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ON THE QUASI-PARTICLE INTERACTIONS IN SPHERICAL NUCLEI

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ABSTRACT

The residual terms that arise from the Bogoliubov-Valatin canonical transformation are studied in angular-momentum representation and applied to some low-energy properties of spherical nuclei. The treatment makes use of the conventional perturbation theory, and its objective is to clarify the roles that each of the interaction terms play, and also to understand how it is that the simplified calculations such as the studies by Kisslinger and Sorensen turn out to be so successful.

First, the Bardeen-Cooper-Schrieffer theory applied to finite nuclei is reexamined, with a constant-pairing-matrix element approximation (G). It is found for a particular system that within coupling strengths of physical significance, there is only one value of G at which the BCS approximation and an exact number-conserving method coincide whereas at smaller and larger values of G, the two methods deviate in opposite directions.

By use of the BCS solutions as the zero-order approximation, the effects of the residual terms on both even-A and odd-A nuclei are analyzed. The results obtained by applying our treatments to single-closed-shell nuclei (N=82 and $52 \leq Z \leq 64$) show that for even-A nuclei, the term H_{22} is sufficient, and it appears that for odd-A nuclei, a first-order treatment of H_{31}, together with a complete account of the H_{22} term, gives good results.

I. INTRODUCTION

Recent developments[1] in nuclear structure studies have opened a door to an easy treatment of complex nuclear systems, which conventional theories could not cope with. In a complex nuclear system, say a nucleus with many particles outside of closed shells (but not as deformed as to justify the use of Bohr-Mottelson Model), the method of calculating energy levels and wave functions gets tremendously complicated, and often gives wrong answers. It is complicated because a complete account of all the possible states leads to an enormous matrix, and it gives wrong answers because approximations inevitably made to simplify the matter are sometimes unjustified. Only recently was it found that one could apply the theories developed in connection with electron gas problems to a nuclear many-body system, if proper care is exercised to take account of the differences between an electron gas and a nuclear system. The theories applied to finite nuclei are the superconductivity theory of Bardeen, Cooper and Schrieffer (BCS)[2] and Bogoliubov[3] and the random-phase approximation (RPA) used for problems of a dense electron gas, which was studied by Sawada[4] and others.[5]

The first theory successfully describes, among others, the energy gap phenomenon in the superconductivity system, and is novel in that it recognizes the role of an anomalous coupling or bound state of particle-particle or hole-hole combination; in other words, the quantities $\langle c_\alpha c_{\alpha'} \rangle_o$ and $\langle c_\alpha^+ c_{\alpha'}^+ \rangle_o$ do not vanish, whereas the usual Hartree-Fock (HF) method neglects such a phenomenon. The difficulty encountered in the conventional perturbation theory when one studies an attractive-fermion system is known to be due to the appearance of this anomalous bound state.[3] The Bogoliubov treatment, which gives exactly the same results as BCS, explicitly eliminates such "dangerous" terms, and from which one obtains an equation giving a set of solutions for independent quasi-particle behavior (see next section). The independent-particle solution of this generalized Hartree-Fock equation[1] (called by Baranger the Hartree-Bogoliubov equation) already contains the pairing effect.

The second theory deals with the treatment of interaction between particles, and describes collective behavior in a many-fermion system. It takes into account the usual scattering terms due to an interaction V_{12}, as well as ground-state correlation (in diagrammatic language, forward-and-backward-going Feynman diagrams). Sawada's treatment was originally intended for an electron gas system, and as it turns out, it leads to a collective solution (plasmon)[6] as well as giving single-particle excitations.

It was found by Anderson[7] that application of RPA to a superconducting system gives most of the elementary excitations having the BCS gap spectrum, and also collective excitations corresponding to longitudinal waves in the neutral Fermi gas (and unperturbed plasma oscillations in a charged gas). Furthermore, for maintaining gauge invariance, the collective excitations are very important.[8]

From these facts, one can immediately see that if one deals with an attractive fermion system, the combination of the two theories mentioned above could naturally lead to a theory free of defects encountered in the old theories;[9] one first transforms to a quasi-particle description to take into account the anmalous coupling, and then treats the interactions between independent quasiparticles.

The nuclear system (specifically the system of finite nuclei) has two major aspects which differentiate it from an electron system:

(a) a nucleus is finite, whereas the electron gas is an infinite system;

(b) a nucleus has a shell structure, whereas the electron gas does not.

Owing to these differences, linear momentum p is not a good quantum number, although angular momentum is (that is, the total angular momentum j in spherical nuclei, and projection of j in axially symmetric nuclei (deformed nuclei).[10]) Despite these differences, we notice two similarities:

(a) An energy gap in the spectra of even-even nuclei, as in superconductivity. This similarity was originally pointed out by Bohr, Mottelson, and Pines,[11] and then thoroughly studied by Belyaev.[12] We

show an example in Fig. 1. Pb^{204} has four particles missing from the closed shell N = 126, and the zero-order energies (30 different levels) would in principle yield all the levels lying between 1.14 MeV and the ground state (zero). The actual spectrum shows only one level in that interval; all the other levels lie above it. Thus there is a gap of about 1 MeV. This trend is observed throughout the even-even nuclei, with very slight change of level density below the gap.

(b) The appearance of a low-lying "collective state" and greatly enhanced electric quadrupole transition from the collective level to 0+ ground state. From a microscopic point of view, this state may be considered as a bound state of a particle and a hole with certain angular momentum j. This bound state arises owing to repeated excitations and de-excitations of many particles of the system, and this phenomenom in a nucleus is comparable to a plasmon in an electron gas.

Thus one is tempted to apply these theories to nuclear problems, but before one does so, it should be noted that whereas such procedures are well justified and rigorous proofs of the validity have been put forth by many people for the elctron gas,[6] it is a completely open question in finite nuclei. No one has yet shown rigorously (a) whether or not BCS approximation (which violates number conservation) causes serious difficulty in finite nuclei, (see Bayman, however.[13] There have been some phenomenological approaches to this question, but no conclusive studies have been published yet), and (b) whether or not RPA (which has a proven domain of validity only in a high-density limit or weak-coupling limit) is really applicable to the nuclear problems. We do not, however, raise a serious question on these points, and in this work we merely adopt the philosophy that the computations by many people, and good results obtained by them, at least partially justify our work. We will examine the first point briefly in the framework of the shell-model picture; the second we do not dwell on.

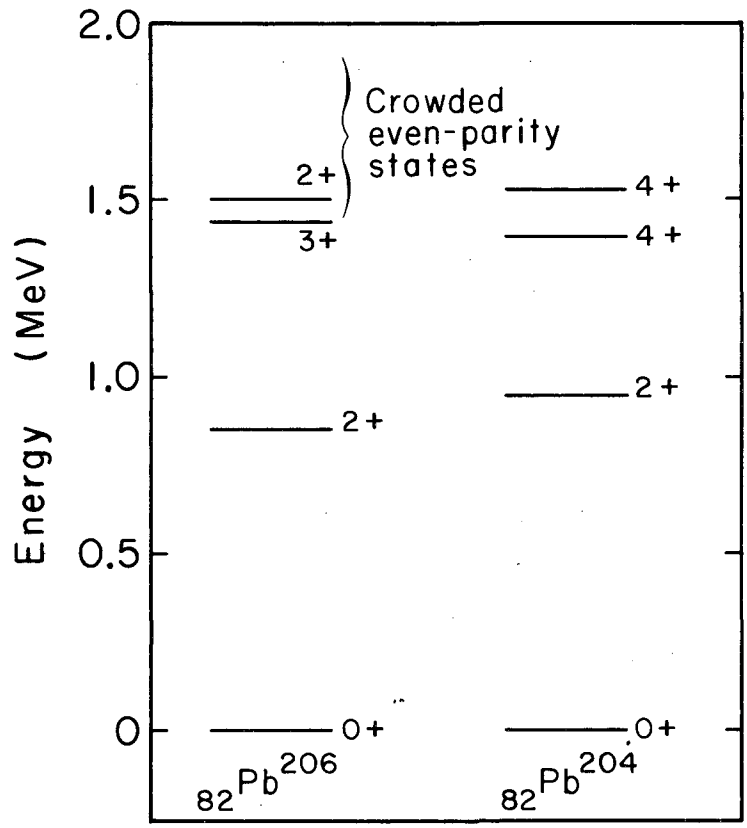


Fig. 1. Low-lying states in Pb nuclei exhibiting a gap.

Let us now see what one does in considering nuclear properties. One approach is to treat the main part of an interaction as a spherically symmetrical self-consistent field. The intricate details are then explained by treating interactions between particles by perturbation theory to the lowest order (i.e., the usual shell model). Such a treatment has been very successful for simple systems with only a few particles outside closed shells. In the other extreme, for very complicated systems, one adopts the Bohr-Mottelson collective model,[10] in which one introduces a deformed field that arises essentially from the residual interactions, and in which the particles follow the well in a self-consistent manner. It is well known that the simple shell model fails to describe collective properties and the Bohr-Mottelson model tends to overemphasize them.

Even though there are some conjectures that an energy gap may be exhibited by the shell model alone, neither thory has yet demonstrated any success in describing the gap. This indicates a similar situation for the nuclear system and for the electron system: the pairing effect cannot be given by perturbation theory alone nor by introducting further coordinates. There always remains that part of an interaction which is not taken into account. This is believed to correspond to the short-range interaction that gives rise to the pairing effect in nuclei. Collective phenomena in nuclei, on the other hand, are fairly well described by the Bohr-Mottelson model. One would prefer, however, to have a theory that describes both single-particle and collective behavior on an equal footing. We see that such a theory is already in use in the electron gas problems.

The sutdies made by Belyaev[12] and others[14] on finite nuclei show that the pairing effect (short-range effect) and long-range effect compete in a complicated way; the former corresponds to the independent quasi-particle picture and the latter to the effect of the residual terms. Baranger[1] is one of the first to take advantage of the understanding

gained in electron gas problems to put the nuclear theory on a more rigorous footing. The RPA which Baranger uses is valid in nuclear problems only if n_o/ν is fairly small, where n_o is the number of quasi particles in the ground state, and ν is the number of available states. Under this condition, the two-quasi-particle operators have the boson commutation relation $[\eta_i\eta_k, \eta_{i'}^{+}\eta_{k'}^{+}] = \delta_{ii'}\delta_{kk'}$, where η is a quasi particle operator. The ground state is composed of amplitudes for different numbers of quasiparticles . Therefore $\eta_i|\Psi_0\rangle \neq 0$. Also the excited two-quasi-particle states are made not only of two-quasi-particle states but also states having various numbers of quasiparticles . Thus $\langle\Psi_{ex}|A_{il}|\Psi_0\rangle$ does not vanish where $A_{il} = \eta_i\eta_l$, and $|\Psi_{ex}\rangle$ = an excited state. This property, in fact, is believed to lead to a coherent contribution, producing the collective nature. (This treatment is understood to be equivalent to the selective summation of forward-and-backward-going perturbation diagrams, though this equivalence is not proven in the nuclear problem.)

Equally successful, though less satisfying from a theoretical point of view, is the work of Kisslinger and Sorenson (KS).[15] Their work relies on the introduction of a quadrupole field, Q, thereby connecting the collective model and the microscopic theory. Their success is an impressive justification for the use of these theories.

The last but the most important point in practical calculation -- the advantage that the quasi-particle scheme has over the conventional methods -- is the ease with which one can calculate the low-lying nuclear properties and also estimate the order of approximations that can be made in the actual calculations.

We aim in this work to study in more detail the roles that each of the residual terms plays. We shall see that in nuclei it is sufficient to sum the forward-going diagrams to describe low-lying states without invoking the term "collective states"; this in fact corresponds to diagonalizing $H_0 + H_{22}$. (See Sec. III.) If this is not sufficient -- as may be the case with the large enhancement of B(E2) values from

some of the low-lying 2+ states -- then we may use simple perturbation theory to take into account the effects of higher number states. For the description of single-particle behavior, we show that KS's procedure of coupling single-particle to "phonon" excitation may be justified from more rigorous treatment of the residual terms H_{31} and H_{22} (Sec. III).

In Sec. II, we recapitulate the Bogoliubov-Valatin transformation; we give the essential features of the pairing correlation and the explicit forms of the operators we need. We then study the validity of the BCS gap equation and of the assumption that is usually adopted, the constancy of the pairing interaction. We proceed next to derive matrix elements of H_{22}, H_{40}, and H_{31}, and obtain formulas for the electric multipole transitions and other single-particle properties. The improvements, the application of field-theoretic techniques, and the comparisons with other works are also discussed.

Finally we apply the equations developed to some of the nuclei with the closed shell at 82 neutrons and a few particles outside the 50-proton closed shell (Sec. IV). We use this region mainly for the reason of convenience, since we can describe this region with relatively few available levels. The equations derived are sufficiently general for use in other complex regions. For the force, we take a general central force with which we calculate such quantities as energy levels, B(E2) values, magnetic g factors, and quadrupole moments.

II. PAIRING INTERACTION

A. Canonical Transformations

We start with the second-quantized Hamiltonian,

$$H = \sum_{\alpha} \epsilon_{\alpha} c_{\alpha}^{+} c_{\alpha} + \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | \bar{V} | \gamma\delta \rangle \; c_{\alpha}^{+} c_{\beta}^{+} c_{\delta} c_{\gamma}, \tag{II-1}$$

where $c_{\alpha}^{+}, c_{\alpha}$ are creation and annihilation operators with the usual anticommutation relations

$$\left\{ c_{\alpha}^{+}, c_{\beta} \right\} = \delta_{\alpha\beta} \; . \tag{II-2}$$

We follow the convention that the subscripts α, β, etc. denote all the quantum numbers so that in momentum space we have $\alpha = (p_a, \sigma_a)$, in the j-j coupling scheme $\alpha = (j_a m_a)$, and in deformed nuclei $\alpha = \Omega_a$, etc.; also that a,b, etc. denote all except magnetic quantum numbers. In Eq. (II-1), the operators C's have the following interpretations:

$$\left.\begin{matrix} c_{\alpha}^{+} \\ c_{\alpha} \end{matrix}\right\} \quad \alpha \in F \qquad \begin{cases} \text{destruction of hole} \\ \text{creation of hole} \end{cases} \tag{II-3}$$

$$\left.\begin{matrix} c_{\alpha}^{+} \\ c_{\alpha} \end{matrix}\right\} \quad \alpha \notin F \qquad \begin{cases} \text{creation of particle} \\ \text{destruction of particle,} \end{cases}$$

where F = Fermi sea.

The derivation of the Hamiltonian, Eq. (1), is given in many textbooks, and discussed by Valatin[16] in a rigorous mathematical sense. We can take $\langle\alpha\beta|\bar{V}|\gamma\delta\rangle$ to be real, and to have the following properties:

$$\langle\alpha\beta|\bar{V}|\gamma\delta\rangle = \langle\beta\alpha|\bar{V}|\delta\gamma\rangle = -\langle\beta\alpha|\bar{V}|\gamma\delta\rangle = -\langle\alpha\beta|\bar{V}|\delta\gamma\rangle \ . \tag{II-4}$$

Furthermore, the factor $1/4$ in front of the potential term, and (II-4) together imply that it is already antisymmetrized. That is,

$$\langle\alpha\beta|\bar{V}|\gamma\delta\rangle = \langle\alpha\beta|V|\gamma\delta\rangle - \langle\alpha\beta|V|\delta\gamma\rangle \ . \tag{II-5}$$

We also have

$$\langle\alpha\beta|\bar{V}|\gamma\delta\rangle^* = \langle\gamma\delta|\bar{V}|\alpha\beta\rangle = \langle\alpha\beta|\bar{V}|\gamma\delta\rangle \ . \tag{II-6}$$

Let us now introduce a canonical transformation which mixes particles and holes, and impose conditions so that now the diagonal part of the Hamiltonian contains more information than was in the original.[3] The most general linear transformation of such nature is

$$\eta_i^+ = \sum_\alpha \left(A_{i\alpha} C_\alpha^+ + B_{i\alpha} C_\alpha \right) \ . \tag{II-7}$$

By imposing the same commutation rules as the original particle operators (II-2), one then obtains conditions to be satisfied by the coefficients $A_{i\alpha}$ and $B_{i\alpha}$. We note that this transformation mixes particles and holes as well as angular momenta, and destroys the conservation of particle numbers. The mixing of angular momenta, which is the natural procedure for the descriptions of nuclear deformation, is too high a price to pay, however, in the spherical nuclei; the experimental evidence as well as theoretical convenience demands

the conservation. Therefore instead of using (II-7), we specialize with a transformation which conserves angular momentum, though still not conserving the number of particles; it is the simplest Bogoliubov-Valatin transformation, which was originally invented for Bose systems (for example, ^4H_e):

$$C_\alpha^+ = \mu_\alpha \eta_\alpha^+ + s_\alpha v_\alpha \eta_{-\alpha} \quad , \tag{II-8}$$

where u and v are real coefficients, having the normalization condition

$$u_\alpha^2 + v_\alpha^2 = 1. \tag{II-9}$$

The s_α here has the following interpretations: in momentum space and deformed nuclei, $s_\alpha v_\alpha = v_{-\alpha} = -v_\alpha$, and in spherical representation, $s_\alpha = (-)^{j_a - m_a}$. The Bogoliubov formulation of superconductivity was originally presented in this context. Now, substituting (II-8) into (II-1), and rearranging the terms (see Baranger[1]), we obtain a new Hamiltonian

$$H = H_{00} + H_{11} + H_{20} + H_I \quad , \tag{II-10}$$

where H_{00} is a constant independent of quasi-particle operators, H_{11} is a diagonal term in a new number operator, H_{20} contains terms that create or destroy only two particles, and H_I describe interactions between quasiparticles.

They are given by

$$H_{00} = \sum_{\alpha} \left[v_\alpha^2 \left(\epsilon_\alpha - \lambda_\alpha - \frac{u_\alpha}{2} \right) - \frac{1}{2} u_\alpha v_\alpha \Delta_\alpha \right] ,$$

$$H_{11} = \sum_{\alpha} \left[(u_\alpha^2 - v_\alpha^2) (\epsilon_\alpha - \lambda_\alpha - \mu_\alpha) + 2 u_\alpha v_\alpha \Delta_\alpha \right] \eta_\alpha^+ \eta_\alpha ,$$

$$H_{20} = \sum_{\alpha} s_\alpha \left[u_\alpha v_\alpha (\epsilon_\alpha - \lambda_\alpha - \mu_\alpha) + \frac{1}{2} (v_\alpha^2 - u_\alpha^2) \Delta_\alpha \right] (\eta_\alpha^+ \eta_{-\alpha}^+ + \eta_{-\alpha} \eta_\alpha) \qquad \text{(II-11)}$$

and $$H_I = \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | \bar{V} | \gamma\delta \rangle \, N(c_\alpha^+ c_\beta^+ c_\delta c_\gamma) ,$$

where $$\Delta_\alpha = -\frac{1}{2} \sum_{\beta} s_\alpha s_\beta \langle \alpha - \alpha | \bar{V} | \beta - \beta \rangle \, u_\beta v_\beta ,$$

$$\mu_\alpha = -\sum_{\beta} \langle \alpha\beta | \bar{U} | \alpha\beta \rangle \, v_\beta^2 ,$$

and $$\langle \alpha\beta | \bar{U} | \alpha\beta \rangle = \frac{1}{2} \left[\langle \alpha - \beta | \bar{V} | \alpha - \beta \rangle + \langle \alpha\beta | \bar{V} | \alpha\beta \rangle \right]$$

The operator N "normal-orders" the operators inside the parenthesis. The reason why H_I comes in normal ordered form is that the contracted terms are already included in the preceding terms. For example, the two operators A and B can be put as $AB = N(AB) - \overline{AB}$

and the "contracted" term can be absorbed in H_{00} , H_{11} , or H_{20} . (See reference 17 for the definitions of "normal ordering" and" contraction.")

Let us now show the explicit forms of H_I , since we shall need them for the main part of this work. Substituting (II-8) into (II-11d), we have

$$H_I = \frac{1}{4}\sum_{\alpha\beta\gamma\delta} \langle\alpha\beta|\bar{V}|\gamma\delta\rangle N\{(u_\alpha\eta_\alpha^+ + s_\alpha v_\alpha \eta_{-\alpha})(u_\beta\eta_\beta^+ + s_\beta v_\beta \eta_{-\beta}) \times$$

$$\times (u_\delta\eta_\delta + s_\delta v_\delta \eta_{-\delta}^+)(u_\gamma\eta_\gamma + s_\gamma v_\gamma \eta_{-\gamma}^+)\} \quad . \tag{II-12}$$

Multiplying out, normal-ordering, changing indices and using the symmetry relations (II-4), (II-5), (II-6), we obtain

$$H_I = H_{22} + H_{31} + H_{40} \quad ,$$

$$\text{where} \quad H_{31} = \frac{1}{2}\sum_{\alpha\beta\gamma\delta} \langle\alpha\beta|\bar{V}|\gamma\delta\rangle s_\delta (u_\alpha u_\beta u_\gamma v_\delta - v_\alpha v_\beta v_\gamma u_\delta)\eta_\alpha^+\eta_\beta^+\eta_{-\delta}^+\eta_\gamma + \text{h.c},$$

$$H_{22} = \frac{1}{4}\sum_{\alpha\beta\gamma\delta} \{\langle\alpha\beta|\bar{V}|\gamma\delta\rangle (u_\alpha u_\beta u_\gamma u_\delta + v_\alpha v_\beta v_\gamma v_\delta) - 4\langle\alpha_-\gamma|V|_-\beta\delta\rangle s_\beta \tag{II-13}$$

$$\times s_\gamma u_\alpha u_\delta v_\beta v_\gamma\} \eta_\alpha^+\eta_\beta^+\eta_\delta\eta_\gamma \quad ,$$

$$H_{40} = \frac{1}{4}\sum_{\alpha\beta\gamma\delta} \langle\alpha\beta|\bar{V}|\gamma\delta\rangle s_\gamma s_\delta u_\alpha u_\beta v_\gamma v_\delta \eta_\alpha^+\eta_\beta^+\eta_{-\delta}^+\eta_{-\gamma} + \text{h.c.} \quad ,$$

where h.c. means Hermitian conjugate. The significance of these terms is discussed in the subsequent sections.

A partial diagonalization of the new Hamiltonian can be obtained if we eliminate the term H_{20}, which Bogoliubov calls the compensation of "dangerous diagrams," by the condition

$$\langle 0|\eta_\alpha \eta_{-\alpha} H_{20}|0\rangle = 0 \ , \tag{II-14}$$

or in other words set the coefficient of the operator $(\eta_\alpha^+ \eta_\alpha^+ + \eta_\alpha \eta_\alpha)$ equal to zero, while neglecting H_I terms altogether. The condition (II-14) is the lowest consideration, and the higher-order correction is briefly discussed in Sec. IIB.

The condition (II-14) gives, then, the equations to determine the coefficients u and v:

$$u_\alpha^2 = \frac{1}{2}\left(1 + \frac{\epsilon_\alpha - \lambda}{E_\alpha}\right) ,$$

$$v_\alpha^2 = 1 - u_\alpha^2 \ , \tag{II-15}$$

$$N = \sum_\alpha v_\alpha^2 \ ,$$

with $$E_\alpha = \{(\epsilon_\alpha - \lambda)^2 + \Delta_\alpha^2\}^{\frac{1}{2}} \ .$$

The last equation for N (i.e., number of particles in the system) insures that the number of particles is approximately conserved, from which the Lagrangian multiplier λ is determined.[12] With Eqs. (II-14) and (II-15), the Hamiltonian can be written as a diagonal term and nondiagonal terms in the quasi-particle representation:

$$H = H_0 + H_I \quad ,$$

$$H_0 = \sum_\alpha E_\alpha \eta_\alpha^+ \eta_\alpha \quad , \tag{II-16}$$

where H_I is given in (II-13). In this form, the H_0 already contains the extra pairing effect Thus

$$H_0|\alpha\rangle = E_\alpha|\alpha\rangle \quad ,$$

$$H_0|\alpha\beta\gamma\ldots\rangle = (E_\alpha + E_\beta + E_\gamma + \ldots)\ |\alpha\beta\gamma\ldots\rangle \quad , \tag{II-17}$$

where $|\alpha\rangle = \eta_\alpha^+|\hat{0}\rangle$, $|\alpha\beta\rangle = \eta_\alpha^+\eta_\beta^+|\hat{0}\rangle$, etc.

and where $|\hat{0}\rangle$ is a "new vacuum" (the so-called BCS ground state). Note that the new vacuum has the following properties:

$$\langle\hat{0}|c_\alpha c_\alpha|\hat{0}\rangle = v_\alpha^2 \quad ,$$

$$\langle\hat{0}|c_\alpha^+ c_\beta^+|\hat{0}\rangle = s_\alpha v_\alpha u_\alpha \delta(\alpha,-\beta) \quad ,$$

since $\eta_\alpha^+|\hat{0}\rangle = |\alpha\rangle$, $\eta_\alpha|\hat{0}\rangle = 0$,

$$\langle\hat{0}|\eta_\alpha\eta_\beta^+|\hat{0}\rangle = \delta(\alpha,\beta) \qquad \text{(II-18)}$$

B. BCS Approximation

It has already been mentioned that the canonical transformation of the form (II-8) mixes particles and holes, and hence a particle-annihilation operator C operating on the new ground state does not vanish The relation given by Eq (II-18), then, destroys the conservation of particle numbers in the system To remedy this shortcoming of the theory, we have used a Lagrangian multiplier λ; we determine λ by imposing the condition $N = \langle\Sigma_\alpha C_\alpha^+ C_\alpha\rangle$. It can be seen that the average quadratic fluctuation in the number of particles is given by

$$\langle N^2\rangle - \langle N\rangle^2 \approx \sum_\alpha \frac{\Delta_\alpha^2}{\Delta_\alpha^2 + (\epsilon_\alpha-\lambda)^2} \quad . \qquad \text{(II-19)}$$

If the state λ is far distant from λ , then $|\epsilon_\alpha-\lambda|\rangle\rangle\Delta$; then the contribution to the sum in Eq. (II-19) diminishes as $\frac{\Delta_\alpha}{(\epsilon_\alpha-\lambda)} \longrightarrow 0$. Now near the Fermi surface $\epsilon_\alpha \approx \lambda$; therefore $\frac{\Delta_\alpha^2}{(\epsilon_\alpha-\lambda)^2+\Delta_\alpha^2} \approx 1$. Thus the fluctuation may be rather significant if we have many states near the Fermi surface. In finite nuclei, a difference of number of particles is expected to be rather serious,

since in this approximation the nucleus we are looking at does not have a fixed N, and moreover is a mixture of neighboring nuclei. In electron systems the uncertainty of N does not cause serious trouble, for here we are dealing with a large number of particles. It has been shown, however, by Belyaev that for highly degenerate systems Eq (II-19) may be written as

$$\langle N^2 \rangle - \langle N \rangle^2 \approx 2N(1 - \frac{N}{2\Omega}) \quad , \tag{II-20}$$

where Ω is pair degeneracy, and that for as few as two particles in the system, admixture is restricted only to the neighboring even-even nuclei. It has also been noted that physically observable quantities are not significantly affected by such uncertainty.

Kerman et al.[18] have studied the goodness of the quasi-particle model in spherical nuclei by diagonalizing the BCS reduced Hamiltonian and looking at the spectra. Also Soloviev and his co-workers[19] have investigated the difference in $\langle N \rangle$ (expectation value of number operator) between the BCS method and an exact method, and found that both treatments show satisfactory agreement.

To see how good BCS approximation is, let us study this problem a little further. To simplify the matter, we consider a two-level model, each level of which has $(2j + 1)$ degeneracy. We study the ground state by the BCS approximation and by the exact diagonalization of the reduced Hamiltonian--both of which adopt a constant matrix element for the pairing-type interaction--and finally by the exact diagonalization with a nonconstant realistic force. We first introduce the formulas involved in the methods.

1. BCS Equation with a Constant Matrix Element

For a constant G, Δ_α in Eq. (II-11e) becomes Δ ; hence [Note that $\alpha = (j_a, m_a)$] we get

$$\frac{G}{2} \sum_a (j_a + \frac{1}{2}) / E_a = -1 \quad , \tag{II-21}$$

where $$E_a = ((\epsilon_a - \lambda)^2 + \Delta^2)^{\frac{1}{2}}$$

$$\Delta = -G \sum_a (j_a + \frac{1}{2}) u_a v_a \quad ,$$

$$u_a^2 = \frac{1}{2}(1 + \frac{\epsilon_a - \lambda}{E_a}) \quad ,$$

and $$v_a^2 = 1 - u_a^2 \quad , \tag{II-22}$$

The simultaneous solution of Eq. (II-21) together with the number equation

$$N = \sum_\alpha \langle C_\alpha^+ C_\alpha \rangle_0 = \sum_\alpha v_\alpha^2 = \sum_a (2j_a + 1) v_a^2 \tag{II-23}$$

gives λ and Δ ; Eq (II-22) then gives u^2 and v^2 .

Thus the pair distribution is given by

$$\langle \frac{N}{2} \rangle_a = (j_a + \frac{1}{2}) v_a^2 \qquad \text{(II-24)}$$

for state a.

2. Exact Diagonalization of Reduced Hamiltonian (EDRH)

Let us first write the BCS reduced Hamiltonian in angular momentum representation:

$$H_{red} = \sum_{\alpha} \epsilon_\alpha c_\alpha^+ c_\alpha + \frac{G}{4} \sum_{\alpha\beta} c_\alpha^+ c_{-\alpha}^+ c_{-\beta} c_\beta \quad , \qquad \text{(II-25)}$$

where $\alpha = (j_a, m_a)$, $-\alpha = (j_a, -m_a)$, etc , $G < 0$

As is obvious, H_{red} has the interaction term that shows explicitly the particle-particle or hole-hole coupling of opposite angular momentum projection quantum number. It implies $(\alpha, -\alpha)$ type of coupling. Following the spin-wave method introduced by Anderson[7] and applied to nuclear problems by Kerman et al.[18], we introduce the spin operators

$$S_+ = \sum_a S_+(j_a) = \sum_{\alpha>0} c_\alpha^+ c_{-\alpha}^+ \quad ,$$

$$S- = \sum_a S-(j_a) = \sum_{\alpha>0} c_{-\alpha} c_\alpha \quad ,$$

$$S_z = \frac{1}{2}\sum_{\alpha>0}\{C^+_\alpha C_\alpha - C_{-\alpha}C^+_{-\alpha}\} = \sum_a S_z(j_a) = \frac{1}{2}\sum_\alpha C^+_\alpha C_\alpha - \frac{1}{2}\sum_a \Omega(j_a) \quad , \tag{II-26}$$

where $\Omega(j_a) = j_a + \frac{1}{2}$, the pair degeneracy of the j_a state.

It can be shown, by using the anticommutation rule of C operators, that S+, S- and S_z obey the same commutation rules as the usual angular momentum operators do. Substitution of (II-26) into (II-25) gives immediately

$$H_{red} = 2\sum_a \epsilon_a S_z(j_a) + \sum_a \epsilon_a \Omega(j_a) + GS+S- \quad . \tag{II-27}$$

We note that in the strong coupling limit

$$H^{strong}_{red} \approx GS+S- \tag{II-28}$$

and is diagonal. On the other hand, in the weak coupling limit

$$H^{weak}_{red} \approx 2\sum_a \epsilon_a S_z(j_a) + \sum_a \epsilon_a \Omega(j_a) \quad , \tag{II-29}$$

a result one would expect. Thus one could take a state that is diagonal in either H^{weak}_{red} or H^{strong}_{red} for a zero-order wave function.

Choosing the state that is diagonal in H_{red}^{weak}, let us construct a wave function involving two levels. Introduce the notations

$$S(j_a)^2 |\sigma^a, \sigma_0^a> = \sigma^a(\sigma^a+1)|\sigma^a, \sigma_0^a> , \qquad \text{(II-30)}$$

$$S_z(j_a)|\sigma^a, \sigma_0^a> = \sigma_0^a|\sigma^a, \sigma_0^a> ,$$

where the superscript a labels the level; i.e., $a = j_a$. Then, for two-level systems,

$$|\sigma^a\sigma_0^a;\sigma^b\sigma_0^b> = \sum_{\sigma\sigma_0} C(\sigma^a\sigma^b\sigma;\ \sigma_0^a\sigma_0^b\sigma_0)\ |\sigma^a\sigma^b:\sigma\sigma_0> , \qquad \text{(II-31)}$$

where C is the well-known Clebsch-Gordan coefficient.

In the representation (II-31), we have

$$\sum_\alpha \epsilon_\alpha S_z(j_\alpha)|\ \sigma^a\sigma_0^a;\ \sigma^b\sigma_0^b> = (\epsilon_a\sigma_0^a + \epsilon_b\sigma_0^b)\ |\ \sigma^a\sigma_0^a;\ \sigma^b\sigma_0^b\ > ,$$

$$\sum_\alpha \epsilon_\alpha\Omega(j_\alpha)\ |\sigma^a\sigma_0^a;\ \sigma^b\sigma_0^b> = (\epsilon_a\Omega(j_a) + \epsilon_b\Omega(j_b)\ |\sigma^a\sigma_0^a;\ \sigma^b\sigma_0^b> .$$

(II-32)

Using the well-known relations,

$$S_+S_- = S^2 - S_z^2 + S_z \quad ,$$

$$S_-S_+ = S^2 - S_z^2 - S_z \quad ,$$

and $S_-S_+ = S_+S_- - 2S_z$, we can easily compute off-diagonal term to give

$$\langle \sigma^a \sigma_0^a : \sigma^b \sigma_0^b | GS_+S_- | \sigma^a \sigma_0^{a'} ; \sigma^b \sigma_0^{b'} \rangle \tag{II-33}$$

$$= G \sum_{\sigma, \sigma_0} C(\sigma^a \sigma^b \sigma ; \sigma_0^a \sigma_0^b \sigma_0) \, C(\sigma^a \sigma^b \sigma ; \sigma_0^{a'} \sigma_0^{b'} \sigma_0) \, [\sigma(\sigma+1) - \sigma_0(\sigma_0-1)] \, .$$

Of course this term contributes also to the diagonal term, since all the terms with $\sigma_0^a + \sigma_0^b = \sigma_0^{a'} + \sigma_0^{b'}$ are nonvanishing. Using Eqs. (II-32) and (II-33), we can diagonalize the matrix of H_{red} exactly. With the system of eight particles with two levels of $a = g_{7/2}$ and $b = d_{5/2}$ we can assign quantum numbers as

$$| \sigma^a \sigma_0^a : \sigma^b \sigma_0^b \rangle$$

$$(7/2^8)_0 (5/2^0)_0 \; | 22 \; ; \; 1.5 \; - 1.5 \rangle$$

$$(7/2^6)_0 (5/2^2)_0 \; | 21 \; ; \; 1.5 \; - 0.5 \rangle$$

$$(7/2^2)_0 (5/2^6)_0 \; | 2\text{-}1 \; ; \; 1.5 \; 1.5 \rangle$$

$$\sigma_0 = 0.5 \; ; \; \sigma = 3.5, \; 2.5, \; 1.5, \; 0.5 \quad . \tag{II-34}$$

Using (II-27), (II-32), and (II-33) we obtain the explicit form of the Hamiltonian matrix:

$$H = \begin{bmatrix} 4G & 2\sqrt{3}G & 0 & 0 \\ 2\sqrt{3}G & 2\epsilon_b + 9G & 2\sqrt{6}G & 0 \\ 0 & 2\sqrt{6}G & 4\epsilon_b + 10G & 3\sqrt{2}G \\ 0 & 0 & 3\sqrt{2}G & 6\epsilon_b + 7G \end{bmatrix} \tag{II-35}$$

Note that states which differ in numbers in a particular shell by more than two particles are not connected, since the interaction term is of two-body scattering nature, and hence can differ at most by 2. This is not obvious offhand, since the off-diagonal term for each σ does not vanish; it vanishes, however, when summed over σ. In (II-35), the zero-order single-particle energy is measured from ϵ_a, i.e., $\epsilon_a = 0$.

Suppose we have diagonalized the matrix and obtained the eigenfunctions; we are then interested in the computation of number distributions. Let us write the eigenfunction as

$$\Psi_\alpha = \sum_i C_i^\alpha \, | \sigma^a \sigma_0^a(i) \; ; \; \sigma^b \sigma_0^b(i) \rangle \tag{II-36}$$

where the sum i runs over all states; for example, for $i = 1$, $\sigma_0^a(1) = 2$, $\sigma_0^b(1) = -1.5$, and so on according to the list given above.

From the last of Eq. (II-26) we have

$$2S_z(j_q) = N_q - \Omega(j_q)$$

or

$$N_q = 2S_z(j_q) + \Omega(j_q) \quad . \tag{II-37}$$

Define the pair operator by $P_q = \frac{N_q}{2}$. Then we have $P_q = \hat{S}(j_q) + \Omega(j_q)/2$. Calculating the expectation value of this operator, we have

$$\langle \Psi^\alpha | P_q | \Psi^\alpha \rangle = \sum_{il} C_i C_l \langle \sigma^a \sigma_0^a(i) \; ; \; \sigma^b \sigma_0^b(i) | P_q | \sigma^a \sigma_0^a(1); \sigma^b \sigma_0^b(1) \rangle . \tag{II-38}$$

Off-diagonal terms vanish, since

$$\langle \sigma^a \sigma_0^a \; ; \; \sigma^b \sigma_0^b | P_q | \sigma^a \sigma_0^a \; ; \; \sigma^b \sigma_0^b \rangle = (\sigma_0^a + \frac{\Omega(j_a)}{2}) \, \delta(q,a) + (\sigma_0^b + \Omega(j_b)/2) \, \delta(q,b) \quad . \tag{II-39}$$

We finally have

$$\langle \Psi^\alpha | P_q | \Psi^\alpha \rangle = \sum_i (C_i^\alpha)^2 \, (\sigma_0(i) + \frac{\Omega(j_i)}{2}) \, \delta(i,q). \tag{II-40}$$

3. Exact Diagonalization of Full Hamiltonian (EDFH)

In order to compare the BCS method and EDRH, with an exact treatment of the Hamiltonian, Eq. (II-1), we desire to diagonalize it to calculate the number distributions in a realistic case. This, in full generality, is a formidable task, but if we take the same two-level system, and confine ourselves to seniority zero state, then the Hamiltonian matrix turns out to be rather simple. It is difficult to compare those two, however, since an "exact" treatment necessarily includes more interactions than the two approximate solutions. The pairing solutions are known to contain only higher harmonics in the expansion of force in terms of spherical harmonics whereas a general force does contain both high and low harmonics. In this sense, two considerations may give quite different information.

It is, however, of interest to see how much information (though qualitative) the BCS solution does contain, since what we consider later is precisely the question of improving our treatment beyond the simple theory of the BCS type.

Let us consider seniority-zero states spanned by the basic two levels $a = g_{7/2}$ and $b = d_{5/2}$. Because of the capacity of each level, the complete set in such subspace is provided by

$$(a^8)_0, (a^6)_0(b^2)_0, (a^4)_0(b^4)_0, (a^2)_0(b^6)_0 \qquad \text{(II-41)}$$

Let $\Psi(a^\alpha b^\beta J)$ denote a completely antisymmetrized state of a particles and b particles, and let $[a^\alpha(\nu_a J_a)b^\beta(\nu_b J_b); \nu J]$ be a state function that describes completely antisymmetrized states in a and b particles. It is easy to deduce that the matrix element (two-body) taken between completely antisymmetrized wave

functions is equivalent to that between partially antisymmetrized amplitudes (see de-Shalit and Talmi[20]). Thus

$$\langle\Psi(a^{\alpha}b^{\beta}J)|\sum_{i<k}^{\alpha+\beta}V_{ik}|\Psi(a^{\alpha}b^{\beta}J)\rangle=\langle a^{\alpha}(\nu_a J_a)b^{\beta}(\nu_b J_b)J|\sum_{i<k}^{\alpha+\beta}V_{ik}|a^{\alpha}(\nu_a' J_a')b^{\beta}(\nu_b' J_b')J\rangle$$

$$= \langle a^{\alpha}\nu_a J_a|\sum_{i<k}^{\alpha}V_{ik}|a^{\alpha}\nu_a' J_a'\rangle + \langle b^{\beta}\nu_b J_b|\sum_{i<k}^{\beta}V_{ik}|b^{\beta}\nu_b' J_b'\rangle \qquad \text{(II-42)}$$

$$+ \langle a^{\alpha}(\nu_a J_a)\, b^{\beta}\,(\nu_b J_b)\, J|\sum_{i=1}^{\alpha}\sum_{k=\alpha+1}^{\alpha+\beta}V_{ik}|a^{\alpha}(\nu_a' J_a')b^{\beta}(\nu_b' J_b')J\rangle$$

In particular we obtain, for seniority-zero states;

$$\langle a^{\alpha}0|\sum_{i<k}^{\alpha}V_{ik}|a^{\alpha}0\rangle=\frac{\alpha(2j_a+3-\alpha)}{2(2j_a+1)}\langle j_a^2\,0|V|j_a^2\,0\rangle$$

$$+\frac{\alpha(\alpha-2)}{(2j_a+1)(2j_a-1)}\sum_{\substack{J>0\\ \text{even}}}\langle j_a^2\,J|V|j_a^2\,J\rangle\,(2J+1) \qquad \text{(II-43)}$$

and $$\langle a^{\alpha}(0)b^{\beta}(0)0|\sum_{i=1}^{\alpha}\sum_{k=\alpha+1}^{\alpha+\beta}V_{ik}|a^{\alpha}(0)b^{\beta}(0)0\rangle$$

$$= \frac{2\alpha\beta}{(2j_a+1)(2j_b+1)}\sum_{J}(2J+1)\langle j_a j_b J|V|j_a j_b J\rangle \qquad \text{(II-44)}$$

Equation (II-44) is a special case of the center-of-mass theorem.[20]

The only nonvanishing off-diagonal terms are those which differ in particle numbers by at most two units, with the total number $\alpha + \beta = \text{const}$. This is because V_{ij} is a two-body operator; this point was also observed in the EDRH matrix shown previously.

