrow[d, "\mathrm{id}"] \\
e \arrow[r] & e \arrow[r, "i(\infty)"'] & K \arrow[r, "\mathrm{id}"'] & K
\end{tikzcd}$$

Since the pasting on the left is equal to the pasting on the right, we need to show the base change morphism of the pasting on the right is an isomorphism when evaluated at X. In that pasting, the left square is homotopy exact because right adjoints are homotopy final (see [Gro13, Proposition 1.24]) and the middle is homotopy exact by *Der*4. Thus, it is enough to show that the base change morphism of the rightmost square is an isomorphism evaluated at X. But this base change morphism, is exactly the counit $k_! k^* \to \mathrm{id}$. Note that k is fully faithful, hence so is $k_!$ (see [Gro13, Proposition 1.26]); since $X = f_! Y \cong k_! \overline{f}_! Y$ lies in the essential image of $k_!$ this finishes the proof. $\square$

Given a category I, let $I^{\triangleright 2}$ denote the pushout:

$$\begin{tikzcd}
I \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & I^{\triangleright} \arrow[d] \\
I^{\triangleright} \arrow[r] & I^{\triangleright 2}
\end{tikzcd}$$

The resulting category looks like:

$$\begin{tikzcd}
I \arrow[r] \arrow[d] & \infty_1 \\
\infty_2 &
\end{tikzcd}$$

i.e. is what we get if we freely adjoint two "terminal" objects relative to I where one of them is named ∞_1 and the other ∞_2. Let $I^{\triangleright\square}$ be the category defined as the pushout:

(5.2.3)
$$\begin{tikzcd} e \arrow[r, "\infty"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & I^{\triangleright} \arrow[d, "i"] \\ \square \arrow[r, "k"'] & I^{\triangleright\square} \end{tikzcd}$$

where the left vertical morphism classifies the vertex $(0,0)$. Essentially we freely attach a "terminal square" relative to I. Pictorially $I^{\triangleright\square}$ looks like:

$$\begin{tikzcd} \mathrm{I} \arrow[dr] & & \\ & (0,0) \arrow[r] \arrow[d] & (0,1) \arrow[d] \\ & (1,0) \arrow[r] & (1,1) \end{tikzcd}$$

Proposition 5.2.4. *Let $\mathbb{D}$ be a triangulated derivator. Consider a distinguished triangle*

$$x \xrightarrow{f} y \to z \to \Sigma x$$

in $\mathbb{D}(e)$ such that $x \cong (\pi_I)_! X$ for some category I and object $X \in \mathbb{D}(I)$. Then there is $Z \in \mathbb{D}(I^{\triangleright 2})$ such that $Z|I \cong X$, $Z_{\infty_1} = 0$, $Z_{\infty_2} \cong y$ and $(\pi_{I^{\triangleright 2}})_! Z \cong z$.

Proof. We have a fully faithful inclusion $i_2 : I^{\triangleright} \to I^{\triangleright 2}$:

$$\begin{tikzcd} \mathrm{I} \arrow[d] & & \mathrm{I} \arrow[r] \arrow[d] & \infty_1 \\ \infty \arrow[rr, phantom, "\xrightarrow{i_2}"] & & \infty_2 & \end{tikzcd}$$

which includes ∞ as ∞_2. Similarly, we have a fully faithful inclusion $i_1 : I^{\triangleright} \to I^{\triangleright 2}$ which includes ∞ as ∞_1. We also have the canonical inclusion $t : I \to I^{\triangleright}$ and a fully faithful inclusion $j : I^{\triangleright 2} \to I^{\triangleright\square}$ which is the identity on I and includes ∞_1, ∞_2 as $(0,1)$ and $(1,0)$ respectively:

$$\begin{tikzcd} \mathrm{I} \arrow[dr] \arrow[ddr] & & & \mathrm{I} \arrow[dr] & & \\ & \infty_1 & & & (0,0) \arrow[r] \arrow[d] & (0,1) \arrow[d] \\ & \infty_2 \arrow[rr, phantom, "\xrightarrow{j}"] & & & (1,0) \arrow[r] & (1,1) \end{tikzcd}$$

