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Author
O'Neill, Jason

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Combinatorics of intersecting set systems

A dissertation submitted in partial satisfaction of the

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in

Mathematics

by

Jason O'Neill

Committee in charge:

Professor Jacques Verstraëte, Chair
Professor Fan Chung Graham
Professor Shachar Lovett
Professor Andrew Suk

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The dissertation of Jason O'Neill is approved, and it is acceptable in quality and form for publication on microfilm and electronically.

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VITA

2017 Bachelor of Science, University of California Los Angeles

2019 Candidate of Philosophy, University of California San Diego

2022 Doctor of Philosophy, University of California San Diego

PUBLICATIONS

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ABSTRACT OF THE DISSERTATION

Combinatorics of intersecting set systems

by

Jason O'Neill

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Professor Jacques Verstraëte, Chair

In this dissertation, we examine various problems in extremal set theory, which typically entails maximizing the size of a collection of subsets, or set family, given intersection constraints. For instance, the classical Erdős-Ko-Rado theorem (1961) establishes the largest set family of size k subsets of an n element set which is *intersecting* (i.e. has the property that any two sets have at least one element in common). The largest such intersecting family is the collection of all size k subsets that contain a fixed element, which is commonly referred to as *the star*. Answering a question of Erdős, Ko and Rado, Hilton and Milner (1967) determined the largest intersecting set family which is not isomorphic to a sub-collection of the star. This dissertation settles a conjecture of Hilton and Milner (1967) on the largest set family, for each integer $d \geq 3$, of size k subsets of an n element set which has the property that any d sets have at least one element in common and is not isomorphic to a sub-collection of the star. We also consider various combinatorial results on set families with restricted intersection properties. In particular, we prove generalizations of both the *Bollobás two family* theorem and the *Oddtown and Eventown* theorems.

Chapter 1

Introduction

This dissertation studies various problems regarding intersecting collections of subsets of a finite set. Problems in *extremal set theory* typically involve maximizing the size of a collection of subsets, or set family, given set theoretic constraints. Perhaps the most famous theorem in the area is the Erdős-Ko-Rado [EKR61] Theorem on the size of the largest set family which is *intersecting* (i.e. any two members from the set family have at least one element in common). Also within this general framework are the foundational Oddtown and Eventown results, which demonstrate the *linear algebra method*, and establish the largest set family for which all pairwise intersections contain an even number of elements and the sets themselves have odd and even number of elements respectively.

There are also other results in this dissertation which fall outside the typical framework of maximizing the size of a set family subject to set theoretic constraints. In particular, given an integer k at least three, this dissertation explores k set family analogs which are generalizations of Bollobás [Bol65] *two family theorem* and Babai and Frankl's [BF20] *Bipartite Oddtown theorem*. Finally, we also explore a problem on determining how *small* a set family can be provided it satisfies various set theoretic constraints and is maximal with respect to these constraints. We will now establish commonly used notation in order to give a more accurate description of the main results of this dissertation.

1.1 Frequently used notation

We first describe some of the main objects of study in the field:

- Interval notation: Given positive integers $n \geq m$, we use the notation $[m, n] = \{m, m+1, \ldots, n\}$ and in the case where $m = 1$, we let $[n] = [1, n]$.

- Non-uniform set systems: Given a positive integer $n \geq 1$, we let $2^{[n]} = \{A \subseteq [n]\}$ denote the collection of subsets of $[n]$. We refer to collections $\mathcal{A} \subseteq 2^{[n]}$ as set systems and use calligraphic font to denote set systems

- k-uniform set systems: Given positive integers $n \geq k \geq 1$, we let $\binom{[n]}{k} = \{A \subseteq [n] : |A| = k\}$. If a set systems $\mathcal{F} \subseteq 2^{[n]}$ also satisfies the property that $|F| = k$ for all $F \in \mathcal{F}$, then we refer to $\mathcal{F} \subseteq \binom{[n]}{k}$ as a k-uniform set system

Given a particular subset $A \subseteq [n]$, we use the following notation to describe objects which correspond to the underlying set:

- Characteristic vector: Given $A \subseteq [n]$, we let $v_A \in \mathbb{F}_n$ be given by $(v_A)_k = 1$ if and only if $k \in A$

- Set Complement: Given a subset $A \subseteq [n]$, let $A^c := [n] \setminus A$ be the complement of A in $[n]$. Further given a set family $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq 2^{[n]}$, let $\mathcal{A}^c = \{A_1^c, \ldots, A_m^c\} \subseteq 2^{[n]}$ denote the set of corresponding complements.

- Link set system: Given $A \subseteq [n]$ and $\mathcal{F} \subseteq 2^{[n]}$, we let $\mathcal{F}(A) = \{B \subseteq [n] : B \cup A \in \mathcal{F}\}$ denote the link of A in $\mathcal{F}$ and we refer to $|\mathcal{F}(A)|$ as the *degree* of A in $\mathcal{F}$

- Linear functional: Given $A \subseteq [n]$, let $\chi_A : \mathbb{F}_2^n \to \mathbb{F}_2$ be given by the inner product $w \mapsto \langle w, v_A \rangle$

- Perpendicular Space: Given the vector space $\mathbb{F}_2^n$ with the standard inner product and vector subspace U, let $U^\perp = \{v \in \mathbb{F}_2^n : \langle v, u\rangle = 0 \ \forall u \in U\}$.

We also use the following asymptotic notation and numerical expressions:

- For positive integers $k \leq n$, let $(n)_{(k)} = (n)(n-1)\cdots(n-k+1)$ denote the falling factorial.

- Asymptotic notation: For functions $f, g : \mathbb{N} \to \mathbb{R}^+$, $f = o(g)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f = O(g)$ and $g = \Omega(f)$ if there exists $c > 0$ such that $f(n) \leq cg(n)$ for all $n \in \mathbb{N}$. If $f = O(g)$ and $g = O(f)$, we write $f = \Theta(g)$.

We now briefly describe the main results proven in this dissertation.

