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Abstract

The behavior of a transverse electromagnetic wave propagating in the direction of a uniform magnetic field in a fully ionized plasma is examined. Linearized kinetic equations with collision terms of the Krook-Bhatnager-Gross type extended by Liboff to include interspecies collisional effects are used in the solution of a spatial boundary-value problem. The region of ion cyclotron resonance is closely investigated, and the transition of the dispersion relation from the low-temperature collision-dominated regime to the high-temperature regime is observed. It is found that moments of the equations are adequate in the collision-dominated regime, but the kinetic equation for the ions must be used at higher temperatures. At these higher temperatures a complete solution of the problem requires numerical work near the source plane. Far from the source, explicit solutions for the fields can be written.

I. Introduction

This paper is concerned with the behavior of plane transverse waves in plasma propagating in the direction of a uniform magnetic field. These waves may be decomposed into two types: a right-handed circularly polarized wave whose vectors rotate in the same sense as the gyration of the electrons of the plasma about the uniform magnetic field; a left-handed wave whose vectors rotate in the sense in which the positive ions gyrate. The left-handed wave is of interest to the controlled fusion programs. At frequencies close to the ion cyclotron frequency, this wave becomes damped and gives its energy to random motion of the plasma, i.e., heats it. Stix has investigated this wave in high temperature plasma where collisions are infrequent;[1,2,3] Engelhardt, in low-temperature plasma where the thermal effects such as viscosity are unimportant.[4] Both authors consider the spatial dependence of the wave to be of the form e^{ikz}.

Assuming a time dependence of the form $e^{-i\omega t}$ with ω real, we consider the problem of a wave propagating in a plasma that fills the semi-infinite space $z > 0$. There is a uniform magnetic field in the plasma perpendicular to the plane boundary $z = 0$. The magnetic field of the wave is given as a boundary condition at $z = 0$. Because of the existence of damping there is no disturbance at $z = \infty$.

The solution of this boundary-value problem shows that the simple form, e^{ikz}, adequately describes the wave only when the thermal effects are small. Criteria are determined for the smallness of thermal effects and the adequacy of simpler solutions.

The result of this study will be a continuous observation of the properties of the waves from low temperature, where collisions are important and thermal effects unimportant; through intermediate temperatures, where the thermal properties of the plasma become important; to high temperature, where collisions

are infrequent and thermal effects are responsible for the damping of the wave. This thermal damping (called cyclotron damping by Stix[2]) is analogous to the Landau damping that occurs for longitudinal waves.

II. Kinetic Equation

Let $f_j = f_j^0 + f_j^1$, where the Maxwell distribution function for the unperturbed particles of type j is

$$f_j^0 = (n_0/\pi^{3/2} a_j^3) \exp(-v^2/a_j^2) ,$$

with $a_j = (2T_0/m_j)^{1/2}$, the most probable speed of particle j, where $j = i, e$ for ions and electrons, respectively, and T_0 is the temperature of the unperturbed plasma in energy units. Both ions and electrons have the unperturbed density n_0. The linearized kinetic equation is then

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \omega_{jc}\, \vec{v}\times\hat{z}\cdot\vec{\nabla}_{\vec{v}}\right) f_j^1 + \frac{q_j}{m_j}\vec{E}\cdot\vec{\nabla}_v f_j^0 = \left.\frac{\partial f_j}{\partial t}\right)_{coll}$$

where $\omega_{jc} = q_j B_0/m_j c$, the gyro frequency of particle j, and B_0 is the magnitude of the uniform magnetic field, which is in the $\hat{z}$ direction.

We use collision forms developed by Liboff,[5] which are extensions of the Krook, Bhatnager, and Gross[6] model. According to this model, the collision form is a term that would cause the distribution function to relax to a local Maxwellian,

$$f_{LM} = \frac{n_0 + n(\vec{r},t)}{\left\{\frac{2\pi}{m}\,[T_0 + T(r,t)]\right\}^{3/2}} \exp\left\{\frac{-m[\vec{v} - \vec{u}(\vec{r},t)]^2}{2[T_0 + T(\vec{r},t)]}\right\},$$

in the absence of external forces. The quantities $n(\vec{r},t)$, $u(\vec{r},t)$, and $T(\vec{r},t)$ are the perturbations produced by the wave in the density, average velocity, and temperature of a species. Liboff includes terms corresponding to the tendency of interactions between the ions and electrons to reduce differences in their average velocities and temperatures. His expression for the ions is

$$\left.\frac{\partial f_i}{\partial t}\right)_{coll} = -\nu_i \left\{ f_i^{\,1} - f_i^{\,0}\left[\frac{n_i}{n_0} + \frac{2}{a_i^{\,2}}(\vec{u}_i \cdot \vec{v}) + \frac{T_i}{T_0}\left(\frac{v^2}{a_i^{\,2}} - \frac{3}{2}\right)\right]\right\}$$

$$- \nu_1 \frac{m_{ie}}{T_0} f_i^{\,0}\, \vec{v} \cdot (\vec{u}_i - \vec{u}_e) - \frac{\nu_2 f_i^{\,0}}{T_0}(T_i - T_e)\left(\frac{v^2}{a_i^{\,2}} - \frac{3}{2}\right).$$

The term in braces in the collision expression is simply $f_{total} - f_{LM}$, where f_{LM} is expressed in terms of f^0 by means of a Taylor expansion in the perturbed quantities. The term m_{ie} is the reduced mass.

To obtain the collision term for the electrons, simply interchange the subscripts "i" and "e." Henceforth, the electron equation will be omitted. The term ν_i, the ion collision frequency, represents the rate at which the ion distribution function approaches a local Maxwellian; ν_e has the analogous meaning for electrons; ν_1, the momentum-transfer collision frequency, represents the rate at which the difference of the average velocities of the two species, $\vec{u}_i - \vec{u}_e$, approaches zero; ν_2, the energy-transfer collision frequency, represents the rate at which the difference of the temperatures of the two species, $T_i - T_e$, approaches zero. The term ν_2 is smaller than ν_1 by about the mass ratio of electrons to ions.

Numerical work by Spitzer using the Boltzmann equation leads to the estimate for the momentum transfer collision frequency[7]

$$\nu_1 = \frac{3.7\, n_0 \ln \Lambda}{T_0^{\,3/2}}\ \mathrm{sec}^{-1},$$

where

$$\Lambda = \frac{1.24 \cdot 10^4\, T_0^{\,3/2}}{\sqrt{n_0}}\ .$$

The dimension of n_0 is cm^{-3} and the dimension of T_0 is °K. This number is obtained from Spitzer's value for the plasma resistivity, η, by comparing his definition of η to our definition of ν_1. This collision frequency is applicable to the case of high magnetic field, $\omega_{ec} >> \nu_1$, for relative velocities perpendicular to the magnetic field, which is our case of interest. For relative velocities parallel to the field or if $\omega_{ec} << \nu_1$, about half the above value is correct.

If one then chooses Spitzer's self-collision frequency[7] for ν, one has $\nu_e = 1.03\ \nu_1$ for electrons. For ions, one has $\nu_i = (m_e/m_i)^{1/2}\nu_e$.

We substitute the collision term into our linearized kinetic equation. Since $\vec{\nabla}_{\vec{v}} f^0 = \frac{-2\vec{v}}{a^2} f^0$, we have

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \omega_{ci}\,\vec{v}\times\hat{z}\cdot\vec{\nabla}_{\vec{v}} + \nu_i\right) f_i^1 = f_i^0 \left\{\frac{2e}{m_i a_i^2}\,\vec{E}\cdot\vec{v}\right.$$

$$+ \nu_i\left[\frac{n_i}{n_0} + \frac{2}{a_i^2}(\vec{u}_i\cdot\vec{v}) + \frac{T_i}{T_0}\left(\frac{v^2}{a_i^2} - \frac{3}{2}\right)\right]$$

$$\left. - \frac{\nu_1 m_{ie}}{T_0}\,v\cdot(\vec{u}_i - \vec{u}_e) - \frac{\nu_2}{T_0}(T_i - T_e)\left(\frac{v^2}{a_i^2} - \frac{3}{2}\right)\right\} \qquad (1)$$

for the ions, and a similar equation for the electrons.

Choosing cylindrical coordinates for the velocity vector, we find the magnetic field term simplifies. For

$$\vec{v} = v_\perp \cos\phi\,\hat{x} + v_\perp \sin\phi\,\hat{y} + v_z\hat{z}$$

we have

$$\vec{v}\times\hat{z}\cdot\vec{\nabla}_{\vec{v}} f = -\frac{\partial f}{\partial\phi}\ .$$

The macroscopic vectors, $\vec{E}$ and $\vec{u}$, appearing on the right-hand side of Eq. (1) are put in the form

$$\vec{E} = E_+(\hat{x} + i\,\hat{y}) + E_-(\hat{x} - i\,\hat{y}) + E_z\,\hat{z}\,.$$

Assuming no spatial variation in the x-y plane, we have

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} - \omega_{ic}\frac{\partial}{\partial \phi} + \nu_i\right) f_i^{\,1} = D_+\,e^{i\phi} + D_-\,e^{-i\phi} + D_z\,, \tag{2}$$

where

$$D_\pm = \frac{2v_\perp f_i^{\,0}}{a_i^{\,2}}\left[\frac{eE_\pm}{m_i} + \left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right)u_{i\pm} + \nu_1\frac{m_{ie}}{m_i}u_{e\pm}\right]$$

and

$$D_z = f_i^{\,0}\left\{\frac{2v_z}{a_i^{\,2}}\left[\frac{eE_z}{m_i} + \left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right)u_{iz} + \nu_1\frac{m_{ie}}{m_i}u_{ez}\right]\right.$$

$$\left. + \nu_i\frac{n_i}{n_0} + \frac{[(v^2/a_i^{\,2}) - 3/2]}{T_0}\left(\nu_i T_i - \nu_2[T_i - T_e]\right)\right\}.$$

A similar equation results for the electrons.

The form of the right-hand side of Eq. (2) suggests a separation of $f_i^{\,1}$ into the form

$$f_i^{\,1} = f_{i+}(z,t,v_\perp,v_z)e^{i\phi} + f_{i-}(z,t,v_\perp,v_z)e^{-i\phi} + f_{i0}(z,t,v_\perp,v_z)\,.$$

Equating coefficients of like exponentials in Eq. (2) produces

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} + \nu_i \mp i\omega_{ic}\right) f_{i\pm} = D_\pm$$

and

$$\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z} + \nu_i\right) f_{i0} = D_z\,.$$

To complete the equations, expressions for $\vec{E}$, $\vec{u}$, n, and T are needed:

$$n = \int f^1 \, d\vec{v} = 2\pi \iint v_\perp \, dv_\perp \, dv_z \, f_0 \, ;$$

since $T = \dfrac{P_0 - nT_0}{n_0}$,

$$T = \frac{2T_0}{3n_0} \int \left(\frac{v^2}{a^2} - 3/2 \right) f^1 \, d\vec{v} = \frac{4\pi \, T_0}{3n_0} \iint v_\perp \, dv_\perp \, dv_z \, f_0 \left(\frac{v^2}{a^2} - 3/2 \right) ;$$

$$n_0 \, u_z = \int v_z \, f^1 \, d\vec{v} = 2\pi \iint v_\perp \, dv_\perp \, v_z \, dv_z \, f_0 \, ;$$

$$n_0 \, u_\pm = \int \frac{v_\perp}{2} \, e^{\mp i\phi} \, f^1 \, d\vec{v} = \pi \iint v_\perp^{\,2} \, dv_\perp \, dv_z \, f_\pm \, .$$

Maxwell's equations furnish $\vec{E}$ in terms of $\vec{u}$:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E_\pm = \frac{4\pi}{c^2} \frac{\partial J_\pm}{\partial t} = \frac{4\pi \, en_0}{c^2} \frac{\partial}{\partial t} \left(u_{i\pm} - u_{e\pm} \right) .$$

For E_z, the equations

$$\frac{\partial E_z}{\partial z} = 4\pi \, e(n_i - n_e)$$

and

$$n_0 \frac{\partial u_z}{\partial z} + \frac{\partial n}{\partial t} = 0$$

are sufficient.

The equations uncouple into three sets of equations for the three independent sets of quantities:

$$f_0 \, , \; E_z, \; u_z, \; n, \; T;$$

$$f_+ \, , \; E_+, \; u_+;$$

$$f_- \, , \; E_- \, , \; u_- \, .$$

The first set corresponds to the longitudinal wave. This problem was first considered by Landau,[8] neglecting collisions and ion motion and has since been the subject of many papers.