The off-diagonal term is given by

$$\langle a^{\alpha}(0)b^{\beta}(0)0|V|a^{\alpha-2}(0)b^{\beta+2}(0)0\rangle =$$

$$\frac{1}{2}\left[\frac{\alpha(\beta+2)(2j_a+3-\alpha)(2j_b+1-\beta)}{(2j_a+1)\ (2j_b+1)}\right]^{1/2} \langle j_a^2 0|V|j_b^2 0\rangle \quad . \tag{II-45}$$

Now if we take a δ-function type interaction; i.e.,

$$V(\vec{r}_1,\vec{r}_2) = \left[V_0(1-\sigma_1\ \sigma_2)/4\right]\delta(\vec{r}_1-\vec{r}_2) \tag{II-46}$$

(we confine ourselves to one kind of particles, either protons or neutrons), and use

$$\langle j_1 j_2 J|V(\delta)P_S|j_1' j_2' J\rangle = -\Theta(j_1 j_1' \ell_1 \ell_1')\ \{[j_1]\ [j_2]\ [j_1']\ [j_2']\}^{\frac{1}{2}}$$

$$C(j_1 j_2 J;\tfrac{1}{2}-\tfrac{1}{2}\ 0)\ C(j_1' j_2' J;\tfrac{1}{2}-\tfrac{1}{2}\ 0)\ \frac{F^0(12\,1'2')}{2(2J+1)}\ \frac{1}{2}\left(1+\Theta(\ell_1\ell_1' J)\right) , \tag{II-47}$$

with

$$F^0(1\,2\,1'\,2') = \frac{V_0}{4\pi}\left(\frac{\pi}{\beta}\right)^{\frac{3}{2}}\int_0^\infty r^2 dr\; R_1 R_2 R_1' R_2'$$

and

$$\Theta(1\,2\,1'\,2') = (-)^{j_1+j_2+j_1'+j_2'}, \quad P_s = \frac{1}{4}(1-\sigma_1\cdot\sigma_2)$$

and

$$\langle j_a^2 0|VP_s|j_a^2 0\rangle = 4F^0(aa)$$

$$\langle j_b^2 0|VP_s|j_b^2 0\rangle = 3F^0(bb)$$

$$\langle j_a^2 0|VP_s|j_b^2 0\rangle = (48)^{\frac{1}{2}}\,\frac{F^0(ab)}{2}$$

$$\sum_{J>0}(2J+1)\,\langle j_a^2 J|VP_s|j_a^2 J\rangle = 12F^0(aa)$$

$$\sum_{J}(2J+1)\,\langle j_a j_b J|VP_s|j_a j_b J\rangle = 12F^0(ab)$$

$$\sum_{J>0}(2J+1)\,\langle j_b^2 J|VP_s|j_b^2 J\rangle = 6F^0(bb)\quad,$$

where

$$F^0(ab) \equiv F^0(abab), \quad \text{etc.}\,,$$

we can put the Hamiltonian matrix in the form:

$$H=\begin{bmatrix} 16F^0(aa) & 2\sqrt{3}\,F^0(ab) & 0 & 0 \\ 2\sqrt{3}\,F^0(ab) & 2\epsilon_b+12F^0(aa)+3F^0(bb)+6F^0(ab) & 2\sqrt{6}\,F^0(ab) & 0 \\ 0 & 2\sqrt{6}\,F^0(ab) & 4\epsilon_b+8F^0(aa)+6F^0(bb)+8F^0(ab) & 3\sqrt{2}\,F^0(ab) \\ 0 & 0 & 3\sqrt{2}\,F^0(ab) & 6\epsilon_b+4F^0(aa)+9F^0(bb)+6F^0(ab) \end{bmatrix} \qquad \text{(II-48)}$$

Figures 2,3, and 4 give the results of $\langle \frac{N}{2} \rangle$ computed for $g_{7/2}$ with the BCS and EDRH methods, using only two degenerate levels, $g_{7/2}$ and $d_{5/2}$. We have also computed $\langle \frac{N}{2} \rangle_{g_{7/2}}$ with the EDFH method; the result is given in Fig. 5. Notice in Figures 2,3, and 4 that at $0 < |G| < 0.16$, and $0.5 > |G| > 0.16$, the agreement gets poorer. The two curves seem to meet at $G = 0.16$, which is slightly lower than what KS used in their first calculation. This probably is the only place of agreement that is physically meaningful. Theoretically, though, it is expected that as $(-G) \rightarrow \infty$, the two curves come together again as they do in the figure. Let us consider this in an analytical way. In our case, it is obvious that as $(-G) \rightarrow \infty$, the BCS solution yields an equal probability of occupation for each magnetic substate of j_a and j_b. In other

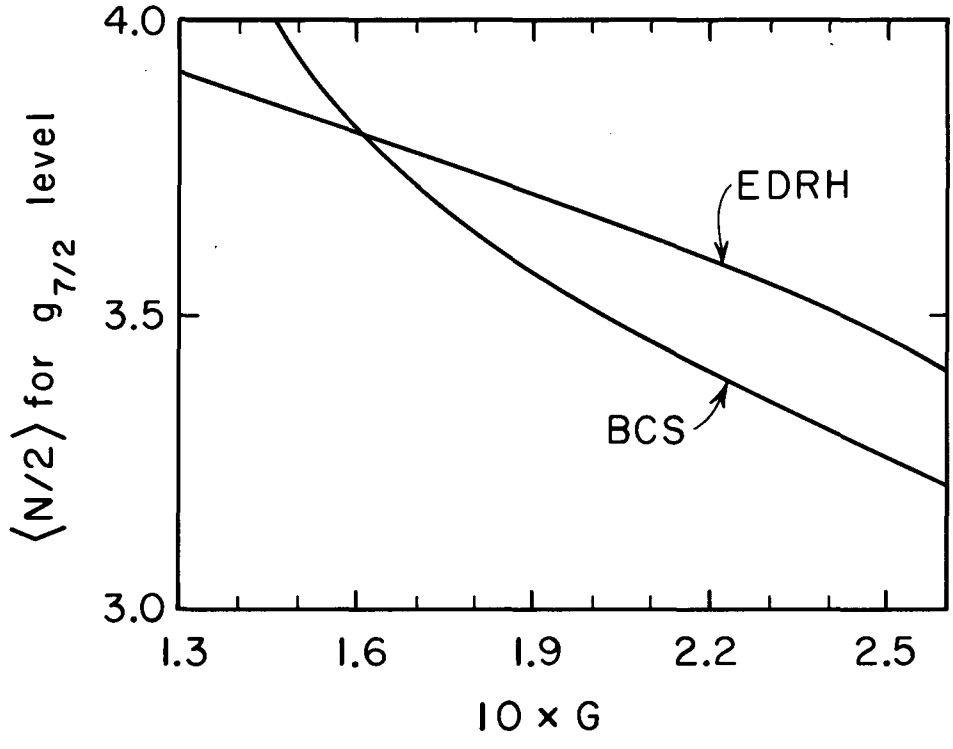


Fig. 2. Pair occupation probability $\langle N/2 \rangle$ for $1g_{7/2}$ level computed by the BCS (Bardeen-Cooper-Schrieffer) and EDRH (exact diagonalization of reduced Hamiltonian) methods, using two levels $1g_{7/2}$, and $2d_{5/2}$ separated by 1 MeV and with a constant G.

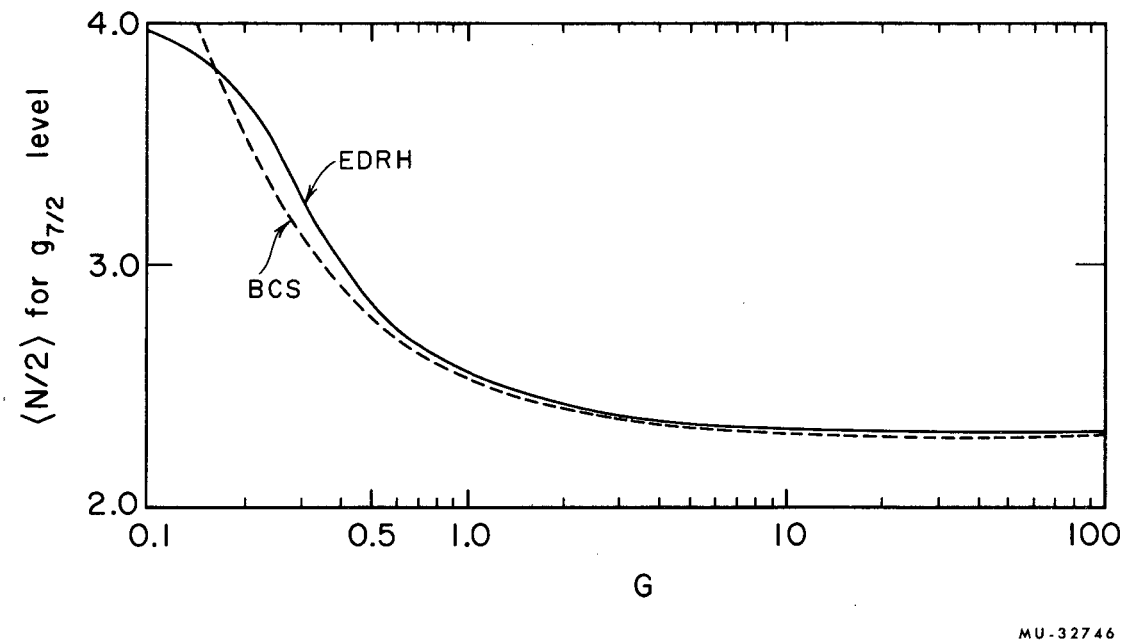


Fig. 3. Pair occupation probability $\langle N/2 \rangle$ for the $1g_{7/2}$ level computed by the BCS and EDRH methods, using two levels, $1g_{7/2}$ and $2d_{5/2}$, separated by 1 MeV and with a larger G.

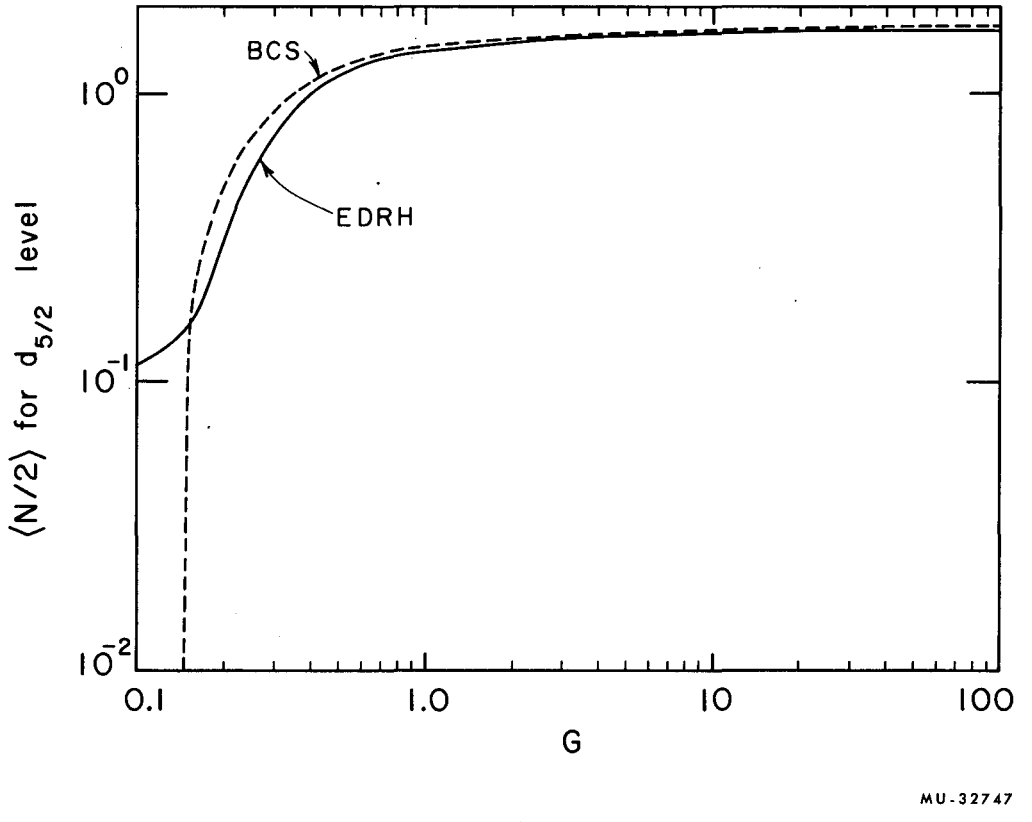


Fig. 4. Pair occupation probability $\langle N/2 \rangle$ for the $2d_{5/2}$ level computed by the BCS and EDRH methods, using two levels $1g_{7/2}$ and $2d_{5/2}$ separated by 1 MeV and with constant G.

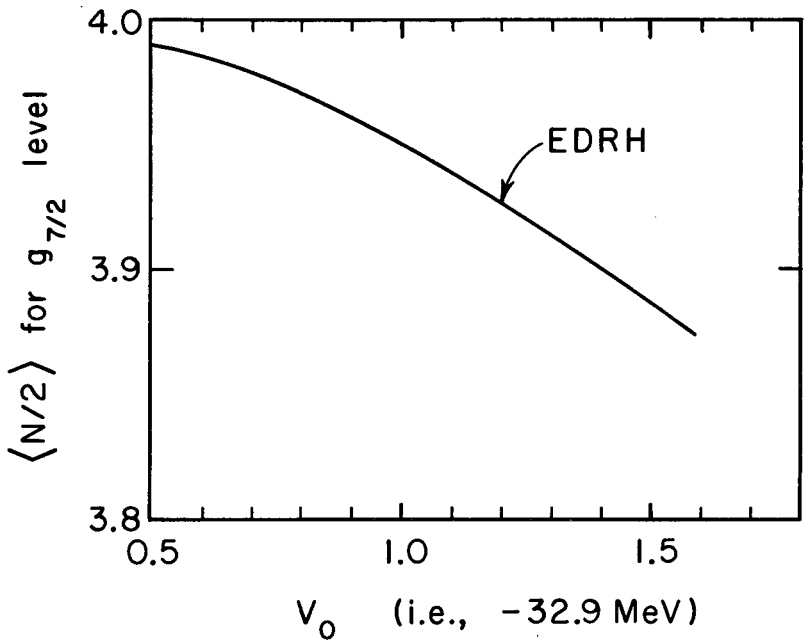


Fig. 5. Pair occupation probability $\langle N/2 \rangle$ for the $1g_{7/2}$ level computed by the EDRH method, using two levels, $1g_{7/2}$ and $2d_{5/2}$, separated by 1 MeV, and with V_0 = -32.9x MeV, $\beta^{-1/2}$ = 1.732 F and $\nu^{-1/2}$ = 2.288 F, where V_0 = potential depth, $\beta^{-1/2}$ = range of the force, and $\nu^{-1/2}$ = oscillator parameter. (For definitions, see Sec. IV.C.)

words $v_a^2 = v_b^2 = \frac{4}{7}$. Therefore $\langle \frac{N}{2} \rangle$ for $j_a(=7/2)$ is given by $\frac{4}{7} \times 4 = \frac{16}{7}$. Now, as $(-G) \to \infty$, we can take H_{red}^{strong} given by Eq. (II-28), which is diagonal in a coupled scheme σ, σ_0 are good quantum numbers . Thus the eigenfunction of this Hamiltonian is

$$|\sigma;\sigma_0\rangle = \sum_{\sigma_0^a \sigma_0^b} C(\sigma_0^a \sigma^b \sigma; \sigma_0^a \sigma_0^b \sigma_0) \; |\sigma^a \sigma_0^a\rangle \; |\sigma^b \sigma_0^b\rangle$$

with the eigenvalue

$$H_{red}^{strong} |\sigma,\sigma_0\rangle = GS+S- \; |\sigma,\sigma_0\rangle = G(\sigma(\sigma+1)-\sigma_0(\sigma_0-1)) \; |\sigma,\sigma_0\rangle \quad .$$

Since σ_0 is fixed (i e., $\sigma_0 = 0.5$) and $G < 0$, the state with the largest σ is the ground state. Hence $\sigma = 3.5$. Taking the number operator given by Eq. (II-37), we have

$$\langle \sigma,\sigma_0 | \frac{N_a}{2} | \sigma,\sigma_0 \rangle = \frac{1}{2} \sum_{\sigma_0^a \sigma_0^b} C(\sigma^a \sigma^b \sigma : \sigma_0^a \sigma_0^b \sigma_0) \; \langle \sigma^a \sigma_0^a | N_a | \sigma^a \sigma_0^a \rangle = \frac{\Omega_a}{2}$$

$$+ \sum_{\sigma_0^a \sigma_0^b} C(\sigma^a \sigma^b \sigma : \sigma_0^a \sigma_0^b \sigma_0)^2 \; \sigma_0^a \quad .$$

Now, using appropriate Clebsch-Gordan coefficients and the list of quantum numbers (II-3), it is trivial algebra to get

$$\frac{1}{2}\langle N_a\rangle = 2 + 2/7 = 16/7 \ ,$$

which coincides exactly with the prediction of the BCS equations.

The diagonalization of the full Hamiltonian with a finite-range force does not show the rapid mixing of various states as the BCS and EDRH do, while a δ- function force mixes the states somewhat faster. In both cases, the $\lim_{(-V_0)\to\infty} \frac{1}{2}\langle N\rangle_{g_{7/2}}$ does not seem to coincide with $16/7$. This as discussed below is not surprising.

We should note that both the exact diagonalization of the reduced Hamiltonian and the full Hamiltonian (in seniority zero scheme) in principle include more interactions than the BCS method does. For example, EDRH contains the information that the BCS describes, and furthermore contains the part describing interactions between seniority zero states with interaction strength G. On the other hand, the BCS approximation, as is well known, neglects the residual terms, and in fact, the inclusion of such terms may lower the gap. Lowering of gap Δ implies increase of v^2 for a state below the Fermi surface, and thus higher $\langle\frac{N}{2}\rangle$ for that state. Thus the goodness of the independent-quasi-particle picture -- that is, the negligibility of residual terms dropped in BCS approximation -- should be determined by comparing it with the EDRH results. The success of an independent-particle model, which is the cornerstone in the simple shell model, lies in the fact that the residual terms are only small perturbative effects, the main physical description being provided by the zeroth-order term.

Thus agreement of BCS method and EDRH at a certain coupling constant G determined by some other prescription would imply the validity of both the independent-quasi-particle picture and the approximate number conservation.

The exact diagonalization of the full Hamiltonian with finite range cannot be used to test the validity of the BCS method, since the former takes into account long-range effects as well as short-range effects. The long-range effects responsible for self-consistent field ($q = 0$) and quadrupole deformations ($q = 2$) may bring down some states below the energy gap, and thus the effective Δ may become smaller. This is precisely what is observed in Fig. 5, where effectively $v^2(g_{7/2})$ stays close to 1 for all force strengths.

The above argument implies, then, that the pairing effect may be well described by the BCS approximation, but the neglected part of the interactions should be treated by considering the residual terms. This problem is considered in a later section from a more general approach (such as general force, etc.). In the remaining part of this section, we briefly discuss qualitative features of that part of the pairing interaction which is neglected in BCS approximation but is included in the EDRH.

We have already pointed out that the gap equation resulting from BCS's approximation contains only a first-order effect. To include higher-order contributions, one may write the gap equation as Γ

$$\Delta_\alpha = -\sum_\beta s_\alpha s_\beta \langle \alpha\alpha|\Gamma|\beta\beta\rangle ,$$

where we have replaced $\bar{V}$ matrix by a Γ matrix which includes all the higher-order terms. Schrieffer considered the possibility of approximating Γ by a t matrix in the superconducting

system.[21] The physical significance of this procedure is not clear yet.

Let us do a simple perturbation calculation to see what effect the higher-order terms have. We assume for simplicity that $\bar{V}$ is replaced by a constant matrix element G.

Following Tolmachev and Tiablikov[22], we find, from the second-order "compensation equation,"

$$\langle 0|\eta_\nu \eta_{-\nu} H_{20}|0\rangle + \langle 0|\eta_\nu \eta_{-\nu} H_{31} \frac{1}{H_0 - E_0} H_{40}|0\rangle = 0 . \qquad \text{(II-50)}$$

an improved gap equation,

$$\Delta_\alpha = -G\sum_\beta u_\beta v_\beta + G^2 \sum_{\beta\gamma\delta} \frac{u_\beta v_\beta (u_\gamma v_\gamma u_\delta v_\delta - u_\gamma^2 v_\gamma^2)}{|\tilde{\epsilon}_\alpha| + |\tilde{\epsilon}_\beta| + |\tilde{\epsilon}_\gamma| + |\tilde{\epsilon}_\delta|} , \qquad \text{(II-51)}$$

and an energy formula in a self-consistent field,

$$\omega_\alpha = \tilde{\epsilon}_\alpha - G^2 \sum_{\beta\gamma\delta} \frac{[u_\beta^2 v_\gamma^2 - u_\beta u_\gamma v_\beta v_\gamma]\ (u_\delta^2 - v_\delta^2)}{|\tilde{\epsilon}_\alpha| + |\tilde{\epsilon}_\beta| + |\tilde{\epsilon}_\gamma| + |\tilde{\epsilon}_\delta|} . \qquad \text{(II-52)}$$

With Eqs. (51) and (52), the equation determining u and v is

$$2\omega_\alpha u_\alpha v_\alpha - \tilde{\Delta}_\alpha (u_\alpha^2 - v_\alpha^2) = 0 \qquad \text{(II-53)}$$

which yields

$$u_\alpha^2 = \frac{1}{2}\left[1 + \frac{\omega_\alpha}{\sqrt{\omega_\alpha^2 + \tilde{\Delta}_\alpha^2}}\right] \quad , \quad v_\alpha^2 = 1 - u_\alpha^2 \quad . \tag{II-54}$$

These constitute a complex set of nonlinear equations to be solved. We can, however, discuss qualitatively what changes occur without actually solving them.

Let us write Eq. (II-51) in a different way:

$$\tilde{\Delta}_\alpha = \sum_\beta \left(-G + G^2 \sum_{\gamma\,\delta} \frac{(u_\gamma v_\gamma u_\delta v_\delta - u_\gamma^2 v_\delta^2)}{|\tilde{\epsilon}_\alpha| + |\tilde{\epsilon}_\beta| + |\tilde{\epsilon}_\delta| + |\tilde{\epsilon}_\gamma|}\right) u_\beta v_\beta \quad . \tag{II-55}$$

Now, noticing that the term involving G^2 is much smaller than G, we may replace $|\tilde{\epsilon}_\beta|$ by some average value, thus removing the dependence on β. If we do so, we can redefine an effective interaction constant G_{eff}, and get

$$\tilde{\Delta}_\alpha = -G_{eff} \sum_\beta u_\beta v_\beta \quad , \tag{II-56}$$

$$G_{eff} = G + G^{(2)} = G - G^2 \sum_{\gamma\,\delta} \frac{(u_\gamma v_\gamma u_\delta v_\delta - u_\gamma^2 v_\delta^2)}{|\tilde{\epsilon}_\alpha| + |\langle\tilde{\epsilon}_\beta\rangle| + |\tilde{\epsilon}_\delta| + |\tilde{\epsilon}_\gamma|} \quad . \tag{II-57}$$

Thus the higher-order effect does not change the essential feature of the theory; it only renormalizes the effective coupling constant, with a small change from unrenormalized G. In fact if one puts into Eq. (II-57) the values obtained by the usual BCS approximation it turns out that in the region where the agreement of the BCS results with EDRH results is worse,

$$G^{(2)} \geq 0$$

Hence we can see that the inclusion of higher-order terms lowers the effective $(-G)$, and therefore Δ. On the other hand, as $(-G)$ goes to infinity, $v_\alpha = v_\beta$ for all α,β and hence $G^{(2)} \to 0$. Thus at very high G, the higher-order contribution vanishes (presumably to every order). This result is in complete agreement with the previous conclusions, and with Figs 2, 3, and 4.

C. Gap Equations and Self-Energy Term

Now that we have studied some features of the superconductivity theory and in particular the BCS approximation in the lowest order, let us see in detail the relevant equations in the representation we have chosen to adopt. We start with

$$\Delta_\alpha = -\frac{1}{2}\sum_\beta \langle \alpha\alpha|\bar{V}|\beta\beta\rangle \; u_\beta v_\beta s_\alpha s_\beta \qquad \text{(II-58a)}$$

and

$$\hat{\epsilon} = \epsilon_\beta + \sum_\gamma \langle \beta\gamma|\bar{U}|\beta\gamma\rangle \; v_\gamma^2 \quad . \qquad \text{(II-58b)}$$

In the completely general derivations of these equations, we are not forced to assume that only total-momentum-zero coupling contributes in the equation nor that the interaction has any cutoff parameters. The former feature has also been emphasized by Anderson et al. in the generalized BCS theory of the electron system;[23] they consider the possibility of coupling in the angular momentum $\ell \neq 0$. These considerations are motivated by the results of Brueckner et al. in liquid ^3H_e ; they find a condensation into an $\ell = 2$ state[24]. In our case, it can easily be seen that only $J = 0$ pairing is permissible, which will be shown shortly. First we rewrite Eqs. (II-58a) and (II-58b) in angular momentum representation, and consider the validity of the constant-matrix approximation in this representation, the relationship between the gap matrix element and the self-energy term. Then we shall show a simple identity in a special case.

Let us start by putting Eqs. (II-58a) and (II-58b) in j-j representation for use in spherical nuclei. It is convenient to write the matrix elements in an invariant form (invariant under rotation and reflection):

$$\sum_{\beta} s_\alpha s_\beta u_\beta v_\alpha \langle \alpha\text{-}\alpha | \bar{V} | \beta\text{-}\beta \rangle = \sum_{\beta J} s_\alpha s_\beta \, C(j_a j_a J \; ; \; m_a\text{-}m_a \; 0) \qquad \text{(II-59)}$$

$$\times \; C(j_b j_b J \; ; \; m_b\text{-}m_b 0) \; G \; (aabbJ) u_b v_b \quad .$$

Similarly,

$$\mu_a = -\sum_{\beta} \langle \alpha\beta | \bar{U} | \alpha\beta \rangle \; v_\beta^2 = -\sum_{\beta} \langle \alpha\beta | \bar{V} | \alpha\beta \rangle \; v_\beta^2$$

$$= -\sum_{j_b m_b} \sum_{JM} C(j_a j_b J \; ; \; m_a m_b M)\; C(j_a j_b J \; ; \; m_a m_b M)\; G(ababJ)\; v_b^2$$

$$= -\sum_{j_b J} \left(\frac{2J+1}{2j_a+1}\right) G(ababJ)\;\; v_b^2 \quad . \tag{II-60}$$

Here we have used the convenient notation of Baranger[1], and will continue to do so from now on:

$$G(abcdJ) = \langle abJM|\bar{V}|cdJM\rangle$$

where $$\langle abJM|\bar{V}|cdJM\rangle \;\; = \tag{II-61}$$

$$\langle abJM|V|cdJM\rangle - \Theta(cdJ)\; \langle abJM|V|dcJM\rangle \quad .$$

and $\langle abJM|V|cdJM\rangle$ is the usual shell-model two-body matrix element. (Note that the G used here is equivalent to $-2G$ as used by Baranger.)

In Eq. (II-59) the dependence on m_b is solely in the second C-G coefficient, and $s_\beta = (-)^{j_b - m_b}$. Since we have $C(j_b j_b 0 : m_b - m_b 0) = (-)^{j_b - m_b} (2j_b+1)^{\frac{1}{2}}$,

we may write

$$\sum_{m_b} s_\beta\; C(j_b j_b J \; ; \; m_b - m_b 0) = (2j_b+1)^{\frac{1}{2}} \sum_{mb} C(j_b j_b J \; ; \; m_b - m_b 0) \;\times$$
$$C(j_b j_b 0 \; ; \; m_b - m_b 0) = \delta(J,0)\; (2j_b+1)^{\frac{1}{2}} \quad . \tag{II-62}$$

Thus we have shown that only the $J = 0$ component contributes.

This result can be explained in slightly different language. In group theory, the combination $\sum_m \phi^j_m (-)^{j-m} \phi^j_{-m}$ transforms as a symplectic group, and the resultant representation has only $J = 0$. This corresponds to a seniority-zero state.

Substituting (II-59) into (II-58a), we obtain

$$\Delta_a = -\frac{1}{2}\sum_b \frac{(2j_b+1)^{\frac{1}{2}}}{(2j_a+1)^{\frac{1}{2}}} u_b v_b \, G(aabb0) \quad . \tag{II-63}$$

If we take a δ-function interaction, Eq. (II-63) is further reduced to

$$\Delta_a = -\frac{1}{2}\sum_b (j_b+\tfrac{1}{2}) \frac{F^0(ab)}{E_b}\Delta_b = -\sum_b (j_b+\tfrac{1}{2}) F^0(ab) u_b v_b \tag{II-64}$$

or $$\sum_{m_b} \langle \alpha\text{-}\alpha | \bar{V} | \beta\text{-}\beta \rangle s_\alpha s_\beta = (2j_b+1) F^0(ab) \quad ,$$

where $F^0(ab)$ is the radial integral defined by

$$F^0(ab) = -c\int_0^\infty dr \, r^2 R_a^2 R_b^2 \quad . \tag{II-65}$$

We have collected all the constants in C.

In the approximate nuclear calculations, one usually sets the radial matrix element to some average value; if we do so, $F^0 \to G$, $G < 0$ where G includes all the other factors such as A^{-1} dependence, etc.; then

$$\Delta_\alpha = -\frac{1}{2}G\sum_b (j_b+\frac{1}{2})\ \frac{\Delta_b}{E_b} \quad . \tag{II-66a}$$

Furthermore, $\Delta_a = \Delta_b = \Delta$.

Hence we obtain

$$-\ \frac{1}{2}\ G\sum_b\ (j_b+\frac{1}{2})E_b^{-1} = 1\ . \tag{II-66b}$$

Thus a δ-function type of interaction (which is a good approximation for the pairing type, since we believe it to be of short range) and some averaged radial integral (which is not well known anyhow) immediately lead to the constant-matrix-element equation.

Now we demonstrate that in the δ-function limit, the self-energy term becomes identical with the gap matrix element;i.e.,

$$\langle\alpha\text{-}\alpha|\bar{V}|\beta\text{-}\beta\rangle s_\alpha\ s_\beta = \langle\alpha\beta|\bar{U}|\alpha\beta\rangle \quad . \tag{II-67}$$

In δ-function approximation, we can write [from Eq. (II-47)]

$$G(ababJ) = (2j_a+1)(2j_b+1)\ C(j_a j_b J\ ;\ \tfrac{1}{2}\text{-}\tfrac{1}{2}\ 0)^2\ \frac{F^0(abab)}{(2J+1)} \tag{II-68}$$

If we substitute this into the equation

$$\sum_{m_b}\langle\alpha\beta|\bar{U}|\alpha\beta\rangle = \sum_J\ \frac{2J+1}{2j_a+1}\ G\ (ababJ) \tag{II-69}$$

and sum over J, we get immediately

$$\sum_{m_b} \langle \alpha\beta | \bar{U} | \alpha\beta \rangle = (2j_b+1)\, F_0(abab) \quad . \tag{II-70}$$

Therefore, comparing Eqs. (II-64) and (II-70), we see that Eq. (II-67) holds.

Physically the relation (II-67) does not have much meaning, since, as pointed out by Belyaev, we expect that although higher harmonics of the force ($k>2$) contribute to the pairing type of interaction, the self-consistent field effect should come from lower harmonics. This then implies that δ - force may not be proper to use for both cases. On the other hand, the same finite-range force may be employed for both, since the self-energy term picks out mainly the $k=0$ component, while the gap term receives the largest contribution from $k > 2$. In practical computations, this is probably the best procedure for obtaining consistency in the theory.

A final remark concerns the role of the self-energy term. If ϵ_α is theoretically calculable, then we are not justified in putting the self-energy term to zero. In such case, Eq. (II-58b) yields a Hartree-Fock energy, and together with the gap equation, a self-consistent equation. Suppose we choose ϵ_α's from an experimental result of a nucleus closest to a doubly magic nucleus; this set of ϵ_α's is precisely what one would obtain by the H-F equation. If this is well known, then the self-energy correction for other nuclei in the same region may amount to some sort of variation in the spacing of ϵ_α's as more particles are filled in the available shells. In such a case, an account of the μ term is perhaps necessary. But if one starts with ϵ_α as parameters, then μ correction does not have any significance, for we can choose ϵ_α's such that the self-energy correction

is implicitly taken care of. (For further discussion on this point, see Nilsson and Prior[25].)

In general, the self-energy term given by Eq. (II-60) depends largely upon the radial integral $R(abab)$. Now, if this function is not very sensitive to a change of states a and b, then we can approximately replace $\bar{G}(ababJ)$ by some function which depends somehow only on a ; thus

$$\mu_a = \xi(a) \sum_b (2j_b + 1)\, v_b^2 \quad . \tag{II-71}$$

Equation (II-71) in turn implies that the correction term is expected to be approximately same for all states, provided

$$\xi(a) \approx \xi(b) \approx \cdots$$

A numerical computation seems to bear out this point in practice. (For more details, see Sec. (IV.)

III. TREATMENT OF QUASI-PARTICLE INTERACTIONS

A. Preliminary Remarks

So far we have dealt with the picture of independent behavior of the quasi-particles. In so doing, we have neglected the residual terms completely; in Sec. IIB, we have, however, considered the effect of these terms on the ground-state properties. We have seen that this refinement leads to renormalization of the interaction matrix element, and change of u and v. For the excited states, however, mere modification of u's and v's does not explain some of the features we have discussed in the introduction, and which we shall treat in the main part of this work. This is so because the higher-order correction does not touch on the longer-range part of the interaction, and hence the important part of interaction is still left intact. From order-of-magnitude arguments, the first-order contribution of such terms could be given an upper limit of $\bar{G}/2$, where $\bar{G}$ is an average interaction strength, if we replace the complete Hamiltonian with a reduced one. When we consider this problem more carefully, we realize that the strongly collective effect observed in some of the nuclei should receive contributions from a wide range of terms and hence the first-order-perturbation estimate is not sufficient.

In the following subsections, we aim to study the effects of these residual terms by perturbation theory. The starting point is the independent quasi-particle behavior, which we take as zero order, and we sum suitable subsets of the higher-order perturbation terms. Closed forms are hard to obtain if we use a completely general force. Where possible, we make a somewhat drastic approximation (for the sake of qualitative analysis) to make contact with the results of KS[15] and of Baranger.[1]

To aid in understanding notations, we list the following definitions:

One-particle state: $|i\rangle = \eta_i^+ |0\rangle$. (III-1)

Two-particle vector-coupled state:

$$|(il)J\rangle = N(il) \sum_{m_i m_l} C(j_i j_l J;\ m_i m_l M)\ \eta_i^{+} \eta_l^{+} |\hat{0}\rangle \ . \qquad \text{(III-2)}$$

Three-particle vector-coupled state:

$$|(il)_I k; f\rangle = N(ilkIf) \sum_{m_i m_l m_k} \sum_{M_I} C(j_i j_l J_I;\ m_2 m_l M_I)$$

$$\times\ C(J_I j_k j_f;\ M_I m_k m_f)\ \eta_i^{+} \eta_l^{+} \eta_k^{+} |\hat{0}\rangle \ . \qquad \text{(III-3)}$$

Four-particle vector-coupled state:

$$|(ilkg);\ II'J\rangle = N(ilkg) \sum_{m_i m_l M_I} \sum_{m_k m_g M_I'} C(j_i j_l J_I;\ m_i m_l M_I)$$

$$\times\ C(j_k j_g J_I';\ m_k m_g M_I')\ C(J_I J_I' J;\ M_I M_I' M)\ \eta_i^{+} \eta_l^{+} \eta_k^{+} \eta_g^{+} |\hat{0}\rangle \qquad \text{(III-4)}$$

The N's are normalization constants of two-particle, three-particle, and four-particle states, respectively; $|\hat{0}\rangle$ is the usual quasi-particle vacuum; C is the Clebsch-Gordan coefficient. From now on, if ambiguity does not arise, we will usually omit j; for example,

$$C(ilJ;\ m_i m_l M) \equiv C(j_i j_l J;\ m_i m_l M),$$

$$W(ilkg;\ JJ') \equiv W(j_i j_l j_k j_g;\ JJ')$$

$$\langle ilJ|V|kgJ\rangle \equiv \langle j_i j_l J|V|j_k j_g J\rangle,$$

$$\langle il|V|kg\rangle \equiv \langle j_i m_i j_l m_l|V|j_k m_k j_g m_g\rangle$$

$$\eta_i \equiv \eta_{j_i m_i}$$

$$E_i \equiv E_{j_i} \qquad \text{etc.}$$

The states defined by (III-3) and (III-4) comprise a complete set of orthonormal states provided j_i, j_l, j_k and j_g are all different. If they are not all different, states with different intermediate angular momentum quantum numbers may not be orthogonal to each other. In such cases, we are forced to use the fractional-parentage coefficient technique. Consider n particles in j shell. Then a completely antisymmetric state with a correct orthonormality condition can be written as[26]

$$|n\alpha J\rangle = \frac{1}{\sqrt{n}} \sum_{J_I M_I \beta} \sum_{m} \langle j^{n-1}(\beta J_I) j\ J|\}\ j^n \alpha J\rangle$$

$$C(J_I jJ;\ M_I m M)\ \eta_{jm}^{+}\ |(n-1)\beta J_I\rangle\ ,$$

where $|n\alpha J\rangle$ is a state of n quasiparticles in the j shell with an extra label α; $\langle \| \rangle$ is a fractional-parentage coefficient introduced in a previous section. For $j \leq 7/2$, the label α is unnecessary. It is easy to see that a repeated application of the relation given above yields all the correct wave functions in terms of the creation operator η^+. For example, a three-quasi-particle state may be written as

$$|(j^3)j_f\rangle = \frac{1}{\sqrt{6}} \sum_{\substack{J_I M_I \beta \\ m_1 m_2 m_3}} C(J_I j j_f;\ M_I m_1 m_f)\ C(jjJ_I;\ m_2 m_3 M_I)$$

$$\langle j^2(\beta J_I) j j_f \| j^3 j_f\rangle \times \eta_j^{+}{}_{m_1}\ \eta_j^{+}{}_{m_2}\ \eta_j^{+}{}_{m_3}|0\rangle\ ,$$

which is in the same form as (III-3) except for the sum over the intermediate states. Since we eventually sum over the intermediate states in the discussions given in the subsequent sections, it is clear that we could use the forms of (III-3) and (III-4) with the understanding that whenever we have more than two particles in the same j shell, we eliminate redundant states, using the prescription given above and the seniority scheme. This procedure is equivalent to redefining the normalization constant in the final result.

In our discussion, as we shall see in Sec. (IV), only $j \leq 7/2$ states are of importance, and hence in case more than two particles occupy one j shell, there is only one independent state[20] and it is easy to choose the correct state. For example, the $[g_{7/2}^4]_{J=0}$ configuration has only the seniority zero state, and use of the prescription shows that $[(g_{7/2}^2)_{J=0}(g_{7/2}^2)_{J=0}]_0$ is the allowed and correct state. It is easy to show that $[(g_{7/2}^2)_J(g_{7/2}^2)_J]_0$ for J=0,2,4 are all equivalent, and thus it is a trivial problem to correctly include the contribution from such configuration. On the other hand, $[(h_{11/2}^4)]_0$ has two independent states (seniority zero and four), and can be easily taken into account in the manner discussed.