1.2 Intersecting families

In this section, we explore the classical literature pertaining to *intersecting set families.* A set family $\mathcal{F} \subseteq 2^{[n]}$ is called *intersecting* if for all $F_1, F_2 \in \mathcal{F}$, the sets F_1 and F_2 have at least one element in common (i.e. $F_1 \cap F_2 \neq \emptyset$). The study of intersecting set families was initiated by the foundational Erdős-Ko-Rado Theorem [EKR61]:

Theorem 1.2.1 (Erdős-Ko-Rado [EKR61])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting family. Then $|\mathcal{F}| \leq 2^{n-1}$. Further, if $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

Proof. Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting family. It then follows that $\mathcal{F}^c \cap \mathcal{F} = \emptyset$ as if $A \in \mathcal{F}^c \cap \mathcal{F}$, then $A, A^c \in \mathcal{F}$; a contradiction. Hence, as $\mathcal{F} \cup \mathcal{F}^c \subseteq 2^{[n]}$ and $|\mathcal{F}^c| = |\mathcal{F}|$, the result follows by noting that

$$2|\mathcal{F}| = |\mathcal{F}| + |\mathcal{F}^c| = |\mathcal{F} \cup \mathcal{F}^c| \leq |2^{[n]}| = 2^n.$$

For the second statement, we let $\mathcal{C}_n$ denote the collection of long cycle permutations of $[n]$ arranged in a circle. Given $\mathcal{F} \subseteq \binom{[n]}{k}$ which is intersecting and $\sigma \in \mathcal{C}_n$, let $\mathcal{F}_\sigma$ denote the sets in $\mathcal{F}$ which appear as cyclically consecutive elements of σ. It then follows that $|\mathcal{F}_\sigma| \leq k$. Without loss of generality, let $\sigma = (1, 2, \ldots, n)$ and suppose $[k] \in \mathcal{F}_\sigma$. Setting $A_i = [i, i+k-1]$ and utilizing $n \geq 2k$, it is straightforward to see that $A_i \in \mathcal{F}_\sigma$ implies that $i \geq n-k+2$ or $i \leq k$. Consider the pair $\{A_{n-k+i}, A_i\}$ for $i \in [2, k]$. As $A_{n-k+i} \cap A_i = \emptyset$ and $\mathcal{F}$ is an intersecting family, $|\mathcal{F}_\sigma \cap \{A_{n-k+i}, A_i\}| \leq 1$. We then recover the desired bound $|\mathcal{F}_\sigma| \leq k$. A double counting argument now gives

$$|\mathcal{F}| \cdot k!(n-k)! = \sum_{F \in \mathcal{F}} |\{\sigma \in \mathcal{C}_n : F \in \mathcal{F}_\sigma\}| = \sum_{\sigma \in \mathcal{C}_n} |\{F \in \mathcal{F} : F \in \mathcal{F}_\sigma\}| \leq (n-1)! \cdot k.$$

Dividing both sides by $k!(n-k)!$, we recover

$$|\mathcal{F}| \leq \frac{(n-1)! \cdot k}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n-1}{k-1}. \qquad \square$$

The proof of the second statement was noted by Katona [Kat72] and uses a method commonly referred to as *Katona's circle method*. The original proof by Erdős, Ko and Rado used a method referred to as *shifting* together with an induction argument.

An interesting component of Theorem 1.2.1 is that there are two substantially different intersecting families of size 2^{n-1}. First, we may take the set family $\mathcal{A}_1 = \{A \subseteq [n] : 1 \in A\}$ – this is intersecting as each pairwise intersection contains the element

1. Second, for odd n, we may take the set family $\mathcal{A}_2 = \{A \subseteq [n] : |A| \geq \frac{n-1}{2}\}$ – this is intersecting as the subsets are too large to have two of them be pairwise disjoint and hence they necessarily contain at least one element in common. However, in the k-uniform case we have a drastically different situation where all of intersecting families of size *nearly* $\binom{n-1}{k-1}$ have similar structure.

The set family $\mathcal{F}_1 = \{F \in \binom{[n]}{k} : 1 \in F\}$ is an intersecting family of size $\binom{n-1}{k-1}$. It is straightforward to see that one can remove a few sets from $\mathcal{F}_1$ and still have an intersecting family of size *nearly* $\binom{n-1}{k-1}$. To rule out this trivial example, Erdős, Ko and Rado defined a set family $\mathcal{F}$ to be *non-trivial* if $\bigcap_{A \in \mathcal{F}} A = \emptyset$. In other words, in a non-trivial set family, for each $i \in [n]$, there exists $X_i \in \mathcal{F}$ so that $i \notin X_i$. Erdős, Ko and Rado asked for the maximum size of a non-trivial intersecting family $\mathcal{F}$ of k-element subsets of $[n]$. This question was answered by Hilton and Milner [HM67].

Theorem 1.2.2 (Hilton-Milner)**.** *Let $n > 2k$ and $k \geq 3$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a non-trivial intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

This may be viewed as a *stability* version of the Erdős-Ko-Rado Theorem, in the sense that an intersecting family of size larger than the bound in Theorem 1.2.2 is necessarily a subfamily of the extremal intersecting family – i.e. there a point in the intersection of all sets in the family. Further, there is a non-trivial intersecting family obtaining the bound in Theorem 1.2.2:

$$\mathcal{HM}(k, 2) = \{[2, k+1]\} \cup \{A \in \binom{[n]}{k} : 1 \in A, A \cap [2, k+1] \neq \emptyset\}.$$

Hilton and Milner [HM67] considered an extension of Theorem 1.2.2 to d-wise intersecting families: a set family $\mathcal{F} \subseteq 2^{[n]}$ is *d-wise intersecting* if any set of d distinct sets in $\mathcal{F}$ have non-empty intersection, with the case $d = 2$ corresponding to intersecting families. In the case where $\mathcal{F} \subseteq 2^{[n]}$ is a non-trivial d-wise intersecting family, there is a natural extension to the family of all sets which contain a fixed element. Observe that the following set family is non-trivial d-wise intersecting:

$$\mathcal{A}(d) = \{A \subseteq [n] : |A \cap [d+1]| \geq d\}.$$

Brace and Daykin [BD71] proved that $\mathcal{A}_d$ is the largest such set family:

Theorem 1.2.3. *Let $\mathcal{A} \subseteq 2^{[n]}$ be a non-trivial d-wise intersecting family. Then $|\mathcal{A}| \leq |\mathcal{A}(d)| = (d+2)2^{n-d-1}$.*

In the case of k-uniform non-trivial d-wise intersecting families, there are two natural extensions to the set families $\mathcal{A}(d)$ and $\mathcal{HM}(k,2)$. These set families were constructed by Hilton and Milner [HM67]. Moreover, Hilton and Milner [HM67] conjectured that for large enough n the extremal non-trivial d-wise intersecting family of k-sets in $[n]$, up to isomorphism, is one of the following two families:

$$\begin{aligned} \mathcal{A}(k,d) &= \{A \in \tbinom{[n]}{k} : |A \cap [d+1]| \geq d\} \\ \mathcal{HM}(k,d) &= \{A \in \tbinom{[n]}{k} : [d-1] \subseteq A, A \cap [d, k+1] \neq \emptyset\} \cup \{[k+1] \setminus \{i\} : i \in [d-1]\}. \end{aligned}$$

In Chapter 2, we prove the conjecture of Hilton and Milner (1967):

Theorem 1.2.4 (O'N-Verstraëte)**.** *Let k, d be integers with $2 \le d < k$. Then there exists $n_0(k,d)$ such that for $n \ge n_0(k,d)$, if $\mathcal{F}$ is a non-trivial, d-wise intersecting family of k-element subsets of $[n]$, then*

$$|\mathcal{F}| \le \max\{|\mathcal{HM}(k,d)|, |\mathcal{A}(k,d)|\}.$$

The defining property of non-trivial families $\mathcal{F} \subseteq 2^{[n]}$ is the fact that for each element $i \in [n]$, there exists at least one set $X_i \in \mathcal{F}$ so that $i \notin X_i$. This notion of avoiding singletons can be extended to avoiding sets of larger size. Given a set system $\mathcal{F} \subseteq 2^{[n]}$, the *transversal number*, denoted $\tau(\mathcal{F})$, is the minimum positive integer s so that there exists a set $S \in \binom{[n]}{s}$ where $S \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. As such, given a set family $\mathcal{F} \subseteq 2^{[n]}$ with $\tau(\mathcal{F}) = s+1$, for each $S \in \binom{[n]}{s}$, there exists $X_S \in \mathcal{F}$ with $S \cap X_S = \emptyset$.