The second set corresponds to the electron cyclotron wave. This wave has been studied by Shafranov, neglecting collisions and the effect of ion motion.[9] Platzman and Buchsbaum extended his work to include collisions,[10] but considered the unperturbed distribution function to be of the form

$$f^0 = N/(v_z^2 + a^2)^2 ,$$

rather than Maxwellian, for simplicity in the numerical work. The quantities N and a are normalization factors chosen to give rise to a specified density and temperature. The collision form used by Platzman and Buchsbaum is simply

$$\left.\frac{\partial f_e}{\partial t}\right)_{coll} = - \nu f_e^1 .$$

The neglect of ion motion reduces the Liboff collision term to

$$\left.\frac{\partial f_e}{\partial t}\right)_{coll} = - \nu_e f_e^1 + \frac{2f_e^0}{a_e^2}(\vec{u}_e \cdot \vec{v}) \left(\nu_e - \nu_1 \frac{m_{ie}}{m_e}\right),$$

since $n_e = T_e = 0$ for the transverse electron cyclotron wave. Since $\nu_e \approx \nu_1 (m_{ie}/m_e)$, the second term on the right-hand side is negligible and we see that the collision form used by Platzman and Buchsbaum is adequate for their case of interest. However, their form is completely inadequate for a treatment of the ion cyclotron wave because of the importance of the electron motion and because

$$\nu_i \not\approx \nu_1 (m_{ie}/m_i) .$$

The Liboff expression is necessary for an adequate treatment of a two-species plasma when both species are perturbed.

We are interested in the third set, which corresponds to the ion cyclotron wave. Dropping the minus-sign subscripts, we have the following set of equations:

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + \nu_i + i\,\omega_{ic}\right) f_i = \frac{2 f_i^{\,0} v_\perp}{a_i^{\,2}} \left[\frac{e}{m_i} E + \left(\nu_i - \frac{m_{ie}}{m_i}\nu_i\right) u_i + \frac{m_{ie}}{m_i}\nu_i u_e\right];$$

$$\left(\frac{\partial}{\partial t} + v_z \frac{\partial}{\partial z} + \nu_e - i\,\omega_{ec}\right) f_e = \frac{2 f_e^{\,0} v_\perp}{a_e^{\,2}} \left[\frac{-eE}{m_e} + \left(\nu_e - \frac{m_{ie}}{m_e}\nu_i\right) u_e + \frac{m_{ie}}{m_e}\nu_i u_i\right];$$

$$n_0\, u_{i,e} = \pi \iint v_\perp^{\,2}\, dv_\perp\, dv_z\, f_{i,e}\ ;$$

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E = \frac{4\pi e n_0}{c^2}\frac{\partial}{\partial t}\,(u_i - u_e)\ .$$

We have defined ω_{ec} to be positive.

For any perturbed quantity, $P(z,t) = P(z)\, e^{i\omega t}$, we have the mathematical condition

$$P(z) \neq 0 \quad \text{for } z \geqq 0\ ,$$
$$P(z) = 0 \quad \text{for } z < 0\ ,$$

since we are interested in the determination of the disturbance in the region $z > 0$ in terms of its value at $z = 0$. The appropriate transform is the one-sided Fourier transform, which is identical in theory to the Laplace transform.[11] Define

$$P(k) = \frac{1}{2\pi}\int_0^\infty P(z)\, e^{-ikz}\, dz.$$

The inverse transform is

$$P(z) = \int_{-\infty - i\gamma}^{\infty - i\gamma} dk\, e^{ikz}\, P(k)\ ,$$

where γ is chosen so that the contour in the k plane is below all singularities of the integrand. This insures that $P(k)$ exists and $P(z) = 0$ for $z<0$. The requirement that no disturbance exist in the limit of infinite distance from the boundary means that $P(k)$ is regular in the lower half k plane including the real k axis because of collisional damping. We therefore may take γ to be zero.

Taking the transform of the equations, we have

$$\left(-k^2 + \frac{\omega^2}{c^2}\right) E = \frac{-i\,\omega\, 4\pi\, e\, n_0}{c^2}\,(u_i - u_e) + \frac{i\,k\,E_b}{2\pi} + \frac{E_b'}{2\pi}, \tag{3}$$

$$\left(-i\omega + ikv_z + i\omega_{ic} + \nu_i\right) f_i = \frac{2f_i^{\,0} v_\perp}{a_i^{\,2}}\left[\frac{e}{m_i} E + \left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right) u_i + \frac{m_{ie}\nu_1}{m_i} u_e\right] + \frac{v_z f_{ib}}{2\pi},$$

$$\left(-i\omega + ikv_z - i\omega_{ec} + \nu_e\right) f_e = \frac{2f_e^{\,0} v_\perp}{a_e^{\,2}}\left[\frac{-e}{m_e} E + \left(\nu_e - \frac{m_{ie}}{m_e}\nu_1\right) u_e + \frac{m_{ie}\nu_1}{m_e} u_i\right] + \frac{v_z f_{eb}}{2\pi},$$

and again

$$n_0\, u_{i,e} = \pi \iint v_\perp^{\,2}\, dv_\perp\, dv_z\, f_{i,e}\,.$$

Now all perturbed quantities are the Fourier transforms and are functions of k. The quantities with subscript b are the boundary values:

$$f_b = f(z = 0),$$

$$E_b = E(z = 0),$$

and

$$E_b' = \lim_{z\to 0} \frac{\partial E(z)}{\partial z} = E'(z = 0)\,.$$

Solving for u_i, we have

$$u_i = \frac{\pi i}{n_0}\int v_\perp^2 dv_\perp dv_z \frac{\left\{\frac{2f_i^0 v_\perp}{a_i^2}\left[\frac{eE}{m_i}+\left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right)u_e\right] u_i + \frac{m_{ie}}{m_i}\nu_1 u_e\right] + \frac{v_z f_{ib}}{2\pi}\right\}}{\omega - \omega_{ic} + i\nu_i - kv_z}$$

$$= -iG_i\left[\frac{eE}{m_i}+\left(\nu_i - \frac{m_{ie}}{m_i}\nu_1\right)u_i + \frac{m_{ie}}{m_i}\nu_1 u_e\right] + \bar{u}_i ,$$

where

$$G_i = \frac{-2}{n_0 a_i^2}\int \frac{dv_z\, v_\perp^3 dv_\perp f_i^0}{\omega - \omega_{ic} - i\nu_i - kv_z}$$

and

$$\bar{u}_i = \frac{i}{2n_0}\int \frac{dv_z\, dv_\perp\, v_\perp^2 v_z f_{ib}}{\omega - \omega_{ic} + i\nu_i - kv_z} . \tag{4}$$

Similar results obtain for u_e. Solving for $u_i - u_e$, we find

$$u_i - u_e = \frac{\left[\bar{u}_i - i\,(G_i eE/m_i)\right](1+i\nu_e G_e) - \left[\bar{u}_e + i(G_e eE/m_e)\right](1+i\nu_i G_i)}{(1+i\nu_i G_i)(1+i\nu_e G_e) - i\nu_1 m_{ie}\left[(G_i/m_i)(1+i\nu_e G_e) + (G_e/m_e)(1+i\nu_i G_i)\right]} . \tag{5}$$

Now

$$G_i = \frac{-2}{\pi^{1/2} a_i^5}\int_{-\infty}^{\infty} dv_z \int_0^{\infty} dv_\perp \frac{v_\perp^3 \exp[-(v_z^2 + v_\perp^2)/a_i^2]}{\omega - \omega_{ic} + i\nu_i - kv_z} .$$

Performing the $v_\perp$ integration and letting $t = v_z/a_i$, we have

$$G_i = \frac{1}{ka_i}\,\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty} \frac{dt\, \exp(-t^2)}{t - \Phi_i} ,$$

where

$$\Phi_i = \frac{\omega - \omega_{ic} + i\nu_i}{k a_i} .$$

In the definition of the transforms, we have k real. Therefore, for $k > 0$, we have $\mathrm{Im}\,\Phi_i > 0$, and for $k < 0$, we have $\mathrm{Im}\,\Phi_i < 0$. Upon evaluation of the inverse transformations later, $G(k)$ will be analytically continued off the real k axis. For the positive real k axis, this analytic continuation is effected by moving the path of integration of the above integral in the t plane so as to be always below the pole at $t = \Phi_i$. In the analytic continuation of $G(k)$ from the negative real axis the path of integration must remain above the pole at $t = \Phi_i$. The t - plane contours are illustrated in Fig. 1.

Although $G(k)$ has different definitions depending on the sign of k, it is continuous at $k = 0$ and $k = \infty$:

$$\lim_{k\to 0^-} G_i(k) = \lim_{k\to 0^+} G_i(k) = G_i(0) = -1/(\omega - \omega_{ic} + i\nu_i) .$$

$$\lim_{k\to \infty^+} G_i(k) = \lim_{k\to \infty^-} G_i(k) = G_i(\infty) = 0$$

The function defined on, and its analytic continuation from, the positive real k axis is called G^+. It is related to $Z(\Phi)$, the Plasma Dispersion Function, which is tabulated in Fried and Conte.[12] Now

$$\frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{dt\, \exp(-t^2)}{t - \Phi} = Z(\Phi)$$

when the contour is below the pole. We then have

$$G_i^+ = (1/ka_i)\, Z(\Phi_i),$$

and similarly,

$$G_e^+ = (1/ka_e)\, Z(\Phi_e),$$

where

$$\Phi_e = (\omega + \omega_{ec} + i\nu_e)/ka_e .$$

The function defined on, and its analytic continuation from, the negative real k axis is called G^-. From Fig. 1 and the definition of G we see that we

have

$$G^-(k) = G^+(k) - \frac{2i\sqrt{\pi}}{ka} \exp(-\Phi^2)$$

and

$$G^-(k) = G^+(-k) .$$

Since we are interested in obtaining $J(E)$, we use the relation

$$J = en_0(u_i - u_e),$$

and from Eq. (5) obtain

$$J = \frac{-i\, n_0\, e^2}{D} E\left[\frac{G_i}{m_i}(1 + i\nu_e G_e) + \frac{G_e}{m_e}(1 + i\nu_i G_i)\right]$$

$$+ \frac{e\, n_0}{D}\left[\bar{u}_i (1 + i\nu_e G_e) - \bar{u}_e (1 + i\nu_i G_i)\right] ,$$

where

$$D = (1 + i\nu_i G_i)(1 + i\nu_e G_e) - i\nu_i m_{ie}\left[(G_i/m_i)(1 + i\nu_e G_e) + (G_e/m_e)(1 + i\nu_i G_i)\right] .$$

We thus have

$$J(k) = \sigma(k)\, E(k) + \bar{J}(k, f_{ib}, f_{eb}) .$$

Substituting the expression for $J(k)$ into Eq. (3), we obtain an expression for $E(k)$ in terms of the boundary conditions,

$$E(k) = \frac{(-i\, k\, E_b - E_b')/2\pi/ + (4\pi\, i\omega/c^2)\, \bar{J}(k)}{k^2 - (\omega^2/c^2) n^2(k)} ,$$

where

$$n^2(k) = 1 + \frac{4\pi i}{\omega}\sigma(k)$$

$$= 1 + \frac{\omega_{pi}^2}{\omega D} G_i(1 + i\nu_e G_e) + \frac{\omega_{pe}^2}{\omega D} G_e(1 + i\nu_i G_i) .$$

Finally, $E(z)$ is given by the inverse transform

$$E(z) = \int_{-\infty}^{+\infty} \frac{dk e^{ikz}\left[(-ikE_b - E_b')/2\pi + (4\pi\, i\omega/c^2)\bar{J}(k)\right]}{k^2 - (\omega^2/c^2)\, n^2(k)} . \qquad (6)$$

It appears that in addition to E_b and E_b', $\mathcal{J}(k)$ must be given to specify the solution. The expression for $\mathcal{J}(k)$ to be used is not known, since it depends on the unknown quantities, f_{ib} and f_{eb}. However, by using certain symmetry properties, we may express the integral involving $\mathcal{J}$ in terms of known quantities. We use the assumption of specular reflection of particles at the boundary,

$$f_b(v_z) = f_b(-v_z) ,$$

which shows that the integrand of (4) is even in v_z:

$$u_i = \frac{i}{2n_0}\int \frac{dv_z\, dv_\perp\, v_\perp^2\, v_z(\omega - \omega_{ic} + i\nu_i + kv_z)f_{ib}}{(\omega - \omega_{ic} + i\nu_i)^2 - k^2 v_z^2}$$

$$= \frac{ik}{2n_0}\int \frac{dv_z\, v_z^2\, dv_\perp\, v_\perp^2\, f_{ib}}{(\omega-\omega_{ic} + i\nu_i)^2 - k^2 v_z^2} .$$

Thus

$$\mathcal{J} = \frac{iek}{2D}\int dv_z\, v_z^2\, dv_\perp v_\infty^2 \left[\frac{f_{ib}(1 + i\nu_e G_e)}{(\omega-\omega_{ic}+i\nu_i)^2 - k^2 v_z^2} - \frac{f_{eb}(1 + i\nu_i G_i)}{(\omega+\omega_{ec}+i\nu_e)^2 - k^2 v_z^2}\right].$$

The specular reflection assumption removes the part of $\mathcal{J}$ that is even in k. The remainder will be chosen so as to satisfy the second boundary condition at $z = \infty$.