In the following subsections, we do not make any distinction between the states with more than two particles in the same j shell and the states with all the particles in different shells. One should understand, however, that whenever the complication discussed above arises, the redundant states are eliminated by choosing the appropriate normalization constant ($j \leqslant 7/2$) and proper states ($j > 7/2$).

B. Effect of H_{22}: Energy Level of Two-Quasi-Particle State

The explicit form of H_{22} was given in Eq. (II-13). We understand this sort of term fairly well from the knowledge obtained in ordinary particle representation. This interaction conserves the number of particles, and describes the scattering of two quasi particles. On can see that in the limit $u \to 1$, $v \to 0$ (particle state) or $v \to 1$, $u \to 0$ (hole state), H_{22} reduces to the usual two-body operator, while the rest of the H_I terms vanish. Obviously this term gives the largest contribution to the even-particle system. If we neglect the rest of the terms, the Hamiltonian has the form

$$H = H_0 + H_{22} \ . \tag{III-5}$$

Let us first consider the even-particle system. Taking the basis function to be the two-particle state, we have

$$H_0 |(il)I\rangle = (E_i + E_l) |(il)I\rangle \ . \tag{III-6}$$

Now expand the wave function in the basis function, (III-2).

$$\Psi_{\alpha I} = \sum_{i \leqslant l} C^{\alpha}_{ilI} |(il)I\rangle \ . \tag{III-7a}$$

Let us use the definition

$$\sum_{(il)} \equiv \sum_{i \leq l}$$

Then $H\Psi \equiv \omega\Psi$ leads to

$$(H_0 + H_{22}) \sum_{(il)} C^{\alpha}_{ilI} |(il)I\rangle = \sum_{(il)} (E_i + E_l)\, C^{\alpha}_{ilI} |(il)I\rangle$$

$$+ \sum_{(il)} C^{\alpha}_{ilI} H_{22} |(il)I\rangle = \omega(\alpha) \sum_{(il)} C^{\alpha}_{ilI} |(il)I\rangle \,, \qquad \text{(III-7b)}$$

or

$$(E_k + E_g)\, C^{\alpha}_{kgI} + \sum_{(il)} C^{\alpha}_{ilI} \langle (kg)I | H_{22} | (il)I\rangle = \omega(\alpha)\, C^{\alpha}_{kgI} \,.$$

Moving the right-hand side to the left, we have

$$C^{\alpha}_{kgI} = \frac{1}{\omega(\alpha) - (E_k + E_g)} \sum_{(il)} \langle (kg)I | H_{22} | (il)I\rangle\, C^{\alpha}_{ilI} \,. \qquad \text{(III-8)}$$

Equation (III-8) is a homogeneous linear integral equation, which has a solution only if the determinant vanishes. It has a closed-form solution only if the H_{22} matrix element is separable. Let us assume

$$\langle (kg)I | H_{22} | (il)I\rangle \approx -\lambda\, D(kgI)\, D(ilI), \qquad \text{(III-9)}$$

where we let D include normalization constant.

Then Eq. (III-8) can be written as

$$C^{\alpha}_{kgI} = \frac{\lambda\, D(kgI)}{(E_k + E_g) - \bar{\omega}(\alpha)} \sum_{(il)} D(ilI)\, C^{\alpha}_{ilI} \,. \qquad \text{(III-10)}$$

The solution for C^{α}_{kgI} is easily found. Since the normalization is defined as

$$\sum_{(kg)} C^{\alpha *}_{kgI}\, C^{\alpha}_{kgI} = \sum_{(kg)} (C^{\alpha}_{kgI})^2 = 1 \,, \qquad \text{(III-11)}$$

we have

$$\sum_{(kg)} (C^{\alpha}_{kgI})^2 = \sum_{(kg)} \left(\frac{\lambda\, D(kgI)}{E_k + E_g - \bar{\omega}(\alpha)} \right)^2 \left(\sum_{(il)} D(ilI)\, C^{\alpha}_{ilI} \right)^2 .$$

Therefore

$$\sum_{(il)} D(ilI)\, C^{\alpha}_{ilI} = \lambda^{-1} \left(\sum_{(kg)} \frac{D^2(kgI)}{(E_k + E_g - \bar{\omega}(\alpha))^2} \right)^{-1/2} .$$

Hence

$$C^{\alpha}_{kgI} = \frac{D(kgI)}{E_k + E_g - \bar{\omega}(\alpha)} \sum_{(il)} \left(\frac{D^2(ilI)}{(E_k + E_l - \bar{\omega}(\alpha))^2} \right)^{-1/2} . \qquad \text{(III-12)}$$

From Eq. (III-10) we obtain a dispersion formula[27] which is of the form close to the dispersion formula obtained by RPA (see Baranger):

$$\sum_{(kg)} \frac{D^2(kgI)}{E_k + E_g - \bar{\omega}(\alpha)} = \lambda^{-1} . \qquad \text{(III-13)}$$

The dispersion formula from the RPA is

$$\sum_{(kg)} \frac{2D^2(kgI)(E_k + E_g)}{(E_k + E_g)^2 - \bar{\omega}(\alpha)^2} = \lambda^{-1} \qquad \text{(III-14)}$$

or $$\sum_{(kg)} \frac{D^2(kgI)}{E_k + E_g - \bar{\omega}} + \sum_{(kg)} \frac{D^2(kgI)}{E_k + E_g + \bar{\omega}} = \lambda^{-1} .$$

To compare (III-13) and (III-14) let us consider two functions,

$$f_1(\bar{\omega}) = \sum_{(kg)} \frac{D^2(kgI)}{E_k + E_g - \bar{\omega}} \qquad \text{(III-15)}$$

and $$f_2(\bar{\omega}) = \sum_{(kg)} \frac{2D^2(kgI)}{(E_k + E_g - \bar{\omega})} \frac{E_k + E_g}{E_k + E_g + \bar{\omega}} .$$

For a given $\bar{\omega}$ with $(E_k + E_g)_{min} > \bar{\omega} > 0$, where $(E_k + E_g)_{min}$ denotes the lowest of $(E_k + E_g)$ for all k and g, then $f_2(\bar{\omega}) \geq f_1(\bar{\omega}) \geq 0$. This implies that the solutions of Eqs. (III-13) and (III-14) (denoted respectively by $\bar{\omega}_1$ and $\bar{\omega}_2$) have the inequality $\bar{\omega}_1 \geq \bar{\omega}_2$ for small $\bar{\omega}$. Thus extreme collective states cannot be described by the H_{22} term alone. The lower the state to be computed, the less correct the solution would be. However, as ω approaches $(E_k + E_g)$, $\frac{2(E_k + E_g)}{E_k + E_g + \omega} \approx 1$ and the two methods become

equivalent. For $(E_k + E_g)_{max} > \omega > (E_k + E_g)_{min}$ no unambiguous conclusions can be drawn; yet higher-lying states are expected to be described well by Eq. (III-13).

The foregoing discussion is rather too schematic. For a force more general than separable, the picture is not so simple. In the remaining part of this section, we work with a general force. The starting point is Eq. (III-8), and we first derive the H_{22} matrix element between two-quasi-particle states. Let the initial and final states be $|(kf)I\rangle$ and $|(il)I\rangle$ respectively. Taking the matix element between these states, we have

$$\langle (il)I|H_{22}|(kf)I\rangle = 1/4 \sum_{\alpha\beta\gamma\delta} \sum_{m_i m_l m_k m_f} N(il)N(kf)$$

$$\{\langle \alpha\beta|\bar{v}|\gamma\delta\rangle \ (u_\alpha u_\beta u_\gamma u_\delta + v_\alpha v_\beta v_\gamma v_\delta)$$

$$-4\langle \alpha-\gamma|\bar{v}|\ \beta\delta\rangle \ s_\beta s_\gamma u_\alpha u_\delta v_\beta v_\gamma\} \ C(ilI;\ m_i m_l M_I)\ C(kfI;\ m_k m_f M_I)$$

$$\times \langle \eta_l \eta_i \eta_\alpha^+ \eta_\beta^+ \eta_\delta \eta_\gamma \eta_k^+ \eta_f^+ \rangle_0 \qquad \text{(III-16)}$$

Here $\langle\ \rangle_0$ denotes vacuum expectation value.[17] Noting the relation

$$\langle \eta_l \eta_i \eta_\alpha^+ \eta_\beta^+ \eta_\delta \eta_\gamma \eta_k^+ \eta_f^+ \rangle_0 = \langle \eta_l \eta_i \eta_\alpha^+ \eta_\beta^+ \rangle_0 \langle \eta_\delta \eta_\gamma \eta_k^+ \eta_f^+ \rangle_0 \qquad \text{(III-16a)}$$

and

$$\langle \eta_l \eta_i \eta_\alpha^{\pm} \eta_\beta^{+} \rangle_0 = \delta(i\alpha)\delta(l\beta) - \delta(i\beta)\delta(l\alpha) \qquad \text{(III-16b)}$$

we have the expansion of Eq. (III-16a):

$$\langle\ \rangle_0 = \delta(i\alpha)\ \delta(l\beta)\ \delta(k\gamma)\ \delta(f\delta) + \delta(i\beta)\ \delta(l\alpha)\ \delta(k\delta)\ \delta(f\gamma)$$

$$- \delta(i\alpha)\ \delta(l\beta)\ \delta(k\delta)\ \delta(f\gamma) - \delta(i\beta)\ \delta(l\alpha)\ \delta(k\gamma)\ \delta(f\delta)\ . \qquad \text{(III-17)}$$

Substituting (III-17) into (III-16), we can immediately sum over α,β,γ and δ to obtain

$$\langle (il)I | H_{22} | (kf)I \rangle = -\sum_{m_i m_l} \sum_{m_k m_f} N(il)N(kf)\ C(ilI;\ m_i m_l M_I)\ C(kfI;\ m_k m_f M_I)$$

$$\times \left[\langle li | \bar{V} | kf \rangle\ W(likf) - \langle l\text{-}k | \bar{V} | i\text{-}f \rangle\ s_i s_k\ (\chi(likf) + \chi(ilfk)) \right.$$

$$\left. + \langle l\text{-}f | \bar{V} | i\text{-}k \rangle\ s_i s_f\ (\chi(lifk) + \chi(ilkf)) \right] , \qquad \text{(III-18)}$$

where we defined

$$W(likf) = u_l u_i u_k u_f + v_l v_i v_k v_f$$

and

$$\chi(likf) = u_l v_i v_k u_f\ . \qquad \text{(III-19)}$$

In deriving Eq. (18), we have used the symmetry relations given in Eqs. (II-4), (II-5), and (II-6). Let us here define two new functions G and F as

$$\langle \alpha\beta | \bar{V} | \gamma\delta \rangle = \langle \alpha\beta | V | \gamma\delta \rangle - \langle \alpha\beta | V | \delta\gamma \rangle$$

$$= \sum_{JM} C(abJ;\ m_a m_b M)\ \{C(cdJ;\ m_c m_d M)\ \langle abJ | V | CdJ \rangle$$

$$- C(cdJ;\ m_c m_d M)\ \langle abJ | V | CdJ \rangle\}$$

$$= \sum_{JM} C(abJ;\ m_a m_b M)\ C(cdJ;\ m_c m_d M)\ G(abcdJ) \qquad \text{(III-20)}$$

and

$$F(abcdJ) = -\sum_{J'} (2J' + 1)\ W(adbc;\ J'J)\ G(dabcJ') \ . \qquad \text{(III-21)}$$

From symmetry properties of $\langle \alpha\beta | \bar{V} | \gamma\delta \rangle$ (real, symmetric), follow the relations

$$G(abcdJ) = -\theta(abJ)\ G(bacdJ) = -\theta(cdJ)\ G(abdcJ)$$

$$= \theta(abcd)\ G(badcJ) = G(cdabJ), \qquad \text{(III-22)}$$

from which we observe

$$F(abcdJ) = F(cdabJ') = \theta(abcd)\ F(badcJ') \ . \qquad \text{(III-23)}$$

With (III-20) and (III-21), Eq. (III-18) finally becomes

$$\langle (il)I|H_{22}|(kf)I\rangle = N(il)\,N(kf)\,[(u_i u_l u_k u_f + v_i v_l v_k v_f)\,G(ilkfJ)$$

$$+ (u_l u_f v_i v_k + u_i u_k v_l v_f)\,F(ilkfJ) - \theta(kfJ)(u_l u_k v_i v_f$$

$$+ u_i u_f v_l v_k)\,F(ilfkJ)]\ . \qquad \text{(III-24)}$$

Note that the first term corresponds to particle-particle interaction, the second and last to particle-hole interaction.

For a given form of interaction force, G and F can in principle be calculated; we write G and F explicitly for a Gaussian type of interaction in Appendix B. Here we shall be content with just writing down the results for the sake of completeness in our discussion. Confining ourselves to only one kind of particle (protons or neutrons only), we decompose the term into a spin-independent part and a spin-dependent part by

$$\langle (il)I|H_{22}|(kf)I\rangle = M^{(0)} + M^{(\sigma)}\ ,$$

$$M^{(0)} = N(il)\,N(kf)\,[(2j_i+1)(2j_l+1)]^{1/2}\,v_0\,a\,\theta(j_i j_l I)$$

$$\times \sum_q \Big[(u_i u_k - \theta(q) v_i v_k)(u_l u_f - \theta(q) v_l v_f)\,C(q j_l j_f;\ 0\ \tfrac{1}{2}\ \tfrac{1}{2})$$

$$C(q j_i j_k;\ 0\ \tfrac{1}{2}\ \tfrac{1}{2})$$

$$\times\ W(j_i j_l j_k j_f;\ Iq)\ R_q(ilkf) + \theta(J)(u_i u_f - \theta(q)v_i v_f)(u_l u_k - \theta(q)v_l v_k)$$

$$\times\ C(q j_l j_k;\ 0\ \tfrac{1}{2}\ \tfrac{1}{2})\ C(q j_i j_f;\ 0\ \tfrac{1}{2}\ \tfrac{1}{2})\ W(j_i j_l j_f j_k;\ Iq)\ R_q(ilfk) + \theta(j_l j_f)$$

$$\times \left(\frac{2j_k+1}{2j_i+1}\right)1/2\ \ \frac{\delta(q,I)}{2q+1}\ \ (u_f v_k + \theta(q)u_k v_f)(u_l v_i + \theta(q)u_i v_l)$$

$$\times\ C(q j_l j_i;\ 0\ \tfrac{1}{2}\ \tfrac{1}{2})\ C(q j_k j_f;\ 0\ \tfrac{1}{2}\ \tfrac{1}{2})\ R_q(lfik)\Big] \qquad \text{(III-25)}$$

and

$$M^{(\sigma)} = +\,2\,N(il)\,N(kf)\,v_0\,b\,\lambda\,\theta(j_i j_k I)\sum_q \Big((u_i u_k + \theta(q)v_i v_k)(u_l u_f$$

$$+\ v_l v_f \theta(q))\ C(q\ell_l\ell_f;\ 000)\ C(q\ell_i\ell_k;\ 000)\ (Z_1 + \tfrac{1}{2}\,Z_1^{(0)})\ R_q(ilkf)$$

$$+\ \theta(I)\ (u_i u_f + \theta(q)v_i v_f)(u_l u_k + \theta(q)v_l v_k)\ C(q\ell_l\ell_f;\ 000)\ C(q\ell_i\ell_k;\ 000)$$

$$\times\ (Z_2 + \tfrac{1}{2}Z_2^{(0)})\ R_q(ilfk) + 3\theta(j_i j_k I)(u_l v_i - \theta(q)u_i v_l)(u_f v_k - \theta(q)u_k v_f)$$

$$\times\ C(q\ell_i\ell_l;\ 000)\ C(q\ell_k\ell_f;\ 000)\ Z_3\ R_q(lfik)\Big), \qquad \text{(III-26)}$$

where

$$R_q(ilkf) = \iint Ri^*(r_1)R_l^*(r_2)\ V_q(\vec{r}_1,\vec{r}_2)\ R_k(r_1)\ R_f(r_2)\ r_1^2 r_2^2\ dr_1 dr_2,$$

$$\lambda = [(2j_i+1)(2j_i+1)(2j_k+1)(2j_f+1)(2\ell_i+1)(2\ell_l+1)]^{1/2} ,$$

$$Z_1 = \theta(\ell_i\ell_l)\ W(j_i j_l \ell_i \ell_l;\ I\ \tfrac{1}{2})\ W(j_k j_f \ell_k \ell_f;\ I\ \tfrac{1}{2})\ W(\ell_i \ell_l \ell_k \ell_f;\ I\ \tfrac{1}{2})$$

$$Z_1^{(0)} = \theta(\ell_i\ell_l)\ W(j_i j_l j_k j_f;\ Iq)\ W(j_l j_f \ell_l \ell_f;\ q\ \tfrac{1}{2})\ W(j_i j_k \ell_i \ell_k;\ q\ \tfrac{1}{2}) ,$$

$$Z_2 = \theta(\ell_i\ell_l)\ W(j_i j_l \ell_i \ell_l;\ I\ \tfrac{1}{2})\ W\ (j_k j_f \ell_k \ell_f;\ I\ \tfrac{1}{2})\ W(\ell_i \ell_l \ell_f \ell_k;\ I\ \tfrac{1}{2})$$

$$Z_2^{(0)} = \theta(\ell_i\ell_l)\ W(j_i j_l j_f j_k;\ I\ \tfrac{1}{2})\ W(j_l j_k \ell_l \ell_k;\ q\ \tfrac{1}{2})\ W(j_i j_f \ell_i \ell_f;\ q\ \tfrac{1}{2}),$$

$$Z_3 = \left(\frac{2\ell_k+1}{2\ell_l+1}\right)^{1/2} \begin{Bmatrix} 1 & q & I \\ \frac{1}{2} & \ell_i & j_i \\ \frac{1}{2} & \ell_l & j_l \end{Bmatrix} \begin{Bmatrix} 1 & q & I \\ \frac{1}{2} & \ell_k & j_k \\ \frac{1}{2} & \ell_f & j_f \end{Bmatrix} , \qquad \text{(III-27)}$$

and a and b are coefficients of spin-independent and spin-dependent parts respectively, and v_0 is the depth of the potential ($v_0<0$).

Let us now consider what is the meaning of separability of the interaction which we have used in deriving the dispersion formula Eq. (III-13). Consider Eq. (III-24). A general-force matrix element involves a radial integral which is not in general separable. Let us assume, however, that the radial part is of the form

$$v_q(r_1,r_2) = \phi_q(r_1)\ \phi_q(r_2) . \qquad \text{(III-28)}$$

which was used by Elliott[28] in introducing the quadrupole-quadrupole

interaction. If we specialize, in particular, in a long-range component of the force, the first term in (III-24) can be neglected, since - it being a particle-particle matrix element — is much smaller than the rest of the terms. Then the second and third terms contain a large term which is completely separable for a single q (say $q = 2$), as can be seen from the last term of Eqs. (III-26) and (III-27). In this case, we can write (neglect the M^{σ} part)

$$\langle (il)I | H_{22} | (kf)I \rangle \approx - \lambda D(ilI)\, D(kfI), \qquad \text{(III-29)}$$

where

$$D(ilI) = \frac{(-)^{j_1+1/2}}{2I+1} C(j_i j_1 I;\ \tfrac{1}{2} - \tfrac{1}{2}\ 0)[(2j_i+1)(2j_1+1)]^{1/2} (u_i v_1 + v_i u_1)$$

$$R_I(il)\ N(il). \qquad \text{(III-30)}$$

Here

$$R_I(il) = \int_0^{\infty} dr \phi_I(r) r_1^{\,2}\ R_i(r)\ R_1(r)$$

with some constant λ. With a Q-Q force of the form ($q=2$)

$$V(\vec{r}_1, \vec{r}_2) = - \frac{5}{4\pi} \chi\, r_1^{\,2} r_2^{\,2}\ Y_2 \cdot Y_2\ , \qquad \text{(III-31)}$$

we have

$$D(i12) = (-)^{j_1+1/2} \left(\frac{\chi}{20\pi}\right)^{1/2} N(i1)\, R_2(i1)\, [(2j_i+1)(2j_1+1)]^{1/2}\, C(j_i j_1 2; \tfrac{1}{2}-\tfrac{1}{2}\ 0), \tag{III-32}$$

with

$$R_2(i1) = \int_0^\infty r^4\, R_i(r)\, R_1(r)\, dr\ .$$

We have seen that only a part of the matrix element is separable and that the matrix element contains particle-particle and particle-hole type interactions whether it be a general force or Q-Q force. Thus picking out a single dominant component corresponds to the separable interactions. For a long-range interaction, this approximation turns out to be rather good, as was shown by KS.[15] However, in a real nucleus, the discarded terms are not in fact negligibly small, and are by no means easy to estimate in general, for they vary from one nucleus to the other. In a subsequent section, we will apply the general formulas derived to calculate excited states in some even-even nuclei.

C. Matrix Elements of Single-Particle Operators

Let us write the wave function obtained from solving the eigenvalue equation (III-8) with the matrix element given by Eqs. (III-25) and (III-26) as

$$\Psi_{\alpha J} = \sum_{(i1)} C^{\alpha}_{i1J} |(i1)J\rangle\ . \tag{III-33}$$

We shall drive equations for $B(E2;\ J \rightarrow J')$ and the magnetic g factor of an excited state of two quasiparticles.

1. Electric Multipole Transition

By definition,

$$B(E\lambda) = \sum_{M_f \mu} |\langle \Psi_f | \mathcal{M}(E\lambda,\mu) | \Psi_i \rangle|^2 , \qquad \text{(III-34)}$$

where

$\mathcal{M}(E\lambda,\mu)$ for $\lambda = 2$ is

$$\mathcal{M}(E2,\mu) = \int r^2 Y_{2\mu}\ (\theta,\phi)\ \rho(r)\ d^3r$$

$$= e \sum_{m_s} \int r^2 Y_{2\mu}\ (\theta,\phi) \phi^*(\vec{r},m_s)\ \phi(\vec{r}_1 m_s) d^3r \qquad \text{(III-35)}$$

Letting

$$\phi(\vec{r}_1 m_s) = \sum_{\alpha} \langle \vec{r} m_s | \alpha \rangle\ C_\alpha ,$$

we have

$$\mathcal{M}(E2,\mu) = e \sum_{m_s} \sum_{\alpha\beta} \int r^2 Y_{2\mu}(\theta,\phi)\ \langle \alpha | \vec{r} m_s \rangle^* \langle \vec{r} m_s | \beta \rangle\ C_\alpha^+ C_\beta\ d^3r$$

$$= e \sum_{\alpha\beta} \langle \alpha | r^2 Y_{2\mu}\ (\theta,\phi) | \beta \rangle\ C_\alpha^+ C_\beta \qquad \text{(III-36)}$$

After the Bogolyubov transformation, the relevant operator is

$$\mathcal{M}(E2,\mu) = e\sum_{\alpha\beta} \langle\alpha|r^2Y_{2\mu}|\beta\rangle \left[(u_\alpha u_\beta - v_\alpha v_\beta)\ \eta^+_\alpha\eta_\beta + \right.$$

$$\left. s_\beta u_\alpha v_\beta\ \eta^+_\alpha\eta^+_{-\beta} + s_\alpha u_\beta v_\alpha\ \eta_{-\alpha}\eta_\beta \right] . \qquad \text{(III-37)}$$

Obviously the first operator connects states with the same number of quasi-particles, and the second and third are for transitions between the states differing in number by two units. We consider two cases:

(a) Transitions between two-quasi-particle states

Using the wave function (III-33) and the first operator of (III-37), we obtain

$$\langle\Psi_{FJ'}|\mathcal{M}(E2,\mu)|\Psi_{IJ}\rangle = e\sum_{(il)}\sum_{(i'l')} C^F_{i'l'J'}C^F_{ilJ}\ \langle(i'l')|\mathcal{M}(E2,\mu)|(il)J\rangle$$

$$= e\sum_{(il)(i'l')} N(il)N(i'l')\ C^F_{i'l'J'}C^I_{ilJ}C(j_ij_lJ;\ m_im_lM)\ C(ji'jl'J';\ m_i'm_l'M')$$

$$\langle\alpha|r^2Y_{2\mu}|\beta\rangle\ (u_\alpha u_\beta - v_\alpha v_\beta)\ \langle\eta_i'\eta_l'\eta_\beta\eta_i^+\eta_l^+\rangle_0$$

$$= e\sum_{(il)(i'l')} N(il)N(i'l')\ C^F_{i'l'J'}C^I_{ilJ}\sum_{\text{all } m's} C(j_ij_lJ;\ m_im_lM)\Big[C(j_ij_l'J';m_im_l'M')$$

$$\langle l'|r^2Y_{2\mu}|l\rangle(u_l'u_l - v_l'v_l)\ \delta(i,i') + C(j_i'j_lJ';m_i'm_lM')\ \langle i'|r^2Y_{2\mu}|i\rangle$$

$$(u_i'u_i - v_i'v_i)\,\delta(l,l') - C(j_i'j_iJ';m_i'm_iM')\,\langle i'|r^2Y_{2\mu}|l\rangle\,(u_i'u_l - v_i'v_l)$$

$$\delta(i,l') - C(j_lj_l'J';\, m_lm_l'M')\,\langle l'|r^2Y_{2\mu}|i\rangle\,(u_l'u_i - v_l'v_i)\,\delta(i',l)\Big] \quad . \tag{III-38}$$

We can eliminate the sum over m's by the usual Racah algebra, and when we finish a straightforward computation, we finally get

$$B(E2;\, J \to J') = e^2(2J'+1)\,\Big|\sum_{(il)(i'l')} \theta(j_ij_l')\,N(il)N(i'l')\,C^F_{i'l'J'}\,C^I_{ilJ}$$

$$\Big[Q(l'l)(u_l,u_l-v_l,v_l)\,W(JJ'j_lj_l';2j_i)\,\delta(j_i,j_i') + Q(i'i)(u_i'u_i-v_i'v_i)$$

$$W(JJ'j_ij_i';2j_l)\,\delta(j_l,j_l') + Q(i'l)(u_i'u_l-v_i'v_l)\,W(JJ'j_lj_i';2j_i)\,\delta(j_i,j_l')$$

$$+Q(l'i)(u_l'u_i-v_l'v_i)\,W(JJ'j_ij_l';\,2j_l)\delta(j_i',j_l)\Big]\Big|^2 ,$$

where

$$Q(il) = \langle i\|r^2Y_2\|l\rangle \quad . \tag{III-39}$$

We have left one sum undone in this formula since, as it stands, it is more convenient for use in practical calculations. If we relax the restrictions on the sum by the replacement

$$\sum_{(il)(i'l')} \to 1/4 \sum_{ili'l'} ,$$

then formally Eq. (III-39) may be reduced to

$$B(E2) = e^2(2J'+1)\Big| \sum_{ili'l'} \theta(j_i j_l')\, N(il)\, N(i'l')\, C^F_{i'l'J'}\, C^I_{ilJ}$$

$$\langle i' \| r^2 Y_2 \| l \rangle\, (u_i' u_l - v_i' v_l)\, W(JJ' j_l j_i';\, 2j_i)\, \delta(j_i, j_i') \Big|^2 . \tag{III-40}$$

Substituting the coefficients C^{α}_{ilJ} (supposing we obtain solutions by a separable force), we get

$$B(E2;J \to J') = e^2(2J'+1)\left[\sum_{(il)} \frac{D^2(ilJ)}{(E_i+E_l-\omega(J))^2}\right]^{-1} \times \left[\sum_{(il)} \frac{D^2(ilJ')}{(E_i+E_l-\omega(J'))^2}\right]^{-1}$$

$$\times \left(\sum_{ili'l'} N(il)N(i'l') \frac{D(ilJ)\, D(i'l'J')}{(E_i+E_l-\omega(J))(E_i'+E_l'-\omega(J'))}\, Q(i'l')(u_i' u_l - v_i' v_l) \right.$$

$$\left. W(JJ' j_l j_i';\, 2j_i)\, \delta(j_i, j_i') \right)^2 . \tag{III-41}$$

Note that this quantity is greatly lessened by the (uu-vv) factor, the absolute value of which is much less than unity if pairing is important (if no pairing, the factor goes to unity), and also by the Racah coefficients.* Thus the greater the pairing effect, the smaller the B(E2)

*[Strictly speaking, the absolute value of the (uu-vv) factor is not always less than unity if pairing is important. The contributions from levels far away from λ will have (uu-vv) $\approx$ 1. But then the amplitude of mixing (C^{α}_{ilJ}) will be very small. Perhaps we should say (uu-vv)C is small.]

becomes. We would then expect that the electric quadrupole transition between low-lying two-quasi-particle states is highly retarded, as is experimentally observed (for example, $4+ \rightarrow 2+$ transition in Ce^{140}). We shall see this clearly in the numerical calculations presented in a later section.

(b) Transition from two-quasi-particle states to the ground state

If the ground state is considered to be a quasi-particle vacuum, then transition to the ground state from an excited two-particle state can be obtained from the term with two annihilation operators, i.e.,

$$\mathcal{M}(E2,\mu) = e \sum_{\alpha\beta} \langle \alpha | r^2 Y_{2\mu} | \beta \rangle \; s_\alpha v_\alpha u_\beta \; \eta_{-\alpha} \eta_\beta \; . \qquad \text{(III-42)}$$

In a way analogous to the derivation of Eq. (III-39), we can easily arrive at

$$\langle 0 | \mathcal{M}(E2,\mu) | \Psi \rangle = (-)^{M_J+1} \; e \, \frac{N(il)}{(2J+1)^{1/2}} \; \delta(J,2) \; \delta(\mu,-M_I)$$

$$\times \, Q(il) \; (u_i v_l + v_i u_l) \; . \qquad \text{(III-43)}$$

[Note that the $J \rightarrow 0+$ transition for $J \neq 2+$ is forbidden according to Eq. (III-43).]

Hence

$$B(E2;\; 2+ \rightarrow 0+) = e^2 \left(\sum_{(il)} N(il) \; C_{il2} \; Q(il) \; (u_i v_l + v_i u_l) \right)^2 \; . \qquad \text{(III-44)}$$

In comparison with B(E2; $J \rightarrow J'$) this quantity is large, since $(u_i v_l + v_i u_l)$ is of order unity, and the absence of the Racah coefficient would make this quantity much larger than the former. Furthermore, we notice the following features. For a J, we can choose the phase and configurations such that the solution C^{α}_{kgJ} of

$$(E_k + E_g)\, C^{\alpha}_{kgJ} + \sum_{(il)} \langle (kg)J|H_{22}|(il)J\rangle\, C^{\alpha}_{ilJ} = \omega_L\, C^{\alpha}_{kgJ} \qquad \text{(III-45)}$$

for the lowest eigenvalue ω_L has the property of

$$-\theta(il)\, C^{\alpha}_{ilJ} > 0 \ .$$

Such two-quasi-particle basis vectors with single-particle states, for example, of 1 $g_{7/2}$, 2 $d_{5/2}$, 2 $d_{3/2}$ and 1 $h_{11/2}$ are $|d^2_{5/2}\rangle$, $|g^2_{7/2}\rangle$, $|d^2_{5/2}\rangle$, $|h^2_{11/2}\rangle$, $|d_{5/2}g_{7/2}\rangle$, $|d_{5/2}d_{3/2}\rangle$ and $|g_{7/2}d_{3/2}\rangle$ for even-parity states, and the phase convention used for the radial integral (the harmonic oscillator type) is that given by Morse and Feshbach.[29] We wish to investigate what the sign of $C^{\alpha}_{ilJ}\, Q(il)$ in Eq. (III-44) is. To do so, let us note

$$\langle n_i \ell_i|r^2|n_l \ell_l\rangle = (-)^{n_i - n_l}\, |\langle n_i \ell_i|r^2|n_l \ell_l\rangle| \ ,$$

where

$$n = \frac{N-\ell}{2} + 1$$

and

$$\mathrm{sgn}(\langle i|Y_2|1\rangle) = (-)^{\xi}$$

where

$$\xi = \delta(j_i, j_1-1) + \delta(j_i, j_1) \ .$$

This relation can be easily obtained if one examines the explicit form of the Clebsch-Gordan coefficient resulting from the reduced matrix element. We collect the signs arising from the various factors for the wave function of interest (i.e., the lowest 2+ state):

	C^{α}_{il2}	$\langle n_i \|r^2\| n_1\rangle$	$\langle i\|r^2 Y_2\|1\rangle$	$C^{\alpha}_{il2}\, Q(il)$
$d^2_{5/2}$	+	+	-	-
$g^2_{7/2}$	+	+	-	-
$d^2_{3/2}$	+	+	-	-
$h^2_{11/2}$	+	+	-	-
$d_{5/2}\ g_{7/2}$	-	-	-	-
$d_{5/2}\ d_{3/2}$	-	+	+	-
$g_{7/2}\ d_{3/2}$	+	-	+	-

Thus we obtain the same signs for the factor $C^{\alpha}_{il2}\, Q(il)$, and since we have $(u_i v_1 + v_i u_1) > 0$ and $N(il) > 0$ for all i and 1, we get a coherent contribution to Eq. (III-44). In this respect, we expect the quantity B(E2) for 2+ to 0+ transition to be much larger than B(E2) single-particle strength.

Let us consider Eq. (III-44) in Q-Q approximation. We can write, in the notation introduced in Eq. (III-29),

$$B(E2;\ 2+ \rightarrow 0+) = e^2 \Big| \sum_{(il)} D(il2)\ C^{\alpha}_{il2} \Big|^2 . \qquad \text{(III-46)}$$

We see this if we recall $D(il2) = N(il)\ Q(il)\ (u_i v_l + v_i u_l)$. Now, using Eq. (III-12), we finally have

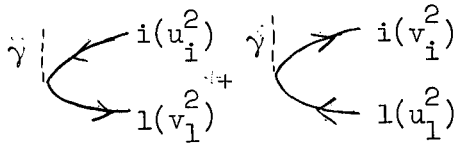
$$B(E2;\ 2+ \rightarrow 0+) = \frac{e^2}{\lambda^2} \Big(\sum_{(il)} \frac{D^2(il2)}{(E_i + E_l - \omega(2+))^2} \Big)^{-1} . \qquad \text{(III-47)}$$

Let us consider briefly the physical implication of Eq. (III-44) or Eq. (III-47). One can visualize the transition process better if we describe it in the following way. Consider the diagram

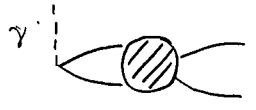
which describes a particle filling a hole, which results in emission of a γ quantum. (For the sake of convenience, we have drawn this and the following diagrams sideways. One should interpret them from right to left.) This process usually involves in the usual particle operators $C^+_p\ C_m$, where $p \in F$, $m \notin F$ (F=Fermi sea); i.e., $C^+_p\ C_m$ describes annihilation of a hole and a particle. This process in low-energy nuclear phenomena can be interpreted as a particle dropping into a hole in a partly filled shell.

If we examine Eq. (III-44), we find that this is precisely the kind of process we are considering. Recalling that a quasiparticle is partly a hole and partly a particle (with probability of v^2 and u^2 respectively), we see that the factor $(u_i v_l + v_i u_l)$ describes the same

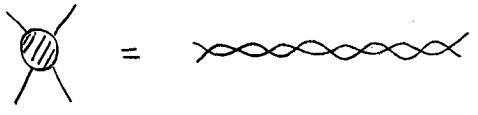
process as the diagram. In fact, we may depict it as a "direct" and an "exchange" term of the form



where the factor in parenthesis denotes the weight of hole and particle characters. Since we have effectively included all the two-particle interactions in Eqs. (III-44) and (III-47), our result then corresponds to the following diagram,



where



describes successive scattering between two particles.

2. Magnetic g Factor

In this section, we show that if we confine ourselves to a two-quasi-particle subspace, we get the same magnetic g factor as one would in the ordinary particle formulation. This, then, implies that our method corresponds merely to configuration-mixing calculation in the ordinary shell-model approach, as far as magnetic moment is concerned. Magnetic moment is defined by

$$\mu_J = Jg_J = \frac{\langle \hat{\mu}_{op} \cdot \vec{J} \rangle}{(J + 1)}$$

or
$$g_J = \frac{\langle \hat{\mu}_{op} \cdot \vec{J} \rangle}{J\,(J+1)} \quad , \tag{III-48}$$

which can be obtained from the relation

$$(\alpha jm' \,|\vec{\mu}|\, \alpha jm) = (jm' \,|j|\, jm)\, \frac{\langle \alpha jj \,|\vec{\mu} \cdot \vec{J}|\, \alpha jj \rangle}{j(j+1)} \tag{III-49}$$

In second quantized form in quasi-particle space the operator $\hat{\mu}_{op}$ is in the same form as the electric transition operator. However, in the absence of a different particle component, the u and v factors appear with an opposite sign i.e.,

$$\hat{\mu}_{op} = \sum_{\alpha\beta} \langle \alpha \,|\mu_{op}|\, \beta \rangle (u_\alpha u_\beta + v_\alpha v_\beta)\; \eta^+_\alpha \eta_\beta \; . \tag{III-50}$$

This is due to the time-reversal property of the $\hat{\mu}_{op}$ operator

$$T\, \hat{\mu}_{op}\, T^{-1} = -\, \hat{\mu}_{op} \; . \tag{III-51}$$

Hence

$$\langle \alpha \,|\mu_{op}|\, \beta \rangle = \langle T\alpha \,|T\mu_{op}\, T^{-1}|\, T\beta \rangle^* = -\langle -\beta \,|\mu_{op}|\, -\alpha \rangle \; . \tag{III-52}$$

If we take the specific form of the μ operator as

$$\mu_{op} = \sum_\gamma \; g_{j_\gamma}\, \vec{j}_\gamma$$

and $$\hat{\mu}_{op} = \sum_{\alpha\beta\gamma} \langle\alpha| g_{j_\gamma} \frac{\vec{j}}{j_\gamma} |\beta\rangle (u_\alpha u_\beta + v_\alpha v_\beta)\ \eta^+_\alpha\ \eta_\beta\ , \qquad \text{(III-53)}$$

and take the expectation value of $\hat{\mu}$ with the wave function,

$$\Psi_{\alpha J} = \sum_{(ik)} C^{\alpha}_{ikJ} |(ik)J\rangle$$

we see only diagonal terms appear. That is, if we put (III-53) into (III-48), we have the operator $\vec{j}_i \cdot \vec{J}$, and $\vec{j}_k \cdot \vec{J}$ etc. Now,

$$\vec{j}_i \cdot \vec{J} = 1/2\ (\vec{J}^2 + \vec{j}_i^{\,2} - \vec{j}_k^{\,2})\ .$$

Hence

$$\vec{j}_i \cdot \vec{J} = 1/2\ (J(J+1) + j_i(j_i+1) - j_k(j_k+1))\ . \qquad \text{(III-54)}$$

Equation (III-48) then immediately leads to

$$\mu_J = \frac{1}{2(J+1)} \sum_{(ik)} \left(C^{\alpha}_{ikJ}\right)^2 [(g_i+g_k)\ J(J+1) + (g_i-g_k)(j_i(j_i+1) - j_k(j_k+1))]\ , \qquad \text{(III-55)}$$

since $u_i^2 + v_i^2 = 1$ for all i.

The fact that two formalisms (qusi-particle and ordinary shell-model approaches) are equivalent arises specifically from the time-reversal property, Eq. (III-52), and diagonal character of the operator. The only difference comes, however, from the coefficients C^{α}_{ikJ}, which will

in general be different in two cases. It is well known in configuration-mixing calculations that a small admixture of other components can change the magnetic moment a great amount. Thus it is expected that the magnetic moment computed with our formula will differ from that calculated by the usual methods from just such an effect -- that is, the difference in the wave function.

D. Effect of H_{40}

We have so far discussed a number-conserving interaction. In such an interaction, the ground state remains as a quasi-particle vacuum. For the collective behavior, one knows that further correlations in the ground state are sometimes very important. In the particle representation, the quasi-particle vacuum is already a correlated state; it is precisely this feature that gives rise to the pairing effect. Experimental findings in nuclei show that the electric quadrupole transition probability [or equivalently the B(E2)] for the transition from the first 2+ to the ground state 0+ is usually much larger than single-particle estimates. As we shall see later, results [on B(E2)] are still somewhat unsatisfactory in many cases, even when many configuration mixtures are introduced to the excited state. This suggests that perhaps a ground-state correlation might be required to increase the theoretical values. This enhancement of electromagnetic transition probabilities due to the ground-state correlation is found in the RPA approach and also in the schematic perturbation calculation with an appropriate form of matrix elements involved. However, a perturbation calculation with a general force, which we present in Sec. IV, shows that the correlations are not appreciable in our case and that the effect appears to differ considerably in different regions.

