Prescribed Mean Curvature Systems

by

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Abstract

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Let (M, g) be a Riemannian manifold of dimension greater or equal to 3, and let $\Sigma \subset M$ be an immersed hypersurface with prescribed mean curvature. I study the geometry of Euler-Lagrange equations in this particular context. This includes a characterization of those prescribed mean curvature systems that are Euler-Lagrange, and I prove that these are locally conformally equivalent to basic systems. Finally, I study Emmy Noether's theorem for first-order conservation laws in the special case of minimal surfaces in Riemannian 3-manifolds. In particular, I am able to identify which conservation laws do arise from symmetries of the system in the sense of Noether and which ones do not.

“Ja, Ja, Ja, Ja, Ja, Nee, Nee, Nee, Nee, Nee”
(Joseph Beuys, 1968)

Für Mama und Papa

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Acknowledgments

First and foremost, I would like to thank my adviser Robert L. Bryant. All the foundational material needed for this thesis I learnt from him. Without Robert's insight, I would have been in the dark throughout my analysis of prescribed mean curvature systems.

Next, I would like to thank Thomas Mettler who introduced me to the topic of prescribed mean curvature systems over a macchiato one afternoon. Thomas informed me, that he had established the existence of *framed* integral manifolds of the prescribed mean curvature system (in other words a result similar to Theorem 2.1.1), and that he had worked out the condition for a prescribed mean curvature system to be locally Euler-Lagrange (i.e. Proposition 2.2.3). Since we carried out our analyses independently, both results are marked with an asterisk.

The past years have not only taught me about prescribed mean curvature systems, I also learnt a lot about myself. Today I look back at a journey that has taken me first across the Atlantic Ocean, and then across North America. A journey of highs and lows, of set backs, self-doubts and unexpected discoveries. I am grateful for the cultural stimuli I received along the way, for the breathtaking landscapes I got to see, and grateful to the composers and musicians who provided the soundtrack. But most importantly, I am grateful to my family and friends for the emotional support and encouragement.

Chapter 1

Introduction

1.1 Background and Motivation

At the outset, the mathematical problem discussed in this text is motivated by three questions. First, there is a classical differential geometry problem that asks whether or not one can always immerse a hypersurface with prescribed mean curvature in a Riemannian manifold. This problem has been studied from various angles, for example in [12] for prescribed mean curvature surfaces in bounded domains in Euclidean space, or in [21] where the authors consider embeddings of prescribed mean curvature hyperspheres in Euclidean space. I will prove here that locally one can always immerse a hypersurface with real-analytic prescribed mean curvature in a Riemannian manifold. In this case, the mean curvature is thought of as a function defined over the unit sphere bundle of the ambient manifold. I will explain, by means of the example of soliton solutions of the mean curvature flow, why this is in fact a natural definition.

Next, there is a classical variational problem: given some functional, one is interested in the stationary solutions, or in other words, the solutions of the associated Euler-Lagrange equation. More akin to the content of this text is in fact the reverse problem: how can I recognize a given quantity as being the solution of the Euler-Lagrange equation of some functional?

To see how these first two problems are related, one may consider the example of a minimal surface in $\mathbb{R}^3$. Dating back to the 18th century, mathematicians like Euler and Lagrange defined these as being the solutions of the Euler-Lagrange equation of the area functional. On the other hand, a minimal surface is also an immersed surface in $\mathbb{R}^3$ with trivial prescribed mean curvature. More generally, the variational problem that I study here can then be described as follows: I will characterize those prescribed mean curvature hypersurfaces in Riemannian manifolds that are indeed the solutions of Euler-Lagrange equations of associated functionals.

Finally, according to a theorem of Emmy Noether, if a variational problem is invariant under the action of a group of symmetries, then each symmetry gives rise to a conservation law. I will consider this result in the case of prescribed mean curvature hypersurfaces. More specifically, I will characterize conservation laws corresponding to minimal surfaces in Riemannian 3-manifolds.

The mathematical framework used to attack the problems listed above was provided by Robert L. Bryant and Phillip Griffiths in the 1990s and has been summarized in the monograph [5]. There they give an outline of the geometry of Euler-Lagrange equations in the context of Monge-Ampère systems. This includes a characterization of Euler-Langrange systems of Monge-Ampère type as well as a generalization of the Noether theorem to this particular context.

Definition 1.1.1. A *Monge-Ampère system* is an exterior differential system $(M, \mathcal{E})$ where M is a contact manifold of dimension $2n{+}1$ with contact form θ, and where $\mathcal{E}$ is given by $\mathcal{E} = (\theta, \Psi)$ for some n-form Ψ.

By reformulating the prescribed mean curvature equation as a system of Monge-Ampère type, I am then able to adapt the findings of Bryant and Griffiths to this particular example.

This text is divided into the following parts: Chapter 1 introduces the main definitions and established results used throughout the text. I will also present two examples, minimal hypersurfaces and solitons of the mean curvature flow, that turn out to be linked in an interesting way.

In Chapter 2, I will address the existence question of prescribed mean curvature surfaces, and I will give a characterization of those systems that are Euler-Lagrange. In particular, I will prove that these are locally equivalent to what is referred to as *basic* systems, and I will explain why this result cannot be reformulated globally.

Chapter 3 contains a complete analysis of conservation laws (of first order) corresponding to minimal surfaces in a Riemannian 3-manifold. In particular, I will show which of these arise in the sense of Noether's Theorem and which ones do not.

Finally, since I am applying the theory of exterior differential systems as the principal toolkit to solve most of the above problems, this text is accompanied by a short appendix on exterior differential systems that summarizes some of the key ideas and results.

1.2 The Notion of a Prescribed Mean Curvature System

To begin, I let (M, g) be an oriented Riemannian manifold of dimension $(n{+}1)$ where, in particular, n is assumed to be at least 2. Throughout this text I shall use the index range

$$\begin{array}{ccccc} 0 & \leq & a,b,c,d & \leq & n \\ 1 & \leq & i,j,k,l & \leq & n. \end{array}$$

To the manifold M one can associate the right-principal SO(n+1)-bundle of positively oriented, orthonormal frames

$$\begin{array}{ccc} \mathrm{SO}(n{+}1) & \longrightarrow & \mathcal{F}_+(M) \\ & & \big\downarrow \pi \\ & & M \end{array}$$

where $\mathcal{F}_+(M)$ consists of elements $y = (x, e_0, \ldots, e_n)$ such that $x \in M$ and the (e_a) form a g-orthonormal, positively oriented basis for T_xM. Moreover, the projection $\pi : \mathcal{F}_+(M) \to M$ onto the base is given by $\pi(y) = x$. The orthonormal frame bundle $\mathcal{F}_+(M)$ has the structure of a smooth manifold whose dimension is given by

$$m = (n+1) + \binom{n+1}{2}.$$

This text contains a significant number of computations involving differential forms. A natural set of 1-forms defined on $\mathcal{F}_+(M)$ arises in the following way: At each point $y \in \mathcal{F}_+(M)$ one can define the *tautological 1-forms* ω^a via the formulae

$$\omega^a(\xi) = g(\pi_*(\xi), e_a)$$

for $\xi \in T_y\mathcal{F}_+(M)$. Moreover I let ω^b_a denote the corresponding Levi-Civita connection forms with the property that

$$\omega^b_a + \omega^a_b = 0.$$

The collection (ω^a, ω^b_a) then gives a canonical coframing of $\mathcal{F}_+(M)$, which satisfies the structure equations of Élie Cartan:

$$\begin{aligned} d\omega^a &= -\omega^a_b \wedge \omega^b \\ d\omega^b_a &= -\omega^b_c \wedge \omega^c_a + \Omega^b_a \\ &= -\omega^b_c \wedge \omega^c_a + \tfrac{1}{2} R^b_{acd}\omega^c \wedge \omega^d. \end{aligned}$$

Here R^b_{acd} denotes the Riemann curvature tensor with the usual symmetries, namely

$$\begin{aligned} R^b_{acd} + R^a_{bcd} &= 0 \\ R^b_{acd} + R^b_{adc} &= 0 \\ R^b_{acd} - R^c_{dba} &= 0 \end{aligned}$$

and of course R^b_{acd} also satisfies the Bianchi identity

$$R^b_{acd} + R^b_{cda} + R^b_{dac} = 0.$$

On the frame bundle $\mathcal{F}_+(M)$ I will define the following forms:

$$\begin{aligned}\omega =&\omega^1 \wedge \ldots \wedge \omega^n \\ \Omega =&\omega^0 \wedge \ldots \wedge \omega^n \\ \omega_{(i)} =&(-1)^{i-1}\omega^1 \wedge \ldots \wedge \hat{\omega}^i \wedge \ldots \wedge \omega^n \\ \psi =& -\omega_0^i \wedge \omega_{(i)}\end{aligned} \tag{1.1}$$

My convention for the *mean curvature* of Σ is that I will take H to be the sum (rather than the average) of the principal curvatures of Σ. This has the following consequence:

Lemma 1.2.1. *An integral manifold $F : \Sigma \to \mathcal{F}_+(M)$ of the exterior differential system $(\mathcal{F}_+(M), (\omega^0))$ satisfies $F^*\psi = H\omega$ where, by abuse of notation*[1]*, ω denotes the volume form of Σ.*

Proof. Since $F : \Sigma \to \mathcal{F}_+(M)$ satisfies $\omega^0 = 0$, one has $d\omega^0 = 0$, and so by Cartan's Lemma there exist functions $h_{jk} = h_{kj} \in C^\infty(\Sigma)$ for which $\omega_k^0 = h_{kj}\omega^j$. These functions h_{jk} can be identified with the second fundamental form of Σ as they define, at each point $p \in \Sigma$, a linear map $W(p) : T_p\Sigma \to T_p\Sigma$ whose matrix in the orthonormal basis $(e_1, \ldots, e_n)$ is given by (h_{jk}). By definition, the trace of this matrix is the mean curvature H of Σ at p. I can now compute

$$\begin{aligned}\psi &= -\omega_0^i \wedge (-1)^{i-1}\omega^1 \wedge \ldots \wedge \hat{\omega}^i \wedge \ldots \wedge \omega^n \\ &= h_{ij}\omega^j \wedge (-1)^{i-1}\omega^1 \wedge \ldots \wedge \hat{\omega}^i \wedge \ldots \wedge \omega^n \\ &= \sum_{i=1}^n h_{ii}\omega^1 \wedge \ldots \wedge \omega^n \\ &= H\omega\end{aligned}$$

as required. □

Using this fact, I can now introduce a first example of what shall later be referred to as a *prescribed mean curvature system.* Here $M = \mathbb{R}^{n+1}$ endowed with the standard metric, and Σ^n is compact without boundary, and $F_0 : \Sigma \to \mathbb{R}^{n+1}$ is a smooth immersion.

Definition 1.2.2. The surface $\Sigma_0 = F_0(\Sigma)$ *evolves under the mean curvature flow* if there exists a family of smooth immersions $\{F_t\}$ with corresponding hypersurfaces $\Sigma_t = F_t(\Sigma) \subset \mathbb{R}^{n+1}$ so that

$$\begin{cases} \frac{\partial F_t}{\partial t}(p) &= -\vec{H}(p,t) \\ F_0(p) &= f \end{cases} \tag{1.2}$$

where f is a smooth function, and where $\vec{H}$ is the mean curvature vector defined as the product of the mean curvature with the outward unit normal.

[1] Whenever the context allows I will omit writing pull-backs explicitly.

Rephrasing this condition in the language of exterior differential systems, I define an ideal

$$\mathcal{M} = (dt \wedge \omega^0, dt \wedge \psi + \Omega)$$

on the manifold $\mathcal{F}_+(\mathbb{R}^{n+1}) \times \mathbb{R}$. An $(n+1)$-dimensional integral manifold of $\mathcal{M}$ on which $dt \wedge \omega \neq 0$ corresponds to a 1-parameter family of hypersurfaces moving by the mean curvature flow (1.2).

The term *soliton* refers to a solution of (1.2) which moves under a 1-parameter subgroup of the symmetry group of M. To make this more precise, I will let $(G, \cdot)$ denote the symmetry group acting on $\mathcal{F}_+(\mathbb{R}^{n+1}) \times \mathbb{R}$. The group action must preserve $\mathcal{M}$, and so G comprises the rigid Euclidean motions, time translation and dilation.

Now let $J \subset \mathbb{R}$ be an interval. If for a curve $g : J \to G$ there exists f so that

$$F_t(p) = g(t) \cdot f(g(t) \cdot p)$$

is a solution of (1.2), f is called a *soliton*. In view of the exterior differential system $(\mathcal{F}_+(\mathbb{R}^{n+1}) \times \mathbb{R}, \mathcal{M})$, one may then re-interpret this definition. Since integral manifolds are insensitive to re-parameterization one has:

Definition 1.2.3. An n-dimensional integral manifold f is a *soliton* if there exists a curve $g : J \to G$ for which $F_t(p) = g(t) \cdot f(p)$ is an $(n+1)$-dimensional integral manifold of $(\mathcal{F}_+(\mathbb{R}^{n+1}) \times \mathbb{R}, \mathcal{M})$.

In [1] Bryant observes that if $V(t)$ denotes the vector field induced by the flow of an element of $\mathfrak{g} = T_e G$, then for an integral manifold F_t as defined above the vector $V(t)(f(p))$ and the subspace $f_*(T_p\Sigma)$ must span an integral element for all $p \in \Sigma, t \in J$. Consequently, Bryant develops the following criterion:

Proposition 1.2.4 (Bryant)**.** *F_t as defined above is an integral manifold of $\mathcal{M}$ if and only if for each $t \in J$, f is an integral manifold of the system $\mathcal{M}_{V(t)}$ that is generated by the forms $V(t) \lrcorner \varphi$ where the $V(t)$ are the vector fields associated to the 1-parameter subgroups of G, and where φ ranges over the elements of $\mathcal{M}$.*

I will assume that the symmetry vector field $V(t)$ is of the form

$$V(t) = \tfrac{\partial}{\partial t} + \tilde{W}$$

where $\tilde{W}$ is the lift of a Killing vector field W to $\mathcal{F}_+(\mathbb{R}^{n+1})$. The ideal $\mathcal{M}_{V(t)}$ is generated by the elements

$$\left(\tfrac{\partial}{\partial t} + \tilde{W}\right) \lrcorner (dt \wedge \omega^0) = \omega^0 - \omega^0 \left(\tfrac{\partial}{\partial t} + \tilde{W}\right) dt$$

and

$$\left(\tfrac{\partial}{\partial t} + \tilde{W}\right) \lrcorner (dt \wedge \psi + \Omega) = \psi - \psi \left(\tfrac{\partial}{\partial t} + \tilde{W}\right) dt + \tilde{W} \lrcorner \Omega.$$

Since I am interested in solitons f, I will assume $t = 0$ which reduces the ideal to

$$\mathcal{M}_V = (\omega^0, \psi + \tilde{W} \lrcorner \Omega).$$

Given that $\tilde{W} \lrcorner \Omega = \omega^0(\tilde{W})\omega - (\tilde{W} \lrcorner \omega) \wedge \omega^0$, and the fact that the latter portion is already contained in the ideal, I can simplify $\mathcal{M}_V$ further and write

$$\mathcal{M}_V = (\omega^0, \psi + \omega^0(\tilde{W})\omega) \tag{1.3}$$

$\mathcal{M}_V$ is now an ideal defined on $\mathcal{F}_+(\mathbb{R}^{n+1})$ alone. In fact, one can think of it as an ideal on $\mathcal{F}_+(\mathbb{R}^{n+1})/\mathrm{SO}(n)$ by quotienting out rotations in the $(e_1, \ldots, e_n)$ frame. The n-dimensional integral manifolds now correspond to hypersurfaces in $\mathbb{R}^{n+1}$ whose mean curvature is measured by $-\omega^0(\tilde{W})$ and as such depends on the angle $\tilde{W}$ makes with the tangent plane. The resulting exterior differential system

$$(\mathcal{F}_+(\mathbb{R}^{n+1})/\mathrm{SO}(n), \mathcal{M}_V)$$

is a system of Monge-Ampère type since, as I will illustrate shortly, $\mathcal{F}_+(\mathbb{R}^{n+1})/\mathrm{SO}(n)$ is a contact manifold of dimension $(2n+1)$ with contact form ω^0, and $\mathcal{M}_V$ is evidently generated by ω^0 and an n-form. It is therefore a particular example of what will henceforth be referred to as a *prescribed mean curvature system.*

To spell out the precise definition of a prescribed mean curvature system I will now return to the more general setting:

$$\begin{array}{ccc} \mathrm{SO}(n+1) & \longrightarrow & \mathcal{F}_+(M) \\ & & \downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

Embedding $\mathrm{SO}(n) \subset \mathrm{SO}(n+1)$ via

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in \mathrm{SO}(n) \right\}$$

gives rise to a natural identification of $\mathcal{F}_+(M)/\mathrm{SO}(n)$ with the unit sphere bundle $\mathcal{S}(M)$ of M which is furnished by the map

$$\nu : \mathcal{F}_+(M) \to \mathcal{S}(M)$$

mapping

$$(x, e_0, ..., e_n) \mapsto (x, e_0).$$

Since by definition

$$\omega^0(\xi) = g(\pi_*(\xi), e_0)$$

on $\mathcal{F}_+(M)$ it is evident that ω^0 is ν-basic, that is, it is the pull-back of a unique, globally defined 1-form on $\mathcal{S}(M)$ which will also be denoted by ω^0. Moreover, since on $\mathcal{F}_+(M)$,

$$\begin{aligned}\omega^0 \wedge (d\omega^0)^n &= \omega^0 \wedge (-\omega^0_k \wedge \omega^k)^n \\ &= -n\omega^0 \wedge \bigwedge_{k=1}^{n} \omega^0_k \wedge \omega^k \\ &\neq 0.\end{aligned}$$

Given that pull-back via the submersion $\nu : \mathcal{F}_+(M) \to \mathcal{S}(M)$ is injective one has:

Lemma 1.2.5. *The 1-form ω^0 generates a line subbundle $I \subset T^*\mathcal{S}(M)$ which defines a contact structure on the $(2n+1)$-dimensional manifold $\mathcal{S}(M)$.*

Furthermore, by construction:

Lemma 1.2.6. *The forms ω, Ω and ψ defined in (1.1) are all ν-basic over $\mathcal{S}(M)$, that is, they are the pull-backs of well-defined forms on $\mathcal{S}(M)$ (which will be denoted by the same Greek letters).*

Proof. A differential form α is ν-basic if and only if both α and $d\alpha$ are ν-semi-basic, that is, for any vertical vector X one has $X \lrcorner \alpha = 0$ and $X \lrcorner d\alpha = 0$. Clearly $\omega^0, d\omega^0, \psi, \omega$ and Ω are semi-basic. Since $d\omega = -\omega^0 \wedge \psi$ and $d\Omega = 0$ I can immediately conclude that both $d\omega$ and $d\Omega$ are semi-basic. To prove the lemma it therefore remains to show that $d\psi$ is semi-basic.

One can expand $d\omega_{(i)}$ as

$$d\omega_{(i)} = \omega^k_i \wedge \omega_{(k)} + \omega^0 \wedge \omega^k_0 \wedge \hat{\omega}_{i,k}$$

where

$$\hat{\omega}_{i,k} = \begin{cases} 0 & i = k \\ \pm\omega^1 \wedge \ldots \wedge \hat{\omega}^i \wedge \ldots \wedge \hat{\omega}^k \wedge \ldots \wedge \omega^n & i \neq k \end{cases}$$

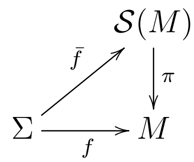
Consequently

$$\begin{aligned} d\psi &= -d\omega^i_0 \wedge \omega_{(i)} + \omega^i_0 \wedge d\omega_{(i)} \\ &= (\omega^i_k \wedge \omega^i_0 - \frac{1}{2} R^i_{0ab} \omega^a \wedge \omega^b) \wedge \omega_{(i)} + \omega^i_0 \wedge \omega^k_i \wedge \omega_{(k)} + \omega^i_0 \wedge \omega^0 \wedge \omega^k_0 \wedge \hat{\omega}_{i,k} \\ &= \omega^i_0 \wedge \omega^0 \wedge \omega^k_0 \wedge \hat{\omega}_{i,k} - \sum_i R^i_{00i} \Omega \end{aligned}$$

which is evidently ν-semi-basic. □

This shows in particular that the exterior differential system $(\mathcal{S}(\mathbb{R}^{n+1}), \mathcal{M}_V)$ giving rise to solitons of the mean curvature flow is indeed a well-defined Monge-Ampère system as previously claimed. It therefore falls precisely into the category of a prescribed mean curvature system (with the *prescription function* being $-\omega^0(\tilde{W})$).

More generally, my goal is to understand how the condition that a hypersurface is immersed in M with prescribed mean curvature can be understood from the perspective of an exterior differential system. A key ingredient is the following observation: if $f : \Sigma \to M$ is an oriented, immersed hypersurface, then I can lift Σ into $\mathcal{S}(M)$ via a map $\bar{f}$



defined as

$$\bar{f} : p \mapsto (x, e_0)$$

where $x = f(p)$, and where e_0 is the unique outward unit normal of Σ at $x \in f(\Sigma) \subset M$, i.e. the unique vector at x normal to $f_*(T_p\Sigma)$. The latter condition implies that

$$(\bar{f})^*\omega^0 = 0.$$

More precisely:

Proposition 1.2.7. *$\bar{f} : \Sigma \to \mathcal{S}(M)$ is an n-dimensional integral manifold of the exterior differential system $(\mathcal{S}(M), (\omega^0))$. Conversely, if $F : \Sigma \to \mathcal{S}(M)$ is an n-dimensional integral manifold of $(\mathcal{S}(M), (\omega^0))$ that is transverse to the projection $\pi : \mathcal{S}(M) \to M$, then $F = \bar{f}$, where $f = \pi \circ F$ is an immersion of a hypersurface in M.*

A general prescribed mean curvature system is then defined in the following way:

Definition 1.2.8. A *prescribed mean curvature system* refers to the Monge-Ampère system $(\mathcal{S}(M), \mathcal{I}_P)$ where $\mathcal{I}_P$ denotes the differential ideal

$$\mathcal{I}_P = (\omega^0, \psi - P\omega)$$

and where the *prescription function* P is defined to be a smooth, real-valued function on $\mathcal{S}(M)$. In the special case where P is defined on M alone, the corresponding prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ will be referred to as *basic*.

A natural question is whether or not (at least locally) I can always immerse a hypersurface in M with prescribed mean curvature. Or, equivalently, one can ask whether or not there exist n-dimensional integral manifolds for the system $\mathcal{I}_P$. In the next chapter I will show that this is indeed the case, namely:

- In the real analytic category $(\mathcal{S}(M), \mathcal{I}_P)$ has integral manifolds of dimension n.