We assume that the value of E_b' is known. This is one boundary condition. The second boundary condition is $E(z = \infty) = 0$. This second condition is equivalent to the requirement that $E(k)$ be regular in the lower half plane. With the k integration contour on the real axis, the requirement is equivalent to the statement that $E(z) = 0$ for $z < 0$. That is, for $z < 0$, the contour must be closed in the lower half k plane. Since $E(k)$ is regular in the lower half plane, the result of the integration, $E(z)$, is zero.

Returning to the definition of G, we see that $G(k) = G(-k)$. Therefore, by inspection of their definitions, we have

$$n^2(k) = n^2(-k)$$

and

$$\mathcal{J}(k) = -\mathcal{J}(-k).$$

Using these properties of $n^2(k)$ and $\mathcal{J}(k)$ and evaluating Eq. (6) for $z < 0$, we have

$$0 = \int_{-\infty}^{\infty} \frac{dk e^{-ik|z|} \left[(-ikE_b - E_b')/2\pi - (4\pi i\omega/c^2)\, \mathcal{J}(-k) \right]}{k^2 - (\omega^2/c^2)\, n^2(-k)} .$$

Replacing the dummy variable k by $-k$, we have

$$0 = \int_{-\infty}^{\infty} \frac{dk e^{ik|z|} \left[(ikE_b - E_b')/2\pi - (4\pi i\omega/c^2)\, \mathcal{J}(k) \right]}{k^2 - (\omega^2/c^2)\, n^2(k)} .$$

Therefore

$$\int_{-\infty}^{\infty} \frac{dk e^{ik|z|} \left[(ik\, E_b/2\pi) - (4\pi i\omega/c^2)\, \mathcal{J}(k) \right]}{k^2 - (\omega^2/c^2)\, n^2(k)}$$

$$= \int_{-\infty}^{\infty} \frac{(E_b'/2\pi)\, dk e^{ik|z|}}{k^2 - (\omega^2/c^2) n^2(k)} .$$

Now for $z > 0$, $|z| = z$ and

$$E(z) = \int_{-\infty}^{\infty} \frac{dk e^{-ik|z|} [(-ikE_b/2\pi) + (4\pi i\omega/c^2) J(k) (-E_b'/2\pi)]}{k^2 - (\omega^2/c^2) n^2(k)}$$

$$= \int_{-\infty}^{\infty} \frac{dk e^{-ik|z|} [-E_b'/2\pi - E_b'/2\pi]}{k^2 - (\omega^2/c^2) n^2(k)} .$$

The result is

$$E(z) = \frac{-E_b'}{\pi} \int_{-\infty}^{\infty} \frac{dk e^{ikz}}{k^2 - (\omega^2/c^2) n^2(k)} \quad \text{for } z \geq 0, \tag{7}$$

$$E(z) = 0 \qquad \text{for } z < 0 .$$

We have essentially chosen the boundary values, E_b and $J(f_{ib}, f_{eb})$ so as to eliminate the solutions that grow rather than damp with z.

We are interested in the wave magnetic field, which is simply related to the spatial derivative of the electric field by one of the Maxwell equations. If we write

$$\vec{B} = B(\hat{y} + i\,\hat{x}) e^{-i\omega t} ,$$

we have

$$\vec{\nabla} \times E(\hat{x} - i\,\hat{y}) e^{-i\omega t} = \frac{-i}{c} \frac{\partial}{\partial t} B(\hat{y} + i\,\hat{x}) e^{-i\omega t} .$$

Hence,

$$B = \frac{-ic}{\omega} \frac{\partial E}{\partial z} . \tag{8}$$

Using Eqs. (7) and (8), we obtain the equations

$$B(z) = \frac{B_b}{\pi i} \int_{-\infty}^{\infty} \frac{k dk e^{ikz}}{k^2 - (\omega^2/c^2) n^2(k)} \quad \text{for } z \geq 0, \tag{9}$$

$$B(z) = 0 \qquad \text{for } z < 0 .$$

To obtain $B(z)$ by contour integration, we must analytically continue $n^2(k)$ off the real k axis. Since the integrand of Eq. (9) contains both G_i and G_e, there are two branch cuts in the upper half k plane: one separating the functions G_i^+ and G_i^-; the other separating the functions G_e^+ and G_e^-. These cuts separate the upper k plane into three regions. Both cuts extend from $k = 0$ to $k = \infty$. $G^+(k)$ is the analytic continuation of $G(k)$ from the positive real k axis into the complex k plane. $G^-(k)$ is the analytic continuation of $G(k)$ from the negative real k axis into the complex k plane. To maintain $G(k)$ single-valued in the k plane, we must cut the k plane along some path between $k = 0$ and $k = \infty$. We therefore see that each cut in the k plane separates the region of the k plane where we use $G^+(k)$ from the region of the k plane where we use $G^-(k)$.

Since $G(k = \infty) = 0$, $n^2(k = \infty) = 1$, and we find that the integrand in Eq. (9) vanishes on a semicircle at infinite k. We may therefore add this semicircle to our original contour of integration, the real k axis, without changing the result. We then shrink the resultant closed contour to as small an area as possible, being careful not to cross any poles of the integrand or the branch cuts. A typical situation is shown in Fig. 2. The determination of the positions of poles and cuts is explained later.

Denoting the three regions by the numbers 1, 2, and 3, we have three corresponding different functions $n_1^2(k)$, $n_2^2(k)$, and $n_3^2(k)$ in the integrand. Corresponding to Fig. 2, we have

$$n_1^2 = n^2(G_i^+, G_e^-),$$

$$n_2^2 = n^2(G_i^-, G_e^-),$$

and

$$n_3^2 = n^2(G_i^+, G_e^+).$$

Assume that there are N_1 poles of the function

$$H_1(k) = \frac{k e^{ikz}}{k^2 - (\omega^2/c^2) n_1^2(k)} \tag{10}$$

in region 1, N_2 poles of $H_2(k)$ in region 2, and N_3 poles of $H_3(k)$ in region 3, where H_2 and H_3 are defined as in Eq. (10) in terms of $n_2^2(k)$ and $n_3^2(k)$. Let

$$B_1 = 2B_b \sum_{a=1}^{N_1} [\text{Res } H_1(k); K_a] ,$$

where

$$[\text{Res } H_1(k); k_a] = \frac{k_a \exp(ik_a z)}{\left[\frac{d}{dk}(k^2 - (\omega^2/c^2) n_1^2(k)\right]_{k=k_a}} ,$$

when k_a is a simple zero of the denominator of $H_1(k)$. B_2 and B_3 have similar definitions for regions 2 and 3. The general solution is then

$$B(z) = B_1 + B_2 + B_3 + B_{12} + B_{31} .$$

The contribution B_{12} arises from the integral along the branch cut separating regions 1 and 2. It is expressible in terms of the difference of $H_1(k)$ and $H_2(k)$. We then have

$$B_{12} = \int_{C_{12}} g_{12}(k)\, dk \exp\left[ikz - \left(\frac{\omega - \omega_{ic} + i\nu_i}{k a_i}\right)^2\right], \tag{11}$$

where the part of the integrand that is a relatively weak function of k is

$$g_{12}(k) = \frac{2B_b\, \omega_{pi}^2 \omega (1 + i\nu_e G_e^-)^2}{\pi^{1/2} c^2 a_i D(G_i^+, G_e^-) [k^2 - (\omega^2/c^2) n_1^2(k)] D(G_i^-; G_e^-) [k^2 - (\omega^2/c^2) n_2^2(k)]}$$

and where the contour C_{12} is along the cut between regions 1 and 2 from $k = 0$ to $k = \infty$.

Since the exponential part of the integrand drops sharply to zero at both end points of the contour, the integral can be approximated by the method of steepest descents.[11] The result is

$$B_{12} = \pi^{1/2} \frac{dk}{d\sigma} \exp\left[f(k_{si})\right] ,$$

where

$$\frac{dk}{d\sigma} = \left[\frac{-f''(k_{si})}{2}\right]^{-1/2} ,$$

$$f(k) = ikz - \left(\frac{\omega-\omega_{ic} + i\nu_i}{ka_i}\right)^2 + \ln g(k) ,$$

and k_{si}, the saddle point, is determined by

$$f'(k_{si}) = iz + \frac{2}{k_{si}}\left(\frac{\omega-\omega_{ic}+i\nu_i}{k_{si}a_i}\right)^2 + \frac{g'(k_{si})}{g(k_{si})} = 0.$$

The steepest-descent approximation requires that the contour, C_{12}, be along a specified path from $k = 0$ through the saddle point, k_{si} to $k = \infty$. Thus the position of the branch cut in the k plane is chosen so that the resulting integral for B_{12} (Eq. 11) can be evaluated by the method of steepest descent. Since $g(k)$ is relatively slowly varying, an approximation to k_{si} is obtained by ignoring the term

$$\frac{g'(k_{si})}{g(k_{si})}$$

to obtain

$$k_{si} = \left[\frac{2i(\omega-\omega_{ic} + i\nu_i)}{a_i^2 z}\right]^{1/3} . \tag{12}$$

Again ignoring the dependence of $g(k)$, we find that the criterion of validity of the steepest-descent approximation,

$$\left| \frac{f'''(k_{si})}{[f''(k_{si})]^{3/2}} \right| << 1 ,$$

leads to the requirement

$$\frac{4}{6^{1/2}} \left| \frac{k_{si}\, a_i}{\omega - \omega_{ic} + i\, \nu_i} \right| << 1 . \tag{13}$$

Using Eq. (12), we find that the method of steepest descent is adequate at large distances from the boundary. The criterion is

$$z >> \frac{8.7\, a_i}{|\omega - \omega_{ic} + i\, \nu_i|} .$$

The result is then

$$B_{12} = \frac{2B_b \omega_{pi}^2 \omega k_{si}^2 \exp\left[-3\left(\frac{\omega-\omega_{ic}+i\,\nu_i}{k_{si}\, a_i}\right)^2\right]}{3^{1/2}\, c^2 (\omega-\omega_{ic}+i\,\nu_i)}$$

$$\times \left\{ \frac{(1+i\,\nu_e\, G_e^-)^2}{D(G_i^+, G_e^-)[k^2-(\omega^2/c^2)n_1^2]\, D(G_i^-, G_e^-)[k^2-(\omega^2/c^2)n_2^2]} \right\}_{k=k_{si}}$$

A similar result is obtained for B_{31} :

$$B_{31} = \frac{2B_b \omega_{pe}^2 \omega k_{se}^2 \exp\left[-3\left(\frac{\omega+\omega_{ec}+i\nu_e}{k_{se}\, a_e}\right)^2\right]}{3^{1/2}\, c^2 (\omega+\omega_{ec}+i\,\nu_e)}$$

$$\times \left\{ \frac{(1+i\,\nu_i\, G_i^+)^2}{D(G_i^+, G_e^-)[k^2-(\omega^2/c^2)n_1^2]\, D(G_i^+, G_e^+)[k^2-(\omega^2/c^2)n_3^2]} \right\}_{k=k_{se}}$$

where

$$k_{se} = \left[\frac{2i(\omega+\omega_{ec}+i\nu_e)^2}{a_e^2 z} \right]^{1/3} .$$

These results are correct when requirement (13) and its analog for the electrons are satisfied. The exponential parts of the expression for B_{31} and B_{12} are then very small. The term B_{31} for the electron branch cut is much smaller than the term B_{12} for the ion branch cut for frequencies satisfying the criterion

$$\omega < \sqrt{\omega_{ic}\,\omega_{ec}} .$$

This criterion is satisfied by the frequencies in our range of interest. B_{31} may therefore be neglected. The electron thermal effects which occur in B_{12} and n^2 are also negligible for the frequencies and wavelengths of interest. This can be explicitly shown by consideration of the function G_e^-, which contains these effects. Since the wave numbers of interest are such that $\mathrm{Im}\ \Phi_e < 0$, G_e^- has the asymptotic expansion for large argument

$$G_e^- \approx \frac{-1}{\omega+\omega_{ec}+i\nu_e} \left[1 + \frac{1}{2\Phi_e^2} + \frac{3}{4\Phi_e^4} + \frac{3\cdot 5}{8\Phi_e^6} + \cdots \right] . \qquad (14)$$

Since the wavelengths of interest are such that

$$1/\epsilon_e = |\Phi_e|^2 >> 1 ,$$

the first term of the expansion is kept and the rest discarded. It can be shown[13] that the expansion (14) corresponds to the use of the truncated set of equations for the moments of the distribution function. Each term of the expansion corresponds to the retention of another moment of a hierarchy of moments. The term ϵ_e is the measure of the electron thermal effects. When these effects are small, the expansion or the electron moment equations may be used. Replacing G_e^- by the first term in the expansion and neglecting the electron branch cut is equivalent to using the moment

equation for the electron flow velocity and neglecting electron viscosity and the higher moments. Since

$$B_{31} \propto \exp(-3\Phi_e^{\,2}) ,$$

where

$$\mathrm{Re}\ \Phi_e^{\,2} > 0,$$

the contribution of the electron branch cut is not expandable in terms of $1/\Phi_e$ and is therefore unobtainable from the truncated moment equations.

