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Modified Mean Curvature Flow in Fuchsian Manifolds

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MODIFIED MEAN CURVATURE FLOW IN FUCHSIAN MANIFOLDS

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by

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Abstract

Modified Mean Curvature Flow in Fuchsian Manifolds

by

Yuk Shing Lam

The equidistant surfaces in a Fuchsian manifold form a canonical foliation by constant mean curvature surfaces. It is therefore natural to ask whether a prescribed leaf of this foliation can be reached by evolving an arbitrary graphical surface. We answer this question affirmatively using the modified mean curvature flow.

Let $(M^3, \overline{g}) = \left(\mathbb{R} \times \Sigma, dr^2 + \cosh^2(r)\, g_0\right)$ be a Fuchsian manifold, where (Σ, g_0) is a closed hyperbolic surface. For any $\sigma \in (-2, 2)$, we consider the flow $\frac{\partial F}{\partial t} = -(H - \sigma)\nu$ starting from an arbitrary smooth closed geodesic graph over the central totally geodesic surface. We prove that the flow exists smoothly for all time, remains graphical, and converges smoothly to the unique equidistant surface $\Sigma(r_\sigma)$, where $r_\sigma = \operatorname{arctanh}\left(\frac{\sigma}{2}\right)$, whose mean curvature is σ.

The initial graph is required only to be strictly graphical; no smallness assumption or quantitative angle–height pinching condition is imposed. The main difficulty is that strict graphicality alone does not provide a uniform bound for the reciprocal of the angle function. We first derive a time-dependent gradient estimate, sufficient to preserve graphicality and exclude finite-time singularities. We then obtain polynomial curvature control and use a blow-up argument to establish a time-independent gradient bound. The resulting uniform curvature estimates, together with the barrier structure supplied by the equidistant foliation, lead to smooth convergence to $\Sigma(r_\sigma)$.

To Min Pui Lam

and

in memory of Chi Kam Lam

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Chapter 1.

Introduction

Constant mean curvature surfaces holds an important place in geometric analysis. They arise as critical points of area among variations that preserve enclosed volume, with minimal surfaces appearing as the special case of zero mean curvature. Their study therefore extends beyond the existence of individual solutions. One would also like to understand whether a distinguished constant mean curvature (CMC) surface is unique, whether it is stable under perturbation, and whether CMC surfaces with different prescribed curvatures fit together to describe the ambient geometry. These questions are particularly natural in hyperbolic three-manifolds, where the global geometry of the manifold is closely related to the behavior of its embedded minimal and CMC surfaces.

Geometric evolution equations provide a dynamical approach to these questions. Under mean curvature flow, a hypersurface moves in the normal direction with speed given by its mean curvature, so minimal surfaces are stationary solutions. The flow can improve the geometry of suitable initial hypersurfaces, but its behavior depends strongly on both the initial data and the ambient space. Huisken's theorem for compact convex hypersurfaces in Euclidean space [12] and the graphical theory of Ecker and Huisken [5,6] are fundamental examples in which geometric assumptions lead to strong long-time or asymptotic control. In more general settings, however, singularities or loss of graphicality may prevent the flow from reaching a stationary surface.

Ordinary mean curvature flow is naturally adapted to the minimal case. To seek a surface with a prescribed nonzero mean curvature, one instead adds a constant forcing term. The resulting modified mean curvature flow is the negative gradient flow of an area–volume functional, and its stationary points are precisely the surfaces having the prescribed mean curvature. Lin and Xiao introduced this flow in hyperbolic space and used it to obtain convergence results for star-shaped hypersurfaces with prescribed asymptotic boundary [14]. This parabolic viewpoint raises a stability question that is stronger than mere existence: how large is the class of initial surfaces whose evolution exists for all time and converges to a given CMC surface?

Fuchsian manifolds provide a natural compact setting in which to examine this question. Such a manifold contains a unique closed totally geodesic surface, and its normal translates form an explicit foliation by totally umbilic CMC surfaces. Thus, for every admissible prescribed mean curvature, the target surface is already known. What remains nontrivial is its dynamical stability. Starting from an arbitrary smooth geodesic graph over the central surface, one must show that the modified flow remains graphical, exists for all time, and approaches the appropriate leaf. The principal difficulty is that the scalar graphical equation remains uniformly parabolic only while the angle between the evolving surface and the normal foliation stays uniformly positive.

We now describe the setting more precisely. Let (Σ, g_0) be a closed hyperbolic surface and let

$$(M^3, \bar{g}) = \left(\mathbb{R} \times \Sigma, dr^2 + \cosh^2(r) g_0\right)$$

be the associated Fuchsian manifold. The central slice $\Sigma = \Sigma(0)$ is totally geodesic, while the equidistant leaves $\Sigma(r) = \{r\} \times \Sigma$ have mean curvature $2 \tanh r$ with the orientation and trace convention used here. Hence, for $\sigma \in (-2, 2)$, the unique leaf with mean curvature σ is $\Sigma(r_\sigma)$,where $r_\sigma := \operatorname{arctanh}\left(\frac{\sigma}{2}\right)$.

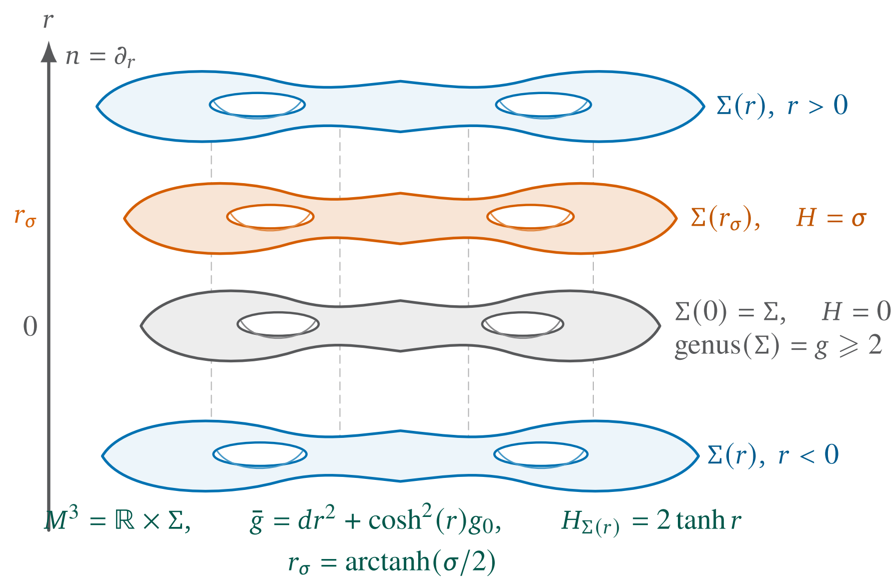

Figure 1.1.: Foliation of a Fuchsian manifold.
The central slice $\Sigma(0)$ is totally geodesic, and every leaf $\Sigma(r)$ is totally umbilic with mean curvature $2\tanh r$. The prescribed value $\sigma \in (-2, 2)$ selects the stationary leaf $\Sigma(r_\sigma)$, where $r_\sigma = \operatorname{arctanh}(\sigma/2)$. The target is drawn above $\Sigma(0)$ for $\sigma > 0$; the case $\sigma < 0$ is obtained by reflection. Every leaf is diffeomorphic to the same closed surface Σ of genus $g \geq 2$. Two handles are shown as the representative case $g = 2$; additional handles are understood when $g > 2$. The perspective drawing is schematic and is not an isometric embedding.

The evolution considered in this paper is

$$\frac{\partial F}{\partial t} = -(H - \sigma)\nu,$$

whose stationary points have mean curvature σ. If the evolving surface is written as a graph

$$S(t) = \{(u(x,t), x) : x \in \Sigma\},$$

then its graphicality is measured by

$$\Theta := \langle \nu, \partial_r \rangle, \qquad w := \Theta^{-1} = \sqrt{1 + \frac{|Du|^2_{g_0}}{\cosh^2(u)}}.$$

Controlling the height u does not by itself control w: a graph may remain in a bounded slab while its tangent planes become nearly vertical. The main analytical problem is therefore to prevent degeneration of the angle.

The closest earlier result is the ordinary mean curvature flow theorem of Huang, Lin, and Zhang [8]. If the initial surface lies within distance a_0 of Σ, they assumed the quantitative condition

$$\min_{S_0} \Theta_0 \geq \tanh(a_0)$$

and proved long-time existence, preservation of graphicality, and continuous convergence to Σ, with smooth convergence when the inequality is strict [8, Theorem 1.2]. This hypothesis is stronger than graphicality: compactness gives $\min\limits_{S_0} \Theta_0 > 0$ for every smooth closed graph, but it does not impose any relation between this minimum and the height. Consequently, the earlier theorem does not cover graphs that are steep relative to their distance from the central slice.

Angle estimates are likewise central in the broader theory of graphical mean curvature flow. Ecker and Huisken developed the Euclidean nonparametric estimates [5,6]; Unterberger proved all-time existence for entire locally Lipschitz radial graphs in hyperbolic space and obtained convergence under an additional bounded-height assumption [17]; and Allmann, Lin, and Zhu established all-time existence and preservation of radial graphicality for the corresponding modified flow, while also showing that convergence fails without further hypotheses [1]. Results for warped products by Borisenko–Miquel [2] and Huang–Zhang–Zhou [11] treat related graphical settings, but their ambient structures or initial assumptions do not imply the unrestricted Fuchsian statement sought here.

The main contribution of this paper is to prove that every smooth closed geodesic graph lies in the basin of attraction of the prescribed Fuchsian CMC leaf. For each $\sigma \in (-2, 2)$, the modified flow exists for all time, remains graphical, and converges smoothly to $\Sigma(r_\sigma)$. When $\sigma = 0$, this removes the height–angle restriction in [8]; for $\sigma \neq 0$, it gives the corresponding global stability result for every nonminimal leaf of the Fuchsian CMC foliation. No small-gradient, convexity, mean-convexity, or closeness assumption is imposed.

This Fuchsian result also provides a symmetric reference point for the CMC foliation problem in quasi-Fuchsian geometry. Uhlenbeck proved uniqueness of the minimal surface and global normal parametrization under the principal curvature bound defining the almost-Fuchsian class [16]. More recently, Choudhury, Mazzoli, and Seppi constructed a unique monotone CMC foliation for quasi-Fuchsian manifolds sufficiently close to the Fuchsian locus [3], while Huang, Lin, and Zhang used the modified flow to obtain CMC leaves for a specified almost-Fuchsian subclass and combined them with end foliations to produce a global foliation [9]. In that setting the CMC leaves are generally not equidistant from the minimal surface. The present paper isolates the exact Fuchsian model and resolves the global flow problem for unrestricted smooth graphical initial data.

We briefly describe the proof. Evolving equidistant leaves provide barriers that keep the height uniformly bounded and force exponential C^0 convergence to r_σ. After the flow is written as a quasilinear scalar equation for u, its linearized operator yields a differential inequality for w. The maximum principle then controls w on every finite time interval, preventing finite-time loss of graphicality and giving long-time existence. To make this control uniform in time, the quantity $\Psi = \cosh(u)\, w$ first produces a linear gradient bound, while the evolution of the second fundamental form gives polynomial curvature growth. A blow-up sequence for an allegedly unbounded gradient would therefore have a nonflat normalized limit. Yet the exponential height decay forces the rescaled surfaces into a slab collapsing to a plane, a contradiction. Uniform gradient and curvature estimates then imply smooth convergence.

The paper is organized as follows. Section 2 develops the graphical formulation and height estimates. Section 3 introduces the linearized operator. Section 4 proves the finite-time gradient estimate and long-time existence. Section 5 establishes the uniform estimates and smooth convergence. The main result is as follows.

Theorem 1. Let M^3 be a Fuchsian manifold and let Σ be the unique closed totally geodesic surface in M^3. For any constant $\sigma \in (-2, 2)$ and any initial smooth closed surface $S_0 \subseteq M^3$ that is a geodesic graph over Σ, the modified mean curvature flow (2.3) starting from S_0 exists for all time, remains a geodesic graph over Σ, and converges smoothly as $t \to \infty$ to the equidistant surface $\Sigma(r_\sigma)$, where $r_\sigma = \operatorname{arctanh}(\frac{\sigma}{2})$.

Chapter 2.

Preliminaries

In this chapter, we establish the notations and recall the basic geometric features of Fuchsian manifolds. We also define the modified mean curvature flow and the precise notion of graphical hypersurfaces in this ambient space.

§ 2.1. The Ambient Fuchsian Geometry

Let Σ be a closed surface of genus $g \geqslant 2$ equipped with a hyperbolic metric g_0. A Fuchsian manifold M^3 is constructed as the warped product $\mathbb{R} \times \Sigma$. We equip M^3 with the standard warped product metric:

$$\bar{g} = dr^2 + \cosh^2(r) g_0, \tag{2.1.1}$$

where $r \in \mathbb{R}$ serves as the coordinate along the real line. Under this metric, M^3 is a complete 3-manifold with constant sectional curvature -1.

The level sets $\Sigma(r) = \{r\} \times \Sigma$ form a global equidistant foliation of M^3. The slice $\Sigma(0) = \Sigma$ is the unique totally geodesic surface in the manifold. For any $r \neq 0$, the slice $\Sigma(r)$ is a totally umbilic surface with principal curvatures equal to $\tanh(r)$.

We denote the Levi-Civita connection on M^3 by $\overline{\nabla}$. The unit normal vector field to the foliation $\Sigma(r)$ is given by $\mathbf{n} := \frac{\partial}{\partial r}$. A highly useful geometric feature of the Fuchsian manifold is the existence of the conformal vector field $V = \cosh(r)\mathbf{n}$. It can be easily checked that for any tangent vector field X on M^3, the covariant derivative of the normal field $\mathbf{n}$ satisfies (see [8, Lemma 2.1]):

$$\overline{\nabla}_X \mathbf{n} = \tanh(r)(X - \bar{g}(X, \mathbf{n})\mathbf{n}). \tag{2.1.2}$$

§ 2.2. Modified Mean Curvature Flow and Graphical Surfaces

Let S be a closed surface. We consider a smooth family of immersions $F : S \times [0, T) \to M^3$. The modified mean curvature flow is defined by the initial value problem:

$$\begin{cases} \frac{\partial}{\partial t} F(x, t) = -(H(x, t) - \sigma)\nu(x, t), \\ F(\cdot, 0) = F_0, \end{cases} \tag{2.2.1}$$

where $H(x, t)$ is the mean curvature of the evolving surface $S(t) = F(S, t)$, $\nu(x, t)$ is the chosen unit normal vector field on $S(t)$, and $\sigma \in (-2, 2)$ is a given real constant. Here, we adopt the convention that the mean curvature is the trace of the second fundamental form.

To analyze the geometry of $S(t)$ relative to the foliation of the ambient space, we define the height function $u(x, t)$ as the signed distance from the point $F(x, t)$ to the totally geodesic reference surface Σ:

$$u(x, t) = r(F(x, t)). \tag{2.2.2}$$

The relative orientation of the surface $S(t)$ is measured by the angle function $\Theta(x, t)$, defined as the inner product between the upward-pointing vertical vector field $\mathbf{n}$ and the surface normal ν:

$$\Theta(x,t) = \bar{g}(\mathbf{n},\nu)(F(x,t)). \tag{2.2.3}$$

With these quantities established, we formalize the definition of a graphical surface in this setting.

Definition 1. A closed, smooth surface $S_0 \subseteq M^3$ is defined as a geodesic graph over the totally geodesic surface Σ if there exists a constant $c_0 > 0$ such that the angle function satisfies $\Theta_0 = \bar{g}(\mathbf{n},\nu_0) \geqslant c_0$ everywhere on S_0.

If $\Theta(x,t) \in (0,1]$ everywhere, the evolving surface $S(t)$ can be represented globally by the height function $r = u(x,t)$ over the reference surface Σ. A surface is a constant height slice $\Sigma(r)$ if and only if $\Theta \equiv 1$.

Let D, ∇, and $\bar{\nabla}$ denote the Levi-Civita connections on the fixed reference surface (Σ, g_0), the evolving surface $(S(t), g(t))$, and the ambient manifold $(M^3, \bar{g})$, respectively. At any *fixed* time t, we parameterize the evolving surface

$$\Sigma_t := S(t)$$

as a graph over Σ via the immersion $F(x) = (u(x), x)$ for $x \in \Sigma$. Choosing a local coordinate system (x^1, x^2) on Σ with the coordinate frame $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right\}$, the covariant derivatives of the height function $u : \Sigma \to \mathbb{R}$ are denoted by:

$$u_i = D_i u, \qquad u_{ij} = D_i D_j u, \qquad \text{and} \qquad |Du|^2 := |Du|^2_{g_0} = (g_0)^{ij} u_i u_j.$$

Then

$$F_i = \frac{\partial F}{\partial x^i} = \frac{\partial u}{\partial x^i}\frac{\partial}{\partial r} + \frac{\partial}{\partial x^i} = u_i \frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}.$$

In the following, we denote $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{\bar{g}}$. Then the induced metric is

$$\begin{aligned} g_{ij} &= \left\langle F_i, F_j \right\rangle \\ &= \left\langle u_i \frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}, u_j \frac{\partial}{\partial r} + \frac{\partial}{\partial x^j} \right\rangle \\ &= u_i u_j \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle + u_i \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial x^j} \right\rangle + u_j \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial r} \right\rangle + \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle . \end{aligned}$$

Since the ambient metric is $\bar{g} = dr^2 + \cosh^2(r) g_0$, we have

$$\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1, \quad \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial x^i} \right\rangle = 0, \quad \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \cosh^2(r)(g_0)_{ij}.$$

Thus, evaluating at $r = u$,

$$g_{ij} = u_i u_j + \cosh^2(u)(g_0)_{ij}. \tag{2.2.4}$$

On the other hand, the inverse metric is

$$g^{ij} = \frac{1}{\cosh^2(u)} \left((g_0)^{ij} - \frac{u^i u^j}{|Du|^2 + \cosh^2(u)} \right). \tag{2.2.5}$$

The proof is included in Appendix A under Lemma 6. Let

$$w := \sqrt{1 + \frac{|Du|^2}{\cosh^2(u)}} \tag{2.2.6}$$

be the gradient function. Since $w^2 \cosh^2(u) = \cosh^2(u) + |Du|^2$, we have

$$g^{ij} = \frac{1}{\cosh^2(u)} \left((g_0)^{ij} - \frac{u^i u^j}{w^2 \cosh^2(u)} \right). \tag{2.2.7}$$

For the Levi-Civita connection, we have

$$\bar{\nabla}_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r} = \tanh(r) \left(\frac{\partial}{\partial r} - \bar{g} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \frac{\partial}{\partial r} \right) = 0, \tag{2.2.8}$$

$$\bar{\nabla}_{\frac{\partial}{\partial r}} \frac{\partial}{\partial x^j} = \bar{\nabla}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial r} = \tanh(r) \frac{\partial}{\partial x^j}, \tag{2.2.9}$$

and

$$\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \cosh(r)\sinh(r)(g_0)_{ij} \frac{\partial}{\partial r}. \tag{2.2.10}$$

The proof is included in Appendix A under Lemma 7.

The outward normal vector to the evolving surface $S(t)$ is

$$\nu = \frac{1}{w}\left(\frac{\partial}{\partial r} - \frac{u^k}{\cosh^2(u)}\frac{\partial}{\partial x^k}\right). \tag{2.2.11}$$

See Appendix A under Lemma 8 for the proof.

Taking the inner product of the basis vectors with ν yields the relation between the angle function $\Theta(x,t)$ (see (2.2.3)) with the gradient function w:

$$\Theta = \left\langle \frac{\partial}{\partial r}, \nu \right\rangle = \frac{1}{w}, \tag{2.2.12}$$

$$\begin{aligned} \left\langle \frac{\partial}{\partial x^i}, \nu \right\rangle &= \frac{1}{w}\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial r} - \frac{u^k}{\cosh^2(u)}\frac{\partial}{\partial x^k} \right\rangle \\ &= -\frac{1}{w\cosh^2(u)} u^k \cosh^2(u)(g_0)_{ik} = -\frac{u_i}{w}. \end{aligned} \tag{2.2.13}$$

Figure 2.1 shows how the graphical condition and the angle function relate.