If we examine the residual terms, we notice that the configuration mixing of the ground state with other states could be done only through the H_{40} term. The "new Tamm-Dancoff" method used in nuclei by Baranger

and others[1] effectively does include 4, 8, 12, etc. particle states in the ground state; thus the introduction of a new operator and a new quasi-particle-correlated vacuum by $Q_B|\Psi_0\rangle = 0$ then redefines the ground state (Q_B is an operator describing collective behavior. Thus $Q_B^+|\Psi_0\rangle = |B\rangle$ corresponds to the collective state denoted by B.) If desired, this chain of approximation obviously can continue, depending upon how good an approximation is desired. One can see that in such a way, we are badly violating number conservation, and furthermore the exclusion principle would become very important.

In single-closed-shell nuclei, the energy gap is about 1 MeV, and hence the lowest-order configuration mixing involves ≈ 4-MeV states. In perturbation theory, the ground state undergoes an energy change $\frac{\bar{G}^2}{\Delta E} \approx \frac{\bar{G}^2}{4}$, where $\bar{G}$ is an average interaction matrix element of H_{40}. Thus if the perturbation theory is any good at all, we can assume that the lowest-order term suffices.

Let us in this section calculate the first-order change in the ground-state wave function, and then compute, in particular, contributions to B(E2) values from four-quasi-particle components for 2+ state to 0+ ground-state transition. As a zeroth-order approximation, the wave function for the total angular momentum zero may be written as

$$|4\rangle \equiv |(ab)_J(cd)_J;\ 0\rangle = N(abcdJ) \sum_{\text{all}m} \sum_{M} \frac{(-)^{J-M}}{(2J+1)^{1/2}}$$

$$\times C(j_a j_b J;\ m_a m_b M)\ C(j_c j_d J;\ m_c m_d M)\ \eta_a^+ \eta_b^+ \eta_c^+ \eta_d^+ |\hat{0}\rangle \ . \qquad \text{(III-56)}$$

Let us be reminded that the subscript a on the operator η_a contains all the quantum numbers such as n_a, l_a, j_a, m_a, whereas "a" stand for j_a alone if it is used in the other quantities.

Making use of the operator given in Sec. II (Eq. II-13) and Wick algebra, we can easily derive (see Appendix A)

$$\langle 4|H_{40}|\hat{0}\rangle = -N(2J+1)^{-1/2} \sum (-)^{J-M} C(j_a j_b J;\ m_a m_b M)\ C(j_c j_d J;\ m_c m_d M)$$

$$\times [s_c s_d \langle ab|\bar{V}|-c-d\rangle\ (X(acdb) + X(cabd)) + s_b s_c \langle ad|\bar{V}|-b-c\rangle$$

$$\times (X(abcd) + X(badc)) - s_b s_d \langle ac|V|-b-d\rangle (X(abdc) + X(bacd))]$$

$$= -N\ (2J+1)^{1/2}\ \theta(J)\ [(X(acdb) + X(cabd))\ G(abcdJ) - (X(abcd) + X(badc))$$

$$\times F(abcdJ) + (X(abdc) + X(bacd))\ F(abdcJ)\ \theta(dcJ)]\ . \qquad \text{(III-57)}$$

With the force constant, $V_0 < 0$.
Here G, F, X (Note: $\chi(abcd) = u_a v_b v_c u_d$) are the same as defined previously, and N(abcdJ) is a normalization constant.

Once we compute the matrix element given by Eq. (III-57), the first-order wave function can easily be determined by

$$\Psi_0 = \phi_0 + \sum_r \frac{V_{r0}}{E_0 - E_r}\ \phi_r\ , \qquad \text{(III-58)}$$

where $\phi_0 = |\hat{0}\rangle$ and $\phi_r = |4_r\rangle$, which is a four-quasi-particle wave function, and where r stands for a set of four indices denoting a four-quasi-particle state.

Let us consider how we can introduce such states; take a system with several j levels with certain degeneracy -- j_a, j_b, j_c and j_d. If we assume the interaction conserves angular momentum, we can, for one

representation, take a set of states as $[(ab)_J\ (cd)_J;\ 0]$,

where $a = b,\ c = d,\ a \neq c$, etc.,

or a, b, c, and d are all different. (III-59)

In (III-59), the states are completely antisymmetric states which are assured in our case by the use of second-quantized form. [Of course, the states with several particles (>2) in one j shell require more careful treatment, but as we have discussed before, the essential point of our discussion is not changed at all even with the inclusion of such states.] In the present case, J is an extra label which may designate degenerate states in zero order, but may give separate states in the presence of an interaction between the quasiparticles (i.e., H_{22}). Note that we have

$$H_0 |(ab)_J(cd)_J;\ J'\rangle = (E_a+E_b+E_c+E_d)\,|(ab)_J(cd)_J;J'\rangle$$

for all J, and J' . (III-60)

Now, a different coupling in (III-59) would give another complete set of states. For example, we may take $[(ac)_J(bd)_J;\ 0]$ instead of $[(ab)_J(cd)_J;\ 0]$, etc. It is well known in angular momentum theory, however, that the first set is not independent of the second set. It can be shown that one representation is related to the other by a unitary transformation. In fact, they are related by

$$[(ab)_{J_1}(cd)_{J_2};J'] = \sum_{J_1'\ J_2'} \{[J_1][J_2][J_1'][J_2']\} \begin{Bmatrix} a & c & J_1' \\ b & d & J_2' \\ J_1 & J_2 & J \end{Bmatrix} [(ac)_{J_1'}(bd)_{J_2'};J],$$

(III-61)

where $[J] = (2J + 1)$,

which is of the form

$$\Psi' = U\Psi \ , \qquad \text{(III-62)}$$

where

$$UU^{+} = 1 \ .$$

Therefore, we are at liberty to choose any set of states we like, but once we choose one, we have to use it consistently.

Let us now determine the contribution of four-quasi-particle states to the 2+ to ground-state electric quadrupole transition probability. This is obtained by use of the operator Eq. (III-37) with two creation operators; in deriving the equation, we take the two-particle states to be those defined in Eq. (III-7). That is, this state is obtained by diagonalizing H_{22} in the subspace of two-particle states. We are interested in the quantity $\langle 4|\mathcal{M}(E2)|\Psi_{2+}\rangle$. For the sake of clarity, let us use the notations

(i) a, b, c, d to denote quantum states of four quasiparticles,

(ii) m, n to denote two-quasi-particle quantum numbers.

Thus

$$|4\rangle \equiv |abcd;\ JJ0\rangle \equiv |(ab)_J(cd)_J\rangle \ ,$$

$$|2\rangle \equiv |(mn)J_I\rangle;\ \Psi_{2+} = \sum_{(m,n)} C_{mn}\ |(mn)2+\rangle \ , \qquad \text{(III-63)}$$

omitting other indices in C.

Define

$$S_{nm}(pq) = C_{pq}\delta_{mp}\delta_{nq} - \theta(j_p j_q)\, C_{qp}\delta_{mq}\delta_{np} \; ; \qquad \text{(III-64)}$$

then a four-quasi-particle state [abcd; JJO] contributes (for a general J_I)

$$\langle 4 | \mathcal{M}(E2) | \Psi_{J_I} \rangle = - \frac{e}{5} N \left[(2J+1)(2J_I+1)\right]^{1/2} \theta(J_I M_I)\, \delta(M_I, -\mu) \sum_{(m,n)} \Big\{ [S_{mn}(ab)\, N(ab)$$

$$\times (u_c v_d + v_c u_d)\, Q(cd) + S_{mn}(cd)\, N(cd)\, (u_a v_b + v_a u_b)\, Q(ab)]\, \delta(J,2)/5$$

$$-\theta(ad) \Big\{ [(S_{mn}(ac)\, N(ac)\, (u_b v_d + v_b u_d)\, Q(bd) + S_{mn}(bd)\, N(bd)\, (u_a v_c + v_a u_c)$$

$$\times Q(ac)]\, W(abcd;\, JJ_I) + [S_{mn}(bc)\, N(bc)\, (u_a v_d + v_a u_d)\, Q(ad)$$

$$+ S_{mn}(ad)\, N(ab)\, (u_b v_c + v_b u_c)\, Q(bc)]\, W(abdc;\, JJ_I) \Big\} \Big\} \; , \qquad \text{(III-65)}$$

$$\text{or} \quad \langle 4 | \mathcal{M}(E2) | \Psi_{J_I} \rangle = - \frac{e}{5} N(abcdJ) \left[(2J+1)(2J_I+1)\right]^{1/2} \theta(J_I M_I)\, \delta(M_I, -\mu)$$

$$\times \sum_{(m,n)} \Big\{ [S_{mn}(ab)\, N(ab)\, D'(cd2) + S_{mn}(cd)\, N(cd)\, D'(ab2)]\, \delta(J,2)/5 \; - \theta(ad) \Big\{ [$$

$$S_{mn}(ac)\, N(ac)\, D'(bd2) + S_{mn}(bd)\, N(bd)\, D'(ac2)]\, W(abcd;\, JJ_I) + [S_{mn}(bc)\, N(bc)$$

$$D'(ad2) + S_{mn}(ad)\, N(ad)\, D'(bc2)]\, W(abdc;\, JJ_I) \Big] \Big\} \Big\} \; , \qquad \text{(III-66)}$$

where we use the definition

$$D'(ab2) = \frac{D(ab2)}{N(ab)} , \qquad \text{(III-67)}$$

and D was previously defined.

Now the total contribution to the transition-matrix element from four-quasi-particle states (noninteracting) may be obtained from

$$\Delta M = -\sum_{(ab)}\sum_{(cd)} \frac{\langle (ab)_J (cd)_J | \mathcal{M}(E2) | \Psi_{J_I} \rangle \langle (ab)_J (cd)_J | H_{40} | \hat{0} \rangle}{E_a + E_b + E_c + E_d} . \qquad \text{(III-68)}$$

In practice, Eq. (III-68) need not be simplified further, since each term describes a specific physical process. However, under the sum of Eq. (III-68) one may redefine the summation index, which leads to a formally simplified expression. A factor 1/4 may replace the summation condition (ab) (cd) by an independent sum of each index. For the sake of simplification, let us keep only the terms without Racah coefficients. Then, for $J_I = 2+$, we have

$$\langle abcdJ | \mathcal{M}(E2) | \Psi_{2+} \rangle \approx -\frac{e}{5} N(abcd2)\ \theta(M_I)\ \delta(M_I, -\mu)$$

$$\times [(1+\delta_{ab})\ C_{ab}\ N(ab)\ D'(cd2) + (1+\delta_{cd})\ N(cd)\ C_{cd}\ D'(ab2)] . \qquad \text{(III-69)}$$

Let us now see what we obtain if we take a separable Q-Q force. Putting the matrix element of H_{40} in the form

$$\langle 4 | H_{40} | \hat{0} \rangle \approx -\lambda\ N(abcd2)\ \sqrt{5}\ D'(ab2)\ D'(cd2) , \qquad \text{(III-70)}$$

we get

$$\Delta M \approx -2\lambda e\,\theta(M_I)\,\delta(M_I,-\mu)\sum_{(ab)(cd)}\frac{N^2(abcd2)}{E_a+E_b+E_c+E_d}\,C_{ab}\,(D'(cd2))^2\,D(ab2))\ . \tag{III-71}$$

(Notice that once we choose J = 2, no ambiguity in the wave functions arises. Choosing this particular set of states is an approximation, and furthermore, we are limiting our subspace within which there are no redundant states.)

We have used in (III-70) and (III-71) the definition

$$D(ab2) = N(ab)\,D'(ab2) = N(ab)\langle a\|r^2Y_2\|b\rangle(u_a v_b + v_a u_b)\ .$$

The coefficients C_{ab} have already been determined, i.e.,

$$C_{ab} = \frac{D(ab2)}{E_a+E_b-\omega}\left[\sum_{(il)}\frac{D^2(il2)}{(E_i+E_l-\omega)^2}\right]^{-1/2}\ .$$

Then

$$\Delta M = -2\lambda e\,\theta(M_I)\,\delta(M_I,-\mu)\left[\sum_{(il)}\frac{D^2(il2)}{(E_i+E_l-\omega)^2}\right]^{-1/2}$$

$$\times\sum_{(ab)}\sum_{(cd)}\frac{N^2(abcd2)\;D^2(ab2)\;D^2(cd2)}{N(cd)^2(E_a+E_b+E_c+E_d)(E_a+E_b-\omega)}\ . \tag{III-72}$$

Combining with M_0 (Eq. III-47), we get

$$M = M_0+\Delta M \approx -\frac{e}{\sqrt{5}\lambda}\,\theta(M_I)\,\delta(M_I,-\mu)\left[\sum_{(il)}\frac{D^2(il2)}{(E_i+E_l-\omega)^2}\right]^{-1/2}$$

$$\times\left[1+2\sqrt{5}\,\lambda^2\sum_{(ab)(cd)}\frac{N^2(abcd2)\;D^2(ab2)\;D^2(cd2)}{(E_a+E_b+E_c+E_d)(E_a+E_b-\omega)N(cd)^2}\right]. \qquad \text{(III-73)}$$

Therefore we have

$$B(E2;\ 2+\ \rightarrow\ 0+) \approx \frac{e^2}{\lambda^2}\left[\sum_{(il)}\frac{D^2(il2)}{(E_i+E_l-\omega)^2}\right]^{-1}\ [1+\zeta]^2\ ,$$

where

$$\zeta = 2\sqrt{5}\,\lambda^2\sum_{(ab)(cd)}\frac{N^2(abcd2)\;D^2(ab2)\;D^2(cd2)}{N(cd)^2(E_a+E_b+E_c+E_d)(E_a+E_b-\omega)}\ . \qquad \text{(III-74)}$$

In general the sum cannot be simplified further, owing to the factor $\frac{N^2(abcd2)}{E_a+E_b+E_c+E_d}$ (in this particular case (i.e., Q-Q force), it is a good approximation to take $N(abcd2) \approx N(ab2)\;N(cd2)$); let us, however, assume that this factor is approximately constant and set it equal to some average value $\langle\frac{N^2}{E}\rangle$. Then

$$\sum_{(ab)(cd)}\langle\frac{N^2}{E}\rangle\,\frac{D^2(ab2)}{E_a+E_b-\omega}\;D'^2(cd2) = \langle\frac{N^2}{E}\rangle\sum_{(cd)}D'^2(cd2)\sum_{(ab)}\frac{D^2(ab2)}{E_a+E_b-\omega}\ . \qquad \text{(III-75)}$$

Furthermore

$$\sum_{(ab)} \frac{D^2(ab2)}{E_a+E_b-\omega} = \lambda^{-1} . \qquad \text{(III-76)}$$

The last equality comes from the result of Eq. (III-13). Now, using Eqs. (III-74), (III-75), and (III-76), we obtain

$$\zeta = 2\sqrt{5}\,\langle\frac{N^2}{E}\rangle \sum_{(cd)} \lambda D'^2(cd2) . \qquad \text{(III-77)}$$

A brief discussion is due on Eq. (III-77). This equation takes into account the interaction between two-quasi-particle states and also ground-state correlation with four-quasiparticles; thus the ground state contains noninteracting four-particle states. An improvement in such a framework is obvious. We might consider the correlations in the excited states (i.e., two-quasi-particle states correlated with six-quasi-particle states, etc.), and interaction between four-particle states in the ground state. An approximate inclusion of all these effects leads to the random-phase approximation (RPA). We should remark that neglect of interactions between four-quasi-particle states may not be valid, for a part of an interaction can be thought of as a coupling of two phonons; as we show in a subsequent section, the interactions between three-quasi-particle states in the odd-A case lead to an effective result of a particle coupling with a phonon state. This effect turns out to be rather important for the description of odd-A nuclei.[15] If we were to consider many quasi-particle states far separated from λ, the argument that N^2/E is constant would not be valid. It is true, however, that E behaves as a cutoff factor and hence the main contribution comes from levels very near the gap. Thus if we were to choose only the states near each other -- and, since the contribution is the largest when there are many low-lying states close to each other (matrix diagonalization shows that this is true) -- then $\langle\frac{N^2}{E}\rangle \approx$ constant.

Let us compute roughly what the second term of Eq. (III-74) contributes. We take Ce^{140} and use KS's parameters. For more general computations, we reserve a later section (see Sec. IV). The relevant single-particle levels and KS's parameters are as given in Table I.

Table I. BCS solutions for Ce^{140} in constant G (used by KS)

State	E	U	V	E
$g_{7/2}$	0.0	0.559	0.829	0.991
$d_{5/2}$	1.0	0.885	0.466	1.125
$d_{3/2}$	2.0	0.965	0.206	1.87
$h_{11/2}$	2.0	0.965	0.206	1.87
$s_{1/2}$	2.0	0.965	0.206	1.87

Let us, for simplicity, neglect the last three levels, and work with only $g_{7/2}$ and $d_{5/2}$ levels. The goodness of this approximation is discussed later. Then the lowest solution to the dispersion formula Eq. (III-13), using the KS value $X \equiv \chi \frac{5}{4\pi} \langle j|r^2|j\rangle^2 = 0.98$, is found to be 1.57 MeV. Note that we have $\lambda = \chi \frac{5}{4\pi}$ and e=2 for proton systems. Substitution of this value into Eq. (III-47) -- that is,

$$B(E2) \approx \frac{e^2}{\lambda^2} \left(\sum_{il} \frac{D^2(il2)}{(E_i + E_l - \omega)^2} \right)^{-1} \quad --$$

gives $B(E2) \approx 0.16 \times 10^{-48}$ Cm^4, about half of experimental value (0.36×10^{-48} Cm^4). Computation of the second term and final B(E2) is shown in Table II.

Table II. Contributions to ζ [Eq. (III-74)] from individual configurations.

a	b	$\lambda D'^2(ab2)$	Configuration	ζ
$g_{7/2}$	$g_{7/2}$	0.320	$(g^2_{7/2})_2$ $(g^2_{7/2})_2$	0.084
			$(g^2_{7/2})_2$ $(g_{7/2}d_{5/2})_2$	0.060
$g_{7/2}$	$d_{5/2}$	0.252		
			$(g^2_{7/2})_2(d^2_{5/2})_2$	0.038
$d_{5/2}$	$d_{5/2}$	0.183	$(d^2_{5/2})_2(d^2_{5/2})_2$	0.008
			Total	0.190

Thus $B(E2) \approx 0.16 \times 10^{-48} \times (1 + 0.190)^2 \approx 0.23 \times 10^{-48}$ Cm^4, about 64% of experimental value.

Let us now compare $B(E2;\ 2+ \rightarrow 0+)$ and $B(E2;\ 4+ \rightarrow 2+)$; from Eqs. (III-41) and (III-74), we have

$$\frac{B(E2;\ 4+ \rightarrow 2+)}{B(E2;\ 2+ \rightarrow 0+)} = 5\lambda^2 \left[f^2 \sum_{(il)} \frac{D^2(il4)}{(E_i+E_l-\omega(4+)^2} \right]^{-1}$$

$$\times \left[\sum \frac{N^{4+}(il)\ N^{2+}(km)\ D(il4)\ D(km2)}{[E_i+E_l-\omega(4+)\ (E_k+E_m-\omega(2+))} Q(kl)\ (u_iu_l-v_iv_l)\ W(42j_lj_k;2j_i)\delta_{ik} \right]^2 . \tag{III-78}$$

Here $f > 1$, and the quantity under the absolute square is lessened by (uu-vv) factor and Racah coefficient. Therefore, it is easy to see that the ratio should indeed be very small, as is the case experimentally. This phenomenon is further examined numerically in a subsequent section.

E. Effect of the H_{31} on Single-Particle Behavior

Thus far we have confined our study to even-particle systems. Whether it is meaningful to treat single-particle behavior in a similar manner is not well understood. We know, however, that the independent quasi-particle description fails to account for many aspects of experimental results in odd-A nuclei. Attempts have been made by KS to include higher-order corrections,[15] and they find a fair success in fitting experimental results. KS first determine two-quasi-particle state wave functions by means of a Lagrangian multiplier method, using a completely separable force (Q-Q model), and after finding the phonon state, they in turn study the coupling of a single particle to such phonon states. Schrieffer[30] used a Green's-function method to calculate the effect of collective excitations on single-particle behavior, and found that if a certain condition is fulfilled (i.e., $E_i \approx E_L \pm E_k$, where E_i and E_k are independent single quasi-particle energies, and E_L is the energy of the collective state of Lth mode), then a single-particle state undergoes a large energy shift. His consideration was also confined to a separable force. If his conjecture is correct -- though it has not yet been investigated in detail in connection with experiments -- then the usual procedure to fix parameters on the basis of the spectra of odd-A nuclei may not be valid. This then requires a self-consistency program, and a really rigorous investigation should necessarily follow the self-consistent calculation. In the conventional nuclear problems, there are many parameters that are not understood theoretically, and such a study would present a formidable task.

Granted that there is such a large energy shift, it is still a reasonable assumption to take

$$\Delta E \ll 2E \qquad \text{(III-79)}$$

In the first place, if statement (III-79) is not true, then a perturbation calculation that we intend to do is not valid at all, and in the second place (and the most crucial point!) the nicety of the independent-particle description is completely lost.

In this section, we assum (III-79) and study how one can improve the description of single-particle behavior beyond the independent particle treatment.

We can see from the form of interaction operators that only the H_{31} term contributes in the lowest order to the change of wave functions in the subspace of odd-number particle systems ("couples" one quasi-particle state to three-quasi-particle states). Taking this term into account to a higher order, then, implies a consideration of odd-particle states with number of particles greater than three. In other words, the correction to the wave function can be obtained by

$$\Psi_i = \eta_i^+|\hat{0}\rangle + \sum_{pqr} \chi_3^i (pqr)\, \eta_p^+\eta_q^+\eta_r^+|\hat{0}\rangle + \sum_{pqrst} \chi_5^i (pqrst)\, \eta_p^+\eta_q^+\eta_r^+\eta_s^+\eta_t^+|\hat{0}\rangle + \cdots , \tag{III-80}$$

where χ_3^i , χ_5^i , ... denote the amplitudes of three-particle, five-particle, etc. states, respectively. In angular-momentum representation, χ contains geometrical factors (i.e., Clebsch-Gordan Coefficients). To higher orders (>1st), the χ function also contains interactions between the quasiparticles of greater number among themselves through the term H_{22}. This further consideration, as we shall indicate shortly, corresponds to the introduction of phonons in a sense similar to that used by KS, provided we take some simplifying assumptions. The χ can contain, in addition, amplitudes due to interactions of the H_{40} type, which connect the one-particle state to the five-particle state to first order. Inclusion of all these terms is tremendously tedious. Fortunately, we may

simplify our treatment by dropping the terms involving five or more quasi-particles on the energetic ground; five-particle states lie at approximately 5 MeV, and hence physically these states are not expected to have much effect on the low-energy properties in which we are interested. In other words, the energy denominator in the perturbation series is large compared with the matrix element. Thus we confine our consideration to only one- and three-quasi-particle states. The validity of this approximation is discussed further at the end of this section.

Then we have

$$\Psi_i \approx \eta_i^+|0\rangle + \sum_{pqr} \chi_3^i (pqr)\, \eta_p^+\eta_q^+\eta_r^+|\hat{0}\rangle \ . \qquad \text{(III-81)}$$

Using the zero-order wave functions for the one-particle and three-particle states given in Sec. (III-A), and the algebra used so far for the matrix elements of H_{22} and H_{40} , we easily obtain

$$\langle 3|H_{31}|1\rangle \equiv \langle (il)J_I k; f|H_{31}|f\rangle$$

$$= -N(ilkJ_I)\left(\frac{2J_I+1}{2j_f+1}\right)^{1/2} [\Delta(likf)\ G(ilkfJ_I)$$

$$-\Delta(lkif)\ F(ilkfJ_I) + \Delta(iklf)\ \theta(kfJ_I)\ F(ilfkJ_I)] \ , \qquad \text{(III-82)}$$

where

$$\Delta(ilkf) = u_i u_l v_k u_f - v_i v_l u_k v_f \ , \qquad \text{(III-83)}$$

and $N(ilkJ_I)$ is a normalization constant for a zeroth-order three-quasi-particle state. J_I acts as an extra label that designates complete states with j_f. Equation (III-82) then gives rise to the first-order correction in wave functions:

$$|\Psi_f\rangle = |f\rangle + \sum_{(ilk)} \frac{\langle 3|H_{31}|1\rangle}{E_f - E_i - E_l - E_k} |(il)J_I k; f\rangle . \qquad \text{(III-84)}$$

1. Consideration of the Collective Effect

Equation (III-84) takes into account a complete set of non-interacting three-quasi-particle states. The next step for improvement is, then, an introduction of interactions among those states. This can be done by diagonalizing H_{22} in the subspace of three-particle states; i.e., in a way similar to that for two-particle states already discussed, we expand an interacting three-quasi-particle states in terms of complete set of unperturbed states,

$$|\phi_f^\alpha\rangle = \sum_{(ilk)} \sum_{J_I} C^f_{ilkJ_I} |(il)J_I k; f\rangle , \qquad \text{(III-85)}$$

from which we get an equation for the determination of the coefficients,

$$(E_i + E_l + E_k)\, C^f_{ilkJ_I} + \sum_{(pqr)} C^f_{pqrJ_I} \langle (il)J_I k;\ f|H_{22}|(pq)J_I r; f\rangle \approx W(\alpha)\, C^f_{ilkJ_I} .$$

(III-86)

In Eq. (III-85) we have neglected the matrix elements $\langle (il)J_I k; f|H_{22}|(pq)J_I' r; f\rangle$ with $J_I \neq J_I'$. We discuss this point later on.

Now, substituting in Eq. (III-84), we have

$$|\Psi_f\rangle = |f\rangle + \sum_{\alpha} \frac{\langle \phi_f^{\alpha} | H_{31} | f \rangle}{E_f - E_{\alpha}} \phi_f^{\alpha} \qquad \text{(III-87)}$$

The sum α goes over all states within J_I, and then over all J_I (allowed). Equation (III-87), then, takes into account a main portion of collective effect. This point was discussed already. Numerical computation indeed shows that except for the 0+ state, this prescription turns out to be sufficiently good.

2. Approximation in the Matrix Element of H_{22}

Let us now consider in detail the matrix element appearing in Eq. (III-86). It has the form

$$\langle 3' | H_{22} | 3 \rangle = 1/4 N(ilkJ_I) N(pqrJ_I') \sum_{\text{all } m} \sum_{M_I M_I'} C(j_i j_l J_I; m_i m_l M_I) C(J_I j_k j_f; M_I m_k m_f)$$

$$\times C(j_p j_q J_I'; m_p m_q M_I') C(J_I' j_r j_f; M_I' m_r m_f) \sum_{\alpha\beta\gamma\delta} \{ \langle \alpha\beta | \bar{V} | \gamma\delta \rangle \ (u_\alpha u_\beta u_\gamma u_\delta + v_\alpha v_\beta v_\gamma v_\delta)$$

$$- 4 \langle \alpha - \gamma | \bar{V} | -\beta\delta \rangle \ u_\alpha u_\delta v_\gamma v_\beta \} \ \langle \eta_r \eta_q \eta_p \eta_\alpha^+ \eta_\beta^+ \eta_\delta \eta_\gamma \eta_i^+ \eta_l^+ \eta_k^+ \rangle_0 \qquad \text{(III-88)}$$

Since H_{22} is a two-body operator, one of the three initial particles should propagate free of interaction at each point. This gives rise to nine possible contractions. For example, we can have contractions of the form

$$\langle \eta_r \eta_q \eta_p \ H_{22} \ \eta_i^+ \eta_l^+ \eta_k^+ \rangle_0 = \langle \eta_r \eta_k^+ \rangle_0 \langle \eta_q \eta_p \ H_{22} \ \eta_i^+ \eta_l^+ \rangle_0 \qquad \text{(III-88a)}$$

or

$$\langle \overbrace{\eta_r \eta_q \eta_p \; H_{22} \; \eta_i^+ \eta_l^+} \eta_k^+ \rangle_0 = - \langle \eta_r \eta_l^+ \rangle_0 \langle \eta_q \eta_p \; H_{22} \; \eta_i^+ \eta_k^+ \rangle_0 \; , \qquad \text{(III-88b)}$$

and so on.

The contraction (III-88a) yields

$$\langle (il) J_I | H_{22} | (pq) J_I' \rangle \; \delta(J_I, J_I') \; \delta(p,r) / N(il) N(pq) \; . \qquad \text{(III-88c)}$$

We get this because two of the Clebsch-Gordan coefficients are eliminated by means of the orthonormality relation. The rest of the contractions, however, give much more complex forms involving higher-j symbols. This complexity can be avoided if we notice that such terms multiplied by extra geometrical factors are often much smaller than (III-88c), and we drop them. In practice, this approximation may not be valid, but one can show that this is exactly the same as the coupling of a single particle to phonon states, if we consider this together with H_{31} in odd-particle systems. Another way of explaining this approximation is that we neglect the antisymmetrization of one particle with the rest of the particles (in which two particles are completely antisymmetrized). In this case we may destroy the unitarity of the transformation. [By this, we mean the unitary transformation of three-particle wave functions similar to Eq. (III-61). The only difference is the appearance of a Racah coefficient in place of the 9-j symbol (reference 20, p. 518)].

We have already given an explicit form of Eq. (III-88c). In the case of the separable potential, we may write

$$\langle 3' | H_{22} | 3 \rangle \approx -\lambda N(ilkJ_I) \; N(pqrJ_I) \; D'(ilJ_I) \; D'(pqJ_I) \; \delta(k,r) \; . \qquad \text{(III-89)}$$

Substitution of Eq. (III-89) into Eq. (III-86) (eigenvalue equation) gives a dispersion formula,

$$\sum_{(ilk)} \frac{\lambda\, N^2(ilkJ_I)\, D^2(ilJ_I)}{E_i+E_l+E_k-\omega} = 1 , \qquad \text{(III-90)}$$

which is very similar to the equation for the two-particle case except for the normalization. In this particular case, we obtain, for the expansion coefficient,

$$C^f_{ilkJ_I} = \frac{\lambda N(ilkJ_I)}{E_i+E_l+E_k-\omega} \left(\sum \frac{\lambda^2 N^2(ilkJ_I)\, D'^2(ilJ_I)}{(E_i+E_l+E_k-\omega)^2} \right)^{-1/2} . \qquad \text{(III-91)}$$

If we consider only a Q-Q force, $J_I = 2$ states alone contribute. Equation (III-91) then completely determines the wave function (III-85) within the approximation we have adopted.

Equation (III-91) is still a formal solution, since we have to solve, in practice, for ω from the dispersion formula, where α runs over all available states. As was pointed out previously, ω cannot be expressed in an analytical form. In order to express a physical quantity solely with known quantities, let us approach this problem with straightforward perturbation calculation. This we do under the assumption given by (III-79).

3. Perturbation Method

For this discussion, we adopt the Wigner-Brillouin (W-B) perturbation scheme, which turns out to be, in this case, more convenient than the Rayleight-Schrödinger expansion. This, we note, is due to a particular way of combining terms in the series. The wave function expansion is given by

$$\Psi_\alpha = \phi_\alpha + \sum \frac{\langle\beta|V|\alpha\rangle}{E - \bar{\epsilon}_\beta}\phi_\beta + \sum \frac{\langle\delta|V|\beta\rangle\langle\beta|V|\alpha\rangle}{(E - \bar{\epsilon}_\beta)(E - \bar{\epsilon}_\delta)}\phi_\delta + \cdots ,$$

(III-92)

where $E = \epsilon_\alpha + \Delta E$, i.e., "exact" energy of state α. The ΔE corresponds to an energy correction computed up to an infinite order. Now suppose $V = H_{22} + H_{31}$. Then we can apply Eq. (III-92) to our problem. We need only to replace

$$\phi_\alpha \quad \text{with} \quad |\alpha\rangle = \eta_\alpha^+ |\hat{0}\rangle ,$$

$$\bar{\phi}_\beta, \ \bar{\phi}_\delta \text{ etc with } |3\rangle = |(il)J_I k;\beta\rangle, \ \delta \text{ instead of } \beta \text{ etc.},$$

$$E \text{ with } E_\alpha + \Delta E ,$$

$$\bar{\epsilon}_\beta \text{ with } E_i + E_l + E_k . \qquad \text{(III-93)}$$

Then the first V in (III-92) is replaced by H_{31} , the second term by $\langle\delta|H_{22}|\beta\rangle\langle\beta|H_{31}|\alpha\rangle$, and so on. If the condition (III-79) holds, we have

$$E - \bar{\epsilon}_\beta = E_\alpha + \Delta E - E_i - E_l - E_k \approx E_\alpha - E_i - E_l - E_k .$$

It should be noted that our discussion is again restricted only to one- and three-particle states, neglecting higher number states. Equation (III-92)can then be rewritten more explicityly as

$$|\Psi_\alpha\rangle = |\alpha\rangle + \sum \frac{\langle(il)Ik;\alpha|H_{31}|\alpha\rangle}{E_\alpha - E_i - E_l - E_k}|(il)Ik;\alpha\rangle$$

$$+ \sum_{(ilk)}\sum_{(pqr)} \frac{\langle(pq)Ir;\alpha|H_{22}|(il)Ik;\alpha\rangle\langle(il)Ik;\alpha|H_{31}|\alpha\rangle}{(E_\alpha - E_p - E_q - E_r)(E_\alpha - E_i - E_l - E_k)}|(pq)Ir;\alpha\rangle$$

$$+ \cdots \qquad \text{(III-94)}$$

with the summation restrictions already pointed out. R-S expansion, on the other hand, is more complex, owing to the appearance of normalization correction terms at each order.

4. Single-Particle Opeartor: Quadrupole Moment

Let us apply the wave functions obtained in Eqs. (III-85), (III-81), and (III-94) to the calculation of a single-particle matrix element--in particular, a quadrupole moment.

The operator has the form (in quasi-particle operators)

$$Q_{op} = \sum_{\alpha\beta} \langle\alpha|\hat{Q}|\beta\rangle \{(u_\alpha u_\beta - v_\alpha v_\beta)\ \eta^+_\alpha \eta_\beta + s_\beta u_\alpha v_\beta \eta^+_\alpha \eta^+_\beta + s_\alpha u_\beta v_\alpha\ \eta_\alpha \eta_\beta\}, \tag{III-95}$$

with $\hat{Q} = \left(\frac{16\pi}{5}\right)^{1/2} e\, r^2\, Y_{20}$.

The quadrupole moment is defined by $Q = \langle\Psi_f|Q_{op}|\Psi_f\rangle_{m_f=j_f}$.

From Eq. (III-87), we have

$$\langle\Psi_f|Q_{op}|\Psi_f\rangle = \langle f|Q_{op}|f\rangle + 2\sum_\alpha \frac{\langle\phi^\alpha_f|H_{31}|f\rangle}{E_f - E_\alpha}\ \langle f|Q_{op}|\phi^\alpha_f\rangle + \sum_{\alpha\beta} \frac{\langle\phi^\alpha|H_{31}|f\rangle\ \langle\phi^\beta_f|H_{31}|f\rangle}{(E_f - E_\alpha)(E_f - E_\beta)}\ \langle\phi^\beta_f|Q_{op}|\phi^\alpha_f\rangle\ . \tag{III-96}$$

We shall in the following consideration neglect the last term, since it is expected to be small compared with the other terms.

The first term contributes

$$\langle f|Q_{op}|f\rangle = \langle f|\hat{Q}|f\rangle\ (u_f^2 - v_f^2)\ , \tag{III-97}$$

with

$$Q^{(0)} = \langle f|Q_{op}|f\rangle_{m_f=j_f} = \begin{pmatrix} j_f & 2 & j_f \\ -j_f & 0 & j_f \end{pmatrix} \langle f\|\hat{Q}\|f\rangle \ (u_f^2 - v_f^2)$$

and

$$\langle f\|\hat{Q}\|f\rangle = \left(\frac{16\pi}{5}\right)^{1/2} e \ \langle f|r^2|f\rangle \ \langle j_f\|Y_2\|j_f\rangle \ ,$$

where

$$\langle \alpha|r^2|\beta\rangle = \int_0^\infty R_\beta(r) R_\alpha(r) r^4 \, dr \ .$$

The second term in Eq. (III-96) involves $\langle f|Q_{op}|(il)Ik;f\rangle$ which yields

$$\langle f|Q_{op}|(il)Ik;f\rangle = N(ilkI)[(2j_f+1)(2I+1)]^{1/2} \begin{pmatrix} j_f & j_f & 2 \\ m_f & -m_f & 0 \end{pmatrix}$$

$$\times \ [(l\|\hat{Q}\|k)(u_l v_k + v_l u_k) \ W(ffIk;2I) \ \delta_{fi} - \theta(j_i j_f I)$$

$$\times \ (i\|\hat{Q}\|k) \ (u_i v_k + v_i u_k) \ W(ffik;2I) \ \delta_{fl} + (i\|\hat{Q}\|l)$$

$$\times \ (u_i v_l + v_i u_l) \ \delta_{I,2} \ \delta_{fk}/5 \ ] \ . \qquad \text{(III-98)}$$

If we assume that the terms with Racah coefficients are much smaller than the others, then we can make an approximation by dropping the first and second terms to get

$$\langle l|Q_{op}|3\rangle \approx (2j_f+1)^{1/2} \ \frac{N(ilkI)}{\sqrt{5}} \begin{pmatrix} j_f & j_f & 2 \\ m_f & -m_f & 0 \end{pmatrix} (i\|\hat{Q}\|l)(u_i v_l + v_i u_l) \delta(I,2) \delta_{fk} \ . \qquad \text{(III-99)}$$

Hence

$$Q_a^{(1)}(ilk) = Q^{(0)}(2j_f+1)^{1/2} \frac{N(ilkI)}{\sqrt{5}} \frac{(i\|\hat{Q}\|1)(u_i v_1 + v_i u_1)}{(f\|\hat{Q}\|f)(u_f^2 - v_f^2)} \delta(I,2)\delta(j_f,j_k). \tag{III-100}$$

We have used the subscript "a" to indicate an approximation, in contradistinction to $Q^{(1)}(ilk)$, which does not contain any approximation in the matrix element. Let us write down, for future reference, formulas for two approaches:

(a) Approach No. 1 [wave function given by (III-87)] :

$$Q \approx Q^{(0)} + 2\sum_{\alpha} \sum_{(ilk)I} (C^f_{ilkI})^2 \frac{\langle (il)Ik;f|H_{31}|f\rangle}{E_f - E_\alpha} Q^{(1)}(ilk) . \tag{III-96a}$$

(b) Approach No. 2 [wave function given by (III-94)] :

$$Q \approx Q^{(0)} + 2\sum \frac{\langle (il)Ik;f|H_{31}|f\rangle}{E_f - E_i - E_1 - E_k} Q^{(1)}(ilk)$$

$$+ 2\sum \frac{\langle (pq)Ir;f|H_{22}|(il)Ik;f\rangle\langle (il)Ik;f|H_{31}|f\rangle}{(E_f - E_p - E_q - E_r)(E_f - E_i - E_1 - E_k)} \times Q^{(1)}(pqr)+\ldots . \tag{III-96b}$$

In approach No. 1, we have coefficients C^f from the system of equations given by (III-86) and, in the case of the separable force, by (III-91). The matrix element of H_{31} also has a dominant term which is separable, and hence we can obtain Q in a closed form.