We construct the object $Z \in \mathbb{D}(I^{\triangleright 2})$: we have[3] $x \cong (t_! X)_\infty = (\pi_{I^\triangleright})_! t_! X$. By adjunction, the morphism $x \to y$ induces a morphism $t_! X \to (\pi_{I^\triangleright})^* y$. Let $F \in \mathbb{D}(I^\triangleright \times [1])$ be a lift of that morphism (which exists because $\mathbb{D}$ is strong). Consider the functor $I^\triangleright \to I^\triangleright \times [1]$ which maps every object i of I to $(i, 0)$ and maps ∞ to $(\infty, 1)$; let $Q \in \mathbb{D}(I^\triangleright)$ be the restriction of F along this functor. Note that by construction $Q|_I \cong X$ and $Q_\infty \cong y$. Set $Z = (i_2)_* Q$. Since right Kan extensions along fully faithful functors are fully faithful [Gro13, Proposition 1.20] the counit $(i_2)^* Z = (i_2)^*(i_2)_* Q \to Q$ is an isomorphism. In particular $Z|_I \cong X$ and $Z|_{\infty_2} \cong y$. Furthermore, since i_2 is a sieve we have $Z_{\infty_1} = 0$ and we only have to check the colimit of Z is isomorphic to z.

Let $k : \square \to I^{\triangleright\square}$ be the bottom horizontal functor in the defining square (5.2.3). Note that the colimit of $Z \in \mathbb{D}(I^{\triangleright 2})$ is isomorphic to the colimit of $W = j_! Z \in \mathbb{D}(I^{\triangleright\square})$; since $I^{\triangleright\square}$ has $k(1, 1)$ as a terminal object, it is enough to prove that $W_{k(1,1)} \cong z$.

Consider the functor $i : I^\triangleright \to I^{\triangleright\square}$ appearing as the right vertical map in the defining square (5.2.3) for $I^{\triangleright\square}$. The induced functor $I \to (I^{\triangleright\square} - i(\infty) \downarrow i(\infty))$ becomes the identity $I \to I$ under the identification $(I^{\triangleright\square} - i(\infty) \downarrow i(\infty)) \cong I$; in particular it has a left adjoint. Moreover $i(\infty)$ is not in the image of j. By Lemma 5.2.2, since W is in the image of $j_!$, we see immediately that $i^* W$ is a colimiting cocone; hence $W_{k(0,0)} \cong x$.

Finally, since $k(1, 1)$ is terminal in $I^{\triangleright\square}$, we note that

$$(I^{\triangleright\square} - k(1, 1) \downarrow k(1, 1)) \cong I^{\triangleright\square} - k(1, 1)$$

Under this identification, the induced functor $\ulcorner \to (I^{\triangleright\square} - k(1, 1) \downarrow k(1, 1))$ is simply the inclusion of $\ulcorner$ as a full subcategory of $I^{\triangleright\square} - k(1, 1)$; this has a left adjoint (collapsing everything in I to $(0, 0)$). Moreover, $k(1, 1) \notin \operatorname{im}(j)$, and W is in the image of $j_!$, Lemma 5.2.2 (or [Gro13, Lemma 3.11]) shows that $k^* W$ is cocartesian. Thus $W_{k(1,1)}$ fits into a cocartesian square with underlying diagram:

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[d] \\ 0 \arrow[r] & W_{k(1,1)} \end{tikzcd}$$

By construction of the triangulations in stable strong derivators, this means we have an exact triangle $x \to y \to W_{k(1,1)} \to \Sigma x$ in $\mathbb{D}(e)$; hence $W_{k(1,1)} \cong z$. □

[3]See [Gro16, Proposition 2.3] for the first isomorphism; the second follows from the fact that a colimit over a category with a terminal object is just evaluation at the terminal object [Gro13, Lemma 1.19].