In this language of transversal numbers, a set family $\mathcal{F} \subseteq 2^{[n]}$ is non-trivial if and only if $\tau(\mathcal{F}) \ge 2$. Further, a (non-empty) set family $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting if and only if $\tau(\mathcal{F}) \le k$ (where each set $F \in \mathcal{F}$ has the defining property). There has been substantial research on intersecting families (i.e. $d = 2$) with $\tau(\mathcal{F}) \ge 3$ and it would interesting to explore these larger transversal problems in the d-wise intersecting setting:

Problem 1.2.5. *Given $n > k > d \ge 3$ integers, what is the maximum size of a d-wise intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ with transversal number at least three?*

In this setting, the set family $\mathcal{A}^3(k,d) = \{A \in \binom{[n]}{k} : |A \cap [2d+1]| \ge 2d-1\}$ is a d-wise intersecting family with $\tau(\mathcal{A}^3(k,d)) = 3$. Further, it is not hard to see that $\mathcal{A}_3(k,d)$ is a natural generalization of $\mathcal{A}(k,d)$. Moreover, there exists a natural generalization of the set family $\mathcal{HM}(k,d)$ in this setting. Let $\mathcal{HM}^3(k,d) := \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \mathcal{F}_3$ where

$$\begin{aligned}
\mathcal{F}_1 &= \left\{A \in \binom{[n]}{k} : [2d-3] \subseteq A, e \subseteq A \text{ for some } e \in \binom{[2d-2, k+2]}{2}\right\} \\
\mathcal{F}_2 &= \{[k+2] \setminus e\}_{e \in \binom{[2d-3]}{2}} \\
\mathcal{F}_3 &= \{A \in \binom{[n]}{k} : |A \cap [2d-3]| = 2d-4, |A \cap [2d-2, k+2]| = k-2d+4\}
\end{aligned}$$

We now conjecture the following analog of Theorem 1.2.4 to the setting with transversal number at least three:

Conjecture 1.2.6. *Let n be sufficiently large and $d \geq 3$ with $2d-1 \leq k$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a d-wise intersecting family with $\tau(\mathcal{F}) \geq 3$, then $|\mathcal{F}| \leq \max\{|\mathcal{A}^3(k,d)|, |\mathcal{HM}^3(k,d)|\}$*

Let us now explain the condition of $2d-1 \leq k$ in Conjecture 1.2.6. Suppose $2d-1 > k$ and hence $k \leq 2d-2$ and that $\mathcal{F} \subseteq \binom{[n]}{k}$ is d-wise intersecting with $\tau(\mathcal{F}) \geq 3$. Without loss of generality, we may assume $[k] \in \mathcal{F}$. We may now find $e_1, \ldots, e_{d-1} \in \binom{[k]}{2}$ so that $e_1 \cup \cdots \cup e_{d-1} = [k]$. For each e_i, there exists $X_i \in \mathcal{F}$ so that $e_i \cap X_i = \emptyset$ by the condition $\tau(\mathcal{F}) \geq 3$. It now follows that $[k] \cap X_1 \cap \cdots \cap X_{d-1} = \emptyset$; which contradicts the d-wise intersecting condition on $\mathcal{F}$.

In the non-uniform setting, the Brace-Daykin theorem determines the size of the largest d-wise intersecting family with transversal number at least two. It would be interesting to explore an analogous statement for transversal number at least three:

Problem 1.2.7. *Given $n > d \geq 3$ integers, what is the maximum size of a d-wise intersecting family $\mathcal{F} \subseteq 2^{[n]}$ with transversal number at least three?*

There is a very natural candidate for Problem 1.2.5 which we conjecture is the largest such set family:

Conjecture 1.2.8. *Let n be sufficiently large and $d \geq 3$. If $\mathcal{F} \subseteq 2^{[n]}$ is a d-wise intersecting family with $\tau(\mathcal{F}) \geq 3$, then $|\mathcal{F}| \leq |\mathcal{A}^3(d)|$ where $\mathcal{A}^3(d)$ is defined as:*

$$\mathcal{A}^3(d) = \{A \subseteq [n] : |A \cap [2d+1]| \geq 2d-1\}.$$

Intersecting set systems are required to have non-empty pairwise intersections. However, there are also interesting combinatorial results when you require the pairwise intersections to all have an even number of elements in common.

1.3 Set families with few intersections of odd size

In this section, we explore the classical Oddtown and Eventown problems on set systems for which all pairwise intersections have an even number of elements. A set family $\mathcal{A} \subseteq 2^{[n]}$ follows *oddtown rules* if the sizes of all sets in $\mathcal{A}$ are odd and distinct pairs of sets from $\mathcal{A}$ have an even number of elements in common.

Theorem 1.3.1 (Berlekamp [Ber69] and Graver [Gra75])**.** *Let $\mathcal{A} \subseteq 2^{[n]}$ satisfy oddtown rules. Then $|\mathcal{A}| \leq n$.*

Proof. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\} \subseteq 2^{[n]}$ be a set family which satisfies oddtown rules. For each A_i, let $v_i = v_{A_i}$ denote the characteristic vector of A_i in $\mathbb{F}_2^n$. Further, let $\langle \cdot, \cdot \rangle$ denote the standard inner product in $\mathbb{F}_2^n$. It then follows that $\langle v_i, v_i \rangle = 1$ for all $i \in [m]$ as $|A_i|$

is odd. It also follows that $\langle v_i, v_j \rangle = 0$ for all $i \neq j \in [m]$ as $|A_i \cap A_j|$ is even. We now claim that the set $\{v_1, \ldots, v_m\}$ is linearly independent in $\mathbb{F}_2^n$. Let $0 = \epsilon_1 v_1 + \cdots + \epsilon_m v_m$ and observe that

$$0 = \langle \epsilon_1 v_1 + \cdots + \epsilon_m v_m, v_i \rangle = \epsilon_i \langle v_i, v_i \rangle = \epsilon_i.$$

As such, it follows that $m \leq n$ as desired. $\square$

There are many constructions of n sets which satisfy oddtown rules and hence show that the above theorem is best possible. We highlight the following constructions which will play a role in later discussions:

$$\mathcal{O}_1 = \{\{i\} : 1 \leq i \leq n\} \quad \text{and} \quad \mathcal{O}_2 = \{A \in \binom{[n]}{3} : A \subseteq [4i-3, 4i]\}$$

where $|\mathcal{O}_2| = n$ when $4|n$.