Let us consider the approach No. 2. We show below that this sum can be done in a closed form, if we choose the separable Q-Q force. If we follow KS, and take the Q-Q force, we readily obtain

$$\langle 3|H_{31}|1\rangle \approx N(\text{ilff2})\frac{\sqrt{5}\,\chi}{20\pi}\theta\,(j_1 j_f)[(2j_i+1)(2j_1+1)(2j_f+1)]$$

$$\times (u_i v_1+v_i u_1)(u_f^2-v_f^2)C(j_i j_1 2;\tfrac{1}{2}-\tfrac{1}{2}0)\;C(j_f j_f 2;\tfrac{1}{2}-\tfrac{1}{2})\;\langle i|r^2|1\rangle\;\langle f|r^2|f\rangle\;\delta_{kf}$$

(III-101)

and

$$\langle 3'|H_{22}|3\rangle \approx N(\text{ilff2})\frac{\chi}{20\pi}\theta(1q)[(2j_i+1)(2j_1+1)(2j_p+1)(2j_q+1)]C(j_p j_q 2;\tfrac{1}{2}-\tfrac{1}{2}\,0)$$

$$\times C(j_i j_1 2;\tfrac{1}{2}-\tfrac{1}{2}\,0)(u_i v_1+v_i u_1)(u_p v_q+v_p u_q)\langle i|r^2|1\rangle\;\langle p|r^2|q\rangle\quad .$$

(III-102)

Substitution of Eqs. (III-101) and (III-102) into (III-96b) gives

$$Q \approx Q^{(0)} + 2Q^{(0)}\,\xi + \cdots$$

$$= Q^{(0)}\left[1 + \frac{2\xi}{1-\xi}\right],\ \xi < 1, \qquad \text{(III-103)}$$

where

$$\xi = \frac{\chi}{20\pi}\sum_{(i1)}\frac{(u_i v_1+v_i u_1)^2}{E_i+E_1}\quad N(\text{ilf2})^2\quad (2j_i+1)(2j_1+1)C(j_i j_1 2;\tfrac{1}{2}-\tfrac{1}{2}\,0)^2$$

$$\times\ \langle i|r^2|1\rangle^2$$

(III-104)

5. Energy Shift

In deriving Eqs. (III-103) and (III-104), we have neglected the energy shift ΔE in the denominator of the series. One could, of course, include the ΔE term in a straightforward way. All we have to do is to replace the denominator $(E_i + E_1)$ by $(E_i + E_1 - \Delta E)$. Provided $\Delta E \leqslant 0$ -- which, we show shortly, is the case in the approximation we have taken-- ξ as defined by Eq. (III-104) is the upper limit. This implies that the Q calculated with Eq. (III-103) is always larger than a more exact value.

Let us briefly consider the energy shift ΔE. If we take the matrix elements of H_{31}' and H_{22} as given by Eqs. (III-101) and (III-102), and sum over all the terms in the W-B energy expansion, we get

$$\Delta E = - \frac{\lambda_1^2 \pi(\Delta E)}{1 - \lambda\pi(\Delta E)} , \qquad \text{(III-105)}$$

where

$$\pi(\Delta E) = \sum_{(il)} \frac{N^2(ilf2)\, D'^2(il2)}{E_i + E_1 - \Delta E} ,$$

$$\lambda_1 = \sqrt{5}\ (2j_f+1)^{1/2} \langle f|r^2|f\rangle\, C(j_f j_f 2; \tfrac{1}{2} -\tfrac{1}{2}\ 0)\ (u_f^2 - v_f^2)\lambda .$$

Equation (III-105) can be solved for ΔE by an iteration method. Rewriting it as

$$\Delta E = (\lambda\Delta E - \lambda_1^2)\, \pi(\Delta E) ,$$

we get (to the second power of λ in both the denominator and the numerator)

$$\Delta E \approx - \frac{\lambda_1^2 \pi(0)}{1 - \lambda\pi(0) - \lambda_1^2\pi'(0)} , \qquad \text{(III-106)}$$

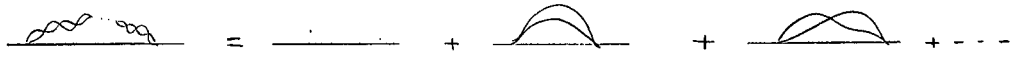
where

$$\pi'(0) = \left[\frac{\partial \pi (\Delta E)}{\partial (\Delta E)} \right]_{\Delta E = 0} .$$

Since $\pi'(0) < 0$, $\pi(0) > 0$ and $\lambda \pi(0) < 1$, we can easily see that $\Delta E < 0$. Furthermore, $|\Delta E| < 1$ and $\ll (E_i + E_1)$ for all i, 1, provided $\pi(0)$ is small compared with unity. Of course, looking at Eq. (III-106), one might suspect that there could occur large ΔE when the denominator happens to be small (this point was discussed before[30]), but just where this occurs is not understood.

6. Discussion

Let us discuss Eq. (III-103) in a different way. For this purpose, we employ graphs similar to Feynman diagrams. The first term in Eq. (III-94) corresponds to an unperturbed propagator, while higher-order terms are obtained by introducing various perturbations. Such a consideration corresponds to a modification of self-energy diagrams, and is given by the following:



The second diagram describes the H_{31} interaction, where the intermediate three particles are not interacting; the third, fourth, and so on describe interactions between two of the three particles. In our derivation, this repeated interaction does not double back, and hence, the number of quasiparticles at any time is never more than three. This is a further simplification from the RPA, in which such doubling is included. Furthermore, the initial single-particle state maintains its identity throughout the interactions, and thus we essentially neglect antisymmetrization with the interacting particles. We have already asserted that these effects are smaller. The chain of "bubbles" may be thought to give rise to a phonon state (in particular the 2+ phonon state), and the whole approach as a coupling of a particle to a phonon.

An obvious improvement over our approximation is to take into account the sum of all diagrams describing interactions between the phonon and one-particle state at an intermediate stage. In practice, this may turn out to be very important.

Let us now compare Eq. (III-103) with the KS result. In the notation given above, KS obtain

$$Q_{KS} = Q^{(0)} \left[1 + \frac{2\xi'}{1-2\xi'}\right] ,$$

where $\xi' = \xi(N^2 = \frac{1}{2})$ (that is, if we set $N^2 = \frac{1}{2}$ we obtain ξ'). Q_{KS} and our Q have similar forms, and in fact, numerically almost the same results. It should be mentioned that our treatment has the advantage of computations with a general force, and showing clearly the kinds of approximations made to arrive at a closed expression.

To illustrate the point raised above, we compute Eq. (III-103) for La^{139}, using exactly the same parameters as KS used. The values used by KS are

$$\lambda = 0.22 ,$$
$$\Delta = 0.88 ,$$

	u	v	E
$g_{7/2}$	0.615	0.788	0.907
$d_{5/2}$	0.912	0.410	1.176
$h_{11/2}, s_{1/2}, d_{3/2}$	0.974	0.227	1.985

With $\lambda = \frac{5}{4\pi} \chi \langle r^2 \rangle_u = 0.98$, we obtain $\xi = 0.35$. Thus $\frac{2\xi}{1-\xi} = 1.1$. The contributions from each state are given in Table III.

Table III. Contribution to quadrupole moment

State	$g_{7/2}^2$	$g_{7/2}\, d_{5/2}$	$d_{5/2}^2$	$h_{11/2}^2$	$d_{3/2}^2$	$(d_{3/2}\, d_{5/2})$	$g_{7/2}\, d_{3/2}$
ξ/λ	0.24	0.01	0.03	0.02	0.001	0.01	0.06

Compare our $\xi = 0.35$ with KS's $\xi' = 0.27$. With their equation they obtain the correction term $\frac{2\xi'}{1-2\xi'^2} = 1.2$. As can be seen above, the dominant contribution comes from $g_{7/2}$, with the rest of the states contributing only about 25%. The main difference between ξ and ξ' is in the normalization factor which in our example is always greater than 1/2. Thus all the terms are multiplied by a kind of weight factor, which on the average gives $\xi \geq \xi'$.

It should be remarked here that the phases are coherent; that is, within the approximation, our consideration manifestly shows that all the contributions come with the same sign. One would expect completely general treatment also to contribute coherently. But this is not quite correct, since with a general force the discarded terms are sometimes of the same size as the term kept to derive the closed form (and sometimes with different signs). This point is further considered in Sec. IV.J in connection with the application of the more general form of the quadrupole moment, Eq. (III-96a).

F. Comparison With Other Approaches

In this subsection, we compare the results we have already discussed with the Green's function treatment of the H_{22} and H_{40} terms. For H_{22}, we follow Schrieffer's work[30] and for $H_{22} + H_{40}$ we follow Galitskii.[31]

There are two approaches in Green's function method to deal with the pairing-type interaction. One is to start with Green's functions built up with particle operators, introducing an extra Green's function in addition to the familiar one. This extra one is invented to take into account the anomalous coupling mentioned in the Introduction, and has the form

$$F_{\nu\nu'}(t,t') = \langle \hat{0} | T(c_\nu(t) c_{\nu'}(t')) | \hat{0} \rangle \quad , \tag{III-107}$$

or its h.c., $F^{+}_{\nu\nu'}(t,t') = \langle \hat{\tilde{0}} | T(c^{+}_{\nu'}(t') c^{+}_{\nu}(t)) | \hat{\tilde{0}} \rangle$,

where $|\hat{\tilde{0}}\rangle$ is an <u>exact</u> vacuum. T is the usual time-ordering operator. Notice that $|\hat{\tilde{0}}\rangle \rightarrow |\hat{0}\rangle \equiv |0\rangle_{BCS}$ does not make the Green's function vanish, whereas for $|\hat{\tilde{0}}\rangle \rightarrow |0\rangle$ (particle vacuum) we get back to a normal fermion system. Gorkov[33] has used this function to derive an equation of superconductivity, and Belyaev[34] extended Gorkov's ideas to two-particle Green's functions, introducing two more functions in addition to the normal ones, and using Low-Gell-Mann[35,36] integral equations invented originally to treat bound states in relativistic field theory.

The other line of approach is to transform to quasi-particle language in the way discussed, and to build Green's functions in the new operator; the zeroth-order Green's function already contains the pairing effects. This method has been used by Schrieffer,[30] Galitskii,[31] and Cheston and Kobe.[37] The last authors derive complex coupled equations in the spirit of Martin-Schwinger's formalism,[38] and eventually have to choose only certain large terms to make a contact with physical quantities.

Schrieffer's approach is the simplest among them, and leads to the same conclusion as ours. It will be easy to see from the following discussions that the conclusions are indeed the same.

Let us define a one-quasi-particle Green's function by

$$G_{\nu}(\tau) = \langle \hat{\tilde{0}} | (\eta_{\nu}(\tau) \eta^{+}_{\nu}(0)) | \hat{\tilde{0}} \rangle \ . \qquad \text{(III-108)}$$

In angular momentum representation, p labels a state of j_p ; the time at which the particle p is created is taken as zero. Here $\eta_{\nu}(\tau)$ is in a Heisenberg representation; i.e., $\eta_{\nu}(\tau) = e^{iH\tau} \eta^{s}_{\nu} e^{-iH\tau}$, where η^{s}_{ν} is a Schrödinger operator. In order to go over to the time-independent formalism, we take the Fourier transform

$$G_{\nu} \equiv G_{\nu}(\epsilon) = \int_{-\infty}^{\infty} G_{\nu}(\tau) e^{i\epsilon\tau} \, d\tau \ . \qquad \text{(III-109)}$$

By replacing $|\hat{\tilde{0}}\rangle \rightarrow |\hat{0}\rangle$ we obtain the zeroth-order Green's function,

$$G^{(0)}(\epsilon) = \int_{-\infty}^{\infty} G_\nu^{(0)}(\tau)\, e^{i\epsilon\tau} = \int_0^{\infty} \langle \hat{0}|\eta_\nu(\tau)\eta_\nu^+(0)|\hat{0}\rangle\, e^{i\epsilon\tau}\, e^{-\xi|\tau|}\, d\tau \,, \tag{III-110}$$

where the exponential $e^{-i\xi|\tau|}$ is a cutoff factor, introduced to make the integration meaningful. One puts $\xi = 0$ at the end of integration. Note that by definition, $\xi \rightarrow 0^+$, to make $e^{-i\xi|\tau|}$ damp out as $|\tau|$ becomes large. To discuss two-particle excitations, one could introduce two-particle Green's functions and determine equations of motion similar to Low-Gell-Mann equations. A similar result may be obtained by the t-matrix method, which we use. In order to do so, let us neglect, as we did in Sec. (III-B), all the interaction terms except H_{22} ; we get

$$V \equiv H_{22} = \frac{1}{4} \sum_{\alpha\beta\gamma\delta} P(\alpha\beta\gamma\delta)\, \eta_\alpha^+ \eta_\beta^+ \eta_\delta \eta_\gamma \tag{III-111}$$

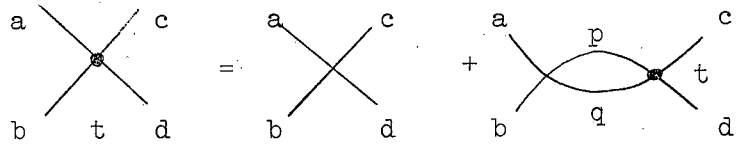
where P is given (see Eq. (II-13)) by

$$P(\alpha\beta\gamma\delta) = \langle \alpha\beta|\bar{V}|\gamma\delta\rangle\, (u_\alpha u_\beta u_\gamma u_\delta + v_\alpha v_\beta v_\gamma v_\delta) - 4\langle \alpha\text{-}\gamma|\bar{V}|\text{-}\beta\delta\rangle s_\beta s_\gamma u_\alpha u_\delta v_\beta v_\gamma \,.$$

The t matrix is given symbolically (with correct signs and factor) by

$$t_{ab;cd} = v_{ab;cd} - v_{ab;pq}\, G_p\, G_q\, t_{pq;cd} \,. \tag{III-112}$$

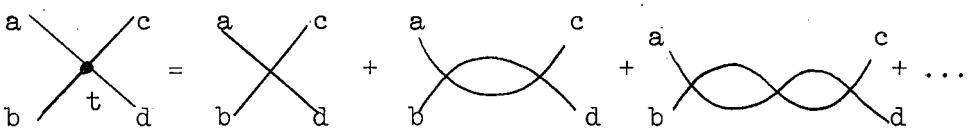
This is an integral equation, which takes into account multiple scattering effects. Graphically, Eq. (III-112) reads as



(III-113)

(This diagrammatic representation corresponds to Hugenholtz's[39]).

If expanded in perturbation series, these are:



(III-114)

In these diagrams, arrows are not shown, since lines do not correspond either to a particle or to a hole. In fact, it is a linear combination of a particle and a hole. Both the direct and exchange terms are included, it should be understood, in the diagrams.

Now define (from Eq. III-24)),

$$v_{ab;cd} = N(ab)N(cd)\,[(u_a u_b u_c u_d + v_a v_b v_c v_d)\,G(abcdJ) + (u_b u_d v_a v_c + u_a u_c v_b v_d)\,F(abcdJ) - \theta(cdJ)\,(u_a u_d v_b v_c + u_v u_c v_a v_d)\,F(abdcJ)\,].$$

(III-115)

We rewrite to Eq. (III-112) in an explicit form:

$$t_{ab;cd} = v_{ab;cd} - i\int \frac{d\epsilon}{2\pi} \sum_{ef} v_{ab;ef}\, G_e^{(0)}(\epsilon+\omega)\, G_f^{(0)}(-\epsilon)\, t_{ef;cd}.$$

(III-116)

In nuclear interactions, v does not contain a retardation effect, which implies that v is independent of ϵ. The expansion of (III-116) would make each term immediately integrable, and the solution for t is the sum of the infinite series. In Eq. (III-116), G is in principle an exact Green's function, but one usually computes it by replacing G by $G^{(0)}$ obtained in Eq. (III-110). As was the case with Eq. (III-8), the integral equation (III-116) is soluble exactly only if v is separable.

Let us take, as usual,

$$v_{ab;cd} = -\lambda\, D(abJ)\, D\,(cdJ). \tag{III-117}$$

Then we obtain

$$t_{ab;cd} = -\frac{\lambda D(abI)D(cdI)}{1 + L(\omega)}\ , \tag{III-118}$$

where

$$L(\omega) = \sum_{(ef)} \frac{D(efI)^2}{\omega - (E_e+E_f)+i\xi}\ . \tag{III-119}$$

As is well known, the pole of the t matrix gives an excitation spectrum; i.e.,

$$1 + \lambda\, L\,(\omega) = 0$$

or

$$\sum_{(ef)} \frac{\lambda\, D(efI)^2}{E_e + E_f - \omega} = 1\ , \tag{III-120}$$

which is in exact agreement with Eq. (III-13) previously obtained. Physically the t matrix obtained by Eq. (III-118) corresponds to an effective interaction arising from successive scatterings between two particles. This effective interaction is, in practice, sufficient for not too strongly collective states. To extend our treatment to RPA effects, it is more convenient to deal with equations of motion; hence we briefly give a sketch of derivations and arguments involved in it.

Introduce two-quasi-particle Green's functions by

$$K_1(ab;cd) = \langle \hat{\hat{0}} | T(\eta_a(t_a)\ \eta_b(t_b)\ \eta_c^+\,(t_c)\ \eta_d^+\,(t_d)) | \hat{\hat{0}} \rangle$$

and

$$K_2(ab;cd) = \langle \hat{\hat{0}} | T(\eta_a^+(t_a)\ \eta_b^+(t_b)\ \eta_c^+(t_c)\ \eta_d^+(t_d)) | \hat{\hat{0}} \rangle \ , \tag{III-121}$$

and split the function K_1 and K_2, letting $t_a, t_b \rangle t_c, t_d$ into the forms

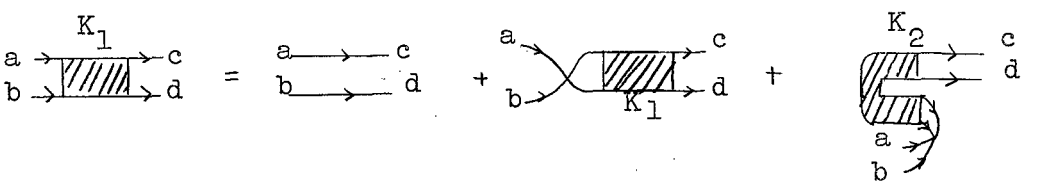
$$K_1 = \sum_\nu \langle \hat{\hat{0}} | T(\eta_a(t_a)\eta_b(t_b)) | \nu \rangle \langle \nu | T(\eta_c^+(t_c)\eta_d^+(t_d)) | \hat{\hat{0}} \rangle$$

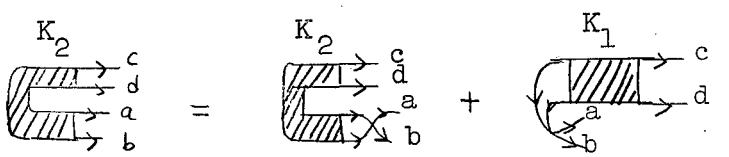
$$= \sum_\nu \psi(ab\nu)\ \bar{\phi}\ (cd\nu) \ ,$$

$$K_2 = \sum_\nu \langle \hat{\hat{0}} | T(\eta_a^+(t_a)\ \eta_b^+(t_b)) | \nu \rangle \langle \nu | T(\eta_c^+(t_c)\eta_d^+(t_d)) \hat{\hat{0}} \rangle$$

$$= \sum_\nu \phi\ (ab\nu)\ \bar{\phi}\ (cd\nu) \ . \tag{III-122}$$

The integral equations (generalized from the Low-Gell-Mann equation) are given in diagrammatic notations by





(III-123b)

In a symbolic notation, we can write (analogously to the t-matrix equation)

$$K_1(ab;cd) = G_a G_b \delta_{ac} \delta_{bd} - i G^{(0)}_{ap} G^{(0)}_{bq} [v_{pq;rs} K_1(rs;cd) + u_{pq;rs} K_2(rs;cd)] \qquad \text{(III-124a)}$$

and

$$K_2(ab;cd) = -i \bar{G}^{(0)}_{ap} \bar{G}^{(0)}_{bq} [v_{pq;rs} K_2(rs;cd) + u_{pq;rs} K_1(rs;cd)] , \qquad \text{(III-124b)}$$

where v corresponds to H_{22} and u to H_{40}. The matrix element u has a different normalization in this case, since we are not in a perturbation-theory framework; u is given by

$$u_{ab;cd} = N(ab)N(cd)[(u_a u_b v_c v_d + v_a v_b u_c u_d) G(abcdJ) - (u_a u_d v_b v_c + v_a v_d u_b u_c) \times F(abcdJ) + \theta(cdJ) (u_a u_c v_b v_d + v_a v_c u_b u_d) F(abdcJ)].$$

Owing to the opposite sense of propagation, we have $\bar{G}^{(0)}(\epsilon) = G^{(0)}(-\epsilon)$. Now, following the usual convention, dropping the inhomogeneous term in Eq. (III-124) for bound-state problems, and substituting the relations (III-122) into Eq. (III-124), we obtain

$$\psi(ab\nu) = - i G_{ap} G_{bq} [v_{pq;rs} \psi(rs\nu) + u_{pq;rs} \phi(rs\nu)],$$

$$\phi(ab\nu) = - i \bar{G}_{ap} \bar{G}_{bq} [v_{pq;rs} \phi(rs\nu) + u_{pq;rs} \psi(rs\nu)]$$

The sum over p,q reduces the equation further, since $G^{(0)}_{ap} = G^{(0)}_{a}\ \delta_{ap}$. If we integrate over ϵ as indicated in Eq. (III-116), we get

$$\psi(ab\nu) = -\ \frac{1}{E_a+E_b-\omega} \sum_{rs} \left[v_{ab;rs}\ \psi(rs\nu) + u_{ab;rs}\ \phi(rs\nu)\right] ,$$

$$\psi(ab\nu) = -\ \frac{1}{E_a+E_b+\omega} \sum_{rs} \left[v_{ab;rs}\ \phi(rs\nu) + u_{ab;rs}\ \psi(rs\nu)\right] . \qquad \text{(III-125)}$$

If we define

$$P(abrs\nu) = (E_a + E_b)\ \delta_{ar}\ \delta_{bs} + v_{ab;rs}$$

and

$$R(abrs\nu) = u_{ab;rs} ,$$

we obtain the same equation as Baranger's, i.e.,

$$\begin{pmatrix} P & R \\ -R & -P \end{pmatrix} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = \epsilon \begin{pmatrix} \psi \\ \phi \end{pmatrix} . \qquad \text{(III-126)}$$

As was shown by Baranger, this equation leads to the well-known dispersion formula, when we choose the largest terms in the quadrupole force, i.e.:

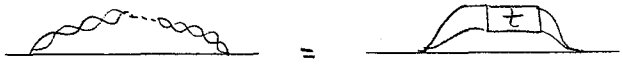
$$\sum_{(ab)} \frac{2D^2(abI)(E_a+E_b)}{(E_a+E_b)^2 - \omega^2} = \frac{1}{\lambda} . \qquad \text{(III-127)}$$

Equation (III-126) or (III-127) obtained from diagrammatic equations lends itself easily to a comparison with the result obtained from the t-matrix method. If in Eqs. (III-124a) and (III-124b) we neglect the H_{40} contribution, then the third term of (III-124a), and all of (III-124b) cannot appear. Then it becomes identical with

Schrieffer's t-matrix result. Thus the t-matrix approximation and our diagonalization of H_{22} essentially neglect four-particle correlation (or in perturbation theory, the contribution from six-particle states). In nuclear problems, the question of how large this contribution is can be determined solely from numerical analysis.

The discussion on the effect of collective excitation on single-particle behavior in the framework of Green's function follows a similar line of consideration; hence we only mention here the conclusions one might expect from it.

In diagrams, the Schrieffer approximation is just a self-energy correction due to successive scattering of two particles; i.e.,



This is considered for a physical problem in Sec. IV-H. An obvious self-energy correction of RPA is to replace the t matrix by the RPA diagrams.

Just how much this approximation with general force improves over t-matrix approximation has not been investigated yet, so that we cannot say much about it in nuclear problems.

G. Comments

1. Spurious States

Because of the lack of number conservation, there arises a spurious state with $J = 0$. As was pointed out by Baranger (see also Anderson, reference 7), the RPA method separates out this nonphysical solution from physical ones.

The number operator is not diagonal in the quasi-particle vacuum, and furthermore has a form

$$N \propto \sum_{j_a m_a} (2j_a+1)^{-1/2} u_a v_a \left[C(aa0;\, m_a - m_a 0)\ (\eta^+_{-\alpha}\eta^+_{\alpha} + \eta_{\alpha}\eta_{-\alpha})\right]. \tag{III-128}$$

This operator, acting on the vacuum, does not give back the vacuum but a state with angular momentum J = 0. This is a spurious state arising from nonconservation of number of particles.

Consider a two-quasi-particle state with angular momentum, say , J:

$$|(pq)J\rangle = \sum C(pqJ;\, m_p m_q M)\ \eta^+_p \eta^+_q |\hat{0}\rangle. \tag{III-129}$$

Taking the dot product, we get

$$\langle (pq)J\,|N|\hat{0}\rangle = \sum_{j_a} \sum_{m_p m_q m_a} C(pqJ;m_p m_q M)\ C(aa0;m_a - m_a 0) \times \frac{u_a v_a}{(2j_a+1)^{1/2}}\ \langle \eta_p \eta_q \eta^+_{\alpha}\eta^+_{\alpha}\hat{0}\rangle \,. \tag{III-130}$$

Thus

$$\langle pqJ\,|N|0\rangle = 2\,\frac{u_a v_a}{(2j_a+1)^{1/2}}\ \delta(p,q)\ \delta(J,0)\ \delta(p,a). \tag{III-131}$$

which shows that all the states with $J \neq 0$ are free of spurious components, but the $J = 0$ states with two particles in a j shell contain spurious components. In general the diagonal matrix element of V_{12} for a pure configuration is very large, and with additional contributions from spurious components, such a state is expected to be depressed tremendously. This is indeed the case, as will be seen in Fig. 12.

It is easy to see that a three-quasi-particle state is completely free of spurious components. What is the situation with the four-quasi-particle state? Note that

$$N^2|\hat{0}\rangle \neq 0 \ .$$

Thus a four-quasi-particle state $[(a^2)_J(b^2)_J 0]$ again contains spurious components. We argue, however, that it is not too serious in this case. The dot product with a four-quasi-particle state yields

$$\langle 4|N^2|\hat{0}\rangle \sim \frac{u_a^2 v_a^2}{(2j_a+1)} \ll 1 \ ,$$

since the factor $(uv) < 1$ and $(2j_a+1)$ is large. Thus neglect of spurious component is not expected to make much difference to a four-quasi-particle state.

In the RPA method, these spurious components are eliminated from physical states. However, this is assured only when the approximate boson commutation rule mentioned in the Introduction has validity. If one does not drop the terms other than the Kronecker δ, such nice separation cannot be obtained. This implies, then, that if $\frac{\nu}{n}$ is not large, the spurious components will persist.

2. Fixed-Particle-Number Method

To remedy this malady, especially for the $J = 0$ two-quasi-particle state, one could employ the method of projecting out of the BCS wave function that part which conserves number of particles. Except for states $J = 0$, this procedure does not seem to introduce much improvement. The fixed-particle-number method has not yet been applied to residual interactions, and it is certainly a question worth an investigation. However, in this paper, we are more interested in states with $J > 0$, and do not feel that those states require finer refinements.

IV. APPLICATIONS

A. Preliminary Remarks

A schematic model has been extensively used by KS and others to numerically compute some interesting features of single-closed-shell nuclei. Subsequently there has also been some work on the application of more general interactions to 50-proton nuclei (Sn),[40] and to lead isotopes.[41] Kisslinger and Sorensen[15b] extended their calculations to more complex systems including proton-neutron interactions in the long-range part of the force. Their model is again a pairing-plus-quadrupole type of interaction, but treats the problem by means of the RPA. Though phenomenological, the fit with experiment is quite impressive. They show that with one set of single-particle energies, and assuming a systematic variation of levels as a function of mass, one could obtain a good understanding of the trends over all regions of spherical nuclei. In their treatment, the parameters are determined from experiments, but consistent use of these parameters removes to a certain degree arbitrariness in their calculations. Although they have used the RPA in the latter calculations, they come to the conclusion that the previous calculation,[15a] using the adiabatic approximation, is quite good for the description of the vibrational levels of spherical nuclei. Their conclusion is valid, one should note, only for the very low-lying states, in which in fact the particles may be thought of as moving adiabatically in a vibrational potential. Since the adiabatic approximation imposes the condition $\omega \ll 2E$, where E is a quasi-particle energy, higher-lying states apparently require more sophisticated treatments. Even though the RPA treatment removes such a condition, and enables one to study low-lying as well as high-lying states without difficulty, the study by KS still leaves one unsatisfied for the following reasons:

1. The constant pairing matrix-element approximation is rather too crude in finite nuclei. Though approximately the same, a difference of matrix

elements calculated with a nonconstant force between two different states is by no means trivial. One can observe from a calculation done with a general force that the diagonal terms are in general twice as large as off-diagonal terms (see Table V).

2. The validity of neglecting components of force other than the quadrupole part is rather dubious. True, as we have shown before, the main contribution often comes from the q=2 part; furthermore, systematics of vibrational states also seem to support the simple picture of a dominant quadrupole force; however, each component of force contributes in a complex way and one would like to know how a general force, such as the forces used in shell-model calculations, can be used consistently. There is neither proof nor disproof at present that a quasiparticle should interact in a characteristic way similar to the usual particle interactions, but in the works by Baranger[1] and Arvieu and Véneroni,[40] and also in our previous sections, it is shown that a completely consistent treatment may be employed. Thus numerical computation can be started with one general force with certain parameters; both gap equations and the interaction between quasiparticles can be calculated with them. It was also pointed out by Baranger that spurious states can be eliminated only if we adopt this general treatment.

We have shown that the only way one could reach KS's results is through neglect of the term G, and some components of F terms in the matrix elements of the residual terms (describing quasi-particle interactions). The omission of the G term, first of all, introduces a violation of commutation rules that, according to Baranger, guarantee a separation of spurious states from physical states. Furthermore, such a procedure -- that is, the neglect of the G term, and the selection of the q=2 term from F -- leaves a force too far removed from the interaction that is known to be present between two nucleons and that appears to be responsible for the observed nuclear properties; recently a shell-model calculation has been done for heavy spherical nuclei to show (by using a force close to free-nucleon interactions) that with the inclusion of tensor force, one could

calculate low-lying states and fit with experiments fairly well.[42] This makes us expect that a similar force should be used for the computations of quasi-particle behavior, since the matrix element of a two-body operator in quasi-particle representation is simply equivalent to a combination of particle-particle and particle-hole matrix elements of the usual shell-model type.

Furthermore, the KS force is useful only for 2+ states [note the δ (J,q) factor in the separable part of the matrix element]; thus if one wanted to compute L + states (such as 4+, 6+, etc.), one would be compelled to introduce 2^Lth pole term in the force components. With all these states, then, are associated separate coupling constants, and these would present a rather embarassing complexity. Another point which was raised by Arvieu[40] is that the quadrupole force or any other completely separable 2^Lth pole force (due to separation of the radial part) is not invariant under translation.

3. In a more general treatment, it may not be true in general that the contributions to physical quantities from all the states come with the same sign. We have shown in our previous discussion that a separable force of special type does indeed contribute coherently to give a collective state. One may ask whether this holds true in a general consideration. Many schematic works[27] have shown that a "collective" solution is obtained when all the perturbation terms add up with the same sign to give a final result that cannot be reached with only a few terms of the perturbation series. Still no one has yet shown that this is true in a general case. We can only look at a few simple diagrams and deduce a general feature from them. It was shown already that the coherent sum results only when some selective terms are summed. In fact the neglected terms may be of opposite sign -- depending upon what states and what kinds of matrix elements we are considering -- so that, though small, they in some cases cancel away some of the contributions.

We now discuss these three points in detail with the help of numerical computations done in the region at the 82-neutron closed shell,

and a few particles beyond the 50-proton closed shell. The calculations are not intended to be extensive; our intention is to understand more deeply what meanings the approximations mentioned have.

B. Single-Particle Levels

As was discussed previously, one should really start with a set of single-particle energy spacings obtained through a self-consistent scheme. This approach, however, is beyond the scope of this work. A much used approach is instead to take those levels obtained experimentally for a nucleus with one proton, say, outside of doubly closed shells. For neutron levels (with a proton closed shell) much information is available. Unfortunately in the region (proton shells) we are considering, almost no experimental results are available. (It should be pointed out here that using a set of experimental energies is not strictly a valid procedure. The experimental values are what one hopes to fit by calculation.)

In the absence of better knowledge, we choose ϵ_α's that are essentially the same as KS's values. It is only an academic problem to find a better set of ϵ_α's, for a little better agreement with experiments with such a set does not have any theoretical (nor, in fact, practical) significance. Thus only a small variation was made in our calculation so that approximate fit could be made with the known levels in odd-A nuclei in this region. We take 1 $g_{7/2}$, 2 $d_{5/2}$, 2 $d_{3/2}$, 1 $h_{11/2}$, 1 $h_{9/2}$, and 3 $s_{1/2}$ with ϵ_α's as given in Table IV. Set I corresponds to that used by Tamura and Udagawa[14] with the spacing between $g_{7/2}$ and $d_{5/2}$ exactly the same as KS's value of 1 MeV; Set III has an increased level spacing between $d_{5/2}$ and $h_{11/2}$ and a lower spacing between $d_{5/2}$ and $g_{7/2}$. This choice turns out to give better spacing between the ground state and the first excited state in La^{139} and Pr^{141}. Set II is changed slightly from Set III. The separation of 1 MeV between $d_{5/2}$ and $g_{7/2}$ seems to be better than anything smaller, on the ground of theoretical level density

Table IV. (a) Unperturbed single-particle energies(ϵ_α) used for computations of quasi-particle energies and wave functions.
(b) Force constants, and ranges used with (a)

			Set I	Set II	Set III
a	ϵ_α	$g_{7/2}$	0	0	0
		$d_{5/2}$	1.00	1.00	0.75
		$d_{3/2}$	2.88	2.90	2.90
		$s_{1/2}$	3.20	3.40	3.40
		$h_{11/2}$	2.18	2.60	2.60
		$h_{9/2}$	∞	5.40	5.40
b	V_o		-32.9X MeV	-32.44y MeV	-32.44y MeV
	$\beta^{-1/2}$		1.732F	1.755F	1.755F
		X = 1.0, 1.15, 1.20			
		y = 1.0, 1.05, 1.1			

in the multiplets arising from these two single-particle states. We will observe these points in connection with numerical results we obtain.

C. Range of Force

We take the interaction between nucleons to be of a Gaussian form, and thus the radial dependence is given by

$$f(r) = e^{-\beta r^2} \qquad \text{(IV-1)}$$

where $\beta^{-1/2}$ in Fermis corresponds to the range of the interaction. The values we have used are $\beta^{-1/2} = 1.755$ for Sets II and III and 1.732 for Set I. These values have been extensively used in shell-model calculations. A somewhat shorter range (≈ 1.55) was used by Arvieu et al.[40] for calculations of quasi-particle interactions in the tin region.

In the interaction matrix element, the radial part is dependent on the product $B = \frac{\nu}{\beta}$, where ν is defined below. For the radial wave function, we choose the harmonic-oscillator wave function. It is believed that the harmonic-oscillator wave function is a fairly good approximation for finite nuclei except perhaps for heavy nuclei, for which a square-well type of correction is needed for higher orbital angular momenta. (See S. G. Nilsson, reference 10, for a discussion on this point.)

The radial wave function has the form

$$R_{n\ell}(r) = N_{n\ell}\, e^{-\frac{\nu}{2}r^2}\, r^{\ell}\, L_{n+\ell+1/2}^{\ell+1/2}(\nu r^2)\,, \qquad \text{(IV-2)}$$

where L is the associated Laguerre polynomial, defined as

$$L^{\ell+1/2}_{n+\ell+1/2}(\nu r^2) = \sum_{k=0}^{n} (-)^k 2^k \binom{n}{k} \frac{(2\ell+1)!!}{(2\ell+k+1)!!} (\nu r^2)^k \ . \qquad \text{(IV-3)}$$

Here ν, which is needed for the computations, is determined from the harmonic-oscillator spacing given by

$$\hbar\omega = \frac{\hbar^2 \nu}{m} \approx 41\ A^{-1/3}\ \text{MeV} \ . \qquad \text{(IV-4)}$$

D. Force Strength

In the absence of n-p interactions, only singlet-even and triplet-odd parts of the central force contribute. Thus our force is*

$$V = (V_{SE}P_{SE} + V_{TO}P_{TO})\ f(r_{12}) = V_{SE}\ (P_{SE} + t\ P_{TO})\ f(r_{12})$$

where V_{SE} and V_{TO} are singlet-even and triplet-odd force strength respectively, and P's are projection operators. In principle, one could use a force more general than above. A sophisticated shell-model calculation would involve a force more close to a free nucleon-nucleon force. For example, Kim[42] finds that the tensor force is essential for some spectra

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*See Appendix B for definitions.

which the central force fails to explain; i.e., the inversion of 1- and 0- state in Bi^{210}. There are other details that might be considered, such as spin-orbit force, etc. We neglect such fine points and take only the central force.

We take the depth of the potential close to $V_o = -33$ and the range ≈ 1.73. This is what True and Ford[43] and Glendenning[44] used in their shell-model calculations. We have determined V_o by computing odd-even mass differences in BCS approximations. Theoretical mass difference is given approximately by $P \approx 2\ E_{min}$, where E_{min} is the lowest odd-A quasi-particle energy. One could improve the estimate by using a four-point formula given[25] by

$$P(Z,N) = 1/2(-E(Z+1,N) + 3E(Z,N) - 3E(Z-1,N) + E(Z-2,N)) ,$$

where E is the BCS ground-state energy. For comparison with experiments, one could use the above formula with E designating experimental total binding energy, which is usually determined by mass measurements or β-decay energetics. We give calculated and experimental odd-even mass differences in Fig. 6 computed with t (triplet-to-singlet force ratio) = 0. It turns out that taking $2\ E_{min}$ is a rather good approximation for P (odd-even mass difference), and we used this estimate. The experimental points correspond to P estimated by the four-point formula using the binding energies given by Wapstra.[45] [Some authors prefer to use a three-point formula which is defined by

$$P(Z,N) = 2E(Z,N) - E(Z-1,N) - E(Z+1,N) .$$

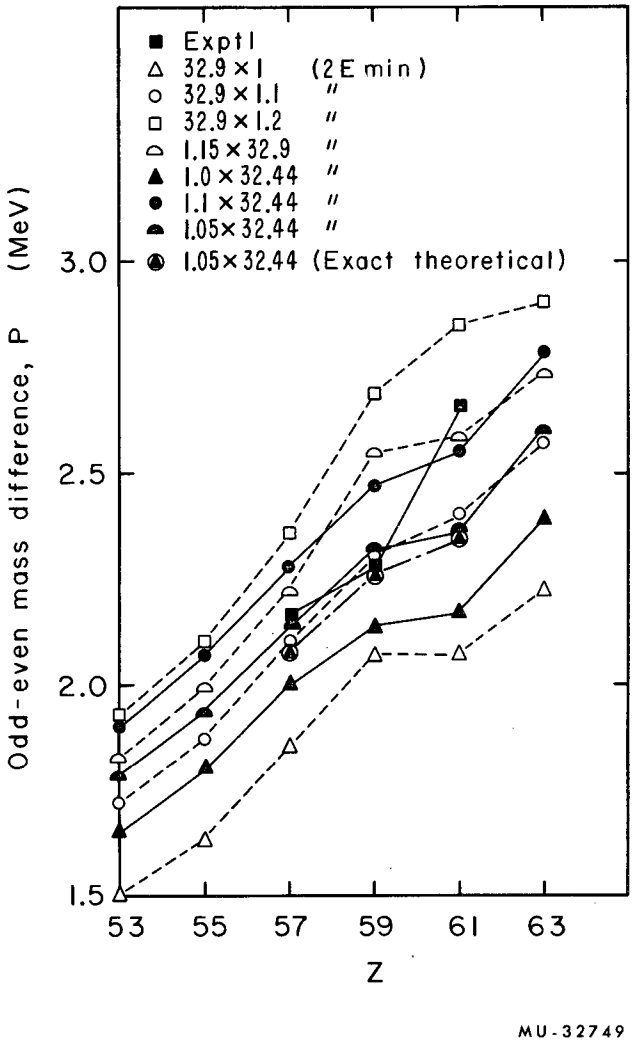

Fig. 6. Odd-even mass differences in 82-neutron nuclei. Theoretical values are obtained by $2E_{min}$.

- - - Set I;
——— Set III;
■——■ experimental points (estimated from the four-point equation with total binding energy);
◭ theoretical values obtained by using the complete formula of BCS ground-state energy, including self-energy term (y=1.05, Set III). Notice that the approximation $2E_{min}$ is in excellent agreement with the exact theoretical value.

