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Author

Chernikov, Artem

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MODEL THEORY, KEISLER MEASURES AND GROUPS

ARTEM CHERNIKOV

Traditionally model theory investigates definable sets in models of first-order theories. Towards this purpose, it is often more instructive to study types, i.e. ultrafilters on the boolean algebra of definable sets. While the compact space of types contains the same information by Stone duality, working with types and realizing them in larger saturated models allows to work with "generic" points of definable sets (e.g., non-standard analysis), and provides an important method at the core of both pure and applied model theory.

Types can be viewed as a special case of *Keisler measures*, or finitely additive probability measures on the Boolean algebra of definable sets (so a type is a Keisler measure taking values in $\{0, 1\}$). Given the general mathematical importance of measure theory, it is remarkable in retrospective that these measures have received little attention in model theory until recently. They were first considered in H.J.Keisler, "Measures and forking", Annals of Pure and Applied Logic 45 (1987), very insightfully generalizing aspects of Shelah's forking from types in stable theories to measures in NIP theories. Some follow up work in the ω-categorical setting appeared in the early 90's, by Albert and Ensley, but seemed to go largely unnoticed. Karpinski and Macyntire, concerned with applications to neural networks, studied definability properties of certain Keisler measures in o-minimal structures and in the p-adics in "Connections between model theory and algebraic and analytic geometry", 149–177, Quad.Mat., 6, Dept. Math., Seconda Univ. Napoli, Caserta, 2000. Recent decades, however, witnessed resurgence of analytic objects in model theory, and Keisler measures in particular, motivated by several intertwined lines of research. We review three consecutive papers that played a crucial role in these developments.

Ehud Hrushovski, Ya'acov Peterzil and Anand Pillay, "Groups, measures, and the NIP", Journal of the American Mathematical Society 21.2 (2008): 563-596. In this influential paper the authors complete a proof of the extensively studied Pillay's conjecture on groups in o-minimal theories. Towards this purpose, they carry out foundational work of developing the theory of generics for certain groups in NIP theories admitting invariant Keisler measures.

The class of NIP theories (i.e., theories with No Independence Property) was introduced by Shelah in his work on the classification program. A theory T is *NIP* if for every partitioned formula $\phi(x, y)$, there is no model of T in which we can find tuples $(a_i : i \in \omega)$ and $(b_s : s \subseteq \omega)$ such that $\phi(a_i, b_s)$ holds if and only if $i \in s$. Typical examples of NIP theories are stable and o-minimal theories, and in recent years NIP theories have become a very active domain of study in their own right.

A theorem of Shelah demonstrates that every saturated NIP group G admits the smallest type-definable subgroup of bounded index, denoted G^{00} (which can be thought of as an abstract analogue of the subgroup of "infinitesimals"). The quotient G/G^{00} is naturally equipped with the structure of a compact topological group, with respect to the logic topology. In the o-minimal case, it is in fact a compact Lie group. Pillay's conjecture predicted that if G is a definably compact group in a saturated o-minimal expansion of a field, then the dimension of G/G^{00} as a Lie group equals the dimension of G as a definable set in an o-minimal structure — a surprising connection between the pure lattice of definable subsets of groups in o-minimal theories and real Lie groups.

These considerations are put into the abstract setting of *fsg* groups in NIP theories (groups with *finitely satisfiable generics*, i.e. groups that admit a global type every translate of which is finitely satisfiable in some fixed small model). The theory of generics (a well-behaved notion of largeness for definable sets) in stable groups plays a crucial role in applications, and in this paper it is generalized from stable groups to the larger class of fsg groups, connecting generic types, G^{00} and stabilizers. A crucial ingredient is the demonstration that fsg groups are *definably amenable*, i.e. that they admit a left invariant Keisler measure on their definable subsets. E.g., the rotation group $\mathrm{SO}(3,\mathbb{R})$ is definably amenable, while there is no invariant measure on *all* subsets by the Banach-Tarski paradox. Such a measure is constructed by lifting the Haar measure on the compact group G/G^{00} by averaging with respect to a global generic type, and relies crucially on the fact that finitely satisfiable types in NIP theories are *Borel definable* (a weakening of the definability of types in stable theories).