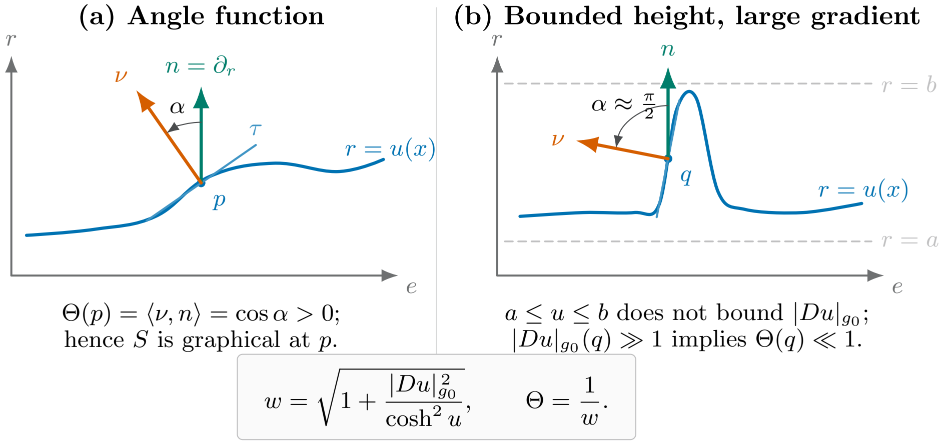

Figure 2.1.: Local normal sections showing the graphical condition and angle function. Since $\Theta = \langle \nu, n \rangle = 1/w$, bounded height does not by itself control the gradient: one may have $a \leq u \leq b$ while Θ approaches zero.

Moreover, the second fundamental form and its index raising, computed in Appendix A under Lemma 9, are

$$h_{ij} = \frac{1}{w}\left(\cosh(u)\sinh(u)(g_0)_{ij} + 2\tanh(u)u_i u_j - u_{ij}\right) \tag{2.2.14}$$

and

$$h^{ij} = \frac{1}{w}\left[\tanh(u)g^{ij} + \frac{\tanh(u)}{w^4\cosh^4(u)}u^i u^j - g^{ik}g^{j\ell}u_{k\ell}\right]. \tag{2.2.15}$$

Now the mean curvature is

$$\begin{aligned}
H &= g^{ij}h_{ij} \\
&= \frac{1}{w}\left(\cosh(u)\sinh(u)g^{ij}(g_0)_{ij} + 2\tanh(u)g^{ij}u_i u_j - g^{ij}u_{ij}\right) \\
&= \frac{1}{w}\left[\tanh(u)\left(3 - \frac{1}{w^2}\right) - g^{ij}u_{ij}\right].
\end{aligned}$$

The proof is in Appendix A under Lemma 10.

Also, the evolution equation of u is

$$\begin{aligned}\frac{\partial u}{\partial t} &= -(H-\sigma)\Theta \\ &= -\frac{H-\sigma}{w} \\ &= \frac{1}{w^2}g^{ij}u_{ij} - \frac{\tanh(u)}{w^2}\left(3-\frac{1}{w^2}\right) + \frac{\sigma}{w}.\end{aligned} \tag{2.2.16}$$

This is a degenerate second order quasi-linear parabolic PDE. Therefore a uniform upper bound on w implies the angle function $\Theta \in (0,1]$ everywhere (see (2.2.3) and (2.2.12)) and the evolving surface $S(t)$ exists for all time and can be represented globally by the height function $u(\cdot,t)$ over the reference surface Σ, see also Remark 1.

The following lemma serves as the Squeezing Lemma (analogous to [8, Theorem 3.1]) for the modified mean curvature flow, see also [9, Theorem 4.3].

Lemma 1 (Height Estimate)**.** Let $u \in C^{2,1}(\Sigma \times [0,T))$ be a smooth solution to the modified mean curvature flow equation (2.2.16) with constant $\sigma \in (-2,2)$ and initial height satisfying $r_{min} \leqslant u(\cdot,0) \leqslant r_{max}$. Then, as long as the modified mean curvature flow exists, the height function $u(\cdot,t)$ of the evolving surface satisfies:

$$r_-(t) \leqslant u(\cdot,t) \leqslant r_+(t), \tag{2.2.17}$$

where $r_-(t)$ and $r_+(t)$ are the solutions to the ODE

$$\frac{dr}{dt} = \sigma - 2\tanh(r(t)), \tag{2.2.18}$$

with initial conditions $r_-(0) = r_{min}$ and $r_+(0) = r_{max}$, respectively. In particular, we have

$$|r_\pm(t) - r_\sigma| \leqslant Me^{-\frac{4-\sigma^2}{2}t}, \tag{2.2.19}$$

for all $t \geqslant 0$, where $r_\sigma := \tanh^{-1}\left(\frac{\sigma}{2}\right)$ and $M > 0$ is a constant depending only on σ, r_{min}, and r_{max}.

Proof. The equidistant surfaces $\Sigma(r)$ of the Fuchsian manifold M^3 from the central reference surface Σ form a global foliation of M^3. Each fiber $\Sigma(r)$ is totally umbilic with constant principal curvatures equal to $\tanh(r)$.

If we evolve an equidistant surface $\Sigma(r_0)$ under the modified mean curvature flow with $\sigma \in (-2, 2)$, it remains an equidistant surface $\Sigma(r(t))$ for all $t \geqslant 0$, and its height $r(t)$ evolves according to the ODE:

$$\frac{dr}{dt} = -(H - \sigma) = \sigma - 2\tanh(r(t)). \tag{2.2.20}$$

By assumption, the initial surface S_0 lies in between two equidistant surfaces $\Sigma(r_{min})$ and $\Sigma(r_{max})$. By the avoidance principle for the modified mean curvature flow, the evolving surface $S(t)$ must remain in between the evolving surfaces $\Sigma(r_-(t))$ and $\Sigma(r_+(t))$ for as long as the flow exists. This yields the uniform height bound $r_-(t) \leqslant u(\cdot, t) \leqslant r_+(t)$.

Solving (2.2.20) explicitly, we have

$$|\sinh(r_\pm(t) - r_\sigma)| e^{\frac{\sigma}{2} r_\pm(t)} = C_\pm e^{-\frac{4-\sigma^2}{2} t}, \tag{2.2.21}$$

where $C_\pm = |\sinh(r_\pm(0) - r_\sigma)| e^{\frac{\sigma}{2} r_\pm(0)}$ are constants determined by the initial conditions. Because the flows of equidistant surfaces are monotonic, $r_\pm(t)$ is bounded between its initial value and r_σ. Consequently, the factor $e^{-\frac{\sigma}{2} r_\pm(t)}$ is uniformly bounded. This implies that as $t \to \infty$, the term $|\sinh(r_\pm(t) - r_\sigma)|$ decays exponentially to zero. Since there exists a constant $c > 0$ such that $|\sinh(x)| \geqslant c|x|$ for x in a bounded interval, we have

$$|r_\pm(t) - r_\sigma| \leqslant M e^{-\frac{4-\sigma^2}{2} t},$$

for all $t \geqslant 0$, where $M > 0$ is a constant depending only on the initial data. □

The barrier comparison established in Lemma 2.2 is summarized in Figure 2.2.

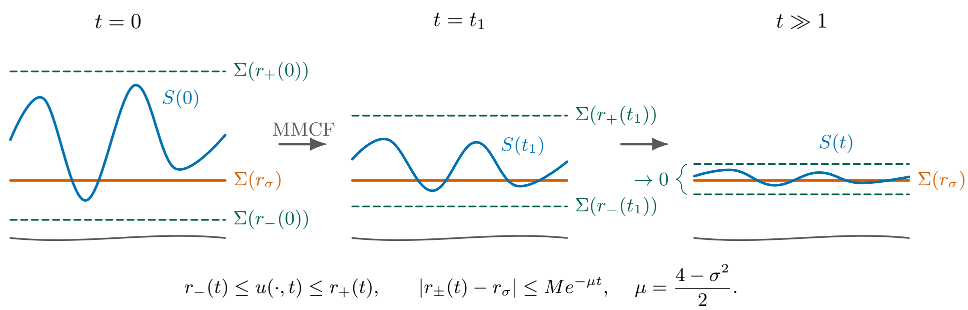

Figure 2.2.: Height squeezing by evolving equidistant barriers.

The lower and upper barriers converge exponentially to the target leaf $\Sigma(r_\sigma)$, forcing the evolving graph to converge to $\Sigma(r_\sigma)$ in height.

Chapter 3.

Linearized Operator

In this chapter, we linearize the evolution equation for the height function $u \in C^{2,1}(\Sigma \times [0, T])$. The resulting linearized operator will be applied to the gradient function w in the subsequent chapters to establish a uniform gradient bound for u. From (2.2.16), u satisfies a fully nonlinear parabolic equation of the form

$$\mathcal{P}[u] = \frac{\partial u}{\partial t} - \mathcal{F}(D^2u, Du, u) = 0, \tag{3.0.1}$$

where the spatial operator $\mathcal{F}$ is explicitly given by

$$\mathcal{F}(D^2u, Du, u) = \frac{1}{w^2} g^{ij} u_{ij} - \frac{\tanh(u)}{w^2}\left(3 - \frac{1}{w^2}\right) + \frac{\sigma}{w}. \tag{3.0.2}$$

To define the linearized operator, we introduce a one-parameter family of variations $u_\epsilon = u + \epsilon\phi$, where $\phi \in C^\infty(\Sigma)$, and compute the Fréchet derivative

$$\mathcal{L}_u[\phi] = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \mathcal{P}[u + \epsilon\phi].$$

Since the covariant derivative is linear, the variations of Du and D^2u are given by $D\phi$ and $D^2\phi$, respectively. Because $\mathcal{F}$ depends on the gradient Du and the function u through the highly nonlinear composite terms w and g^{ij}, we define the background-dependent geometric coefficients via partial derivatives:

$$\alpha^{ij} := \frac{\partial \mathcal{F}}{\partial u_{ij}}, \qquad \beta^k := \frac{\partial \mathcal{F}}{\partial u_k}, \qquad \gamma := \frac{\partial \mathcal{F}}{\partial u},$$

so that the *linearized* operator takes the precise form

$$\mathcal{L}_u[\phi] = \frac{\partial \phi}{\partial t} - \alpha^{ij} D_i D_j \phi - \beta^k D_k \phi - \gamma \phi.$$

Here, it is clear that the principal symbol is

$$\alpha^{ij} = \frac{1}{w^2} g^{ij}.$$

The nonlinear equation is parabolic because $\alpha^{ij}(x,t) = \frac{1}{w^2(x,t)} g^{ij}(x,t)$ is strictly positive definite on $\Sigma \times [0,T)$. The drift vector $\beta^k = \frac{\partial \mathcal{F}}{\partial u_k}$ and zeroth-order reaction term γ can be computed explicitly through the method of first variation (in Appendix B under Lemma 11) as

(3.0.3)
$$\beta^k = -\frac{2u^k}{w^4 \cosh^4(u)} (g_0)^{ij} u_{ij} - \frac{2(g_0)^{ik} u^j u_{ij}}{w^4 \cosh^4(u)} + \frac{4u^k u^i u^j u_{ij}}{w^6 \cosh^6(u)} + \frac{2\tanh(u) u^k}{w^4 \cosh^2(u)} \left(3 - \frac{2}{w^2}\right) - \frac{\sigma u^k}{w^3 \cosh^2(u)}$$

and

(3.0.4)
$$\gamma = -\frac{2\tanh(u)}{w^4 \cosh^2(u)} (g_o)^{ij}\; u_{ij} + \frac{4\tanh(u)\, u^i u^j}{w^6 \cosh^4(u)} u_{ij} - \frac{\operatorname{sech}^2(u)}{w^2}\left(3 - \frac{1}{w^2}\right) + \left(3 - \frac{2}{w^2}\right) \frac{2\tanh^2(u)}{w^4}(w^2 - 1) + \frac{\sigma \tanh(u)}{w^3}(w^2 - 1).$$

The derivation of the uniform gradient bounds will rely on the differential part of the linearized operator (comprising both the second-order diffusion and first-order drift terms) given by:

$$\mathscr{L} := \frac{\partial}{\partial t} - \alpha^{ij} D_i D_j - \beta^k D_k, \tag{3.0.5}$$

where the principal symbol $\alpha^{ij} = \frac{1}{w^2} g^{ij}$ guarantees uniform parabolicity as long as the gradient w remains bounded, and the drift vector β^k is exactly determined by the background geometry and the Hessian u_{ij} as derived above.

Chapter 4.

Time-dependent Gradient Bound and Long-time Existence

In this chapter, we utilize the differential part of the linearized operator $\mathcal{L}_u$ derived in chapter 3, namely, $\mathcal{L}$ in (3.0.5), to establish a time-dependent gradient bound for the height function along the flow, which yields the long-time existence of the flow. We begin by computing the time derivative for the gradient function $w = \sqrt{1 + \frac{|Du|^2}{\cosh^2(u)}}$ along the flow, and the proof is included in Appendix B under Lemma 12.

$$
\begin{aligned}
\frac{\partial w}{\partial t} = \frac{u^k}{w\cosh^2(u)}\Bigg[& -\frac{2}{w^3} w_k\, g^{ij} u_{ij} - \frac{2\tanh(u)}{w^2\cosh^2(u)}\, u_k\, (g_0)^{ij} u_{ij} \\
& + \frac{4\tanh(u)}{w^4\cosh^4(u)}\, u_k\, u^i u^j u_{ij} - \frac{2}{w^4\cosh^4(u)}\, u^i_k\, u^j u_{ij} \\
& + \frac{2}{w^5\cosh^4(u)}\, w_k\, u^i u^j u_{ij} + \frac{1}{w^2} g^{ij} D_k u_{ij} \\
& - \operatorname{sech}^2(u)\, \frac{1}{w^2}\left(3 - \frac{1}{w^2}\right) u_k + \frac{2\tanh(u)}{w^3}\left(3 - \frac{2}{w^2}\right) w_k - \frac{\sigma}{w^2} w_k \Bigg] \\
& - \frac{w^2-1}{w}\tanh(u)\left[\frac{1}{w^2} g^{ij} u_{ij} - \frac{\tanh(u)}{w^2}\left(3 - \frac{1}{w^2}\right) + \frac{\sigma}{w}\right].
\end{aligned} \tag{4.0.1}
$$

Next, we move onto the gradient bound.

§ 4.1. Time-Dependent Exponential Gradient Bound

Lemma 2 (Gradient Evolution Inequality)**.** Let $u \in C^{2,1}(\Sigma\times[0,T))$ be a smooth solution to the modified mean curvature flow (2.2.16). There exists a constant $C_0 > 0$, depending only on the background metric g_0 on Σ and the C^0 bound of u in Lemma 1, such that the gradient function w satisfies the differential inequality:

$$\mathscr{L}w \leqslant C_0 w - \frac{1}{2w\cosh^2(u)}\alpha^{ij}(g_0)^{k\ell}u_{ik}u_{j\ell} + \frac{1}{w}\alpha^{ij}w_iw_j + 4\tanh(u)\alpha^{ij}u_iw_j, \tag{4.1.1}$$

where $\mathscr{L}$ is the linear parabolic operator from (3.0.5). In particular, we have

$$\mathscr{L}w \leqslant C_0 w + \frac{1}{w}\alpha^{ij}w_iw_j + 4\tanh(u)\alpha^{ij}u_iw_j. \tag{4.1.2}$$

Proof. To compute w_k, we differentiate $w^2 = 1 + \frac{u^\ell u_\ell}{\cosh^2(u)}$:

$$\begin{aligned} 2ww_k &= \frac{2u^\ell u_{\ell k}}{\cosh^2(u)} - 2\tanh(u)\frac{u^\ell u_\ell}{\cosh^2(u)}u_k \\ w_k &= \frac{u^\ell u_{\ell k}}{w\cosh^2(u)} - \frac{\tanh(u)}{w}\frac{u^\ell u_\ell}{\cosh^2(u)}u_k \\ &= \frac{u^\ell u_{\ell k}}{w\cosh^2(u)} - \frac{\tanh(u)}{w}(w^2-1)u_k. \end{aligned} \tag{4.1.3}$$

Differentiating once more yields the Hessian $w_{ij} = D_jw_i$:

$$w_{ij} = D_j\left(\frac{u^\ell u_{\ell i}}{w\cosh^2(u)}\right) - D_j\left(\frac{\tanh(u)}{w}(w^2-1)u_i\right).$$

The first term gives:

$$\begin{aligned} D_j\left(\frac{u^\ell u_{\ell i}}{w\cosh^2(u)}\right) = &\left[-\frac{w_j}{w^2\cosh^2(u)} - \frac{2\tanh(u)}{w\cosh^2(u)}u_j\right]u^\ell u_{\ell i} \\ &+ \frac{1}{w\cosh^2(u)}(u^\ell_j u_{\ell i} + u^\ell D_ju_{\ell i}). \end{aligned}$$

Expanding the second term yields:

$$
\begin{aligned}
D_j\left(\frac{\tanh(u)}{w}(w^2-1)u_i\right) &= \left(\frac{\operatorname{sech}^2(u)}{w}u_j - \frac{\tanh(u)}{w^2}w_j\right)(w^2-1)u_i \\
&\quad + \frac{\tanh(u)}{w}\,u_i(2ww_j) + \frac{\tanh(u)}{w}(w^2-1)u_{ij} \\
&= \left(\frac{w^2-1}{w}\right)\operatorname{sech}^2(u)\,u_iu_j + \left(\frac{w^2+1}{w^2}\right)\tanh(u)\,u_iw_j \\
&\quad + \frac{\tanh(u)}{w}(w^2-1)u_{ij}.
\end{aligned}
$$

Putting these together, we have

$$
\begin{aligned}
w_{ij} &= \frac{1}{w\cosh^2(u)}\left(u^\ell_j u_{\ell i} + u^\ell D_j u_{\ell i}\right) - \frac{w_j}{w^2\cosh^2(u)}u^\ell u_{\ell i} - \frac{2\tanh(u)}{w\cosh^2(u)}u_ju^\ell u_{\ell i} \\
&\quad - \frac{\operatorname{sech}^2(u)}{w}(w^2-1)u_iu_j - \tanh(u)\left(1+\frac{1}{w^2}\right)u_iw_j - \frac{\tanh(u)}{w}(w^2-1)u_{ij}.
\end{aligned}
\tag{4.1.4}
$$

We now evaluate $\mathscr{L}w = \frac{\partial w}{\partial t} - \alpha^{ij}w_{ij} - \beta^k w_k$. Taking derivative of the flow equation (3.0.1): $u_t = \mathcal{F}(D^2u, Du, u)$ yields $D_ku_t = \alpha^{ij}D_ku_{ij} + \beta^\ell u_{\ell k} + \gamma u_k$, where $\gamma = \frac{\partial \mathcal{F}}{\partial u}$. Along with (B.0.4) that gives $\frac{\partial w}{\partial t} = \frac{u^k}{w\cosh^2(u)}(\alpha^{ij}D_ku_{ij} + \beta^\ell u_{\ell k} + \gamma u_k) - \frac{\tanh(u)}{w}(w^2-1)u_t$, we get

$$
\mathscr{L}w = \frac{u^k}{w\cosh^2(u)}(\alpha^{ij}D_ku_{ij} + \beta^\ell u_{\ell k} + \gamma u_k) - \frac{\tanh(u)}{w}(w^2-1)u_t - \alpha^{ij}w_{ij} - \beta^k w_k.
$$