A slight caveat when it comes to computations is the fact that $\mathcal{S}(M)$ does not carry a natural coframe, unlike the frame bundle $\mathcal{F}_+(M)$. Because of this fact, most computations will therefore be performed on $\mathcal{F}_+(M)$ instead of $\mathcal{S}(M)$. Suppose then one were interested in n-dimensional integral manifolds of the contact system $(\mathcal{S}(M), I)$ for which $\omega \neq 0$ (these are often referred to as *Legendre submanifolds*). It is clear that n-dimensional integral manifolds of the "pull-back system" $(\mathcal{F}_+(M), I)$ will give rise to such Legendre submanifolds via ν. However, due to the presence of Cauchy characteristics, n-dimensional integral manifolds for $(\mathcal{F}_+(M), I)$ are "framed" integral manifolds in that at each point one is left with a choice of frame due to the $\mathrm{SO}(n)$ action on the $(e_1, \ldots, e_n)$ frame. In accounting for this extra freedom the Cartan characters will not agree with those for $(\mathcal{S}(M), I)$. This dichotomy will be addressed in detail in the following chapter, in particular in the proof of Theorem 2.1.1. These calculations will also make use of the following technical fact which follows immediately from the definition of ν:

Lemma 1.2.9. *For each 1-form $\eta \in \Omega^1(\mathcal{S}(M))$ there exist functions $A_0, A_i, B_j \in C^\infty(\mathcal{F}_+(M))$ so that, when pulled-back to $\mathcal{F}_+(M)$,*

$$\eta = A_0\omega^0 + A_i\omega^i + B_j\omega^j_0.$$

1.3 Towards the Calculus of Variations

In order to draw a connection between prescribed mean curvature systems and the calculus of variations, I will once again assume that $M = \mathbb{R}^{n+1}$. As before, I will let I denote the contact ideal on the unit sphere bundle $\mathcal{S}(\mathbb{R}^{n+1})$. This time I am interested in minimal surfaces in $\mathbb{R}^{n+1}$, so consequently the prescribed mean curvature must satisfy $P = 0$, resulting in the basic prescribed mean curvature system $(\mathcal{S}(\mathbb{R}^{n+1}), \mathcal{I}_0)$ with

$$\mathcal{I}_0 = (\omega^0, \psi).$$

Then, if N is a compact, n-dimensional integral manifold of $(\mathcal{S}(\mathbb{R}^{n+1}), \mathcal{I}_0)$, it corresponds to a minimal surace in $\mathbb{R}^{n+1}$. On the other hand, N must be, amongst the family $\{N_t\}$ of compact Legendre submanifolds of $(\mathcal{S}(\mathbb{R}^{n+1}), I)$, one that minimizes the area functional $\mathcal{F}_\omega$ defined as

$$\mathcal{F}_\omega(N_t) = \int_{N_t} \omega. \tag{1.4}$$

In other words, N solves the Euler-Lagrange equation arising from the first variation of the functional (1.4) above. This raises the following question: for a general Riemannian manifold M, and a general prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$, when are the n-dimensional integral manifolds stationary for some functional like the one above? When this is indeed the case, the system will be referred to as *Euler-Lagrange*. More precisely:

Definition 1.3.1. A prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ is called *Euler-Lagrange* if there exists an n-form Λ on $\mathcal{S}(M)$ so that amongst compact Legendre submanifolds N of $\mathcal{S}(M)$, the n-dimensional integral manifolds of $(\mathcal{S}(M), \mathcal{I})$ are stationary for

$$\mathcal{F}_\Lambda(N) = \int_N \Lambda.$$

The n-form Λ is commonly referred to as a *Lagrangian* of $(\mathcal{S}(M), \mathcal{I}_P)$. It should be noted that if $\Lambda' \in \Omega^n(\mathcal{S}(M))$ were to differ from Λ by either an element of the contact ideal I or an exact form, then

$$\mathcal{F}_\Lambda(N) = \mathcal{F}_{\Lambda'}(N).$$

This notion of equivalence of two Lagrangians suggests one studies the de Rham complex $(\bar{\Omega}^*, \bar{d})$ consisting of the spaces $\bar{\Omega}^n = \Omega^n / I^n$ together with the induced exterior derivative. The fact that $\mathcal{S}(M)$ is a contact manifold implies that $d\Lambda \in I^{n+1}$. Therefore the equivalence class $[\Lambda]$ must be a member of the characteristic cohomology group

$$\bar{H}^n = H^n(\bar{\Omega}^*, \bar{d}).$$

In [5], the authors examine the long exact sequence

$$\cdots \longrightarrow H^n_{dR}(\mathcal{S}(M)) \longrightarrow \bar{H}^n \xrightarrow{\delta} H^{n+1}(I) \longrightarrow H^{n+1}_{dR}(\mathcal{S}(M)) \longrightarrow \cdots$$

resulting from the short exact sequence

$$0 \longrightarrow I^* \longrightarrow \Omega^*(\mathcal{S}(M)) \longrightarrow \bar{\Omega}^* \longrightarrow 0$$

They prove:

Theorem 1.3.2 (Bryant, Griffiths, Grossman)**.** *Any class $[\Theta] \in H^{n+1}(I)$ has a unique global representative $\Theta \in I^{n+1}$ satisfying*

(*i*) $d\Theta = 0$*;*

(*ii*) $\Theta \equiv 0 \mod I$*.*

So in particular, for any given Lagrangian Λ, there must exist, amongst members of the class $\delta([\Lambda])$, a unique Θ with the property that $\Theta \equiv 0 \mod I$. Such a Θ corresponding to a Lagrangian Λ is called the *Poincaré-Cartan form* of Λ. The significance of Θ lies in the following result: by writing

$$\Theta = -\omega^0 \wedge \Psi$$

for $\Psi \in \Omega^n(\mathcal{S}(M))$ and, for a fixed boundary variation $F : N \times [0,1] \to \mathcal{S}(M)$, leting $\mathcal{F}_\Lambda$ denote the functional

$$\mathcal{F}_\Lambda(N_t) = \int_{N_t} \Lambda \tag{1.5}$$

one has:

Theorem 1.3.3 (Bryant, Griffiths, Grossman)**.** *A Legendre submanifold $f : N \to \mathcal{S}(M)$ is stationary for* (1.5)*, i.e.* $\frac{d}{dt}\big|_{t=0} \mathcal{F}_\Lambda(N_t) = 0$ *for* $F|_{t=0} = f$*, if and only if it is an integral manifold of the exterior differential system* (ω^0, Ψ) *on* $\mathcal{S}(M)$*.*

In the special case of the minimal hypersurface system on $\mathbb{R}^{n+1}$, the Lagrangian Λ is given by $\Lambda = \omega$, while the Poincaré-Cartan form Θ is given by $\Theta = d\omega = -\omega^0 \wedge \psi$, i.e., $\Psi = \psi$ in this particular case.

For a general prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$, I will, proceeding in an analogous fashion, write

$$\Psi = \psi - P\omega,$$

and I will refer to

$$\Theta = -\omega^0 \wedge \Psi$$

as the "candidate" Poincaré-Cartan form. Following the discussion surrounding Theorem 1.3.2, the prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ is Euler-Lagrange provided that Θ is exact.

In the next chapter I will discuss what it means for a prescribed mean curvature system to be *locally Euler-Lagrange*. In particular, I will show:

- Any basic prescribed mean curvature system is locally Euler-Lagrange.

The central goal of Chapter 2 will then be to prove:

- A prescribed mean curvature system that is locally Euler-Lagrange is equivalent to a basic system under conformal scaling of the ambient metric.

1.4 Conservation Laws and the Noether Theorem

In the classical calculus of variations one associates to solutions of the Euler-Lagrange equation quantities which are constant along such solutions, so-called *classical conservation laws*. A theorem of Emmy Noether asserts that such conservation laws are derived from symmetries of the system.

Following the approach furnished in [5], the notion of such quantities constant along solutions in the context of Monge-Ampère systems, and in particular for prescribed mean curvature systems, takes the following shape:

Definition 1.4.1. A *conservation law of first-order* for $(\mathcal{S}(M), \mathcal{I}_P)$ is a $(n-1)$-form φ with the property that $d\varphi \in \mathcal{I}_P$.

Given that I will only consider conservation laws of first-order in this text, the "first-order" designation will dropped from here on. Moreover, one automatically obtains trivial conservation laws by considering elements of $\mathcal{I}_P^{n-1}$ or $(n-1)$-forms that are already exact on

$\mathcal{S}(M)$. These trivial conservation laws will be ruled out by insisting that a conservation law be *proper*, i.e. that it is an element of the space

$$\bar{\mathcal{C}} = H^{n-1}(\Omega^*(\mathcal{S}(M))/\mathcal{I}_P)/\pi(H^{n-1}_{dR}(\mathcal{S}(M))$$

where π is described by the long exact sequence

$$\cdots \longrightarrow H^{n-1}_{dR}(\mathcal{S}(M)) \xrightarrow{\pi} H^{n-1}(\Omega^*(\mathcal{S}(M))/\mathcal{I}_P) \longrightarrow H^n(\mathcal{I}_P) \longrightarrow H^n_{dR}(\mathcal{S}(M)) \longrightarrow \cdots$$

So from this point on, a conservation law is assumed to be both of first order and proper. In order to state Noether's theorem the way it is presented in [5], I start with an Euler-Lagrange prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ with Lagrangian Λ and Poincaré-Cartan form Θ. As before, I will let I represent the contact ideal for $\mathcal{S}(M)$. The first step is to make sense of the notion "symmetries of the system" as there are several candidates. These are:

(i) $\mathfrak{g}_{[\Lambda]} = \{V \in \Gamma(T\mathcal{S}(M)) : \mathcal{L}_V I \subset I, \quad \mathcal{L}_V[\Lambda] = 0\}$ symmetries of the Lagrangian;

(ii) $\mathfrak{g}_{\Theta} = \{V \in \Gamma(T\mathcal{S}(M)) : \mathcal{L}_V \Theta = 0\}$ symmetries of the Poincaré-Cartan form;

(iii) $\mathfrak{g}_{\mathcal{I}_P} = \{V \in \Gamma(T\mathcal{S}(M)) : \mathcal{L}_V \mathcal{I}_P \subset \mathcal{I}_P\}$ symmetries of the ideal.

By viewing $\bar{\mathcal{C}}$ as a subset of $H^n(\mathcal{I})$ via inclusion, one has:

Theorem 1.4.2 (Noether)**.** *The linear isomorphism* $\eta : \mathfrak{g}_{\Theta} \to H^n(\mathcal{I}_P)$ *given by* $\eta(V) = V \lrcorner \Theta$ *maps the subalgebra* $\mathfrak{g}_{[\Lambda]}$ *into* $\bar{\mathcal{C}}$*.*

In [5] the authors give an illustration of Theorem 1.4.2 by computing conservation laws for minimal surfaces and constant mean curvature surfaces in $\mathbb{R}^{n+1}$ (see [5], pp. 35-43 for details). I will reproduce one very easy subcase for a minimal surface $\Sigma^2 \subset \mathbb{R}^3$. The ideal in question is then $\mathcal{I}_0 = (\omega^0, \psi)$ while $\Lambda = \omega = \omega^1 \wedge \omega^2$ and $\Theta = d\omega$. Clearly, Θ has to preserve Euclidean motions, so for V I may take a translation vector field

$$V = v_0 e_0 + v_1 e_1 + v_2 e_2$$

lifted to the frame bundle $\mathcal{F}_+(\mathbb{R}^3)$. The coefficients must satisfy

$$\begin{aligned} dv_0 &= +v_1\omega_0^1 + v_2\omega_0^2 \\ dv_1 &= -v_0\omega_0^1 + v_2\omega_1^2 \\ dv_2 &= -v_0\omega_0^2 - v_1\omega_1^2 \end{aligned} \tag{1.6}$$

Then I have

$$V \lrcorner \Theta = v_2\omega^0 \wedge \omega_0^1 - v_1\omega^0 \wedge \omega_0^2 - v_0\psi \in \mathcal{I}_0$$

and by applying (1.6), I see that $V \lrcorner \Theta$ is precisely $d(v_2\omega^1 - v_1\omega^2)$, i.e. the conservation law corresponding to the symmetry V is given by

$$\varphi = v_2\omega^1 - v_1\omega^2.$$

In [5] the authors highlight that for the inclusions

$$\mathfrak{g}_{[\Lambda]} \subset \mathfrak{g}_\Theta \subset \mathfrak{g}_{\mathcal{I}_P}$$

the latter may be strict, as furnished by the example of minimal hypersurfaces in Euclidean space, where $\mathcal{I}_0$ remains invariant under dilation while Θ does not. That is to say, one might expect the existence of additional conservation laws not captured by the Theorem of Noether. This is precisely the starting point of my investigation in the final chapter where I address the following question: Does the theorem of Noether furnish a complete set of conservation laws for minimal surfaces in a Riemannian 3-manifold? I will prove:

- The answer is yes when M has non-trivial curvature. However, in the case when M is flat, there exist additional conservation laws.

Chapter 2

Analysis of the Monge-Ampère System

2.1 Local Existence

In this chapter I will analyze the local structure of a prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$. An integral manifold of dimension n for this system on which $\omega \neq 0$ would correspond to an immersed, oriented hypersurface in M whose mean curvature is measured by the function P. Consequently, the first question to be addressed here is whether or not such integrals exist in the first place. I claim:

Theorem 2.1.1 (*)**.** *The prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ is involutive. In the real analytic category local n-dimensional integral manifolds exist and depend on 2 functions of $(n-1)$ variables.*

Proof. The proof employs Cartan's Test as well as the Cartan-Kähler Theorem. As indicated earlier, all computations are performed on $\mathcal{F}_+(M)$, and so in particular (the pull-back of) P is now assumed to be a function that is constant on fibers of $\nu : \mathcal{F}_+(M) \to \mathcal{S}(M)$. In order to get the correct count of Cartan characters, I will make the following modification: Recall that $m = (n+1) + \binom{n+1}{2}$. Furthermore I will define $m' = n + \binom{n}{2}$. Rather than looking for "framed" integral manifolds of dimension n of the pull-back system for which $\omega \neq 0$ (and these do exist) one searches for integral manifolds of dimension m' subject to the enlarged independence condition that

$$\omega_+ = \omega \wedge \bigwedge_{i<j} \omega_i^j$$

be non-vanishing. The proof is divided into the following three steps: first I will show that every m'-dimensional integral element is ordinary. Next I argue that the hypotheses of Cartan's Test are met in order to prove involutivity. Finally, I evoke the Cartan-Kähler Theorem to complete the proof.

Let $G_{m'}(T\mathcal{F}_+(M),\omega_+)$ denote the set of m'-planes in $T\mathcal{F}_+(M)$ on which ω_+ is non-vanishing. $G_{m'}(T\mathcal{F}_+(M),\omega_+)$ can be given the structure of a smooth manifold of dimension $m+(n+1)m'$ as follows: let $y \in \mathcal{F}_+(M)$, and let $(U,(y_1,\ldots,y_m))$ be a coordinate neighborhood in $\mathcal{F}_+(M)$ containing y. Set

$$G_{m'}(TU,\omega_+) = \{E \in G_{m'}(T_y\mathcal{F}_+(M)) : y \in U, \omega_+|_E \neq 0\}.$$

With the usual index range, define smooth functions $p_i, p^k_l, q^j_i, q^{jk}_l$ on $G_{m'}(TU,\omega_+)$ via the equations

$$\begin{aligned}\omega^0 - p_i(E)\omega^i - p^k_l(E)\omega^l_k &= 0\\ \omega^j_0 - q^j_i(E)\omega^i - q^{jk}_l(E)\omega^l_k &= 0.\end{aligned}$$

Then clearly, since $\omega^j_k + \omega^k_j = 0$, one has

$$p^k_l + p^l_k = 0, \tag{2.1}$$

and also, for all j,

$$q^{jk}_l + q^{jl}_k = 0. \tag{2.2}$$

$G_{m'}(TU,\omega_+)$ together with the functions $y_1,\ldots,y_m,p_i,p^k_l,q^j_i,q^{jk}_l$ then defines a smooth coordinate chart on $G_{m'}(T\mathcal{F}_+(M),\omega_+)$.

Now let $V_{m'}(\mathcal{I}_P,\omega_+) \subset G_{m'}(T\mathcal{F}_+(M),\omega_+)$ denote the space of m'-dimensional integral elements of $\mathcal{I}_P$ for which $\omega_+ \neq 0$. Then $E \in V_{m'}(\mathcal{I}_P,\omega_+)$ if and only if

$$\begin{aligned} p_i &= 0\\ p^k_l &= 0\\ q^j_i - q^i_j &= 0\\ q^{jk}_l &= 0\\ \sum_i q^i_i + P &= 0.\end{aligned} \tag{2.3}$$

These equations are smooth and independent, hence $V_{m'}(\mathcal{I}_P,\omega_+)$ is cut out cleanly as a submanifold of $G_{m'}(T\mathcal{F}_+(M),\omega_+)$. So by definition any element $E \in V_{m'}(\mathcal{I}_P,\omega_+)$ is ordinary. Moreover

$$\operatorname{codim}\left[V_{m'}(\mathcal{I}_P,\omega_+), G_{m'}(T\mathcal{F}_+(M),\omega_+)\right] = \tfrac{1}{2}(n^3+n^2+2).$$

At $y \in \mathcal{F}_+(M)$, I define a flag of integral elements

$$(0)_y = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_{m'} \subset T_y\mathcal{F}_+(M)$$

as follows: let W_i, W_j^0 denote the vector fields dual to ω^i, ω_0^j. Then set

$$X_i = W_i + q_i^j W_j^0$$

where the q_i^j satisfy the conditions (2.2) and (2.3). Furthermore let

$$X_{n+1} = W_2^1 \quad , \quad \dots \quad , \quad X_{m'} = W_n^{n-1}$$

with W_l^k representing the vector fields dual to ω_k^l, and I will continue to assume $l > k$. Finally let

$$E_k = \langle X_1, ..., X_k \rangle.$$

Recall that for a k-dimensional integral element E_k the corresponding polar space $H(E_k)$ is defined to be the space of integral extensions to the next higher dimension, that is

$$H(E_k) = \{v \in T_y\mathcal{F}_+(M) : v \lrcorner \phi(X_1, \dots, X_k) = 0 \quad \forall \phi \in \mathcal{I}_P^{k+1}\}.$$

As is customary, for $k < m'$, I set

$$c_k = \operatorname{codim}\left[H(E_k), T_y\mathcal{F}_+(M)\right]$$

and

$$c_{m'} = m - m' = n + 1.$$

One immediately has the chain of inclusions

$$T_y\mathcal{F}_+(M) \supset H(E_0) \supset ... \supset H(E_{m'}) \supset E_{m'}.$$

According to Cartan's Test, an m'-dimensional integral element $E \in V_{m'}(\mathcal{I}_P)$ is the terminus of an ordinary flag provided that

$$c_0 + c_1 + ... + c_{m'-1} = \operatorname{codim}\left[V_{m'}(\mathcal{I}_P, \omega_+), G_{m'}(T\mathcal{F}_+(M), \omega_+)\right].$$

Now

$$H(E_0) = \{v \in T_y\mathcal{F}_+(M) : \omega^0(v) = 0\}$$

and consequently

$$c_0 = 1. \tag{2.4}$$

Next

$$H(E_1) = \{v \in T_y\mathcal{F}_+(M) : v \lrcorner \phi(X_1) = 0 \quad \forall \phi \in \mathcal{I}_P^2\}.$$

Clearly $\mathcal{I}^2$ is generated by

$$d\omega^0 = -\omega_i^0 \wedge \omega^i$$

and

$$\beta \wedge \omega^0$$

where $\beta \in \Omega^1(\mathcal{F}_+(M))$, and so according to the latter equation one must have

$$\omega^0(v) = 0$$

while the former implies

$$\omega_1^0(v) + \sum_i q_1^i \omega^i(v) = 0$$

so that

$$c_1 = 2. \tag{2.5}$$

Similarly for $k < n-1$, first note that $v \in H(E_k)$ if $v \in H(E_{k-1})$ and if in addition v satisfies the equation

$$\omega_k^0(v) + \sum_i q_k^i \omega^i(v) = 0$$

and consequently

$$c_k = k + 1. \tag{2.6}$$

Now for the case $k = n - 1$, note that $\mathcal{I}_P^n$ is made up of forms $\alpha \wedge \omega^0$, $\beta \wedge d\omega^0$ for $\alpha \in \Omega^{n-1}(\mathcal{F}_+(M))$ and $\beta \in \Omega^{n-2}(\mathcal{F}_+(M))$ and also $\psi - P\omega$. So, for $v \in H(E_{n-1})$, not only do I have, for $k \leq n-1$,

$$\omega^0(v) = 0$$

and

$$\omega_k^0(v) + \sum_i q_k^i \omega^i(v) = 0,$$

but from $v \lrcorner (\psi - P\omega)(X_1, \ldots, X_{n-1}) = 0$ it follows that

$$(-1)^n \omega_0^n(v) - q_n^n \omega^n(v) = 0.$$

All these equations are linearly independent, so

$$c_{n-1} = n + 1. \tag{2.7}$$

There are no additional equations when $k > n-1$ so that for $n - 1 < k < m'$ I also have

$$c_k = n + 1. \tag{2.8}$$

Combining (2.4) - (2.8) I can then conclude that

$$c_0 + c_1 + ... + c_{m'-1} = \tfrac{1}{2}(n^3 + n^2 + 2) = \operatorname{codim}\left[V_n(\mathcal{I}_P, \omega), G_n(T\mathcal{F}_+(M), \omega)\right].$$

This proves the second step, namely involutivity of the system.

Finally, if one now assumes $\mathcal{I}_P$ is real analytic, then by the Cartan-Kähler Theorem one has the existence of an integral manifold of $\mathcal{I}_P$ at y whose tangent space at y is E, which is what I wanted to show. Also, since the last non-zero Cartan character is

$$s_{n-1} = c_{n-1} - c_{n-2} = 2$$

the local m'-dimensional integral manifolds of $(\mathcal{F}_+(M), \mathcal{I}_P)$ (or equivalently the local n-dimensional integral manifolds of $(\mathcal{S}(M), \mathcal{I}_P)$) depend on 2 functions of $(n-1)$ variables as claimed. □

2.2 The Local Structure of Euler-Lagrange Systems

Having established local existence for prescribed mean curvature systems, I will now study the local structure of those prescribed mean curvature systems that are (locally) Euler-Lagrange. The first goal is to understand the conditions imposed on the prescription function P that will ensure that $(\mathcal{S}(M), \mathcal{I}_P)$ has the property that it is locally Euler-Lagrange. Thanks to a criterion established in [5], this task is reduced to a straight-forward computation. To lay out this computation, I will consider a prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ and, as before, I will set $\Psi = \psi - P\omega$, and I will continue to refer to $\Theta = -\omega^0 \wedge \Psi$ as the "candidate" Poincaré-Cartan form.