With the electron thermal effects ignored we have

$$n^2(k) = 1 + \frac{(\omega_{pi}^{\,2}/\omega)\,(\omega+\omega_{ec})G_i - (\omega_{pe}^{\,2}/\omega)\,(1 + i\nu_i\, G_i)}{(1+i\nu_i G_i)(\omega+\omega_{ec}) - i\nu_i m_{ie}\,[(G_i/m_i)(\omega+\omega_{ec}) - (1+i\nu_i G_i)/m_e]} . \tag{15}$$

There is now no electron branch cut and therefore no region 3. $B_{31} = B_3 = 0$. Region 1 now includes what was region 3.

III. Solution by Expansion

The asymptotic expansion of G_i is[12]

$$G_i^{\pm} \approx \frac{i\sqrt{\pi}}{ka_i}\,\gamma^{\pm}\exp(-\Phi_i^{\,2}) - \frac{1}{(\omega-\omega_{ic}+i\nu_i)}\left[1 + \frac{1}{2\Phi_i^{\,2}} + \frac{3}{4\Phi_i^{\,4}} + \cdots + \frac{(2s-1)(2s-3)\cdots 3\cdot 1}{(2\Phi_i^{\,2})^s} + \cdots\right] \tag{16}$$

where

$\gamma^+ = 2,\ \gamma^- = 0$ for $\mathrm{Im}\ \Phi_i < 0$,

$\gamma^+ = 1,\ \gamma^- = -1$ for $\mathrm{Im}\ \Phi_i = 0$,

$\gamma^+ = 0,\ \gamma^- = -2$ for $\mathrm{Im}\ \Phi > 0$.

Using the moment equations rather than the kinetic equation leads to the above expansion with the exception of the first term. Then the expansion is the same for

G_i^+ and G_i^-. Keeping higher moments in the moment approach is equivalent to keeping higher terms in this expansion. However, the use of the truncated moment equations precludes any knowledge of the existence of the branch cut, since these equations lead to the same expression for $G_i(k)$ for all k. Since $G_i(k)$ appears to be single-valued, no cut appears and B_{12} is nonexistent.

The moment equations, then, may be incorrect for two reasons: First, B_{12}, the branch cut contribution which is unobtainable from the moment equations, may be significant. Second, the expansion (16) diverges for any finite Φ_i. According to the theory of asymptotic expansions,[14] the best numerical approximation to G_i is obtained by the use of a finite number of terms of the expansion. The error is of the order of magnitude of the last term used. Therefore, the number of moments that should be retained for a quantitatively accurate result depends on the magnitude of

$$\Phi_i = \frac{\omega-\omega_{ic} + i\nu_i}{ka_i} ,$$

which is not known until the problem is solved, that is, $k(\omega)$ is found. The retention of too many moments leads to inaccurate results.

We shall first solve the problem by keeping just one or two terms of the expansion of G_i. The smallness parameter of the expansion is

$$\epsilon_i = \frac{ka_i}{(\omega-\omega_{ic} + i\nu_i)}^2 .$$

We define

$$Y = 1 + (\omega-\omega_{ic} + i\nu_i)\, G_i .$$

The expression for n^2 is now

$$n^2 = 1 - \frac{\omega_p^2 - \frac{Y}{\omega}\left[\omega_{pi}^2(\omega+\omega_{ec}) - \omega_{pe}^2\, i\nu_i\right]}{(\omega-\omega_{ic})(\omega+\omega_{ec}) + i\nu_i\omega + iY\left[(\nu_i - (m_{ie}/m_i)\nu_i)(\omega+\omega_{ec}) + i\nu_i\nu_i(m_{ie}/m_e)\right]} . \qquad (17)$$

Retention of just the first term of the expansion reduces Y to zero. We then have

$$n^2 = k_0^{\,2}\omega^2/c^2 = 1 - \omega_p^{\,2}/[(\omega-\omega_{ic})(\omega+\omega_{ec})+i\nu_1\omega] , \tag{18}$$

and the solution

$$B(z) = B_b\, e^{ik_0 z}, \qquad I_m(k_0) > 0 \tag{19}$$

This is the result obtained by using moment equations and neglecting ion viscosity and higher moments. The retention of the first two terms yields

$$Y \approx -\frac{1}{2\Phi_i^{\,2}} = \frac{-k^2 a_i^{\,2}}{2(\omega-\omega_{ic}+i\,\nu_i)^2} .$$

Substitution into Eq. (17) yields

$$n^2 = 1 - \frac{\omega_p^{\,2}\left[1 - \dfrac{k_-^{\,2}T_0}{(m_i+m_e)\omega(\omega-\omega_{ic}+i\nu_i)}\right]}{\left\{(\omega-\omega_{ic})(\omega+\omega_{ec})+i\nu_1\omega\right\}\left\{1 - \dfrac{k_-^{\,2}T_0[m_e(\omega+\omega_{ec})+i\,m_{ie}\,\nu_1]}{m_i m_e[\omega-\omega_{ic}+i\,\nu_i]\,[(\omega-\omega_{ic})(\omega+\omega_{ec})+i\,\nu_1\omega]}\right\}} , \tag{20}$$

exhibiting the lowest-order thermal correction, which causes the index of refraction to be dependent on k in addition to ω.

The solution is now

$$B(z) = B_b[A_1 e^{ik_1 z} + A_2 e^{ik_2 z}] , \tag{21}$$

where

$$A_1 = (\delta^2 - k_1^{\,2})/(k_2^{\,2} - k_1^{\,2}) ,$$

$$A_2 = 1 - A_1 = (\delta^2 - k_2^{\,2})/(k_1^{\,2} - k_2^{\,2}) ,$$

and

$$\delta^2 = \frac{m_i\, m_e[(\omega-\omega_{ic})(\omega+\omega_{ec}) + i\,\nu_1\omega]\,[\omega-\omega_{ic}+i\,\nu_i]}{T_0[m_e(\omega+\omega_{ec})+i\,m_{ie}\,\nu_1]} .$$

This is the result obtained by using moment equations and keeping the ion viscosity but neglecting ion heat flow and higher moments. (The solutions of the dispersion relation, k_1 and k_2, have no relation to the regions 1 and 2 previously mentioned.)

The importance of collisions to the applicability of the moment equations at resonance is apparent from the form of ϵ_i. The wavelength must be larger than the mean free path for ion collisions, that is, $1/k > a_i/\nu_i$. When this criterion is satisfied, the heat flow may be neglected. However, the viscosity may still not be negligible. The criterion for neglecting ion viscosity is more stringent in the resonance region. For $\nu_i < \omega_{ec}$, the criterion is seen from the denominator of (20) to be $\epsilon_i' << 1$, where

$$\epsilon_i' = \frac{k^2 T_o}{\nu_i m_{ie} \nu_i} \approx \frac{1}{2}\sqrt{\frac{m_i}{m_e}}\,\frac{k^2 a_i^2}{\nu_i^2} = \frac{1}{2}\left(\frac{m_i}{m_e}\right)\left|\epsilon_i(\omega=\omega_{ic})\right|^{1/2}.$$

Therefore for cases where $|\epsilon_i(\omega=\omega_{ic})| << 1$ and heat flow is negligible we may find $\epsilon_i' \gtrsim 1$, indicating that ion viscosity is not negligible and may significantly alter the results of the cold-plasma theory.

In order to display the behavior of the solutions, we choose values of density and magnetic field representative of a wave experiment[15] conducted at the Lawrence Radiation Laboratory, Berkeley. These are

$$n_o = 3.5\times10^{14}\ \mathrm{cm}^{-3}$$

and

$$B_o = 1.09\times10^{4}\ \text{gauss}.$$

We then have the following values for a deuterium plasma:

$$\omega_p = 1.06 \times 10^{12}\ \mathrm{sec}^{-1};$$

$$\omega_{ec} = 1.92 \times 10^{11}\ \mathrm{sec}^{-1};$$

$$\omega_{ic} = 5.24 \times 10^{7}\ \mathrm{sec}^{-1}.$$

For $T_0 = 2\times10^4$ °K, we have $\nu_1 = 3.45\times10^9$ sec^{-1} and $\nu_i = 5.9\times10^7$ sec^{-1}.

Figure 3 is a plot of the trajectories of k_1 and k_2 in the complex k plane for this case; k_0 in the figure is the result obtained when viscosity is neglected. Let $\Omega = \omega/\omega_{ic}$. The quantities k_0 and k_1 are plotted for Ω values between $\Omega = 0.5$ and $\Omega = 2.0$, while k_2 is plotted for $0.95 \leq \Omega \leq 1.05$. Beyond this region, k_2 becomes too large to neglect heat flow and higher moments. That is,

$$\left| \frac{k_2^{\,2}\, a_i^{\,2}}{(\omega-\omega_{ic}+i\,\nu_i)^2} \right|$$

becomes comparable to unity, nullifying the validity of the expansion and therefore the moment approach. Where this is so, the coefficient of the k_2 wave, A_2, becomes negligibly small. ($|A_2|$ is plotted as a function of Ω in the lower left section of Fig. 3.) Therefore, the behavior of k_2 is not known where it is not needed. At $\Omega = 1.015$, however, $|A_2| = 0.35$. At $T_0 = 2\times10^4$, therefore, the viscosity is at the threshold of importance.

So for $0.95 \leq \Omega \leq 1.05$, the expression (21) is necessary. Beyond this region, expression (19) suffices.

For order-of-magnitude estimates of damping, the viscosity may be ignored and expression (19) used.

At lower temperatures, it is found that the k_2 wave may be entirely ignored. The coefficient, A_1, remains essentially unity, $k_1 \approx k_0$, and k_2 recedes to infinity corresponding to zero damping length.

We now derive the criterion for neglecting the thermal effects in terms of the density, temperature, and magnetic field of the plasma. We have

$$k^2\, T_0/\nu_i\, m_{ie}\, \nu_1 \ll 1.$$

We choose for k^2 the value which is obtained from the nonviscous dispersion relation, (18) when $\omega \approx \omega_{ic}$. We have

$$k_0^2(\omega_{ic}) = \omega_{ic}^2/c^2 + (i\,\omega_{ic}\,\omega_p^2)/\nu_1\,c^2 \quad .$$

The first term on the right, which corresponds to the displacement current, may be ignored in our region of interest, where

$$\omega_p^2/\omega_{ic}\,\nu_1 >> 1 \, .$$

Substitution produces the criterion

$$\epsilon^{00} = \frac{\omega_{ic}\,\omega_p^2}{\nu_1\,c^2} \quad \frac{T_0}{\nu_i\,\nu_1 m_{ie}} = \frac{3\times10^3 B_0 T_0^{11/2}}{n_0^2 (\ln\Lambda)^3} << 1 .$$

Inserting the value for parameters pertaining to Fig. 3, we obtain the value $0.28 < 1$. Figure 3 then represents a case in which the criterion is barely satisfied. Noticing that the criterion is heavily temperature-dependent, we examine the case when $T_0 = 3\times10^4\,°K$. We then obtain the value $2.1 > 1$. Now the viscosity must be kept. Figure 4 is a plot of the trajectories of k_1 and k_2 for this case. The trajectory of k_0 is included for comparison and $|A_2|$ is plotted as before. The coefficient $|A_1|$ is approximately $1-|A_2|$, and is therefore not plotted in Figs. 3 and 4.

Now it is found that, for $0.9 \leq \Omega \leq 1.1$, expression (21) is necessary. It is also found that the wave k_1 no longer identifies with k_0; $k_1 \approx k_0$ at $\Omega = 0.9$, but $k_2 \approx k_0$ at $\Omega = 1.1$. A_2 varies from negligibly small values near $\Omega = 0.9$ to nearly unity at $\Omega = 1.05$; A_1 and A_2 are about equal at $\Omega = 1.01$. Now, no order-of-magnitude estimates can be made for the damping at resonance by considering one wave alone. The disturbance is expressed in terms of two wave forms.