Noting that the item contained in the first term $\frac{u^k}{w\cosh^2(u)}\beta^\ell u_{\ell k}$ and the last item $-\beta^k w_k$, they can be combined once indices k and ℓ are swapped:

$$
\begin{aligned}
\frac{u^k}{w\cosh^2(u)}\beta^\ell u_{\ell k} - \beta^k w_k &= \frac{u^\ell}{w\cosh^2(u)}\beta^k u_{k\ell} - \beta^k w_k \\
&= \beta^k\left(\frac{u^\ell}{w\cosh^2(u)}u_{k\ell} - w_k\right) \\
&= \beta^k\frac{\tanh(u)}{w}\left(w^2-1\right)u^k,
\end{aligned}
$$

in which the last equal sign is justified by using (4.1.3). Therefore, we now have

$$\begin{aligned}\mathscr{L}w &= \frac{u^k}{w\cosh^2(u)}(\alpha^{ij}D_k u_{ij} + \gamma u_k)\\ &\qquad - \frac{\tanh(u)}{w}(w^2-1)u_t - \alpha^{ij}w_{ij} + \beta^k \frac{\tanh(u)}{w}\left(w^2-1\right)u^k\end{aligned}$$

Next, we expand $-\alpha^{ij}w_{ij}$. Using again (4.1.3), we have

$$u^\ell u_{\ell i} = w\cosh^2(u)w_i + \tanh(u)\cosh^2(u)(w^2-1)u_i. \tag{4.1.5}$$

Contracting the second and third terms in (4.1.4) with $-\alpha^{ij}$ yields:

$$\begin{aligned}&-\alpha^{ij}\left[-\frac{w_j}{w^2\cosh^2(u)}u^\ell u_{\ell i} - \frac{2\tanh(u)}{w\cosh^2(u)}u_j u^\ell u_{\ell i}\right]\\ &= \frac{1}{w^2\cosh^2(u)}\alpha^{ij}w_j\left[w\cosh^2(u)w_i + \tanh(u)\cosh^2(u)(w^2-1)u_i\right]\\ &\quad + \frac{2\tanh(u)}{w\cosh^2(u)}\alpha^{ij}u_j\left[w\cosh^2(u)w_i + \tanh(u)\cosh^2(u)(w^2-1)u_i\right]\\ &= \frac{1}{w}\alpha^{ij}w_i w_j + \frac{\tanh(u)}{w^2}(w^2-1)\alpha^{ij}u_i w_j\\ &\qquad + 2\tanh(u)\alpha^{ij}u_j w_i + \frac{2\tanh^2(u)}{w}(w^2-1)\alpha^{ij}u_i u_j.\end{aligned}$$

The expansion of $-\alpha^{ij}w_{ij}$ also contains the term $+\tanh(u)(1+\frac{1}{w^2})\alpha^{ij}u_i w_j$, which is from the second last term in (4.1.4) when contracted with $-\alpha^{ij}$. Adding this to the above, the term involving $\alpha^{ij}u_i w_j$ becomes:

$$\tanh(u)\left[\left(1-\frac{1}{w^2}\right) + 2 + \left(1+\frac{1}{w^2}\right)\right]\alpha^{ij}u_i w_j = 4\tanh(u)\alpha^{ij}u_i w_j.$$

Therefore,

$$
\begin{aligned}
\mathscr{L}w =& \frac{u^k}{w\cosh^2(u)}(\alpha^{ij}D_k u_{ij} + \gamma u_k) - \frac{\tanh(u)}{w}(w^2-1)u_t \\
&- \alpha^{ij}\Bigg(\frac{1}{w\cosh^2(u)}\left(u_j^\ell u_{\ell i} + u^\ell D_j u_{\ell i}\right) - \frac{\operatorname{sech}^2(u)}{w}(w^2-1)u_i u_j \\
&- \frac{\tanh(u)}{w}(w^2-1)u_{ij} - 4\tanh(u)u_i w_j\Bigg) \\
&+ \frac{1}{w}\alpha^{ij}w_i w_j + \frac{2\tanh^2(u)}{w}(w^2-1)\alpha^{ij}u_i u_j + \beta^k\frac{\tanh(u)}{w}(w^2-1)u_k.
\end{aligned}
$$

We now define

$$
\begin{cases}
\mathcal{A} := -\frac{\tanh(u)}{w}(w^2-1)u_t + \alpha^{ij}\frac{\tanh(u)}{w}(w^2-1)u_{ij} \\
\mathcal{B} := \frac{\operatorname{sech}^2(u)}{w}(w^2-1)\alpha^{ij}u_i u_j + \frac{2\tanh^2(u)}{w}(w^2-1)\alpha^{ij}u_i u_j \\
\mathcal{C} := \frac{u^k}{w\cosh^2(u)}\alpha^{ij}(D_k u_{ij} - D_j u_{ik}) \\
\mathcal{D} := \frac{\gamma u_k u^k}{w\cosh^2(u)} + \beta^k\frac{\tanh(u)}{w}(w^2-1)u_k \\
\mathcal{Q} := \frac{1}{w\cosh^2(u)}\alpha^{ij}u_i^\ell u_{j\ell} \\
\mathcal{L}_{trace} := -\frac{2\tanh(u)}{w}(w^2-1)\alpha^{ij}u_{ij}
\end{cases}
$$

such that

$$
\mathscr{L}w = (\mathcal{A}+\mathcal{B}) + \mathcal{C} + \mathcal{D} - \mathcal{Q} + \frac{1}{w}\alpha^{ij}w_i w_j + 4\tanh(u)\alpha^{ij}u_i w_j\ . \tag{4.1.6}
$$

Using (2.2.16), $u_t = \frac{1}{w^2}g^{ij}u_{ij} - \frac{\tanh(u)}{w^2}\left(3 - \frac{1}{w^2}\right) + \frac{\sigma}{w}$ and substituting $\frac{1}{w^2}g^{ij} = \alpha^{ij}$ yields

$$
\begin{aligned}
\mathcal{A} &= -\frac{\tanh(u)}{w}(w^2-1)u_t + \alpha^{ij}\frac{\tanh(u)}{w}(w^2-1)u_{ij} \\
&= -\frac{\tanh(u)}{w}(w^2-1)\alpha^{ij}u_{ij} + \frac{\tanh^2(u)(w^2-1)}{w^3}\left(3-\frac{1}{w^2}\right) \\
&\quad - \frac{\sigma\tanh(u)(w^2-1)}{w^2} + \alpha^{ij}\frac{\tanh(u)}{w}(w^2-1)u_{ij} \\
&= \frac{\tanh^2(u)(w^2-1)}{w^3}\left(3-\frac{1}{w^2}\right) - \frac{\sigma\tanh(u)(w^2-1)}{w^2} \qquad (4.1.7)
\end{aligned}
$$

As w, $\tanh^2(u)$, w^2-1, and $3-\frac{1}{w^2}$ are non-negative,

$$\mathcal{A} \leq \frac{\tanh^2(u)(w^2-1)}{w^3}\left(3-\frac{1}{w^2}\right)+\frac{|\sigma\tanh(u)|(w^2-1)}{w^2}.$$

Moreover, we have $|\tanh(u)|\leq 1$ and $|\sigma|\leq 2$, so

$$\begin{aligned}
\mathcal{A} &\leq \frac{w^2-1}{w^3}\left(3-\frac{1}{w^2}\right)+\frac{2(w^2-1)}{w^2} \\
&\leq \frac{w^2-0}{w^3}(3-0)+\frac{2(w^2-0)}{w^2} \\
&= \frac{3}{w}+2\,.
\end{aligned}$$

Given $w\geq 1$, we have $\frac{1}{w}\leq 1$, so

$$\mathcal{A}\leq 5\leq 5w. \tag{4.1.8}$$

Using (B.0.1): $\alpha^{ij}=\frac{(g_0)^{ij}}{w^2\cosh^2(u)}-\frac{u^iu^j}{w^4\cosh^4(u)}$ and $|Du|^2_{g_0}=\cosh^2(u)(w^2-1)$, we have

$$\begin{aligned}
\alpha^{ij}u_i &= \frac{u^j}{w^2\cosh^2(u)}-\frac{|Du|^2_{g_0}u^j}{w^4\cosh^4(u)} \\
&= \frac{u^j}{w^2\cosh^2(u)}\left(1-\frac{w^2-1}{w^2}\right) \\
&= \frac{u^j}{w^4\cosh^2(u)}. \qquad (4.1.9)
\end{aligned}$$

Contracting this with u_j, we get

$$\begin{aligned}
\alpha^{ij}u_iu_j &= \frac{(g_0)^{ij}u_iu_j}{w^2\cosh^2(u)}-\frac{u^iu^ju_iu_j}{w^4\cosh^4(u)} \\
&= \frac{|Du|^2}{w^2\cosh^2(u)}-\frac{|Du|^4}{w^4\cosh^4(u)} \\
&= \frac{1}{w^4}\cdot\frac{|Du|^2}{\cosh^2(u)}\cdot\left(w^2-\frac{|Du|^2}{\cosh^2(u)}\right) \\
&= \frac{1}{w^4}(w^2-1). \qquad (4.1.10)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathcal{B} &= \frac{\operatorname{sech}^2(u)}{w}(w^2-1)\alpha^{ij}u_iu_j + \frac{2\tanh^2(u)}{w}(w^2-1)\alpha^{ij}u_iu_j \\
&= \frac{\operatorname{sech}^2(u)}{w}(w^2-1)\frac{1}{w^4}(w^2-1) + \frac{2\tanh^2(u)}{w}(w^2-1)\frac{1}{w^4}(w^2-1) \\
&= \frac{(w^2-1)^2}{w^5}\left(\operatorname{sech}^2(u) + 2\tanh^2(u)\right)
\end{aligned}
\tag{4.1.11}
$$

Note that $\operatorname{sech}^2(u) + 2\tanh^2(u) = 1 + \tanh^2(u) \leqslant 1$, so we get

$$\mathcal{B} \leq \frac{2(w^2-1)^2}{w^5} \leq \frac{2(w^2-0)^2}{w^5} = \frac{2}{w}.$$

Similarly by $\frac{1}{w} \leq 1 \leq w$,

$$\mathcal{B} \leq 2w \tag{4.1.12}$$

Now by (4.1.8) and (4.1.12), we have

$$\mathcal{A} + \mathcal{B} \leqslant 7w. \tag{4.1.13}$$

Moving onto the third-order covariant derivatives $C = \frac{u^k}{w\cosh^2(u)}\alpha^{ij}(D_ku_{ij} - D_ju_{ik})$, we need to work with the Ricci identity. For a manifold with constant sectional curvature K, we have

$$R(X,Y)Z = K\left(\langle Y,Z\rangle X - \langle X,Z\rangle Y\right).$$

In a Fuchsian manifold, $K = -1$, and we can let

$$X = \partial_k, \qquad Y = \partial_j, \qquad Z = \partial_i,$$

and obtain

$$
\begin{aligned}
R(\partial_k,\partial_j)\partial_i &= (-1)\left(\langle\partial_j,\partial_i\rangle\partial_k - \langle\partial_k,\partial_i\rangle\partial_j\right) \\
&= -\left[(g_0)_{ji}\,\partial_k - (g_0)_{ki}\,\partial_j\right].
\end{aligned}
$$

Therefore,

$$\begin{aligned} R_{kjiq} &= \left\langle R(\partial_k, \partial_j)\partial_i, \partial_q \right\rangle_{g_0} \\ &= -\left[(g_0)_{ji}(g_0)_{kq} - (g_0)_{ki}(g_0)_{jq} \right]. \end{aligned}$$

Contracting this with $(g_0)^{pq}$,

$$R_{kji}^{\ \ \ p} = -\left[(g_0)_{ji}\,\delta_k^p - (g_0)_{ki}\,\delta_j^p \right].$$

Contracting this with u_p,

$$R_{kji}^{\ \ \ p} u_p = -(g_0)_{ji}u_k + (g_0)_{ki}u_j.$$

Substituting this into C and using the definition $\alpha^{ij} = \frac{1}{w^2}g^{ij}$, we evaluate

$$\begin{aligned} C &= \frac{1}{w^3\cosh^2(u)} g^{ij}u^k(u_j(g_0)_{ki} - u_k(g_0)_{ij}) \\ &= \frac{1}{w^3\cosh^2(u)} g^{ij}(u_iu_j - |Du|^2(g_0)_{ij}) \\ &= -\frac{w^2-1}{w^3\cosh^2(u)}, \end{aligned} \tag{4.1.14}$$

where we used (A.0.5): $g^{ij}u_iu_j = \frac{w^2-1}{w^2}$, (A.0.4): $g^{ij}(g_0)_{ij} = \frac{2}{\cosh^2(u)} - \frac{|Du|^2}{w^2\cosh^4(u)}$ and $|Du|^2 = \cosh^2(u)(w^2-1)$. Therefore,

$$C \leq 0 \tag{4.1.15}$$

Now, using (3.0.4) and multiplying this by $\frac{u^ku_k}{w\cosh^2(u)} = \frac{w^2-1}{w}$ we get

$$\begin{aligned} \frac{\gamma u_ku^k}{w\cosh^2(u)} =& \left(-\frac{2\tanh(u)}{w^4\cosh^2(u)}(g_0)^{ij}u_{ij} + \frac{4\tanh(u)}{w^6\cosh^4(u)}u^iu^ju_{ij} \right)\frac{w^2-1}{w} \\ & - \frac{\operatorname{sech}^2(u)(w^2-1)}{w^3}\left(3 - \frac{1}{w^2}\right) - \frac{2\tanh^2(u)(w^2-1)^2}{w^5}\left(3 - \frac{2}{w^2}\right) \\ & + \frac{\sigma\tanh(u)(w^2-1)^2}{w^4}. \end{aligned}$$

Multiplying β^k in (3.0.3) by $\frac{\tanh(u)}{w}(w^2-1)u_k$ and using $u^k u_k = \cosh^2(u)(w^2-1)$, we have

$$\begin{aligned}
\beta^k \frac{\tanh(u)}{w}(w^2-1)u_k =& -\frac{2\tanh(u)(w^2-1)^2}{w^5\cosh^2(u)}(g_0)^{ij}u_{ij} - \frac{2\tanh(u)(w^2-1)}{w^5\cosh^4(u)}u^i u^j u_{ij} \\
&+ \frac{4\tanh(u)(w^2-1)^2}{w^7\cosh^4(u)}u^i u^j u_{ij} \\
&+ \frac{2\tanh^2(u)(w^2-1)^2}{w^5}\left(3-\frac{2}{w^2}\right) - \frac{\sigma\tanh(u)(w^2-1)^2}{w^4}.
\end{aligned}$$

Note that the $\tanh^2(u)$ and σ terms in $\frac{\gamma u_k u^k}{w\cosh^2(u)}$ and $\beta^k\frac{\tanh(u)}{w}(w^2-1)u_k$ cancel out. Now grouping the $(g_0)^{ij}u_{ij}$ terms yields:

$$-\frac{2\tanh(u)(w^2-1)}{w^5\cosh^2(u)}\left[1+(w^2-1)\right](g_0)^{ij}u_{ij} = -\frac{2\tanh(u)(w^2-1)}{w^3\cosh^2(u)}(g_0)^{ij}u_{ij}.$$

Grouping the $u^i u^j u_{ij}$ terms yields:

$$\frac{2\tanh(u)(w^2-1)}{w^7\cosh^4(u)}\left[2-w^2+2(w^2-1)\right]u^i u^j u_{ij} = \frac{2\tanh(u)(w^2-1)}{w^5\cosh^4(u)}u^i u^j u_{ij}.$$

Therefore,

$$\begin{aligned}
\mathcal{D} =& \frac{\gamma u_k u^k}{w\cosh^2(u)} + \beta^k\frac{\tanh(u)}{w}(w^2-1)u_k \\
=& -\frac{2\tanh(u)(w^2-1)}{w^3\cosh^2(u)}(g_0)^{ij}u_{ij} + \frac{2\tanh(u)(w^2-1)}{w^5\cosh^4(u)}u^i u^j u_{ij} \\
&- \frac{\operatorname{sech}^2(u)(w^2-1)}{w^3}\left(3-\frac{1}{w^2}\right) \\
=& -\frac{2\tanh(u)(w^2-1)}{w}\left(\frac{(g_0)^{ij}}{w^2\cosh^2(u)} - \frac{u^i u^j}{w^4\cosh^4(u)}\right)u_{ij} \\
&- \frac{\operatorname{sech}^2(u)(w^2-1)}{w^3}\left(3-\frac{1}{w^2}\right) \\
=& -\frac{2\tanh(u)(w^2-1)}{w}\alpha^{ij}u_{ij} - \frac{\operatorname{sech}^2(u)(w^2-1)}{w^3}\left(3-\frac{1}{w^2}\right),
\end{aligned} \tag{4.1.16}$$

where in the last equal sign we have used (B.0.1) for α^{ij}. Note that $\operatorname{sech}^2(u)$, w^2-1,

w^3, and $3 - \frac{1}{w^2}$ are all non-negative, so

$$\mathcal{D} \leqslant -\frac{2\tanh(u)(w^2-1)}{w}\alpha^{ij}u_{ij} = \mathcal{L}_{trace} \tag{4.1.17}$$

Combining (4.1.13), (4.1.15), and (4.1.17), we yield

$$\mathscr{L}w \leqslant 7w + \mathcal{L}_{trace} - \mathcal{Q} + \frac{1}{w}\alpha^{ij}w_i w_j + 4\tanh(u)\alpha^{ij}u_i w_j \tag{4.1.18}$$

Now for the primary positive semi-definite quadratic Hessian term,

$$\mathcal{Q} = \frac{1}{w\cosh^2(u)}\alpha^{ij}u_i^{\ell}u_{j\ell} = \frac{1}{w\cosh^2(u)}\alpha^{ij}(g_0)^{k\ell}u_{ik}u_{j\ell} \ , \tag{4.1.19}$$

we first note that (4.1.9) gives $\alpha^{ij}u_i = \frac{u^j}{w^4\cosh^2(u)}$. Thus, the gradient vector Du is an eigenvector of α^{ij} with respect to g_0 with eigenvalue:

$$\lambda_{\|} = \frac{1}{w^4\cosh^2(u)}.$$

Now let V be a tangent vector such that $\langle V, Du\rangle_{g_0} = u_i V^i = 0$. Contracting α^{ij} with V_j yields:

$$\begin{aligned}\alpha^{ij}V_j &= \frac{1}{w^2\cosh^2(u)}(g_0)^{ij}V_j - \frac{1}{w^4\cosh^4(u)}u^i(u^jV_j)\\ &= \frac{1}{w^2\cosh^2(u)}V^i.\end{aligned}$$

Thus, any vector orthogonal to Du is an eigenvector of α^{ij} with eigenvalue:

$$\lambda_{\perp} = \frac{1}{w^2\cosh^2(u)}. \tag{4.1.20}$$

Therefore, the minimum and maximum eigenvalues of α^{ij} are

$$\lambda_{\min} = \lambda_{\|} = \frac{1}{w^4\cosh^2(u)} \quad \text{and} \quad \lambda_{\max} = \lambda_{\perp} = \frac{1}{w^2\cosh^2(u)}\,.$$

By definition, $w\cosh^2(u)\,Q$ is the trace of a positive semi-definite 2-tensor $M = M_{ij} = (g_0)^{k\ell}u_{ik}u_{j\ell}$ against α^{ij}. To see that M is positive semi-definite, we contract it with an arbitrary tangent vector V:

$$M_{ij}V^iV^j = (g_0)^{k\ell}(u_{ik}V^i)(u_{j\ell}V^j) = (g_0)^{k\ell}W_kW_\ell = |W|^2_{g_0} \geqslant 0\,,$$

where $W_k = u_{ik}V^i$. Therefore, because M_{ij} is positive semi-definite, its trace against the positive semi-definite metric α^{ij} is bounded below by the product of the minimum eigenvalue of α and the g_0-trace of M:

$$\begin{aligned}
Q =&\frac{1}{w\cosh^2(u)}\alpha^{ij}M_{ij} \geqslant \frac{1}{w\cosh^2(u)}\lambda_{\min}(\alpha)\mathrm{tr}_{g_0}(M)\\
=&\frac{1}{w\cosh^2(u)}\lambda_{\min}(\alpha)(g_0)^{ij}M_{ij} = \frac{1}{w\cosh^2(u)}\lambda_{\min}(\alpha)(g_0)^{ij}(g_0)^{k\ell}u_{ik}u_{j\ell}\\
=&\frac{1}{w^5\cosh^4(u)}|D^2u|^2_{g_0} \geq 0.
\end{aligned} \tag{4.1.21}$$