Definition 2.2.1. The prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ is said to be *locally Euler-Lagrange* if there exists a closed n-form which, as an element of the ring $\Omega^n(\mathcal{S}(M))$, is locally expressible as a multiple of Θ.

It should be noted that, since

$$d\omega^0 \wedge \Psi = -\omega^0_k \wedge \omega^k \wedge \left[-\omega^i_0 \wedge \omega_{(i)} - P\omega\right] = 0,$$

the $(n-1)$-form Ψ is, by definition, primitive modulo the contact ideal. This fact allows me to adapt the criterion established by Bryant, Griffiths and Grossman (see [5], Theorem 1.2 for details) to the case of a prescribed mean curvature system as follows:

Theorem 2.2.2 (Bryant, Griffiths, Grossman)**.** *The prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ is locally Euler-Lagrange if and only if the form Θ satisfies*

$$d\Theta = \varphi \wedge \Theta$$

where $\varphi \in \Omega^1(\mathcal{S}(M))$ has the property that $d\varphi \equiv 0 \mod (\omega^0)$.

In order to carry out this computation, I will work on $\mathcal{F}_+(M)$ and assume that dP is of the form

$$dP = A_0\omega^0 + A_i\omega^i + B_j\omega^j_0$$

for $A_0, A_i, B_j \in C^\infty(\mathcal{F}_+(M))$. Moreover, I will write

$$\begin{aligned} dA_0 &= C_{00}\omega^0 + C_{0k}\omega^k + D_{0k}\omega^k_0 + \tfrac{1}{2}D^l_{0k}\omega^k_l \\ dA_i &= C_{i0}\omega^0 + C_{ik}\omega^k + D_{ik}\omega^k_0 + \tfrac{1}{2}D^l_{ik}\omega^k_l \\ dB_j &= S_{j0}\omega^0 + S_{jk}\omega^k + T_{jk}\omega^k_0 + \tfrac{1}{2}T^l_{jk}\omega^k_l \end{aligned}$$

where $C_{ak}, D_{ak}, D^l_{ak}, S_{ja}, T_{jk}, T^l_{jk} \in C^\infty(\mathcal{F}_+(M))$ and in particular

$$\begin{aligned} D^l_{ak} + D^k_{al} &= 0 \\ T^l_{jk} + T^k_{jl} &= 0. \end{aligned}$$

The condition $d^2P = 0$ imposes constraints on the coefficient functions A_0, A_i and B_j. Calculation yields that I must have

$$\begin{aligned} C_{ja} - C_{aj} &= B_i R^i_{0ja} \\ D^j_{ik} &= \delta_{ij} A_k - \delta_{ik} A_j \\ D_{ij} &= S_{ji} - A_0 \delta_{ij} \\ S_{k0} &= D_{0k} - A_k \\ D^k_{0j} &= 0 \\ T^k_{ij} &= \delta_{ik} B_j - \delta_{ij} B_k \\ T_{jk} - T_{kj} &= 0 \end{aligned}$$

and so

$$\begin{aligned} dA_0 &= C_{00}\omega^0 + C_{0k}\omega^k + D_{0k}\omega^k_0 \\ dA_i &= \big(C_{0i} + B_j R^j_{0i0}\big)\omega^0 + C_{ik}\omega^k + S_{ki}\omega^k_0 - A_0\omega^i_0 + A_k\omega^k_i \\ dB_j &= \big(D_{0j} - A_j\big)\omega^0 + S_{jk}\omega^k + T_{jk}\omega^k_0 + B_k\omega^k_j \end{aligned}$$

To determine a suitable φ for which $d\Theta = \varphi \wedge \Theta$, I first note that

$$\begin{aligned} d\Theta &= -d\big(\omega^0 \wedge (\psi - P\omega)\big) \\ &= -d\omega^0 \wedge \psi + P d\omega^0 \wedge \omega + \omega^0 \wedge d\psi - \omega^0 \wedge dP \wedge \omega - P\omega^0 \wedge d\omega \\ &= B_j \omega^j_0 \wedge \Omega \end{aligned} \tag{2.9}$$

On the other hand, assuming that φ, when pulled back to $\mathcal{F}_+(M)$, is of the form

$$\varphi = J_0\omega^0 + J_i\omega^i + K_j\omega^j_0$$

then implies that

$$\begin{aligned} \varphi \wedge \Theta &= -(J_0\omega^0 + J_i\omega^i + K_j\omega^j_0) \wedge \omega^0 \wedge (\psi - P\omega) \\ &= -J_i\omega^i \wedge \omega^0 \wedge \psi - K_j\omega^j_0 \wedge \omega^0 \wedge \psi + K_j\omega^j_0 \wedge \omega^0 \wedge P\omega \\ &= -J_j\omega^j_0 \wedge \Omega - K_j\omega^j_0 \wedge \omega^0 \wedge \psi + K_j P\omega^j_0 \wedge \Omega \end{aligned} \tag{2.10}$$

By setting (2.9) and (2.10) equal I can then conclude that $K_j = 0$ and $J_j = -B_j$, so that

$$\phi = J_0\omega^0 - B_j\omega^j.$$

Taking exterior derivative yields

$$\begin{aligned} d\varphi =& dJ_0 \wedge \omega^0 + J_0 d\omega^0 - dB_j \wedge \omega^j - B_j d\omega^j \\ =& dJ_0 \wedge \omega^0 + J_0 d\omega^0 - \big((D_{0j} - A_j)\omega^0 + S_{jk}\omega^k + T_{jk}\omega^k_0 + B_k\omega^k_j\big) \wedge \omega^j + B_j\omega^j_a \wedge \omega^a \\ =& dJ_0 \wedge \omega^0 + J_0 d\omega^0 - (D_{0j} - A_j)\omega^0 \wedge \omega^j - S_{jk}\omega^k \wedge \omega^j - T_{jk}\omega^k_0 \wedge \omega^j - B_k\omega^k_j \wedge \omega^j \\ & + B_j\omega^j_0 \wedge \omega^0 + B_j\omega^j_k \wedge \omega^k \\ \equiv& - S_{jk}\omega^k \wedge \omega^j - T_{jk}\omega^k_0 \wedge \omega^j \mod (\omega^0) \end{aligned} \tag{2.11}$$

According to Theorem 2.2.2, I can now conclude:

Proposition 2.2.3 (*)**.** *Using the notation established above, the prescribed mean curvature system* $(\mathcal{S}(M), \mathcal{I}_P)$ *is locally Euler-Lagrange provided that* $S_{jk} = S_{kj}$ *and* $T_{jk} = \delta_{jk} \cdot F$ *for some smooth function* F*.*

It is worth pointing out that if P were defined on M alone, one would in particular have $B_j = 0$ for all j, and so any basic prescribed mean curvature system is immediately locally Euler-Lagrange. I claim that conversely a locally Euler-Lagrange system is, in a sense, basic. More precisely:

Theorem 2.2.4. *A prescribed mean curvature system that is locally Euler-Lagrange is locally conformally equivalent to a basic prescribed mean curvature system.*

The proof calls for several intermediate results, beginning with the observation that φ above can be made into an honest closed form on $\mathcal{F}_+(M)$ as opposed to only being closed modulo the contact ideal. I claim:

Lemma 2.2.5. *The* 1*-form* $\varphi = F\omega^0 - B_j\omega^j$ *is closed on* $\mathcal{F}_+(M)$*.*

Proof. I will mimic a computation from [5]: Equation (2.11) shows that in particular

$$d(-B_j\omega^j) \equiv -F d\omega^0 \mod (\omega^0).$$

And so, for some $\alpha \in \Omega^1$,

$$\begin{aligned} d(-B_j\omega^j) &= -F d\omega^0 + \alpha \wedge \omega^0 \\ &= d(-F\omega^0) + (\alpha + dF) \wedge \omega^0. \end{aligned}$$

Consequently, if I set $\tilde{\alpha} = \alpha + dF$, I then have that

$$d(-B_j\omega^j + F\omega^0) = \tilde{\alpha} \wedge \omega^0.$$

It remains to show that the right-hand side is equal to zero. I first take the exterior derivative on both sides. Then

$$0 = d\tilde{\alpha} \wedge \omega^0 - \tilde{\alpha} \wedge d\omega^0$$

which implies that

$$\tilde{\alpha} \wedge d\omega^0 \equiv 0 \mod (\omega^0).$$

It now suffices to observe that the map $(\wedge d\omega^0) : \Omega^k(\mathcal{F}_+(M)) \to \Omega^{k+2}(\mathcal{F}_+(M))$ is injective, whence $\tilde{\alpha} \wedge \omega^0 = 0$ which completes the proof. □

This fact allows me to rephrase the condition of $(\mathcal{S}(M), \mathcal{I}_P)$ being locally Euler-Lagrange as an exterior differential system in the following way: Given that φ is closed, the Poincaré Lemma ensures that locally there exists a function u for which

$$du = F\omega^0 - B_j\omega^j \tag{2.12}$$

On the manifold $\mathcal{F}_+(M) \times \mathbb{R}^{4+2n}$, I then define two 1-forms

$$\begin{aligned}\Phi_0 &= dP - A_0\omega^0 - A_i\omega^i - B_j\omega_0^j \\ \Phi_1 &= du - F\omega^0 + B_j\omega^j\end{aligned}$$

generating the differential ideal $\mathcal{P} = (\Phi_0, \Phi_1)$.

Proposition 2.2.6. *The linear Pfaffian system $(\mathcal{F}_+(M) \times \mathbb{R}^{4+2n}, \mathcal{P})$ is involutive. In the real analytic category, local integral manifolds depend on 2 functions of $(n{+}1)$ variables.*

Proof. Taking exterior derivatives yields

$$d\begin{pmatrix}\Phi_0 \\ \Phi_1\end{pmatrix} \equiv -\begin{pmatrix} dA_0 & dA_i & dB_j & 0 \\ dF & -dB_i & 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^0 \\ \omega^i \\ \omega_0^j \\ \omega_l^k\end{pmatrix} + T \mod (\Phi_0, \Phi_1),$$

where the torsion term T is given by

$$T = \begin{pmatrix} A_0\omega_k^0 \wedge \omega^k + A_i\omega_a^i \wedge \omega^a + B_j\left[\omega_k^j \wedge \omega_0^k - \frac{1}{2}R_{0ab}^j\omega^a \wedge \omega^b\right] \\ F\omega_k^0 \wedge \omega^k - B_j\omega_a^j \wedge \omega^a \end{pmatrix}.$$

To see whether torsion can be absorbed, make the substitutions

$$\begin{aligned}\alpha_0 &= dA_0 - \left(C_{00}\omega^0 + C_{0k}\omega^k + D_{0k}\omega_0^k + \tfrac{1}{2}D_{0k}^l\omega_l^k\right) \\ \alpha_i &= dA_i - \left(C_{i0}\omega^0 + C_{ik}\omega^k + D_{ik}\omega_0^k + \tfrac{1}{2}D_{ik}^l\omega_l^k\right) \\ \beta_j &= dB_j - \left(S_{j0}\omega^0 + S_{jk}\omega^k + T_{jk}\omega_0^k + \tfrac{1}{2}T_{jk}^l\omega_l^k\right) \\ \zeta &= dF - \left(U_0\omega^0 + U_k\omega^k + V_k^0\omega_0^k + \tfrac{1}{2}V_k^l\omega_l^k\right)\end{aligned}$$

with the obvious skew-symmetries. Then

$$d\Phi_0 = -\begin{pmatrix}\alpha_0 & \alpha_i & \beta_j & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^0 \\ \omega^i \\ \omega_0^j \\ \omega_l^k\end{pmatrix},$$

provided that

$$\begin{aligned}
C_{ja} - C_{aj} &= B_i R^i_{0ja} \\
D^j_{ik} &= \delta_{ij} A_k - \delta_{ik} A_j \\
D_{ij} &= S_{ji} - A_0 \delta_{ij} \\
S_{k0} &= D_{0k} - A_k \\
D^k_{0j} &= 0 \\
T^k_{ij} &= \delta_{ik} B_j - \delta_{ij} B_k \\
T_{jk} - T_{kj} &= 0.
\end{aligned}$$

Consequently

$$\begin{aligned}
d\Phi_1 = &- [\zeta + U_0\omega^0 + U_k\omega^k + V^0_k\omega^k_0 + \tfrac{1}{2}V^l_k\omega^k_l] \wedge \omega^0 \\
&+ [\beta_i + (D_{0i} - A_i)\omega^0 + S_{ik}\omega^k + T_{ik}\omega^k_0 + B_k\omega^k_i] \wedge \omega^i \\
&+ F\omega^0_k \wedge \omega^k - B_j\omega^j_a \wedge \omega^a \\
= &- \zeta \wedge \omega^0 \\
&+ (U_i + D_{0i} - A_i)\omega^0 \wedge \omega^i \\
&+ (V^0_k + B_k)\omega^0 \wedge \omega^k_0 \\
&+ (\tfrac{1}{2}V^l_k)\omega^0 \wedge \omega^k_l \\
&+ (-S_{ik})\omega^i \wedge \omega^k \\
&+ (-T_{ik} + \delta_{ik}F)\omega^i \wedge \omega^k_0 \\
&+ (-B_k + B_k)\omega^i \wedge \omega^k_i,
\end{aligned}$$

so that additionally

$$\begin{aligned}
U_i &= -D_{0i} + A_i \\
S_{ik} - S_{ki} &= 0 \\
V^0_k &= -B_k \\
V^l_k &= 0 \\
T_{ik} &= \delta_{ik} \cdot F.
\end{aligned}$$

Under these hypotheses,

$$d\begin{pmatrix} \Phi_0 \\ \Phi_1 \end{pmatrix} = -\begin{pmatrix} \alpha_0 & \alpha_i & \beta_j & 0 \\ \zeta & -\beta_i & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^0 \\ \omega^i \\ \omega^j_0 \\ \omega^k_l \end{pmatrix},$$

and so $s_0 = \ldots = s_{n+1} = 2$ while s_{n+2} and all higher characters are 0. Consequently,

$$1s_1 + 2s_2 + \ldots + (n+1)s_{n+1} = (n+1)(n+2).$$

The result will now follow from Cartan's test provided I am able to show that this number equals the fiber dimension of $V_m(\mathcal{P})$. Write

$$\begin{aligned}
\alpha_0 &= a_{00}\omega^0 + a_{0k}\omega^k + a^0_{0k}\omega^k_0 + \tfrac{1}{2}a^l_{0k}\omega^k_l \\
\alpha_i &= a_{i0}\omega^0 + a_{ik}\omega^k + a^0_{ik}\omega^k_0 + \tfrac{1}{2}a^l_{ik}\omega^k_l \\
\beta_j &= b_{j0}\omega^0 + b_{jk}\omega^k + b^0_{jk}\omega^k_0 + \tfrac{1}{2}b^l_{jk}\omega^k_l \\
\zeta &= f_0\omega^0 + f_k\omega^k + f^0_k\omega^k_0 + \tfrac{1}{2}f^l_k\omega^k_l
\end{aligned}$$

with the usual skew-symmetry assumptions on a^l_{0k}, a^l_{ik}, b^l_{jk} and f^l_k. The equation $d\Phi_0 = 0$ imposes

$$\begin{aligned}
a_{0k} &= a_{k0} \\
a^0_{0k} &= b_{k0} \\
a^l_{0k} &= 0 \\
a_{ik} &= a_{ki} \\
a^0_{ik} &= b_{ki} \\
a^l_{ik} &= 0 \\
b^0_{jk} &= b^0_{kj} \\
b^l_{jk} &= 0,
\end{aligned}$$

while the equation $d\Phi_1 = 0$ implies that further

$$\begin{aligned}
f_k &= b_{k0} \\
f^0_k &= 0 \\
f^l_k &= 0 \\
b_{ik} &= b_{ki},
\end{aligned}$$

leaving precisely $(n+1)(n+2)$ degrees of freedom in the fiber, which completes the proof. □

Proposition 2.2.6 suggests that for a prescribed mean curvature system that is locally Euler-Lagrange, the function P locally determines, and is determined by, two functions that are defined entirely on M. In order to establish one direction of this claim, I let s denote the map

$$\begin{array}{ccc}
\mathcal{F}_+(M) & \xrightarrow{\ s\ } & \bar{\mathcal{F}}_+(M) \\
& {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\bar{\pi}} & \\
& M &
\end{array}$$

that scales a g-orthonormal frame (e_a) by a conformal factor $e^{\frac{u}{n}}$ with u as in (2.12) above. The resulting frame $(\bar{e}_a)$ over M is then orthonormal with respect to the scaled metric $\bar{g} = e^{-\frac{2u}{n}} g$. I will let $\bar{\omega}^a$, $\bar{\omega}^a_b$ to be the canonical coframing on $\bar{\mathcal{F}}_+(M)$. Then, under pull-back by s to $\mathcal{F}_+(M)$ one has:

Lemma 2.2.7.

$$\begin{aligned} \bar{\omega}^a &= e^{-\frac{u}{n}}\omega^a \\ \bar{\omega}^j_0 &= \omega^j_0 - \tfrac{1}{n}\left(F\omega^j + B_j\omega^0\right) \end{aligned}$$

Proof. The first formula follows straight from the definition since for $\xi \in T_y\mathcal{F}_+(M)$

$$s^*(\bar{\omega}^a)(\xi) = \bar{\omega}^a(s_*\xi) = \bar{g}(\bar{\pi}_* s_* \xi, \bar{e}_a) = \bar{g}(\pi_*\xi, \bar{e}_a) = e^{-\frac{u}{n}}\omega^a.$$

Taking exterior derivative of the equation above in the case when $a = 0$ and applying the structure equations then implies that $\bar{\omega}^0_j \wedge \omega^j + \frac{1}{n}u_j\omega^0 \wedge \omega^j - \omega^0_j \wedge \omega^j = 0$, where I have written $du = u_a\omega^a$. Cartan's Lemma assures the existence of functions $h_{jk} = h_{kj}$ so that

$$\bar{\omega}^0_j - \omega^0_j = h_{jk}\omega^k - \tfrac{1}{n}u_j\omega^0.$$

Now, both $\bar{\omega}^0_j$ and ω^0_j are entries of $(n{+}1) \times (n{+}1)$ skew-symmetric matrices, so the same skew-symmetry must hold for the right-hand side above. Recalling that $u_0 = F$ gives the desired result. ☐

Analogously to (1.1) in the previous chapter, I define differential forms

$$\begin{aligned} \bar{\omega} &= \bar{\omega}^1 \wedge \ldots \wedge \bar{\omega}^n \\ \bar{\Omega} &= \bar{\omega}^0 \wedge \ldots \wedge \bar{\omega}^n \\ \bar{\omega}_{(i)} &= (-1)^{i-1}\bar{\omega}^1 \wedge \ldots \wedge \hat{\bar{\omega}}^i \wedge \ldots \wedge \bar{\omega}^n \\ \bar{\psi} &= -\bar{\omega}^i_0 \wedge \bar{\omega}_{(i)} \end{aligned}$$

on $\bar{\mathcal{F}}_+(M)$. These can, according to Lemma 2.2.7, be expressed in terms of their $\mathcal{F}_+(M)$ counterparts as follows:

$$\begin{aligned} \bar{\omega} &= e^{-u}\omega \\ \bar{\Omega} &= e^{-\frac{n+1}{n}u}\Omega \\ \bar{\omega}_{(i)} &= e^{-\frac{n-1}{n}u}\omega_{(i)} \\ \bar{\psi} &= e^{-\frac{n-1}{n}u}\left(\psi + F\omega + \tfrac{1}{n}\sum_i B_i\omega^0 \wedge \omega_{(i)}\right). \end{aligned} \tag{2.13}$$

To show how two functions defined on the base M determine a locally Euler-Lagrange prescribed mean curvature system, I will start with a basic prescribed mean curvature system (with respect to the metric $\bar{g}$) whose corresponding differential ideal will be denoted by

$$\bar{\mathcal{I}}_{\bar{P}} = (\bar{\omega}^0, \bar{\psi} - \bar{P}\bar{\omega}).$$

Since $\bar{\mathcal{I}}_{\bar{P}}$ is basic, $\bar{P}$ is defined entirely on M, and so the corresponding Poincaré-Cartan form

$$\bar{\Theta} = -\bar{\omega}^0 \wedge \left(\bar{\psi} - \bar{P}\bar{\omega}\right)$$

must satisfy $d\bar{\Theta} = 0$. When pulled back to $\mathcal{F}_+(M)$ however, $\bar{\Theta}$ takes the form

$$\begin{aligned}\bar{\Theta} &= -e^{-\frac{u}{n}}\omega^0 \wedge \left(e^{-\frac{(n-1)}{n}u}\left(\psi + F\omega + \tfrac{1}{n}B_i\omega^0 \wedge \omega_{(i)}\right) - \bar{P}e^{-u}\omega\right)\\ &= -e^{-u}\omega^0 \wedge \left(\psi - \left(-F + e^{-\frac{u}{n}}\bar{P}\right)\omega\right).\end{aligned}$$

It follows, given two functions $u, \bar{P}$ on M, that by defining

$$P = -F + e^{-\frac{u}{n}}\bar{P}, \tag{2.14}$$

the corresponding prescribed mean curvature system

$$(\mathcal{S}(M), \mathcal{I}_P)$$

defined with respect to the metric g and with Poincaré-Cartan form $\Theta = -\omega^0 \wedge (\psi - P\omega)$ has the property that

$$d\Theta = \varphi \wedge \Theta$$

for $\varphi = du$, and is therefore, by definition, locally Euler-Lagrange.

Proof of Theorem 2.2.4. The argument above shows how $u, \bar{P}$ determine a locally Euler-Lagrange system. To prove the theorem, it remains to argue that every prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ that is locally Euler-Lagrange arises this way. By computing

$$dP = A_0\omega^0 + A_i\omega^i + B_j\omega_0^j$$

I have an explicit description for the functions B_j. Next, I declare F to be the function that makes the form $F\omega^0 - B_j\omega^j$ exact. There is a unique such F, for if $G\omega^0 - B_j\omega^j$ were also exact, $(F - G)\omega^0$ would then be an exact multiple of ω^0. The form ω^0, however, is a contact form, and so necessarily $F - G = 0$. Finally, the differential equation $du = F\omega^0 - B_j\omega^j$ determines u (up to a constant), and from the formula $P = -F + e^{-\frac{u}{n}}\bar{P}$ I can then recover $\bar{P}$ as desired. □

The decomposition of the prescription function of a locally Euler-Lagrange system given by (2.14) suggests the following definition:

Definition 2.2.8. Let $(\mathcal{S}(M), \mathcal{I}_P)$ be a locally Euler-Lagrange prescribed mean curvature system. The system will be referred to as *locally conformally minimal* if there exists a choice of u and $\bar{P}$ in 2.14 such that $\bar{P} = 0$.