We illustrate the wave form of $B_-(z)$ at resonance on Fig. 5 for the temperature $T = 3\times10^4$ °K. The upper plot is Real$[A_1 \exp(ik_1 z)]$, the contribution of k_1 to the total solution. The middle plot is Real$[A_2 \exp(ik_2 z)]$. The lower plot is Real$[A_1 \exp(ik_1 z) + A_2 \exp(ik_2 z)]$, the complete solution as given by Eq. (21). For comparison, we include Real$[\exp(ik_0 z)]$, the solution without viscosity. Since

$$\vec{B} = B_-(z)\,(\hat{y} + i\,\hat{x})e^{-i\omega t},$$

we are plotting the component of $\vec{B}$ in the direction $(\hat{y}\cos\omega t + \hat{x}\sin\omega t)$ at time t as a function of z. Referring to the lowest plot, we find that both waves are severely damped, but the wave that includes the viscosity effects does not decrease as abruptly as the other wave. The viscosity acts to reduce the shear caused by the spatial variation of the wave field.

The question arises as to the effect of the next term of the expansion (heat flow) on these solutions. Will a third wave form arise? Including heat flow necessitates solving a cubic equation for $k^2(\omega)$. Since heat flow is unimportant unless we have $\epsilon_i \gtrsim 1$, in which case the moment expansion is invalid, nothing is gained by its inclusion. Instead, the moment expansion is abandoned. In the next section, the problem is done without the expansion. This method is necessary for $\epsilon_i \gtrsim 1$. Solution of Eq. (20) shows $\epsilon_i \approx 1$ in the resonance region for $T_0 = 10^5$ °K. Therefore, for temperatures of this order and higher and for the previously mentioned values of density and magnetic field, the expansion method cannot be used.

No simple criterion exists for the determination of ϵ_i in general. A coarse criterion is obtained by again using $k_0^2(\omega_{ic})$ in the expression for ϵ_i. At $\omega = \omega_{ic}$, we have

$$\epsilon_i^0 = \frac{\omega_{ic}\,\omega_p^2}{\nu_i\,c^2}\;\frac{a_i^2}{\nu_i^2} = \frac{10^2\,B_0\,T_0^{11/2}}{n_0^2(\ln\Lambda)^3}\,.$$

Here $T_0 = 10^5$ corresponds to $\epsilon_i^0 = 28$. The coarse criterion is too pessimistic. Figure 4 shows $|k_1(\omega_{ic})| < |k_0(\omega_{ic})|$. Lacking a simple expression for the pertinent values of k_1 or k_2, $\epsilon_i^0 < 1$ will be considered a sufficient condition for using the moment equations and neglecting heat flow. If $\epsilon_i^0 > 1$ but not by much, it would be worthwhile to neglect heat flow and check the value of ϵ_i pertaining to the solutions k_1 and k_2 where they are important.

IV. Solution Without Expansion

We shall now solve the problem without expanding G_i. With the electron thermal effects ignored and using the functions

$$Y^{\pm} = 1 + (\omega - \omega_{ic} + i\nu_i)G_i^{\pm},$$

we have

$$B_{12} = \frac{2B_b\,\omega_{pi}^2\,\omega(\omega-\omega_{ic}+i\nu_i)(\omega+\omega_{ec})^2 k_s^2 \exp\left\{-3\,[(\omega-\omega_{ic}+i\nu_i)/k_s a_i]^2\right\}}{3^{1/2}c^2\,\chi(Y^+(k_s))[k_s^2-(\omega^2/c^2)n^2(Y^+(k_s))]\,\chi(Y^-(k_s))[k_s^2-(\omega^2/c^2)n^2(Y^-(k_s))]}$$

where

$$\chi(Y) = (\omega-\omega_{ic})(\omega+\omega_{ec}) + i\nu_i\omega + iY\left\{\nu_i(\omega+\omega_{ec}) - m_{ie}\nu_i[(\omega+\omega_{ec}/m_i) - (i\nu_i/m_e)]\right\}$$

and

$$n^2(Y) = 1 - \frac{\omega_p^2 - (Y/\omega)[\omega_{pi}^2(\omega+\omega_{ec}) - i\,\omega_{pe}^2\,\nu_i]}{\chi(Y)}.$$

The "i" subscript on k_s has been dropped. We use Y^+ in region 1 and Y^- in region 2 when looking for zeros of the function

$$k^2 - (\omega^2/c^2)n^2(Y)$$

which correspond to the poles of the integrand of Eq. (9).

Since the boundary between regions 1 and 2 is the steepest descent contour, the contour's position in the k plane must be known in relation to the poles. The contour is defined by $\mathrm{Im}\, f(k) = \mathrm{Im}\, f(k_s)$.

Numerical work shows the contour to leave the origin at the angle $a = \arg(\omega-\omega_{ic} + i\nu_i)$ and tend to infinity at the angle $\pi/2$ with or without the inclusion of the term $\ln g(k)$ in the expression for $f(k)$. With $g(k)$ excluded, the contour has an asymptote at $k = (3/2)\,\mathrm{Re}\, k_s$. An example of the contour was sketched in Fig. 2. The asymptote and the contour can be affected by $g(k)$. We define

$$z_{near} = \left| \frac{50\, a_i}{\omega-\omega_{ic} + i\,\nu_i} \right|$$

At this value of z, we find that the contour path and asymptote are radically changed by the addition of $g(k)$ when the path passes close to a pole. This is precisely the situation for which the path must be accurate and hence $g(k)$ must be retained. As z increases, the relative effect of $g(k)$ decreases.

For $z < z_{near}$, the steepest-descent approximation becomes inaccurate; z_{near} is then the smallest distance from the boundary for which the steepest-descent contour and the value of B_{12} are known to a reasonable degree of accuracy (about 10%

For the parameters

$$n_0 = 3.5\times10^{14}\ \mathrm{cm}^{-3},$$

$$B_0 = 1.09\times10^{4}\ \text{gauss},$$

and

$$T_0 = 2\times10^{4}\,{}^\circ\mathrm{K} \text{ and } 3\times10^{4}\,{}^\circ\mathrm{K}$$

previously used, the results are very nearly identical to those previously obtained. We find only two poles, whose trajectories follow those outlined in Figs. 3 and 4. Defining

$$B_1 = a_1 B_b \exp(ik_1 z) ,$$

$$B_2 = a_2 B_b \exp(ik_2 z)$$

and

$$B_{12} = a_{12}(z) B_b ,$$

we find that the coefficients a_1 and a_2 agree to 1% with the coefficients A_1 and A_2. The sum $a_1 + a_2$ is unity to a good approximation. This means that B_{12} is negligible at $z = 0$, since

$$B_b = B_1(0) + B_2(0) + B_{12}(0)$$

requires

$$a_{12}(0) = 1 - a_1 - a_2 . \tag{22}$$

At z_{near}, $a_{12} \approx 10^{-20}$; z_{near} is quite small. Its maximum value is about 3 cm at $\omega = \omega_{ic}$. Thus B_{12} is completely negligible and the expansions (electron viscosity and ion heat flow neglected) previously used are adequate for these low temperatures.

The next case we consider is

$$T_0 = 10^5 \cdot K,$$

where we found the criterion for validity of the expansion to be violated. At this temperature we still find two poles, k_1 and k_2. However, their trajectories, which appear in Fig. 6, show that their magnitudes are smaller than those of the lower-temperature case of Fig. 4. A third pole, labeled k_3, also appears when $\Omega > 1.1$. Its trajectory is shown in Fig. 6 for $1.1 \leq \Omega \leq 1.2$. The trajectory is not carried to higher Ω because this pole does not contribute to the solution. It is a pole of the function

$$\frac{k e^{ikz}}{k^2 - (\omega^2/c^2) n^2(Y^+)} ,$$

and it occurs to the left of the steepest-descent contours for z_{near} and $z_{far} = 2z_{near}$, which are sketched in Fig. 6 for $\Omega = 1.2$. Only poles of this function which lie to the right of the contour contribute to $B(z)$. Notice that the contour

approaches the imaginary axis with increasing z. At some $z >> z_{far}$, the steepest-descent contour will be on the other side of k_3 at $\Omega = 1.2$. Then k_3 will be part of the complete solution. However, at this distance (≈ 10 meters) the contribution of this pole to the solution is infinitesimal, since its damping length is 1 cm.

We therefore have only two distinct waves from the poles with exponential spatial dependence. The branch-cut contribution, B_{12}, which was negligible at lower temperatures, is now on the threshold of importance. The coefficients a_1 and a_2 are plotted in Fig. 6 along with $a_{12}(0)$. The maximum value of $a_{12}(0)$ occurs near resonance, where $a_{12}(0) \approx 0.2$. Thus $B(z)$ may still be approximated by the two exponential solutions. However, the moment equations incorrectly describe these solutions. They must be obtained by the kinetic treatment.

The final case we consider is

$$T_0 = 5\times10^5 \cdot K .$$

This case is representative of the low-collision-frequency regime in which B_{12} is significant. We again find two poles. Their trajectories are plotted in Fig. 7. A check of the results using a collisionless theory shows essentially the same results for k_1, k_2, a_1, a_2 and therefore $a_{12}(z = 0)$. At $T_0 = 5\times10^5 \cdot K$, we have

$$\nu_i = 7.7\times10^5 \text{ sec}^{-1} << \omega_{ic} = 5.24\times10^7 \text{ sec}^{-1} .$$

Thus collisions are negligible when $\omega \neq \omega_{ic}$. At resonance the function G_i in the index of refraction has the argument $\Phi_i = i\nu_i/ka_i$. Since $|ka_i| = 1.7\times10^7 \text{ sec}^{-1}$ for both k_1 and k_2 near resonance, we have $|\Phi_i| = 0.045 << 1$. Now, for small Φ_i,

$$G_i^{\pm} = \pm \frac{i\sqrt{\pi}}{k a_i} - \frac{2}{k a_i}\left[\Phi_i + O(\Phi_i)^2\right] .$$

The leading term is independent of ν_i. So for $\nu_i << \omega_{ic}$ or for $\nu_i << |a_i k_{resonance}|$ the dielectric constant, Eq. (17), may be replaced by the simpler result obtained in a collisionless theory,

$$n^2(k) = 1 + (\omega_{pi}^2/\omega)\, G_i(k) - \omega_{pe}^2/\omega(\omega+\omega_{ec}) ,$$

with very little change in the solutions B_1 and B_2. The criterion for neglecting collisions at resonance is

$$\left| \frac{\nu_i}{k\, a_i} \right| << 1.$$

We now obtain the value of k at resonance. Replacing G by the first term of the small Φ expansion, we have

$$\frac{k^2 c^2}{\omega_{ic}^2} = n^2 = 1 \pm \frac{i\sqrt{\pi}\,\omega_{pi}^2}{\omega_{ic}\, k\, a_i} - \frac{\omega_{pe}^2}{\omega_{ic}(\omega_{ic} + \omega_{ec})} .$$

Ignoring $\omega_{ic} << \omega_{ec}$ in the third term, we have

$$k^2 = \frac{\omega_{ic}^2}{c^2} + \frac{\omega_{ic}\,\omega_{pi}^2}{c^2}\left(\frac{\pm i/\pi}{k\, a_i} - \frac{1}{\omega_{ic}}\right),$$

since

$$\frac{\omega_{pe}^2}{\omega_{ec}} = \frac{\omega_{pi}^2}{\omega_{ic}} .$$

Since $\omega_{pi} >> \omega_{ic}$, the first term is negligible compared with the third. The third term is not negligible compared with the second. It is about half the magnitude of the second term in this case. This indicates that the contribution of the electron current to the dielectric constant that produces this term should be kept even at ion cyclotron resonance. For the purpose of obtaining an order-of-magnitude approximation to k, we ignore it here and obtain

$$k^3 = \frac{\pm i\,\omega_{ic}\,\omega_{pi}^{\,2}\,\pi^{1/2}}{c^2\,a_i} ,$$

so

$$|k_{1,2}|^3 \approx \omega_{ic}\,\omega_{pi}^{\,2}\,\pi^{1/2}/c^2\,a_i \quad .$$

Our criterion is then

$$\left|\frac{\nu_i}{k\,a_i}\right|^3 = \frac{c^2\,\nu_i^{\,3}}{a_i^{\,2}\,\omega_{ic}\,\omega_{pi}^{\,2}\sqrt{\pi}} \approx \frac{n_0^{\,2}\,(\ln\Lambda)^3}{3\,B_0\,T_0^{\,11/2}} << 1 \qquad (23)$$

for ignoring collisions in obtaining B_1 and B_2.

This criterion is of no use for B_{12}. The branch-cut contribution is heavily dependent on collisions. Near resonance we have

$$z_{near} = 50\,a_i/\nu_i$$

and

$$k_s(z_{near}) = i\,\nu_i/a_i\,(25)^{1/3} \approx i\,\nu_i/3\,a_i \quad .$$

Therefore $\Phi_i(k_s) = 3$ at resonance. We then have $G_i(\Phi_i)$ replaced by its asymptotic value for large argument to obtain, $G_i^{*} \approx -1/i\,\nu_i$.

In contrast to the case of the poles where $\left|\frac{\nu_i}{k\,a_i}\right| << 1$ and G_i is independent of ν_i, we have $\left|\frac{\nu_i}{k_s a_i}\right| > 1$, and G_i is now dependent on ν_i.