Now we evaluate the remaining term $\mathcal{L}_{trace} = -\frac{2\tanh(u)(w^2-1)}{w}\alpha^{ij}u_{ij}$ in (4.1.18). Since α^{ij} is positive-definite and symmetric, let S^{ij} be the unique positive-definite symmetric square root of α^{ij} such that $\alpha^{ij} = \sum_{k=1}^2 S^{ik}S^{kj}$. We define the 2×2 matrix $M = M^{k\ell} = S^{ik}u_{ij}S^{j\ell}$ and rewrite

$$\alpha^{ij}u_{ij} = \sum_{k=1}^{2} S^{ik}S^{kj}u_{ij} = \sum_{k=1}^{2} S^{ik}u_{ij}S^{j\ell}\delta^k_\ell = \sum_{k=1}^{2} M^{k\ell}\delta^k_\ell = \mathrm{tr}(M).$$

We apply the trace inequality

$$(\mathrm{tr}(M))^2 \leq 2\mathrm{tr}(M^2),$$

which yields

$$(\alpha^{ij}u_{ij})^2 \leq 2\mathrm{tr}(M^2) = 2\sum_{\ell=1}^{2}\sum_{k=1}^{2}(S^{ik}u_{ij}S^{j\ell})(S^{p\ell}u_{pq}S^{qk})$$
$$=2\alpha^{qi}u_{ij}\alpha^{jp}u_{pq} = 2\alpha^{ik}\alpha^{j\ell}u_{ij}u_{k\ell}, \tag{4.1.22}$$

where we used $S^{j\ell} = S^{\ell j}$, $\sum_{\ell=1}^{2} S^{j\ell}S^{p\ell} = \alpha^{jp}$ and $\sum_{k=1}^{2} S^{qk}S^{ik} = \alpha^{qi}$. Because the max eigenvalue of α^{ij} is $\frac{1}{w^2\cosh^2(u)}$ (see (4.1.20)), by (4.1.22) and the definition of Q we have

$$(\alpha^{ij}u_{ij})^2 \leqslant \frac{2}{w^2\cosh^2(u)}\alpha^{ik}(g_0)^{j\ell}u_{ij}u_{k\ell} = \frac{2}{w}Q.$$

Therefore,

$$|\mathcal{L}_{trace}| \leqslant \frac{2(w^2-1)}{w}|\alpha^{ij}u_{ij}| \leqslant 2w\sqrt{\frac{2}{w}Q} = 2\sqrt{4w}\sqrt{\frac{Q}{2}} \leqslant \frac{Q}{2} + 4w\,,$$

where we applied Young's inequality in the last step. Combining these estimates into (4.1.18), we obtain

$$\mathscr{L}w \leqslant 11w - \frac{Q}{2} + \frac{1}{w}\alpha^{ij}w_iw_j + 4\tanh(u)\alpha^{ij}u_iw_j.$$

Discarding the negative-definite term $-\frac{Q}{2}$, we yield

$$\mathscr{L}w \leqslant C_0w + \frac{1}{w}\alpha^{ij}w_iw_j + 4\tanh(u)\alpha^{ij}u_iw_j.$$

□

Corollary 1 (Time-Dependent Exponential Gradient Bound)**.** Let $u \in C^{2,1}(\Sigma \times [0,T))$ be a smooth solution to the modified mean curvature flow (2.2.16). The gradient function w satisfies the time-dependent exponential bound:

$$w(x,t) \leqslant \left(\sup_{x\in\Sigma} w(x,0)\right)e^{C_0t} \qquad \forall (x,t) \in \Sigma \times [0,T),$$

where the constant $C_0 > 0$ is given in Lemma 2. Consequently, the flow cannot develop finite-time gradient singularities, ensuring that the smooth solution can be extended for all time $t \in [0, \infty)$. Moreover, the flow converges exponentially in C^0 to the equidistant surface $\Sigma(r_\sigma)$ where $r_\sigma = \tanh^{-1}\left(\frac{\sigma}{2}\right)$ is from Lemma 1.

Proof. We define the vector field $\tilde{\beta}^k$:

$$\tilde{\beta}^k = \beta^k + \frac{1}{w}\alpha^{ik}w_i + 4\tanh(u)\alpha^{ik}u_i$$

and the corresponding linear parabolic operator

$$\tilde{\mathscr{L}} = \frac{\partial}{\partial t} - \alpha^{ij}D_iD_j - \tilde{\beta}^k D_k.$$

The inequality (4.1.2) can be rewritten as:

$$\tilde{\mathscr{L}}w \leqslant C_0 w.$$

Now, let $\tilde{w}(x,t) = e^{-C_0 t}w(x,t)$. We have

$$\begin{aligned}
\tilde{\mathscr{L}}\tilde{w} &= \frac{\partial}{\partial t}(e^{-C_0 t}w) - \alpha^{ij}D_iD_j(e^{-C_0 t}w) - \tilde{\beta}^k D_k(e^{-C_0 t}w) \\
&= -C_0 e^{-C_0 t}w + e^{-C_0 t}\frac{\partial w}{\partial t} - e^{-C_0 t}\alpha^{ij}w_{ij} - e^{-C_0 t}\tilde{\beta}^k w_k \\
&= e^{-C_0 t}(\tilde{\mathscr{L}}w - C_0 w) \leqslant 0.
\end{aligned}$$

Because $\tilde{\mathscr{L}}$ is a uniformly parabolic operator with continuous coefficients, the standard parabolic maximum principle guarantees that the spatial maximum of the sub-solution $\tilde{w}$ is non-increasing in time. Therefore, for all $t \in [0, T)$, we have:

$$e^{-C_0 t}w(x,t) = \tilde{w}(x,t) \leqslant \sup_{x\in\Sigma}\tilde{w}(x,0) = \sup_{x\in\Sigma}w(x,0),$$

which yields the desired result by multiplying both sides by $e^{C_0 t}$. $\square$

Chapter 5.

Uniform Gradient Bound and Convergence

§ 5.1. The case $\sigma = 0$

We now use the test function

$$\Psi(x,t) = \cosh(u(x,t))w(x,t)$$

to derive an improved linear bound on the gradient function w, and in particular a uniform gradient bound in the case of mean curvature flow ($\sigma = 0$).

Theorem 2. Let $u \in C^{2,1}(\Sigma \times [0,T))$ be a smooth solution to the modified mean curvature flow (2.2.16) with $\sigma \in (-2,2)$. There exist constants $C_1, C_2 > 0$, depending only on the background metric g_0 on Σ, the C^0 bound of u in Lemma 1, the constant σ, and the initial gradient bound $\|w(\cdot,0)\|_\infty$, such that:

$$w(x,t) \leqslant C_1(1+t) \qquad \forall (x,t) \in \Sigma \times [0,T). \tag{5.1.1}$$

Moreover, when $\sigma = 0$ we have

$$w(x,t) \leqslant C_2 \qquad \forall (x,t) \in \Sigma \times [0,T). \tag{5.1.2}$$

Proof. We compute the linear parabolic operator $\mathscr{L} = \partial_t - \alpha^{ij}D_iD_j - \beta^k D_k$ from (3.0.5) acting on the test function $\Psi = \cosh(u)w$. To do so, we first compute:

$$\Psi_k = \sinh(u)u_k w + \cosh(u)w_k,$$
$$\Psi_t = \sinh(u)u_t w + \cosh(u)w_t,$$
$$\Psi_{ij} = \cosh(u)u_i u_j w + \sinh(u)u_{ij}w + \sinh(u)(u_i w_j + u_j w_i) + \cosh(u)w_{ij}.$$

Substituting these into $\mathscr{L}$ yields

$$\mathscr{L}\Psi = \cosh(u)\mathscr{L}w + w\sinh(u)\mathscr{L}u - w\cosh(u)\alpha^{ij}u_i u_j - 2\sinh(u)\alpha^{ij}u_i w_j. \tag{5.1.3}$$

Define the spatial maximum function $\Psi_{\max}(t) = \sup_{x\in\Sigma}\Psi(x,t)$. By the compactness of the closed 2-surface Σ, for each $t \in [0,T)$, there exists a point $x_t \in \Sigma$ such that $\Psi(x_t,t) = \Psi_{\max}(t)$. At this spatial maximum point (x_t,t), the gradient vanishes ($\Psi_k(x_t,t) = 0$), which implies that at (x_t,t) we have

$$w_k = -\tanh(u)wu_k. \tag{5.1.4}$$

Furthermore, the Hessian matrix of Ψ is negative semi-definite ($\Psi_{ij}(x_t,t) \leqslant 0$). Because $\alpha^{ij} = \frac{1}{w^2}g^{ij}$ is strictly positive-definite, we have $-\alpha^{ij}\Psi_{ij}(x_t,t) \geqslant 0$. Thus, using Hamilton's trick we have

$$\frac{d}{dt}\Psi_{\max}(t) \leqslant \mathscr{L}\Psi(x_t,t).$$

Substituting (5.1.4) into the cross term in (5.1.3), we have

$$-2\sinh(u)\alpha^{ij}u_i w_j = -2\sinh(u)\alpha^{ij}u_i(-\tanh(u)wu_j) = 2w\frac{\sinh^2(u)}{\cosh(u)}\alpha^{ij}u_i u_j\,,$$

Combining this with the $u_i u_j$ term in (5.1.3), we obtain:

$$\mathscr{L}\Psi = \cosh(u)\mathscr{L}w + w\sinh(u)\mathscr{L}u + w\left(\frac{2\sinh^2(u) - \cosh^2(u)}{\cosh(u)}\right)\alpha^{ij}u_i u_j. \tag{5.1.5}$$

Next, we evaluate $\mathscr{L}u = u_t - \alpha^{ij}u_{ij} - \beta^k u_k$. Using the flow equation (2.2.16), we have

$$u_t - \alpha^{ij}u_{ij} = -\frac{\tanh(u)}{w^2}\left(3 - \frac{1}{w^2}\right) + \frac{\sigma}{w}. \tag{5.1.6}$$

To compute $\beta^k u_k$, we contract β^k with u_k and use $|Du|^2 = u^k u_k = \cosh^2(u)(w^2 - 1)$ to get:

$$\begin{aligned}
\beta^k u_k =& \Bigg(- \frac{2u^k}{w^4\cosh^4(u)}(g_0)^{ij}u_{ij} - \frac{2(g_0)^{ik}u^j u_{ij}}{w^4\cosh^4(u)} + \frac{4u^k u^i u^j u_{ij}}{w^6\cosh^6(u)} \\
&+ \frac{2\tanh(u)u^k}{w^4\cosh^2(u)}\left(3 - \frac{2}{w^2}\right) - \frac{\sigma u^k}{w^3\cosh^2(u)}\Bigg)u_k \\
=& - \frac{2(w^2-1)}{w^4\cosh^2(u)}(g_0)^{ij}u_{ij} - \frac{2u^i u^j u_{ij}}{w^4\cosh^4(u)} + \frac{4(w^2-1)u^i u^j u_{ij}}{w^6\cosh^4(u)} + P_\beta \\
=& - \frac{2(w^2-1)}{w^2}\left[\frac{(g_0)^{ij}}{w^2\cosh^2(u)} - \frac{u^i u^j}{w^4\cosh^4(u)}\right]u_{ij} - \frac{2 - 2\left(\frac{w^2-1}{w^2}\right)}{w^4\cosh^4(u)}u^i u^j u_{ij} + P_\beta \\
=& - \frac{2(w^2-1)}{w^2}\alpha^{ij}u_{ij} - \frac{2}{w^6\cosh^4(u)}u^i u^j u_{ij} + P_\beta,
\end{aligned} \tag{5.1.7}$$

where we used (B.0.1) to get $\alpha^{ij}u_{ij} = \frac{(g_0)^{ij}}{w^2\cosh^2(u)}u_{ij} - \frac{u^i u^j}{w^4\cosh^4(u)}u_{ij}$ and

$$P_\beta = \frac{2\tanh(u)(w^2-1)}{w^4}\left(3 - \frac{2}{w^2}\right) - \frac{\sigma(w^2-1)}{w^3} = \frac{6\tanh(u)}{w^2} - \frac{\sigma}{w} + O(w^{-3}),$$

as $w \longrightarrow \infty$.

Subtracting (5.1.7) from (5.1.6) yields:

$$\mathscr{L}u = \frac{2(w^2-1)}{w^2}\alpha^{ij}u_{ij} + \frac{2}{w^6\cosh^4(u)}u^i u^j u_{ij} + P_u, \tag{5.1.8}$$

where

$$P_u = -P_\beta - \frac{\tanh(u)}{w^2}\left(3 - \frac{1}{w^2}\right) + \frac{\sigma}{w} = -\frac{9\tanh(u)}{w^2} + \frac{2\sigma}{w} + O(w^{-3}) \quad \text{as } w \longrightarrow \infty.$$

Next, substituting (5.1.4) into the formula for w_k from (4.1.3), we have

$$w_k = \frac{u^\ell u_{\ell k}}{w\cosh^2(u)} - \frac{\tanh(u)}{w}(w^2-1)u_k = -\tanh(u)wu_k$$

This implies $u^\ell u_{\ell k} = -\tanh(u)\cosh^2(u)u_k$. Contracting it with u^k yields that at the maximum point (x_t, t), we have

$$u^i u^j u_{ij} = u^k(u^\ell u_{\ell k}) = -\tanh(u)\cosh^2(u)|Du|^2 = -\tanh(u)\cosh^4(u)(w^2-1).$$

Substituting this into $w\sinh(u)\mathscr{L}u$ using (5.1.8), we get

(5.1.9)
$$w\sinh(u)\left[\frac{2}{w^6\cosh^4(u)}u^i u^j u_{ij}\right] = -\frac{2\sinh^2(u)(w^2-1)}{w^5\cosh(u)} = O(w^{-3}) \quad \text{as } w\to\infty.$$

Now we use the identity for $\mathscr{L}w$ (proof in Appendix B under Lemma 13):

$$\begin{aligned}\mathscr{L}w = &-\frac{1}{w\cosh^2(u)}\alpha^{ij}(g_0)^{k\ell}u_{ik}u_{j\ell} - \frac{2\tanh(u)(w^2-1)}{w}\alpha^{ij}u_{ij}\\ &-\frac{3(w^2-1)}{w^3\cosh^2(u)} + \frac{1}{w}\alpha^{ij}w_iw_j + 4\tanh(u)\alpha^{ij}u_iw_j\\ &+\frac{\tanh^2(u)(w^2-1)(5w^2-3)}{w^5} - \frac{\sigma\tanh(u)(w^2-1)}{w^2}.\end{aligned} \tag{5.1.10}$$

At the spatial maximum point (x_t, t), we substitute (5.1.4) into the gradient cross-terms of (5.1.10):

$$\frac{1}{w}\alpha^{ij}w_iw_j = \frac{1}{w}\alpha^{ij}\big(-\tanh(u)wu_i\big)\big(-\tanh(u)wu_j\big) = w\tanh^2(u)\alpha^{ij}u_iu_j,$$
$$4\tanh(u)\alpha^{ij}u_iw_j = 4\tanh(u)\alpha^{ij}u_i\big(-\tanh(u)wu_j\big) = -4w\tanh^2(u)\alpha^{ij}u_iu_j.$$

Summing these evaluates to $-3w\tanh^2(u)\alpha^{ij}u_iu_j$. Applying (4.1.10): $\alpha^{ij}u_iu_j = \frac{w^2-1}{w^4}$, this reduces to:

$$-3w\tanh^2(u)\left(\frac{w^2-1}{w^4}\right) = -\frac{3\tanh^2(u)(w^2-1)}{w^3}.$$

Combining this with the $\frac{\tanh^2(u)(w^2-1)(5w^2-3)}{w^5}$ term from (5.1.10) yields:

$$-\frac{3\tanh^2(u)(w^2-1)}{w^3}+\frac{\tanh^2(u)(w^2-1)(5w^2-3)}{w^5}=\frac{\tanh^2(u)(w^2-1)(2w^2-3)}{w^5}.$$

Therefore, evaluated at the maximum point (x_t, t), $\mathscr{L}w$ simplifies to:

$$\begin{aligned}\mathscr{L}w =& -\frac{1}{w\cosh^2(u)}\alpha^{ij}(g_0)^{k\ell}u_{ik}u_{j\ell}-\frac{2\tanh(u)(w^2-1)}{w}\alpha^{ij}u_{ij}\\ &-\frac{3(w^2-1)}{w^3\cosh^2(u)}+\frac{\tanh^2(u)(w^2-1)(2w^2-3)}{w^5}-\frac{\sigma\tanh(u)(w^2-1)}{w^2}.\end{aligned}\tag{5.1.11}$$

Now we substitute (5.1.8) and (5.1.11) into (5.1.5). First, we consider the two Hessian trace terms involving $\alpha^{ij}u_{ij}$:

$$\begin{aligned}\mathrm{Trace}_{\mathscr{L}w} &:= \cosh(u)\left[-\frac{2\tanh(u)(w^2-1)}{w}\alpha^{ij}u_{ij}\right]=-\frac{2\sinh(u)(w^2-1)}{w}\alpha^{ij}u_{ij},\\ \mathrm{Trace}_{\mathscr{L}u} &:= w\sinh(u)\left[\frac{2(w^2-1)}{w^2}\alpha^{ij}u_{ij}\right]=\frac{2\sinh(u)(w^2-1)}{w}\alpha^{ij}u_{ij}.\end{aligned}$$

Therefore, these two trace terms sum to zero at the maximum point (x_t, t) in (5.1.5).

Combining these using (5.1.8), (5.1.9), (5.1.11), and (4.1.10), we obtain:

$$\begin{aligned}\mathscr{L}\Psi =& \cosh(u)\mathscr{L}w + w\sinh(u)\mathscr{L}u + w\left(\frac{2\sinh^2(u)-\cosh^2(u)}{\cosh(u)}\right)\alpha^{ij}u_iu_j\\ =& \cosh(u)\Big(-\frac{1}{w\cosh^2(u)}\alpha^{ij}(g_0)^{k\ell}u_{ik}u_{j\ell}-\frac{3(w^2-1)}{w^3\cosh^2(u)}\\ &+\frac{\tanh^2(u)(w^2-1)(2w^2-3)}{w^5}-\frac{\sigma\tanh(u)(w^2-1)}{w^2}\Big)\\ &+ w\sinh(u)\Big(-\frac{9\tanh(u)}{w^2}+\frac{2\sigma}{w}\\ &+O(w^{-3})\Big)+w\left(\frac{2\sinh^2(u)-\cosh^2(u)}{\cosh(u)}\right)\frac{w^2-1}{w^4}\\ =& -\cosh(u)Q+\sigma\sinh(u)-\frac{6\sinh^2(u)+4}{w\cosh(u)}+O(w^{-2}),\end{aligned}$$

where Q is the positive semi-definite quadratic Hessian from (4.1.21). Note that the primary obstruction here is the dominating term $\sigma\sinh(u)$ when w is large. For $\sigma = 0$, we have:

$$\frac{d}{dt}\Psi_{\max}(t) \leqslant \mathscr{L}\Psi(x_t, t) \leqslant -\frac{6\sinh^2(u)+4}{w\cosh(u)} + O(w^{-2}).$$

By Lemma 1, $\cosh(u)$ is uniformly bounded from above (and trivially bounded from below by 1). Therefore, there exists a sufficiently large constant $\bar{C} > 0$ such that whenever $\Psi_{\max}(t) > \bar{C}$, we have $\frac{d}{dt}\Psi_{\max}(t) < 0$. Arguing as in the proof of Lemma 1, we have

$$\Psi(x,t) \leqslant \max\left\{\sup_{x\in\Sigma}\Psi(x,0), \bar{C}\right\} := C_\Psi,$$

which, since $w \leqslant \Psi$, immediately implies $w(x,t) \leqslant C_2$.