An interesting subfamily of systems that are locally conformally minimal arises when the underlying prescribed mean curvature system is assumed to correspond to solitons of the mean curvature flow of hypersurfaces in $\mathbb{R}^{n+1}$. It is a well-known fact (see for example [13] or [16] for details) that translating solitons of the mean curvature flow are minimal hypersurfaces with respect to a scaled metric. In order to apply Theorem 2.2.4, I will return to the prescribed mean curvature system

$$\mathcal{M}_V = (\omega^0, \psi + \omega^0(\tilde{W})\omega) \tag{2.15}$$

established in the previous chapter. As before, $\tilde{W}$ refers to the lift of the vector field W to $\mathcal{F}_+(\mathbb{R}^{n+1})$, and $\omega^0(\tilde{W})$ is the pull-back of a well-defined function on $\mathcal{S}(\mathbb{R}^{n+1})$ which, for the sake of keeping notation simple, I will also denote by $\omega^0(\tilde{W})$. In [15] Hungerbühler and Mettler refer to integral manifolds of the system (2.15) as *W-pseudosolitons.* They prove[2]:

Theorem 2.2.9 (Hungerbühler, Mettler)**.** *The W-pseudosoliton is equivalent to a minimal hypersurface if and only if W is a gradient vector field.*

I will merely illustrate how a local version of this result can be read off immediately from Theorem 2.2.4 by the following example: I let g be the standard metric on $\mathbb{R}^{n+1}$, and further I assume $\mathbb{R}^{n+1}$ has coordinates $x = (x^1, \ldots, x^{n+1})$. If $W = \frac{\partial}{\partial x^1}$ represents translation in the x^1-direction, then $W = \nabla u$ where $u : \mathbb{R}^{n+1} \to \mathbb{R}$ given by $u(x) = x^1$. Writing $du = F\omega^0 - B_j\omega^j$ (for some coefficient functions F, B_j) then F must be, by definition, equal to $\omega^0(\tilde{W})$. According to Theorem 2.2.4 (using the same notation as in the proof) I can then deduce that $P = -F$ and consequently $\bar{P} = 0$. In other words, the W-pseudosoliton corresponding to the system

$$\mathcal{M}_V = (\omega^0, \psi + \omega^0(\tilde{W})\omega) = (\omega^0, \psi - P\omega)$$

must be locally equivalent to the minimal hypersurface prescribed mean curvature system

$$\bar{\mathcal{I}}_P = (\bar{\omega}^0, \bar{\psi})$$

with respect to the scaled metric $\bar{g} = e^{-\frac{2}{n}x^1}g$.

The converse statement is also true, for if one started with a minimal hypersurface system with $\bar{P} = 0$, then necessarily $P = -F$ and, unravelling the definitions, W would correspond to the gradient of the function u with $du = F\omega^0 - B_j\omega^j$.

[2]Their theorem, as well as the discussion here, holds for hypersurfaces of any oriented Riemannian manifold and not just $\mathbb{R}^{n+1}$.

2.3 A Global Obstruction

Given that Theorem 2.2.9 is a global result, one might wonder whether or not there exists a global recoding of Theorem 2.2.4. It turns out that this is not the case as there is a natural obstruction to Theorem 2.2.4 being a global result. This obstruction lies within the function u found in the proof, or more precisely, in the cohomology class

$$[F\omega^0 - B_j\omega^j] \in H^1_{dR}(M)$$

for u may not be well-defined globally. In this section I will construct a prescribed mean curvature system that is locally Euler-Lagrange for which this cohomology class is non-trivial.

First I let $M = \mathbb{R}^3\backslash\{0\}$ endowed with the standard metric. For $\lambda > 1$, $f : M \to M$ shall denote the function $f(x) = \lambda x$. Then $\mathcal{S}(M) = M \times S^2$, and an element of $\mathcal{S}(M)$ will consequently be written as (x, v). There is a natural group action by a group G, say, acting on $\mathcal{S}(M)$via scaling by λ and rotations by R, say, as follows:

$$\lambda \cdot (x, v) = (\lambda x, v)$$
$$R \cdot (x, v) = (Rx, Rv)$$

One can then show that the 1-form θ defined via

$$\theta = \tfrac{1}{\sqrt{x \cdot x}} v \cdot dx$$

is a G-invariant contact form on $\mathcal{S}(M)$ with the property that an oriented hypersurface $\Sigma^2 \to M$ whose outward unit normal at x is given by v is a Legendre submanifold of $(\mathcal{S}(M), (\theta))$. Conversely, every Legendre submanifold that is transverse to the projection $\mathcal{S}(M) \to M$ gives rise to such an oriented immersed hypersurface $\Sigma^2 \to M$.

Next, I seek natural G-invariant 2-forms on $\mathcal{S}(M)$. The potential candidate

$$\tfrac{1}{2} v \cdot (dx \times dx)$$

pulls back to Σ to give the area form ω, say, while the 2-form

$$-v \cdot (dv \times dx)$$

pulls back to $H\omega$. However, these 2-forms are only invariant under rotations but not scaling. Therefore I define

$$\Upsilon_0 = \tfrac{1}{2(x \cdot x)} v \cdot (dx \times dx)$$
$$\Upsilon_1 = -\tfrac{1}{\sqrt{x \cdot x}} v \cdot (dv \times dx).$$

Finally I set

$$\mathcal{I} = (\theta, \Upsilon_1 - P\Upsilon_0).$$

Then I claim:

Proposition 2.3.1. *There exists a choice of P for which the prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I})$ has the property that it is locally Euler-Lagrange, but that it cannot be globally equivalent to a basic prescribed mean curvature system.*

To see why this is true, I first point out that there is a natural surjection

$$\mathrm{SO}(3) \times \mathbb{R}^+ \times S^1 \to \mathcal{S}(M)$$

given by

$$((e_1, e_2, e_3), r, \phi) \mapsto (re_1, \cos(\phi)e_1 + \sin(\phi)e_2),$$

where $e_3 = e_1 \times e_2$. It turns out to be favorable to perform computations on $\mathrm{SO}(3) \times \mathbb{R}^+ \times S^1$. To do this, I let η_i^j denote the left-invariant 1-forms of SO(3) so that

$$de_i = \eta_i^j e_j \tag{2.16}$$
$$d\eta_i^j = -\eta_k^j \wedge \eta_i^k. \tag{2.17}$$

Equations (2.16) and (2.17) then allow me to rewrite θ, Υ_0 and Υ_1 as

$$\begin{aligned}
\theta &= \cos(\phi)\tfrac{dr}{r} + \sin(\phi)\eta_1^2 \\
\Upsilon_0 &= \cos(\phi)\eta_1^2 \wedge \eta_1^3 - \sin(\phi)\tfrac{dr}{r} \wedge \eta_1^3 \\
\Upsilon_1 &= -2\cos^2(\phi)\eta_1^2 \wedge \eta_1^3 - \cos^2(\phi)d\phi \wedge \eta_1^3 - \sin(\phi)\cos(\phi)\eta_1^2 \wedge \eta_2^3 - \sin^2(\phi)d\phi \wedge \eta_1^3 \\
&\quad - \sin^2(\phi)\eta_1^2 \wedge \eta_1^3 + \sin(\phi)\cos(\phi)\tfrac{dr}{r} \wedge \eta_1^3 + \sin^2(\phi)\tfrac{dr}{r} \wedge \eta_2^3.
\end{aligned}$$

If Φ denotes the 2-form $\Phi = \Upsilon_1 - P\Upsilon_0$, and Θ denotes the candidate Poincaré-Cartan form $\Theta = -\theta \wedge \Phi$, I claim:

Lemma 2.3.2. *There exists a choice of $P = P(\phi)$ for which $d\Theta = \frac{dr}{r} \wedge \Theta$.*

Proof of Proposition 2.3.1. First of all, the form Φ is primitive modulo the contact ideal, as

$$\begin{aligned}
d\theta \wedge \Phi &= \sin(\phi)d\phi \wedge \tfrac{dr}{r} \wedge \eta_1^2 \wedge \eta_1^3 \\
&= (\cos(\phi)\tfrac{dr}{r} + \sin(\phi)\eta_1^2) \wedge d\phi \wedge \tfrac{dr}{r} \wedge \eta_1^3 \\
&\equiv 0 \mod (\theta)
\end{aligned}$$

Therefore, assuming Lemma 2.3.2 for the time being, this particular choice of P then has the property that $d\Theta = \frac{dr}{r} \wedge \Theta$, and thus $d(r^{-1}\Theta) = 0$. By Theorem 2.2.2 this means that for this particular P, $(\mathcal{S}(M), \mathcal{I})$ is locally Euler-Lagrange. Since Θ is defined entirely in terms of forms that are invariant under G, it is still a well-defined, closed Poincaré-Cartan form for the quotient space $S^1 \times S^2$. The function r^{-1}, however, does not drop to the quotient by f. Equivalently, the form $\frac{dr}{r}$, while well-defined for the quotient, cannot be globally exact as claimed. □

It now remains to prove:

Proof of Lemma 2.3.2. Computing on $\mathrm{SO}(3) \times \mathbb{R}^+ \times S^1$ I have that

$$\begin{aligned}\Theta =&(\cos^3(\phi)+\sin^2(\phi)\cos(\phi))\tfrac{dr}{r}\wedge d\phi\wedge\eta_1^3\\&+(2\cos^3(\phi)+2\sin^2(\phi)\cos(\phi)+P)\tfrac{dr}{r}\wedge\eta_1^2\wedge\eta_1^3\\&+(\sin(\phi)\cos^2(\phi)+\sin^3(\phi))\tfrac{dr}{r}\wedge\eta_1^2\wedge\eta_2^3\\&-(\sin(\phi)\cos^2(\phi)+\sin^3(\phi))d\phi\wedge\eta_1^2\wedge\eta_1^3\end{aligned}$$

and so

$$\tfrac{dr}{r}\wedge\Theta = -(\sin(\phi)\cos^2(\phi)+\sin^3(\phi))\tfrac{dr}{r}\wedge d\phi\wedge\eta_1^2\wedge\eta_1^3$$

whereas

$$d\Theta = -(2\sin(\phi)\cos^2(\phi)+2\sin^3(\phi)-P')d\phi\wedge\tfrac{dr}{r}\wedge\eta_1^2\wedge\eta_1^3$$

Equating the forms $\frac{dr}{r}\wedge\Theta$ and $d\Theta$ and solving the ordinary differential equation for P suggests one takes $P = -3\cos(\phi)$, which completes the proof. □

Chapter 3

Conservation Laws for Minimal Surfaces

3.1 The General Set-Up

In this chapter, I will examine conservation laws for minimal surfaces in a Riemannian 3-manifold. As the underlying prescribed mean curvature system is Euler-Lagrange, Theorem 1.4.2 guarantees that every symmetry of the metric will give a conservation law. The motivation for this chapter is then to answer whether or not all classical conservation laws for minimal surfaces arise this way.

Throughout this chapter, n is assumed to be 2. The prescribed mean curvature system corresponding to a minimal surface $\Sigma^2 \subset M^3$ takes the form

$$(\mathcal{S}(M), \mathcal{I}_0)$$

with the ideal $\mathcal{I}_0$ being generated as

$$\mathcal{I}_0 = (\omega^0, \psi).$$

With this notation in place the main result of this chapter can be formulated as follows:

Theorem 3.1.1. *Conservation laws for the minimal surface prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_0)$ come from symmetries of the metric except when M is flat.*

This theorem has an immediate consequence. If one started with a prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ in dimension $n = 2$ that was locally Euler-Lagrange as well as locally conformally minimal (for example a soliton of the mean curvature flow), then, taking u to be the function as in (2.14), one has:

Corollary 3.1.2. *If $(\mathcal{S}(M), \mathcal{I}_P)$ is locally conformally minimal, all conservation laws come from symmetries of the metric unless $(M, \bar{g} = e^{-u}g)$ happens to be flat.*

Proof. This is an immediate consequence of Theorems 3.1.1 and 2.2.4. □

To prove Theorem 3.1.1, I will set up an exterior differential system whose integral manifolds correspond to conservation laws of the system $(\mathcal{S}(M), \mathcal{I}_0)$. I will show that this system is not involutive, and that prolongation results in non-absorbable torsion terms. Setting these equal to zero will impose that one of the following cases must occur:

- M must be flat; the special case when M is a space form will be analyzed separately in the following section.
- The linear Pfaffian system obtained after the second prolongation implies the existence of a corresponding symmetry of the metric.

To begin, a conservation law for $(\mathcal{S}(M), \mathcal{I}_0)$ is, by definition, a 1-form $\varphi \in \Omega^1(\mathcal{S}(M))$ with the property that $d\varphi \in \mathcal{I}_0$ (and of course φ is assumed to be neither contained in $\mathcal{I}_0$ nor exact). Equivalently, one can (locally) think of a conservation law as being a closed 2-form that is contained in $\mathcal{I}_0$, and the latter perspective is the one I will adapt throughout this chapter.

Clearly $\mathcal{I}_0^2$ is generated algebraically by the forms $d\omega^0, \psi$, and $\omega^0 \wedge \gamma$ where $\gamma \in \Omega^1(\mathcal{S}(M))$. Now, since for any function f,

$$f d\omega^0 = d(f\omega^0) - df \wedge \omega^0,$$

one can trade off any multiple of $d\omega^0$ (modulo an exact form) with a multiple of ω^0. Consequently, I define

$$\alpha = u_1\omega^0 \wedge \omega^1 + u_2\omega^0 \wedge \omega^2 + u_3\omega^0 \wedge \omega_0^1 + u_4\omega^0 \wedge \omega_0^2 + u_5\psi$$

for some functions $u_1, \ldots, u_5$. Given that α is expressed purely in terms of forms that are semi-basic with respect to the bundle projection $\nu : \mathcal{F}_+(M) \to \mathcal{S}(M)$, one may as well assume that α lives on $\mathcal{F}_+(M)$, for any closed form of this kind will be well-defined on $\mathcal{S}(M)$.

On the manifold $\mathcal{F}_+(M) \times \mathbb{R}^5$, I let $\mathcal{K}$ be the differential ideal generated by $\kappa = d\alpha$. The existence of a closed 2-form $\alpha \in \mathcal{I}$ is then encoded by the exterior differential system $(\mathcal{F}_+(M) \times \mathbb{R}^5, \mathcal{K})$ subject to the independence condition

$$\Omega_+ = \Omega \wedge \omega_0^1 \wedge \omega_0^2 \wedge \omega_1^2 \neq 0.$$

Given that one expects conservation laws to impose conditions on the metric, the following should not be surprising:

Lemma 3.1.3. *The exterior differential system $(\mathcal{F}_+(M) \times \mathbb{R}^5, \mathcal{K})$ is not involutive.*

Proof. Analogously to Theorem 2.1.1, I define $G_6(T(\mathcal{F}_+(M)\times\mathbb{R}^5),\Omega_+)$ to be the set of 6-planes on which Ω_+ is non-vanishing. $G_6(T(\mathcal{F}_+(M)\times\mathbb{R}^5),\Omega_+)$ is then a manifold of dimension 41 that can locally be described as follows: At a point $y\in\mathcal{F}_+(M)\times\mathbb{R}^5$, let $(U,(y_1,\ldots,y_6,u_1,\ldots,u_5))$ be a coordinate neighborhood containing y. Let

$$G_6(TU,\Omega_+)=\{E\in G_6(T_y(\mathcal{F}_+(M)\times\mathbb{R}^5)):y\in U,\quad \Omega_+|_E\neq 0\}$$

and let the smooth functions $p_{10},\ldots,p_{55}$ on $G_6(TU,\Omega_+)$ be defined via the equations

$$\begin{aligned}
du_1-p_{10}(E)\omega^0-p_{11}(E)\omega^1-p_{12}(E)\omega^2-p_{13}(E)\omega^1_0-p_{14}(E)\omega^2_0-p_{15}(E)\omega^2_1&=0\\
du_2-p_{20}(E)\omega^0-p_{21}(E)\omega^1-p_{22}(E)\omega^2-p_{23}(E)\omega^1_0-p_{24}(E)\omega^2_0-p_{25}(E)\omega^2_1&=0\\
du_3-p_{30}(E)\omega^0-p_{31}(E)\omega^1-p_{32}(E)\omega^2-p_{33}(E)\omega^1_0-p_{34}(E)\omega^2_0-p_{35}(E)\omega^2_1&=0\\
du_4-p_{40}(E)\omega^0-p_{41}(E)\omega^1-p_{42}(E)\omega^2-p_{43}(E)\omega^1_0-p_{44}(E)\omega^2_0-p_{45}(E)\omega^2_1&=0\\
du_5-p_{50}(E)\omega^0-p_{51}(E)\omega^1-p_{52}(E)\omega^2-p_{53}(E)\omega^1_0-p_{54}(E)\omega^2_0-p_{55}(E)\omega^2_1&=0.
\end{aligned}\tag{3.1}$$

Then $G_6(TU,\Omega_+)$, together with the y's, u's and p's, defines a coordinate chart for $G_6(T(\mathcal{F}_+(M)\times\mathbb{R}^5),\Omega_+)$. An integral element $E\in V_6(\mathcal{K},\Omega_+)$ is then characterized by the equations

$$\begin{aligned}
p_{12}-p_{21}-u_3R^1_{012}-u_4R^2_{012}-u_5R^1_{001}-u_5R^2_{002}&=0\\
p_{13}-p_{31}&=0\\
p_{14}-p_{41}-p_{50}&=0\\
p_{15}-u_2&=0\\
p_{23}-p_{32}+p_{50}&=0\\
p_{24}-p_{42}&=0\\
p_{25}+u_1&=0\\
p_{34}-p_{43}-2u_5&=0\\
p_{35}-u_4&=0\\
p_{45}+u_3&=0\\
p_{53}-u_4&=0\\
p_{51}+u_2&=0\\
p_{52}-u_1&=0\\
p_{54}+u_3&=0\\
p_{55}&=0.
\end{aligned}\tag{3.2}$$

Consequently, any $E\in V_6(\mathcal{K},\Omega_+)$ is ordinary and

$$\operatorname{codim}\left[V_6(\mathcal{K},\Omega_+),G_6(T(\mathcal{F}_+(M)\times\mathbb{R}^5),\Omega_+)\right]=15.$$

At $y \in \mathcal{F}_+(M) \times \mathbb{R}^5$, I define a flag

$$(0)_y = E_0 \subset E_1 \subset ... \subset E_6 \subset T_y(\mathcal{F}_+(M) \times \mathbb{R}^5)$$

as follows: First, let $W_0, \ldots, W_5$ be the vector fields dual to the 1-forms $\omega^0, \omega^1, \ldots, \omega_1^2$ respectively and write

$$\partial^1 = \frac{\partial}{\partial u_1} \ , \ \ldots \ , \ \partial^5 = \frac{\partial}{\partial u_5}.$$

Next, set

$$X_0 = W_0 + p_{10}\partial^1 + \ldots + p_{50}\partial^5$$
$$\vdots$$
$$X_5 = W_5 + p_{15}\partial^1 + \ldots + p_{55}\partial^5$$

where the p's satisfy Equations (3.2). Finally, for $k \geq 1$ set

$$E_k = \langle X_0, ..., X_{k-1} \rangle.$$

By definition

$$c_k = \text{codim}\left[H(E_k), T_y(\mathcal{F}_+(M) \times \mathbb{R}^5)\right]$$

when $k < 6$, while

$$c_6 = 5.$$

Clearly $\mathcal{K}^1 = \mathcal{K}^2 = (0)$ and therefore

$$c_0 = c_1 = 0. \tag{3.3}$$

Next, consider $\mathcal{K}^3$. From the condition $v \in H(E_2)$, one obtains the polar equation

$$(du_1 - p_{10}\omega^0 - p_{11}\omega^1 - p_{12}\omega^2 - p_{13}\omega_0^1 - p_{14}\omega_0^2 - u_2\omega_1^2)(v) = 0$$

and hence

$$c_2 = 1. \tag{3.4}$$

$\mathcal{K}^4$ is generated algebraically by forms $\gamma_1 \wedge \kappa$ for $\gamma \in \Omega^1$. Choosing $\gamma = \omega^0$ then implies that in addition one must have

$$(du_2 - p_{20}\omega^0 - (p_{12} - u_3R^1_{012} - u_4R^2_{012} - u_5R^1_{001} - u_5R^2_{002})\omega^1 - p_{22}\omega^2 - p_{23}\omega_0^1 - p_{24}\omega_0^2 + u_1\omega_1^2)(v) = 0$$

and so

$$c_3 = 2. \tag{3.5}$$

For $\mathcal{K}^5$ one considers forms $\gamma_2 \wedge \kappa$ for $\gamma_2 \in \Omega^2$. Taking $\gamma_2 = \omega^1 \wedge \omega^2$ gives rise to the additional polar equation

$$(du_3 - p_{30}\omega^0 - p_{13}\omega^1 - (p_{23} + p_{50})\omega^2 - p_{33}\omega_0^1 - p_{34}\omega_0^2 - u_4\omega_1^2)(v) = 0$$

while $\gamma_2 = \omega^0 \wedge \omega^1$ furnishes the equation

$$(du_5 - p_{50}\omega^0 + u_2\omega^1 - u_1\omega^2 - u_4\omega_0^1 + u_3\omega_0^2)(v) = 0$$

and so

$$c_4 = 4. \tag{3.6}$$

Finally, for $\mathcal{K}^6$, setting $v \lrcorner \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \kappa(X_0, \ldots, X_4) = 0$ shows there is one more polar equation, namely

$$(du_4 - p_{40}\omega^0 - (p_{14} - p_{50})\omega^1 - p_{24}\omega^2 - (p_{34} - 2u_5)\omega_0^1 - p_{44}\omega_0^2 + u_3\omega_1^2)(v) = 0$$

and so

$$c_5 = 5. \tag{3.7}$$

It follows from (3.3) - (3.7) that

$$c_0 + \ldots + c_5 = 12 \neq 15 = \text{codim}\left[V_6(\mathcal{K}, \Omega_+), G_6(T(\mathcal{F}_+(M) \times \mathbb{R}^5), \Omega_+)\right]$$

as claimed. □

As a result, one has to prolong. I let $\mathcal{K}^{(1)}$ denote the pull-back of the contact ideal on $G_6(T(\mathcal{F}_+(M) \times \mathbb{R}^5), \Omega_+)$ to the manifold $(\mathcal{F}_+(M) \times \mathbb{R}^5)^{(1)} = V_6(\mathcal{K}, \Omega_+)$. In other words

$$\mathcal{K}^{(1)} = (\theta_1, \ldots, \theta_5)$$

where

$$\begin{aligned}
\theta_1 =& du_1 - p_{10}\omega^0 - p_{11}\omega^1 - p_{12}\omega^2 - p_{13}\omega_0^1 - p_{14}\omega_0^2 - u_2\omega_1^2 \\
\theta_2 =& du_2 - p_{20}\omega^0 - (p_{12} - u_3R^1_{012} - u_4R^2_{012} - u_5R^1_{001} - u_5R^2_{002})\omega^1 - p_{22}\omega^2 - p_{23}\omega_0^1 - p_{24}\omega_0^2 + u_1\omega_1^2 \\
\theta_3 =& du_3 - p_{30}\omega^0 - p_{13}\omega^1 - (p_{23} + p_{50})\omega^2 - p_{33}\omega_0^1 - p_{34}\omega_0^2 - u_4\omega_1^2 \\
\theta_4 =& du_4 - p_{40}\omega^0 - (p_{14} - p_{50})\omega^1 - p_{24}\omega^2 - (p_{34} - 2u_5)\omega_0^1 - p_{44}\omega_0^2 + u_3\omega_1^2 \\
\theta_5 =& du_5 - p_{50}\omega^0 + u_2\omega^1 - u_1\omega^2 - u_4\omega_0^1 + u_3\omega_0^2
\end{aligned} \tag{3.8}$$