Thus at higher temperatures, at which the criterion (23) is satisfied, the collisions must still be kept for the investigation of the branch-cut contribution when this contribution is evaluated by the method of steepest descent.

For this low-collision regime, the significance of the branch cut is further illustrated by the following occurrence. Referring to Fig. 7, we see that the trajectory of k_2 now remains near the imaginary axis throughout the frequency range of interest.

For $\omega < \omega_{ic}$, the branch cut is in the upper left quarter plane for all z, and k_2 is on another sheet; k_2 therefore makes no contribution to the solution. For $\omega > \omega_{ic}$, the branch cut is in the upper right quarter plane for all z, and k_2 contributes to the solution. Of course, the total solution varies continuously through this apparently discontinuous change in the results. Since B_{12} is evaluated at z_{near}, a distance at which the contribution of k_2 to the solution is less than 10^{-38} for $\omega \approx \omega_{ic}$, the presence or absence of k_2 is imperceptible. At $z = 0$, the presence or absence of k_2 is important, since it has a coefficient, $a_2 \approx 0.7$ for $\omega \approx \omega_{ic}$. Since $a_1 \approx 0.65$ for $\omega \approx \omega_{ic}$, by Eq. (22) we must have $a_{12}(0) = 0.35$ for $\omega < \omega_{ic}$ and $a_{12}(0) = -0.35$ for $\omega > \omega_{ic}$. This illustrates the futility of attempting to attribute independence to each of the three terms B_1, B_2, and B_{12}. The existence of B_1 and B_2 as solutions and the value of B_{12} are wholly dependent on the choice of the position of the branch cut in the k plane.

Thus B_{12} is significant near resonance in the low-collision-frequency regime. However, the steepest-descent approximation we have used does not give us the form of B_{12} at small z. The smallest distance at which the approximation is valid, z_{near}, is of the order of meters for $T_0 = 5\times10^5\,°K$ and $\omega \approx \omega_{ic}$. At this distance B_1 and B_2 are less than $10^{-38}\,B_b$, and $B_{12} \approx 10^{-16}\,B_b$. The steepest-descent approximation determines B_{12} accurately only where it is small. This result has been useful at low temperatures ($T_0 \lesssim 10^5\,°K$), where it demonstrated that B_{12} could be neglected compared with $B_1 + B_2$. At these higher temperatures, however, it will be necessary to abandon the steepest-descent approximation near resonance in order to study the behavior of B_{12} at reasonable distances from the boundary. A numerical integration of the complex integral in Eq. (11) would have to be performed. We have not attempted this numerical analysis. In this low-collision regime, in which $\nu_i << \omega_{ic}$, the effects of collisions may not be adequately represented by the Liboff collision model we have used. The reasons for this are given in the concluding section.

V. Summary, Conclusions, and Suggestions for Further Work

We have shown that collisional effects on the ion cyclotron wave allow the wave to be described in terms of a cold plasma theory (that is, via the moment equations with zero pressure tensor) when the criterion

$$\epsilon^{00} = \frac{\omega_{ic}\,\omega_p^{\,2} T_0}{\nu_i^{\,2}\,\nu_i\,c^2 m_{ie}} \approx \frac{3 \cdot 10^3 B_0 T_0^{\,11/2}}{n_0^{\,2} (\ln \Lambda)^3} << 1$$

is satisfied. The dimensions are

T_0 in ° Kelvin, n_0 in cm^{-3}, and B_0 in gauss.

The thermal effects may be included solely through the components of the ion-pressure tensor that lead to viscosity if the resultant waves have $|k\,a_i|/\nu_i < 1$ at resonance. A coarse and pessimistic criterion is

$$\left(\frac{|k\,a_i|}{\nu_i}\right)^2 \approx \epsilon^0 = \frac{2\omega_{ic}\,\omega_p^{\,2} T_0}{\nu_i\,\nu_i^{\,2}\,c^2 m_i} \approx \frac{10^2 B_0 T_0^{\,11/2}}{n_0^{\,2} (\ln \Lambda)^3} \lesssim 1 .$$

If $\epsilon^0 \approx 1$, the resultant waves might still have

$$|k\,a_i| / \nu_i < 1.$$

If not, the moment equation approach must be abandoned. Under these conditions, the use of even higher moments than the pressure tensor is of no help, since the addition of each higher moment is equivalent to keeping another term in an asymptotic expansion of the plasma dispersion function. Since the asymptotic expansion is invalid for $|k\,a_i|/\nu_i \gtrsim 1$, the moment expansion will then be incorrect.

Using the kinetic approach, we showed that the solution of a boundary-value problem for the waves contains a new term which can be important near resonance at low collision frequencies. This term has the exponential dependence

$$\exp\left[-3\left(\frac{\omega-\omega_{ic}+i\,\nu_i}{2i\,a_i}\right)^{2/3} z^{2/3}\right]$$

at large z, which led Shafranov,[9] who discovered a similar term by using a collisionless theory for the electron cyclotron wave, to call it the dominant term near resonance. We have shown that this term is neither dominant nor negligible near resonance for low collision frequencies, and that its value is negligibly small and strongly dependent on collisions at large z, where the steepest-descent method is valid.

According to our kinetic model, collisions have no effect on waves with dependence e^{ikz} near resonance if $\nu_i << |k\,a_i|$. Using the collisionless theory to estimate k at resonance, we find the criterion for neglecting collisions to be

$$\epsilon^1 = \frac{m_i c^2 \nu_i^3}{2\sqrt{\pi}\,T_0\,\omega_{ic}\omega_{pi}^2} \approx \frac{n_0^2 (\ln \Lambda)^3}{3\,B_0 T_0^{11/2}} << 1 .$$

Figure 8 is a logarithmic plot of the lines $\epsilon^{00} = 1$, $\epsilon^0 = 1$, and $\epsilon^1 = 1$ as a function of density and temperature for the case $B_0 = 10^4$ gauss. The four cases we have treated are marked by circles.

The region below the line $\epsilon^{00} = 1$ consists of the values of n_0 and T_0 for which the thermal effects may be ignored. Between this line and the line $\epsilon^0 = 1$, the thermal effects may be introduced by ion viscosity alone. Thus below $\epsilon^0 = 1$, the moment equation approach with heat flow neglected is valid. Above the line $\epsilon^0 = 1$, the kinetic treatment must be used. Above the line $\epsilon^1 = 1$, the collisions have no effect on the waves with e^{ikz} dependence according to the collisional model we use. In this region, the new term becomes important, but is inadequately described without numerical analysis.

Further work will be necessary for the region above the line $\epsilon^{i} = 1$. This region where numerical analysis will be necessary is also the region where the relaxation collision model of Liboff,[5] which we have used, may be insufficient for the description of collisional effects. J. P. Dougherty[16] has recently introduced a model Fokker-Planck equation for the collisions of a single species of particle. His model, which necessitates solving a differential equation in velocity space, requires numerical analysis. He shows that if $\nu_i << \omega_{ic}$ his model predicts larger effects of ion-ion collisions when applied to ionospheric radar scattering than does a simpler model of the form we use. For $\nu_i \gtrsim \omega_{ic}$, both models give similar results.

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FIGURE LEGENDS

Fig. 1. Relation of contour and pole in t plane.

Fig. 2. Integration contour in k plane.

Fig. 3. Trajectories of k_1 and k_2. $T_0 = 2\times10^4$ °K. ——— k_1, k_2; --- k_0.

Fig. 4. Trajectories of k_1 and k_2. $T_0 = 3\times10^4$ °K. ——— k_1, k_2; --- k_0.

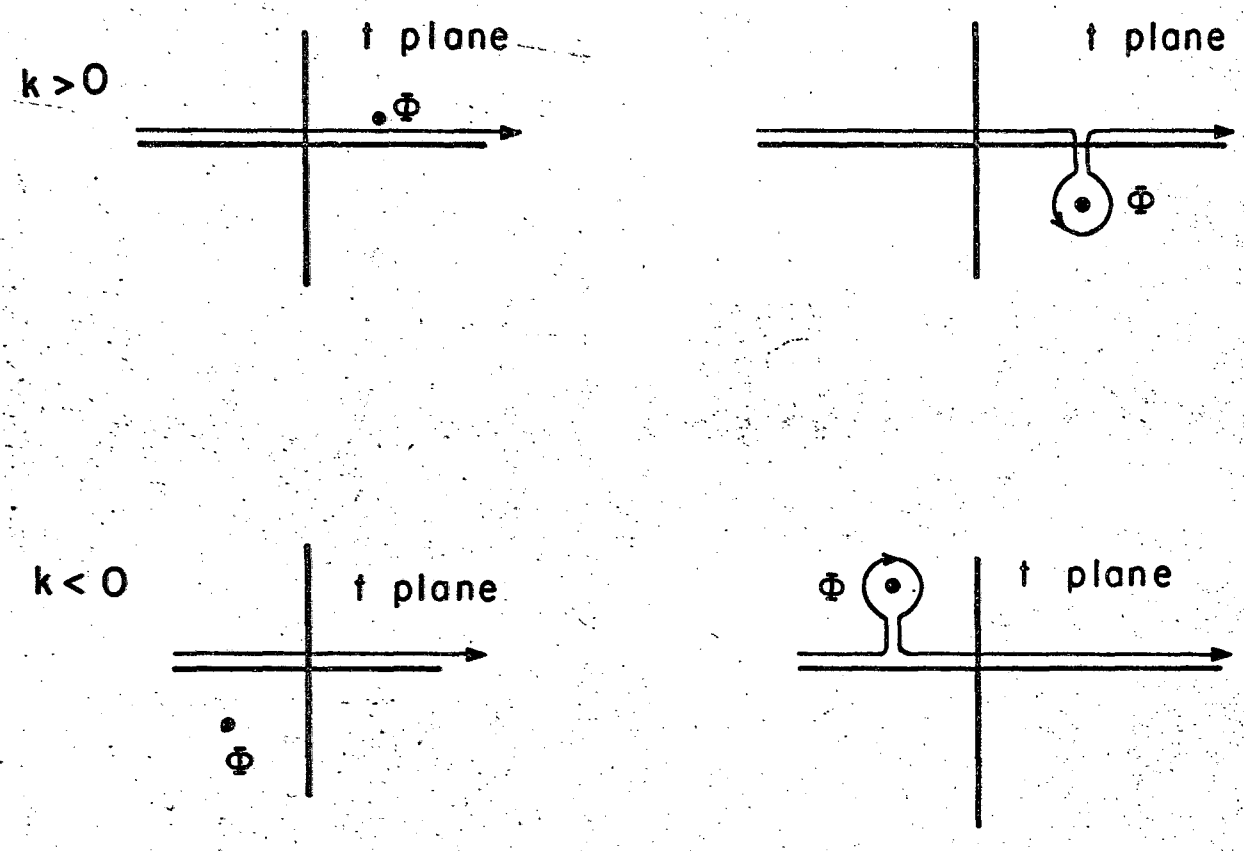
Fig. 5. Wave forms of B at resonance. $T_0 = 3\times10^4$ °K. For lowest drawing only,

——— $\mathrm{Re}\,[A_1 \exp(ik_1 z) + A_2 \exp(ik_2 z)]$; --- $\mathrm{Re}\,\exp(ik_0 z)$.

Fig. 6. Trajectories of k_1 and k_2 and sample branch cuts. $T_0 = 10^5$ °K.

Fig. 7. Trajectories of k_1 and k_2. $T_0 = 5\times10^5$ °K.

Fig. 8. Regions of validity for methods of solution.