For general $\sigma \in (-2,2)$, Lemma 1 again guarantees that $\sigma\sinh(u)$ is uniformly bounded, and thus $\frac{d}{dt}\Psi_{\max}(t) \leqslant \tilde{C}$. Integrating this from $t=0$ yields:

$$w(x,t) \leqslant \Psi(x,t) \leqslant \Psi_{\max}(0) + \tilde{C}t \leqslant C_1(1+t). \qquad \square$$

§ 5.2. The general case $\sigma \in (-2,2)$

We first recall the evolution equations of the gradient function w and the squared norm of the second fundamental form $|A|^2$ under the modified mean curvature flow.

Lemma 3. Let $u \in C^{2,1}(\Sigma\times[0,T))$ be a smooth solution to the modified mean curvature flow (2.2.16) with $\sigma \in (-2,2)$. Then the gradient function w satisfies the evolution equation:

$$\begin{aligned}(\partial_t - \Delta)w &= -w|A|^2 + 2w^2\tanh(u)H + 2\tanh(u)\langle\nabla w, \mathbf{n}\rangle \\ &\quad - \frac{w^2-1}{w\cosh^2(u)} - 2w\tanh^2(u) - \frac{2}{w}|\nabla w|^2 - \sigma\tanh(u)(w^2-1)\,,\end{aligned} \tag{5.2.1}$$

where A is the second fundamental form of the evolving surface Σ_t and $\mathbf{n} = \frac{\partial}{\partial r}$.

Proof. By [8, Theorem 3.6] and its proof, the evolution equation for $\Theta = \frac{1}{w}$ under the modified mean curvature flow is given by:

$$\begin{aligned}(\partial_t - \Delta)\Theta &= |A|^2\Theta - 2\tanh(u)H + 2\tanh(u)\langle \nabla\Theta, \mathbf{n}\rangle \\ &\quad + \frac{\Theta(1-\Theta^2)}{\cosh^2(u)} + 2\tanh^2(u)\Theta + \sigma\tanh(u)(1-\Theta^2).\end{aligned} \tag{5.2.2}$$

Since $w = \Theta^{-1}$, we have $\nabla w = -w^2\nabla\Theta$ and:

$$(\partial_t - \Delta)w = -w^2(\partial_t - \Delta)\Theta - \frac{2}{w}|\nabla w|^2.$$

Multiplying the right-hand side of (5.2.2) by $-w^2$ term-by-term, we obtain:

$$\begin{aligned} -w^2\left(|A|^2 w^{-1}\right) &= -w|A|^2, \\ -w^2\left(-2\tanh(u)H\right) &= 2w^2\tanh(u)H, \\ -w^2\left(2\tanh(u)\langle -w^{-2}\nabla w, \mathbf{n}\rangle\right) &= 2\tanh(u)\langle \nabla w, \mathbf{n}\rangle, \\ -w^2\left(\frac{w^{-1}(1-w^{-2})}{\cosh^2(u)}\right) &= -\frac{w^2-1}{w\cosh^2(u)}, \\ -w^2\left(2\tanh^2(u)w^{-1}\right) &= -2w\tanh^2(u), \\ -w^2\left(\sigma\tanh(u)(1-w^{-2})\right) &= -\sigma\tanh(u)(w^2-1)\,, \end{aligned}$$

which yields (5.2.1). □

Lemma 4. Let $u \in C^{2,1}(\Sigma\times[0,T))$ be a smooth solution to the modified mean curvature flow (2.2.16) with $\sigma \in (-2, 2)$. Then the squared norm of the second fundamental form $|A|^2$ of the evolving surface Σ_t satisfies the following evolution equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 = -2|\nabla A|^2 + 2|A|^4 + 4(|A|^2 - H^2) - 2\sigma\mathrm{tr}(A^3) - 2\sigma H. \tag{5.2.3}$$

Proof. By [8, Theorem 2.2], the evolution equation for $|A|^2$ under the standard mean curvature flow is given by:

$$\left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 = -2|\nabla A|^2 + 2|A|^4 + 4(|A|^2 - H^2). \tag{5.2.4}$$

Following Huisken's [13, Lemma 3.3 and Theorem 3.4], under a normal variation $\partial_t F = f\nu$ for a smooth function f, the induced metric g_{ij}, its inverse g^{ij}, and the components of the second fundamental form of the evolving surface satisfy the evolution equations:

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= 2fh_{ij}, \\
\frac{\partial}{\partial t} g^{ij} &= -2fh^{ij}, \\
\frac{\partial}{\partial t} h_{ij} &= -\nabla_i \nabla_j f + fh_{il}h^l_j + f\overline{R}_{0i0j},
\end{aligned}$$

where $\overline{R}_{0i0j} = -\langle \bar{R}(\nu, \partial_i F)\partial_j F, \nu\rangle$ denotes the ambient Riemann curvature tensor evaluated in the normal direction.

Compared with (5.2.4), the additional contributions to the evolution equation of $|A|^2$ under the modified mean curvature flow arise from $\frac{\partial}{\partial t}$ acting on $|A|^2$. Because the time derivative operator is linear with respect to the normal variation speed, the evolution under the modified mean curvature flow $\partial_t F = -(H - \sigma)\nu$ decomposes into the standard mean curvature flow contribution ($f = -H$) and the contribution from the constant forcing term ($f = \sigma$). For this forcing contribution, denoting its corresponding variation by δ_σ, we have:

$$\begin{aligned}
&\delta_\sigma g^{ij} = -2\sigma h^{ij}, \\
&\delta_\sigma h_{ij} = \sigma h_{il}h^l_j + \sigma\overline{R}_{0i0j} = \sigma h_{il}h^l_j - \sigma g_{ij},
\end{aligned}$$

where we used $\overline{R}_{0i0j} = -g_{ij}$ since the ambient space M^3 is a Fuchsian manifold equipped with a hyperbolic metric. We then evaluate the variation $\delta_\sigma|A|^2 = \delta_\sigma(g^{ik}g^{jl}h_{ij}h_{kl})$:

$$\begin{aligned}
\delta_\sigma|A|^2 &= 2(\delta_\sigma g^{ik})h_i^l h_{kl} + 2g^{ik}g^{jl}(\delta_\sigma h_{ij})h_{kl} \\
&= 2(-2\sigma h^{ik})(h^2)_{ik} + 2h^{ij}\big(\sigma(h^2)_{ij} - \sigma g_{ij}\big) \\
&= -4\sigma\mathrm{tr}(A^3) + 2\sigma\mathrm{tr}(A^3) - 2\sigma H \\
&= -2\sigma\mathrm{tr}(A^3) - 2\sigma H. \qquad (5.2.5)
\end{aligned}$$

Combining (5.2.5) with (5.2.4) yields (5.2.3). □

We next establish a polynomial growth estimate for the second fundamental form of the evolving surfaces, following the ideas from [5].

Lemma 5. Let $u \in C^{2,1}(\Sigma\times[0,T))$ be a smooth solution to the modified mean curvature flow (2.2.16) with $\sigma \in (-2,2)$. There exists a constant $C > 0$, depending only on the background metric g_0 on Σ, the C^0 bound of u in Lemma 1, the constant σ, and the initial gradient bound $\|w(\cdot,0)\|_\infty$, such that the second fundamental form of the evolving surface Σ_t satisfies

$$|A(x,t)| \leqslant C(1+t)^4, \qquad \forall (x,t) \in \Sigma\times[0,\infty).$$

Proof. Fix an arbitrary time $T > 0$. By Theorem 2, on $\Sigma\times[0,T]$ we have
$w \leq C_1(1+T) := C_T$. We define the test function $Q = |A|^2\Phi(w)$ with $\Phi(w) = w^2 e^{c_T w^2}$, where $c_T = \frac{1}{2C_T^2}$ such that $c_T w^2 \leqslant \frac{1}{2}$ on $\Sigma\times[0,T]$.

We first compute:

$$\Phi'(w) = \left(\frac{2}{w} + 2c_T w\right)\Phi(w), \qquad (5.2.6)$$

$$\Phi''(w) = \left(-\frac{2}{w^2} + 2c_T\right)\Phi(w) + \left(\frac{2}{w} + 2c_T w\right)^2\Phi(w). \qquad (5.2.7)$$

Using (5.2.1), the evolution equation of $\Phi(w)$ is given by:

$$
\begin{aligned}
(\partial_t - \Delta)\Phi &= \Phi'(\partial_t - \Delta)w - \Phi''|\nabla w|^2 \\
&= -w\Phi'|A|^2 - \left(\frac{2}{w}\Phi' + \Phi''\right)|\nabla w|^2 + \Phi'\mathcal{E}_w,
\end{aligned} \tag{5.2.8}
$$

where

$$\mathcal{E}_w = 2w^2\tanh(u)H + 2\tanh(u)\langle\nabla w, \mathbf{n}\rangle - \frac{w^2-1}{w\cosh^2(u)} - 2w\tanh^2(u) - \sigma\tanh(u)(w^2-1).$$

Using (5.2.8) and Lemma 4, the evolution equation of Q is given by:

(5.2.9)

$$
\begin{aligned}
(\partial_t - \Delta)Q =& \Phi(\partial_t - \Delta)|A|^2 + |A|^2(\partial_t - \Delta)\Phi - 2\nabla|A|^2 \cdot \nabla\Phi \\
=& \Phi\left(-2|\nabla A|^2 + 2|A|^4 + 4(|A|^2 - H^2) - 2\sigma\mathrm{tr}(A^3) - 2\sigma H\right) \\
&+ |A|^2\left(-w\Phi'|A|^2 - \left(\frac{2}{w}\Phi' + \Phi''\right)|\nabla w|^2 + \Phi'\mathcal{E}_w\right) - 2\nabla|A|^2 \cdot \nabla\Phi.
\end{aligned}
$$

Because $\Sigma \times [0,T]$ is compact, Q achieves its maximum at some point (x_0, t_0) and without loss of generality we can assume $t_0 \in (0,T]$. At (x_0,t_0), we have $\nabla Q = 0$ (or equivalently $\nabla|A|^2 = -|A|^2\frac{\Phi'}{\Phi}\nabla w$ or $\nabla|A| = -\frac{|A|}{2}\frac{\Phi'}{\Phi}\nabla w$) and $(\partial_t - \Delta)Q \geqslant 0$. Substituting $\nabla|A|^2 = -|A|^2\frac{\Phi'}{\Phi}\nabla w$ into the cross-term in (5.2.9) at (x_0,t_0) yields:

$$-2\nabla|A|^2 \cdot \nabla\Phi = -2\left(-|A|^2\frac{\Phi'}{\Phi}\nabla w\right) \cdot (\Phi'\nabla w) = 2|A|^2\frac{(\Phi')^2}{\Phi}|\nabla w|^2.$$

Applying Kato's inequality $|\nabla|A|| \leqslant |\nabla A|$, we have

$$-2\Phi|\nabla A|^2 \leqslant -2\Phi|\nabla|A||^2 = -2\Phi\left(\frac{|A|^2}{4}\frac{(\Phi')^2}{\Phi^2}|\nabla w|^2\right) = -\frac{1}{2}|A|^2\frac{(\Phi')^2}{\Phi}|\nabla w|^2.$$

We collect all gradient terms from (5.2.9) into a single expression $\mathcal{G}$:

$$\begin{aligned}\mathcal{G} &\leqslant -\frac{1}{2}|A|^2\frac{(\Phi')^2}{\Phi}|\nabla w|^2 - |A|^2\left(\frac{2}{w}\Phi' + \Phi''\right)|\nabla w|^2 + 2|A|^2\frac{(\Phi')^2}{\Phi}|\nabla w|^2 \\ &= |A|^2|\nabla w|^2\left[\frac{3}{2}\frac{(\Phi')^2}{\Phi} - \frac{2}{w}\Phi' - \Phi''\right] \\ &= |A|^2|\nabla w|^2\Phi\left[\frac{3}{2}y^2 - \frac{2}{w}y - \left(-\frac{2}{w^2} + 2c_T + y^2\right)\right] \\ &= -2c_T\Phi(1-c_Tw^2)|A|^2|\nabla w|^2 \leq -c_T\Phi|A|^2|\nabla w|^2\,,\end{aligned}$$

where $y = \frac{2}{w} + 2c_T w$ and we used (5.2.6), (5.2.7) and $1 - c_T w^2 \geq \frac{1}{2}$ by the choice of c_T. Therefore, at the maximum (x_0, t_0), using again (5.2.6) and (5.2.9), we have

$$0 \leq (\partial_t - \Delta)Q \leqslant -2c_T w^2\Phi|A|^4 - c_T\Phi|A|^2|\nabla w|^2 + \Phi'\mathcal{E}_w|A|^2 + \Phi\mathcal{O}(|A|^3). \tag{5.2.10}$$

Now we re-write

$$\mathcal{E}_w = 2\tanh(u)\langle\nabla w, \mathbf{n}\rangle + \tilde{\mathcal{E}}_w,$$

where $\tilde{\mathcal{E}}_w = 2w^2\tanh(u)H - \frac{w^2-1}{w\cosh^2(u)} - 2w\tanh^2(u) - \sigma\tanh(u)(w^2-1)$. Because $|u| \leqslant C_0$ by Lemma 1 and $|\mathbf{n}| \leqslant 1$, we have

$$\begin{aligned}\left|2\tanh(u)\Phi'|A|^2\langle\nabla w, \mathbf{n}\rangle\right| &\leq C\Phi'|A|^2|\nabla w| = C\left(\frac{2}{w} + 2c_T w\right)\Phi|A|^2|\nabla w| \\ &\leq 3C\Phi|A|^2|\nabla w| = \left(\sqrt{2c_T\Phi}|A||\nabla w|\right)\left(\frac{3C}{\sqrt{2c_T}}\sqrt{\Phi}|A|\right) \\ &\leq c_T\Phi|A|^2|\nabla w|^2 + \frac{9C^2}{4c_T}\Phi|A|^2.\end{aligned}$$

where we used (5.2.6), $c_T w^2 \leqslant \frac{1}{2}$ (so that $2c_T w = \frac{2c_T w^2}{w} \leqslant \frac{1}{w} \leq 1$ since $w \geqslant 1$) and Young's inequality. The term $c_T\Phi|A|^2|\nabla w|^2$ above cancels with the $-c_T\Phi|A|^2|\nabla w|^2$ term in (5.2.10).

Now note that the leading term in $\tilde{\mathcal{E}}_w$ is $2w^2\tanh(u)H = \mathcal{O}(w^2|A|)$. Using (5.2.6) and multiplying $\tilde{\mathcal{E}}_w$ by $\Phi'|A|^2$ yields a bound by

$$\Phi\left(\frac{2}{w}+2c_T w\right)O(w^2|A|^3)=\Phi O((w+c_T w^3)|A|^3).$$

Since $w \leqslant C_T$ and $c_T = \frac{1}{2C_T^2}$, we have $w + c_T w^3 \leqslant C_T + \frac{1}{2}C_T = \frac{3}{2}C_T$. Thus, (5.2.10) at the maximum point (x_0, t_0) reduces to:

$$0 \leqslant -2c_T w^2\Phi|A|^4 + C_3 C_T \Phi|A|^3 + \frac{9C^2}{4c_T}\Phi|A|^2,$$

where $C_3 > 0$ is a constant depending only on $|u| \leq C_0$ and $\sigma \in (-2,2)$. Therefore, using again $w \geq 1$, we have $|A(x_0,t_0)| \leq C_4 C_T^3$ (since $\frac{1}{2c_T} = C_T^2$). Noting that $\Phi(w) \leq C_T^2 e^{\frac{1}{2}}$, we have

(5.2.11)
$$|A(x,t)|^2 \leq Q_{\max} = |A(x_0,t_0)|^2\Phi(w(x_0,t_0)) \leqslant \left(C_4C_T^3\right)^2\left(C_T^2 e^{\frac{1}{2}}\right) = C_5(1+T)^8\,,$$

for all $(x,t) \in \Sigma \times [0,T]$. Since T was chosen arbitrarily and the inequality holds for all $t \leqslant T$, we conclude $|A(x,t)|^2 \leqslant C_5(1+t)^8$, completing the proof. □

Remark 1. The proof of Lemma 5 also yields that if the gradient function satisfies a time-independent bound $w \leqslant C_1$, then the norm of the second fundamental form satisfies a uniform bound $|A| \leqslant C_0$ along the flow. A more detailed proof is included in Appendix B under Lemma 14. It then follows directly from the standard induction argument of Huisken [13, Lemma 7.2] (see also [12]) that all higher-order covariant derivatives of A are uniformly bounded (noting that the constant normal forcing $\sigma\nu$ in the flow only generates extra terms of the form $\sigma \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A$ and $\sigma\nabla^m \overline{R}$ in the evolution equation for $|\nabla^m A|^2$).

5.2.1. Uniform gradient bound via blow-up

We establish a uniform gradient bound by using the exponential C^0 convergence of the flow to the equidistant surface $\Sigma(r_\sigma)$ by Corollary 1, combined with a Type II curvature blow-up argument.

Theorem 3. Let $u \in C^{2,1}(\Sigma \times [0,\infty))$ be a smooth solution to the modified mean curvature flow (2.2.16) with $\sigma \in (-2,2)$. There exists a constant $C > 0$, depending only on the background metric g_0 on Σ, the C^0 bound of u in Lemma 1, the constant σ, and the initial gradient bound $\|w(\cdot,0)\|_\infty$, such that:

$$w(x,t) \leqslant C\,, \qquad \forall (x,t) \in \Sigma \times [0,\infty).$$

Proof. We proceed by contradiction. Assume the uniform gradient bound fails. Then by Theorem 2, we have

$$\limsup_{t\to\infty} \max_{x\in\Sigma} w(x,t) = \infty. \tag{5.2.12}$$

We claim that we must also have

$$\limsup_{t\to\infty} \max_{x\in\Sigma} |A|(x,t) = \infty.$$

Suppose for contradiction that $|A(x,t)| \leqslant K$ uniformly for all $x \in \Sigma$ and $t > 0$. Then following Huisken [13, Lemma 7.2] (see also Remark 1), all higher-order covariant derivatives of A are also uniformly bounded: $|\nabla^m A| \leqslant C_m$. Since the evolving surfaces Σ_t are graphs over a compact surface Σ, uniform bounds on A and all its covariant derivatives, combined with the uniform height bound (Lemma 1), imply that the height function $u(\cdot,t)$ is uniformly bounded in $C^k(\Sigma)$ for all integers $k \geqslant 0$. Moreover, by Lemma 1, the flow converges exponentially in $C^0(\Sigma)$ to the equidistant surface $u \equiv r_\sigma = \operatorname{arctanh}(\sigma/2)$. Standard Gagliardo-Nirenberg interpolation inequalities guarantee that a sequence uniformly bounded in $C^2(\Sigma)$ and converging in $C^0(\Sigma)$ must also converge in $C^1(\Sigma)$. Consequently, the spatial gradient $Du(\cdot, t)$ converges uniformly to 0 and $w = \sqrt{1 + |Du|^2_{g_0}/\cosh^2(u)}$ converges uniformly to 1 everywhere on Σ as $t \to \infty$, which contradicts the assumption (5.2.12).

Therefore, there exists a sequence of times $t_k \to \infty$ and points $x_k \in \Sigma$ such that:

$$A_k := |A(x_k, t_k)| = \max_{x\in\Sigma,\, t\leqslant t_k} |A(x,t)| \to \infty.$$

Note that the points (x_k, t_k) achieving this maximum of $|A|$ need not coincide with the points maximizing the gradient w.