A closer inspection of the equations above, in particular the ω_0^2 and ω_0^1 coefficients for θ_3 and θ_4 respectively, suggests that one make a change of variables. This is motivated by the fact that calculations are being performed on $\mathcal{F}_+(M)$, i.e., the circle bundle over the unit sphere bundle $\mathcal{S}(M)$. Consequently, one expects calculations to be invariant under the circle action. And so it seems natural to chose notation so that it will split up equivariantly under this action. Consequently, I replace

$$p_{34} \mapsto p_{34} + u_5,$$

which will simplify the upcoming equations significantly. For θ_3 and θ_4 this means that these terms now take the form

$$\begin{aligned}\theta_3 =& du_3 - p_{30}\omega^0 - p_{13}\omega^1 - (p_{23}+p_{50})\omega^2 - p_{33}\omega_0^1 - (p_{34}+u_5)\omega_0^2 - u_4\omega_1^2 \\ \theta_4 =& du_4 - p_{40}\omega^0 - (p_{14}-p_{50})\omega^1 - p_{24}\omega^2 - (p_{34}-u_5)\omega_0^1 - p_{44}\omega_0^2 + u_3\omega_1^2.\end{aligned}$$

The analysis of the resulting linear Pfaffian system $((\mathcal{F}_+(M)\times\mathbb{R}^5)^{(1)}, \mathcal{K}^{(1)})$ will inevitably involve covariant derivatives of the curvature tensor. This calls for some additional notation. One thing that sets a Riemannian 3-manifold apart from those of higher dimension is the fact that in dimension 3 the Riemannian curvature tensor R is completely determined by the Ricci curvature tensor Ric, which is defined to be the contraction

$$R_{ab} = R^c_{acb}.$$

R_{ab} then defines a symmetric 3-by-3 matrix which, when written out, takes the from

$$\begin{pmatrix} -R^1_{001} - R^2_{002} & -R^2_{012} & R^1_{012} \\ -R^2_{012} & -R^1_{001} - R^2_{112} & -R^1_{002} \\ R^1_{012} & -R^1_{002} & -R^2_{002} - R^2_{112} \end{pmatrix}$$

At this point it is helpful to call upon a little representation theory of the Lie group SO(3). As is customary, I will let $\mathcal{H}_m$ denote the irreducible representations of SO(3). That is, for each m, $\mathcal{H}_m$ is the vector space of homogeneous, harmonic, complex-valued polynomials of degree m on $\mathbb{R}^3$. Its dimension is then given by $2m+1$.

The Riemann curvature tensor in dimension 3 can be decomposed as an element of $\mathcal{H}_0 \otimes \mathcal{H}_1$ as follows: I will choose respective bases $\{\mathfrak{a}\}$ and $\{\mathfrak{a}_{ab}\}$ of $\mathcal{H}_0$ and $\mathcal{H}_1$ by letting $\mathfrak{a}$ be the scalar curvature part, that is

$$\mathfrak{a} = \mathrm{tr}_g Ric = -2(R^1_{001} + R^2_{002} + R^2_{112})$$

and by writing $\mathfrak{a}_{ab}$ as the trace-free Ricci curvature part, i.e.

$$\mathfrak{a}_{ab} = R_{ab} - \tfrac{1}{3}\mathfrak{a} I_3.$$

Under these hypotheses, θ_2 now becomes

$$\theta_2 = du_2 - p_{20}\omega^0 - (p_{12} - u_3\mathfrak{a}_{02} + u_4\mathfrak{a}_{01} + u_5(\mathfrak{a}_{00} + \tfrac{1}{3}\mathfrak{a}))\omega^1 - p_{22}\omega^2 - p_{23}\omega_0^1 - p_{24}\omega_0^2 + u_1\omega_1^2.$$

Moreover, the Cartan structure equations for the connection forms turn into

$$\begin{aligned} d\omega_0^1 &= -\omega_2^1 \wedge \omega_0^2 + (\mathfrak{a}_{22} - \tfrac{1}{6}\mathfrak{a})\omega^0 \wedge \omega^1 - \mathfrak{a}_{12}\omega^0 \wedge \omega^2 + \mathfrak{a}_{02}\omega^1 \wedge \omega^2 \\ d\omega_0^2 &= -\omega_1^2 \wedge \omega_0^1 - \mathfrak{a}_{12}\omega^0 \wedge \omega^1 + (\mathfrak{a}_{11} - \tfrac{1}{6}\mathfrak{a})\omega^0 \wedge \omega^2 - \mathfrak{a}_{01}\omega^1 \wedge \omega^2 \\ d\omega_1^2 &= -\omega_0^2 \wedge \omega_1^0 + \mathfrak{a}_{02}\omega^0 \wedge \omega^1 - \mathfrak{a}_{01}\omega^0 \wedge \omega^2 + (\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})\omega^1 \wedge \omega^2. \end{aligned}$$

The next step is to write down derivatives of the $\mathfrak{a}$ and $\mathfrak{a}_{ab}$ introduced above. In view of the fact that $R \in \Gamma(TM \otimes T^*M \otimes T^*M \otimes T^*M)$ one has that

$$\nabla : R \mapsto \nabla R \in \Gamma(TM \otimes T^*M \otimes T^*M \otimes T^*M) \otimes \Gamma(T^*M).$$

Given that M is 3-dimensional one can identify its cotangent space T^*M with $\mathcal{H}_1$. According to the Clebsch-Gordan formula (see [20], p.70)

$$\mathcal{H}_m \otimes \mathcal{H}_1 = \mathcal{H}_{m-1} \oplus \mathcal{H}_m \oplus \mathcal{H}_{m+1} \tag{3.9}$$

as a representation space. In the case of the Riemann curvature tensor, (3.9) in conjunction with the Bianchi identity then implies that ∇R must take values in $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Moreover, the $\mathcal{H}_2$-piece corresponds to the Cotton tensor, which vanishes if and only if the manifold M is locally conformally flat.

To $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ I will associate respective bases $\{\mathfrak{b}_a\}$, $\{\mathfrak{b}_{ab}\}$ and $\{\mathfrak{b}_{abc}\}$ which I define as follows: Let

$$d\mathfrak{a} = \mathfrak{b}_a \omega^a.$$

Next let ϵ_{abc} be skew-symmetric in all indices and declare $\epsilon_{012} = 1$. Up to some constant λ one can then write

$$d\mathfrak{a}_{ab} = \mathfrak{a}_{ac}\omega^a_b + \mathfrak{a}_{cb}\omega^c_a + \left[\mathfrak{b}_{abc} + (\mathfrak{b}_{ad}\epsilon_{dbc} + \mathfrak{b}_{bd}\epsilon_{dac}) + \lambda(\mathfrak{b}_a\delta_{bc} + \mathfrak{b}_b\delta_{ac} - \tfrac{2}{3}\mathfrak{b}_c\delta_{ab})\right]\omega^c.$$

The fact that $d^2\omega^b_a = 0$ determines that $\lambda = \frac{1}{20}$. The structure equations for the $d\mathfrak{a}_{ab}$-terms can now be written explicitly as follows:

$$\begin{aligned}
d\mathfrak{a}_{00} =& (\mathfrak{b}_{000} + \tfrac{1}{15}\mathfrak{b}_0)\omega^0 + (\mathfrak{b}_{001} + 2\mathfrak{b}_{02} - \tfrac{1}{30}\mathfrak{b}_1)\omega^1 + (\mathfrak{b}_{002} - 2\mathfrak{b}_{01} - \tfrac{1}{30}\mathfrak{b}_2)\omega^2 + 2\mathfrak{a}_{01}\omega^1_0 + 2\mathfrak{a}_{02}\omega^2_0 \\
d\mathfrak{a}_{01} =& (\mathfrak{b}_{001} - \mathfrak{b}_{02} + \tfrac{1}{20}\mathfrak{b}_1)\omega^0 + (\mathfrak{b}_{011} + \mathfrak{b}_{12} + \tfrac{1}{20}\mathfrak{b}_0)\omega^1 + (\mathfrak{b}_{012} + \mathfrak{b}_{00} - \mathfrak{b}_{11})\omega^2 \\
& + (\mathfrak{a}_{11} - \mathfrak{a}_{00})\omega^1_0 + \mathfrak{a}_{12}\omega^2_0 + \mathfrak{a}_{02}\omega^2_1 \\
d\mathfrak{a}_{02} =& (\mathfrak{b}_{002} + \mathfrak{b}_{01} + \tfrac{1}{20}\mathfrak{b}_2)\omega^0 + (\mathfrak{b}_{012} + \mathfrak{b}_{22} - \mathfrak{b}_{00})\omega^1 + (\mathfrak{b}_{022} - \mathfrak{b}_{12} + \tfrac{1}{20}\mathfrak{b}_0)\omega^2 \\
& + \mathfrak{a}_{12}\omega^1_0 + (\mathfrak{a}_{22} - \mathfrak{a}_{00})\omega^2_0 - \mathfrak{a}_{01}\omega^2_1 \\
d\mathfrak{a}_{11} =& (\mathfrak{b}_{011} - 2\mathfrak{b}_{12} - \tfrac{1}{30}\mathfrak{b}_0)\omega^0 + (\mathfrak{b}_{111} + \tfrac{1}{15}\mathfrak{b}_1)\omega^1 + (\mathfrak{b}_{112} + 2\mathfrak{b}_{01} - \tfrac{1}{30}\mathfrak{b}_2)\omega^2 - 2\mathfrak{a}_{01}\omega^1_0 + 2\mathfrak{a}_{12}\omega^2_1 \\
d\mathfrak{a}_{12} =& (\mathfrak{b}_{012} + \mathfrak{b}_{11} - \mathfrak{b}_{22})\omega^0 + (\mathfrak{b}_{112} - \mathfrak{b}_{01} + \tfrac{1}{20}\mathfrak{b}_2)\omega^1 + (\mathfrak{b}_{122} + \mathfrak{b}_{02} + \tfrac{1}{20}\mathfrak{b}_1)\omega^2 \\
& - \mathfrak{a}_{02}\omega^1_0 - \mathfrak{a}_{01}\omega^2_0 + (\mathfrak{a}_{22} - \mathfrak{a}_{11})\omega^2_1 \\
d\mathfrak{a}_{22} =& (\mathfrak{b}_{022} + 2\mathfrak{b}_{12} - \tfrac{1}{30}\mathfrak{b}_0)\omega^0 + (\mathfrak{b}_{122} - 2\mathfrak{b}_{02} - \tfrac{1}{30}\mathfrak{b}_1)\omega^1 + (\mathfrak{b}_{222} + \tfrac{1}{15}\mathfrak{b}_2)\omega^2 - 2\mathfrak{a}_{02}\omega^2_0 - 2\mathfrak{a}_{12}\omega^2_1.
\end{aligned}$$

Given that $(\mathfrak{a}_{ab})$ is, by assumption, trace-free one expects $d\mathfrak{a}_{00} + d\mathfrak{a}_{11} + d\mathfrak{a}_{22} = 0$ and the equations above do indeed reflect that.

With this notation in place I will now start the analysis of the system $((\mathcal{F}_+(M)\times\mathbb{R}^5)^{(1)},\mathcal{K}^{(1)})$. Calculation yields

$$d\begin{pmatrix}\theta_5\end{pmatrix}\equiv-\begin{pmatrix}dp_{50} & 0 & 0 & 0 & 0 & 0\end{pmatrix}\wedge\begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega_0^1\\ \omega_0^2\\ \omega_1^2\end{pmatrix}+T_5\quad\text{mod }(\theta_1,\ldots,\theta_5),$$

where the torsion term T_5 is given by

$$\begin{aligned}T_5=&[p_{20}-u_4(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a})-u_3\mathfrak{a}_{12}]\omega^0\wedge\omega^1+[-p_{10}+u_4\mathfrak{a}_{12}+u_3(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a})]\omega^0\wedge\omega^2\\&+[u_2-p_{40}]\omega^0\wedge\omega_0^1+[p_{30}-u_1]\omega^0\wedge\omega_0^2+[-p_{22}-p_{11}-u_4\mathfrak{a}_{02}-u_3\mathfrak{a}_{01}]\omega^1\wedge\omega^2\\&+[2p_{50}-p_{14}-p_{23}]\omega^1\wedge\omega_0^1+[p_{13}-p_{24}]\omega^1\wedge\omega_0^2+[p_{13}-p_{24}]\omega^2\wedge\omega_0^1\\&+[2p_{50}+p_{14}+p_{23}]\omega^2\wedge\omega_0^2+[p_{33}+p_{44}]\omega_0^1\wedge\omega_0^2.\end{aligned}$$

Comparing the coefficients of $\omega^1\wedge\omega_0^1$ and $\omega^2\wedge\omega_0^2$ implies that $V_6(\mathcal{K}^{(1)},\Omega_+)=\varnothing$ unless one imposes that

$$p_{50}=0,$$

and this condition in turn has the consequence that

$$\begin{aligned}p_{20}-u_4(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a})-u_3\mathfrak{a}_{12}&=0\\-p_{10}+u_4\mathfrak{a}_{12}+u_3(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a})&=0\\u_2-p_{40}&=0\\p_{30}-u_1&=0\\-p_{22}-p_{11}-u_4\mathfrak{a}_{02}-u_3\mathfrak{a}_{01}&=0\\p_{13}-p_{24}&=0\\p_{14}+p_{23}&=0\\p_{33}+p_{44}&=0.\end{aligned}\tag{3.10}$$

Restricting to the manifold $\mathcal{F}_+(M)\times\mathbb{R}^{11}$, $\mathcal{K}^{(1)}$ is then generated by

$$\begin{aligned}\theta_1=&du_1-(u_4\mathfrak{a}_{12}+u_3(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a}))\omega^0-p_{11}\omega^1-p_{12}\omega^2-p_{13}\omega_0^1-p_{14}\omega_0^2-u_2\omega_1^2\\\theta_2=&du_2-(u_4(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a})+u_3\mathfrak{a}_{12})\omega^0-(p_{12}-u_3\mathfrak{a}_{02}+u_4\mathfrak{a}_{01}+u_5(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}))\omega^1\\&+(p_{11}+u_4\mathfrak{a}_{02}+u_3\mathfrak{a}_{01})\omega^2+p_{14}\omega_0^1-p_{13}\omega_0^2+u_1\omega_1^2\\\theta_3=&du_3-u_1\omega^0-p_{13}\omega^1+p_{14}\omega^2-p_{33}\omega_0^1-(p_{34}+u_5)\omega_0^2-u_4\omega_1^2\\\theta_4=&du_4-u_2\omega^0-p_{14}\omega^1-p_{13}\omega^2-(p_{34}-u_5)\omega_0^1+p_{33}\omega_0^2+u_3\omega_1^2\\\theta_5=&du_5+u_2\omega^1-u_1\omega^2-u_4\omega_0^1+u_3\omega_0^2.\end{aligned}$$

In order to absorb torsion, I will make the substitutions

$$\begin{aligned}
\xi_{11} &= dp_{11} - p_{110}\omega^0 - p_{111}\omega^1 - p_{112}\omega^2 - p_{113}\omega_0^1 - p_{114}\omega_0^2 - p_{115}\omega_1^2\\
\xi_{12} &= dp_{12} - p_{120}\omega^0 - p_{121}\omega^1 - p_{122}\omega^2 - p_{123}\omega_0^1 - p_{124}\omega_0^2 - p_{125}\omega_1^2\\
\xi_{13} &= dp_{13} - p_{130}\omega^0 - p_{131}\omega^1 - p_{132}\omega^2 - p_{133}\omega_0^1 - p_{134}\omega_0^2 - p_{135}\omega_1^2\\
\xi_{14} &= dp_{14} - p_{140}\omega^0 - p_{141}\omega^1 - p_{142}\omega^2 - p_{143}\omega_0^1 - p_{144}\omega_0^2 - p_{145}\omega_1^2\\
\xi_{33} &= dp_{33} - p_{330}\omega^0 - p_{331}\omega^1 - p_{332}\omega^2 - p_{333}\omega_0^1 - p_{334}\omega_0^2 - p_{335}\omega_1^2\\
\xi_{34} &= dp_{34} - p_{340}\omega^0 - p_{341}\omega^1 - p_{342}\omega^2 - p_{343}\omega_0^1 - p_{344}\omega_0^2 - p_{345}\omega_1^2.
\end{aligned}$$

By investigating the terms p_{130}, p_{140} and p_{340} this would in particular mean that

$$\begin{aligned}
p_{130} &= -p_{11} - u_4\mathfrak{a}_{02} - 2u_3\mathfrak{a}_{01} + (p_{34}-u_5)\mathfrak{a}_{12} + p_{33}(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a})\\
&= p_{11} - u_4\mathfrak{a}_{02} + (p_{34}+u_5)\mathfrak{a}_{12} - p_{33}(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a})\\
p_{140} &= -p_{12} - u_4\mathfrak{a}_{01} - p_{33}\mathfrak{a}_{12} + (p_{34}+u_5)(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a})\\
&= p_{12} + u_4\mathfrak{a}_{01} - p_{33}\mathfrak{a}_{12} + u_5(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}) - (p_{34}-u_5)(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a})\\
p_{340} &= -2p_{14} = 2p_{14}.
\end{aligned}$$

Again $V_6(\mathcal{K}^{(1)}, \Omega_+) = \varnothing$ unless one restricts $\mathcal{K}^{(1)}$ to $\mathcal{F}_+(M) \times \mathbb{R}^8$ by imposing that

$$\begin{aligned}
p_{11} &= -u_3\mathfrak{a}_{01} - u_5\mathfrak{a}_{12} - \tfrac{1}{2}p_{33}(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a})\\
p_{12} &= -u_4\mathfrak{a}_{01} + u_5(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a}) - \tfrac{1}{2}p_{34}(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a})\\
p_{14} &= 0.
\end{aligned}$$

So now

$$\begin{aligned}
\theta_1 =& du_1 - (u_4\mathfrak{a}_{12}+u_3(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a}))\omega^0 + (u_3\mathfrak{a}_{01}+u_5\mathfrak{a}_{12}+\tfrac{1}{2}p_{33}(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}))\omega^1\\
&- (-u_4\mathfrak{a}_{01}+u_5(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a})-\tfrac{1}{2}p_{34}(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}))\omega^2 - p_{13}\omega_0^1 - u_2\omega_1^2\\
\theta_2 =& du_2 - (u_4(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a})+u_3\mathfrak{a}_{12})\omega^0 + (u_3\mathfrak{a}_{02}+u_5(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a})+\tfrac{1}{2}p_{34}(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}))\omega^1\\
&- (-u_4\mathfrak{a}_{02}+u_5\mathfrak{a}_{12}+\tfrac{1}{2}p_{33}(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}))\omega^2 - p_{13}\omega_0^2 + u_1\omega_1^2\\
\theta_3 =& du_3 - u_1\omega^0 - p_{13}\omega^1 - p_{33}\omega_0^1 - (p_{34}+u_5)\omega_0^2 - u_4\omega_1^2\\
\theta_4 =& du_4 - u_2\omega^0 - p_{13}\omega^2 - (p_{34}-u_5)\omega_0^1 + p_{33}\omega_0^2 + u_3\omega_1^2\\
\theta_5 =& du_5 + u_2\omega^1 - u_1\omega^2 - u_4\omega_0^1 + u_3\omega_0^2.
\end{aligned}$$