Alternatively, a collection $\mathcal{A}$ of even-sized subsets of an n element set follows *eventown rules* if all pairs of sets from $\mathcal{A}$ have even-sized intersections.

Theorem 1.3.2 (Berlekamp [Ber69] and Graver [Gra75])**.** *Let $\mathcal{A} \subseteq 2^{[n]}$ satisfy eventown rules. Then $|\mathcal{A}| \leq 2^{\lfloor n/2 \rfloor}$.*

Proof. Let $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq 2^{[n]}$ satisfy eventown rules. As before, we let $v_i = v_{A_i}$ denote the characteristic vector in $\mathbb{F}_2^n$ and consider $W = \operatorname{span}\{v_1, \ldots, v_m\}$. For each $w \in W$, it follows that $\langle w, v_i \rangle = 0$ for all characteristic vectors v_i. Hence, $W \subseteq W^\perp$. Noting the classical linear algebra result that $\dim(W) + \dim(W^\perp) = n$, it thus follows that $\dim(W) \leq \lfloor n/2 \rfloor$. The result then follows as $|W| \leq 2^{\lfloor n/2 \rfloor}$. $\square$

There are non-isomorphic eventown families which have $2^{\lfloor n/2 \rfloor}$ sets. In fact, by examining the proof of Theorem 1.3.2 we immediately obtain:

Corollary 1.3.3. *Let $\mathcal{A} \subseteq 2^{[n]}$ satisfy eventown rules and $|\mathcal{A}| < 2^{\lfloor n/2 \rfloor}$. Then there exists $X \in 2^{[n]}$ so that $\mathcal{A} \cup \{X\}$ satisfies eventown rules.*

As such, any maximal eventown family is maximum. Further, this allows us to construct many non-isomorphic eventown families of size $2^{\lfloor n/2 \rfloor}$. One such eventown construction, given $B_i = \{2i-1, 2i\}$, is as follows:

$$\mathcal{E} = \left\{ \bigcup_{i \in J} B_i : J \in 2^{[\lfloor n/2 \rfloor]} \right\}.$$

The oddtown and eventown problems are foundational results which highlight the linear algebra method [BF20] in extremal combinatorics. There have been numerous extensions and variants of these results in the literature [OV20b, SV18, Vu99, SV05, Vu97, FO83, DFS83]. It is also worth noting that one can equivalently ask about extremal problems where sets of even (odd) size have an odd sized pairwise intersections. Given an extremal oddtown construction and adding an auxiliary element to each set in the family, we recover a set family of sets of even size for which each pair of distinct sets have an odd number of elements in common. This results in a natural duality between the oddtown and eventown problems and those which requires pairwise odd sized intersections. As such, we restrict our study in this dissertation to the classical Oddtown and Eventown problems.

Many problems in extremal combinatorics involve maximizing the size of an object which satisfies combinatorial constraints. In this setting, we necessarily have at least one instance of the forbidden combinatorial constraint if we have one more than the maximum size. Supersaturation problems entail the *minimum* number of instances of the forbidden combinatorial constraint if we have more than the maximum size. The notion of *supersaturation* stems from the fact that, in certain instances, given one more than the extremal amount, we are guaranteed *many* instances of the forbidden property. In other words, the forbidden property saturates as we get slightly more than the maximum.

The first instance of supersaturation in extremal combinatorics comes from triangle-free graphs. Mantel [Man07] proved that any n-vertex triangle-free graph has at most $\lfloor n^2/4 \rfloor$ edges. This result is easily seen to be best possible by taking a complete bipartite graph with part sizes nearly equal. The Erdős-Rademacher problem in extremal combinatorics involves the minimum number of triangles in an n-vertex graph with $\lfloor n^2/4 \rfloor + s$ edges. Erdős [Erd62] proved that there are at least $s \cdot \lfloor n/2 \rfloor$ triangles in an n-vertex graph with $\lfloor n^2/4 \rfloor + s$ edges for $s \leq 3$.

In Chapter 3, we explore the *supersaturation* versions of the oddtown and eventown problems: given slightly more than n odd-sized subsets (resp. $2^{\lfloor n/2 \rfloor}$ even-sized subsets) of an n element set, how many pairs of sets from our collection must have an odd number of elements in common?

For a collection, or set family $\mathcal{A}$, let $\mathrm{op}(\mathcal{A})$ denote the number of distinct pairs of sets $A, B \in \mathcal{A}$ for which $|A \cap B|$ is odd. For convenience, suppose $n = 2k = 4l$ and let $X_1, \ldots, X_l$ be pairwise disjoint sets with $X_i = \{x_{1,i}, x_{2,i}, x_{3,i}, x_{4,i}\}$. Given X_i, let

$A_{2i-1} = \{x_{1,i}, x_{2,i}\}$, $A_{2i} = \{x_{3,i}, x_{4,i}\}$, $B_{2i-1} = \{x_{1,i}, x_{4,i}\}$, and $B_{2i} = \{x_{2,i}, x_{3,i}\}$. Define

$$\mathcal{E}_1 = \left\{ \bigcup_{j \in J} A_j \ : \ J \subseteq [k] \right\} \quad \text{and} \quad \mathcal{E}_2 = \left\{ \bigcup_{j \in J} B_j \ : \ J \subseteq [k] \right\}. \tag{1.1}$$

Observe that both $\mathcal{E}_1$ and $\mathcal{E}_2$ are extremal (i.e. largest) eventown families and moreover for each $X \in \mathcal{E}_2 \setminus \mathcal{E}_1$, $\mathrm{op}(\mathcal{A} \cup \{X\}) = 2^{k-1}$. For $1 \leq s \leq 2^k - 2^l$, let $\mathcal{E}_1(s)$ denote a set family which contains s elements from $\mathcal{E}_2 \setminus \mathcal{E}_1$ together with $\mathcal{E}_1$. It then follows that $|\mathcal{E}_1(s)| = 2^k + s$ and $\mathrm{op}(\mathcal{E}_1(s)) = s \cdot 2^{k-1}$. Our main result is that we are able to show this is best possible when $s = 1$ and $s = 2$:

Theorem 1.3.4. *Let $n \geq 1$. If $\mathcal{A} \subseteq 2^{[n]}$ is a collection of sets of even size with $|\mathcal{A}| \geq 2^{\lfloor n/2 \rfloor} + 1$, then $\mathrm{op}(\mathcal{A}) \geq 2^{\lfloor n/2 \rfloor - 1}$. Moreover, if $\mathcal{A} \subseteq 2^{[n]}$ is a collection of sets of even size with $|\mathcal{A}| \geq 2^{\lfloor n/2 \rfloor} + 2$, then $\mathrm{op}(\mathcal{A}) \geq 2^{\lfloor n/2 \rfloor}$.*