Remark 5.2.5. Let $u : I \to I^{\triangleright 2}$ be the canonical inclusion. A similar argument as in the proof above shows that for $X \in \mathbb{D}(I)$ the underlying diagram of $k^* j_! u_* X \in \mathbb{D}(\square)$ looks like:

$$\begin{tikzcd}
(\pi_I)_! X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma(\pi_I)_! X
\end{tikzcd}$$

Since Σ commutes with colimits we have that $(\pi_I)_! X \cong (\pi_{I^{\triangleright 2}})_! \Sigma^{-1} Y$ where $Y = u_* X$. In other words, if an object $x \in \mathbb{D}(e)$ can be constructed as a colimit of a coherent diagram $X \in \mathbb{D}(I)$, then up a shift it can also be constructed as a colimit of a diagram $Y \in \mathbb{D}(I^{\triangleright 2})$ which is a right extension by zero of X.

Now define diagram categories C_n by iteratively applying the construction $(-)^{\triangleright 2}$ to e. Let C be the full subcategory of the poset $\mathbb{Z}^2$ spanned by $(0, 0)$ and all (i, j) with $i, j \geq 0$, $|i - j| = 1$ and $i \neq j$. Then we can identify each C_n with the full subcategory of C spanned by all pairs of integers neither of which exceeds n. So for instance $C_1 = \ulcorner$, the category C_2 looks like:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[dd] \arrow[dr] & \\
(0,1) \arrow[rr] \arrow[dr] & & (2,1) \\
& (1,2) &
\end{tikzcd}$$

the category C_3 looks like:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (1,0) \arrow[dd] \arrow[dr] & & \\
(0,1) \arrow[rr] \arrow[dr] & & (2,1) \arrow[dd] \arrow[dr] & \\
& (1,2) \arrow[rr] \arrow[dr] & & (3,2) \\
& & (2,3) &
\end{tikzcd}$$

and so on. Then the basic idea of generating the smallest triangulated subcategory containing $\mathcal{I}$ is that by left Kan extensions we can fill in the missing diagonal points, so that each point (n, n) corresponds to an object of $\langle \mathcal{I} \rangle_{n+1}$.

Corollary 5.2.6. *Let $\mathbb{D}$ be a triangulated derivator and let $\mathcal{I}$ be a full pointed subcategory in $\mathbb{D}(e)$. Then any object in $\langle \mathcal{I} \rangle$ is the colimit of a C_n-shaped coherent diagram (for some n) that is pointwise in $\Sigma^*\mathcal{I}$.*

Proof. By Lemma 5.2.1, it suffices to show that every object of $\langle \mathcal{I} \rangle_n$ is the colimit of a diagram of shape C_n that is pointwise in $\Sigma^*\mathcal{I}$. This is clear by induction and Proposition 5.2.4. □

Remark 5.2.7. We remark that in conjunction with Theorem 1.4.8, the above Corollary implies that if $\mathbb{D}$ is a triangulated $\mathbf{Dir}_\mathrm{f}$-derivator then the smallest subcategory of $\mathbb{D}(e)$ that contains $\mathcal{I}$ and is closed under colimits is simply the full replete subcategory $\mathscr{C}$ spanned by colimits of coherent diagrams that are pointwise in $\mathcal{I}$. By contrast, if $\mathbb{D}$ is not triangulated, then we do not know if a similar claim holds. We can of course construct $\mathscr{C}$ by iteratively taking colimits of objects pointwise in $\mathcal{I}$, but the point here is that we can do this in one step.