Now we translate the space-time origin to $p_k = (x_k, u(x_k, t_k), t_k)$ and perform a parabolic rescaling by the factor $\lambda_k = A_k$. Let $u_k = u(x_k, t_k)$. We define the rescaled height function $\tilde{u}_k : \tilde{\Omega}_k \times [\tilde{t}_{start}, 0] \to \mathbb{R}$ by:

$$\tilde{u}_k(\tilde{x}, \tilde{t}) = A_k\left(u(x_k + A_k^{-1}\tilde{x}, t_k + A_k^{-2}\tilde{t}) - u_k\right), \tag{5.2.13}$$

where $\tilde{\Omega}_k = A_k(\Sigma - x_k)$ and $\tilde{t}_{start} = -A_k^2 t_k$. The rescaled surface is the graph $\tilde{\Sigma}_k^{\tilde{t}} = \{(\tilde{x}, \tilde{u}_k(\tilde{x}, \tilde{t})) : \tilde{x} \in \tilde{\Omega}_k\}$.

By definition, the rescaled surfaces satisfy $|\tilde{A}_k(\tilde{x}, \tilde{t})|^2 = A_k^{-2}|A|^2 \leqslant 1$ for all $\tilde{t} \in [\tilde{t}_{start}, 0]$, and attain exactly $|\tilde{A}_k(0,0)| = 1$. We then apply a global rotation $R_k \in SO(3)$ to the ambient space such that $R_k(\nu_k) = \mathbf{n}$, where ν_k is the unit normal of the evolving surface at p_k. Let $\hat{\Sigma}_k^{\tilde{t}} = R_k(\tilde{\Sigma}_k^{\tilde{t}})$. As $A_k \to \infty$, the ambient space flattens to Euclidean space $\mathbb{R}^3$. Because $|\tilde{A}_k(\tilde{x}, \tilde{t})|^2 \leqslant 1$ for all $\tilde{t} \in [\tilde{t}_{start}, 0]$, all higher-order derivatives of $\hat{A}_k$ are uniformly bounded (see Remark 1) which guarantees that a subsequence converges smoothly to an ancient limit flow $\hat{\Sigma}_\infty^{\tilde{t}}$ in $\mathbb{R}^3$, defined for $\tilde{t} \in (-\infty, 0]$, with

$$|\hat{A}_\infty(0,0)| = 1.$$

Now by Lemma 5, we have

$$A_k = \max_{x \in \Sigma,\, t \leqslant t_k} |A(x,t)| \leqslant C(1+t_k)^4.$$

Note that by Lemma 1, the original evolving surface Σ_{t_k} lies within a horizontal slab of width less than $Ce^{-\frac{4-\sigma^2}{2}t_k}$. Under the parabolic rescaling by $\lambda_k = A_k$, the rescaled width of this slab is bounded by

$$A_k \cdot Ce^{-\frac{4-\sigma^2}{2}t_k} \leq C(1+t_k)^4 e^{-\frac{4-\sigma^2}{2}t_k} \longrightarrow 0 \quad \text{as } t_k \longrightarrow \infty.$$

The geometric contradiction in the rescaling argument is illustrated in Figure 5.1.

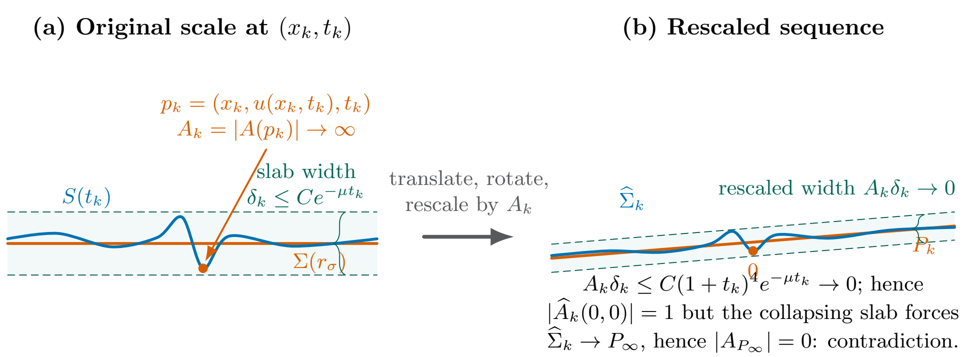

Figure 5.1.: Schematic of the blow-up contradiction.

After rescaling at a point of maximal curvature, the rescaled slab width tends to zero. Consequently, any smooth limit must lie in a plane, which contradicts the normalized curvature condition $|\widehat{A}_k(0,0)| = 1$.

The rotations R_k preserve ambient distances and thus, the smooth ancient limit surface $\hat{\Sigma}_\infty$ must be a flat plane, which contradicts $|\hat{A}_\infty(0,0)| = 1$. Therefore, the assumption (5.2.12) must be false. The gradient $w(x,t)$ is uniformly globally bounded for all time. □

Proof of Theorem 1. By Theorem 3, the gradient function w of the evolving surfaces

Σ_t is globally and uniformly bounded as long as the flow exists. Combined with the uniform height bound from Lemma 1, this guarantees that the second fundamental form and all its higher-order covariant derivatives are uniformly bounded globally in space and time (c.f. Remark 1). This precludes the formation of finite-time singularities, ensuring that the modified mean curvature flow exists smoothly for all time $t \in [0, \infty)$ and that the evolving surfaces remain geodesic graphs over Σ.

By Corollary 1, the height function of the flow converges exponentially in $C^0(\Sigma)$ to the equidistant surface $\Sigma(r_\sigma)$, where $r_\sigma = \operatorname{arctanh}\left(\frac{\sigma}{2}\right)$. Because the height function and all its higher-order derivatives are uniformly bounded for all time, standard interpolation inequalities guarantee that this exponential C^0 convergence elevates to smooth convergence in the C^∞ topology. Therefore, the flow converges smoothly to $\Sigma(r_\sigma)$ as $t \to \infty$. $\square$

Appendices

A. Derivation of the Graphical Formulas

Lemma 6. The computation of the inverse metric is as follows.

$$g^{ij} = \frac{1}{\cosh^2(u)} \left((g_0)^{ij} - \frac{u^i u^j}{|Du|^2 + \cosh^2(u)} \right).$$

Proof. Contracting the identity $\delta_\ell^j = g_{\ell k} g^{kj}$ with $(g_0)^{i\ell}$ gives

$$\begin{aligned}
(g_0)^{i\ell}\delta_\ell^j &= (g_0)^{i\ell} g_{\ell k} g^{kj} \\
(g_0)^{ij} &= (g_0)^{i\ell} \left(u_\ell u_k + \cosh^2(u)(g_0)_{\ell k} \right) g^{kj} \\
&= u^i u_k g^{kj} + \cosh^2(u) g^{ij} \\
&= u^i \alpha^j + \cosh^2(u) g^{ij},
\end{aligned}$$

where $\alpha^j := u_k g^{kj}$, and hence

$$g^{ij} = \frac{1}{\cosh^2(u)} \left((g_0)^{ij} - u^i \alpha^j \right). \tag{A.0.1}$$

To determine α^j, we contract the identity $\delta_\ell^j = g_{\ell k} g^{kj}$ with u^ℓ this time:

$$\begin{aligned}
u^\ell \delta_\ell^j &= u^\ell g_{\ell k} g^{kj} \\
u^j &= u^\ell \left(u_\ell u_k + \cosh^2(u)(g_0)_{\ell k} \right) g^{kj} \\
&= u^\ell u_\ell u_k g^{kj} + \cosh^2(u) u^\ell (g_0)_{\ell k} g^{kj}.
\end{aligned}$$

Note that $u^\ell u_\ell = |Du|^2$ and $u^\ell (g_0)_{\ell k} = u_k$, so we obtain

$$\begin{aligned} u^j &= |Du|^2\alpha^j + \cosh^2(u)\alpha^j \\ &= \alpha^j \left(|Du|^2 + \cosh^2(u)\right). \end{aligned}$$

Therefore

$$u_k g^{kj} = \alpha^j = \frac{u^j}{|Du|^2 + \cosh^2(u)}. \tag{A.0.2}$$

Substituting into (A.0.1), we obtain

$$g^{ij} = \frac{1}{\cosh^2(u)}\left((g_0)^{ij} - \frac{u^i u^j}{|Du|^2 + \cosh^2(u)}\right).$$

□

Lemma 7.

$$\bar{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - \cosh(r)\sinh(r)(g_0)_{ij}\frac{\partial}{\partial r}.$$

Proof. Let X be a tangent vector field on Σ, naturally extended to M^3 such that it commutes with the radial vector field, meaning the Lie bracket $\left[\frac{\partial}{\partial r}, X\right] = 0$. Using (2.1.2) with $\mathbf{n} = \frac{\partial}{\partial r}$, we have

$$\bar{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} = \tanh(r)\left(\frac{\partial}{\partial r} - \bar{g}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}\right) = 0.$$

Furthermore, since X is tangent to Σ, it is orthogonal to the radial direction ($\bar{g}(X, \mathbf{n}) = 0$). Therefore,

$$\bar{\nabla}_{\frac{\partial}{\partial r}}X = \bar{\nabla}_X\frac{\partial}{\partial r} = \tanh(r)\left(X - \bar{g}\left(X, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r}\right) = \tanh(r)X.$$

In particular,

$$\bar{\nabla}_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^j} = \bar{\nabla}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial r} = \tanh(r)\frac{\partial}{\partial x^j}.$$

Next, we compute the covariant derivative $\bar{\nabla}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}$. The metric components in the local coordinate frame $\{r, x^1, x^2\}$ are:

$$\bar{g}_{rr} = 1, \qquad \bar{g}_{ri} = 0, \qquad \bar{g}_{ij} = \cosh^2(r)(g_0)_{ij}.$$

The inverse metric components are:

$$\bar{g}^{rr} = 1, \qquad \bar{g}^{ri} = 0, \qquad \bar{g}^{ij} = \frac{1}{\cosh^2(r)}(g_0)^{ij}.$$

Expanding $\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$ into its radial and spatial components, we have

$$\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \bar{\Gamma}^r_{ij} \frac{\partial}{\partial r} + \bar{\Gamma}^k_{ij} \frac{\partial}{\partial x^k}.$$

For $\bar{\Gamma}^r_{ij}$, we compute:

$$\bar{\Gamma}^r_{ij} = \frac{1}{2} \bar{g}^{rr} \left(\frac{\partial \bar{g}_{jr}}{\partial x^i} + \frac{\partial \bar{g}_{ir}}{\partial x^j} - \frac{\partial \bar{g}_{ij}}{\partial r} \right) = -\cosh(r)\sinh(r)(g_0)_{ij}.$$

For $\bar{\Gamma}^k_{ij}$, where $k \in \{1, 2\}$, we have:

$$\begin{aligned}
\bar{\Gamma}^k_{ij} &= \frac{1}{2} \bar{g}^{k\ell} \left(\frac{\partial \bar{g}_{j\ell}}{\partial x^i} + \frac{\partial \bar{g}_{i\ell}}{\partial x^j} - \frac{\partial \bar{g}_{ij}}{\partial x^\ell} \right) \\
&= \frac{1}{2} \left[\frac{1}{\cosh^2(r)} (g_0)^{k\ell} \right] \left(\cosh^2(r) \frac{\partial (g_0)_{j\ell}}{\partial x^i} + \cosh^2(r) \frac{\partial (g_0)_{i\ell}}{\partial x^j} - \cosh^2(r) \frac{\partial (g_0)_{ij}}{\partial x^\ell} \right) \\
&= \frac{1}{2} (g_0)^{k\ell} \left(\frac{\partial (g_0)_{j\ell}}{\partial x^i} + \frac{\partial (g_0)_{i\ell}}{\partial x^j} - \frac{\partial (g_0)_{ij}}{\partial x^\ell} \right) \\
&= (\Gamma_0)^k_{ij},
\end{aligned}$$

where $(\Gamma_0)^k_{ij}$ are the Christoffel symbols of the metric g_0 on the refence surface Σ. Using that $D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = (\Gamma_0)^k_{ij} \frac{\partial}{\partial x^k}$, we have

$$\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \bar{\Gamma}^k_{ij} \frac{\partial}{\partial x^k} + \bar{\Gamma}^r_{ij} \frac{\partial}{\partial r} = D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \cosh(r)\sinh(r)(g_0)_{ij} \frac{\partial}{\partial r}.$$

□

Lemma 8. The outward normal vector to the evolving surface $S(t)$ is

$$\nu = \frac{1}{w}\left(\frac{\partial}{\partial r} - \frac{u^k}{\cosh^2(u)}\frac{\partial}{\partial x^k}\right).$$

Proof. Recall that

$$F_i = u_i\frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}.$$

Now let $\tilde{\nu} = \frac{\partial}{\partial r} + B^k\frac{\partial}{\partial x^k}$ be an unnormalized normal vector, then we have

$$\begin{aligned}
0 &= \langle \tilde{\nu}, F_i\rangle \\
&= \left\langle \frac{\partial}{\partial r} + B^k\frac{\partial}{\partial x^k}, u_i\frac{\partial}{\partial r} + \frac{\partial}{\partial x^i}\right\rangle \\
&= u_i\left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right\rangle + B^k\left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i}\right\rangle \\
&= u_i + B^k\cosh^2(u)(g_0)_{ki} \\
&= u_i + \cosh^2(u)B_i\,,
\end{aligned}$$

which yields $B_i = -\frac{u_i}{\cosh^2(u)}$, and raising the index gives $B^k = -\frac{u^k}{\cosh^2(u)}$. Thus,

$$\tilde{\nu} = \frac{\partial}{\partial r} - \frac{u^k}{\cosh^2(u)}\frac{\partial}{\partial x^k}$$

with

$$\begin{aligned}
|\tilde{\nu}|^2 &= \langle \tilde{\nu}, \tilde{\nu}\rangle \\
&= \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right\rangle + \frac{u^k u^\ell}{\cosh^4(u)}\left\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\right\rangle \\
&= 1 + \frac{u^k u_k}{\cosh^2(u)} = 1 + \frac{|Du|^2}{\cosh^2(u)} = w^2.
\end{aligned}$$

Therefore, the unit outward normal to $S(t)$ is

$$\nu = \frac{1}{w}\left(\frac{\partial}{\partial r} - \frac{u^k}{\cosh^2(u)}\frac{\partial}{\partial x^k}\right).$$

□

Lemma 9. the second fundamental form h_{ij} and its index raising h^{ij} are

$$h_{ij} = \frac{1}{w}\left(\cosh(u)\sinh(u)(g_0)_{ij} + 2\tanh(u)u_i u_j - u_{ij}\right)$$

and

$$h^{ij} = \frac{1}{w}\left[\tanh(u)g^{ij} + \frac{\tanh(u)}{w^4\cosh^4(u)}u^i u^j - g^{ik}g^{j\ell}u_{k\ell}\right].$$

Proof. First we have

$$\begin{aligned}\bar\nabla_{F_i}F_j &= \bar\nabla_{u_i\frac{\partial}{\partial r}+\frac{\partial}{\partial x^i}}\left(u_j\frac{\partial}{\partial r} + \frac{\partial}{\partial x^j}\right)\\ &= u_i u_j\bar\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r} + u_i\bar\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial x^j} + u_j\bar\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial r} + \bar\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} + (\partial_i u_j)\frac{\partial}{\partial r}.\end{aligned}$$

Using (2.2.8), (2.2.9),(2.2.10) and evaluating at $r = u$, this simplifies to

$$\begin{aligned}\bar\nabla_{F_i}F_j =& u_i\tanh(u)\frac{\partial}{\partial x^j} + u_j\tanh(u)\frac{\partial}{\partial x^i} + D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\\ &- \cosh(u)\sinh(u)(g_0)_{ij}\frac{\partial}{\partial r} + (\partial_i u_j)\frac{\partial}{\partial r}.\end{aligned} \tag{A.0.3}$$

Now we evaluate $h_{ij} = -\langle\bar\nabla_{F_i}F_j, \nu\rangle$ term by term using (A.0.3) and (2.2.11). For the symmetric mixed terms, by (2.2.13), we have

$$\begin{aligned}&-\left\langle u_i\tanh(u)\frac{\partial}{\partial x^j} + u_j\tanh(u)\frac{\partial}{\partial x^i}, \nu\right\rangle\\ &= -u_i\tanh(u)\left(-\frac{u_j}{w}\right) - u_j\tanh(u)\left(-\frac{u_i}{w}\right) = \frac{2\tanh(u)u_i u_j}{w}.\end{aligned}$$

Next note that we have $D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} = (\Gamma_0)^k_{ij}\frac{\partial}{\partial x^k}$. Therefore, along with (2.2.12) and (2.2.13), we obtain:

$$\begin{aligned}-\left\langle D_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} + (\partial_i u_j)\frac{\partial}{\partial r}, \nu\right\rangle &= -(\Gamma_0)^k_{ij}\left(-\frac{u_k}{w}\right) - (\partial_i u_j)\left(\frac{1}{w}\right)\\ &= -\frac{1}{w}\left(\partial_i u_j - (\Gamma_0)^k_{ij}u_k\right) = -\frac{u_{ij}}{w}.\end{aligned}$$

Finally, for the remaining normal term we have

$$-\left\langle -\cosh(u)\sinh(u)(g_0)_{ij}\frac{\partial}{\partial r}, \nu\right\rangle = \frac{\cosh(u)\sinh(u)(g_0)_{ij}}{w}.$$

Combining these yields:

$$h_{ij} = \frac{1}{w}\left(\cosh(u)\sinh(u)(g_0)_{ij} + 2\tanh(u)u_iu_j - u_{ij}\right).$$

Raising the indices twice, we get

$$\begin{aligned}
h^{ij} &= g^{ik}g^{j\ell}h_{k\ell} \\
&= \frac{1}{w}\left(\cosh(u)\sinh(u)g^{ik}g^{j\ell}(g_0)_{k\ell} + 2\tanh(u)(g^{ik}u_k)(g^{j\ell}u_\ell) - g^{ik}g^{j\ell}u_{k\ell}\right).
\end{aligned}$$

For the first term:

$$\begin{aligned}
g^{ik}g^{j\ell}(g_0)_{k\ell} &= g^{ik}\frac{1}{\cosh^2(u)}\left(\delta^j_k - \frac{u^ju_k}{w^2\cosh^2(u)}\right) \\
&= \frac{1}{\cosh^2(u)}g^{ij} - \frac{u^j}{w^2\cosh^4(u)}(g^{ik}u_k) \\
&= \frac{1}{\cosh^2(u)}g^{ij} - \frac{u^iu^j}{w^4\cosh^6(u)}.
\end{aligned}$$

Note that we previously established that $g^{ik}u_k = \frac{u^i}{w^2\cosh^2(u)}$ in (A.0.2). Substituting these back, we have

$$\begin{aligned}
h^{ij} &= \frac{1}{w}\Bigg[\cosh(u)\sinh(u)\left(\frac{1}{\cosh^2(u)}g^{ij} - \frac{u^iu^j}{w^4\cosh^6(u)}\right) \\
&\qquad + 2\tanh(u)\frac{u^i}{w^2\cosh^2(u)}\frac{u^j}{w^2\cosh^2(u)} - g^{ik}g^{j\ell}u_{k\ell}\Bigg] \\
&= \frac{1}{w}\left[\tanh(u)g^{ij} - \frac{\tanh(u)}{w^4\cosh^4(u)}u^iu^j + \frac{2\tanh(u)}{w^4\cosh^4(u)}u^iu^j - g^{ik}g^{j\ell}u_{k\ell}\right] \\
&= \frac{1}{w}\left[\tanh(u)g^{ij} + \frac{\tanh(u)}{w^4\cosh^4(u)}u^iu^j - g^{ik}g^{j\ell}u_{k\ell}\right].
\end{aligned}$$

□

Lemma 10. The mean curvature of the evolving surface is

$$H = \frac{1}{w}\left[\tanh(u)\left(3 - \frac{1}{w^2}\right) - g^{ij}u_{ij}\right].$$