Given $3 \leq s \leq 2^{\lfloor n/2 \rfloor} - 2^{\lfloor n/4 \rfloor}$, we believe that the set family $\mathcal{E}_1(s)$ minimizes the number of pairwise intersections of odd size amongst all families of $2^{\lfloor n/2 \rfloor} + s$ sets of even size and conjecture:

Conjecture 1.3.5. *Let $n \geq 1$. Fix $3 \leq s \leq 2^{\lfloor n/2 \rfloor} - 2^{\lfloor n/4 \rfloor}$. If $\mathcal{A} \subseteq 2^{[n]}$ is a collection of sets of even size with $|\mathcal{A}| \geq 2^{\lfloor n/2 \rfloor} + s$, then $\mathrm{op}(\mathcal{A}) \geq s \cdot 2^{\lfloor n/2 \rfloor - 1}$.*

For oddtown, recall the extremal constructions $\mathcal{O}_1$ and $\mathcal{O}_2$. For $1 \leq s \leq n$, let $\mathcal{O}_1(s)$ be any set family consisting of $\mathcal{O}_1$ together with exactly s sets from $\mathcal{O}_2$. It then follows that $|\mathcal{O}_1(s)| = n + s$ and $\mathrm{op}(\mathcal{O}_1(s)) = 3s$. Hence there are set families with slightly more than n odd sets and few pairwise odd-sized intersection which in some sense demonstrates that supersaturation results for oddtown do not occur. In any case, we are able to prove this is best possible for $s = 1$:

Theorem 1.3.6. *Let $n \geq 1$ and $\mathcal{A} \subseteq 2^{[n]}$ be a collection of sets of odd size with $|\mathcal{A}| \geq n+1$. Then* $\mathrm{op}(\mathcal{A}) \geq 3$*.*

It is also worth noting that there exist extremal examples to Theorem 1.3.6 which do not contain an extremal oddtown family. To see this, let $n = 5$ and take $\mathcal{X} = \{X_1, \ldots, X_6\}$ where $X_1 = \{1,2,3\}$, $X_2 = \{1,4,5\}$, $X_3 = \{1,2,4\}$, $X_4 = \{1,3,5\}$, $X_5 = \{1,3,4\}$, and $X_6 = \{1,2,5\}$. Then $\mathrm{op}(\mathcal{X}) = 3$. Given $2 \leq s \leq n$, we believe that the set family $\mathcal{O}_1(s)$ minimizes the number of pairwise intersections of odd size amongst all families of $n + s$ sets of odd size and conjecture:

Conjecture 1.3.7. *Let $n \geq 1$ and fix $1 \leq s \leq n$. If $\mathcal{A} \subseteq 2^{[n]}$ is a collection of sets of odd size with $|\mathcal{A}| \geq n + s$, then* $\mathrm{op}(\mathcal{A}) \geq 3s$*.*

There are numerous extensions of the oddtown and eventown problems. In particular, Deza, Frankl, and Singhi [DFS83] used *multilinear polynomials* to give an extension of the Oddtown theorem to arbitrary primes:

Theorem 1.3.8 (Deza-Frankl-Singhi [DFS83])**.** *Let p be prime and let $L \subseteq [0, p-1]$ be so that $|L| = s$. If $\mathcal{F} \subseteq 2^{[n]}$ is such that for all $F_1 \neq F_2 \in \mathcal{F}$, $|F_1 \cap F_2| \in L$ and $|F_1| \notin L$, then $|\mathcal{F}| \leq \binom{n}{0} + \cdots + \binom{n}{s}$.*

The bound that Theorem 1.3.8 gives for oddtown (i.e. when $p = 2$ and $s = 1$) is almost tight, but Theorem 1.3.8 also handles many more cases. It would be interesting to extend the supersaturation result in Theorems 1.3.4 and 1.3.6 to the more general p and L setting:

Problem 1.3.9. *Let p be prime and let $L \subseteq [0, p-1]$ with $|L| = s$. What is the minimum number of pairwise intersections not in $L \pmod p$ over all set families $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| \geq \binom{n}{0} + \cdots + \binom{n}{s} + 1$? What is it amongst families with $|\mathcal{F}| \geq \binom{n}{0} + \cdots + \binom{n}{s} + t$ for a positive integer $t > 1$?*

There are also results on families $\mathcal{F} \subseteq \binom{[n]}{k}$ which, modulo a prime p, have restricted pairwise intersections. In this area, the monumental result is the Frankl-Wilson Theorem [FW81] which states:

Theorem 1.3.10 (Frankl-Wilson [FW81])**.** *Let p be prime and let $L \subseteq [0, p-1]$ with $k \notin L \pmod p$ and $|L| = s$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is such that for all $F_1 \neq F_2 \in \mathcal{F}$, $|F_1 \cap F_2| \in L \pmod p$, then $|\mathcal{F}| \leq \binom{n}{s}$.*

Analogous to Problem 1.3.9 asking for a supersaturation version of Theorem 1.3.8, it would also be interesting to explore supersaturation versions of the well studied Frankl-Wilson Theorem:

Problem 1.3.11. *Let p be prime and let $L \subseteq [0, p-1]$ with $k \notin L \pmod p$ and $|L| = s$. What is the minimum number of pairwise intersections not in $L \pmod p$ over all set families $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| \geq \binom{n}{s} + 1$? What is it amongst families with $|\mathcal{F}| \geq \binom{n}{s} + t$ for a positive integer $t > 1$?*

1.4 Cross family intersections

In this section, we explore a natural extension to problems in extremal set theory with intersection constraints on the pairwise intersections from a set family. The sets families $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ are said to be *cross-intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. There is a rich history of research on cross-intersecting set families initiated from Hilton and Milner [HM67] who proved the following:

Theorem 1.4.1 (Hilton-Milner [HM67])**.** *Let $n \geq 2k$. If $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ are cross intersecting, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \binom{n-k}{k} + 1.$$

It is straightforward to see that Theorem 1.4.1 is best possible by taking $\mathcal{A} = \{[k]\}$ and $\mathcal{B} = \{A \in \binom{[n]}{k} : A \cap [k] \neq \emptyset\}$. In addition to being a cross family variant of the Erdős-Ko-Rado theorem, Theorem 1.4.1 is a key lemma in the proof of Theorem 1.2.2. Pyber [Pyb86] proved another classical result on the maximum such product:

Theorem 1.4.2 (Pyber [Pyb86])**.** *Let $n \geq 2k$. If $\mathcal{A} \subseteq \binom{[n]}{k}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ are cross-intersecting, then*

$$|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2$$

In Section 1.3, we explored the classical oddtown problem of set families consisting of sets of odd size for each distinct pairs of sets have an even number of elements in common. The generalization of this classical result to the cross family setting is the *bipartite oddtown theorem*, which appears in Babai and Frankl [BF20, Exercise 1.1.8]:

Theorem 1.4.3 (Babai, Frankl [BF20])**.** *Let* $\mathcal{A} = \{A_1, A_2, \ldots, A_m\} \subset 2^{[n]}$ *and* $\mathcal{B} = \{B_1, B_2, \ldots, B_m\} \subset 2^{[n]}$ *be families where* $|A_i \cap B_i|$ *is odd for all* $i \in [m]$ *and* $|A_i \cap B_j|$ *is even for all* $i \neq j \in [m]$*. Then* $m \leq n$*.*

Proof. Recall that v_{A_i} and v_{B_j} are the characteristic vectors of A_i and B_j respectively. Set A to be the $m \times n$ matrix whose m rows consist of the vectors $v_{A_1}, \ldots, v_{A_m} \in \mathbb{F}_2^n$. Similarly, set B to be the $n \times m$ matrix whose m columns consist of the vectors $v_{B_1}, \ldots, v_{B_m} \in \mathbb{F}_2^n$. The bipartite oddtown conditions implies that $AB = I_m$ where I_m is the $m \times m$ identity matrix. Noting the standard linear algebra fact that $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$, we recover the desired bound that $m \leq n$. $\square$

To see that Theorem 1.4.3 is a generalization of the Oddtown theorem, let $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq 2^{[n]}$ satisfy oddtown rules. We now define $\mathcal{B} = \{B_1, \ldots, B_m\} \subseteq 2^{[n]}$ so that $B_i = A_i$ for all $i \in [m]$. It then follows that $\mathcal{A}$ and $\mathcal{B}$ satisfy the condition of Theorem 1.4.3 and hence the conclusion also implies the Oddtown result. Although the proof of Theorem 1.4.3 is different than the proof of Oddtown theorem, it still falls under the general *linear algebra method*. It is also worth noting that translating the proof of oddtown with slight alterations gives an upper bound of $n + 1$ for the bipartite oddtown problem, where $n+1$ comes from dimension of the vector space of multilinear polynomials of degree at most one (see also the proof of Theorem 1.3.8 [ABS91, DFS83]).

In Section 1.2, we explored an extension of a classical result on pairwise intersections to the d-wise intersecting setting. In this section, we similarly explore an extension of Theorem 1.4.3 to $k \geq 3$ families. We use the letter k to denote the number of set families as opposed to d as our methods involve reducing the problem to a *k-uniform hypergraph* problem for which the notation k is more standard. Our main result is as follows:

Theorem 1.4.4. *Let $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ be set families of an n element set with $\mathcal{A}_j = \{A_{j,i} : 1 \leq i \leq m\}$ where $|\bigcap_{j=1}^{k} A_{j,i_j}|$ is even if and only if $i_1, \ldots, i_k \in [m]$ are all distinct. Then $m = O(n^{1/\lfloor k/2 \rfloor})$.*

By employing a connection to a hypergraph covering problem and a hypergraph extension (see Alon [Alo86], Cioabă-Kündgen-Verstraëte [CKV09], and Leader-Milićević-Tan [LMT18]) of the Graham-Pollak Theorem [GP72], we also show that Theorem 1.4.4 is best possible. Moreover, we prove the following:

Theorem 1.4.5. *Let t, k be integers with $t \geq 2$ and $2t - 2 \leq k$. If $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ are set families of an n element set with $\mathcal{A}_j = \{A_{j,i} : 1 \leq i \leq m\}$ where $|\bigcap_{j=1}^{k} A_{j,i_j}|$ is even if and only if at least t of the i_j are distinct, then $m = O(n^{1/(t-1)})$.*

Theorem 1.4.5 is seen to be best possible by again using a connection to a hypergraph covering problem. In the case where $\mathcal{A} = \mathcal{A}_1 = \cdots = \mathcal{A}_k$, we recover a variation of the classical oddtown problem. For integers $t < k$, a family $\mathcal{A} \subseteq 2^{[n]}$ is said to follow *(k, t)-oddtown rules* if the intersection of any d distinct sets in $\mathcal{A}$ is odd if $d < t$ and even if $t \leq d \leq k$.

Corollary 1.4.6. *Let $t, k \in \mathbb{N}$ be so that $2 \leq t \leq k$ and $2t - 2 \leq k$. If $\mathcal{A} \subseteq 2^{[n]}$ follows (k, t)-oddtown rules, then $|\mathcal{A}| = O(n^{1/(t-1)})$.*

Corollary 1.4.6 is best possible by considering $\mathcal{A} = \{A_i\}_{i\in[n]}$ where $A_i = \{F \in \binom{[n]}{t-1} : i \in F\}$ and where n is so that the binomial coefficients corresponding to the size of the d-wise intersections are odd for $d < t$. When $t < k$ and $2t - 2 > k$, we are able to show that $m = O(n^{1/(k-t+1)})$ (see Section 4.3), and conjecture that a stronger bound holds:

Conjecture 1.4.7. *Let t, k be integers with $t \geq 2$ and $2t-2 > k$. If $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ are set families of an n element set with $\mathcal{A}_j = \{A_{j,i} : 1 \leq i \leq m\}$ where $|\bigcap_{j=1}^{k} A_{j,i_j}|$ is even if and only if at least t of the i_j are distinct, then $m = O(n^{1/\lfloor k/2 \rfloor})$.*

The bound $m = O(n^{1/\lfloor k/2 \rfloor})$ comes from a connection to a hypergraph covering problem (see Section 4.1) and would imply that Conjecture 1.4.7 if true is tight. By adding an element to *every* set in a family, the size of d-wise intersections switch parity and we get "reverse" analogs of Theorems 1.4.4 and 1.4.5.

Many results in extremal set theory regarding even/odd sized intersections have been extended to arbitrary primes p. Theorem 1.3.8 and Theorem 1.3.10 are canonical examples of this in the non-uniform and uniform cases respectively. For $k \geq 3$, Szabó and Vu [SV05] consider k-wise oddtown problems wherein the size of the sets are odd and the sizes of intersections of k distinct sets are even. Grolmusz and Sudakov [FS04] study k-wise (p, L)-intersecting set systems where the size of distinct k-wise intersections are in $L \pmod p$. Similarly, we propose the following extension to Theorem 1.4.4 for primes $p > 2$.

Conjecture 1.4.8. *Let p be prime and $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ be set families of an n element set with $\mathcal{A}_j = \{A_{j,i} : 1 \leq i \leq m\}$ where $|\bigcap_{j=1}^{k} A_{j,i_j}|$ is nonzero modulo p if and only if $i_1, \ldots, i_k \in [m]$ are all distinct. Then $m = O(n^{(p-1)/\lfloor k/2 \rfloor})$.*

Füredi and Sudakov [FS04, Lemma 3.1] used multilinear polynomials to prove if $\mathcal{A} = \{A_1, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, \ldots, B_m\}$ are families of subsets of an n element set where $|A_i \cap B_j|$ is nonzero modulo p if and only if $i \neq j$, then

$$m \leq \sum_{i=0}^{p-1} \binom{n}{i}. \tag{1.2}$$

This verifies Conjecture 1.4.8 when $k = 2$. It then follows that (1.2) is asymptotically best possible by taking $\mathcal{A} = \binom{[n]}{p-1}$ and $\mathcal{B}$ to be the corresponding complements.