Definition of G for k real

Analytic continuations for k complex

Fig. 1

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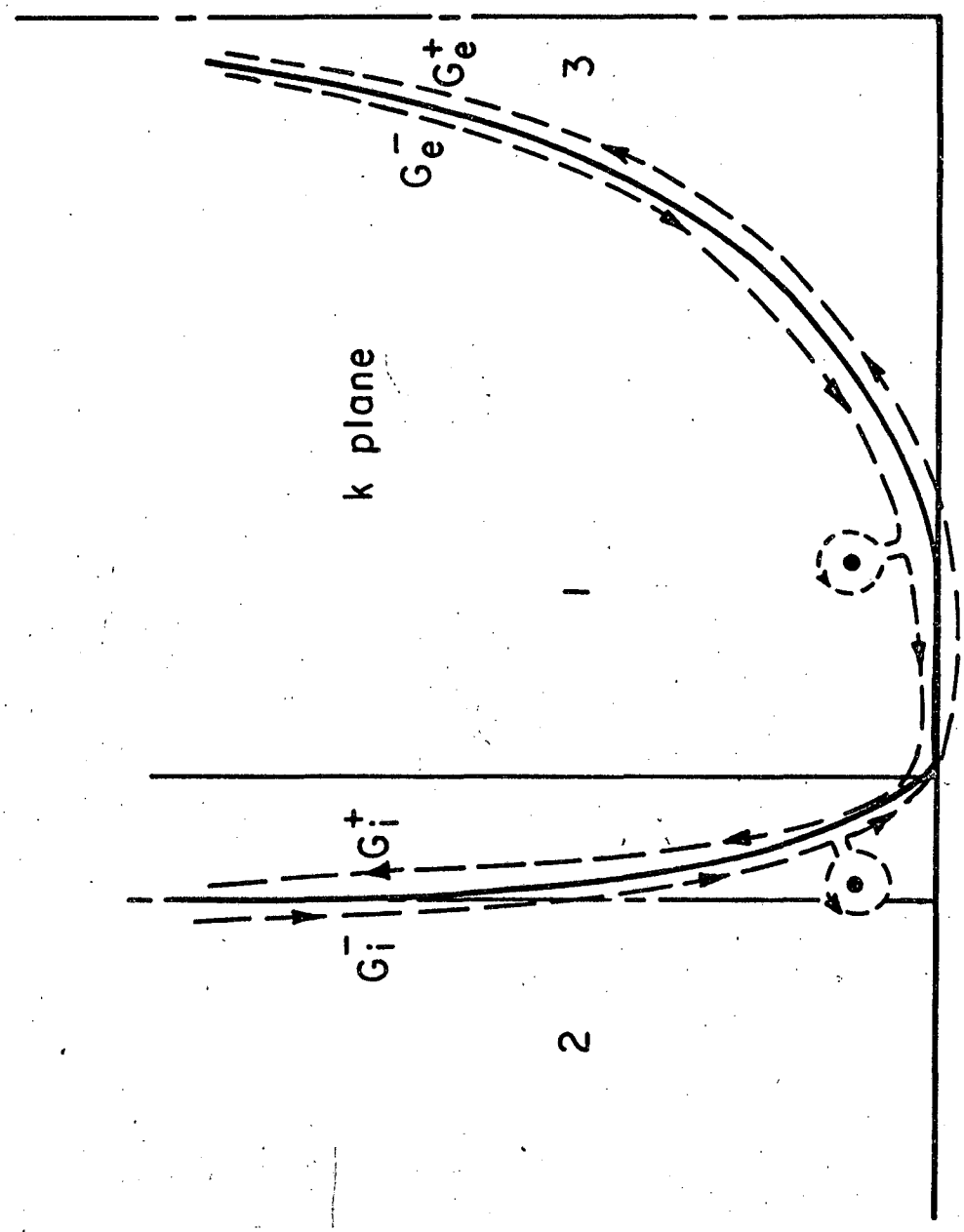

Fig. 2

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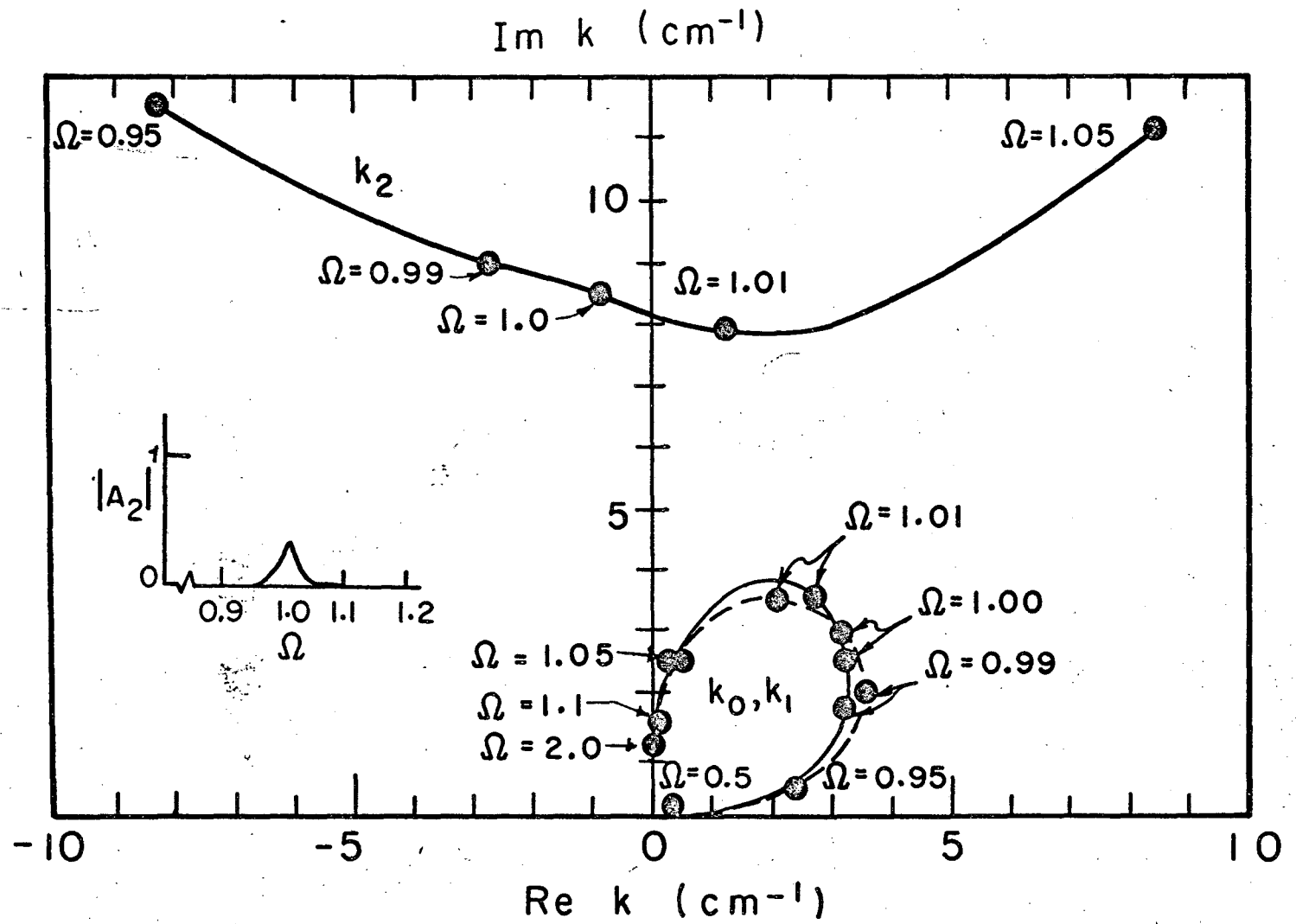

Fig. 3

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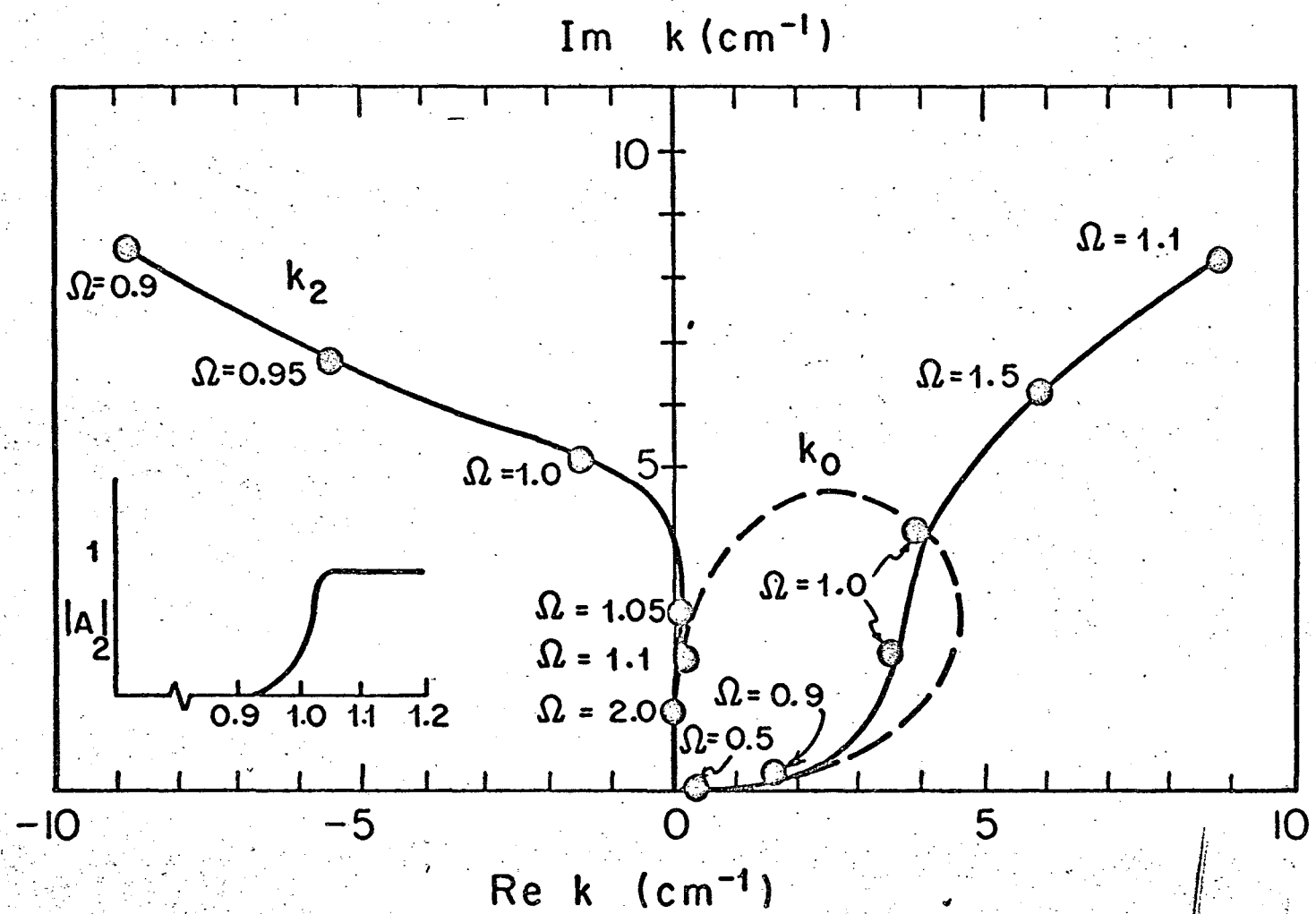

Fig. 4

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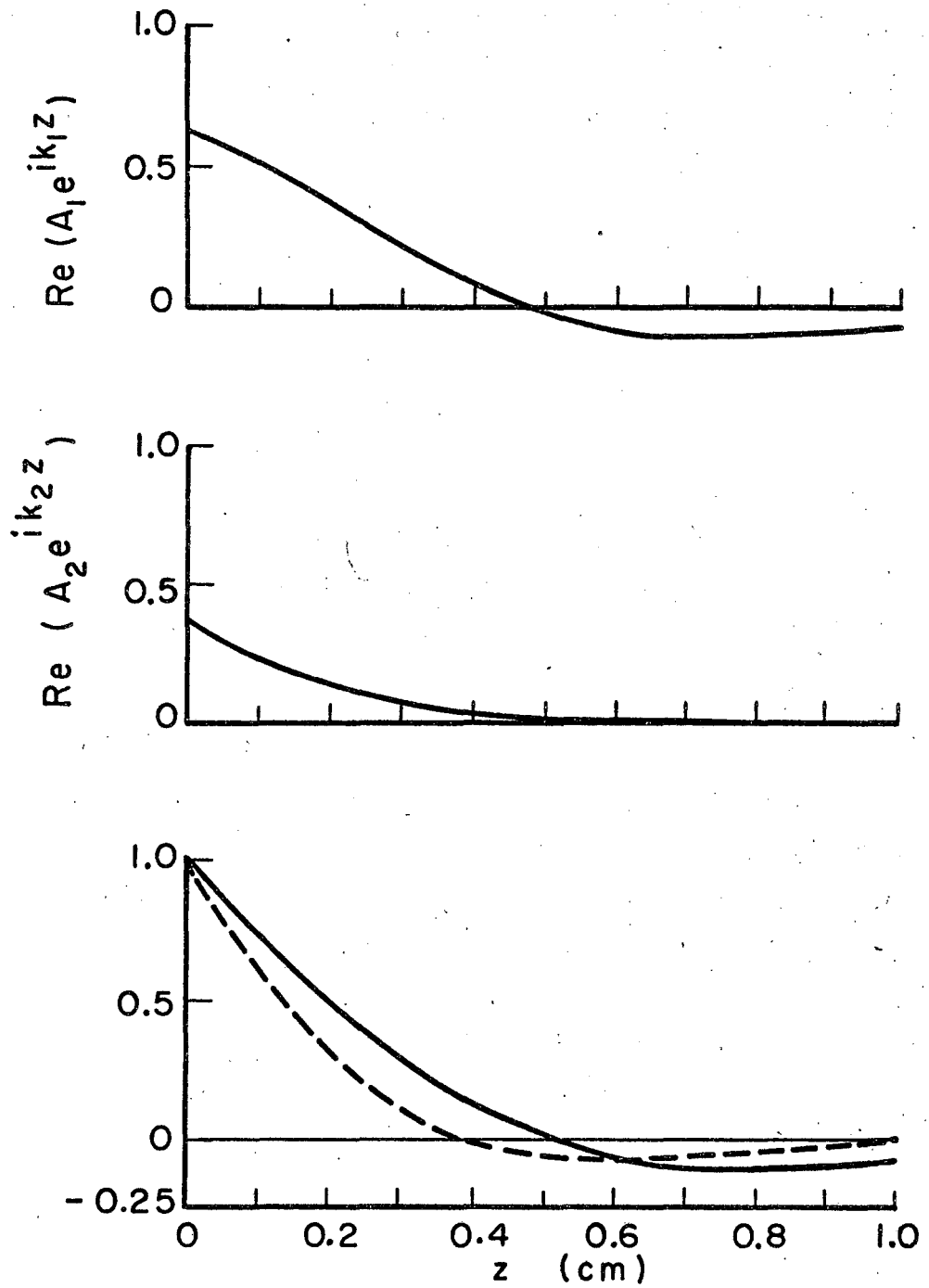