Remark 5.2.8. The proof above makes it clear that if $x \in \langle \mathcal{I} \rangle_n$ then x is the colimit of a coherent diagram $X \in \mathbb{D}(C_n)$ that is pointwise in $\Sigma^*\mathcal{I}$. As already remarked though $\langle \mathcal{I} \rangle_n \subset \langle \mathcal{I} \rangle_{n+1}$, hence x must also be a colimit of some $Y \in \mathbb{D}(C_{n+1})$ that is pointwise in $\Sigma^*\mathcal{I}$. Note that Remark 5.2.5 shows that we can simply take Y as the right extension by zero of $\Sigma^{-1}X$ along the inclusion $C_n \hookrightarrow C_{n+1}$.

We now turn our attention to compactly generated triangulated derivators. A big triangulated derivator will be called compactly generated if $\mathbb{D}(e)$ is compactly generated with its canonical triangulation. By [Hor15, Lemma 3.1.6] all the values of $\mathbb{D}$ are then compactly generated and a generating get for $\mathbb{D}(I)$ is given by $\{i_! g | i \in I, g \in \mathcal{G}\}$ where $\mathcal{G}$ is a generating set for $\mathbb{D}(e)$.

Corollary 5.2.9. *Let $\mathbb{D}$ be a compactly generated big triangulated derivator. Let $\mathcal{G}$ be a set of compact generators of $\mathbb{D}(e)$. Then any compact object $x \in \mathbb{D}(e)$ is a retract of a finite direct colimit of a coherent diagram that is pointwise in $\mathcal{G}$.*

Proof. The full subcategory of compact objects in $\mathbb{D}(e)$ is precisely the thick subcategory generated by $\mathcal{G}$. Lemma 5.2.1 shows that this thick subcategory is precisely the full subcategory of $\mathbb{D}(e)$ generated by direct summands of objects in $\cup \langle \mathcal{G} \rangle_n$. The result then follows from Corollary 5.2.6.

□

Remark 5.2.10. Recall that a quasicategory $\mathscr{C}$ is presentable if and only if it admits small colimits and there is a set $\mathcal{G}$ of compact objects in $\mathscr{C}$ such that any object in $\mathscr{C}$ is a colimit of a diagram in $\mathscr{C}$ that is pointwise in $\mathcal{G}$ (cf. [Lur09, Theorem 5.5.1.1]). Moreover, if $\mathscr{C}$ is a stable infinity category, then this is equivalent to $\mathrm{Ho}(\mathscr{C})$ being compactly generated as a triangulated category (cf. [Lur17, Remark 1.4.4.3]). Thus, it is reasonable to expect that a big triangulated **Cat**-derivator will be compactly generated if and only if we can write every object in $\mathbb{D}(e)$ as a colimit of a coherent diagram of pointwise compact objects. For this it would be enough to show that any object in $\mathrm{Loc}(\mathcal{G})$, the localizing subcategory of $\mathbb{D}(e)$ generated by $\mathcal{G}$, is in fact a colimit of compact objects. The Corollary above is a first step in this direction and future research would be necessary to establish the truth (or failure) of the claims made in this remark.

5.3 Localization of compactly generated triangulated derivators

For this section, let $\mathbb{T}$ be a big triangulated **Cat**-derivator, such that $\mathbb{T}(e)$ is compactly generated. By [Hor15, Lemma 3.1.6], it follows that $\mathbb{T}(J)$ is a compactly generated triangulated category for any category J. Let $\mathscr{E}$ be a full subcategory of $\mathbb{T}(e)$ of compact objects, and let $\mathcal{L}$ be the localizing subcategory it generates. Finally, let $\mathbb{L}$ be the maximal subprederivator of $\mathbb{T}$ associated to $\mathcal{L}$ (cf. Definition 1.4.12).