1.5 The Bollobás set pairs inequality

In Section 1.4, we considered cross-intersecting families and proved bounds on the number of sets from each family provided the set families satisfied underlying intersection properties. Although many of the classical results in the field fit under this guise, there are also instances of classical results in which we can prove much more. One particular instance of this is the Bollobás set pairs inequality or two families theorem [Bol65]:

Theorem 1.5.1. (Bollobás) *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be families of finite sets, such that $A_i \cap B_j \neq \emptyset$ if and only if $i, j \in [m]$ are distinct. Then*

$$\sum_{i=1}^{m} \binom{|A_i \cup B_i|}{|A_i|}^{-1} \leq 1. \tag{1.3}$$

Proof. Set $X = (A_1 \cup B_1) \cup \cdots \cup (A_m \cup B_m)$ to be the ground set and suppose $|X| = n$. For each $i \in [m]$, define

$$\mathcal{C}_i = \{\pi : X \to [n] : \max_{x \in A_i} \pi(x) < \min_{x \in B_i} \pi(x)\}$$

to be the collections of permutations which order all of A_i before B_i. It then follows that $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$. To see this, suppose that $\pi \in \mathcal{C}_i \cap \mathcal{C}_j$ and without loss of generality suppose that π is so that

$$\max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x). \tag{1.4}$$

As $A_i \cap B_j \neq \emptyset$, there exists an element $y \in A_i \cap B_j$. As a result, by definition of $\pi \in \mathcal{C}_i \cap \mathcal{C}_j$ and (1.4),

$$\pi(y) \leq \max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x) < \min_{x \in B_j} \pi(x) \leq \pi(y).$$

It therefore follows that $\pi(y) < \pi(y)$; a contradiction. A straightforward counting argument then gives

$$|\mathcal{C}_i| = \binom{n}{|A_i \cup B_i|} |A_i|!|B_i|!(n - |A_i \cup B_i|)! = n! \cdot \binom{|A_i \cup |B_i|}{|A_i|}^{-1}.$$

Thus, noting that the $\mathcal{C}_i$ are pairwise disjoint, we recover

$$n! \geq \sum_{i=1}^{m} |\mathcal{C}_i| = \sum_{i=1}^{m} n! \cdot \binom{|A_i \cup |B_i|}{|A_i|}^{-1}$$

and the result follows by dividing through by $n!$. $\square$

For convenience, we refer to a pair of families $\mathcal{A}$ and $\mathcal{B}$ satisfying the conditions of Theorem 1.5.1 as a *Bollobás set pair*. Bollobás set pairs are also interesting for their corresponding *one family problem.* A set system $\mathcal{A} \subseteq 2^{[n]}$ is called an *antichain* if for all $A, B \in \mathcal{A}$, $A \nsubseteq B$ and $B \nsubseteq A$ (i.e. no two sets in the family have one contained in the other). Given an antichain $\mathcal{A} \subseteq 2^{[n]}$, it follows that $(\mathcal{A}, \mathcal{A}^c)$ is a Bollobás set pair. An immediate corollary of Theorem 1.5.1 is that the number of sets in a Bollobás set pair is at most $\binom{n}{\lfloor n/2 \rfloor}$. Further, this also implies the classical result of Sperner [Spe28] on the size of the largest antichain. As such, Theorem 1.5.1 is substantially stronger than a the typical extremal set theory result on the maximum number of subsets set families can have if they satisfy an intersections constraint.

The inequality (1.3) is tight, as we may take the pairs (A_i, B_i) to be distinct partitions of a set of size $a + b$ with $|A_i| = a$ and $|B_i| = b$ for $1 \leq i \leq \binom{a+b}{a}$. The latter inequality was proved for $a = 2$ by Erdős, Hajnal and Moon [EHM64], and in general has a number of different proofs [Han64, JP71, Kat74, Lov77a, Lov77b]. A geometric version of Theorem 1.5.1 was proved by Lovász [Lov77a, Lov77b], who showed that if $A_1, A_2, \ldots, A_m$ and $B_1, B_2, \ldots, B_m$ are respectively a-dimensional and b-dimensional subspaces of a linear space and $\dim(A_i \cap B_j) = 0$ if and only if $i, j \in [m]$ are distinct, then $m \leq \binom{a+b}{a}$. Theorem 1.5.1 has been generalized in a number of different directions in the literature [Fra82, F$\ddot{8}$4, KKK15, Lov07, Tal04, Tuz85].

In this dissertation, we give a generalization of Theorem 1.5.1 from the case of two families to $k \geq 3$ families of sets with conditions on the k-wise intersections. For $2 \leq t \leq k$, a *Bollobás (k,t)-tuple* is a sequence $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ of set families $\mathcal{A}_j = \{A_{j,i} : 1 \leq i \leq m\}$ where $\bigcap_{j=1}^{k} A_{j,i_j} \neq \emptyset$ if and only if at least t of the indices $i_1, i_2, \ldots, i_k$

are distinct. We refer to m as the *size* of the Bollobás (k,t)-tuple. Let $[m]_{(t)}$ denote the set of sequences of t distinct elements of $[m]$ and fix a surjection $\phi : [k] \to [t]$. For $\sigma \in [m]_{(t-1)}$, set $\sigma(t) = \sigma(1)$ and define $A_{1,\sigma}(\phi) = \bigcap_{j:\phi(j)=1} A_{j,\sigma(1)}$ and, for $2 \le j \le t$, we define

$$A_{j,\sigma}(\phi) = \bigcap_{h:\phi(h)=j} A_{h,\sigma(j)} \setminus \bigcup_{h=1}^{j-1} A_{h,\sigma}(\phi).$$

Using this notation, we generalize (1.3) as follows:

Theorem 1.5.2. *Let $k \ge t \ge 2$ and $m \ge t$, let $\phi : [k] \to [t]$ be a surjection, and let $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ be a Bollobás (k,t)-tuple of size m. Then*

$$\sum_{\sigma \in [m]_{(t-1)}} \binom{|A_{1,\sigma}(\phi) \cup A_{2,\sigma}(\phi) \cup \cdots \cup A_{t,\sigma}(\phi)|}{|A_{1,\sigma}(\phi)| \;\; |A_{2,\sigma}(\phi)| \;\; \cdots \;\; |A_{t,\sigma}(\phi)|}^{-1} \le 1. \tag{1.5}$$

We show in Chapter 5 that this inequality is tight for all $k \ge t = 2$, but do not have an example to show that this inequality is tight for any $t > 2$.