By setting

$$C = \tfrac{1}{2}(\mathfrak{a}_{00} + \tfrac{1}{3}\mathfrak{a}), \tag{3.11}$$

I can write the tableau as:

$$d\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{pmatrix} \equiv -\begin{pmatrix} 0 & -Cdp_{33} & -Cdp_{34} & dp_{13} & 0 & 0\\ 0 & -Cdp_{34} & Cdp_{33} & 0 & dp_{13} & 0\\ 0 & dp_{13} & 0 & dp_{33} & dp_{34} & 0\\ 0 & 0 & dp_{13} & dp_{34} & -dp_{33} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega_0^1\\ \omega_0^2\\ \omega_1^2\end{pmatrix} + \begin{pmatrix}T_1\\ T_2\\ T_3\\ T_4\\ 0\end{pmatrix} \mod \mathcal{K}^{(1)}_{\text{alg}}$$

where $\mathcal{K}^{(1)}_{\text{alg}}$ denotes the algebraic ideal $(\theta_1, \dots, \theta_5)$, and where the torsion terms $T_1, \dots, T_4$ are given by

$$
\begin{aligned}
T_1 =& [u_1\mathfrak{a}_{01} - u_2\mathfrak{a}_{02} - u_3(\mathfrak{b}_{122} + \mathfrak{b}_{02} + \tfrac{1}{20}\mathfrak{b}_1) + u_4(\mathfrak{b}_{112} - \mathfrak{b}_{01} + \tfrac{1}{20}\mathfrak{b}_2) + u_5(\mathfrak{b}_{012} + \mathfrak{b}_{11} - \mathfrak{b}_{22})\\
&+ p_{13}(\mathfrak{a}_{11} - \mathfrak{a}_{22}) + \tfrac{1}{2}p_{33}(\mathfrak{b}_{000} + \tfrac{2}{5}\mathfrak{b}_0)]\omega^0 \wedge \omega^1\\
&+ [2u_2\mathfrak{a}_{01} + u_3(\mathfrak{b}_{112} + 2\mathfrak{b}_{01} - \tfrac{1}{5}\mathfrak{b}_0) + u_4(-\mathfrak{b}_{111} + \tfrac{1}{10}\mathfrak{b}_1) + u_5(-\mathfrak{b}_{011} + 2\mathfrak{b}_{12} + \tfrac{1}{5}\mathfrak{b}_0)\\
&+ 2p_{13}\mathfrak{a}_{12} + \tfrac{1}{2}p_{34}(\mathfrak{b}_{000} + \tfrac{2}{5}\mathfrak{b}_0)]\omega^0 \wedge \omega^2\\
&+ [-u_3\mathfrak{a}_{01} - u_4\mathfrak{a}_{02} + p_{34}\mathfrak{a}_{12} - \tfrac{1}{2}p_{33}(\mathfrak{a}_{22} - \mathfrak{a}_{11})]\omega^0 \wedge \omega^1_0 + [-p_{33}\mathfrak{a}_{12} - \tfrac{1}{2}p_{34}(\mathfrak{a}_{22} - \mathfrak{a}_{11})]\omega^0 \wedge \omega^2_0\\
&+ [-u_1\mathfrak{a}_{12} + u_2(\mathfrak{a}_{11} - \mathfrak{a}_{00}) + u_3(-\mathfrak{b}_{012} - \mathfrak{b}_{00} + \mathfrak{b}_{11}) + u_4(\mathfrak{b}_{011} + \mathfrak{b}_{12} + \tfrac{1}{20}\mathfrak{b}_0)\\
&+ u_5(\mathfrak{b}_{001} - \mathfrak{b}_{02} + \tfrac{1}{20}\mathfrak{b}_1) - p_{13}\mathfrak{a}_{02} + \tfrac{1}{2}p_{33}(-\mathfrak{b}_{002} + 2\mathfrak{b}_{01} - \tfrac{3}{10}\mathfrak{b}_2)\\
&+ \tfrac{1}{2}p_{34}(\mathfrak{b}_{001} + 2\mathfrak{b}_{02} + \tfrac{3}{10}\mathfrak{b}_1)]\omega^1 \wedge \omega^2\\
&+ [-2p_{33}\mathfrak{a}_{01} + u_5\mathfrak{a}_{02} + u_3(\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})]\omega^1 \wedge \omega^1_0 + [-p_{33}\mathfrak{a}_{02} - p_{34}\mathfrak{a}_{01}]\omega^1 \wedge \omega^2_0\\
&+ [p_{34}(\mathfrak{a}_{00} + \tfrac{1}{3}\mathfrak{a})]\omega^1 \wedge \omega^2_1 + [-2p_{34}\mathfrak{a}_{01} - u_5\mathfrak{a}_{01} + u_4(\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})]\omega^2 \wedge \omega^1_0\\
&+ [p_{33}\mathfrak{a}_{01} - p_{34}\mathfrak{a}_{02}]\omega^2 \wedge \omega^2_0 + [-p_{33}(\mathfrak{a}_{00} + \tfrac{1}{3}\mathfrak{a})]\omega^2 \wedge \omega^2_1 + u_2\omega^1_0 \wedge \omega^2_0,\\
T_2 =& [2u_1\mathfrak{a}_{02} + u_3(-\mathfrak{b}_{222} + \tfrac{1}{10}\mathfrak{b}_2) + u_4(\mathfrak{b}_{122} - 2\mathfrak{b}_{02} - \tfrac{1}{5}\mathfrak{b}_1) + u_5(\mathfrak{b}_{022} + 2\mathfrak{b}_{12} - \tfrac{1}{5}\mathfrak{b}_0)\\
&+ 2p_{13}\mathfrak{a}_{12} + \tfrac{1}{2}p_{34}(\mathfrak{b}_{000} + \tfrac{2}{5}\mathfrak{b}_0)]\omega^0 \wedge \omega^1\\
&+ [-u_1\mathfrak{a}_{01} + u_2\mathfrak{a}_{02} + u_3(\mathfrak{b}_{122} + \mathfrak{b}_{02} + \tfrac{1}{20}\mathfrak{b}_1) - u_4(\mathfrak{b}_{112} - \mathfrak{b}_{01} + \tfrac{1}{20}\mathfrak{b}_2) - u_5(\mathfrak{b}_{012} + \mathfrak{b}_{11} - \mathfrak{b}_{22})\\
&- p_{13}(\mathfrak{a}_{11} - \mathfrak{a}_{22}) - \tfrac{1}{2}p_{33}(\mathfrak{b}_{000} + \tfrac{2}{5}\mathfrak{b}_0)]\omega^0 \wedge \omega^2\\
&+ [p_{33}\mathfrak{a}_{12} + \tfrac{1}{2}p_{34}(\mathfrak{a}_{22} - \mathfrak{a}_{11})]\omega^0 \wedge \omega^1_0 + [-u_3\mathfrak{a}_{01} - u_4\mathfrak{a}_{02} + p_{34}\mathfrak{a}_{12} - \tfrac{1}{2}p_{33}(\mathfrak{a}_{22} - \mathfrak{a}_{11})]\omega^0 \wedge \omega^2_0\\
&+ [u_1(\mathfrak{a}_{00} - \mathfrak{a}_{22}) + u_2\mathfrak{a}_{12} + u_3(-\mathfrak{b}_{022} + \mathfrak{b}_{12} - \tfrac{1}{20}\mathfrak{b}_0) + u_4(\mathfrak{b}_{012} + \mathfrak{b}_{22} - \mathfrak{b}_{00})\\
&+ u_5(\mathfrak{b}_{002} + \mathfrak{b}_{01} + \tfrac{1}{20}\mathfrak{b}_2) + p_{13}\mathfrak{a}_{01} - \tfrac{1}{2}p_{33}(\mathfrak{b}_{001} + 2\mathfrak{b}_{02} + \tfrac{3}{10}\mathfrak{b}_1)\\
&+ \tfrac{1}{2}p_{34}(-\mathfrak{b}_{002} + 2\mathfrak{b}_{01} - \tfrac{3}{10}\mathfrak{b}_2)]\omega^1 \wedge \omega^2\\
&+ [-p_{33}\mathfrak{a}_{02} - p_{34}\mathfrak{a}_{01}]\omega^1 \wedge \omega^1_0 + [-2p_{34}\mathfrak{a}_{02} + u_5\mathfrak{a}_{02} + u_3(\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})]\omega^1 \wedge \omega^2_0\\
&+ [-p_{33}(\mathfrak{a}_{00} + \tfrac{1}{3}\mathfrak{a})]\omega^1 \wedge \omega^2_1 + [p_{33}\mathfrak{a}_{01} - p_{34}\mathfrak{a}_{02}]\omega^2 \wedge \omega^1_0\\
&+ [2p_{33}\mathfrak{a}_{02} - u_5\mathfrak{a}_{01} + u_4(\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})]\omega^2 \wedge \omega^2_0 + [-p_{34}(\mathfrak{a}_{00} + \tfrac{1}{3}\mathfrak{a})]\omega^2 \wedge \omega^2_1 - u_1\omega^1_0 \wedge \omega^2_0,\\
T_3 =& [-u_3\mathfrak{a}_{01} - u_4\mathfrak{a}_{02} + p_{34}\mathfrak{a}_{12} - \tfrac{1}{2}p_{33}(\mathfrak{a}_{22} - \mathfrak{a}_{11})]\omega^0 \wedge \omega^1 + [p_{33}\mathfrak{a}_{12} + \tfrac{1}{2}p_{34}(\mathfrak{a}_{22} - \mathfrak{a}_{11})]\omega^0 \wedge \omega^2\\
&+ [-p_{33}\mathfrak{a}_{02} + (p_{34} + u_5)\mathfrak{a}_{01} - u_4(\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})]\omega^1 \wedge \omega^2 + u_1\omega^1 \wedge \omega^1_0 + u_2\omega^1 \wedge \omega^2_0\\
&- 2p_{34}\omega^1_0 \wedge \omega^2_1 + 2p_{33}\omega^2_0 \wedge \omega^2_1, \text{ and}\\
T_4 =& [-p_{33}\mathfrak{a}_{12} - \tfrac{1}{2}p_{34}(\mathfrak{a}_{22} - \mathfrak{a}_{11})]\omega^0 \wedge \omega^1 + [-u_3\mathfrak{a}_{01} - u_4\mathfrak{a}_{02} + p_{34}\mathfrak{a}_{12} - \tfrac{1}{2}p_{33}(\mathfrak{a}_{22} - \mathfrak{a}_{11})]\omega^0 \wedge \omega^2\\
&+ [-p_{33}\mathfrak{a}_{01} - (p_{34} - u_5)\mathfrak{a}_{02} + u_3(\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})]\omega^1 \wedge \omega^2 + u_1\omega^2 \wedge \omega^1_0 + u_2\omega^2 \wedge \omega^2_0\\
&+ 2p_{33}\omega^1_0 \wedge \omega^2_1 + 2p_{34}\omega^2_0 \wedge \omega^2_1.
\end{aligned}
$$

Obviously, the case when C defined via (3.11) vanishes deserves some special attention. I claim:

Lemma 3.1.4. *If M has the property that $C = 0$ then M must be flat.*

Proof. By assumption $\mathfrak{a}_{00} + \frac{1}{3}\mathfrak{a} = 0$ in all frames and so, by taking exterior derivative,

$$0 = (\mathfrak{b}_{000}+\tfrac{2}{5}\mathfrak{b}_0)\omega^0 + (\mathfrak{b}_{001}+2\mathfrak{b}_{02}+\tfrac{3}{10}\mathfrak{b}_1)\omega^1 + (\mathfrak{b}_{002}-2\mathfrak{b}_{01}+\tfrac{3}{10}\mathfrak{b}_2)\omega^2 + 2\mathfrak{a}_{01}\omega_0^1 + 2\mathfrak{a}_{02}\omega_0^2.$$

In particular

$$\mathfrak{a}_{01} = \mathfrak{a}_{02} = 0,$$

which in turn implies that

$$\mathfrak{a}_{12} = 0,$$

as well as the property that

$$\mathfrak{a}_{00} = \mathfrak{a}_{11} = \mathfrak{a}_{22}.$$

But then by the fact that $(\mathfrak{a}_{ab})$ is trace-free it follows that

$$0 = 3\mathfrak{a}_{00} = \mathfrak{a},$$

which completes the proof. □

The case of minimal surfaces in flat space will be studied explicitly in the next section. For the remainder of this section I will assume that $C \neq 0$. In an attempt to absorb the torsion terms $T_1, \ldots, T_4$ I will substitute

$$\begin{aligned}
\xi_{13} &= dp_{13} - p_{130}\omega^0 - p_{131}\omega^1 - p_{132}\omega^2 - p_{133}\omega_0^1 - p_{134}\omega_0^2 - p_{135}\omega_1^2 \\
\xi_{33} &= dp_{33} - p_{330}\omega^0 - p_{331}\omega^1 - p_{332}\omega^2 - p_{333}\omega_0^1 - p_{334}\omega_0^2 - p_{335}\omega_1^2 \\
\xi_{34} &= dp_{34} - p_{340}\omega^0 - p_{341}\omega^1 - p_{342}\omega^2 - p_{343}\omega_0^1 - p_{344}\omega_0^2 - p_{345}\omega_1^2,
\end{aligned}$$

with

$$\begin{aligned}
p_{130} &= -u_3\mathfrak{a}_{01} - u_4\mathfrak{a}_{02} + p_{34}\mathfrak{a}_{12} - \tfrac{1}{2}p_{33}(\mathfrak{a}_{22}-\mathfrak{a}_{11}) \\
p_{131} &= -p_{33}\mathfrak{a}_{01} - (p_{34}-u_5)\mathfrak{a}_{02} + u_3(\mathfrak{a}_{00}-\tfrac{1}{6}\mathfrak{a}) \\
p_{132} &= p_{33}\mathfrak{a}_{02} - (p_{34}+u_5)\mathfrak{a}_{01} + u_4(\mathfrak{a}_{00}-\tfrac{1}{6}\mathfrak{a}) \\
p_{133} &= -u_1 \\
p_{134} &= -u_2 \\
p_{135} &= 0 \\
p_{330} &= 0 \\
p_{331} &= 0 \\
p_{332} &= 0 \\
Cp_{333} &= -p_{33}\mathfrak{a}_{01} + p_{34}\mathfrak{a}_{02} \\
Cp_{334} &= -p_{33}\mathfrak{a}_{02} - p_{34}\mathfrak{a}_{01} \\
p_{335} &= 2p_{34},
\end{aligned}$$

and

$$\begin{aligned}
p_{340} &= 0\\
p_{341} &= 0\\
p_{342} &= 0\\
Cp_{343} &= -p_{33}\mathfrak{a}_{02} - p_{34}\mathfrak{a}_{01}\\
Cp_{344} &= +p_{33}\mathfrak{a}_{01} - p_{34}\mathfrak{a}_{02}\\
p_{345} &= -2p_{33}.
\end{aligned}$$

This will however leave several non-absorbable torsion terms. Explicitly these are:

$$\begin{aligned}
0 =& p_{33}\mathfrak{a}_{12} + \tfrac{1}{2}p_{34}(\mathfrak{a}_{22}-\mathfrak{a}_{11})\\
0 =& u_1\mathfrak{a}_{01} - u_2\mathfrak{a}_{02} - u_3(\mathfrak{b}_{122}+\mathfrak{b}_{02}+\tfrac{1}{20}\mathfrak{b}_1) + u_4(\mathfrak{b}_{112}-\mathfrak{b}_{01}+\tfrac{1}{20}\mathfrak{b}_2) + u_5(\mathfrak{b}_{012}+\mathfrak{b}_{11}-\mathfrak{b}_{22})\\
&+ p_{13}(\mathfrak{a}_{11}-\mathfrak{a}_{22}) + \tfrac{1}{2}p_{33}(\mathfrak{b}_{000}+\tfrac{2}{5}\mathfrak{b}_0)\\
0 =& 2u_2\mathfrak{a}_{01} + u_3(\mathfrak{b}_{112}+2\mathfrak{b}_{01}-\tfrac{1}{5}\mathfrak{b}_0) + u_4(-\mathfrak{b}_{111}+\tfrac{1}{10}\mathfrak{b}_1) + u_5(-\mathfrak{b}_{011}+2\mathfrak{b}_{12}+\tfrac{1}{5}\mathfrak{b}_0)\\
&+ 2p_{13}\mathfrak{a}_{12} + \tfrac{1}{2}p_{34}(\mathfrak{b}_{000}+\tfrac{2}{5}\mathfrak{b}_0)\\
0 =& -u_1\mathfrak{a}_{12} + u_2(\mathfrak{a}_{11}-\mathfrak{a}_{00}) + u_3(-\mathfrak{b}_{012}-\mathfrak{b}_{00}+\mathfrak{b}_{11}) + u_4(\mathfrak{b}_{011}+\mathfrak{b}_{12}+\tfrac{1}{20}\mathfrak{b}_0)\\
&+ u_5(\mathfrak{b}_{001}-\mathfrak{b}_{02}+\tfrac{1}{20}\mathfrak{b}_1) - p_{13}\mathfrak{a}_{02} + \tfrac{1}{2}p_{33}(-\mathfrak{b}_{002}+2\mathfrak{b}_{01}-\tfrac{3}{10}\mathfrak{b}_2) + \tfrac{1}{2}p_{34}(\mathfrak{b}_{001}+2\mathfrak{b}_{02}+\tfrac{3}{10}\mathfrak{b}_1)\\
0 =& 2u_1\mathfrak{a}_{02} + u_3(-\mathfrak{b}_{222}+\tfrac{1}{10}\mathfrak{b}_2) + u_4(\mathfrak{b}_{122}-2\mathfrak{b}_{02}-\tfrac{1}{5}\mathfrak{b}_1) + u_5(\mathfrak{b}_{022}+2\mathfrak{b}_{12}-\tfrac{1}{5}\mathfrak{b}_0)\\
&+ 2p_{13}\mathfrak{a}_{12} + \tfrac{1}{2}p_{34}(\mathfrak{b}_{000}+\tfrac{2}{5}\mathfrak{b}_0)\\
0 =& u_1(\mathfrak{a}_{00}-\mathfrak{a}_{22}) + u_2\mathfrak{a}_{12} + u_3(-\mathfrak{b}_{022}+\mathfrak{b}_{12}-\tfrac{1}{20}\mathfrak{b}_0) + u_4(\mathfrak{b}_{012}+\mathfrak{b}_{22}-\mathfrak{b}_{00})\\
&+ u_5(\mathfrak{b}_{002}+\mathfrak{b}_{01}+\tfrac{1}{20}\mathfrak{b}_2) + p_{13}\mathfrak{a}_{01} - \tfrac{1}{2}p_{33}(\mathfrak{b}_{001}+2\mathfrak{b}_{02}+\tfrac{3}{10}\mathfrak{b}_1) + \tfrac{1}{2}p_{34}(-\mathfrak{b}_{002}+2\mathfrak{b}_{01}-\tfrac{3}{10}\mathfrak{b}_2).
\end{aligned}$$

Assuming all of the above vanish identically the tableau takes the form

$$d\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{pmatrix} \equiv -\begin{pmatrix} 0 & -C\xi_{33} & -C\xi_{34} & \xi_{13} & 0 & 0\\ 0 & -C\xi_{34} & C\xi_{33} & 0 & \xi_{13} & 0\\ 0 & \xi_{13} & 0 & \xi_{33} & \xi_{34} & 0\\ 0 & 0 & \xi_{13} & \xi_{34} & -\xi_{33} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^1_0\\ \omega^2_0\\ \omega^2_1\end{pmatrix} \quad \mathrm{mod}\ \mathcal{K}^{(1)}_{\mathrm{alg}}$$

It turns out that this system is not involutive, and that further prolongation is necessary. This means adjoining the forms ξ_{13}, ξ_{33} and ξ_{34} to the system and computing their exterior derivatives. The problem in doing so is that some of the coefficients of ξ_{33} and ξ_{34} are only be determined as multiples of C. I will therefore focus on a special case in the next section, namely, where M is a space form, and use this as a test-case before returning to the general scenario.

3.2 A Special Case: Minimal Surfaces in a 3-dimensional Space Form

Throughout this section M is assumed to be a space form, that is M has constant sectional curvature K. In terms of the Riemann curvature tensor this means that

$$R^a_{bcd} = K(\delta^a_c \delta_{bd} - \delta^a_d \delta_{bc})$$

and translating this condition in terms of the notation introduced in the previous section, one has that

$$\mathfrak{a} = 6K$$

while

$$\mathfrak{a}_{ab} = \mathfrak{b}_a = \mathfrak{b}_{ab} = \mathfrak{b}_{abc} = 0.$$

Under these hypotheses the analysis carried out earlier leads to $\mathcal{K}^{(1)} = (\theta_1, \ldots, \theta_5)$ defined on $\mathcal{F}_+(M) \times \mathbb{R}^8$ with

$$\begin{aligned}
\theta_1 &= du_1 + Ku_3\omega^0 + Kp_{33}\omega^1 + K(p_{34}{+}u_5)\omega^2 - p_{13}\omega^1_0 - u_2\omega^2_1 \\
\theta_2 &= du_2 + Ku_4\omega^0 + K(p_{34}{-}u_5)\omega^1 - Kp_{33}\omega^2 - p_{13}\omega^2_0 + u_1\omega^2_1 \\
\theta_3 &= du_3 - u_1\omega^0 - p_{13}\omega^1 - p_{33}\omega^1_0 - (p_{34}{+}u_5)\omega^2_0 - u_4\omega^2_1 \\
\theta_4 &= du_4 - u_2\omega^0 - p_{13}\omega^2 - (p_{34}{-}u_5)\omega^1_0 + p_{33}\omega^2_0 + u_3\omega^2_1 \\
\theta_5 &= du_5 + u_2\omega^1 - u_1\omega^2 - u_4\omega^1_0 + u_3\omega^2_0.
\end{aligned}$$

With the substitutions

$$\begin{aligned}
\xi^K_{13} &= dp_{13} + Ku_3\omega^1 + Ku_4\omega^2 + u_1\omega^1_0 + u_2\omega^2_0 \\
\xi^K_{33} &= dp_{33} - 2p_{34}\omega^2_1 \\
\xi^K_{34} &= dp_{34} + 2p_{33}\omega^2_1,
\end{aligned}$$

all torsion is absorbable and one arrives at the tableau

$$d\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{pmatrix} \equiv -\begin{pmatrix} 0 & -K\xi^K_{33} & -K\xi^K_{34} & \xi^K_{13} & 0 & 0 \\ 0 & -K\xi^K_{34} & K\xi^K_{33} & 0 & \xi^K_{13} & 0 \\ 0 & \xi^K_{13} & 0 & \xi^K_{33} & \xi^K_{34} & 0 \\ 0 & 0 & \xi^K_{13} & \xi^K_{34} & -\xi^K_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^1_0 \\ \omega^2_0 \\ \omega^2_1 \end{pmatrix} \quad \bmod \mathcal{K}^{(1)}_{\text{alg}}.$$

At this point, it is necessary to distinguish between the following two cases:

3.2.1 The case $K \neq 0$

If $K \neq 0$ the dimension of $V_6(\mathcal{K}^{(1)}, \Omega_+)$ equals that of $\mathcal{F}_+(M) \times \mathbb{R}^8$, meaning $(\mathcal{F}_+(M) \times \mathbb{R}^8, \mathcal{K}^{(1)})$ cannot be involutive. Consequently one has to prolong the system by setting

$$\mathcal{K}^{(2)} = (\theta_1, \ldots, \theta_5, \theta_6, \theta_7, \theta_8)$$

with

$$\begin{aligned} \theta_6 &= \xi^K_{13} \\ \theta_7 &= \xi^K_{33} \\ \theta_8 &= \xi^K_{34}. \end{aligned}$$

Calculation yields

$$d\theta_6 = -2Kp_{33}\omega^1 \wedge \omega^1_0 - 2Kp_{34}\omega^1 \wedge \omega^2_0 - 2Kp_{34}\omega^2 \wedge \omega^1_0 + 2Kp_{33}\omega^2 \wedge \omega^2_0,$$

while

$$d\theta_7 = 2Kp_{34}\omega^1 \wedge \omega^2 + 2p_{34}\omega^1_0 \wedge \omega^2_0$$

and

$$d\theta_8 = -2Kp_{33}\omega^1 \wedge \omega^2 - 2p_{33}\omega^1_0 \wedge \omega^2_0.$$

And so $V_6(\mathcal{K}^{(2)}) = \varnothing$ unless one imposes that

$$\begin{aligned} p_{33} &= 0 \\ p_{34} &= 0. \end{aligned}$$

in which case $\theta_7 = \theta_8 = 0$ and therefore, by the Frobenius theorem (see [2], page 27):

Proposition 3.2.1. *The exterior differential system $(\mathcal{F}_+(M) \times \mathbb{R}^6, \mathcal{K}^{(2)})$ is a Frobenius system whence $\mathcal{F}_+(M) \times \mathbb{R}^6$ is foliated by 6-dimensional integral manifolds of $\mathcal{K}^{(2)}$.*

The consequence is that in the $K \neq 0$ case the dimension of the space of conservation laws equals the dimension of the group of symmetries of the space form, namely 6. In particular, this means that the rank 6 Frobenius system $\mathcal{K}^{(2)} = (\theta_1, \ldots, \theta_6)$ represents, in disguise, the Lie algebra of the symmetries.