Proposition 5.3.1. *The prederivator $\mathbb{L}$ is actually a triangulated derivator.*

Proof. Note that since $\mathcal{L}$ is a triangulated subcategory of $\mathbb{T}(e)$, it is closed under homotopy pushouts by Lemma 1.7.4. Since it is also closed under coproducts, it follows by [PS16, Theorem 7.13] and Proposition 1.4.13 that the maximal subprederivator $\mathbb{L}$ of $\mathbb{T}$ associated to $\mathcal{L}$ is a right derivator. Note also that each shift of $\mathbb{L}$ is actually a triangulated derivator when restricted to finite direct categories; this follows by Lemma 1.7.4, by Proposition 1.4.13 and by Theorem 1.4.8. Consider the inclusion $i : \mathbb{L} \to \mathbb{T}$; this is cocontinuous since left Kan extensions in $\mathbb{L}$ are exactly computed by the corresponding ones in $\mathbb{T}$. Since $\mathbb{L}$ is compactly generated, by [Hor15, Theorem 3.2.1], the functor $i : \mathbb{L} \to \mathbb{T}$ admits a right adjoint. By the dual of Lemma 4.4.1, it follows that $\mathbb{L}$ is actually a derivator. To finish the proof, note that the functor $i : \mathbb{L} \to \mathbb{T}$ is exact and apply the dual of Corollary 4.4.4. □

Proposition 5.3.2. *The levelwise Verdier localizations $\mathbb{T}(J)/\mathbb{L}(J)$ assemble to a triangulated derivator $\mathbb{T}/\mathbb{L}$. Moreover the localization morphism $\mathbb{T} \to \mathbb{T}/\mathbb{L}$ admits a right adjoint.*

Proof. By Theorem 4.2.17 and Lemma 4.3.1 it follows that $\mathbb{T}/\mathbb{L}$ is a right derivator[4]. Moreover, by Proposition 4.3.2, each shift of $\mathbb{T}/\mathbb{L}$ is a triangulated derivator when restricted to finite direct categories. Consider the morphism $\gamma : \mathbb{T} \to \mathbb{T}/\mathbb{L}$ which is levelwise given by the canonical quotient functors. Since $\mathbb{T}$ is compactly generated, by [Hor15, Theorem 3.2.1], γ admits a right adjoint. Note that for each category I, since γ_I is a localization of triangulated categories, its right adjoint has to be fully faithful. Now by Lemma 4.4.1 it follows that $\mathbb{T}/\mathbb{L}$ is actually a derivator, which has to be triangulated by Corollary 4.4.4. □

As an application, we now prove a derivator version of the Neeman-Thomason Theorem (cf. [Nee92]).

Theorem 5.3.3. *Let $\mathbb{T}$ be a big* **Cat***-triangulated derivator and $\mathcal{E}$ a set of compact objects in $\mathbb{T}(e)$ closed under arbitrary suspensions. Let $\mathcal{L}$ be the localizing subcategory generated by $\mathcal{E}$ and $\mathbb{L}$ the associated triangulated derivator (cf. 5.3.1) and $\mathbb{S} := \mathbb{T}/\mathbb{L}$ the associated Verdier localization 5.3.2. Then the sequence of exact morphisms:*

$$\mathbb{L} \to \mathbb{T} \to \mathbb{S}$$

yields a sequence of exact morphisms of $\mathbf{Dir}_f$*-derivators:*

$$\mathbb{L}^c \to \mathbb{T}^c \to \mathbb{S}^c$$

The induced morphism:

$$\mathbb{T}^c/\mathbb{L}^c \to \mathbb{S}^c$$

induces an equivalence of $\mathbf{Dir}_f$*-derivators:*

$$(\mathbb{T}^c/\mathbb{L}^c)^\natural \to \mathbb{S}^c$$

where $(-)^\natural$ refers to idempotent completion (cf. Construction 2.2.2 and Theorem 2.3.10).

[4] See the proof of Proposition 4.3.2 for why $\Sigma(\mathcal{L})$ is closed under colimits.