Proof. The mean curvature is

$$\begin{aligned} H &= g^{ij}h_{ij} \\ &= \frac{1}{w}\left(\cosh(u)\sinh(u)g^{ij}(g_0)_{ij} + 2\tanh(u)g^{ij}u_iu_j - g^{ij}u_{ij}\right). \end{aligned}$$

Since Σ is a 2-dimensional surface, we have $(g_0)^{ij}(g_0)_{ij} = 2$. Therefore,

$$\begin{aligned} g^{ij}(g_0)_{ij} &= \frac{1}{\cosh^2(u)}\left((g_0)^{ij}(g_0)_{ij} - \frac{u^iu^j(g_0)_{ij}}{w^2\cosh^2(u)}\right) \\ &= \frac{1}{\cosh^2(u)}\left(2 - \frac{|Du|^2}{w^2\cosh^2(u)}\right). \end{aligned}$$

Using $|Du|^2 = \cosh^2(u)(w^2-1)$, this simplifies to:

$$g^{ij}(g_0)_{ij} = \frac{1}{\cosh^2(u)}\left(2 - \frac{w^2-1}{w^2}\right) = \frac{1}{\cosh^2(u)}\left(1 + \frac{1}{w^2}\right). \tag{A.0.4}$$

We also have

$$\begin{aligned} g^{ij}u_iu_j &= \frac{1}{\cosh^2(u)}\left((g_0)^{ij}u_iu_j - \frac{u^iu^ju_iu_j}{w^2\cosh^2(u)}\right) \\ &= \frac{1}{\cosh^2(u)}\left(|Du|^2 - \frac{|Du|^4}{w^2\cosh^2(u)}\right) \\ &= (w^2-1) - \frac{(w^2-1)^2}{w^2} = \frac{w^2-1}{w^2}. \end{aligned} \tag{A.0.5}$$

Therefore,

$$
\begin{aligned}
H &= \frac{1}{w}\left[\cosh(u)\sinh(u)\frac{1}{\cosh^2(u)}\left(1+\frac{1}{w^2}\right)+2\tanh(u)\frac{w^2-1}{w^2}-g^{ij}u_{ij}\right]\\
&= \frac{1}{w}\left[\tanh(u)\left(1+\frac{1}{w^2}\right)+2\tanh(u)\left(1-\frac{1}{w^2}\right)-g^{ij}u_{ij}\right]\\
&= \frac{1}{w}\left[\tanh(u)\left(3-\frac{1}{w^2}\right)-g^{ij}u_{ij}\right].
\end{aligned}
$$

□

B. Linearization, Evolution of the Gradient Function, and Test Function

Lemma 11. The drift term and the zeroth-order reaction term are

$$\begin{aligned}\beta^k = &-\frac{2u^k}{w^4\cosh^4(u)}(g_0)^{ij}u_{ij} - \frac{2(g_0)^{ik}u^j u_{ij}}{w^4\cosh^4(u)} + \frac{4u^k u^i u^j u_{ij}}{w^6\cosh^6(u)} \\ &+ \frac{2\tanh(u)u^k}{w^4\cosh^2(u)}\left(3-\frac{2}{w^2}\right) - \frac{\sigma u^k}{w^3\cosh^2(u)}\end{aligned}$$

and

$$\begin{aligned}\gamma = &-\frac{2\tanh(u)}{w^4\cosh^2(u)}(g_o)^{ij}\ u_{ij} + \frac{4\tanh(u)\,u^i u^j}{w^6\cosh^4(u)}u_{ij} \\ &- \frac{\operatorname{sech}^2(u)}{w^2}\left(3-\frac{1}{w^2}\right) + \left(3-\frac{2}{w^2}\right)\frac{2\tanh^2(u)}{w^4}(w^2-1) + \frac{\sigma\tanh(u)}{w^3}(w^2-1).\end{aligned}$$

Proof. Let $u_\epsilon = u + \epsilon\phi$. Denote $w_\epsilon = w[u+\epsilon\phi] = \sqrt{1+\frac{|D(u+\epsilon\phi)|^2}{\cosh^2(u+\epsilon\phi)}}$.
Since the background connection D is fixed, we have

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}(u_\epsilon)_i = \phi_i, \qquad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}(u_\epsilon)^i = \phi^i, \qquad \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}(u_\epsilon)_{ij} = \phi_{ij}.$$

Also,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}|D(u+\epsilon\phi)|^2 = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left((g_0)^{jk}(u_\epsilon)_j(u_\epsilon)_k\right) = 2\,(g_0)^{jk}u_j\phi_k = 2u^k\phi_k,$$

and

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \cosh^2(u+\epsilon\phi) = 2\sinh(u)\cosh(u)\,\phi,$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \tanh(u+\epsilon\phi) = \operatorname{sech}^2(u)\,\phi = \frac{1}{\cosh^2(u)}\phi.$$

By chain rule,

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \frac{1}{\cosh^2(u_\epsilon)} = -\frac{1}{\left(\cosh^2(u)\right)^2} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \cosh^2(u_\epsilon) = -\frac{2\sinh(u)}{\cosh^3(u)}\phi.$$

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \frac{1}{\cosh^4(u_\epsilon)} = \frac{2}{\cosh^2(u)} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \frac{1}{\cosh^2(u_\epsilon)} = -\frac{4\sinh(u)}{\cosh^5(u)}\phi.$$

Now we have

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} w_\epsilon^2 &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left[1 + \frac{|Du_\epsilon|^2}{\cosh^2(u_\epsilon)}\right] \\
&= 0 + \frac{2u^k\phi_k}{\cosh^2(u)} - \frac{2\sinh(u)}{\cosh^3(u)}\,\phi\,|Du|^2 \\
&= \frac{2u^k\phi_k}{\cosh^2(u)} - 2\tanh(u)\frac{|Du|^2}{\cosh^2(u)}\,\phi
\end{aligned}$$

Note that $\frac{|Du|^2}{\cosh^2(u)} = w^2 - 1$, so we get

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} w_\epsilon^2 = \frac{2u^k\phi_k}{\cosh^2(u)} - 2\tanh(u)(w^2-1)\phi.$$

Once again by chain rule,

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \frac{1}{w_\epsilon^2} &= -\frac{1}{w^4} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} w_\epsilon^2 \\
&= -\frac{2}{w^4\cosh^2(u)}u^k\phi_k + \frac{2\tanh(u)}{w^4}(w^2-1)\phi.
\end{aligned}$$

by equation (2.2.7), $g^{ij} = \frac{1}{\cosh^2(u)} (g_o)^{ij} - \frac{u^i u^j}{w^2 \cosh^4(u)}$, so

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} g^{ij}[u+\epsilon\phi] &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left[\frac{1}{\cosh^2(u_\epsilon)} (g_o)^{ij} - \frac{u_\epsilon^i u_\epsilon^j}{w_\epsilon^2 \cosh^4(u_\epsilon)}\right] \\
&= -\frac{2\sinh(u)}{\cosh^3(u)} (g_o)^{ij}\,\phi - \frac{u^i u^j}{w^2} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \frac{1}{\cosh^4(u_\epsilon)} \\
&\quad - \frac{u^i u^j}{\cosh^4(u)} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \frac{1}{w_\epsilon^2} - \frac{1}{w^2\cosh^4(u)} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (u_\epsilon^i u_\epsilon^j) \\
&= -\frac{2\sinh(u)}{\cosh^3(u)} (g_o)^{ij}\,\phi - \frac{u^i u^j}{w^2}\left[-\frac{4\sinh(u)}{\cosh^5(u)}\phi\right] \\
&\quad - \frac{u^i u^j}{\cosh^4(u)}\left[-\frac{2}{w^4\cosh^2(u)} u^k\phi_k + \frac{2\tanh(u)}{w^4}(w^2-1)\phi\right] \\
&\quad - \frac{u^i\phi^j + u^j\phi^i}{w^2\cosh^4(u)} \\
&= -\frac{2\tanh(u)}{\cosh^2(u)} (g_o)^{ij}\,\phi + 2\cdot\frac{2\tanh(u)\,u^i u^j}{w^2\cosh^4(u)}\phi \\
&\quad + \frac{2u^i u^j u^k}{w^4\cosh^6(u)}\phi_k - \left(\frac{w^2-1}{w^2}\right)\frac{2\tanh(u)\,u^i u^j}{w^2\cosh^4(u)}\phi - \frac{u^i\phi^j + u^j\phi^i}{w^2\cosh^4(u)}
\end{aligned}$$

As $2 - \frac{w^2-1}{w^2} = \frac{w^2+1}{w^2} = 1 + \frac{1}{w^2}$, combining the second and 4th term yields

$$\begin{aligned}
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} g^{ij}[u+\epsilon\phi] = &-\frac{2\tanh(u)}{\cosh^2(u)} (g_o)^{ij}\,\phi + \left(1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^i u^j}{w^2\cosh^4(u)}\phi \\
&+ \frac{2u^i u^j u^k}{w^4\cosh^6(u)}\phi_k - \frac{u^i\phi^j + u^j\phi^i}{w^2\cosh^4(u)}
\end{aligned}$$

Applying product rule,

$$
\begin{aligned}
&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}g^{ij}[u+\epsilon\phi]\right)\\
&=\frac{1}{w^2}\left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}g^{ij}[u+\epsilon\phi]\right)+g^{ij}\left(\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\frac{1}{w_\epsilon^2}\right)\\
&=\frac{1}{w^2}\Bigg[-\frac{2\tanh(u)}{\cosh^2(u)}(g_o)^{ij}\phi+\left(1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^2\cosh^4(u)}\phi+\frac{2u^iu^ju^k}{w^4\cosh^6(u)}\phi_k\\
&\qquad\qquad-\frac{u^i\phi^j+u^j\phi^i}{w^2\cosh^4(u)}\Bigg]+g^{ij}\left[-\frac{2}{w^4\cosh^2(u)}u^k\phi_k+\frac{2\tanh(u)}{w^4}(w^2-1)\phi\right]\\
&=-\frac{u^i\phi^j+u^j\phi^i}{w^4\cosh^4(u)}+\left[\frac{2u^iu^ju^k}{w^6\cosh^6(u)}-\frac{2u^kg^{ij}}{w^4\cosh^2(u)}\right]\phi_k\\
&\quad+\left[-\frac{2\tanh(u)}{w^2\cosh^2(u)}(g_o)^{ij}+\left(1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)}+\frac{2\tanh(u)}{w^4}(w^2-1)g^{ij}\right]\phi
\end{aligned}
$$

Once again by product rule,

$$
\begin{aligned}
&\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}g^{ij}[u+\epsilon\phi]\,(u_\epsilon)_{ij}\right)\\
&=\frac{1}{w^2}g^{ij}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}(u_\epsilon)_{ij}+u_{ij}\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}g^{ij}[u+\epsilon\phi]\right)\\
&=\frac{1}{w^2}g^{ij}\phi_{ij}-\frac{u^i\phi^j+u^j\phi^i}{w^4\cosh^4(u)}u_{ij}+\left[\frac{2u^iu^ju^ku_{ij}}{w^6\cosh^6(u)}-\frac{2u^ku_{ij}g^{ij}}{w^4\cosh^2(u)}\right]\phi_k\\
&\quad+u_{ij}\left[-\frac{2\tanh(u)}{w^2\cosh^2(u)}(g_o)^{ij}+\left(1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)}+\frac{2\tanh(u)}{w^4}(w^2-1)g^{ij}\right]\phi
\end{aligned}
$$

Particularly in the second item, since $u_{ij}=u_{ji}$, we have

$$(u^i\phi^j+u^j\phi^i)u_{ij}=2u^i\phi^ju_{ij}=2u^iu_{ij}(g_0)^{jk}\phi_k=2u^iu_i^k\phi_k$$

The second item can be simplified into $-\frac{2u^iu_i^k}{w^4\cosh^4(u)}\phi_k$. Combining with the third item, we obtain

$$
\begin{aligned}
&\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}g^{ij}[u+\epsilon\phi]\,(u_\epsilon)_{ij}\right)\\
&=\frac{1}{w^2}g^{ij}\phi_{ij}+\left[-\frac{2u^iu_i^k}{w^4\cosh^4(u)}+\frac{2u^iu^ju^ku_{ij}}{w^6\cosh^6(u)}-\frac{2u^ku_{ij}g^{ij}}{w^4\cosh^2(u)}\right]\phi_k\\
&\quad+u_{ij}\left[-\frac{2\tanh(u)}{w^2\cosh^2(u)}\,(g_o)^{ij}+\left(1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)}+\frac{2\tanh(u)}{w^4}(w^2-1)g^{ij}\right]\phi
\end{aligned}
$$

Now we linearize the part $-\frac{\tanh(u)}{w^2}\left(3-\frac{1}{w^2}\right)+\frac{\sigma}{w}$

$$
\begin{aligned}
&\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(-\frac{\tanh(u_\epsilon)}{w_\epsilon^2}\left(3-\frac{1}{w_\epsilon^2}\right)+\frac{\sigma}{w_\epsilon}\right)\\
&=-\frac{\operatorname{sech}^2(u)\,\phi}{w^2}\left(3-\frac{1}{w^2}\right)-\tanh(u)\;\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}\left(3-\frac{1}{w_\epsilon^2}\right)\right)+\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\frac{\sigma}{w_\epsilon}
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}\left(3-\frac{1}{w_\epsilon^2}\right)\right)&=\left(3-\frac{1}{w^2}\right)\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}\right)+\left(\frac{1}{w^2}\right)\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(3-\frac{1}{w_\epsilon^2}\right)\\
&=\left(3-\frac{1}{w^2}\right)\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}\right)-\left(\frac{1}{w^2}\right)\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}\right)\\
&=\left(3-\frac{2}{w^2}\right)\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\left(\frac{1}{w_\epsilon^2}\right)\\
&=\left(3-\frac{2}{w^2}\right)\left[-\frac{2}{w^4\cosh^2(u)}u^k\phi_k+\frac{2\tanh(u)}{w^4}(w^2-1)\phi\right]
\end{aligned}
$$

Also, since $\frac{d}{d\epsilon}\big|_{\epsilon=0}\frac{1}{w_\epsilon^2}=\frac{2}{w}\;\frac{d}{d\epsilon}\big|_{\epsilon=0}\frac{1}{w_\epsilon}$, we have

$$
\begin{aligned}
\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\frac{\sigma}{w_\epsilon}&=\frac{\sigma w}{2}\;\frac{d}{d\epsilon}\bigg|_{\epsilon=0}\frac{1}{w_\epsilon^2}\\
&=-\frac{\sigma u^k}{w^3\cosh^2(u)}\phi_k+\frac{\sigma\tanh(u)}{w^3}(w^2-1)\phi.
\end{aligned}
$$

The coefficient of ϕ_{ij} is $\alpha^{ij}=\frac{1}{w^2}g^{ij}$, and using (2.2.7), we get

$$
\alpha^{ij}=\frac{(g_0)^{ij}}{w^2\cosh^2(u)}-\frac{u^iu^j}{w^4\cosh^4(u)}\,. \tag{B.0.1}
$$

On the other hand, extracting the coefficient of ϕ_k, we have

$$\begin{aligned}\beta^k =& -\frac{2u^i u_i^k}{w^4\cosh^4(u)} + \frac{2u^i u^j u^k u_{ij}}{w^6\cosh^6(u)} - \frac{2u^k u_{ij} g^{ij}}{w^4\cosh^2(u)}\\ &+\tanh(u)\left(3-\frac{2}{w^2}\right)\frac{2}{w^4\cosh^2(u)}u^k - \frac{\sigma u^k}{w^3\cosh^2(u)}\end{aligned}$$

Using $u_i^k = u_{ij}(g_0)^{jk}$, we can further simplify β^k and we should expand g^{ij} in terms of g_0. Once again, Recall that we have (2.2.7), namely,$g^{ij} = \frac{1}{\cosh^2(u)}(g_0)^{ij} - \frac{u^i u^j}{w^2\cosh^4(u)}$. Therefore, the third term is

$$-\frac{2u^k u_{ij} g^{ij}}{w^4\cosh^2(u)} = -\frac{2u^k}{w^4\cosh^4(u)}(g_0)^{ij}u_{ij} + \frac{2u^k u^i u^j u_{ij}}{w^6\cosh^6(u)}. \tag{B.0.2}$$

Plugging (B.0.2) back, we have the first three terms in the expression of β^k (using $u_i^k = u_{ij}(g_0)^{jk}$):

$$\begin{aligned}&-\frac{2(g_0)^{jk}u^i u_{ij}}{w^4\cosh^4(u)} + \frac{2u^i u^j u^k u_{ij}}{w^6\cosh^6(u)} - \frac{2u^k u_{ij} g^{ij}}{w^4\cosh^2(u)}\\ =& -\frac{2(g_0)^{jk}u^i u_{ij}}{w^4\cosh^4(u)} + \frac{2u^i u^j u^k u_{ij}}{w^6\cosh^6(u)} - \frac{2u^k}{w^4\cosh^4(u)}(g_0)^{ij}u_{ij} + \frac{2u^k u^i u^j u_{ij}}{w^6\cosh^6(u)}\\ =& -\frac{2u^k}{w^4\cosh^4(u)}(g_0)^{ij}u_{ij} - \frac{2(g_0)^{jk}u^i u_{ij}}{w^4\cosh^4(u)} + \frac{4u^k u^i u^j u_{ij}}{w^6\cosh^6(u)}\\ =& -\frac{2u^k}{w^4\cosh^4(u)}(g_0)^{ij}u_{ij} - \frac{2(g_0)^{ik}u^j u_{ij}}{w^4\cosh^4(u)} + \frac{4u^k u^i u^j u_{ij}}{w^6\cosh^6(u)}.\end{aligned}$$

where in the last equality we used $(g_0)^{jk}u^i u_{ij} = (g_0)^{ik}u^j u_{ij}$. Therefore,

$$\begin{aligned}\beta^k =& -\frac{2u^k}{w^4\cosh^4(u)}(g_0)^{ij}u_{ij} - \frac{2(g_0)^{ik}u^j u_{ij}}{w^4\cosh^4(u)}\\ &+\frac{4u^k u^i u^j u_{ij}}{w^6\cosh^6(u)} + \frac{2\tanh(u)u^k}{w^4\cosh^2(u)}\left(3-\frac{2}{w^2}\right) - \frac{\sigma u^k}{w^3\cosh^2(u)}.\end{aligned}$$

Now we extract the coefficient of ϕ and obtain

(B.0.3)

$$\begin{aligned}\gamma = u_{ij}&\left[-\frac{2\tanh(u)}{w^2\cosh^2(u)}(g_o)^{ij} + \left(1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)} + \frac{2\tanh(u)}{w^4}(w^2-1)g^{ij}\right]\\ &-\frac{\operatorname{sech}^2(u)}{w^2}\left(3-\frac{1}{w^2}\right) + \left(3-\frac{2}{w^2}\right)\frac{2\tanh^2(u)}{w^4}(w^2-1) + \frac{\sigma\tanh(u)}{w^3}(w^2-1)\end{aligned}$$

Similarly, this could be simplified with (2.2.7). Particularly, the term involving g^{ij} turns into the follows.

$$\begin{aligned}\frac{2\tanh(u)}{w^4}(w^2-1)g^{ij} &= \frac{2\tanh(u)}{w^4}(w^2-1)\left(\frac{1}{\cosh^2(u)}(g_0)^{ij} - \frac{u^iu^j}{w^2\cosh^4(u)}\right)\\ &= \frac{2\tanh(u)}{w^4\cosh^2(u)}(w^2-1)(g_0)^{ij} - \frac{2\tanh(u)}{w^6\cosh^4(u)}(w^2-1)(u^iu^j)\\ &= \left(\frac{w^2-1}{w^2}\right)\frac{2\tanh(u)}{w^2\cosh^2(u)}(g_0)^{ij} - \left(\frac{w^2-1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)}\\ &= \left(1-\frac{1}{w^2}\right)\frac{2\tanh(u)}{w^2\cosh^2(u)}(g_0)^{ij} + \left(-1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)}\end{aligned}$$