3.2.2 The case $K = 0$

In the case where M is flat, $\mathcal{K}^{(1)}$ is generated by

$$\begin{aligned} \theta_1 &= du_1 - p_{13}\omega^1_0 - u_2\omega^2_1 \\ \theta_2 &= du_2 - p_{13}\omega^2_0 + u_1\omega^2_1 \\ \theta_3 &= du_3 - u_1\omega^0 - p_{13}\omega^1 - p_{33}\omega^1_0 - (p_{34}+u_5)\omega^2_0 - u_4\omega^2_1 \\ \theta_4 &= du_4 - u_2\omega^0 - p_{13}\omega^2 - (p_{34}-u_5)\omega^1_0 + p_{33}\omega^2_0 + u_3\omega^2_1 \\ \theta_5 &= du_5 + u_2\omega^1 - u_1\omega^2 - u_4\omega^1_0 + u_3\omega^2_0, \end{aligned}$$

and with the substitutions

$$\begin{aligned}\xi_{13}^0 &= dp_{13} + u_1\omega_0^1 + u_2\omega_0^2\\ \xi_{33}^0 &= dp_{33} - 2p_{34}\omega_1^2\\ \xi_{34}^0 &= dp_{34} + 2p_{33}\omega_1^2,\end{aligned}$$

the tableau takes the form

$$d\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{pmatrix} \equiv -\begin{pmatrix}0 & 0 & 0 & \xi_{13}^0 & 0 & 0\\ 0 & 0 & 0 & 0 & \xi_{13}^0 & 0\\ 0 & \xi_{13}^0 & 0 & \xi_{33}^0 & \xi_{34}^0 & 0\\ 0 & 0 & \xi_{13}^0 & \xi_{34}^0 & -\xi_{33}^0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega_0^1\\ \omega_0^2\\ \omega_1^2\end{pmatrix} \quad \bmod \mathcal{K}_{\text{alg}}^{(1)}.$$

The bundle $V_6(\mathcal{K}^{(1)}, \Omega_+)$ over $\mathcal{F}_+(M) \times \mathbb{R}^8$ now has fiber dimension 2. Since on any integral manifold $d\theta_1 = d\theta_2 = 0$, the form ξ_{13}^0 must also vanish on integral manifolds. Consequently, I will adjoin the form ξ_{13}^0 to the differential ideal by setting

$$\mathcal{K}_+^{(1)} = (\theta_1, \ldots, \theta_5, \theta_6 = \xi_{13}^0).$$

After rearranging, the corresponding tableau is now

$$d\begin{pmatrix}\theta_3\\ \theta_4\\ \theta_5\\ \theta_6\\ \theta_1\\ \theta_2\end{pmatrix} \equiv -\begin{pmatrix}\xi_{33} & \xi_{34} & 0 & 0 & 0 & 0\\ \xi_{34} & -\xi_{33} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega_0^1\\ \omega_0^2\\ \omega_1^2\\ \omega^0\\ \omega^1\\ \omega^2\end{pmatrix} \quad \bmod (\theta_1, \ldots, \theta_6)$$

and clearly $s_1 = 2$. According to Cartan's Test, $(\mathcal{F}_+(M) \times \mathbb{R}^8, \mathcal{K}_+^{(1)})$ is then involutive. This establishes:

Proposition 3.2.2. *The space of conservation laws for minimal surfaces in Euclidean space depends on* 2 *functions of* 1 *variable.*

The above also shows that, in the case of flat space, the dimension of the space of conservation laws is strictly larger than the group of symmetries. This raises the question where these additional conservation laws come from.

One aspect that distinguishes the case of a minimal surface in flat space from that in an ambient space with non-trivial curvature is that in flat space the Gauss map $\Sigma \to S^2$ is holomorphic. A direct consequence of this fact, obtained by inverting this map, is the Weierstrass representation formula which states that a minimal surface in flat space can be written as the projection of a holomorphic function (see [7], page 117 for details). In contrast, if the ambient space were not flat, for example, if one were to consider $\Sigma^2 \subset S^3$, then given that S^3 has no holomorphic structure, no analogue of the Weierstrass formula could exist in this case.

In the flat case, however, one can now take a local harmonic 1-form on S^2 which, when pulled-back via the Gauss map to Σ, will have the property that its exterior derivative is zero, and therefore defining a conservation law for $(\mathcal{S}(M), \mathcal{I}_0)$. By the virtue of being a harmonic form it depends on two real functions of one variable as one would expect given that $s_1 = 2$.

Aside from conservation laws of this type, there are of course those coming from symmetries of the metric in the sense of Noether, as well as those due to the additional symmetry of the exterior differential system $\mathcal{I}_0$, namely constant dilation.

3.3 The General Case continued

Returning to the general case, I will continue to assume $C \neq 0$. Analogous to the case $K \neq 0$ the corresponding linear Pfaffian system $\mathcal{K}^{(1)}$ with tableau

$$d\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{pmatrix} \equiv -\begin{pmatrix} 0 & -C\xi_{33} & -C\xi_{34} & \xi_{13} & 0 & 0\\ 0 & -C\xi_{34} & C\xi_{33} & 0 & \xi_{13} & 0\\ 0 & \xi_{13} & 0 & \xi_{33} & \xi_{34} & 0\\ 0 & 0 & \xi_{13} & \xi_{34} & -\xi_{33} & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega^1_0\\ \omega^2_0\\ \omega^2_1\end{pmatrix} \quad \operatorname{mod} \mathcal{K}^{(1)}_{\text{alg}}$$

is not involutive, meaning that I will need to prolong. So, defined on the manifold $(\mathcal{F}_+(M) \times \mathbb{R}^5)^{(2)} = V_6(\mathcal{K}^{(1)})$, I have the differential ideal

$$\mathcal{K}^{(2)} = (\theta_1, \ldots, \theta_8),$$

with $\theta_6, \theta_7, \theta_8$ defined as

$$\begin{aligned}\theta_6 &= \xi_{13}\\ \theta_7 &= \xi_{33}\\ \theta_8 &= \xi_{34}.\end{aligned}$$

Taking exterior derivatives of these will produce further non-absorbable torsion terms. Amongst these are, from the coefficients of $\omega^1 \wedge \omega^1_0$ and $\omega^2 \wedge \omega^2_0$ in $d\theta_6$,

$$2p_{33}(\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a}) - \mathfrak{a}_{01}p_{333} + \mathfrak{a}_{02}p_{334} = 0, \tag{3.12}$$

as well as, from $\omega^1 \wedge \omega^2_0$ and $\omega^2 \wedge \omega^1_0$ in $d\theta_6$,

$$2p_{34}(\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a}) - \mathfrak{a}_{01}p_{334} + \mathfrak{a}_{02}p_{333} = 0. \tag{3.13}$$

Multiplying (3.12) and (3.13) through by C (and recalling that C is assumed to be non-zero) then gives the identities

$$p_{33}[(\mathfrak{a}_{00}-\tfrac{1}{6}\mathfrak{a})(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}) + \mathfrak{a}_{01}^2 + \mathfrak{a}_{02}^2] = 0 \tag{3.14}$$
$$p_{34}[(\mathfrak{a}_{00}-\tfrac{1}{6}\mathfrak{a})(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}) + \mathfrak{a}_{01}^2 + \mathfrak{a}_{02}^2] = 0. \tag{3.15}$$

From (3.14) and (3.15), one can immediately deduce that $\mathcal{K}^{(2)}$ has no integral manifolds unless one assumes that either

$$(\mathfrak{a}_{00}-\tfrac{1}{6}\mathfrak{a})(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}) + \mathfrak{a}_{01}^2 + \mathfrak{a}_{02}^2 = 0 \tag{3.16}$$

or

$$p_{33} = p_{34} = 0 \tag{3.17}$$

Lemma 3.3.1. *If M satisfies condition* (3.16) *then M must be flat.*

Proof. Let $f : \mathrm{Sym}_3(\mathbb{R}) \to \mathbb{R}$ be the quadratic function defined on the space of symmetric 3-by-3 matrices as

$$f(A) = (\mathfrak{a}_{00}-\tfrac{1}{6}\mathfrak{a})(\mathfrak{a}_{00}+\tfrac{1}{3}\mathfrak{a}) + \mathfrak{a}_{01}^2 + \mathfrak{a}_{02}^2 \tag{3.18}$$

where $A = (\mathfrak{a}_{ab})$ and $\mathrm{tr}A = \mathfrak{a}$. Given that (3.16) must hold in all frames, proving the lemma amounts to proving that if a matrix A and everything conjugate to A lie in the zero-locus of f, then $A = 0$. So I will assume $f(A) = 0$. By rotation one can then diagonalize A and consequently, according to (3.18), all its eigenvalues equal either $\frac{1}{6}\mathfrak{a}$ or $-\frac{1}{3}\mathfrak{a}$. However, no sum involving three of $\frac{1}{6}\mathfrak{a}$ and $-\frac{1}{3}\mathfrak{a}$ adds up to $\mathfrak{a}$ unless $\mathfrak{a} = 0$. And this in return means that A itself must be the zero matrix as claimed. □

It follows that studying conservation laws in the case (3.16) reduces to Proposition 3.2.2. Consequently I will from now on assume that M is not flat. In other words, for conservation laws to exist I must assume condition (3.17) is satisfied. The corresponding differential system is then of the following form:

$$\mathcal{K}^{(1)} = (\theta_1, \ldots, \theta_5),$$

where

$$\begin{aligned}
\theta_1 =& du_1 - (u_4\mathfrak{a}_{12}+u_3(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a}))\omega^0 + (u_3\mathfrak{a}_{01}+u_5\mathfrak{a}_{12})\omega^1 - (-u_4\mathfrak{a}_{01}+u_5(\mathfrak{a}_{11}-\tfrac{1}{6}\mathfrak{a}))\omega^2 \\
&- p_{13}\omega_0^1 - u_2\omega_1^2 \\
\theta_2 =& du_2 - (u_4(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a})+u_3\mathfrak{a}_{12})\omega^0 + (u_3\mathfrak{a}_{02}+u_5(\mathfrak{a}_{22}-\tfrac{1}{6}\mathfrak{a}))\omega^1 - (-u_4\mathfrak{a}_{02}+u_5\mathfrak{a}_{12})\omega^2 \\
&- p_{13}\omega_0^2 + u_1\omega_1^2 \\
\theta_3 =& du_3 - u_1\omega^0 - p_{13}\omega^1 - u_5\omega_0^2 - u_4\omega_1^2 \\
\theta_4 =& du_4 - u_2\omega^0 - p_{13}\omega^2 + u_5\omega_0^1 + u_3\omega_1^2 \\
\theta_5 =& du_5 + u_2\omega^1 - u_1\omega^2 - u_4\omega_0^1 + u_3\omega_0^2.
\end{aligned} \tag{3.19}$$

Substituting

$$\xi_{13} = dp_{13} + u_3\mathfrak{a}_{01} + u_4\mathfrak{a}_{02}\omega^0 - u_5\mathfrak{a}_{02} - u_3(\mathfrak{a}_{00}-\tfrac{1}{6}\mathfrak{a})\omega^1 + u_5\mathfrak{a}_{01} - u_4(\mathfrak{a}_{00}-\tfrac{1}{6}\mathfrak{a})\omega^2 + u_1\omega_0^1 + u_2\omega_0^2$$

then gives a tableau of the form

$$d\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{pmatrix} \equiv -\begin{pmatrix} 0 & 0 & 0 & \xi_{13} & 0 & 0\\ 0 & 0 & 0 & 0 & \xi_{13} & 0\\ 0 & \xi_{13} & 0 & 0 & 0 & 0\\ 0 & 0 & \xi_{13} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \wedge \begin{pmatrix}\omega^0\\ \omega^1\\ \omega^2\\ \omega_0^1\\ \omega_0^2\\ \omega_1^2\end{pmatrix} \quad \text{mod } \mathcal{K}^{(1)}_{\text{alg}},$$

provided one imposes that the non-absorbable torsion terms given by

$$\begin{aligned}
0 =& u_1\mathfrak{a}_{01} - u_2\mathfrak{a}_{02} - u_3(\mathfrak{b}_{122}+\mathfrak{b}_{02}+\tfrac{1}{20}\mathfrak{b}_1) + u_4(\mathfrak{b}_{112}-\mathfrak{b}_{01}+\tfrac{1}{20}\mathfrak{b}_2) + u_5(\mathfrak{b}_{012}+\mathfrak{b}_{11}-\mathfrak{b}_{22})\\
&+ p_{13}(\mathfrak{a}_{11}-\mathfrak{a}_{22})\\
0 =& 2u_2\mathfrak{a}_{01} + u_3(\mathfrak{b}_{112}+2\mathfrak{b}_{01}-\tfrac{1}{5}\mathfrak{b}_0) + u_4(-\mathfrak{b}_{111}+\tfrac{1}{10}\mathfrak{b}_1) + u_5(-\mathfrak{b}_{011}+2\mathfrak{b}_{12}+\tfrac{1}{5}\mathfrak{b}_0) + 2p_{13}\mathfrak{a}_{12}\\
0 =& - u_1\mathfrak{a}_{12} + u_2(\mathfrak{a}_{11}-\mathfrak{a}_{00}) + u_3(-\mathfrak{b}_{012}-\mathfrak{b}_{00}+\mathfrak{b}_{11}) + u_4(\mathfrak{b}_{011}+\mathfrak{b}_{12}+\tfrac{1}{20}\mathfrak{b}_0)\\
&+ u_5(\mathfrak{b}_{001}-\mathfrak{b}_{02}+\tfrac{1}{20}\mathfrak{b}_1) - p_{13}\mathfrak{a}_{02}\\
0 =& 2u_1\mathfrak{a}_{02} + u_3(-\mathfrak{b}_{222}+\tfrac{1}{10}\mathfrak{b}_2) + u_4(\mathfrak{b}_{122}-2\mathfrak{b}_{02}-\tfrac{1}{5}\mathfrak{b}_1) + u_5(\mathfrak{b}_{022}+2\mathfrak{b}_{12}-\tfrac{1}{5}\mathfrak{b}_0) + 2p_{13}\mathfrak{a}_{12}\\
0 =& u_1(\mathfrak{a}_{00}-\mathfrak{a}_{22}) + u_2\mathfrak{a}_{12} + u_3(-\mathfrak{b}_{022}+\mathfrak{b}_{12}-\tfrac{1}{20}\mathfrak{b}_0) + u_4(\mathfrak{b}_{012}+\mathfrak{b}_{22}-\mathfrak{b}_{00})\\
&+ u_5(\mathfrak{b}_{002}+\mathfrak{b}_{01}+\tfrac{1}{20}\mathfrak{b}_2) + p_{13}\mathfrak{a}_{01}
\end{aligned}$$

vanish identically. While this system is clearly not involutive, it is evident, very much as in the case of flat space, that the form ξ_{13} must vanish on integral manifolds. Consequently I will enlarge the ideal by adding

$$\theta_6 = \xi_{13} \tag{3.20}$$

so that

$$\mathcal{K}^{(1)}_{+} = (\theta_1, \dots, \theta_6).$$

It remains to calculate the exterior derivative of θ_6 which is given by

$$d\theta_6 \equiv [2u_1\mathfrak{a}_{02}-2u_2\mathfrak{a}_{01}+u_3(\mathfrak{b}_{002}-2\mathfrak{b}_{01}-\tfrac{1}{5}\mathfrak{b}_1)-u_4(\mathfrak{b}_{001}+2\mathfrak{b}_{02}-\tfrac{1}{5}\mathfrak{b}_1)+u_5(\mathfrak{b}_{011}+\mathfrak{b}_{12}+\tfrac{1}{20}\mathfrak{b}_0)]\omega^1\wedge\omega^2$$

Under the assumption that this non-absorbable torsion coefficient vanishes identically I can now conclude:

Proposition 3.3.2. *Assuming $p_{33} = p_{34} = 0$ the resulting exterior differential system $\mathcal{K}^{(1)}_{+}$ is then an integrable Frobenius system.*

The missing step in proving Theorem 3.1.1 is now to argue that the conservation laws arising from Proposition 3.3.2 are implied by symmetries of the metric in the sense of Noether. This claim is motivated by the following observation.

Let

$$V = v_0 e_0 + v_1 e_1 + v_2 e_2$$

be a symmetry vector field on M. Then, lifting V to $\mathcal{F}_+(M)$ it is characterized by the fact that

$$\mathcal{L}_V \omega^a = \mathcal{L}_V \omega^a_b = 0.$$

To spell this out:

$$\begin{aligned} 0 = \mathcal{L}_V \omega^0 &= V \lrcorner (d\omega^0) + d(V \lrcorner \omega^0) \\ &= -\omega^0_1(V)\omega^1 + \omega^0_1 v_1 - \omega^0_2(V)\omega^2 + \omega^0_2 v_2 + dv_0, \end{aligned}$$

and hence

$$dv_0 = -\omega^1_0(V)\omega^1 - \omega^2_0(V)\omega^2 + v_1\omega^1_0 + v_2\omega^2_0. \tag{3.21}$$

Similarly, from $\mathcal{L}_V\omega^1 = 0$, one has

$$dv_1 = \omega^1_0(V)\omega^0 - \omega^2_1(V)\omega^2 - v_0\omega^1_0 + v_2\omega^2_1 \tag{3.22}$$

and, from $\mathcal{L}_V\omega^2 = 0$,

$$dv_2 = \omega^2_0(V)\omega^0 + \omega^2_1(V)\omega^1 - v_0\omega^2_0 - v_1\omega^2_1. \tag{3.23}$$

Finally, computing $0 = \mathcal{L}_V\omega^1_0 = V \lrcorner (d\omega^1_0) + d(V \lrcorner \omega^1_0)$ yields

$$\begin{aligned} d(\omega^1_0(V)) =& ((\mathfrak{a}_{22} - \tfrac{1}{6}\mathfrak{a})v_1 - \mathfrak{a}_{12}v_2)\omega^0 + (-(\mathfrak{a}_{22} - \tfrac{1}{6}\mathfrak{a})v_0 + \mathfrak{a}_{02}v_2)\omega^1 \\ &+ (\mathfrak{a}_{12}v_0 - \mathfrak{a}_{02}v_1)\omega^2 - \omega^2_1(V)\omega^2_0 + \omega^2_0(V)\omega^2_1, \end{aligned} \tag{3.24}$$

while $0 = \mathcal{L}_V\omega^2_0$ implies

$$\begin{aligned} d(\omega^2_0(V)) =& (-\mathfrak{a}_{12}v_1 + (\mathfrak{a}_{11} - \tfrac{1}{6}\mathfrak{a})v_2)\omega^0 + (\mathfrak{a}_{12}v_0 - \mathfrak{a}_{01}v_2)\omega^1 \\ &+ (-(\mathfrak{a}_{11} - \tfrac{1}{6}\mathfrak{a})v_0 + \mathfrak{a}_{01}v_1)\omega^2 + \omega^2_1(V)\omega^1_0 - \omega^1_0(V)\omega^2_1, \end{aligned} \tag{3.25}$$

and $0 = \mathcal{L}_V\omega^2_1$ gives

$$\begin{aligned} d(\omega^2_1(V)) =& (\mathfrak{a}_{02}v_1 - \mathfrak{a}_{01}v_2)\omega^0 + (-\mathfrak{a}_{02}v_0 + (\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})v_2)\omega^1 \\ &+ (\mathfrak{a}_{01}v_0 - (\mathfrak{a}_{00} - \tfrac{1}{6}\mathfrak{a})v_1)\omega^2 - \omega^2_0(V)\omega^1_0 + \omega^1_0(V)\omega^2_0. \end{aligned} \tag{3.26}$$

Then, according to Theorem 1.4.2, the conservation law corresponding to V is given by

$$v \lrcorner \Theta$$

where $\Theta = -\omega^0 \wedge \psi$ is the Poincaré-Cartan form. One can easily compute that

$$v \lrcorner \Theta = \omega_0^2(V)\omega_0 \wedge \omega^1 - \omega_0^1(V)\omega^0 \wedge \omega^2 + v_2\omega_0 \wedge \omega_0^1 - v_1\omega^0 \wedge \omega_0^2 - v_0\psi.$$

In terms of the notation introduced earlier in this chapter, this means that

$$\begin{aligned} u_1 &= \omega_0^2(V) \\ u_2 &= -\omega_0^1(V) \\ u_3 &= v_2 \\ u_4 &= -v_1 \\ u_5 &= -v_0. \end{aligned} \tag{3.27}$$

Moreover, from the equations (3.21) - (3.26) one can read off that

$$p_{13} = \omega_1^2(V),$$

and in particular

$$p_{33} = p_{34} = 0.$$

I claim that this argument can be reversed:

Proposition 3.3.3. *If M is non-flat, any conservation law of $(\mathcal{S}(M), \mathcal{I}_0)$ comes from a symmetry of the metric.*

Proof. First I apply Proposition 3.3.2, and so the system $\mathcal{K}_+^{(1)}$ with $(\theta_1, \ldots, \theta_6)$ defined via equations (3.19) and (3.20) is involutive. Consequently I can define a vector field, or more precisely, the lift to $\mathcal{F}_+(M)$ of a vector field

$$V = v_0e_0 + v_1e_1 + v_2e_2$$

on M as follows: First set

$$\begin{aligned} v_0 &= -u_5 \\ v_1 &= -u_4 \\ v_2 &= u_3. \end{aligned} \tag{3.28}$$

If I were able to show that V was a symmetry vector field on M, then (3.28) would uniquely define V downstairs on M. To see this, assume $v_0 = v_1 = v_2 = 0$. Then V would project to M to be the zero vector field. Given that any symmetry vector field is Killing, V would then lift uniquely to be the zero vector field on $\mathcal{F}_+(M)$. Thus, to complete this proof, I need to argue that V, when defined via (3.28) above, satisfies the Killing equation or, more precisely, that it satisfies the conditions given by equations (3.21) - (3.26). As a result, I will further impose that

$$\begin{aligned} \omega_0^1(V) &= -u_2 \\ \omega_0^2(V) &= u_1 \\ \omega_1^2(V) &= p_{13}. \end{aligned}$$

With V defined in this manner, it now follows from equations (3.21) - (3.26) that V is indeed a symmetry vector field. Furthermore, unravelling the definitions, V must be the symmetry giving rise to the conservation law defined by each integral of $\mathcal{K}_+^{(1)}$ as required. □

Proof of Theorem 3.1.1. To prove Theorem 3.1.1 it now suffices to combine Lemmas 3.1.3, 3.1.4, and 3.3.1 as well as Propositions 3.2.2, 3.3.2 and 3.3.3. □

3.4 Future Work: An Outlook

Considering conservation laws for minimal surfaces in Riemannian 3-manifolds is obviously restrictive. Having studied this rather special case, on might then wonder what would happen if these restrictions were loosened. More precisely, this would mean either looking at a prescribed mean curvature system

$$(\mathcal{S}(M), \mathcal{I}_P)$$

with a non-zero prescription function P. Alternatively, one could consider dimensions $n > 2$. And ultimately, one would like to understand the case of $(\mathcal{S}(M), \mathcal{I}_P)$ defined over M of arbitrary dimension.