Proof. By [Nee92, Lemma 2.2], we know that the inclusion $\mathbb{L} \to \mathbb{T}$ preserves compact objects on the base as does the quotient functor $\mathbb{T} \to \mathbb{S}$ (cf. [Nee92, Lemma 2.4]). Thus, we get the claimed sequence of exact morphisms $\mathbf{Dir}_\mathrm{f}$:

$$\mathbb{L}^\mathrm{c} \to \mathbb{T}^\mathrm{c} \to \mathbb{S}^\mathrm{c}$$

Note that $\mathbb{S}$ is idempotent complete as is any big triangulated derivator. Moreover $\mathbb{S}^\mathrm{c}$ is thick in $\mathbb{S}$ (on each level). We deduce that $\mathbb{S}^\mathrm{c}$ as $\mathbb{S}$ is idempotent complete. Hence, the induced morphism:

$$\mathbb{T}^\mathrm{c}/\mathbb{L}^\mathrm{c} \to \mathbb{S}^\mathrm{c}$$

induces a morphism:

$$(\mathbb{T}^\mathrm{c}/\mathbb{L}^\mathrm{c})^\natural \to \mathbb{S}^\mathrm{c}$$

This is an exact morphism of $\mathbf{Dir}_\mathrm{f}$-derivators, and the functor on the bases is an equivalence by the main theorem in [Nee92]. We thus conclude by [Fra96, Proposition 1.5.1]. □

We also get the following derivator version of [Bal16, Theorem 4.2]:

Theorem 5.3.4. *Let $\mathbb{D}$ be a big compactly generated triangulated* **Cat**-*derivator and $M : \mathbb{D} \to \mathbb{D}$ a separable exact monad that preserves coproducts. Then the big triangulated derivator $\mathbb{E} := M\text{-}\mathbf{Mod}_\mathbb{D}$ (cf. 3.4.4) is also compactly generated. If the monad M preserves compact objects on the base, then we also get an equivalence of $\mathbf{Dir}_f$-derivators:*

$$(F_M(\mathbb{D}^c))^\natural \to \mathbb{E}^c$$

where $F_M : \mathbb{D} \to M\text{-}\mathbf{Mod}_\mathbb{D}$ is the free-module morphism (cf. Proposition 3.2.14) and $(-)^\natural$ refers to idempotent completion (cf. Construction 2.2.2 and Theorem 2.3.10).

Proof. This follows from [Bal16, Theorem 4.2] once we note that the monad M preserves compact objects on the base if and only if it preserves pointwise compact objects on all levels, and then the fact that pointwise and levelwise compact objects coincide over finite direct diagrams (cf. Proposition 5.1.4). □

REFERENCES

[B10] Theo Bühler. "Exact categories." *Expo. Math.*, **28**(1):1–69, 2010.

[Bal05] Paul Balmer. "The spectrum of prime ideals in tensor triangulated categories." *J. Reine Angew. Math.*, **588**:149–168, 2005.

[Bal11] Paul Balmer. "Separability and triangulated categories." *Adv. Math.*, **226**(5):4352–4372, 2011.

[Bal15] Paul Balmer. "Stacks of group representations." *J. Eur. Math. Soc. (JEMS)*, **17**(1):189–228, 2015.

[Bal16] Paul Balmer. "The derived category of an étale extension and the separable Neeman-Thomason theorem." *J. Inst. Math. Jussieu*, **15**(3):613–623, 2016.

[BDS15] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. "Restriction to finite-index subgroups as étale extensions in topology, KK-theory and geometry." *Algebr. Geom. Topol.*, **15**(5):3025–3047, 2015.

[Bec13] Falk Beckert. "Dualities and adjunctions of stable derivators.", 2013. (http://www2.math.uni-wuppertal.de/ beckert/Diplomarbeit_Falk_Beckert.pdf).

[BKP89] R. Blackwell, G. M. Kelly, and A. J. Power. "Two-dimensional monad theory." *J. Pure Appl. Algebra*, **59**(1):1–41, 1989.

[BM92] Apostolos Beligiannis and Nikolaos Marmaridis. "On left derived categories of a stable category." In *Proceedings of the Sixth International Conference on Representations of Algebras (Ottawa, ON, 1992)*, volume 14 of *Carleton-Ottawa Math. Lecture Note Ser.*, p. 31. Carleton Univ., Ottawa, ON, 1992.