The proof of Pillay's conjecture proceeds by induction on the o-minimal dimension of G, demonstrating in parallel that definably compact groups in o-minimal theories are precisely the fsg ones. It relies on deep structural results for such o-minimal groups by Berarducci, Edmundo, Starchenko, and others, which allow a decomposition into the commutative and definably simple cases.

Ehud Hrushovski and Anand Pillay, "On NIP and invariant measures", Journal of the European Mathematical Society 13.4 (2011): 1005-1061. This paper continues to study stable-like phenomena in NIP theories. Based on the previous work of Shelah and Poizat, the important role of forking and invariant types in NIP is demonstrated by establishing counterparts of some results for global types from the stable case. In particular, the distinction between type-definable and invariant objects of bounded index is brought into the picture, and it is shown that Lascar strong types coincide with Kim-Pillay strong types over extension bases. The theory is demonstrated to be particularly strong in the case of *generically stable types*. These types generalize stably dominated types that played a crucial role in the study of algebraically closed valued fields by Haskell, Hrushovski and Macpherson, and have many equivalent characterizations in terms of forking.

Additionally, some of the results are generalized to measures. Since the NIP condition is equivalent to saying that all uniformly definable families of sets have finite Vapnik-Chervonenkis dimension, the fundamental *VC-theorem* from statistical learning theory can be applied on the space of types, showing the following: for every Keisler measure μ, every formula $\phi(x,y)$ and every real $\varepsilon > 0$, there are finitely many types $p_1, \ldots, p_n$ such that for every parameter b, we have $\left|\mu(\phi(x,b)) - \frac{|\{1 \leq i \leq n : \phi(x,b) \in p_i\}|}{n}\right| < \varepsilon$. This approximation result provides a systematic way to reduce many questions about measures to types in NIP theories. Using it, Borel definability of invariant measures is established and is used to define non-forking products via integration.

A systematic investigation of generics in definably amenable groups is initiated, establishing appropriate generalizations of the corresponding results in the fsg case. It is shown that G is definably amenable if and only if there is a global strongly f-generic type (i.e. a type such that all of its translates don't fork over some small model), and in this case both G^{00} and G^{∞} are equal to its stabilizer (where G^{∞} is the smallest *invariant* subgroup of bounded index). This provides a counterpart of the compactness of Lascar strong types in the definable group setting. Uniqueness of invariant measures is investigated under various fsg-like assumptions, culminating with the proof of uniqueness of invariant measures in fsg groups in NIP theories.

As a main application, the *compact domination conjecture* from the previous paper is confirmed in the commutative case for definably compact groups in saturated o-minimal expansions of fields. Namely, in the group case, the conjecture says that for $\pi : G \to G/G^{00}$ the canonical surjective homomorphism, for any definable subset X of G, the set $\left\{b \in G/G^{00} : \pi^{-1}(b) \cap X \neq \emptyset \text{ and } \pi^{-1}(b) \cap (G \setminus X) \neq \emptyset\right\}$ has Haar measure 0. I.e., a definable group G is dominated by the compact group G/G^{00} via an abstract "standard part" map. The proof relies on virtually everything known about o-minimal groups including the trichotomy and deep results on decomposition of definably compact groups, and provides considerably more information on what these sets look like. Using it, the general case of compact domination is proved in Hrushovski, Peterzil, Pillay, "On central extensions and definably compact groups in o-minimal structures", Journal of Algebra 327.1 (2011): 71-106.

Ehud Hrushovski, Anand Pillay, and Pierre Simon, "Generically stable and smooth measures in NIP theories", Transactions of the American Mathematical Society 365.5 (2013): 2341-2366. This paper studies generically stable measures in NIP theories, demonstrating their abundance and generalizing the results on generically stable types from the previous paper, and isolates and studies the generic compact domination phenomenon.