However, as pointed out by Nilsson and Prior,[25] the four-point formula seems to have less fluctuation. It turns out that, as Fig. 6 shows, the approximation of taking 2 E_{min} for the theoretical value is rather good (in comparison with an exact evaluation with the four-point formula and with the ground-state energy $\langle H-\lambda N\rangle_o$ including the self-energy term, but neglecting the quasi-particle interactions). Thus we prefer the four-point formula. One should note that the difference between the two ways of evaluation is considerable.] Wapstra's mass table provides only three points in the N = 82 region, and hence it is rather difficult to determine V_o unambiguously. It appears that for gap equations, a pure singlet force with V_o = -1.15 x 32.9 MeV seems to be better than others, and hence we have taken this value to be used for the independent-particle Set I. For Set III, V_o is expected to be smaller than that for Set I, since lowering of $\Delta E(g_{7/2}-d_{5/2})$ increases interactions between two states; thus in the region where the occupation of the particles in those two levels is of primary interest, the effective force strength is renormalized to become somewhat larger. For that case, we have taken V_o = -1.05 x 32.44 MeV, $\beta^{-1/2}$ = 1.755. We have also tried other values of t (for example, -0.5, 0.5, and 1). For negative values of t, the theoretical odd-even mass differences are in general smaller than the experimental values, and for positive values of t, the theory seems to yield a trend opposite to the experimental results; that is, the theoretical value decreases at Z = 59, while the experimental points definitely indicate a gradual increase of P as mass is increased. Also we observed that the energy spacings of single-quasi-particle states were hard to fit with positive t's. Hence we have taken t = 0 and varied V_o. So unless specified otherwise, we shall confine our calculation of the gap solutions to t=0. (This appears to be contradictory to what we obtain in the calculation of two-quasi-particle spectra where $t \approx 1$ seems to be favored. We will say more about this subsequently.)

We have plotted zero-order quasi-particle levels as a function of Z for various force strengths in Figs. 7 and 8. Here again experimental

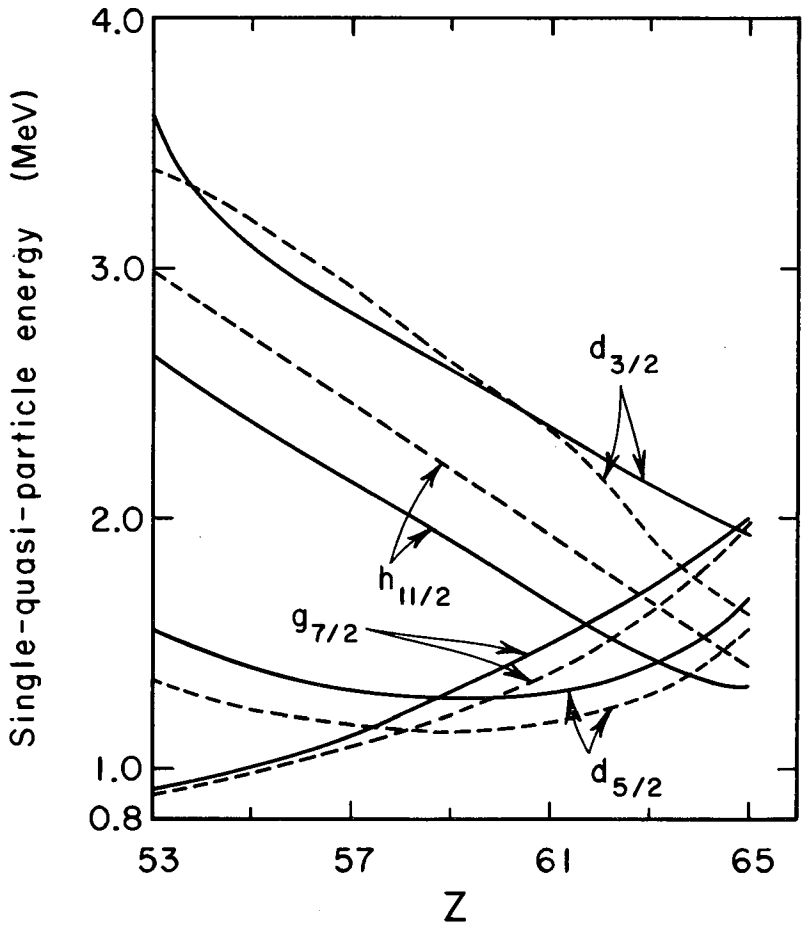

Fig. 7. Theoretical single-quasi-particle energies of 82-neutron odd-A nuclei given in absolute scale.

——— Set I (V_0 = -1.15 × 32.9),
- - - - - Set III (V_0 = -1.05 × 32.44).

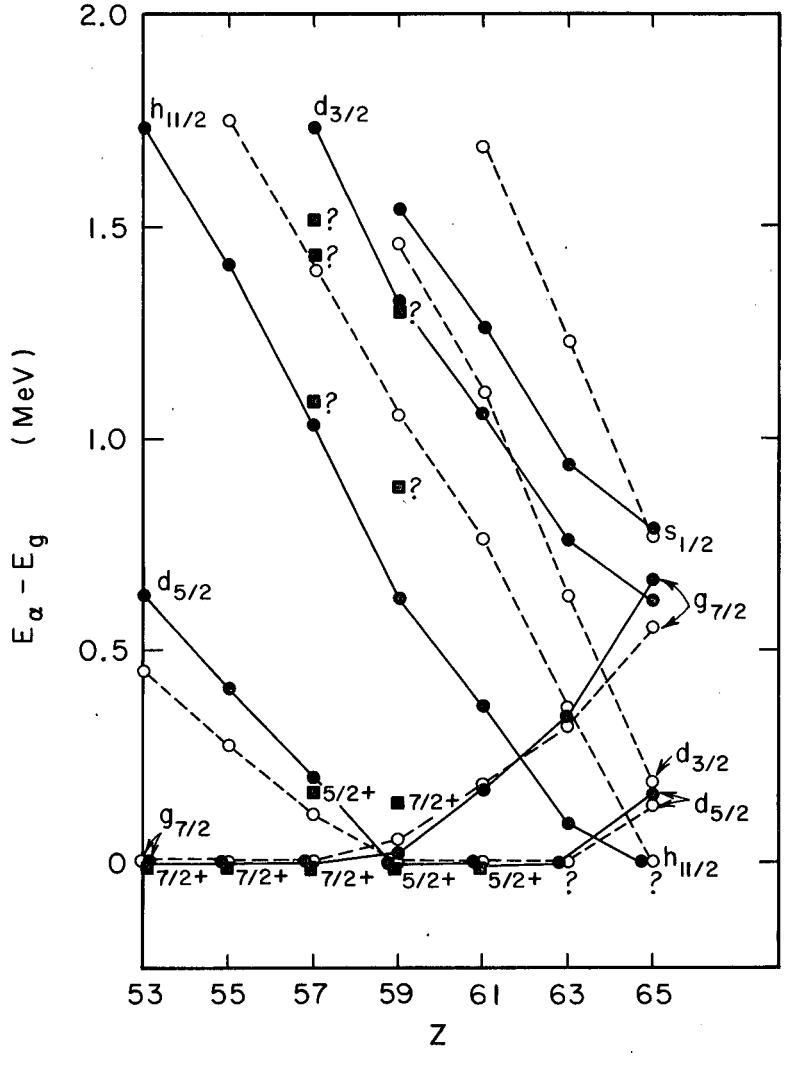

Fig. 8. Theoretical single-quasi-particle energies of 82-neutron odd-A nuclei, given in relative scale (relative to the lowest level).

● ——— Set I;

○ - - - - Set III;

■ experiment.

results are too meager to make any quantitative comparisons. It appears that Set III fits better the spacing of $g_{7/2}$ and $d_{5/2}$ in La^{139} and Pr^{141}, where $g_{7/2}$ and $d_{5/2}$ are inverted from the $g_{7/2}$ ground state in La^{139} to the $d_{5/2}$ ground state in Pr^{141}. Such an inversion also occurs with Set I, and the ground-state spins are consistent with experimental assignments. However, Set I seems to move the whole spectrum toward the nuclei with higher Z. Thus the separation of $d_{5/2}$ and $g_{7/2}$ in La^{139} is larger, and that in Pr^{141} (inverted spectrum) smaller than experiment.

In general, our results show that near $Z \leq 58$ the only low-lying single-quasi-particle states are $g_{7/2}$ and $d_{5/2}$, the other levels being separated by about 1 MeV. First, the relative motion of the $d_{5/2}$ and $g_{7/2}$ levels studied in experiment (in such nuclei as Pr, La, and Cs) turns out to be larger than what Figs. 7 and 8 indicate; secondly, some unidentified levels do not seem to fall on the theoretical curves. However, these features are not to be considered too seriously. As we have emphasized before, the results we have shown do not include the $H_{22} + H_{31}$ interaction (or phonon-particle coupling), which in principle may be very important. [See, in this connection, KS's second paper (reference 15b), where the phonon-particle coupling is considered in computing the single-particle levels in spherical nuclei.] We do not calculate the energy spectra resulting from the inclusion of the $H_{22} + H_{31}$ interaction, but instead use the BCS wave functions to study other properties that include the additional residual interactions. We present these calculations in appropriate places.

E. Solutions of Gap Equations

We give the pairing (gap) matrix elements for various forces in Table V and for various triplet-to-singlet force ratios in Table VI and Fig. 9. Along with them we have also computed the matrix elements that appear in the self-energy term μ. This is given in Table VII and Fig. 10. The self-energy contribution to the single-particle energies, ϵ_α, may be

Table V. Gap matrix elements $\frac{G(aacco)}{[(j_a+1/2)(j_c+1/2)]^{1/2}}$ for various forces with $\nu^{-1/2} = 2.288$ F (μ=o) and $\frac{V_{TO}}{V_{SE}} = 0$. ($\nu^{-1/2}$ is the oscillator parameter).

	$1g_{7/2}^2$	$2d_{5/2}^2$	$2d_{3/2}^2$	$3s_{1/2}^2$	$1h_{11/2}^2$
$g_{7/2}^2$	0.2468[a]	0.1181	0.1181	0.0940	0.1958
	0.3815[b]	0.1993	0.1993	0.1727	0.3155
	0.2366[c]	0.1167	0.1167	0.0954	0.1894
$d_{5/2}^2$		0.3240[a]	0.3241	0.2120	0.1176
		0.4508[b]	0.4508	0.2726	0.1960
		0.2987[c]	0.2987	0.1899	0.1164
$d_{3/2}^2$			0.3241[a]	0.2120	0.1168
			0.4508[b]	0.2726	0.1946
			0.2987[c]	0.1899	0.1156
$s_{1/2}^2$				0.9485[a]	0.0969
				1.2535[b]	0.1686
				0.8607[c]	0.0972
$h_{11/2}^2$					0.1808[a]
					0.2924[b]
					0.1743[c]

In each block

(a) First row: $V_0 = -32.9$ MeV, $\beta^{-1/2} = 1.732$ F

(b) Second row: $V_0 = -133.20$ MeV, $\beta^{-1/2} = 1.018$ F

(c) Third row: $V_0 = -37.50$ MeV, $\beta^{1/2} = 1.550$ F

Table VI. Gap matrix elements $\frac{G(aacco)}{[(j_a+1/2)(j_c+1/2)]^{1/2}}$ for various ratios of triplet-to-single force strength. V_0 =-32.44 MeV, $\beta^{-1/2}$ = 1.755 F, $\nu^{-1/2}$ = 2.288 F.

	$1g_{7/2}^2$	$2d_{5/2}^2$	$2d_{3/2}^2$	$3s_{1/2}^2$	$1h_{11/2}^2$
$g_{7/2}^2$	0.2483[a]	0.1182	0.1183	0.0938	0.1967
	0.2849[b]	0.1119	0.1277	0.0938	0.1717
	0.3583[c]	0.0992	0.1468	0.0938	0.1216
	0.4316[d]	0.0865	0.1658	0.0938	0.0716
	0.2116[e]	0.1245	0.1087	0.0938	0.2217
$d_{5/2}^2$		0.3276	0.3276	0.2151	0.1177
		0.3453	0.3011	0.2151	0.1232
		0.3806	0.2482	0.2151	0.1340
		0.4159	0.1952	0.2151	0.1450
		0.3099	0.3541	0.2151	0.1122
$d_{3/2}^2$			0.3276	0.2151	0.1169
			0.3674	0.2151	0.1088
			0.4468	0.2151	0.0925
			0.4866	0.2151	0.0763
			0.2879	0.2151	0.1250
$s_{1/2}^2$				0.9609	0.0969
				0.9609	0.0969
				0.9609	0.0969
				0.9609	0.0969
				0.9609	0.0969
$h_{11/2}^2$					0.1817
					0.2019
					0.2422
					0.2851
					0.1615

In each block

(a) First row: $\frac{V_{TO}}{V_{SE}} = 0.0$

(b) Second row: $\frac{V_{TO}}{V_{SE}} = 0.2$

(c) Third row: $\frac{V_{TO}}{V_{SE}} = 0.6$

(d) Fourth row: $\frac{V_{TO}}{V_{SE}} = 1.0$

(e) Fifth row: $\frac{V_{TO}}{V_{SE}} = -0.2$

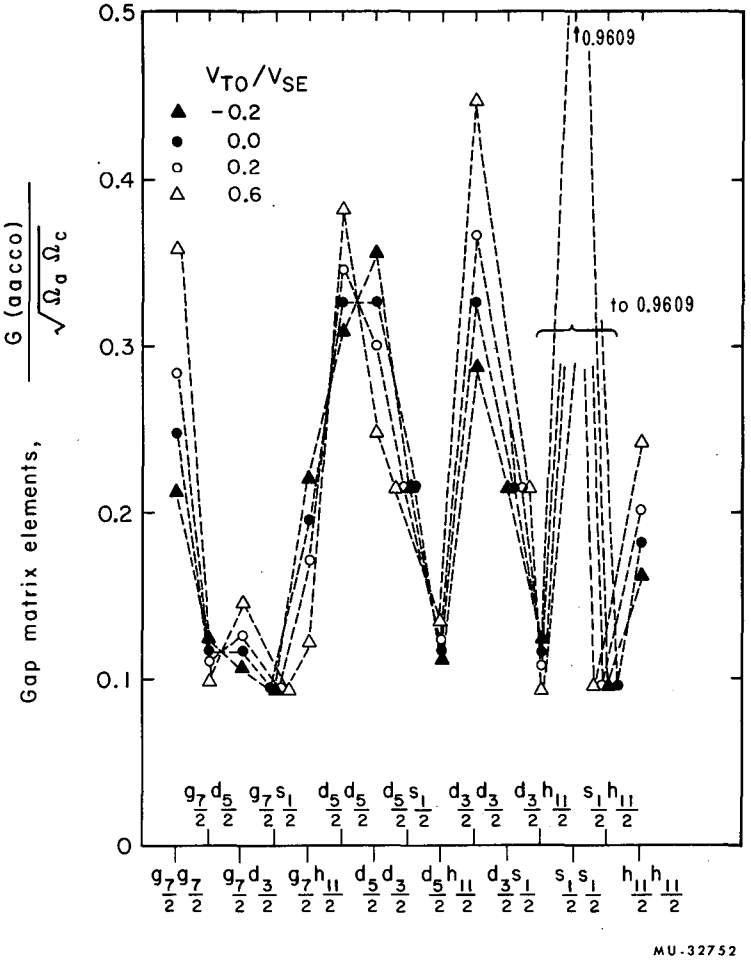

Fig. 9. Gap matrix elements for various triplet-to-singlet force ratios, computed with V_0 = -32.44 MeV, $\beta^{-1/2}$ = 1.755 F, $\nu^{-1/2}$ = 2.288 F.

Table VII. Self-energy matrix elements[a] computed with $V_0 = -32.44$, $\beta^{-1/2} = 1.755$ F. $\frac{V_{TO}}{V_{SE}} = 0$, $\nu^{-1/2} = 2.288$ F. Last three columns correspond to μ [Eq. (II-60)] for X_e^{136}, C_e^{140}, and S_m^{144} with appropriate corrections for $\nu^{-1/2}$.

	$1g_{7/2}$	$2d_{5/2}$	$2d_{3/2}$	$3s_{1/2}$	$1h_{11/2}$	$1h_{9/2}$	μX_e^{136}	μC_e^{140}	μS_m^{140}
$1g_{7/2}$	0.3491	0.3114	0.2752	0.2562	0.4264	0.3012	0.725	1.438	2.143
$2d_{5/2}$		0.4417	0.5408	0.4416	0.2494	0.2766	0.699	1.449	2.236
$2d_{3/2}$			0.3922	0.4416	0.2804	0.2396	0.675	1.445	2.296
$3s_{1/2}$				0.9609	0.2288	0.2288	0.610	1.290	2.031
$1h_{11/2}$					0.2990	0.4000	0.812	1.555	2.231
$1h_{9/2}$						0.2788	0.633	1.259	1.883

a. $\sum_J (2J+1) \frac{G(ababJ)}{(j_a+1/2)(j_b+1/2)}$

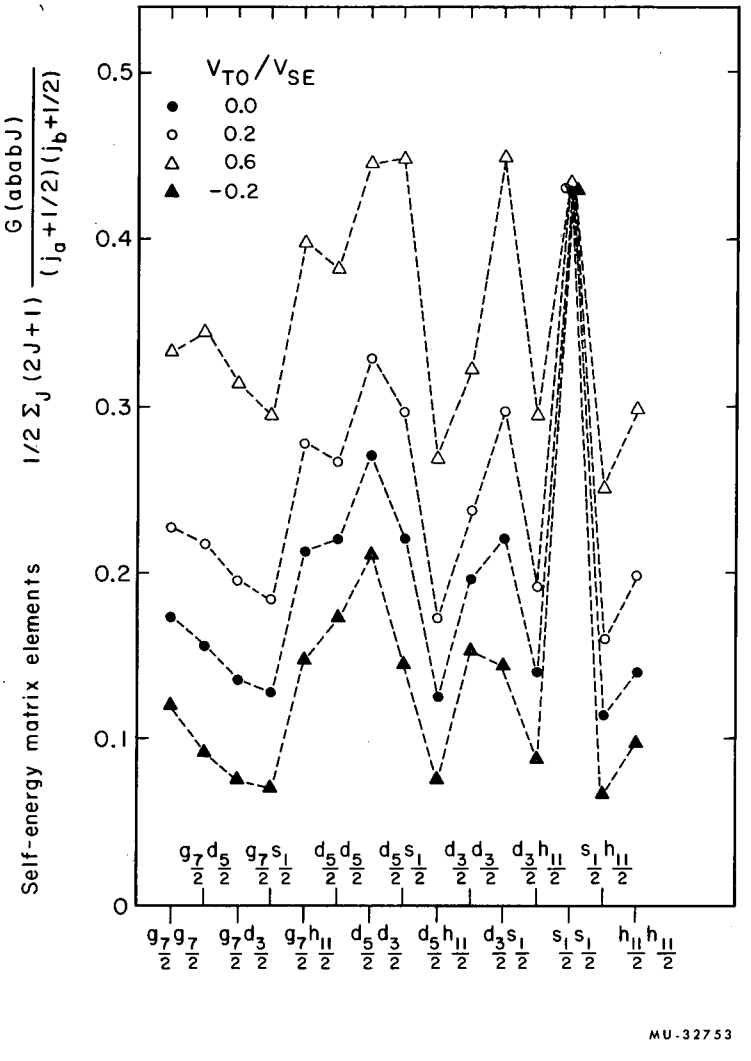

Fig. 10. Self-energy matrix elements for various triplet-to-singlet force ratios, computed with V_0 = -32.44 MeV, $\beta^{-1/2}$ = 1.755 F, $\nu^{-1/2}$ = 2.288 F.

obtained by summing over the products of pair degeneracy, Ω, the matrix elements, $\frac{G(aacco)}{\Omega_a \Omega_c}$, and the occupation probability, v_α^2. Of course, we cannot obtain μ independently of the gap equations, but should solve it self-consistently with them. We exhibit the results in the last column of Table VII for Xe^{136}, Ce^{140}, and Sm^{144}. We can see that, as we have conjectured before, μ is approximately constant for all except the higher-lying levels (for instance, $s_{1/2}$ and $h_{g/2}$). Since these states are relatively unimportant for the nuclei we are considering, such deviation is not expected to modify the spectra considerably. This near constancy of μ is more prominent in heavier nuclei than in light nuclei, since in the former, such fine effects as μ correction are smeared out by an average field, but this cannot be said for light nuclei.

In order to see the effect of self-energy terms on the occupation factor v_α^2, we have solved the gap equation with and without self-energy terms and with all the other parameters fixed. This is shown in Fig. 11. As can be seen, v^2 is changed very little. This implies that setting $\mu=0$ does not cause a serious error, and is perhaps more consistent with the line of approach we have adopted.

Thus one can expect that Set I and Set III with μ give similar qualitative results for most of the nuclei we are studying except for the renormalization of force strength in nuclei with $Z > 58$ and possible differences arising from the separation of $g_{7/2}$ and $d_{5/2}$. This feature has in fact already been observed in odd-even mass differences given in Fig. 6. This statement does not hold for nuclei with larger Z (≈ 64), not because of the importance of μ, but because of the position of the Fermi level (again due to the choice of energy separations).

We give the gap solutions calculated for even-A nuclei of interest with Set I in Appendix C. The solutions are expected to be rather good for nuclei around Ce^{140}, but those for nuclei far away from it should not be taken too literally. One defect of our Set I solutions for heavier nuclei is evident in that α-decay half-width calculation seems to require a substantial decrease of Δ at the Z=64 subshell, whereas our results do

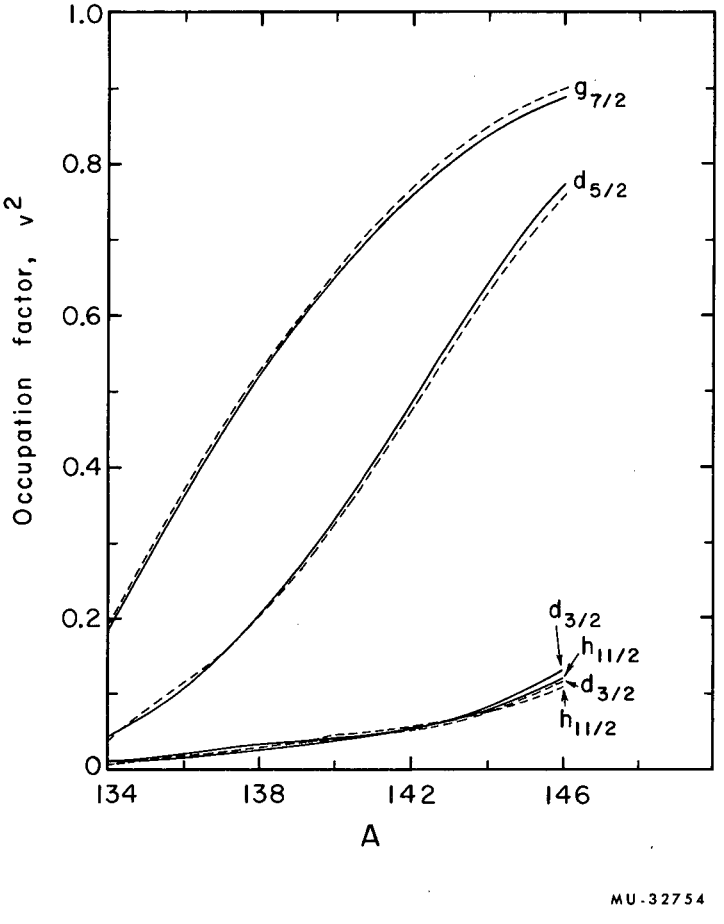

Fig. 11. Effect of self-energy terms on the occupation factor v^2 in 82-neutron nuclei, computed with y = 1.05 in Set III. --- v^2 with self-energy terms, ——— v^2 without self-energy terms.

not show this. This defect can be corrected, however, by adjusting the spacing of $d_{5/2}$ and $h_{11/2}$ levels[46] (a somewhat more detailed remark is given in Sec. IV.k.).

Let us discuss two of the detailed features neglected in the calculation. We have not taken into account the effect of blocking in the excited quasi-particle states. This is essentially an effect of the exclusion principle, and has been studied extensively in deformed nuclei. Nilsson and Prior have calculated the effect of blocking on the odd-even mass differences in the deformed region,[25] and find that the effect is to change $P \approx \Delta$ to $P \approx 1.1\,\Delta$, around 10% increase (P denotes odd-even mass differences and Δ denotes energy gap). Soloviev, on the other hand, has shown a more dramatic effect of blocking by introducing blocking corrections into the excited states (noninteracting two-quasi-particle states) of deformed even-even nuclei; his analysis shows that the correction induces, with a physically reasonable force strength G, a variation of Δ from $\approx$ 20 to nearly 100%.[47] This, of course, does not mean that the overall picture changes with the inclusion of blocking, and owing to the higher-order corrections in the gap matrix elements, it is difficult to make a unique analysis of this problem. However, it seems that a more detailed study than our work should take such effects into account in an appropriate way.

In spherical nuclei, on the other hand, the meaning of blocking is not well understood. Furthermore, a complication arises which destroys the simplicity of the quasi-particle model. To be consistent, one has to resort to blocking for both ground and excited states; in such a case, the canonical transformation method is not adequate, since then the vacuum is no longer invariant. One has to assume a vacuum for excited states different from that for the ground state. In order to maintain simplicity, and since it is not clear whether blocking would give any better results in spherical nuclei, we neglect such an effect in our study.

We have already discussed how good the number distribution is in the BCS approximation (Sec. I). Also, detailed studies have been made on

this problem by various authors with various methods.[13,18] The spuriousness in the number of particles can in principle (and in practice) be eliminated by projecting out of the BCS wave functions only the components describing fixed-particle states, and it turns out that such a procedure in some cases modifies drastically the distribution of particle numbers and matrix elements of single-particle operators. But such a prescription has so far been used only for the ground state, and it is still an open question how one could employ it in interacting quasi-particle systems. We do not use this prescription in our work.

F. Two-Quasi-Particle Spectra

The zero-order wave functions are those given by the BCS procedures; proper angular-momentum coupling and antisymmetrization through second quantization are as given before. Equation (II-8) is then solved by the IBM 7094. Computations are done first by using only $|g_{7/2}{}^2\rangle$, $|d_{5/2}\, g_{7/2}\rangle$, and $|d_{5/2}{}^2\rangle$ configurations, since all the other states lie much higher. A second set of computations uses the independent-particle energy Set I, with no self-energy correction, and the third Set III with the self-energy correction included. These last two include all the two-quasi-particle configurations except those involving $s_{1/2}$ and $h_{9/2}$. We believe that neglect of these states does not cause a serious error in the results, in view of the fact that they are rather high-lying. We give the results in Figs. 12 through 18, and wave functions in Appendix C for some low-lying states (i.e., 2+, 3+, and 4+).

As Fig. 12 shows, the 2+ and 4+ states in Ce^{140} can be fitted best with the triplet-to-singlet force ratio $t \approx 1$. This, as we have mentioned before, is in disagreement with t=0, which seems to be better for the solutions of gap equations. The study by Arvieu et al. also indicates that a better fit with experiment could be obtained with t=1, though their conclusion applies to both the pairing and quasi-particle interactions.[40] As is well known, a repulsive odd-force is necessary for low-energy nuclear

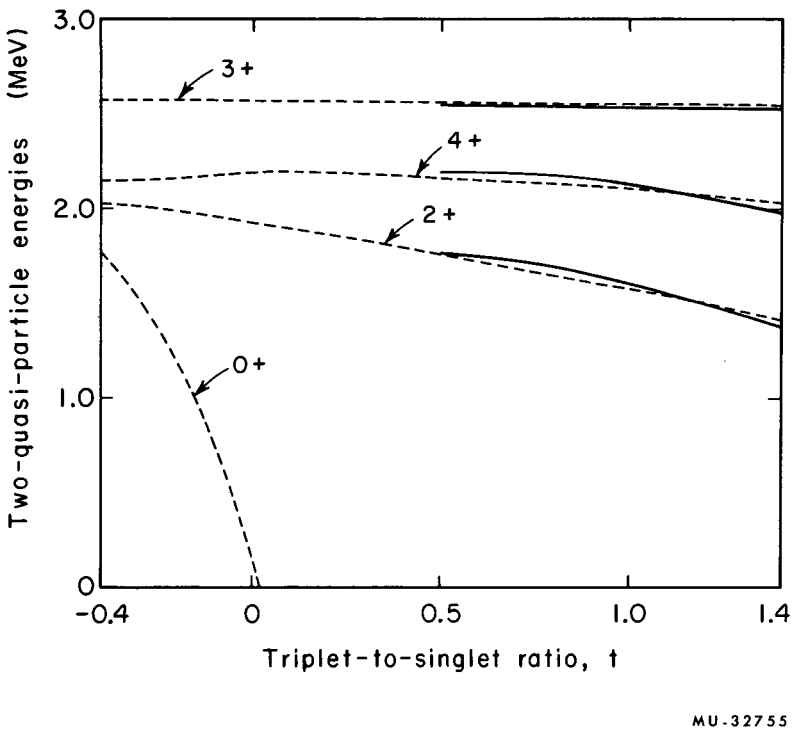

Fig. 12. Theoretical two-quasi-particle spectra for Ce^{140}. For pairing interaction, Set I with x = 1.20 is used, for quasi-particle interaction, only three configurations, $(d_{5/2}{}^2)$, $(d_{5/2}\ g_{7/2})$ and $(g_{7/2}{}^2)$ are taken into account. Solid lines, computed with Set I and x = 1.15 including seven configurations, are included to compare with three-configuration approximation. Curves show depression of levels as a function of triplet-to-singlet ratios.

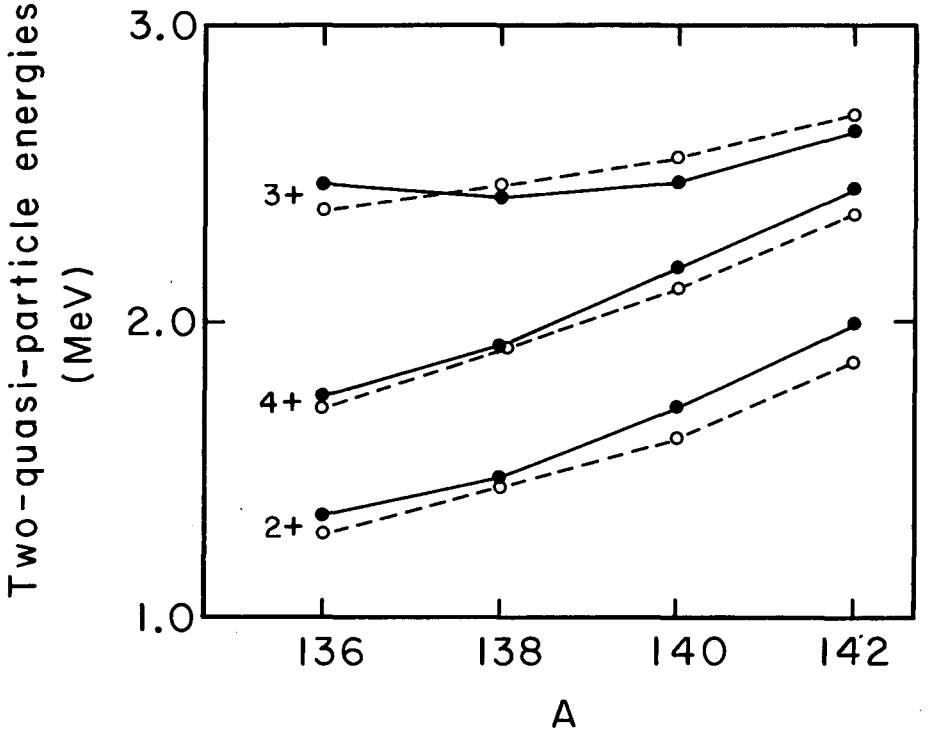

Fig. 13. Theoretical two-quasi-particle spectra for Xe^{136}, Ba^{138}, Ce^{140}, and Nd^{142} with three-configuration approximation. Set I with x = 1.20 and V_{TO}/V_{SE} = t = 1.0 are used. Solid lines correspond to the diagonalization of H_{22} with only q = 2 and 4 components of force, dashed lines to that of H_{22} with all components. V_0 = -1.20 × 32.9 MeV, $\beta^{-1/2}$ = 1.732 F.

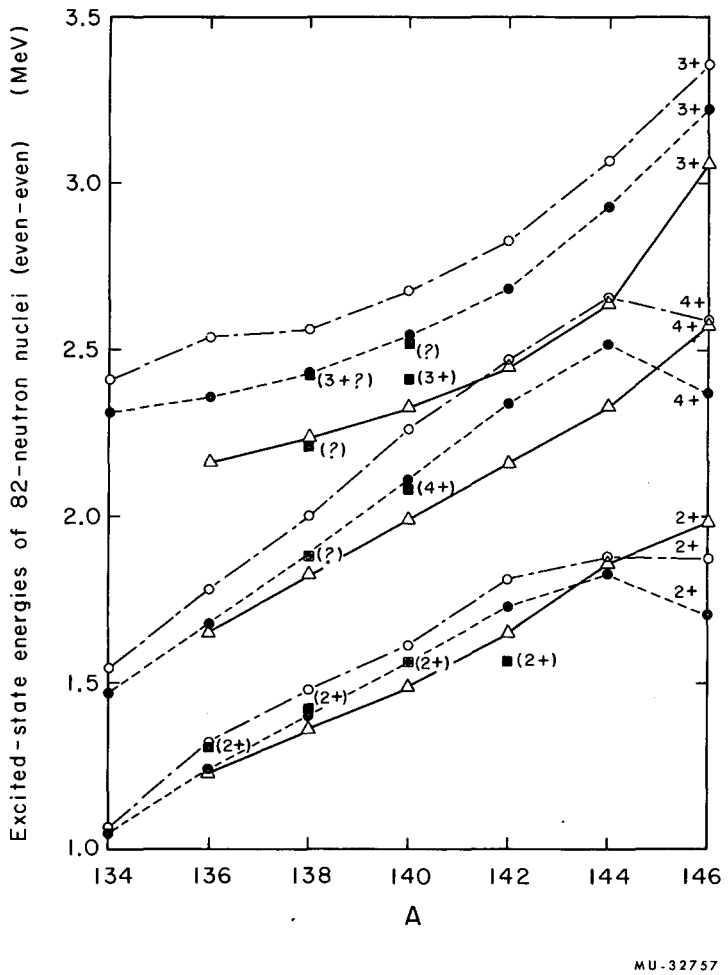

Fig. 14. Theoretical two-quasi-particle spectra for 82-neutron even-A nuclei computed with all seven configurations, t = 1.0. Experimental points are given in solid squares, with known spins in parenthesis. If spins are unknown, or tentatively assigned, question marks are put in the parenthesis.

● ---- 1.15 × 32.9 (Set I) ○ ---- 1.20 × 32.9 (Set I)

△ ---- 1.05 × 32.4 (Set III)

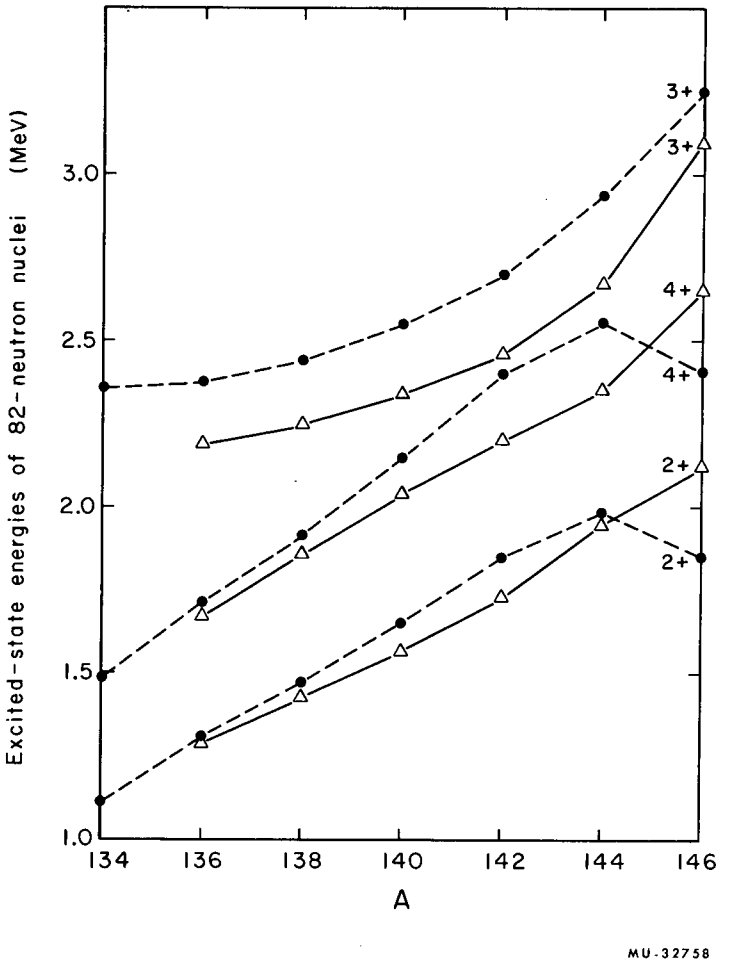

Fig. 15. Theoretical two-quasi-particle spectra for 82-neutron, even-A nuclei computed with all seven configurations, t = 0.8. Experimental points are given in solid squares, with known spins in parenthesis. If spins are unknown, or tentatively assigned, question marks are put in the parenthesis.

△— y = 1.05, Set III.

● x = 1.15, Set I.

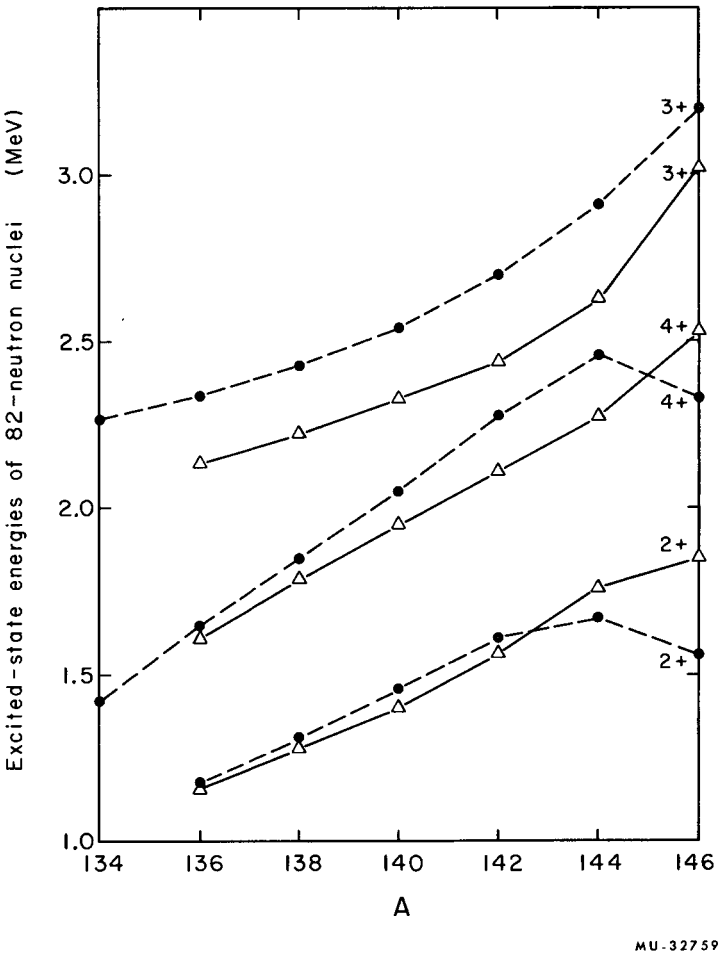

Fig. 16. Theoretical two-quasi-particle spectra for 82-neutron, even-A nuclei computed with all seven configurations, t = 1.2. Experimental points are given in solid squares, with known spins in parenthesis. If spins are unknown, or tentatively assigned, question marks are put in the parenthesis.

△— y = 1.05, Set III.

● x = 1.15, Set I.

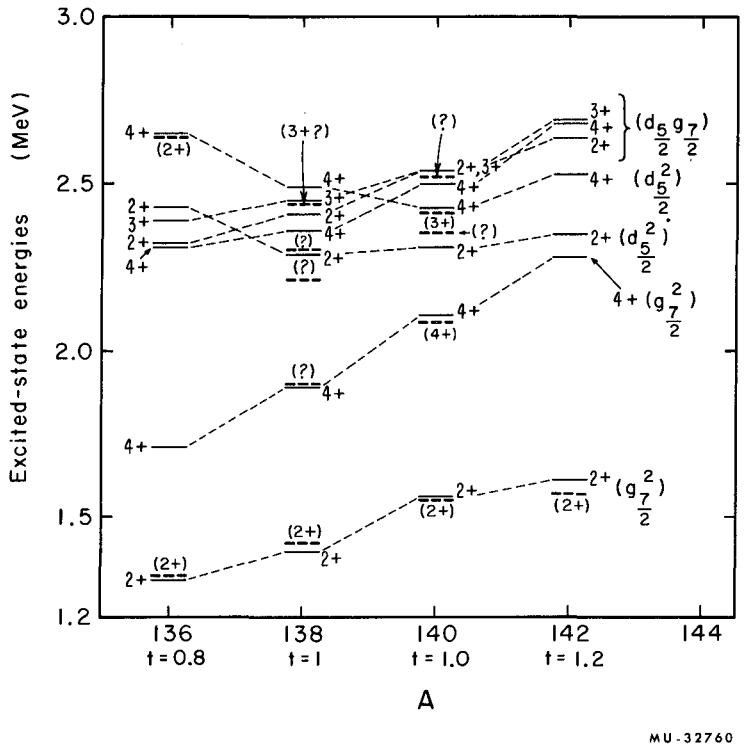

Fig. 17. Theoretical 2+, 3+, and 4+ states in A = 136 to A = 142 with slight changes in t. Differences in t show the renormalization of force as a function of number of particles outside the core.

x = 1.15 with Set I.