[Bor94a] Francis Borceux. *Handbook of categorical algebra. 1*, volume 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Basic category theory.

[Bor94b] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[BS01] Paul Balmer and Marco Schlichting. "Idempotent completion of triangulated categories." *J. Algebra*, **236**(2):819–834, 2001.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[BV07] Alain Bruguières and Alexis Virelizier. "Hopf monads." *Adv. Math.*, **215**(2):679–733, 2007.

[BZ17] Paul Balmer and John Zhang. "Affine space over triangulated categories: a further invitation to Grothendieck derivators." *J. Pure Appl. Algebra*, **221**(7):1560–1564, 2017.

[Car16] Kevin Carlson. "On the ∞-categorical Whitehead theorem and the embedding of quasicategories in prederivators." Preprint, arxiv: 1612.06980v3, 2016.

[Cis03] Denis-Charles Cisinski. "Images directes cohomologiques dans les catégories de modèles." *Ann. Math. Blaise Pascal*, **10**(2):195–244, 2003.

[Cis08] Denis-Charles Cisinski. "Propriétés universelles et extensions de Kan dérivées." *Theory Appl. Categ.*, **20**:No. 17, 605–649, 2008.

[Cis10] Denis-Charles Cisinski. "Catégories dérivables." *Bull. Soc. Math. France*, **138**(3):317–393, 2010.

[Col18] Ian Coley. "Stabilization of derivators revisited." arXiv:1802.04343v2, 2018.

[CT11] Denis-Charles Cisinski and Gonçalo Tabuada. "Non-connective K-theory via universal invariants." *Compos. Math.*, **147**(4):1281–1320, 2011.

[DHK04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy limit functors on model categories and homotopical categories*, volume 113 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.

[Fra96] Jens Franke. "Uniqueness theorems for certain triangulated categories with an Adams spectral sequence." *Preprint*, 1996. Available at: http://www.math.uiuc.edu/K-theory/0139/Adams.pdf.

[GPS14a] Moritz Groth, Kate Ponto, and Michael Shulman. "The additivity of traces in monoidal derivators." *J. K-Theory*, **14**(3):422–494, 2014.

[GPS14b] Moritz Groth, Kate Ponto, and Michael Shulman. "Mayer-Vietoris sequences in stable derivators." *Homology Homotopy Appl.*, **16**(1):265–294, 2014.

[Gro90] Alexandre Grothendieck. "Les Dérivateurs." Manuscript, https://webusers.imj-prg.fr/ georges.maltsiniotis/groth/Derivateurs.html, 1990.

[Gro11] Moritz Groth. "On the theory of derivators (dissertation)." (http://www.math.uni-bonn.de/people/grk1150/DISS/dissertation-groth.pdf), 2011.

[Gro12] Moritz Groth. "Monoidal derivators and additive derivators." arxiv:1203.5071, 2012.

[Gro13] Moritz Groth. "Derivators, pointed derivators and stable derivators." *Algebr. Geom. Topol.*, **13**(1):313–374, 2013.

[Gro16] Moritz Groth. "Revisiting the canonicity of canonical triangulations." arxiv:1602.04846, 2016.

[GZ67] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[Hel88] Alex Heller. "Homotopy theories." *Mem. Amer. Math. Soc.*, **71**(383):vi+78, 1988.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Hor15] Fritz Hörmann. "Fibered Multiderivators and (co)homological descent." arxiv:1505.00974v2, 2015.

[Hor17] Fritz Hörmann. "Enlargement of (fibered) derivators." arXiv:1706.09692v1, 2017.

[Joy02] A. Joyal. "Quasi-categories and Kan complexes." *J. Pure Appl. Algebra*, **175**(1-3):207–222, 2002. Special volume celebrating the 70th birthday of Professor Max Kelly.