The key notion of a *smooth measure* stemming from Keisler's paper is considered systematically. A Keisler measure is smooth if there is a unique way to extend it to a global measure. Generalizing Karpinski and Macintyre, smoothness of any Borel probability measure on the real or p-adic semialgebraic sets is demonstrated. On the one hand, every Keisler measure in an NIP theory can be extended to a smooth one, possibly over a larger model. On the other hand, smooth measures enjoy the following property, called *fim* ("frequency interpretation measures"): a sufficiently long sample of *elements* from the structure uniformly approximates the measure of all sets in a definable family with high probability. Combined, these two facts give a purely model theoretic proof of the approximation of arbitrary measures by types in NIP theories from the previous paper, avoiding the use of the VC-theorem.

The class of measures satisfying fim is more general than that of smooth measures, and corresponds to *generically stable measures*. Generalizing from the case of types, many natural characterizations of generic stability for measures are established (in terms of finite satisfiability and definability, commutativity of non-forking products, etc.). An important ingredient in the proof of these equivalences is a result of Ben Yaacov establishing that the randomization T^R of an NIP theory T is still NIP (where T^R is the first-order theory in the sense of *continuous logic* whose models are random variables in models of T; the type spaces of T^R correspond to the spaces of Keisler measures of T, and provide an alternative setting for studying Keisler measures).

Finally, *generic compact domination* is introduced, generalizing the domination statements from smooth measures to generically stable ones, but now the set being dominated is a space of types rather than a definable or type-definable set. In the definable group version it states the following. Let $S \subseteq S_G(\mathbb{M})$ be the set of global generic types of G, equivalently (under the fsg assumption) the set of types $p \in S_G(\mathbb{M})$ which are G^{00}-invariant. Let X be a definable subset of G, and let h be the Haar measure on the compact group G/G^{00}. The conjecture says that for h-almost every coset gG^{00} of G^{00}, the set $S \cap gG^{00}$ lies entirely inside X or entirely outside X. This condition can be interpreted as "non-fractality" of the topological border of constructible sets in G/G^{00} whose families of translates have finite VC-dimension. A proof is proposed, however it contains a flaw in the group case — the proof of Theorem 4.3 contains a gap. A correct proof of the generic compact domination was provided later in Simon, "VC-sets and generic compact domination", Israel Journal of Mathematics 218.1 (2017).

We conclude with some directions in the rapidly growing body of work largely inspired by the papers under review. The class of *distal theories* was introduced by Simon, as those theories in which every generically stable measure is smooth. Among other things, generalizing the o-minimal case, compact domination holds for arbitrary fsg groups definable in distal theories — a proof is presented in Simon. " A guide to NIP theories", Cambridge University Press, 2015 based on the unpublished work of Hrushovski, Macpherson and Pillay. The proof of the equivalence of various notions of genericity in fsg groups is completed in Hrushovski, Pillay, and Simon. "A note on generically stable measures and fsg groups", Notre Dame Journal of Formal Logic 53.4 (2012): 599-605, and a theory of generics in arbitrary definably amenable groups is developed in Chernikov and Simon, "Definably amenable NIP groups", Journal of the American Mathematical Society 31.3 (2018): 609-641. Many methods and objects of topological dynamics were introduced into the area, pioneered by Newelski, and a fascinating parallel between the work discussed here and the study of *tame* dynamical systems in the sense of Glasner and Megrelishvili has been established. Finally, the study of Keisler measures combined with the ultraproduct method led to a new connection with extremal combinatorics, especially in the context of regularity lemmas for restricted families of graphs, which turn out to be closely related to the compact domination statements discussed above. We refer to Starchenko, S. "NIP, Keisler measures and combinatorics", Séminaire BOURBAKI (2016): 68 for a survey.

Department of Mathematics, University of California Los Angeles, Los Angeles, CA 90095-1555

Email address: chernikov@math.ucla.edu