——— J = theoretical;

- - - (J) = experimental;

▬ ▬ ▬ (?) = experimental with spins unidentified or tentative.

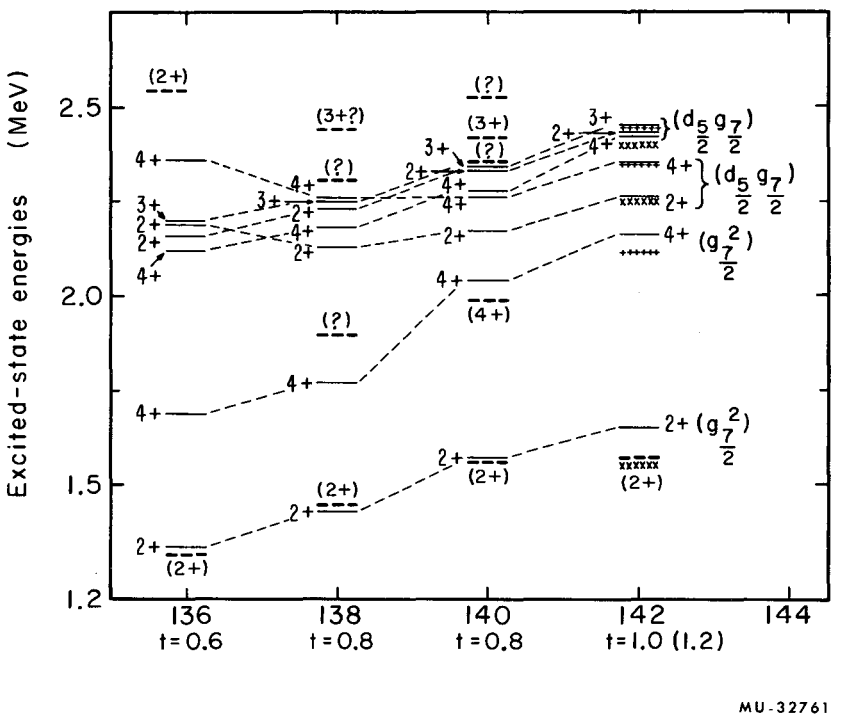

Fig. 18. Theoretical 2+, 3+, and 4+ states in A = 136 to A = 142 with slight changes in t. Differences in t show the renormalization of force as a function of number of particles outside the core.

y = 1.05 with Set III.

——— J = thoretical;

- - - (J) = experimental;

xxxx theoretical 2+ for t in parentheses;

++++ theoretical 2+ for t in parentheses;

▬ ▬ ▬ (?) experimental with spins unidentified or tentative.

properties, and the usual shell-model calculations also include such a component. We leave the answer to the question of why t>0 to Sec. IV.K. Here we shall try to give a feasible explanation why t in the quasi-particle interaction turns out to be larger than that in the pairing interaction in our calculations.

(a). The study by Kim[42] shows that inclusion of a tensor force [in our case, the tensor-triplet-odd (TTO) part] affects mostly the states with J=0, whereas it modifies very little the states with $J>0$ in a system of one kine of particles.

(b). The inclusion of TTO in the analysis of free-nucleon scattering phase shifts with the Brueckner-Gammel-Thaler (BGT) force turns out to be equivalent to taking $t < 0$, say $t \approx -0.4$, although this is strictly true for J=0 only (for $J>0$, this may not hold).

(c) Since, as we have shown before, only J=0 states are involved in pairing interaction, we should perhaps compute the gap solutions with $t \approx -0.4$. This, we mentioned, was not a proper choice of t because of the discrepancy on odd-even mass differences. Let us suppose that for some reason (say, a renormalization of force strength), $t \approx 0$ in the pairing interaction.

(d). We are primarily interested in the states with $J > 0$ arising from the quasi-particle interactions. Then, according to the statement (a), we may neglect TTO.

Now, if we assume that the renormalization acts in the same way (though perhaps not exactly in the same amount) in the pairing and quasi-particle interactions, then the statements (a) through (d) imply that it is reasonable to use a larger t in calculations of two-quasi-particle spectra than for the pairing interactions. This also implies that a two-quasi-particle 0+ state may be determined by looking at $t \approx 0$ rather than at $t \approx 1$ (of course, because of the spuriousness of this state, this procedure is still not good enough, but if one uses the RPA, this is probably what one should do). With this much justification, we shall fron now on take $t \approx 1$ for <u>all quasi-particle interactions</u>.

Not many 4+ states are known experimentally in the 82-neutron region; hence no conclusive comparison can be made. The only observed 4+ state is in Ce^{140}, and it lies 2.08 MeV above the ground state. This level seems to be correctly identified with Set I, x=1.15, and t=1, and its dominant component is $|g_{7/2}^{2}\rangle$ (although the experimental magnetic g factor measurement prefers the $|g_{7/2}\ d_{5/2}\rangle$ configuration; see Sec. IV.I) and the wave function determined here seems to be consistent with the B(E2) for 4+ to 2+ transition (Sec. IV.G).

The 2+ states, however, are known for almost all 82-neutron nuclei; let us consider this state in detail, since it is of interest for microscopic or macroscopic theories because of its collective nature. One can see from Figs. 17 and 18 that both sets of ϵ_α's give "satisfactory" results, if we choose t between 0.6 and 1.4. However, for one t (say, t=1), our theoretical calculation does not reproduce the lowering of the 2+ state in Nd^{142} below or equal to that of Ce^{140}, as is observed in experiment. With due consideration of experimental errors, the lowering of about 40 keV, which is the discrepancy here, may not be significant. Nevertheless, our result shows a monotonic increase (a separation of 100 keV); this trend continues up to Sm^{144} then drops, at Gd^{146}, to about 1.7 MeV (Set I). Theoretically the result of Nd^{142} is rather natural, for the following reasons. With Set I, the spacing between independent quasi-particle states of configurations $g_{7/2}^{2}$ and $d_{5/2}^{2}$ is given by $\Delta E\ (g_{7/2}^{2}-d_{5/2}^{2}) > 0$, and the magnitude is small ($\approx$ 200 keV). But in this case, we have the inequality $|\langle g_{7/2}^{2}|H_{22}|g_{7/2}^{2}\rangle| > |\langle d_{5/2}^{2}|H_{22}|d_{5/2}^{2}\rangle|$, and both matrix elements are attractive. This is in turn due to larger Slater integrals for $g_{7/2}^{2}$ than for $d_{5/2}^{2}$, for $q \geq 2$ (for $q = 0$, about the same), and to u and v factors.

Specifically, we get the results given in Table VIII computed for 2+ in Nd^{142} with V_o =-1.15 x 32.9 MeV, t = 1.0, $\beta^{-1/2}$ = 1.732.

Table VIII. Matrix elements of H_{22} for $g_{7/2}^2$ and $d_{5/2}^2$ in Nd^{142}.

α	$2E_\alpha$	$\langle H_{22} \rangle$	$2E_\alpha + \langle H_{22} \rangle_\alpha$
$(g_{7/2}^2)_{2+}$	2.7500	-0.6193	2.1307
$(d_{5/2}^2)_{2+}$	2.5500	-0.1423	2.4077

Thus, when the interaction H_{22} is turned on, the $(g_{7/2}^2)_{2+}$ comes lower than $(d_{5/2}^2)_{2+}$ even in the first-order perturbation theory. However, $2E_{(g_{7/2}^2)} + \langle H_{22} \rangle_{(g_{7/2}^2)}$ is higher than that in Ce^{140} (which is 1.805 MeV with the same parameters), and the off-diagonal terms are about the same (for example, $\langle (d_{5/2}^2)_2 | H_{22} | (g_{7/2}^2)_2 \rangle$ = -0.2763 in Ce^{140} and -0.2736 in Nd^{142}), the 2+ state in Nd^{142} cannot be expected to be equal to or lower than that in Ce^{140}. In our framework, therefore, the only way to fit the experiment is to have the quasi-particle energy of $(d_{5/2}^2)$ much lower than that of $(g_{7/2}^2)$. This consideration applies to a large extent to the 4+ state, and (as we shall see later) the magnetic g factor of the 4+ state seems to indicate that Ce^{140} is already a large mixture of $(g_{7/2}^2)$, $(d_{5/2}^2)$, and $(d_{5/2}\, g_{7/2})$. This also suggests that our choice of spacing between $g_{7/2}$ and $d_{5/2}$ might be inadequate. Because of these considerations, we have also tried smaller spacings, but such modifications led to wrong odd-even mass differences and the spacing between 2+ and 4+ states in Ce^{140} became worse. There is, however, one way out of this dilemma and that is to invoke a renormalization effect on the force strength. This is not an established fact, but we shall return to this point later in our discussion.

The reason why Sets I and III give a completely different picture at Gd^{146} is that 2+ with I has the dominent component of $(h_{11/2}^2)$, while III has that of $(d_{5/2}^2)$. This is of course due to the choice of $d_{5/2}$-$h_{11/2}$ separation. In case I, the diagonal element of $(h_{11/2}^2)$ is large to give

a depression which is observed in our result. Experimental data[48] on Eu^{147} and Eu^{149} which have 63 protons identify the $h_{11/2}$ level about 500 keV above the ground state $d_{5/2}$. The first excited state is $g_{7/2}$, making us suspect that $|\epsilon h_{11/2} - \epsilon d_{5/2}| > |\epsilon d_{5/2} - \epsilon g_{7/2}|$. Though this statement is not strictly correct -- for, in these nuclei, one has to take into account also the proton-neutron interaction -- it seems likely that one has to take the spacing somewhat larger than what we have in Set I. This of course is the motivation for choosing Set III; however, in the absence of experimental data in this region on the energy levels, B(E2), etc., no conclusion can be drawn here.

We believe that the higher-lying states such as 3+ and 4+ can be fairly well described by H_{22} alone. There is only one experimentally known 4+ level in this region (4+ in Ce^{140}), and it seems that the theoretical spacing between 2+ and 4+ in this nucleus is consistent with experiment. We also test the wave function of the 4+ state in the next section. The experimental 3+ level, which also is observed only in Ce^{140}, is lower than the Set I result (about 150 keV) and higher than the Set III result (about 70 keV). The only low-lying 3+ state is of the configuration $(d_{5/2}\ g_{7/2})_{3+}$, and the other configurations that might perturb the lower 3+ state are $(d_{5/2}\ d_{3/2})_{3+}$ and $(d_{3/2}\ g_{7/2})_{3+}$. But the last two states are very high in energy, and thus one would expect the lowest 3+ to be an almost pure $(d_{5/2}\ g_{7/2})_{3+}$ two-quasi-particle state. This is borne out by the result. The wave function turns out to be (for Set I, t = 1.0)

$$|3+\rangle = 0.99992|(d_{5/2}g_{7/2})_{3+}\rangle + 0.00421|(d_{5/2}d_{3/2})_{3+}\rangle + 0.01172|(d_{3/2}g_{7/2})_{3+}\rangle.$$

Now, since this state is almost pure, the fit with experiment may be obtained by adjusting parameters used for pairing solutions. This could perhaps be put to advantage in determining some of the parameters,

but in practice, we found that if we fitted theory to the experiment for the 3+ state, the spacing between 4+ and 2+ was upset. The difficulty with 3+ states was also observed by Arvieu et al.[38] in Pb^{206}, for which they found the theoretical 3+ level to be about 235 keV higher than the experimental position. Why this is so in two different regions, or whether it is a mere accident, is not understood; additional experimental results would be very useful for an understanding on this point.

G. B(E2) for 4+ to 2+ Transitions

It has been observed experimentally that in Ce^{140}, the electric quadrupole transition from the 2.08-MeV 4+ state to the 1.6-MeV 2+ state is very retarded; in fact[49] $\frac{B(E2)_{sp}}{B(E2)_{exp}} \approx 17$, were $B(E2)_{sp}$ is the single-particle B(E2) unit defined by

$$B(E2)_{sp} = 3 \cdot 10^{-5} A^{4/3} e^2 10^{-48} \text{ Cm}^4 ,$$

or in general 2^λ-pole transitions,

$$B(E\lambda)_{sp} = (2\lambda+1) \frac{e^2}{4\pi} \left(\frac{3}{\lambda+3}\right)^2 R_0^{2\lambda}$$

As we have discussed before, we expect this transition to be considerably slower than the 2+ → 0+ transition because of the reduction factor $(u_a u_b - v_a v_b)$ and the Racah coefficients. However, in view of the fact that $\frac{B(E2)_{sp}}{B(E2)_{th}} \approx 3$, if one calculates with the pure configuration $g_{7/2}^2$, it is obvious that some other mechanisms are required (in addition to the

above factors) to explain this drastic retardation. To see this, we have used the wave functions of two-quasi-particle states tabulated in Appendix C to compute the B(E2) for the $4+ \rightarrow 2+$ transition. The results are shown in Table IX for the choice of parameters as given by Set I. In all nuclei, one observes, the transitions are markedly retarded. Especially Ba^{138} shows that B(E2) is nearly zero. The only possible explanation is that there is a large cancellation in the transition-matrix elements.

It is obvious that, firstly, each matrix element multiplied by appropriate amplitudes is small -- this is due to the $(u_a u_b - v_a v_b)$ factor and Racah coefficients (we might consider this as an interference factor arising from complications in geometry of the physical process); secondly, random signs give rise to cancellations among the transition-matrix elements. Such cancellation is again due to the factor $(u_a u_b - v_a v_b)$, which has the properties

$$(u_a u_b - v_a v_b) \begin{cases} > 0 & \text{if a and b are above } \lambda \text{ (chemical potential),} \\ < 0 & \text{if a and b are below } \lambda, \\ \text{random} & \text{if one is above and the other is below } \lambda, \end{cases}$$

and due to the signs of the amplitudes multiplying each matrix element. It is precisely these mechanisms that makes $\frac{B(E2)_{sp}}{B(E2)_{th}}$ for Ba^{138} so small. The cancellation persists throughout the nuclei (so the ratio is large for almost all nuclei), but predominance of contributions with one sign over the other makes B(E2) for A=134 and A=146 rather large, on the order of $B(E2)_{sp}$.

Our value of $\frac{B(E2)_{sp}}{B(E2)_{th}} = 28$ seems to be considerably larger than the experimental value 17 in Ce^{140}, but at least it shows the correct behavior of large retardation. We may remark here that a drastic approximation of neglecting all the states except $(g_{7/2}^2)$, $(d_{5/2}^2)$, and $(d_{5/2}g_{7/2})$ still maintains the main features of this treatment. In Ce^{140} (See Table X), the ratio is about 14, a little less than the experimental value.

Table IX: B(E2) values for $2+ \rightarrow 0+$ and $4+ \rightarrow 2+$ transitions in units of 10^{-48} cm^4 e^2. Wave functions used in the calculation correspond to those obtained with $V_0 = -32.9\chi$ MeV for $\chi = 1.15$ and 1.20, $\beta^{-\frac{1}{2}} = 1.732$ F, $\frac{V_{TO}}{V_{SE}} = 1.0$, and Set I ($e_{eff} = 2e$).

		$2+ \rightarrow 0+$ transitions				$4+ \rightarrow 2+$ transitions					
	χ	First $2+ \rightarrow 0+$ B(E2)				Second $2+ \rightarrow 0+$ B(E2)		$4+ \rightarrow$ First $2+$			Second $2+ \rightarrow 4+$
		H_{22} [a]	$H_{22}+H_{40}$ [b]	$\frac{B(E2)cal}{B(E2)sp}$ [b]	$\frac{B(E2)exp}{B(E2)sp}$	H_{22} [a]	$\frac{B(E2)cal}{B(E2)sp}$	B(E2)	$\frac{B(E2)sp}{B(E2)cal}$	$\frac{B(E2)sp}{B(E2)cal}$	$\frac{B(E2)sp}{B(E2)cal}$
$_{52}Te^{134}$	1.15	0.101	0.101	4.9				0.993	2.0		56
	1.20	0.103	0.103	5.0				1.05	1.8		
$_{54}Xe^{136}$	1.15	0.178	0.169	8.1		0.0122	0.58	0.203	10.0		415
	1.20	0.180	0.174	8.3				0.233	9.0		
$_{56}Ba^{138}$	1.15	0.228	0.231	10.8	14	0.0103	0.48	0.29×10^{-4}	7.2×10^{4}		49
	1.20	0.237	0.240	11.2				0.29×10^{-2}	7.2×10^{2}		
$_{58}Ce^{140}$	1.15	0.271	0.284	13.1	17	0.0038	0.18	0.079	28	17	14
	1.20	0.283	0.300	13.9				0.052	42		
$_{60}Nd^{142}$	1.15	0.331	0.358	16.1	15	3.02×10^{-4}	0.014	0.084	26		20
	1.20	0.353	0.385	17.3				0.077	29		
$_{62}Sm^{144}$	1.15	0.417	0.458	20.2		0.0007	0.025	0.16×10^{-3}	1.5×10^{4}		17
	1.20	0.436	0.481	21.1				0.026	86		
$_{64}Gd^{146}$	1.15	0.431	0.469	20.3		0.0365	1.58	1.003	2.3		6.6
	1.20	0.453	0.495	21.5				0.879	2.6		

a: the effect of the H_{22} term on the B(E2 ; $2+ \rightarrow 0+$)

b: the effect of the $H_{22} + H_{40}$ on the B(E2 ; $2+ \rightarrow 0+$)

Table X: B(E2:4+→2+), B(E2:2+→0+) and magnetic g factor computed with only three configurations, $(g_{7/2}^2)$, $(d_{5/2}g_{7/2})$, $(d_{5/2})^2$. V_0 =-1.20×32.9 MeV, $\beta^{-1/2}$ = 1.732F, $\frac{V_{TO}}{V_{SE}}$ = 1.0, 1.2, e_{eff} = 2.

A	$\frac{V_{TO}}{V_{SE}}$	$\frac{B(E2)_{sp}}{B(E2)_{4+\rightarrow 2+}}$	$\frac{B(E2)_{2+\rightarrow 0+}}{B(E2)_{sp}}$ [a]	g factor [b]
$_{54}Xe^{136}$	1.0	18.4	6.8	0.80
$_{56}Ba^{138}$	1.0	179	8.1	0.82
	1.2	281	8.2	0.84
$_{58}Ce^{140}$	1.0	13.0	8.4	0.90
	1.2	14.4	8.5	0.92
$_{60}Nd^{142}$	1.0	11.4	8.2	1.24
	1.2	9.80	8.1	1.20

a: Includes the effect of H_{22} only.

b: Using g factor of neighboring isotopes.

Although no experimental results are available in this region, we have also computed B(E2) for $2'+ \rightarrow 4+$ transitions, where 2'+ means the second 2+ state. This is also shown in Table IX. The results $B(E2)_{2'+ \rightarrow 4+}$ are comparable in magnitude of retardation to the $B(E2)_{4+ \rightarrow 2+}$.

H. B(E2) for $2+ \rightarrow 0+$ Transitions

This quantity is of special interest, since experimentally the first $2+ \rightarrow 0+$ ground-state transition even in spherical nuclei is enhanced by more than ten times the single-particle estimate, and a reasonable theory should be able to display this characteristic. It is believed by many poeple that the first 2+ is a collective vibrational state, and hence a large enhancement is expected. In spherical nuclei, the meaning of collective vibration is not well understood, and furthermore an application of Bohr-Mottelson concepts of collective behavior of finite nuclei does not seem to be justified. This large enhancement of B(E2) in spherical nuclei should be explained on the basis of microscopic descriptions; that is, such effects should be regarded as being generated from interactions between particles in the system.

In our approach, we need not introduce the boson approximation for a pair of particles, nor invoke a specific vibrational hypothesis for the state. We merely consider interactions between two particles, and improve the ground state, if necessary, by introducing further correlations into it. Let us now discuss these points in more detail.

Table IX contains the transition probabilities computed with the same parameters as used in two-quasi-particle excitation spectra (i.e., t = 1.0). The equation used is Eq. (III-44). We have already shown in Sec. III.C that $B(E2)_{2+ \rightarrow 0+}$ is much larger than $B(E2)_{4+ \rightarrow 2+}$, just from order-of-magnitude arguments. Furthermore, contributions from various components turn out to be coherent, to give a resultant large matrix element. This is shown in Table XI for come of the nuclei considered.

Table XI. Contributions to $\langle \hat{0} | \mathfrak{M}(E2) | 2+ \rangle \, C^{\alpha}_{ilk}$ from each component of wave functions of two-quasi-particle states.
$V_0 = -1.15 \times 32.9$ MeV, $\beta^{-1/2} = 1.732$F, $\frac{V_{TO}}{V_{SE}} = 1.0$,
and Set Ia. The unit is in $(\frac{\hbar}{M\omega})e_{eff}$.
Last row is added to show contributions from H_{40} interaction

Nuclei / States	Ba^{138}	Ce^{140}	Nd^{142}
$d_{3/2}^2$	0.024	0.037	0.058
$d_{5/2}^2$	0.461	0.768	1.064
$d_{5/2}g_{7/2}$	0.096	0.100	0.090
$g_{7/2}^2$	2.226	1.962	1.536
$h_{11/2}^2$	0.106	0.239	0.594
$d_{5/2}d_{3/2}$	0.061	0.103	0.168
$g_{7/2}d_{3/2}$	0.278	0.317	0.370
sum	3.252	3.526	3.880
H_{40} contribution	0.020	0.099	0.157
Total sum	3.272	3.625	4.037

This is one example in which one could show that the higher-order perturbation series contribute with the same sign to the total quantity. Notice that even though an amplitude for a component is small, the matrix element may turn out to be rather large, and that the total sum from states other than the dominant configuration is usually equal to or greater than the main component.

Still the computed B(E2) is not quite sufficient to fit the experimental results. One can easily guess that this insufficiency will be more marked, the lower-lying the 2+ state is. This is because a very low-lying solution may not be obtained from our treatment.

In order to see what can be done to lessen this defect, we have introduced independent four-quasi-particle states into the ground state. Four-quasi-particle states involve on the average an excitation energy $\approx 4E_\alpha$, where E_α is a single-quasi-particle energy (about 1 MeV). Therefore one would expect the contributions from four-quasi-particle states to be insufficient to make drastic changes in the results already obtained. However, B(E2) is a square of the matrix element; thus even a small contribution is expected to be significant in the studies of finer details. We list contributions to the matrix element from four-quasi-particle states in Table XI together with those from the H_{22} term. The four-quasi-particle states taken into account are given in Table XII.

The trend is rather obvious; as Z is increased, the contributions from both the H_{22} term and the H_{40} term increase. If one examines the matrix element of H_{40} between the ground state and a four-quasi-particle state, one can see that the particle-particle and particle-hole interactions compete, i.e., they come with opposite signs. Since in a pure configuration (i.e., $[j_a^{\,4}]$), we have G>>F, and also KS's studies, which essentially take only the F term, show a coherent contribution to B(E2), it is clear that such a pure configuration yields a repulsive matrix element in $\langle 4 \mid H_{40} \mid \hat{0} \rangle$. On the other hand, configurations of the form $[(j_a^{\,2})_J\ (j_b^{\,2})_J]$ have a large F term, and hence mainly enhance the transition probability. It is thus a reasonable conclusion that a model

Table XII. Four-quasi particle states coupled to ground state for calculation of B(E2;2+→ 0+) (ν = seniority quantum number).

Configuration[a]	Possible ν or J values	Nuclei which take into account these states
$[g_{7/2}^4]_0$	$\nu = 0$	
$[d_{5/2}g_{7/2}^3]_0$	$\nu = 3$	
$[(d_{5/2}^2)_J(g_{7/2}^2)_J]_0$	J = 0,2,4	
$[d_{5/2}^4]_0$	$\nu = 0$	A = 136-146
$[h_{11/4}^4]_0$[b]	$\nu = 0,4$	
$[(h_{11/2}^2)_J(g_{7/2}^2)_J]_0$	J = 0,2,4,6	
$[(h_{11/2}^2)_J(d_{5/2}^2)_J]_0$	J = 0,2,4	
$[(h_{11/2}^2)_J(d_{3/2}^2)_J]_0$	J = 0,2	
$[(d_{3/2}^2)_J(g_{7/2}^2)_J]_0$	J = 0,2	
$[(d_{3/2}^2)_J(d_{5/2}^2)_J]_0$	J = 0,2	

a: The configuration $[d_{5/2}^3 g_{7/2}]_0$, which is missing from the list, does not exist, since the antisymmetrized wave function for any seniority quantum number ν vanishes for a group-theoretical reason.

b: For simplicity, we neglect the $\nu = 4$ state.

calculation of KS amounts to taking only the second type of matrix elements, including the pure configurations in the form $[(j_a{}^2)_J(j_a{}^2)_J]$, with J = 2 (it is easy to see that these are all of a coherent sign).

Now, in the nuclei of lower Z, the only energetically favored states are the first four configurations listed in Table XII. Among these, $[g_{7/2}{}^4]_0$ with seniority zero gives the largest contribution to the transition matrix element, but with an opposite sign to the H_{22} contributions. Thus the four-quasi-particle states either lower the B(E2) or make no significant difference.

However, as more particles are added, all the states listed in Table XII become gradually more important, and the net effect is to increase the B(E2) values, since there are more configurations of the form $[(j_a{}^2)_J(j_b{}^2)_J]$ than pure ones.

Comparison with experimental results (though not abundant enough to be conclusive) shows that the theoretical trend does not appear to be consistent with the experimental one. Since the four-quasi-particle effect is fairly small in general, the main theoretical trend is dictated by the two-quasi-particle interactions (through H_{22}). We have already observed a similar discrepancy between theory and experiment in the two-quasi-particle spectra, and as pointed out there, a renormalization of the force constant seems to be present in both the energy spectra and the electric quadrupole transitions. Though not tabulated here, a calculation with a higher t shows the lowering of the B(E2) for the nuclei with Z > 58 (see Table X). The monotonic increase of B(E2) was also observed by KS in their first paper[15a]; their B(E2) results are, however, larger than the experimental results by almost a factor of two, whereas ours are lower by about 19% (in Ce^{140}, for example). This discrepancy of about 19% is not, we believe, a serious one, in view of the crudity of the parameters used and of the uncertainties in the single-particle energies. Besides, the interaction between four quasiparticles may also be significant; this point needs

further study if the importance of four-quasi-particle states in the ground state is to be understood more rigorously.

Whereas the contribution to B(E2) is small, the effect of four-quasi-particle states on the total ground-state energy $\langle H \rangle_0$ in the lowest order of perturbation theory is expected to be rather considerable; to second order, the energy change $\Delta E < 0$ for all the states, and furthermore states like $[(j_a^{-2})_0 (j_b^{-2})_0]$ have large $\langle 4 \mid H_{40} \mid \hat{0} \rangle$ values. Thus the net result is to lower the ground state by a large amount. To higher orders (in H_{22}), however, the energy denominator gets larger and the energy shift is expected to be reduced (just as in the case of three-quasi-particle effect). For this reason, we do not give the energy shift of the ground state in this work. (Notice that the reason why we have $|\Delta(B(E2))| << |\Delta E|$ is that the configurations involving $[(j_a^{-2})_J (j_b^{-2})_J]$ with j_a or $j_b = h_{11/2}$ do not contribute for J=0 in B(E2)).

As a side interest, we have also computed B(E2) for the second $2+ \rightarrow 0+$ ground-state transition. It is well known experimentally that this transition is highly retarded compared with the first $2+ \rightarrow 0+$ transition. In our method, this transition is not strictly forbidden, whereas in the linearized RPA method, it is forbidden. Our results (Table IX) are indeed consistent with experimental results (for other nuclei, since in the region of our interest, experimental values are not available). KS computed these also by an extended RPA, and one finds that their results are frequently lower by an order of magnitude. We have not much to compare with theirs except Ce^{140}. They find $B(E2)_{2'+ \rightarrow 0+}$ in Ce^{140} to be ≈ 0.0001, while our results give ≈ 0.004. A general trend seems to indicate that as A increases, B(E2) falls to a minimum (at A=142) and rises again to an order of $B(E2)_{sp}$ (at A=146). This rising trend is expected to continue if we approach the deformed region. This feature has also been observed by KS.[15b]

I. Magnetic g Factor of 4+ State

There seem to be no experimental data for magnetic g factors of excited states in this region, except for the 4+ level in Ce^{140}, which was recently measured. As we have tested the wave function of 4+ state for Ce^{140}, in particular by looking at the $4+ \rightarrow 2+$ transition probability, it is also of interest to see how good it is in fitting the magnetic g factor. The results are shown in Fig. 19. In computing the g factor of the 4+ state, the values of g_j for $d_{5/2}$ and $g_{7/2}$ are taken from neighboring odd-A nuclei, and those for $d_{3/2}$ and $h_{11/2}$ are estimated with

$$g_\ell = \left\{ \begin{array}{c} 1 \\ \\ 0 \end{array} \right\} \quad \text{and} \quad g_s = \left\{ \begin{array}{c} 5.585 \\ \\ -3.826 \end{array} \right\} ,$$

where the upper entry corresponds to the proton and the lower to the neutron. Since the configurations involving $d_{3/2}$ and $h_{11/2}$ have very small amplitudes in Ce^{140}, the contributions from them are negligible. Thus the g factor can also tell us something about the degree of mixing among the configurations $| g_{7/2}^2 \rangle$, $| d_{5/2}\, g_{7/2} \rangle$ and $| d_{5/2}^2 \rangle$. Our results, g=0.92, for Ce^{140} is slightly lower than the experimental value of 1.15 measured by Kaplan et al.[50] (Professor D. A. Shirley pointed out to us that the results of Kaplan et al. might be in error.[53] He and Levy[51] obtained $g_{4+} = 1.08 \pm 0.10$. He also quoted the result of Bodenstedt et al.,[52] $g_{4+} = 1.11 \pm 0.04$, which is considered to be the best value. We are indebted to him for informing us on these points.) These authors estimate their experimental error to be 0.08. They conclude that the 4+ level should be $(d_{5/2}\, g_{7/2})$ state, or a mixed configuration with the dominant configuration being $(d_{5/2}\, g_{7/2})$ state (see Table XIII). Our calculation shows, on the other hand, that $(g_{7/2}^2)$ is the dominant state with a mixture of $(d_{5/2}^2)$ and $(d_{5/2}\, g_{7/2})$ (i.e., $| 4+ \rangle \cong 0.30 \, | d_{5/2}^2 \rangle + 0.91 \, | g_{7/2}^2 \rangle - 0.26 \, | d_{5/2}\, g_{7/2} \rangle$). As we

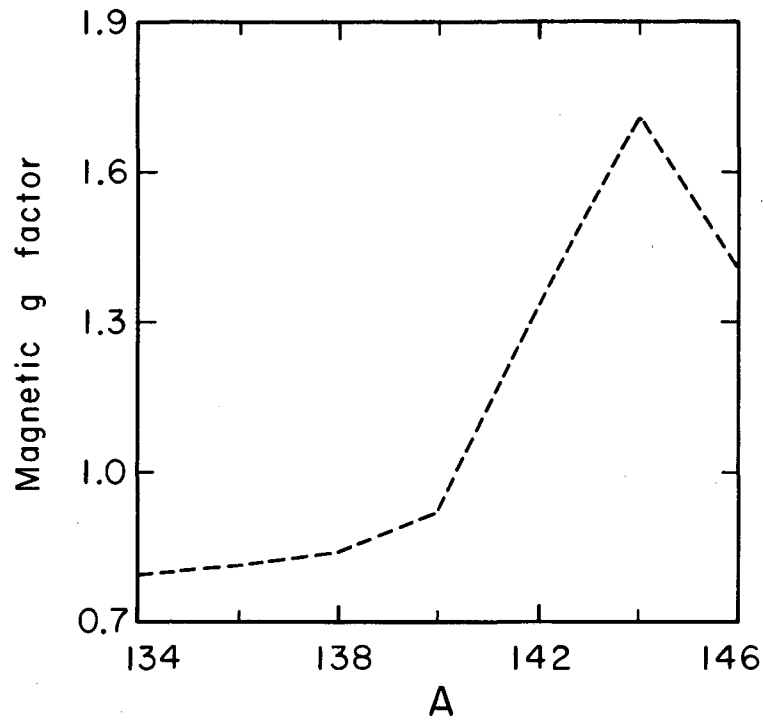

Fig. 19. Magnetic g factor of 4+ states in 82-neutron, even-A nuclei computed with wave functions obtained with x = 1.15, Set I and t = 1.0; g factors for noninteracting configurations $(d_{5/2}^2)$, $(d_{5/2}\ g_{7/2})$, and $(g_{7/2}^2)$ are taken from empirical results of neighboring odd-A nuclei, the rest taken from Schmidt values.

Table XIII. Gyromagnetic ratio (g factor) for two particle configurations in Ce^{140}. The third column corresponds to "effective" gyromagnetic ratio obtained from La^{139} (ground-state spin 7/2) and Pr^{141} (ground-state spin 5/2). (The g factor for Pr^{141} is from the value estimated by Shirley, reference 53.)

Configuration	Gyromagnetic ratio	
	Using Schmidt Values	Using g factor of neighboring odd-A nuclei
$(g^2_{7/2})$	0.49	0.79
$(d^2_{5/2})$	1.92	1.76
$(d_{5/2}g_{7/2})$	0.96	1.17

have argued before, since $g_{7/2}$ is the lowest single-quasi-particle state, and the diagonal matrix element of H_{22} is the largest for $(g_{7/2}^2)$ (with attractive contribution), no inversion of this level with others could occur. Furthermore, from other results [such as the energy levels and B(E2) values], the assumption that $(g_{7/2}^2)$ is the dominant component seems to be better than the g factor might indicate.

It is interesting to note that, as Fig. 20 shows, if one takes the wave functions for $t < 0$ as one does in the conventional shell-model calculations, then agreement gets better; in fact at $t \approx -0.5$, a fair agreement, within experimental error, can be found if contributions from all the components of the wave function are taken into account. In a way, this phenomenon is not too surprising, for as we have mentioned before, the formula for the magnetic g factor is the same (as far as the structure of the equation is concerned) for both the shell model and the quasi-particle model. But since the two methods need not yield the same amplitudes, the magnetic g factor is not expected to be the same at the same t. This discrepancy of t for other results [the energy levels and B(E2)] and for the magnetic g factor implies, then, either the failure of the quasi-particle theory or uncertainty in the single-particle energies we have taken. We tend to believe that the second is to blame, for the following reason: the g factor is very sensitive to configuration mixing (a slight increase of the $|d_{5/2}^2)$ component in the wave function increases it by a large amount), and probably a better choice of ϵ'_αs might bring back t to the conventional value (i.e., $t < 0$).

J. The Effect of Quasi-Particle Interactions on Odd-A Nuclei: Quadrupole Mement

We argued in Sec. III.E that a selective summation of the separable component of the force in the Q-Q limit leads to a result similar to KS's formula. The numerical result computed with our

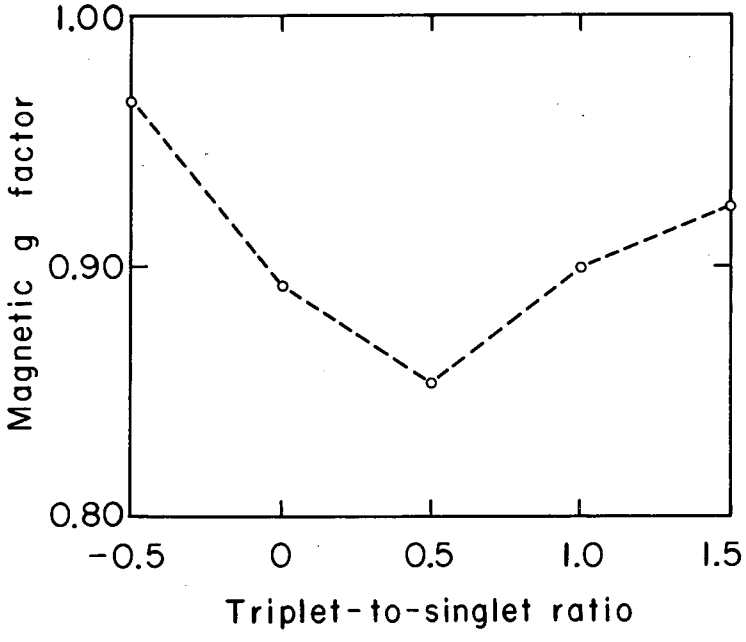

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Fig. 20. Variation of magnetic g factor for Ce^{140} as a function of t. Wave functions used here are those obtained with three-configuration approximation, x = 1.20, Set I, and t = 1.0.

closed-form expression was essentially the same for La^{139} as KS's value, which is $Q = 0.21 \times 10^{-24}$ cm^2. A conclusion that one can draw from such considerations is that the contribution from the residual terms is about the same as or greater than the contribution from a single quasiparticle assumed to be occupying the ground state of odd-A nuclei. Since, in perturbation theory language, the net result from such residual interactions is equivalent to a sum of all the higher-order terms, it is clear that all these terms should add coherently to give such a large value. This also implies that in one region a nucleus with a larger number of particles outside the closed shell should have a quadrupole moment whose magnitude is in general greater than for a nucleus with a smaller number. Thus KS, in their first paper, obtained $|Q(Pr^{141})| > |Q(La^{139})|$, but their result is in gross disagreement with experiment (experimentally $\left|\frac{Q(La^{139})}{Q(Pr^{141})}\right| \approx 3$).

Let us now consider what would happen if we worked with a more general treatment. Firstly, the occupation factors u and v would be altered owing to the general gap matrix element (equivalently nonconsistant Δ); hence the quasi-particle quadrupole moment would be changed. Secondly, the terms neglected in deriving the closed form may not be negligible, and also the presence of such terms will alter the sign with which they contribute. The last point further implies that the so-called "phonon" contribution need not be the same as or greater than the quasi-particle contribution.

These points have been somewhat clarified by KS's second paper.[15b] Their results (in the RPA) show that the single quasi-particle term and the phonon term may not have a same sign (Ir^{191} and Ir^{193}, for example) nor is the phonon contribution as large as the single quasi-particle contribution.

To see these points more clearly, we discuss the effect of H_{31} + H_{22} interactions on the quadrupole moment with a general force used in our previous calculations. There are only three nuclei in the region

of our interest for which the moments are known, and hence we do not go into an extensive computation, but limit ourselves to La^{139} and Pr^{141}. For this purpose, let us take Set I (V_o = -1.15 x 32.9 MeV, $\beta^{-1/2}$ = 1.732 F, t = 1.0) without any adjustment. The gap solutions are listed in Table XIV, and the equation we use (for notations see Sec. III.E) is

$$Q \cong Q^{(0)} + 2 \sum_{\alpha} \sum_{(ilk)I} [C^{\alpha}_{ilkI}]^2 \frac{\langle (il)Ik;f|H_{31}|f\rangle}{E_f - E_\alpha} Q^{(1)}_{(ilk)} ,$$

where E_α is computed from

$$(H_0 + H_{22}) \Psi_\alpha = E_\alpha \Psi_\alpha .$$

In computing the quadrupole moment, we make the following simplifying assumptions: that the largest contribution comes from

(a) I = 2+ states,

(b) k = k' states in diagonalizing the H_{22} matrix whose matrix element then reads

$$\langle (il)Ik;f|H_{22}|(i'l')Ik';f\rangle$$

$$\approx \frac{N(ilkI)\, N(i'l'k'I)}{N(ilI)\, N(i'l'I')} \langle (il)I|H_{22}|(i'l')I\rangle\, \delta(k,k') .$$

Table XIV: Gap solutions for La^{139} and Pr^{141} computed with Set I, $V_0 = -1.15 \times 32.9$ Mev, $\beta^{-\frac{1}{2}} = 1.732$ F, $\frac{V_{TO}}{V_{SE}} = 1.0$.