As discussed earlier, the fact that the Riemannian curvature tensor is completely determined by the Ricci tensor when M has dimension 3 allowed for a fairly neat set of expressions for the covariant derivatives of the curvature tensor. In higher dimensions, these would be considerably more complicated.

Instead of looking at higher dimensions, I will briefly introduce the case when $n = 2$, but now

$$\mathcal{I}_P = (\omega^0, \psi - P\omega).$$

Furthermore, I will assume that P is non-constant. Adapting the same notation as before, I will now consider a 1-form

$$\alpha = u_1\omega^0 \wedge \omega^1 + u_2\omega^0 \wedge \omega^2 + u_3\omega^0 \wedge \omega_0^1 + u_4\omega^0 \wedge \omega_0^2 + u_5(\psi - P\omega).$$

Moreover, I will set

$$\Phi_0 = dP - A_0\omega^0 - A_1\omega^1 - A_2\omega^2 - B_1\omega_0^1 - B_2\omega^2.$$

Analogously to Lemma 3.1.3, conservation laws of $(\mathcal{S}(M), \mathcal{I}_P)$ are then encoded by integrals of the exterior differential system

$$\mathcal{K}_P = (d\alpha, \Phi_0)$$

defined on $\mathcal{F}_+(M) \times \mathbb{R}^{11}$ subject to the independence condition $\Omega_+ \neq 0$.

$G_6(T(\mathcal{F}_+(M) \times \mathbb{R}^{11}), \Omega_+)$ turns out to be a manifold of dimension 83. To give a local description at some point $y \in \mathcal{F}_+(M) \times \mathbb{R}^{11}$ say, I need functions $p_{10}, \ldots, p_{55}$ (defined in an analogous way to (3.1)), as well as functions $r_0, \ldots, r_5, q_{00}, \ldots q_{45}$, where

$$\begin{aligned}
dP - r_0(E)\omega^0 - r_1(E)\omega^1 - r_2(E)\omega^2 - r_3(E)\omega_0^1 - r_4(E)\omega_0^2 - r_5(E)\omega_1^2 &= 0 \\
dA_0 - q_{00}(E)\omega^0 - q_{01}(E)\omega^1 - q_{02}(E)\omega^2 - q_{03}(E)\omega_0^1 - q_{04}(E)\omega_0^2 - q_{05}(E)\omega_1^2 &= 0 \\
&\vdots \\
dB_2 - q_{40}(E)\omega^0 - q_{41}(E)\omega^1 - q_{42}(E)\omega^2 - q_{43}(E)\omega_0^1 - q_{44}(E)\omega_0^2 - q_{45}(E)\omega_1^2 &= 0
\end{aligned}$$

for $E \in G_6(T_y(\mathcal{F}_+(M) \times \mathbb{R}^{11}))$ with $\Omega_+|_E \neq 0$. For E to be an integral element, one needs

$$\begin{aligned}
r_0 - A_0 &= 0 \\
r_1 - A_1 &= 0 \\
r_2 - A_2 &= 0 \\
r_3 - B_1 &= 0 \\
r_4 - B_2 &= 0 \\
r_5 &= 0 \\
q_{01} - q_{10} - B_1 R^1_{001} - B_2 R^2_{001} &= 0 \\
q_{02} - q_{20} - B_1 R^1_{002} - B_2 R^2_{002} &= 0 \\
q_{03} - A_1 - q_{30} &= 0 \\
q_{04} - A_2 - q_{40} &= 0 \\
q_{05} &= 0 \\
q_{12} - q_{21} - B_1 R^1_{012} - B_2 R^2_{012} &= 0 \\
A_0 + q_{13} - q_{31} &= 0 \\
q_{14} - q_{41} &= 0 \\
q_{15} - A_2 &= 0 \\
q_{23} - q_{32} &= 0 \\
A_0 + q_{24} - q_{42} &= 0 \\
A_1 + q_{25} &= 0 \\
q_{34} - q_{43} &= 0 \\
q_{35} - B_2 &= 0 \\
B_1 + q_{45} &= 0
\end{aligned}$$

and

$$
\begin{aligned}
p_{12} - p_{21} - p_{50}P - u_3R^1_{012} - u_4R^2_{012} - u_5R^1_{001} - u_5R^2_{002} - u_5A_0 &= 0 \\
p_{13} - p_{31} &= 0 \\
p_{14} - p_{41} - p_{50} - u_5P &= 0 \\
p_{15} - u_2 &= 0 \\
p_{23} - p_{32} + p_{50} + u_5P &= 0 \\
p_{24} - p_{42} &= 0 \\
p_{25} + u_1 &= 0 \\
p_{34} - p_{43} - 2u_5 &= 0 \\
p_{35} - u_4 &= 0 \\
p_{45} + u_3 &= 0 \\
p_{53} - u_4 &= 0 \\
p_{51} + u_2 - u_5B_1 - u_4P &= 0 \\
p_{52} - u_1 - u_5B_2 + u_3P &= 0 \\
p_{54} + u_3 &= 0 \\
p_{55} &= 0.
\end{aligned}
$$

I can now conclude that any $E \in V_6(\mathcal{K}_P, \Omega_+)$ is ordinary. However, it turns out that $c_0 + \ldots + c_5 = 34 \neq 36 = \operatorname{codim}[V_6(\mathcal{K}_P, \Omega_+), G_6(T(\mathcal{F}_+(M) \times \mathbb{R}^{11}), \Omega_+)]$, and so, just as in the case of minimal surfaces, I will now have to prolong. The resulting linear Pfaffian system is obviously larger than it was in the case of minimal surfaces, and it will therefore produce much more non-absorbable torsion than it did previously.

Appendix A

Exterior Differential Systems

A.1 Statement of the Cartan-Kähler Theorem

The content of this appendix is in no way intended to be original, but merely a condensed summary of the topics from exterior differential systems that are particularly relevant to this text. A much more detailed account of everything that follows can be found in [2].

Definition A.1.1. An *exterior differential system* $(M, \mathcal{I})$ consists of a smooth manifold M together with a graded, homogeneous, two-sided ideal $\mathcal{I} \subset \Omega^*(M)$ that is closed under exterior derivative.

$\mathcal{I}$ is often specified through a list of (differential) generators. So in the case of a prescribed mean curvature system $(\mathcal{S}(M), \mathcal{I}_P)$ with $\mathcal{I}_P = (\omega^0, \psi - P\omega)$, a typical element of $\mathcal{I}_P$ will be of the form $\alpha_1 \wedge \omega^0 + \alpha_2 \wedge d\omega^0 + \alpha_3 \wedge (\psi - P\omega) + \alpha_4 \wedge d(\psi - P\omega)$ for some forms $\alpha_1, \ldots, \alpha_4 \in \Omega^*(\mathcal{S}(M))$.

Definition A.1.2. An *integral manifold* of dimension n of an exterior differential system $(M, \mathcal{I})$ is an immersion $f : N^n \to M$ satisfying $f^*\alpha = 0$ for each $\alpha \in \mathcal{I}$.

Since pull-backs are mostly ignored throughout this text, I will continue to do so now and simply write $\alpha = 0$. In the case of the prescribed mean curvature ideal $\mathcal{I}_P$, a necessary and sufficient condition for Σ to be an integral manifold is then that $\omega^0 = 0$ and $\psi - P\omega = 0$.

Returning to a general exterior differential system $(M, \mathcal{I})$, I will now assume that $f : N^n \to M$ is an integral manifold of dimension n. For $\alpha \in \mathcal{I}$ and $y \in N$, this means in particular that

$$\alpha\big|_y \in \wedge^*(T_y^*N).$$

It follows that vanishing of α at a point $y \in N$ depends only on the tangent space of N at y. This suggests that the search for integral manifolds should begin by looking at the following "infinitesimal integral manifolds":

Definition A.1.3. A linear subspace $E \subset T_xM$ is said to be an *integral element* of $\mathcal{I}$ at x if $\alpha|_E = 0$ for all $\alpha \in \mathcal{I}$.

It is clear that every tangent space of an integral manifold is automatically an integral element. However, not every integral element is necessarily the tangent space of some integral manifold. One of the central questions in the theory of exterior differential systems is then to determine when this will indeed be the case. The answer is provided by the Cartan-Kähler Theorem which, before it can be stated here, calls for more definitions.

I will adhere to the convention set in [2] where M is assumed to be of dimension $n+s$. An n-dimensional integral element $E \subset T_xM$ is then naturally an element of the Grassmannian of n-planes $G_n(T_xM)$. It turns out that the full *Grassmann bundle* $G_n(TM)$ defined as

$$G_n(TM) = \bigcup_{x \in M} G_n(T_xM)$$

is a smooth manifold of dimension $n + s + ns$. An example of such a Grassmann bundle can be found in the proof of Theorem 2.1.1, more precisely in equations (3.1), where its smooth structure is described in detail.

If I let $V_n(\mathcal{I}) \subset G_n(TM)$ denote the closed subset of the n-dimensional integral elements, then I declare an integral element $E \in V_n(\mathcal{I})$ to be *ordinary* if, in a neighborhood of E, $V_n(\mathcal{I})$ is cut out cleanly as a smooth submanifold of $G_n(TM)$ by smooth functions defined by the ideal. For example, in the proof of Theorem 2.1.1, I am able to show that the set of m'-dimensional integral elements $V_{m'}(\mathcal{I}_P, \omega_+)$, on which $\omega_+ \neq 0$, is cut out cleanly as the zero locus of a set of $\frac{1}{2}(n^3 + n^2 + 2)$ linearly independent, smooth functions that are indeed defined by $\mathcal{I}_P$. Therefore these m'-dimensional integral elements are indeed ordinary.

The principle behind the Cartan-Kähler theorem is, loosely speaking, to "build" an integral manifold by starting with a lower-dimensional integral manifold and "thickening" or "extending" it to one of higher dimension. At the infinitesimal level, the question of whether or not an integral element can be extended to one of larger dimension is encoded by the following: For a k-dimensional integral element $E \in V_k(\mathcal{I})$, I define the *polar space* of E to be the set

$$H(E) = \{v \in T_xM : v \lrcorner \alpha(e_1, \ldots, e_k) = 0 \quad \forall \alpha \in \mathcal{I}^{k+1}\}.$$

Here $\mathcal{I}^{k+1} = \mathcal{I} \cap \Omega^{k+1}(M)$, and $(e_1, \ldots, e_k)$ is assumed to be a basis of E. In the proof of Lemma 3.1.3, I also refer to the set of polar equations, which is simply defined to be the annihilator of $H(E)$. The significance of the polar space lies in the fact that $v \in H(E)$ if and only if either $v \in E$ already, or the space $E^+ = E + \mathbb{R}v$ belongs to $V_{k+1}(\mathcal{I})$. Consequently, I define the space of polar extensions of E to be

$$\mathbb{P}(H(E)/E) = \{E^+ \in V_{k+1} : E \subset E^+\}$$

and I let $r(E)$ denote the dimension of this real projective space with the understanding that $r(E) = -1$ if E is maximal, i.e. if E has no higher-dimensional extensions. Thinking of r as a function $r(E) = \dim H(E) - (k+1)$, an ordinary integral element $E_k \in V_k(\mathcal{I})$ is called *regular* if $r : V_k(\mathcal{I}) \to [-1, \infty)$ is locally constant on $V_k(\mathcal{I})$.

At the infinitesimal level, the process of "building up" an integral manifold from lower dimensions is then carried out in the following manner: at $x \in M$ I can build a *flag* of integral elements

$$(0)_x = E_0 \subset E_1 \subset \ldots \subset E_n \subset T_x M$$

where $E_i \in V_i(\mathcal{I})$. Such a flag is called *ordinary* if all E_i are ordinary, and if further the $E_0, \ldots, E_{n-1}$ are regular. In this case:

Theorem A.1.4 (Cartan-Kähler)**.** *If $(M, \mathcal{I})$ is real-analytic, and if $E_n \subset T_x M$ is the terminus of an ordinary flag of integral elements, then there exists an n-dimensional integral manifold of $\mathcal{I}$ that passes through x and whose tangent space at x is E_n.*

The main obstacle is to determine whether or not a given n-dimensional integral element E_n is the terminus of an ordinary flag. When this is indeed the case, one refers to E_n, or equivalently, to the system as being *involutive.* To test for involutivity, one defines

$$c_k = \mathrm{codim}[H(E_k), T_x M]$$

for $k < n$, and

$$c_n = \dim M - n = s.$$

Then:

Theorem A.1.5 (Cartan's Test, Version 1)**.** *Let $(M, \mathcal{I})$ be an exterior differential system with $\mathcal{I}^0 = (0)$. Then for a flag*

$$(0)_x = E_0 \subset E_1 \subset \ldots \subset E_n \subset T_x M$$

of integral elements, one has

$$\mathit{codim}[V_n(\mathcal{I}), G_n(TM)] \geq c_0 + \ldots + c_{n-1},$$

with equality if and only if E_n is involutive.

Having defined the c_k one can also associate to a flag of integral elements the *Cartan characters*

$$s_0 = c_0$$

and

$$s_k = c_k - c_{k-1}$$

for $k > 0$. Their significance lies in the fact that if a real-analytic exterior differential system $(M, \mathcal{I})$ passes Cartan's Test for involutivity, the resulting integral manifold depends locally on s_0 constants, s_1 functions of 1 variable, ..., s_n functions of n variables.

One can obviously reformulate Cartan's Test in terms of the Cartan characters. This formulation turns out to be very convenient in the case of linear Pfaffian systems. And so I will present this other version in the last section.

A.2 The Process of Prolongation

I will now turn to the case when Cartan's Test for involutivity fails. One such example is furnished by the exterior differential system for conservation laws for minimal surfaces $(\mathcal{F}_+(M) \times \mathbb{R}^5, \mathcal{K})$ in Lemma 3.1.3. Here, the original system fails to detect constraints that are imposed on the partial differential equation by higher-order derivatives involving curvature terms. In this case, one refines or "prolongs" the system, simply by adding higher derivative terms. To set the stage, I will return to the Grassmann bundle $G_n(TM)$, and I will assume that $\pi : G_n(TM) \to M$ denotes the bundle projection. Using the same notation as in [2], I can associate to $E \in G_n(TM)$ with $\pi(E) = x$ the perpendicular element $E^\perp \subset T_x^*M$. Then let $C_E = \pi_x^*(E^\perp)$ and let $C \subset T^*G_n(TM)$ be the subbundle

$$C = \bigcup_{E \in G_n(TM)} C_E.$$

I can now define the *contact ideal* $\mathcal{C}$ on $G_n(TM)$ to be the ideal generated by sections of the bundle C. Its significance is the following: an immersion $f : N^n \to M$ has a canonical tangential lift $\bar{f}$ to $G_n(TM)$

$$\begin{array}{ccc} & & G_n(TM) \\ & \overset{\bar{f}}{\nearrow} & \downarrow \pi \\ N & \xrightarrow[f]{} & M \end{array}$$

defined via

$$\bar{f}(x) = f_*(T_xN).$$

Then:

Theorem A.2.1. *$\bar{f} : N \to G_n(TM)$ is an integral manifold of $(G_n(TM), \mathcal{C})$. Conversely, if $F : N^n \to G_n(TM)$ is an integral manifold that is transverse to the fibration π then $F = \overline{\pi \circ F}$.*

A crude description of the prolongation algorithm can then be given as follows: Assuming none of the $E \in V_n(\mathcal{I})$ are involutive, then I will set

$$M^{(1)} = V_n(\mathcal{I})$$

and let

$$\mathcal{I}^{(1)} = \mathcal{C},$$

where $\mathcal{C}$ is now pulled-back to $V_n(\mathcal{I}) \subset G_n(TM)$. This description of the algorithm is crude for a number of reasons. One of them being that $V_n(\mathcal{I})$ is merely a closed subset of $G_n(TM)$. So in practice I may have to restrict $M^{(1)}$ to some open, connected subset of $G_n(TM)$ that is contained in $V_n(\mathcal{I})$.

For example, a description of the generating 1-forms for the prolonged ideal in local coordinates is given in the case of conservation laws for minimal surfaces by Equations (3.8).

The point is that if I can establish the existence of an integral manifold N of $(M^{(1)}, \mathcal{I}^{(1)})$, then according to Theorem A.2.1, N must be the lift of an integral manifold of $(M, \mathcal{I})$. Of course, integral elements of $(M^{(1)}, \mathcal{I}^{(1)})$ may also fail to be involutive, and I may have to repeat this algorithm by passing to $(M^{(2)}, \mathcal{I}^{(2)})$. In the case of conservation laws for minimal surfaces this is indeed necessary.

Subject to certain non-degeneracy hypotheses, the Cartan-Kuranishi theorem asserts that after a finite number of iterations I can either establish involutivity and thus, in the real-analytic category, existence of integral manifolds, or it will happen that eventually $M^{(k)}$ is empty for large k, in which case no integral manifolds can exist. A precise formulation of this statement can be found in [2], pp. 260-265.

A.3 Linear Pfaffian Systems

There is a family of exterior differential systems that deserves special attention, namely those that are generated by sections of some subbundle $I \subset T^*M$. For example, the contact ideal $\mathcal{C}$ on $G_n(TM)$ and therefore any prolongation of an exterior differential system will fall into this category. It turns out that these carry some additional structure that will make them into what is called a *Linear Pfaffian System*. To give a formal definition, I will continue to assume that M has dimension $n+s$. Suppose there exists a coframing

$$\theta^1, \ldots, \theta^k, \eta^1, \ldots, \eta^n, \pi^1, \ldots, \pi^{s-k}$$

so that

$$\mathcal{I} = (\theta^1, \ldots, \theta^k).$$

I will assume further that the $\eta^1, \ldots, \eta^n$ define an *independence condition*, that is, I seek n-dimensional integral manifolds of $(M, \mathcal{I})$ for which

$$\eta = \eta^1 \wedge \ldots \wedge \eta^n \neq 0.$$

Then:

Definition A.3.1. $(M, \mathcal{I})$ as defined above is a *Linear Pfaffian System* with *independence condition* η if, for $j = 1, \ldots, k$, $d\theta^j$ can be expressed as

$$d\theta^j \equiv \sum_{\substack{1 \le t \le s-k \\ 1 \le m \le n}} A^j_{tm} \pi^t \wedge \eta^m + \tfrac{1}{2} \sum_{1 \le l,m \le n} T^j_{lm} \eta^l \wedge \eta^m \mod (\theta^1, \ldots, \theta^k).$$

In the decomposition of $d\theta^j$ above, the first term gives rise to what is known as the *tableau* of the system, while the second term is referred to as the *apparent torsion*. The designation "apparent" lies in the fact that the decomposition is given modulo $(\theta^1, \ldots, \theta^k)$ and so one might, by making a suitable substitution for the π^t-terms, be able to "absorb" this term in the tableau. When this is not the case, however, it is clear that no integral manifolds for which $\eta \neq 0$ can exist. One example of a linear Pfaffian system with non-absorbable torsion is furnished by the prolongation of the system for conservation laws for minimal surfaces.

A much more simple case is the one where $\mathcal{I} = (\theta^1, \ldots, \theta^k)$ and where the θ^j satisfy

$$d\theta^j \equiv 0 \mod (\theta^1, \ldots, \theta^k).$$

This is known as a *Frobenius System*, and one can show that, even in the smooth category, this system has k-dimensional integral manifolds (see [2], p. 27 for details). I can also apply Cartan's Test, since for a flag

$$(0)_x = E_0 \subset E_1 \subset \ldots \subset E_k \subset T_x M$$

with

$$E_k = \{v \in T_x M : \theta^1(v) = \ldots = \theta^k(v) = 0\}$$

one has

$$c_0 = \ldots = c_{k-1} = k$$

and so

$$c_0 + \ldots + c_{k-1} = nk = \text{codim}[V_k(\mathcal{I}), G_k(TM)].$$

Theorem A.1.4 will then assert the existence of k-dimensional integral manifolds provided I assume that $\mathcal{I}$ is real-analytic.

A final fact about linear Pfaffian systems used in the proof of Proposition 2.2.6 is the following: It turns out (see [2], p. 141 for details) that for a linear Pfaffian system $(M, \mathcal{I})$ one can easily read off the Cartan characters from the tableau: For a "generic" choice of the $\eta^1, \ldots, \eta^n$, one has that the number of independent 1-forms in the first l columns of the tableau matrix is equal to the sum $s_1 + s_2 + \ldots + s_l$. As a result, it is preferable to use a formulation of Cartan's test that is given in terms of the Cartan characters. For a flag $(0)_x = E_0 \subset E_1 \subset \ldots \subset E_n \subset T_x M$ I can compute that

$$\begin{aligned} s_1 + 2s_2 + \ldots + ns_n &= (c_1 - c_0) + 2(c_2 - c_1) + \ldots + n(c_n - c_{n-1}) \\ &= -(c_0 + \ldots + c_{n-1}) + ns \\ &\geq -\text{codim}[V_n(\mathcal{I}), G_n(TM)] + ns \\ &= \dim V_n(\mathcal{I}) - (n + s) \\ &= \dim V_n(\mathcal{I})_x \end{aligned}$$

where $V_n(\mathcal{I})_x$ denotes the fiber over $x \in M$. Thus:

Theorem A.3.2 (Cartan's Test, Version 2)**.** *Let* $(M, \mathcal{I})$ *be an exterior differential system with* $\mathcal{I}^0 = (0)$*. Then for a flag*

$$(0)_x = E_0 \subset E_1 \subset \ldots \subset E_n \subset T_x M$$

of integral elements, one has

$$s_1 + 2s_2 + \ldots + ns_n \geq \dim V_n(\mathcal{I})_x,$$

with equality if and only if E_n *is involutive.*

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