Fig. 5

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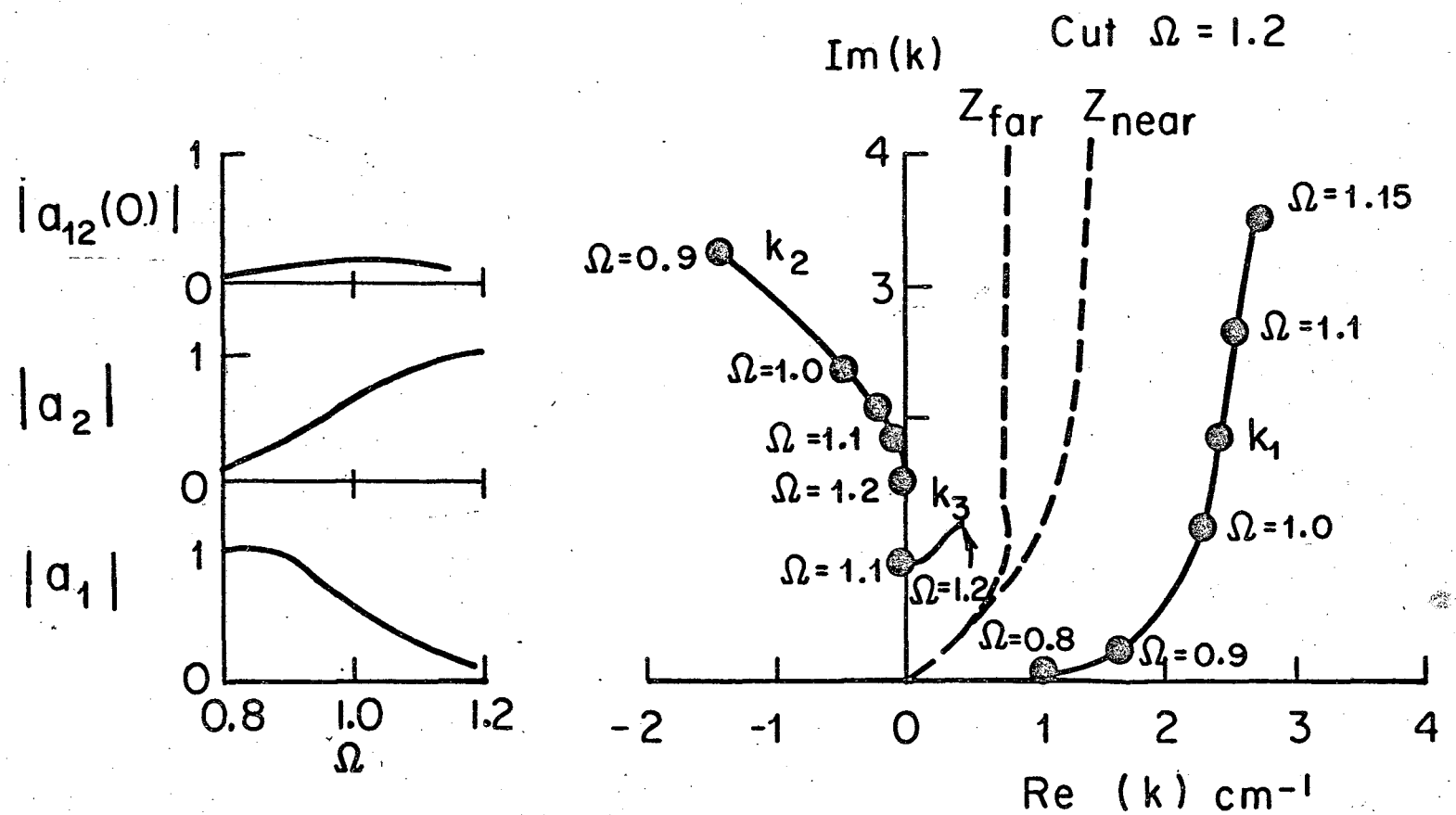

Fig. 6

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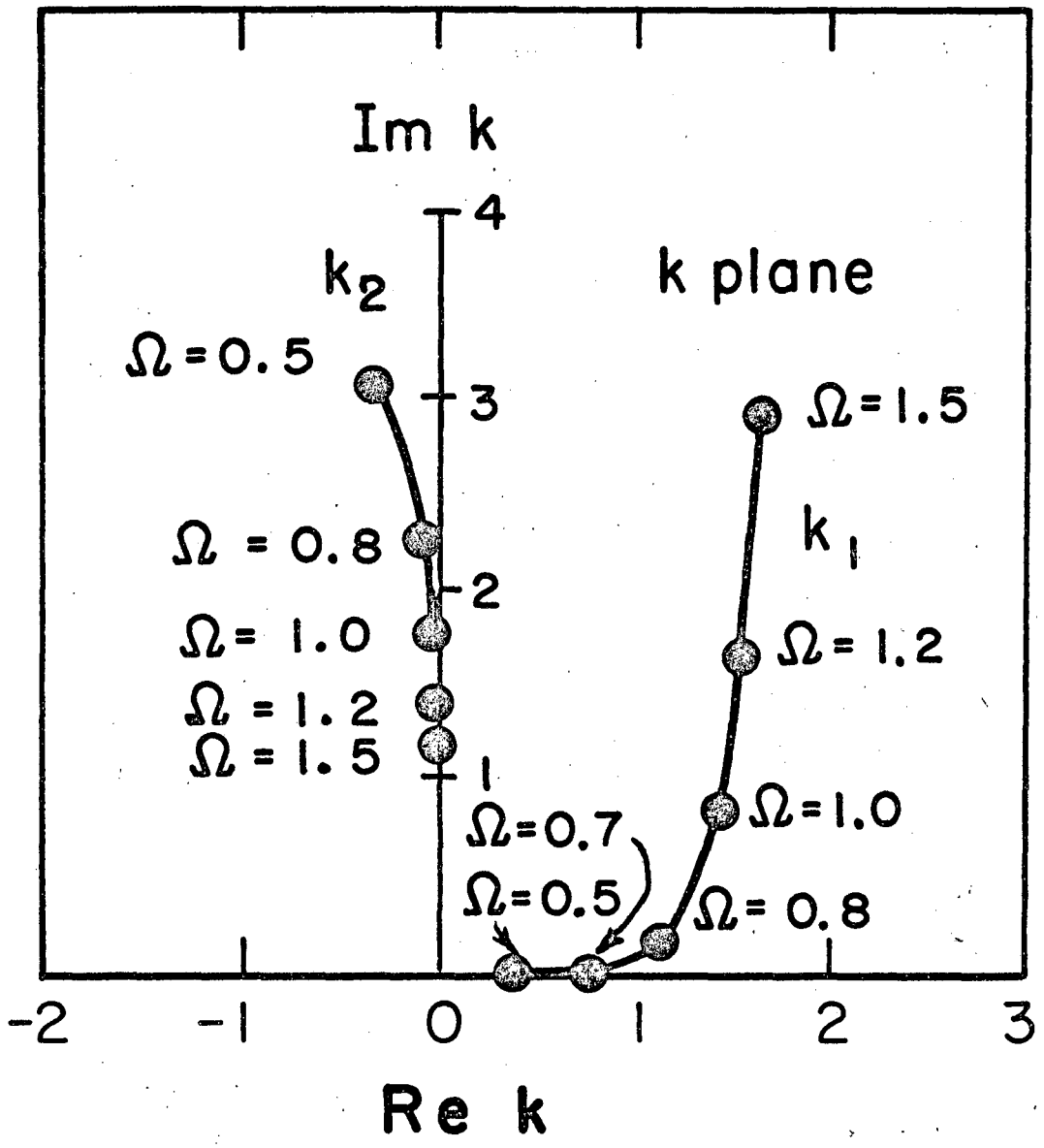

Fig. 7

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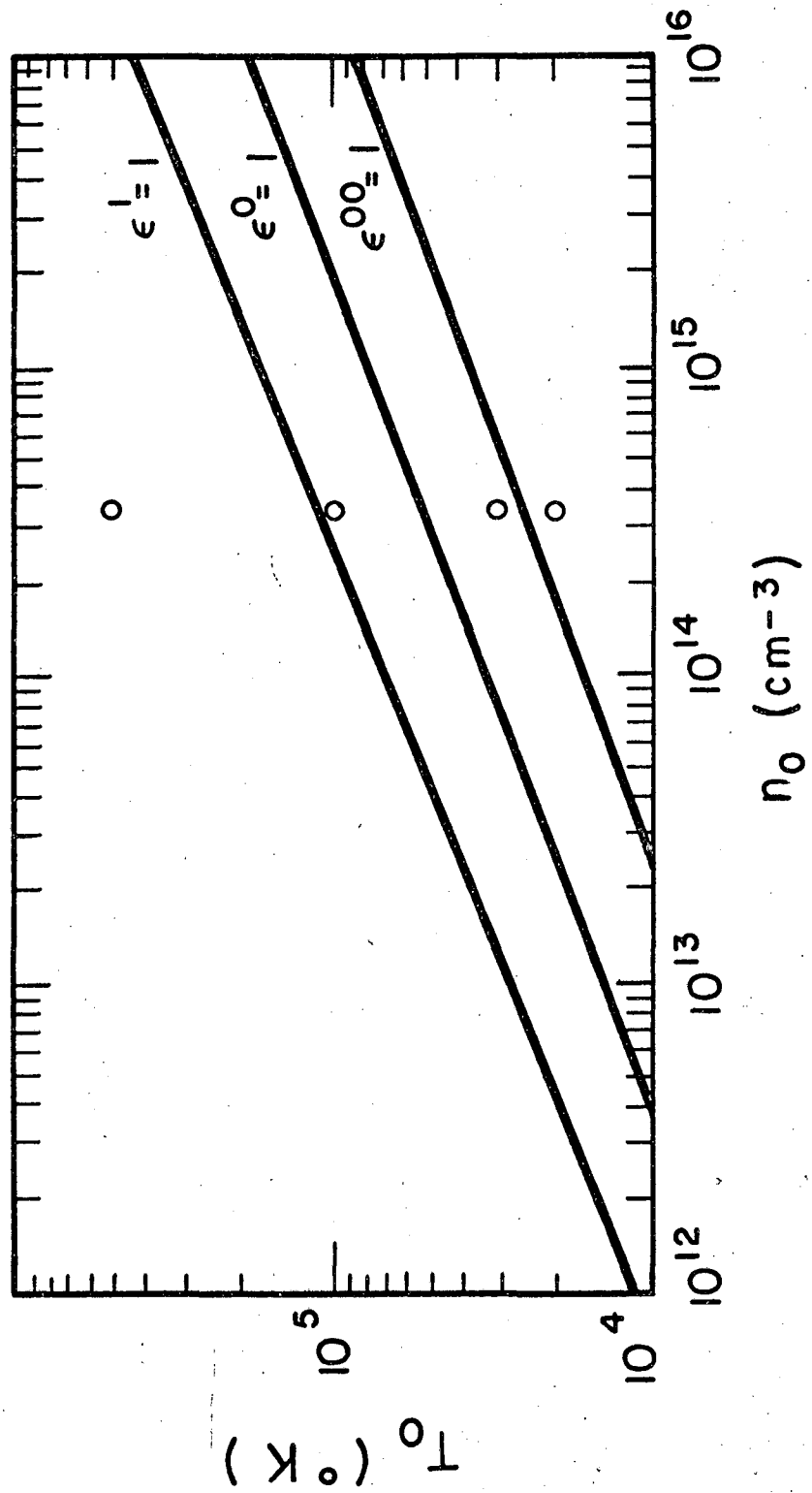

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Fig. 8