Therefore, the items in the bracket in (B.0.3) combine into:

$$\begin{aligned}&-\frac{2\tanh(u)}{w^2\cosh^2(u)}(g_o)^{ij} + \left(1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)} + \frac{2\tanh(u)}{w^4}(w^2-1)g^{ij}\\ &= \left(-1+1-\frac{1}{w^2}\right)\frac{2\tanh(u)}{w^2\cosh^2(u)}(g_o)^{ij} + \left(1+\frac{1}{w^2}-1+\frac{1}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)}\\ &= \left(-\frac{1}{w^2}\right)\frac{2\tanh(u)}{w^2\cosh^2(u)}(g_o)^{ij} + \left(\frac{2}{w^2}\right)\frac{2\tanh(u)\,u^iu^j}{w^4\cosh^4(u)}\\ &= -\frac{2\tanh(u)}{w^4\cosh^2(u)}(g_o)^{ij} + \frac{4\tanh(u)\,u^iu^j}{w^6\cosh^4(u)}\end{aligned}$$

Therefore, we get

$$\begin{aligned}\gamma = &-\frac{2\tanh(u)}{w^4\cosh^2(u)}(g_o)^{ij}\,u_{ij} + \frac{4\tanh(u)\,u^iu^j}{w^6\cosh^4(u)}u_{ij}\\ &-\frac{\operatorname{sech}^2(u)}{w^2}\left(3-\frac{1}{w^2}\right) + \left(3-\frac{2}{w^2}\right)\frac{2\tanh^2(u)}{w^4}(w^2-1) + \frac{\sigma\tanh(u)}{w^3}(w^2-1)\end{aligned}$$

$\square$

Lemma 12. The evolution equation of the gradient function $w = \sqrt{1 + \frac{|Du|^2}{\cosh^2(u)}}$ is

$$
\begin{aligned}
\frac{\partial w}{\partial t} = \frac{u^k}{w\cosh^2(u)} \Bigg[& -\frac{2}{w^3} w_k\, g^{ij} u_{ij} - \frac{2\tanh(u)}{w^2\cosh^2(u)}\, u_k\, (g_0)^{ij} u_{ij} \\
& + \frac{4\tanh(u)}{w^4\cosh^4(u)}\, u_k\, u^i u^j u_{ij} - \frac{2}{w^4\cosh^4(u)}\, u^i_k\, u^j u_{ij} \\
& + \frac{2}{w^5\cosh^4(u)}\, w_k\, u^i u^j u_{ij} + \frac{1}{w^2} g^{ij} D_k u_{ij} \\
& - \operatorname{sech}^2(u)\, \frac{1}{w^2} \left(3 - \frac{1}{w^2}\right) u_k + \frac{2\tanh(u)}{w^3} \left(3 - \frac{2}{w^2}\right) w_k - \frac{\sigma}{w^2} w_k \Bigg] \\
& - \frac{w^2 - 1}{w}\, \tanh(u) \left[\frac{1}{w^2} g^{ij} u_{ij} - \frac{\tanh(u)}{w^2} \left(3 - \frac{1}{w^2}\right) + \frac{\sigma}{w} \right].
\end{aligned}
$$

Proof. We begin with

$$
\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{1}{2w} \frac{\partial}{\partial t}(w^2) \\
&= \frac{1}{2w} \frac{\partial}{\partial t} \left(1 + \frac{|Du|^2}{\cosh^2(u)}\right) \\
&= \frac{1}{2w} \left[\frac{1}{\cosh^2(u)} \frac{\partial}{\partial t}(|Du|^2) + |Du|^2 \frac{\partial}{\partial t} \left(\frac{1}{\cosh^2(u)}\right) \right].
\end{aligned}
$$

Since

$$
|Du|^2 = u^k u_k = (g_0)^{ik} u_i u_k,
$$

by symmetry we have

$$
\begin{aligned}
\frac{\partial}{\partial t}(|Du|^2) &= \frac{\partial}{\partial t} \left[(g_0)^{ik} u_i u_k\right] \\
&= 2(g_0)^{ik} u_i \frac{\partial u_k}{\partial t} \\
&= 2u^k D_k(u_t).
\end{aligned}
$$

Also,

$$\frac{\partial}{\partial t}\left(\frac{1}{\cosh^2(u)}\right) = -\frac{2\tanh(u)}{\cosh^2(u)}\, u_t.$$

Substituting these into the evolution of w, we obtain

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{1}{2w}\left[\frac{2u^k}{\cosh^2(u)} D_k(u_t) - |Du|^2 \frac{2\tanh(u)}{\cosh^2(u)} u_t\right] \\ &= \frac{u^k}{w\cosh^2(u)} D_k(u_t) - \frac{1}{w}\frac{|Du|^2}{\cosh^2(u)}\tanh(u)\, u_t.\end{aligned}$$

Using $\frac{|Du|^2}{\cosh^2(u)} = w^2 - 1$, we conclude

(B.0.4) $$\frac{\partial w}{\partial t} = \frac{u^k}{w\cosh^2(u)} D_k(u_t) - \frac{w^2-1}{w}\tanh(u)\, u_t.$$

Note that by (2.2.16), $u_t = \frac{1}{w^2} g^{ij} u_{ij} - \frac{\tanh(u)}{w^2}\left(3 - \frac{1}{w^2}\right) + \frac{\sigma}{w}$, so we have

(B.0.5) $$D_k(u_t) = D_k\left(\frac{1}{w^2} g^{ij} u_{ij}\right) - D_k\left(\frac{\tanh(u)}{w^2}\left(3 - \frac{1}{w^2}\right)\right) + \sigma D_k\left(\frac{1}{w}\right).$$

We now compute each term separately. For the first term, we have

$$D_k\left(\frac{1}{w^2} g^{ij} u_{ij}\right) = D_k\left(\frac{1}{w^2}\right) g^{ij} u_{ij} + \frac{1}{w^2} D_k(g^{ij})\, u_{ij} + \frac{1}{w^2} g^{ij} D_k(u_{ij}).$$

Since

(B.0.6) $$D_k\left(\frac{1}{w^2}\right) = -\frac{2}{w^3} w_k \quad \text{and} \quad D_k(u_{ij}) = u_{ijk}\ ,$$

we obtain

$$D_k\left(\frac{1}{w^2} g^{ij} u_{ij}\right) = -\frac{2}{w^3} w_k\, g^{ij} u_{ij} + \frac{1}{w^2} D_k(g^{ij})\, u_{ij} + \frac{1}{w^2} g^{ij} u_{ijk}.$$

Next, noting that $D_k(\tanh(u)) = \operatorname{sech}^2(u)\, u_k$ and using (B.0.6),

$$\begin{aligned} D_k\left(\frac{\tanh(u)}{w^2}\left(3-\frac{1}{w^2}\right)\right) =& \operatorname{sech}^2(u)\, u_k\, \frac{1}{w^2}\left(3-\frac{1}{w^2}\right) \\ &+ \tanh(u)\left(-\frac{2}{w^3}w_k\right)\left(3-\frac{1}{w^2}\right) \\ &+ \tanh(u)\,\frac{1}{w^2}\left(\frac{2}{w^3}w_k\right) \\ =& \operatorname{sech}^2(u)\, u_k\, \frac{1}{w^2}\left(3-\frac{1}{w^2}\right) - \frac{2\tanh(u)}{w^3}\left(3-\frac{2}{w^2}\right)w_k. \end{aligned}$$

Lastly we have $D_k\left(\frac{1}{w}\right) = -\frac{1}{w^2}w_k$. Substituting all of these into (B.0.5), we get

(B.0.7)
$$\begin{aligned} D_k(u_t) =& -\frac{2}{w^3}w_k\, g^{ij}u_{ij} + \frac{1}{w^2}D_k(g^{ij})\, u_{ij} + \frac{1}{w^2}g^{ij}u_{ijk} \\ &- \operatorname{sech}^2(u)\,\frac{1}{w^2}\left(3-\frac{1}{w^2}\right)u_k \\ &+ \frac{2\tanh(u)}{w^3}\left(3-\frac{2}{w^2}\right)w_k - \frac{\sigma}{w^2}w_k. \end{aligned}$$

For $D_k(g^{ij})$, by (2.2.7), namely, $g^{ij} = \operatorname{sech}^2(u)\,(g_0)^{ij} - \operatorname{sech}^4(u)\,\frac{u^iu^j}{w^2}$, we have

$$D_k(g^{ij}) = (g_0)^{ij}D_k\left(\operatorname{sech}^2(u)\right) - \frac{u^iu^j}{w^2}\,D_k\left(\operatorname{sech}^4(u)\right) - \operatorname{sech}^4(u)\,D_k\left(\frac{u^iu^j}{w^2}\right).$$

Since $D_k\left(\operatorname{sech}^2(u)\right) = -2\operatorname{sech}^2(u)\tanh(u)\,u_k = -\frac{2\tanh(u)}{\cosh^2(u)}\,u_k$, we get
$D_k\left(\operatorname{sech}^4(u)\right) = 2\operatorname{sech}^2(u)D_k\left(\operatorname{sech}^2(u)\right) = -\frac{4\tanh(u)}{\cosh^4(u)}\,u_k$.
On the other hand, we have

$$D_k\left(\frac{u^iu^j}{w^2}\right) = \frac{1}{w^2}D_k\left(u^iu^j\right) + u^iu^jD_k\left(\frac{1}{w^2}\right) = \frac{u^i_ku^j + u^iu^j_k}{w^2} - \frac{2u^iu^jw_k}{w^3}.$$

Therefore,

$$\begin{aligned} D_k(g^{ij}) =& -2\tanh(u)\cosh^{-2}(u)\,u_k\,(g_0)^{ij} + \frac{4\tanh(u)}{w^2\cosh^4(u)}\,u_k\,u^iu^j \\ &- \frac{1}{w^2\cosh^4(u)}\,(u^i_k\,u^j + u^iu^j_k) + \frac{2}{w^3\cosh^4(u)}\,w_k\,u^iu^j. \end{aligned}$$

Substituting this into $\frac{1}{w^2}D_k(g^{ij})u_{ij}$ and contracting with u_{ij} yields $u^i_k u^j u_{ij} + u^i u^j_k u_{ij} = 2u^i_k u^j u_{ij}$. Thus, equation (B.0.7) becomes:

$$\begin{aligned}
D_k(u_t) = &-\frac{2}{w^3}w_k\, g^{ij}u_{ij} - \frac{2\tanh(u)}{w^2\cosh^2(u)}\, u_k\, (g_0)^{ij}u_{ij} \\
&+ \frac{4\tanh(u)}{w^4\cosh^4(u)}\, u_k\, u^i u^j u_{ij} - \frac{2}{w^4\cosh^4(u)}\, u^i_k\, u^j u_{ij} \\
&+ \frac{2}{w^5\cosh^4(u)}\, w_k\, u^i u^j u_{ij} + \frac{1}{w^2}g^{ij}D_k u_{ij} \\
&- \operatorname{sech}^2(u)\,\frac{1}{w^2}\left(3 - \frac{1}{w^2}\right)u_k + \frac{2\tanh(u)}{w^3}\left(3 - \frac{2}{w^2}\right)w_k - \frac{\sigma}{w^2}w_k.
\end{aligned}$$

Now we can put the expression on the right side and (2.2.16) into (B.0.4).

$$\begin{aligned}
\frac{\partial w}{\partial t} = \frac{u^k}{w\cosh^2(u)}\Bigg[&-\frac{2}{w^3}w_k\, g^{ij}u_{ij} - \frac{2\tanh(u)}{w^2\cosh^2(u)}\, u_k\, (g_0)^{ij}u_{ij} \\
&+ \frac{4\tanh(u)}{w^4\cosh^4(u)}\, u_k\, u^i u^j u_{ij} - \frac{2}{w^4\cosh^4(u)}\, u^i_k\, u^j u_{ij} \\
&+ \frac{2}{w^5\cosh^4(u)}\, w_k\, u^i u^j u_{ij} + \frac{1}{w^2}g^{ij}D_k u_{ij} \\
&- \operatorname{sech}^2(u)\,\frac{1}{w^2}\left(3 - \frac{1}{w^2}\right)u_k + \frac{2\tanh(u)}{w^3}\left(3 - \frac{2}{w^2}\right)w_k - \frac{\sigma}{w^2}w_k\Bigg] \\
&- \frac{w^2-1}{w}\tanh(u)\left[\frac{1}{w^2}g^{ij}u_{ij} - \frac{\tanh(u)}{w^2}\left(3 - \frac{1}{w^2}\right) + \frac{\sigma}{w}\right].
\end{aligned} \tag{B.0.8}$$

□

Lemma 13. The identity when $\mathscr{L}$ is operated on w is as follows.

$$\begin{aligned}
\mathscr{L}w = &- \frac{1}{w\cosh^2(u)}\alpha^{ij}(g_0)^{k\ell}u_{ik}u_{j\ell} - \frac{2\tanh(u)(w^2-1)}{w}\alpha^{ij}u_{ij} \\
&- \frac{3(w^2-1)}{w^3\cosh^2(u)} + \frac{1}{w}\alpha^{ij}w_i w_j + 4\tanh(u)\alpha^{ij}u_i w_j \\
&+ \frac{\tanh^2(u)(w^2-1)(5w^2-3)}{w^5} - \frac{\sigma\tanh(u)(w^2-1)}{w^2}.
\end{aligned}$$

Proof. Recall (4.1.6). We have

$$\mathscr{L}w = (\mathcal{A} + \mathcal{B}) + \mathcal{C} + \mathcal{D} - \mathcal{Q} + \frac{1}{w}\alpha^{ij}w_i w_j + 4\tanh(u)\alpha^{ij}u_i w_j \ .$$

Now we apply (4.1.7), (4.1.11), (4.1.14), (4.1.16), and (4.1.19) and get

$$\begin{aligned}\mathscr{L}w = & \frac{\tanh^2(u)(w^2-1)}{w^3}\left(3 - \frac{1}{w^2}\right) - \frac{\sigma\tanh(u)(w^2-1)}{w^2} \\ & + \frac{\operatorname{sech}^2(u)}{w}(w^2-1)\frac{1}{w^4}(w^2-1) + \frac{2\tanh^2(u)}{w}(w^2-1)\frac{1}{w^4}(w^2-1) \\ & \quad - \frac{w^2-1}{w^3\cosh^2(u)} - \frac{2\tanh(u)(w^2-1)}{w}\alpha^{ij}u_{ij} - \frac{\operatorname{sech}^2(u)(w^2-1)}{w^3}\left(3 - \frac{1}{w^2}\right) \\ & \quad - \frac{1}{w\cosh^2(u)}\alpha^{ij}(g_0)^{k\ell}u_{ik}u_{j\ell} + \frac{1}{w}\alpha^{ij}w_i w_j + 4\tanh(u)\alpha^{ij}u_i w_j.\end{aligned}$$

Extracting the $\tanh^2(u)$-terms yields

$$\begin{aligned} & \frac{\tanh^2(u)(w^2-1)}{w^3}\left(3 - \frac{1}{w^2}\right) + \frac{2\tanh^2(u)}{w}(w^2-1)\frac{1}{w^4}(w^2-1) \\ = & \frac{\tanh^2(u)(w^2-1)}{w^3}\left(\frac{3w^2-1}{w^2}\right) + \frac{2\tanh^2(u)\,(w^2-1)^2}{w^5} \\ = & \frac{\tanh^2(u)(w^2-1)}{w^5}\left[(3w^2-1) + 2(w^2-1)\right] \\ = & \frac{\tanh^2(u)(w^2-1)}{w^5}(5w^2-3)\,,\end{aligned}$$

and extracting the $\operatorname{sech}^2(u)$-terms (along with the $\frac{1}{\cosh^2(u)}$-term) yields

$$\begin{aligned}
&\frac{\operatorname{sech}^2(u)}{w}(w^2-1)\frac{1}{w^4}(w^2-1) - \frac{w^2-1}{w^3\cosh^2(u)} - \frac{\operatorname{sech}^2(u)(w^2-1)}{w^3}\left(3-\frac{1}{w^2}\right) \\
&= \frac{(w^2-1)^2}{w^5\cosh^2(u)} - \frac{w^2-1}{w^3\cosh^2(u)} - \frac{(w^2-1)}{w^3\cosh^2(u)}\left(\frac{3w^2-1}{w^2}\right) \\
&= \frac{w^2-1}{w^5\cosh^2(u)}\left[(w^2-1)-w^2-(3w^2-1)\right] \\
&= \frac{w^2-1}{w^5\cosh^2(u)}(-3w^2) \\
&= \frac{-3(w^2-1)}{w^3\cosh^2(u)},
\end{aligned}$$

which proves the desired identity. □

Lemma 14 (Uniform curvature estimate)**.** Let $F : \Sigma \times [0, T_{\max}) \longrightarrow M$ be a smooth graphical solution of the modified mean curvature flow $\frac{\partial F}{\partial t} = -(H-\sigma)\nu$ in the Fuchsian manifold $(M, \overline{g}) = \left(\mathbb{R}\times\Sigma, dr^2 + \cosh^2(r)g_0\right)$. Suppose that the gradient function satisfies the time-independent estimate $w(x,t) \leq C_1$ for every $(x,t) \in \Sigma \times [0, T_{\max})$. Then there exists a constant $C_6 > 0$, independent of t and $T_{\max}$, such that $\sup_{(x,t)\in\Sigma\times[0,T_{\max})} |A(x,t)| \leq C_6$. The constant C depends only on the fixed geometric data, the uniform height bound, C_1, σ, and $\sup_{\Sigma}|A(\cdot,0)|$. In other words, the norm of the second fundamental form remains uniformly bounded for the entire duration of the flow.

Proof. Fix $T_0 < T_{\max}$, and let (x_0, t_0) be a maximum point of Q on $\Sigma\times[0,T_0]$. The case $t_0 = 0$ is controlled by the smooth initial data. We may therefore assume that $t_0 > 0$.

At (x_0, t_0), the parabolic maximum principle and the calculation in the proof of Lemma 5, with c_T replaced by c, give

$$\begin{aligned}
0 &\leq \left(\frac{\partial}{\partial t} - \Delta\right) Q \\
&\leq -2cw^2\Phi|A|^4 - c\Phi|A|^2|\nabla w|^2 + \Phi' E_w |A|^2 + C\Phi\left(1+|A|^3\right),
\end{aligned}$$

where E_w denotes the lower-order term appearing in the proof of Lemma 5. Here and below, C depends only on the fixed geometric data, the uniform height bound, C_1, and σ, but is independent of T_0.

By the uniform height estimate and the assumption $w \le C_1$, the same estimates used in the proof of Lemma 5 imply

$$|\Phi' E_w|\,|A|^2 \le c\Phi|A|^2|\nabla w|^2 + C\Phi\left(1 + |A|^3\right).$$

Substituting this estimate into the preceding inequality gives

$$0 \le -2cw^2\Phi|A|^4 + C\Phi\left(1 + |A|^3\right).$$

Since $w \ge 1$ and $\Phi > 0$, we obtain $2c|A|^4 \le 2C\left(1 + |A|^3\right)$ at (x_0, t_0). If $|A(x_0, t_0)| \le 1$, then the theorem is trivially true. Otherwise, we can apply $1 + |A|^3 \le 2|A|^3$ on the right side of the inequality and hence obtain $2c|A|^4 \le 2C\left(2|A|^3\right)$, which means

$$|A(x_0, t_0)| \le \frac{2C}{c}.$$

Since (x_0, t_0) is a maximum point of Q, and since Φ is bounded above and below by positive time-independent constants, the cases $t_0 = 0$ and $t_0 > 0$ together yield

$$\sup_{\Sigma\times[0,T_0]} |A|^2 \le \frac{2C}{c}\left(1 + \sup_{\Sigma} |A(\cdot, 0)|^2\right).$$

The right-hand side is a constant independent of T_0. Renaming it C_6, we get

$$\sup_{\Sigma\times[0,T_{\max})} |A| \le C_6.$$

The norm of the second fundamental form remains uniformly bounded along the flow.

$\square$

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