		Δ	λ	U	V	E
La^{139}	$g_{7/2}$	1.09	0,251	0.6224	0.7827	1.114
	$d_{5/2}$	1.08		0.8860	0.4638	1.314
	$d_{3/2}$	1.08		0.9811	0.1935	2.842
	$s_{1/2}$	0.91		0.9888	0.1494	3.087
	$h_{11/2}$	0.95		0.9740	0.2264	2.149
Pr^{141}	$g_{7/2}$	1.14	0,5730	0.5248	0.8512	1.276
	$d_{5/2}$	1.20		0.8170	0.5766	1.274
	$d_{3/2}$	1.20		0.9714	0.2374	2.600
	$s_{1/2}$	1.03		0.9827	0.1851	2.820
	$h_{11/2}$	1.01		0.9611	0.2762	1.896

In principle I = 0+, 2+, 4+ contribute, but Y_{20} has the largest matrix element for I = 2, while I $\neq$ 2 give rather small contributions. Thus the first approximation is expected to be rather good. The second approximation has somewhat doubtful validity, but it is true that our approximation takes the dominant term, and furthermore makes the computation much simpler than taking the full matrix element.

Under these approximations, we obtain the results given in Table XV. Notice that our results are very close to those of KS's second calculations. It is obvious that Q is very sensitive to the $(u^2 - v^2)$ factor. Since our calculation is not extensive, we cannot draw a general conclusion, but we can, however, see the following features:

(a) that the contribution from residual terms is not necessarily as large as or greater than the quasi-particle contribution,

(b) that even though $Q^{(1)}$ may increase as more shells are filled, such contributions are not large enough to upset the feature provided by the single quasi-particle value (for example, the Pr^{141} case).

Let us finally discuss the validity of the approximations by which only the separable terms in the H_{31} and H_{22} matrix elements are kept (for example, neglect of the G term). Some sample calculations of G and F are shown in Table XVI, along with reduction factors multiplying G and F in the matrix element. Now the $(u_i v_1 + v_i u_1)$ factor multiplying the separable component of H_{22} is always greater than the rest; hence in Q-Q force only a small error is introduced in dropping the G term. But such a procedure is rather dangerous for matrix elements of H_{31}, since -- as can be seen in Table XVI, Δ_2 and Δ_3 are usually much smaller than Δ_1 ; thus $\Delta_2 F_1$ and $\Delta_3 F_2$ may be comparable to or smaller than $\Delta_1 G$. In some cases, $\Delta_1 G > \Delta_2 F_1$, $\Delta_3 F_2$. This irregularity is caused by the factor in H_{31} (i.e., $u_a u_b v_c u_d - v_a v_b u_c v_d$). One can see that the magnitude of this factor depends on what combination of states we are considering. This fact also dictates the sign with which each term contributes.

Table XV: Quadrupole moments of La^{139} and Pr^{141} in units of 10^{-24} cm^2; $e_{eff} = 2$.

	Q^0	$Q^{(1)}$	$Q_{theor.}$	$Q_{exp.}$	$Q'_{Quas.}$	Q_{phonon}	$Q_{th.}$
La^{134}	0.0858	0.0216	0.107	0.23	0.12	0.02	0.14
Pr^{141}	-0.0248	-0.0464	-0.0712	-0.07	-0.09	-0.14	-0.10

$Q'_{quas.}$ = KS' value for quasi-particle contribution

Q'_{phonon} = KS' value for phonon contribution

$Q'_{th.}$ = KS' value for total theoretical contributions including δ-function residual interactions (see ref.15b).

Table XVI: Some G and F matrix elements, and reduction factors (uuvu-vvuv) for La^{139}. The parameters are $V_0 = -1.15 \times 32.9$ Mev, $\beta^{-\frac{1}{2}} = 1.732$ F, $\nu^{-\frac{1}{2}} = 2.286$ F, $\frac{V_{TO}}{V_{SE}} = 1$. The equation for the H_{31} matrix element in the notation of this table is

$$\langle 3|H_{31}|1\rangle \sim -[\Delta_1 G - \Delta_2 F_1 + \Delta_3 F_2] .$$

i	l	k	f	I	G	F_1	F_2	Δ_1	Δ_2	Δ_3
$h_{11/2}$	$h_{11/2}$	$g_{7/2}$	$g_{7/2}$	2	0.0337	-0.6030	0.6030	0.4372	-0.0497	-0.0497
$d_{5/2}$	$d_{5/2}$	$g_{7/2}$	$g_{7/2}$	2	-0.0595	-0.3129	0.3129	0.2776	-0.0926	-0.0926
$d_{5/2}$	$g_{7/2}$	$g_{7/2}$	$g_{7/2}$	2	0.0138	-0.0807	0.0807	0.0918	-0.3130	-0.0918
$d_{3/2}$	$d_{3/2}$	$g_{7/2}$	$g_{7/2}$	2	-0.1712	-0.1761	0.1761	0.4507	-0.0428	-0.0428

K. Discussion

In our computations, we have taken a set of single-particle energy levels and pushed through an analysis for the 82-neutron nuclei from A=134 to A=146. Here we discuss several points that we have not clarified in the preceding sections.

One should understand that an analysis of nuclei very near a doubly closed nucleus along the line of the treatment that we have adopted is perhaps only of academic interest (for example, Xe^{134}). First of all, the quasi-particle approach (or the BCS approximation) may be far from valid for a nucleus with only two or three particles outside the core, for in such a system, the nonconservation of particle numbers would cause a strong admixing of neighboring nuclei. Secondly, the low-lying nuclear properties of such a system can be treated more rigorously by means of the ordinary shell model.

As one can see from our results, the ratio $\frac{V_{TO}}{V_{SE}}$ turns out to be close to 1, which differs rather drastically from the conventional values such as the Rosenfeld mixture (t = -0.55), the one-pion-exchange potential (t = -0.33), and the Serber force (t = 0). This somewhat strange feature had first been observed by Arvieu et al.[40] They suggested that the possible explanation for this "anomaly" lies in the renormalization of the force constant that arises from the use of a limited number of levels. If this is so, then, one may ask, why is this not the case in the conventional shell-model calculations where also only a small number of states are in practice taken into account? Thus it appears at first thinking that their argument does not hold. One could eliminate this difficulty by saying that either (a) the fault lies in the incorrect choice of single-particle energies (noninteracting)[54] or (b) the interaction between quasiparticles may not be the same as that between the usual particles. Neither statement has been proved yet, nor has any contrary argument been suggested. Let us assume the second is the valid explanation. Then other features can be justified in terms of

the renormalization of the force constant. For example, A=136 seems to have a better result with t=0.8 than with 1.0. The λ for $A < 140$, say, is much lower than λ for $A = 140$; hence many more levels are effectively taken into account for the former than for the latter, the effect of which is to reduce the ratio. On the other hand, λ for $A > 140$ is higher and the "effective levels" are reduced, from which a larger ratio results.

From the above considerations, it appears that an increase of t is equivalent to increasing the force constant V_0. This is indeed the case with Ce^{140} spectra computed with only $(d_{5/2}{}^2)$, $(d_{5/2}\, g_{7/2})$, and $(g_{7/2}{}^2)$, where a slightly larger V_0 is needed to fit the experimental spectra. Since the renormalization of the force constant applies also to the gap solutions, as pointed out by Belyaev,[12] it may be that the renormalization applied to the quasi-particle interactions as mentioned above and the effect of the tensor force as discussed in Sec. IV.F together bring about the anomalously large and positive t value, and also the differences in t between the pairing and quasi-particle interactions.

At one point in Sec. III.D (in computing $B(E2)_{2+ \rightarrow 0+}$ for Ce^{140} in the Q-Q limit), we have assumed that all the two-quasi-particle states except $(d_{5/2}{}^2)$, $(d_{5/2}\, g_{7/2})$, and $(g_{7/2}{}^2)$ could be neglected. This procedure turns out to be rather good for energy spectra, provided a proper increase of V_0 is taken into account. But since the matrix elements of single-particle operators are sensitive to all the components of a wave function, such quantities as electromagnetic transition probabilities should be computed with all the states including those neglected in Sec. III.D.

One further remark concerns the energy spacing of the single-particle level we have taken. It turns out that in order to explain alpha-decay systematics, one needs to increase the spacing of $d_{5/2}$ and $h_{11/2}$ to around 2 MeV. This is required for a substantial decrease of the energy gap at Z=64. This adjustment would prevent $(h_{11/2}{}^2)$ from

coming down too fast, and would also push up 2+ and 4+ states at Z=64. This feature appears to be more consistent with the alpha-decay systematics than the ones we have used.[46] However, owing to the lack of any experimental information at about Z=64, we have not made this adjustment.

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APPENDICES

A. Derivation of the Matrix Elements of the Residual Terms

We give here a sketch of algebra used to derive the matrix elements of the interaction Hamiltonian. The matrix elements of H_{22}, H_{40}, and H_{31} in appropriate subspaces are computed in a similar manner, and hence we just give one example. Let us consider H_{40}. The operator was given in Sec. II, but for the sake of completeness we rewrite it here:

$$H_{40} = \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | \bar{V} | \gamma\delta \rangle s_\gamma s_\delta u_\alpha u_\beta v_\gamma v_\delta \eta^+_\alpha \eta^+_\beta \eta^+_{-\delta} \eta^+_{-\gamma} + \text{h.c} \quad . \tag{A-1}$$

Denote the quasi-particle vacuum by $|\hat{0}\rangle$, which has the property of

$$\begin{aligned} \eta^+_\alpha |\hat{0}\rangle &= |\alpha\rangle \quad , \\ \eta_\alpha |\hat{0}\rangle &= 0 \quad , \qquad \text{for all } \alpha, \end{aligned} \tag{A-2}$$

then H_{40} operating on the vacuum corresponds to a four-quasi-particle state; thus

$$\langle 4 | H_{40} | \hat{0} \rangle \neq 0 \quad .$$

An unperturbed four-quasi-particle state with total angular momentum coupled to zero can be written as

$$|4\rangle \equiv |(pq)_J (rs)_J ; 0\rangle = \theta(pqrs)(2J+1)^{1/2} N(pqrsJ) \sum_{\substack{m_p m_q \\ m_r m_s \\ M}} (-)^{J-M}$$

$$\times \begin{pmatrix} j_p & j_q & J \\ m_p & m_q & -M \end{pmatrix} \begin{pmatrix} j_r & j_s & J \\ m_r & m_s & -M \end{pmatrix} \eta^+_{p'} \eta^+_{q'} \eta^+_{r'} \eta^+_{s'} |\hat{0}\rangle \quad , \tag{A-3}$$

where a primed symbol $p' \equiv (j_p, m_p)$, otherwise $p \equiv j_p$. Now, taking the matrix element between $|4\rangle$ and $|\hat{0}\rangle$, we get

$$\langle 4|H_{4Q}|\hat{0}\rangle = \theta(pqrs)(2J+1)^{1/2}N(pqrsJ)\sum_{\alpha\beta\gamma\delta}\sum_{\text{all m's}}\langle\alpha\beta|\bar{V}|-\gamma-\delta\rangle(-)^{J-M} \tag{A-4}$$

$$\begin{pmatrix} j_p & j_q & J \\ m_p & m_q & -M \end{pmatrix}\begin{pmatrix} j_r & j_s & J \\ m_r & m_s & -M \end{pmatrix} s_\gamma s_\delta\, u_\alpha u_\beta v_\gamma v_\delta \langle \eta_s, \eta_r, \eta_q, \eta_p, \eta_\alpha^+ \eta_\beta^+ \eta_\delta^+ \eta_\gamma^+ \rangle_0 \quad .$$

The last factor in Eq. (A-4) is a vacuum expectation value, and can be easily evaluated by Wick's algebra.[17] By a simple induction, one can see that the following relation holds[55]:

$$\langle \eta_{\ell_1}\eta_{\ell_2}\cdots\eta_{\ell m}\eta_{k_1}^+\eta_{k_2}^+\cdots\eta_{km}^+\rangle_0 = (-)^{\binom{m}{2}}\,\mathrm{Det}(a_{rs}) \tag{A-5}$$

where $\mathrm{Det}(a_{rs})$ is the m-by-m determinant whose elements are

$$a_{rs} = \langle \eta_{\ell_r}\eta_{k_s}^+\rangle = \delta(\ell_r, k_s) \quad .$$

If we employ this theorem, $\langle\rangle_0$ in Eq. (4) becomes

$$\langle\rangle_0 = \begin{vmatrix} a_{s'\alpha} & a_{s'\beta} & a_{s'\delta} & a_{s'\gamma} \\ a_{r'\alpha} & a_{r'\beta} & a_{r'\delta} & a_{r'\gamma} \\ a_{q'\alpha} & a_{q'\beta} & a_{q'\delta} & a_{q'\gamma} \\ a_{p'\alpha} & a_{p'\beta} & a_{p'\delta} & a_{p'\gamma} \end{vmatrix} \quad . \tag{A-6}$$

In principle, expansion of this determinant completely determines the terms in Eq. (A-4). But no such tedious computation needs be done. The symmetries of $\langle\alpha\beta|\bar{V}|-\gamma-\delta\rangle$; i.e.,

$$\langle\alpha\beta|\bar{V}|-\gamma-\delta\rangle = \langle\beta\alpha|\bar{V}|-\delta-\gamma\rangle = \langle-\gamma-\delta|\bar{V}|\alpha\beta\rangle = -\langle\beta\alpha|\bar{V}|-\gamma-\delta\rangle = -\langle\alpha\beta|\bar{V}|-\delta-\gamma\rangle \,, \tag{A-7}$$

suggest immediately that there can be only three different $\bar{V}$ functions, i.e.,

$$\langle p'q'|\bar{V}|-r'-s'\rangle \quad , \quad \langle p's'|\bar{V}|-q'-r'\rangle \quad , \text{ and } \quad \langle p'r'|\bar{V}|-q'-s'\rangle \quad .$$

Note that no other combination can occur. Now $u_\alpha u_\beta v_\delta v_\gamma \equiv \omega(\alpha\beta\gamma\delta)$, because of its permutation symmetry $\omega(\alpha\beta\gamma\delta) = \omega(\alpha\beta\delta\gamma) = \omega(\beta\alpha\delta\gamma) = \omega(\beta\alpha\gamma\delta)$, would appear in the combination of $(\omega(\alpha\beta\gamma\delta) + \omega(\gamma\delta\alpha\beta))$. This gives, then, a total of six different terms.

One can easily see from examination of the determinant that only $(\omega(\alpha\beta\gamma\delta) + \omega(\gamma\delta\alpha\beta))$, not that with a minus sign, results. Since a 4-by-4 determinant has 24 terms, this implies that a factor 4 ($\frac{24}{6} = 4$) multiplies the whole term, which just cancels the factor 1/4 in Eq. (A-4).

Thus we can immediately write (by inspection)

$$\langle 4|H_{40}|\hat{0}\rangle = -\theta(pqrs)(2J+1)^{1/2} N(pqrsJ) \sum_{\text{all } m's} \sum_{M} (-)^{J-M} \begin{pmatrix} j_p & j_q & J \\ m_p & m_q & -M \end{pmatrix} \begin{pmatrix} j_r & j_s & J \\ m_r & m_s & -M \end{pmatrix}$$

$$\times [\langle p'q'|\bar{V}|-r'-s'\rangle\, s_{r'}\, s_{s'} (\omega(pqrs) + \omega(rspq)) \qquad \text{(A-8)}$$

$$+ \langle p's'|\bar{V}|-q'-r'\rangle\, s_{q'}\, s_{r'} (\omega(psqr) + \omega(qrps))$$

$$+ \langle p'r'|\bar{V}|-q'-s'\rangle\, s_{q'}\, s_{s'} (\omega(prqs) + \omega(qspr))] \quad .$$

Note that $\bar{V}$ functions here are all antisymmetrized; i.e.,

$$\langle p'q'|\bar{V}|r's'\rangle = \langle p'q'|V|r's'\rangle - \langle p'q'|V|s'r'\rangle \quad .$$

At this point, we introduce an invariant function G by

$$\langle p'q'|\bar{V}|r's'\rangle = \theta(pqrs) \sum_{J'M} (2J+1) \begin{pmatrix} j_p & j_q & J' \\ m_p & m_q & -M \end{pmatrix} \begin{pmatrix} j_r & j_s & J' \\ m_r & m_s & -M \end{pmatrix} G(pqrsJ') \quad . \qquad \text{(A-9)}$$

Correspondence between G and ordinary shell-model matrix elements is (see Appendix B)

$$G(pqrsJ) = \langle pqJ|V|rsJ\rangle - \theta(rsJ)\langle pqJ|V|srJ\rangle \quad . \qquad \text{(A-10)}$$

Substitution of (A-9) into Eq. (A-8) gives rise to four 3-j symbols multiplying each G function. The sum over magnetic quantum numbers of four 3-j symbols leads to either a Krönecker delta $\delta(J,J')$ or a 6-j symbol

(see, for example, Edmonds[56]). It is easy to see that the first term in Eq. (A-8) gives $\delta(J,J')$ and the two remaining terms 6-j symbols. When the usual algebra is worked out, we obtain

$$\begin{aligned}\langle 4|H_{40}|\hat{0}\rangle = &- N(pqrsJ)(2J+1)^{1/2}\theta(J)[(\omega(pqrs) + \omega(rspq))G(pqrsJ) \\ &- \theta(qr)(\omega(psqr) + \omega(qrps))\sum_{J'} \theta(J')(2J'+1)\begin{Bmatrix} j_p & j_q & J \\ j_r & j_s & J' \end{Bmatrix} G(qrpsJ') \\ &- \theta(qr)(\omega(prsq) + \omega(sqpr))\sum_{J'} \theta(J')(2J'+1)\begin{Bmatrix} j_p & j_q & J \\ j_s & j_r & J' \end{Bmatrix} G(prsqJ')] \ .\end{aligned} \tag{A-11}$$

If we define an F function by

$$F(pqrsJ) = - \sum_{J'} (2J'+1)\theta(pqrs)\begin{Bmatrix} j_p & j_q & J' \\ j_r & j_s & J \end{Bmatrix} G(spqrJ) \quad ,$$

and use the symmetries of the 6-j symbol, and the fact that

$$G(abcdJ) = - \theta(abJ)G(bacdJ) = - \theta(cdJ)G(abdcJ) = \theta(abcd)G(badcJ) \ ,$$

then we can put Eq. (A-11) into a form of Eq. (III-57) in the main text; i.e.,

$$\begin{aligned}\langle 4|H_{40}|\hat{0}\rangle = &- N(pqrsJ)(2J+1)^{1/2}\theta(J) \\ &\times [G(pqrsJ)(\omega(pqrs) + \omega(rspq)) - F(pqrsJ)(\omega(psqr) + \omega(qrps)) \\ &+ F(pqsrJ)\theta(srJ)(\omega(prqs) + \omega(qspr))] \quad .\end{aligned} \tag{A-12}$$

B. Shell-Model Matrix Elements of a Central Force

To supplement Secs. II and II of the main text, we write down some of the well-known formulas for the matrix elements of central-force operators. We are interested in the quantities $\langle pqJ|V^{\sigma}|rsJ\rangle$ and $\langle pqJ|V^{\sigma}|rsJ\rangle$, which are respectively a spin-independent and a spin-dependent matrix element.

First introduce the usual expansion of $V(r_{12})$ in the Legendre polynomial:

$$V^{o} = V(\vec{r}_1-\vec{r}_2) = \sum_k v_k(r_1,r_2)P_k(\cos\omega) \ ,$$
$$V^{\sigma} = V^{o}\sigma_1\cdot\sigma_2 \quad . \tag{B-1}$$

If we put

$$P_k(\omega) = \sum_\mu (-)^\mu C_{k\mu}(1)\ C_{k-\mu}(2) \quad , \tag{B-2}$$

where

$$C_{k\mu}(1) = \left(\frac{4\pi}{2k+1}\right)^{1/2} Y_{k\mu}(\theta_1,\varphi_1) \quad , \tag{B-3}$$

then a straightforward Racah algebra yields the formulas (for the details of derivation, see Kim[42])

$$\langle pqJ|V^{o}|rsJ\rangle = \sum_k F_k\theta(qsJ)\left[\frac{(2j_k+1)(2j_q+1)(2j_r+1)(2j_s+1)}{2k+1}\right]^{1/2}$$
$$\times C(j_r j_p k;\tfrac{1}{2}-\tfrac{1}{2}\ 0)C(j_s j_q k;\tfrac{1}{2}-\tfrac{1}{2}\ 0)W(prqs;kJ) \tag{B-4}$$

and

$$\langle pqJ|V^{\sigma}|rsJ\rangle = 2\sum_k\{F_k\theta(qsJ)$$
$$\times \frac{[(2j_p+1)(2j_q+1)(2j_r+1)(2j_s+1)(2\ell_p+1)(2\ell_q+1)(2\ell_s+1)(2\ell_r+1)]^{1/2}}{[2k+1]^{1/2}} \tag{B-5}$$
$$\times C(\ell_p\ell_r k;000)C(\ell_q\ell_s k;000)W(\ell_p j_p\ell_q j_q;\tfrac{1}{2}J)W(\ell_r j_r\ell_s j_s;\tfrac{1}{2}J)$$
$$\times W(\ell_p\ell_r\ell_q\ell_s;kJ)\} + \langle pqJ|V^{\sigma}|rsJ\rangle \quad ,$$

where

$$F_k = \int\int R_p^*(r_1)R_q^*(r_2)v_k(r_1,r_2)R_r(r_1)R_s(r_2)r_1^2r_2^2dr_1dr_2$$

is the usual Slater integral. In general, the central force contains singlet-even, singlet-odd, triplet-even, and triplet-odd components. In our case, where only one kind of particles is considered, the triplet-even and singlet-odd force vanish owing to the generalized Pauli principle. In such a case, the total central force is

$$V = V_{SE}(p_S + \frac{V_{TO}}{V_{SE}} p_T)V^o \equiv v_o(p_S + \ell p_T)V^o \quad , \tag{B-6}$$

where

$$p_S = \frac{1 - \sigma_1 \cdot \sigma_2}{4} \quad , \quad p_T = \frac{3 + \sigma_1 \cdot \sigma_2}{4} \quad .$$

Then the matrix element of (B-6) is easily obtained from Eqs. (B-4) and (B-5):

$$\langle pqJ|V|rsJ\rangle = V_o[\langle pqJ|V^o p_S|rsJ\rangle + t\langle pqJ|V^o p_T|rsJ\rangle] \quad . \tag{B-7}$$

The functions G and F (defined in the main text) are related to Eq. (B-7) by

$$G(pqrsJ) = \langle pqJ|V|rsJ\rangle - \theta(rsJ)\langle pqJ|V|srJ\rangle \quad , \tag{B-8}$$

$$F(pqrsJ) = -\Sigma_{J'} W(psqr;J'J)\{\langle spJ'|V|qrJ'\rangle - \theta(qrJ')\langle spJ'|V|rqJ'\rangle\} \quad . \tag{B-9}$$

Delta Function Limit

The matrix element for the δ-function interaction used in Secs. II.B and II.C is trivially obtained from Eqs. (B-4) and (B-5) by setting k = 0 in the Slater integral and summing over k. One obtains

$$\begin{aligned}\langle pqJ|V^o|rsJ\rangle = {} & \frac{1}{2(2J+1)} F_o[(2j_p+1)(2j_q+1)(2j_r+1)(2j_s+1)]^{1/2} \\ & \times C(j_p j_q J;\tfrac{1}{2} - \tfrac{1}{2}\ 0)C(j_r j_s J;\tfrac{1}{2} - \tfrac{1}{2}\ 0) \\ & \times [-\theta(j_p j_r \ell_p \ell_r) + \theta(j_p j_q j_r j_s)\frac{AA'}{4J(J+1)}]\end{aligned} \tag{B-10}$$

and

$$\langle pqJ|V^{\sigma}|rsJ\rangle = \frac{1}{2(2J+1)} F_o[(2j_p+1)(2j_q+1)(2j_r+1)(2j_s+1)]^{1/2}$$

$$\times C(j_p j_q J;\tfrac{1}{2} - \tfrac{1}{2}\ 0)C(j_r j_s J;\tfrac{1}{2} - \tfrac{1}{2}\ 0)\{\theta(j_p j_r \ell_p \ell_r)[1 + 2\theta(\ell_p \ell_q J)]$$

$$+ \theta(j_p j_q j_r j_s)\frac{AA'}{4J(J+1)}]\}\ , \qquad \text{(B-11)}$$

where

$$A = (2j_p+1) + \theta(j_p j_q J)(2j_q+1)\ ,\quad A' = (2j_r+1) + \theta(j_r j_s J)(2j_s+1)\ .$$

If one combines (B-9) and (B-10) into (B-6), then one finds that in the case of one kind of particles, only singlet-even contributes and leads to the equation quoted in Secs. II.B and II.C; i.e.,

$$\langle pqJ|V(\delta)p_S|rsJ\rangle = -\theta(j_p j_r \ell_p \ell_r)\{(2j_p+1)(2j_q+1)(2j_r+1)(2j_s+1)\}^{1/2}$$

$$\times C(j_p j_q J;\tfrac{1}{2} - \tfrac{1}{2}\ 0)C(j_r j_s J;\tfrac{1}{2} - \tfrac{1}{2}\ 0)\frac{F_o}{2(2J+1)}\frac{1}{2}[1 + \theta(\ell_p \ell_r J)]\ .$$

C. Tabulation of Wave Functions

In this appendix, we give the BCS wave functions for even-A nuclei (A = 134 to A = 146), and also the eigenvalues and eigenvectors of the Hamiltonian $H_o + H_{22}$ in the two-quasi-particle subspaces. Most of the results are calculated with x = 1.15 and Set I, and the ones denoted as "Extra" are obtained with y = 1.05 and Set III.

Table C-I. Gap solutions (v_α, E_α, λ, Δ_α) calculated with Set I,

$V_o = -1.15 \times 32.9$ MeV, $\beta^{-1/2} = 1.732$ Fm, $\frac{v_{TO}}{v_{SE}} = 0.$

A	λ		$g_{7/2}$	$d_{5/2}$	$d_{3/2}$	$s_{1/2}$	$h_{11/2}$
		Δ_α	0.7060	0.5861	0.5855	0.4903	0.6000
134	- 0.5224	v_α	0.4501	0.1827	0.0851	0.0654	0.1090
		E_α	0.8783	1.631	3.452	3.755	2.768
		Δ_α	0.9255	0.8263	0.8254	0.6923	0.7944
136	- 0.2162	v_α	0.6215	0.2940	0.1299	0.0998	0.1594
		E_α	0.9504	1.470	3.204	3.486	2.524
		Δ_α	1.047	1.006	1.005	0.8464	0.9082
138	0.0936	v_α	0.7379	0.4066	0.1722	0.1326	0.2038
		E_α	1.051	1.354	2.962	3.220	2.275
		Δ_α	1.116	1.145	1.144	0.9718	0.9797
140	0.4108	v_α	0.8202	0.5208	0.2152	0.1669	0.2502
		E_α	1.189	1.288	2.721	2.954	2.022
		Δ_α	1.161	1.248	1.246	1.075	1.031
142	0.7375	v_α	0.8764	0.6302	0.2604	0.2044	0.3053
		E_α	1.375	1.275	2.479	2.687	1.773
		Δ_α	1.204	1.321	1.319	1.164	1.079
144	1.067	v_α	0.9119	0.7247	0.3093	0.2472	0.3755
		E_α	1.609	1.323	2.242	2.430	1.550
		Δ_α	1.251	1.372	1.370	1.242	1.129
146	1.381	v_α	0.9331	0.7962	0.3618	0.2952	0.4595
		E_α	1.863	1.424	2.030	2.203	1.383

Table C-II. Eigenvalues for even-pairty states computed with $v_o = -1.15 \times 32.9$ MeV, $\beta^{-1/2} = 1.732$ F. Relevant equation is Eq. (III-7b).

J	A	$\frac{v_{TO}}{v_{SE}}$	Eigenvalues in increasing order						
1	134	1.0	2.047	4.461					
	136	1.0	2.219	4.028					
	138	1.0	2.494	3.917					
	140	1.0	2.649	3.774					
	142	1.0	2.777	3.701					
	144	1.0	2.930	3.681					
	146	1.0	3.141	3.690					
2	134	0.8	1.112	2.219	2.652	3.747	4.592	4.656	6.511
		1.0	1.045	2.123	2.600	3.723	4.518	4.553	6.468
	136	0.8	1.308	2.316	2.432	3.751	4.162	4.259	6.028
		1.0	1.243	2.256	2.399	3.744	4.080	4.184	5.986
	138	1.0	1.397	2.287	2.413	3.607	3.753	3.944	5.521
		1.2	1.312	2.247	2.395	3.533	3.725	3.937	5.488
	140	1.0	1.558	2.307	2.535	3.147	3.677	3.867	5.062
		1.2	1.456	2.283	2.522	3.087	3.629	3.913	5.034
	142	1.0	1.741	2.384	2.669	2.782	3.575	3.885	4.607
		1.2	1.614	2.350	2.644	2.765	3.544	3.945	4.585
	144	1.0	1.830	2.378	2.795	2.897	3.516	3.926	4.173
		1.2	1.675	2.350	2.773	2.884	3.501	3.993	4.160
	146	1.0	1.709	2.479	3.112	3.190	3.500	3.798	3.989
		1.2	1.558	2.456	3.065	3.170	3.488	3.811	4.053
3	134	1.0	2.313	4.157	4.775				
	136	1.0	2.361	4.116	4.412				
	138	1.0	2.436	4.085	4.116				
	140	1.0	2.543	3.886	4.064				
	142	1.0	2.697	3.716	4.061				
	144	1.0	2.927	3.605	4.083				
	146	1.0	3.223	3.549	4.126				
4	134	0.8	1.493	2.295	2.897	4.099	4.364	5.046	
		1.0	1.457	2.263	2.829	4.071	4.388	4.996	
	136	0.8	1.713	2.311	2.645	3.963	4.067	4.572	
		1.0	1.687	2.297	2.595	3.944	4.105	4.523	
	138	1.0	1.889	2.359	2.478	3.749	3.980	4.050	
		1.2	1.851	2.340	2.458	3.763	3.958	4.070	
	140	1.0	2.106	2.428	2.501	3.550	3.597	3.935	
		1.2	2.053	2.402	2.520	3.497	3.646	3.974	
	142	1.0	2.342	2.544	2.659	3.099	3.489	3.914	
		1.2	2.276	2.530	2.679	3.059	3.553	3.952	
	144	1.0	2.522	2.652	2.851	2.988	3.451	3.929	
		1.2	2.461	2.619	2.840	2.992	3.530	3.966	
	146	1.0	2.374	2.731	3.131	3.366	3.517	3.974	
		1.2							
5	134	1.0	2.379	4.132					
	136	1.0	2.381	4.139					
	138	1.0	2.425	4.150					
	140	1.0	2.519	4.161					
	142	1.0	2.679	4.179					
	144	1.0	2.927	4.210					
	146	1.0	3.244	4.253					

Table C-III. Eigenvalues and eigenvectors for even-J states computed with v_o = - 1.15 × 32.9 MeV, $\beta^{-1/2}$ = 1.732 F, Set I. Relevant equation is Eq. (III-7b). Extra table corresponds to v_o = - 1.05 × 32.4, $\beta^{-1/2}$ = 1.755 F, Set III.

	A	$\frac{v_{TO}}{v_{SE}}$	$\omega(\alpha)$	$d^2_{5/2}$	$g^2_{7/2}$	$d^2_{3/2}$	$h^2_{11/2}$	$d_{5/2}g_{7/2}$	$d_{5/2}d_{3/2}$	$g_{7/2}d_{3/2}$
J = 2+	134	0.8	1.112	0.0990	0.9880	0.0317	- 0.0106	- 0.0363	- 0.0266	0.1042
			2.219	0.0931	- 0.0558	0.0095	- 0.0502	- 0.9891	- 0.0124	0.0845
			2.652	0.9746	- 0.1065	0.0350	- 0.1122	0.1127	0.0279	0.1078
		1.0	1.045	0.0971	0.9855	0.0344	0.0088	- 0.0621	- 0.0420	0.1115
			2.123	0.1412	- 0.0906	0.0131	- 0.0763	- 0.9738	- 0.0155	0.1315
			2.600	0.9694	- 0.0943	0.0251	- 0.1186	0.1707	0.0026	0.0874
	136	0.8	1.308	0.1833	0.9731	0.0361	0.0165	- 0.0660	- 0.0365	0.1103
			2.316	0.2104	- 0.1104	0.0121	- 0.0619	- 0.9682	- 0.0108	0.0463
			2.432	0.9489	- 0.1686	0.0337	- 0.1012	0.2351	0.0234	0.0627
		1.0	1.243	0.1893	0.9668	0.0392	0.0379	- 0.0969	- 0.0562	0.1182
			2.256	0.3314	- 0.1655	0.0159	- 0.1011	- 0.9188	- 0.0155	0.0889
			2.400	0.9122	- 0.1428	0.0221	- 0.0984	0.3691	- 0.0008	0.0334
	138	1.0	1.397	0.3047	0.9309	0.0459	0.0757	- 0.1111	- 0.0755	0.1213
			2.287	0.9136	- 0.3176	0.0269	- 0.1539	- 0.1960	- 0.0183	0.0376
			2.413	0.2300	0.0421	0.0004	0.0345	0.9713	0.0095	- 0.0266
	140	1.0	1.558	0.4241	0.8710	0.0562	0.1398	- 0.1106	- 0.1032	0.1261
			2.307	0.8887	0.0717	- 0.0027	0.0810	0.9900	0.0109	- 0.0081
			2.535	0.0985	- 0.2341	0.0087	0.9563	- 0.0685	- 0.0968	0.0830
	142	1.0	1.741	0.5337	0.7584	0.0736	0.2899	- 0.1007	- 0.1442	0.1400
			2.384	0.8076	- 0.4095	0.0252	- 0.4212	- 0.0200	- 0.0203	- 0.0363
			2.669	0.1375	- 0.0939	- 0.0002	0.3094	0.9362	- 0.0040	0.0130
		1.2	1.614	0.4958	0.7530	0.0784	0.3374	- 0.1209	- 0.1752	0.1476
			2.350	0.8057	- 0.3364	0.0198	- 0.4770	- 0.0683	- 0.0609	- 0.0385
			2.644	0.2218	- 0.1768	0.0009	0.3769	0.8814	- 0.0229	- 0.0104
	144	1.0	1.830	0.4877	0.5322	0.0932	0.6376	- 0.0710	- 0.1799	0.1623
			2.378	0.7496	0.0629	0.0468	- 0.6507	- 0.0714	- 0.0550	- 0.0213
			2.795	0.4120	- 0.7467	- 0.0062	0.3618	0.3745	0.0046	- 0.0385
		1.2	1.675	0.4469	0.5389	0.1003	0.6495	- 0.0870	- 0.2036	0.1698
			2.350	0.7530	0.0803	0.0429	- 0.6332	- 0.1090	- 0.1065	- 0.0238
			2.773	0.4297	- 0.6876	- 0.0071	0.3464	0.4697	- 0.0123	- 0.0431
	146	1.0	1.709	0.2971	0.3058	0.1033	0.8662	- 0.0389	- 0.1706	0.1633
			2.479	0.8651	0.1988	0.1040	- 0.4170	- 0.0789	- 0.1413	0.0334
			3.112	0.3056	- 0.5532	- 0.0541	0.1713	0.7386	0.1108	- 0.1029
		1.2	1.559	0.3042	0.3346	0.1171	0.8433	- 0.0521	- 0.1930	0.1752
			2.456	0.8406	0.1889	0.0963	- 0.4463	- 0.1119	- 0.1907	0.0207
			3.065	0.3336	- 0.5079	- 0.0548	0.1759	0.7606	0.0694	- 0.1165
J = 2+ (extra)	144	1.0	1.866	0.7222	0.5936	0.0925	0.2340	- 0.1087	- 0.1774	0.1398
			2.425	0.6738	- 0.6352	- 0.0030	- 0.3582	0.0747	0.0276	- 0.0888
			2.581	0.0376	0.0907	- 0.0003	0.1176	0.9882	0.0027	- 0.0053
	146	1.0	1.991	0.4299	0.3481	0.1461	0.7700	- 0.0597	- 0.2063	0.1835
			2.438	0.8083	0.1158	0.1010	- 0.5555	- 0.0769	- 0.0923	0.0082
			2.930	0.2599	- 0.3486	- 0.0910	0.1510	0.8698	0.0951	- 0.1190

Table C-III (cont'd)

	A	$\frac{v_{TO}}{v_{SE}}$	$\omega(\alpha)$	$d^2_{5/2}$	$g^2_{7/2}$	$d^2_{3/2}$	$h^2_{11/2}$	$d_{5/2}g_{7/2}$	$d_{5/2}d_{3/2}$	$g_{7/2}d_{3/2}$
J = 4+	134	0.8	1.493	0.0437	0.9958		- 0.0095	- 0.0568	- 0.0328	0.0448
			2.295	0.0532	- 0.0619		- 0.0181	- 0.9952	- 0.0456	0.0245
			2.897	0.9926	- 0.0447		- 0.0535	0.0609	- 0.0673	0.0409
		1.0	1.458	0.0506	0.9911		- 0.0035	- 0.0982	- 0.0528	0.0513
			2.262	0.0973	- 0.1082		- 0.0352	- 0.9861	- 0.0511	0.0514
			2.829	0.9801	- 0.0484		- 0.0601	0.1133	- 0.1381	0.0392
	136	0.8	1.713	0.0743	0.9917		0.0003	- 0.0873	- 0.0426	0.0404
			2.310	0.0821	- 0.0961		- 0.0164	- 0.9904	- 0.0524	0.0148
			2.645	0.9898	- 0.0696		- 0.0512	0.0931	- 0.0565	0.0324
		1.0	1.687	0.0898	0.9824		0.0059	- 0.1414	- 0.0678	0.0471
			2.297	0.1625	- 0.1605		- 0.0359	- 0.9701	- 0.0618	0.0414
			2.595	0.9699	- 0.0716		- 0.0563	0.1856	- 0.1253	0.0283
	138	1.0	1.889	0.1586	0.9631		0.0173	- 0.1924	- 0.0891	0.0459
			2.359	0.3591	- 0.2464		- 0.0449	- 0.8944	- 0.0840	0.0354
			2.478	0.9077	- 0.0986		- 0.0490	0.3980	- 0.0942	0.0160
	140	1.0	2.106	0.2965	0.9077		0.0343	- 0.2638	- 0.1219	0.0508
			2.428	0.8262	- 0.3932		- 0.0740	- 0.3836	- 0.0978	0.0254
			2.501	0.4589	0.1068		- 0.0130	0.8819	- 0.0039	- 0.0078
	142	1.0	2.342	0.6473	0.6570		0.0611	- 0.3364	- 0.1693	0.0623
			2.544	0.7459	- 0.6195		- 0.1471	0.1952	0.0026	- 0.0120
			2.659	0.0889	0.3854		0.0453	0.9169	0.0274	- 0.0079
		1.2	2.275	0.6082	0.6579		0.0731	- 0.3710	- 0.2201	0.0762
			2.530	0.7548	- 0.6105		- 0.1787	0.1497	- 0.0550	- 0.0123
			2.679	0.1405	0.3771		0.0734	0.9120	- 0.0077	- 0.0293
	144	1.0	2.5219	0.8522	0.3222		0.2579	- 0.2519	- 0.1869	0.0714
			2.6523	0.4169	- 0.2576		- 0.8566	0.1339	0.0742	- 0.0514
			2.8505	0.2702	- 0.2802		0.3573	0.8432	0.0972	- 0.0214
		1.2	2.461	0.7532	0.3886		0.3387	- 0.3075	- 0.2523	- 0.0374
			2.619	0.5897	- 0.2191		- 0.8289	0.0524	- 0.0102	- 0.0471
			2.840	0.3618	- 0.3675		0.3696	0.7698	0.0580	0.0937
	146	1.0	2.374	0.0869	0.1207		0.9751	- 0.0684	- 0.1334	0.0681
			2.731	0.9585	0.1248		- 0.1357	- 0.1446	- 0.1556	0.0470
			3.131	0.2320	- 0.2596		0.1162	0.8857	0.2748	- 0.0724
		1.2	2.331	0.1271	0.1509		0.9621	- 0.0856	- 0.1450	0.0839
			2.693	0.9168	0.1470		- 0.2020	- 0.1947	- 0.2365	0.0564
			3.118	0.3137	- 0.3247		0.1216	0.8599	0.1787	- 0.0994
J = 4+ (extra)	144	1.0	2.326	0.9168	0.2716		0.0248	- 0.2585	- 0.1246	0.0529
			2.565	0.3727	- 0.4133		- 0.0788	0.8187	0.1080	- 0.0462
			2.709	0.1232	- 0.8415		- 0.1172	- 0.5050	0.0828	- 0.0328
	146	1.0	2.594	0.6154	0.1864		0.7079	- 0.1688	- 0.2166	0.0998
			2.661	0.7450	0.0070		- 0.6644	- 0.0576	- 0.0083	- 0.0098
			2.988	0.2213	- 0.2318		0.1661	0.8993	0.2314	- 0.0866

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