[Kel74] G. M. Kelly. "Doctrinal adjunction." In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pp. 257–280. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[Kel07] Bernhard Keller. "Appendice: Le dérivateur triangulé associé à une catégorie exacte." In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pp. 369–373. Amer. Math. Soc., Providence, RI, 2007.

[Kra07] Henning Krause. "Derived categories, resolutions, and Brown representability." In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pp. 101–139. Amer. Math. Soc., Providence, RI, 2007.

[Kra10] Henning Krause. "Localization theory for triangulated categories." In *Triangulated categories*, volume 375 of *London Math. Soc. Lecture Note Ser.*, pp. 161–235. Cambridge Univ. Press, Cambridge, 2010.

[KS74] G. M. Kelly and Ross Street. "Review of the elements of 2-categories." In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pp. 75–103. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[KV87] Bernhard Keller and Dieter Vossieck. "Sous les catégories dérivées." *C. R. Acad. Sci. Paris Sér. I Math.*, **305**(6):225–228, 1987.

[Len17] Tobias Lenz. "Homotopy (Pre-)Derivators of Cofibration Categories and Quasi-Categories." arxiv: 1712.07845, 2017.

[Lor18] Fosco Loregian. "Localization theory for derivators." arXiv:1802.08193v1, 2018.

[LS14] Jichun Liu and Longgang Sun. "Idempotent completion of pretriangulated categories." *Czechoslovak Math. J.*, **64(139)**(2):477–494, 2014.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. "Higher Algebra." (http://www.math.harvard.edu/ lurie/papers/HA.pdf), 2017.

[Mac98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[Mal] Georges Maltsiniotis. "La K-théorie d'un dérivateur triangulé." In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pp. 341–368.

[Mal07] Georges Maltsiniotis. "Le théorème de Quillen, d'adjonction des foncteurs dérivés, revisité." *C. R. Math. Acad. Sci. Paris*, **344**(9):549–552, 2007.

[MR11] Fernando Muro and George Raptis. "A note on K-theory and triangulated derivators." *Adv. Math.*, **227**(5):1827–1845, 2011.

[MR17] Fernando Muro and George Raptis. "K-theory of derivators revisited." *Ann. K-Theory*, **2**(2):303–340, 2017.

[Nee92] Amnon Neeman. "The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel." *Ann. Sci. École Norm. Sup. (4)*, **25**(5):547–566, 1992.

[Nee01] Amnon Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[PS16] Kate Ponto and Michael Shulman. "The linearity of traces in monoidal categories and bicategories." *Theory Appl. Categ.*, **31**:Paper No. 23, 594–689, 2016.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin-New York, 1967.

[Ren09] Olivier Renaudin. "Plongement de certaines théories homotopiques de Quillen dans les dérivateurs." *J. Pure Appl. Algebra*, **213**(10):1916–1935, 2009.

[Rod14a] Beatriz Rodríguez González. "A derivability criterion based on the existence of adjunctions." arxiv:1202.3359v2, 2014.

[Rod14b] Beatriz Rodríguez González. "Realizable homotopy colimits." *Theory Appl. Categ.*, **29**:No. 22, 609–634, 2014.

[Sch02] Marco Schlichting. "A note on K-theory and triangulated categories." *Invent. Math.*, **150**(1), 2002.

[SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[Str72] Ross Street. "The formal theory of monads." *J. Pure Appl. Algebra*, **2**(2):149–168, 1972.

[Tab08] Gonçalo Tabuada. "Higher K-theory via universal invariants." *Duke Math. J.*, **145**(1):121–206, 2008.

[TT90] R. W. Thomason and Thomas Trobaugh. "Higher algebraic K-theory of schemes and of derived categories." In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pp. 247–435. Birkhäuser Boston, Boston, MA, 1990.

[Ver96] Jean-Louis Verdier. "Des catégories dérivées des catégories abéliennes." *Astérisque*, (239):xii+253 pp. (1997), 1996. With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.