For $n \ge k \ge t \ge 2$, let $\beta_{k,t}(n)$ denote the maximum m such that there exists a Bollobás (k,t)-tuple of size m consisting of subsets of $[n]$. Then (1.3) gives $\beta_{2,2}(n) \le \binom{n}{\lfloor n/2 \rfloor}$ which is tight for all $n \ge 2$. Letting $H(q) = -q \log_2 q - (1-q)\log_2(1-q)$ denote the standard binary entropy function, we prove the following theorem:

Theorem 1.5.3. *For $k \ge 3$ and large enough n,*

$$\frac{1}{k} \le \frac{\log_2 \beta_{k,2}(n)}{n} \le H\left(\frac{1}{k}\right) \le \frac{\log_2(ke)}{k}. \tag{1.6}$$

For $k \ge t \ge 3$ and large enough n,

$$\frac{\log_2 e}{\binom{k}{t-1}(t+1)t^{t-1}} \le \frac{\log_2 \beta_{k,t}(n)}{n} \le \frac{2}{\binom{k}{t-1}(t-1)^{t-3}}. \tag{1.7}$$

This determines $\log_2 \beta_{k,2}(n)$ up to a factor of order $\log_2 k$ and $\log_2 \beta_{k,t}(n)$ up to a factor of order t^3. We leave it as an open problem to determine the asymptotic value of $(\log_2 \beta_{k,t}(n))/n$ as $n \to \infty$ for any $k \geq 3$ and $t \geq 2$. A natural source for lower bounds on $\beta_{k,t}(n)$ comes from the probabilistic method – see the random constructions in Section 5.2.1 which establish the lower bounds in Theorem 1.5.3. To prove Theorem 1.5.3, we use a natural connection to hypergraph covering problems.

Theorem 1.5.1 has a wide variety of applications, from saturation problems [Bol65, MS15] to covering problems for graphs [Han64, Orl77], complexity of 0-1 matrices [Tar75], geometric problems [AK85], counting cross-intersecting families [FK18], crosscuts and transversals of hypergraphs [Tuz85, Tuz94, Tuz96], hypergraph entropy [KM88, Sim95], and perfect hashing [FK84, GR19]. In this section, we give an application of our main results to hypergraph covering problems. For a k-uniform hypergraph H, let $f(H)$ denote the minimum number of complete k-partite k-uniform hypergraphs whose union is H. In the case of graph covering, a simple connection to the Bollobás set pairs inequality (1.3) may be described as follows. Let $K_{n,n} \setminus M$ denote the complement of a perfect matching $M = \{x_i y_i : 1 \leq i \leq n\}$ in the complete bipartite graph $K_{n,n}$ with parts $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. If $H_1, H_2, \ldots, H_m$ are complete bipartite graphs in a minimum covering of $K_{n,n} \setminus M$, then let $A_i = \{j : x_i \in V(H_j)\}$ and $B_i = \{j : y_i \in V(H_j)\}$. Setting $\mathcal{A} = \{A_i\}_{i \in [m]}$ and $\mathcal{B} = \{B_i\}_{i \in [m]}$, it is straightforward to check that $(\mathcal{A}, \mathcal{B})$ is a Bollobás set pair, and Theorem 1.5.1 applies to give

$$f(K_{n,n} \backslash M) = \min\{m : \binom{m}{\lceil m/2 \rceil} \geq n\}. \tag{1.8}$$

In a similar way, Theorem 1.5.2 applies to covering complete k-partite k-uniform hypergraphs. Let $K_{n,n,\ldots,n}$ denote the complete k-partite k-uniform hypergraph with parts $X_i = \{x_{ij} : j \in [n]\}$ for $i \in [k]$. Let $H_{k,t}(n)$ denote the subhypergraph consisting of hyperedges $\{x_{1,i_1}, x_{2,i_2}, \ldots, x_{k,i_k}\}$ such that at least t of the indices $i_1, i_2, \ldots, i_k$ are distinct, and set $f_{k,t}(n) = f(H_{k,t}(n))$. Then there is a one-to-one correspondence between Bollobás (k,t)-tuples of subsets of $[m]$ and coverings of $H_{k,t}(n)$ with m complete k-partite k-graphs. We let $\beta_{k,t}(m)$ be the maximum size of a Bollobás (k,t)-tuple of subsets of $[m]$, so that

$$f_{k,t}(n) = \min\{m : \beta_{k,t}(m) \geq n\}. \tag{1.9}$$

This correspondence together with Theorem 1.5.2 will be exploited to prove

$$f_{k,2}(n) \geq \min\{m : \binom{m}{\lceil m/k \rceil} \geq n\} \tag{1.10}$$

which is partly an analog of (1.8). More generally, we prove the following theorem:

Theorem 1.5.4. *For $k \geq 3$ and large enough n,*

$$\frac{k}{\log_2(ke)} \leq \frac{1}{H(\frac{1}{k})} \leq \frac{f_{k,2}(n)}{\log_2 n} \leq k. \tag{1.11}$$

For $k \geq t \geq 3$ and large enough n,

$$\binom{k}{t-1}\frac{(t-1)^{t-3}}{2} \leq \frac{f_{k,t}(n)}{\log_2 n} \leq \frac{(t+1)t^{t-1}}{\log_2 e}\binom{k}{t-1}. \tag{1.12}$$

The bounds on $\beta_{k,t}(n)$ in Theorem (1.5.3) follow immediately from this theorem and (1.9). Equation (1.12) gives the order of magnitude for each $t \geq 3$ as $k \to \infty$, but for $t = 2$, Equation (1.11) has a gap of order $\log_2 k$. From (1.10), we obtain $\beta_{k,2}(n) \leq \binom{n}{\lfloor n/k \rfloor}$.

It is perhaps unsurprising that the asymptotic value of $f_{k,t}(n)/\log_2 n$ as $n \to \infty$ is not known for any $k > 2$, since a limiting value of $f(K_n^k)/\log_2 n$ is not known for any $k > 2$ – see Körner and Marston [KM88] and Guruswami and Riazanov [GR19].

Although there has been substantial research on Bollobás set pairs and various generalization, there are still many possible directions for future research. For example, it would be interesting to explore the problem where we allow for more permissible values for $|A_i \cap B_i|$; i.e. a two family version of Theorem 1.3.8:

Problem 1.5.5. *Let p be prime and $L \subset [0, p-1]$. A set pair $(\mathcal{A}, \mathcal{B})$ is (p, L)-cross intersecting if $\mathcal{A} = \{A_1, A_2, \ldots, A_m\} \subset 2^{[n]}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\} \subset 2^{[n]}$ are so that $A_i \cap B_j \in L \pmod{p}$ if and only if $i, j \in [m]$ are distinct. What is the largest (p, L)-cross intersecting set pair?*

In the next section, we explore one of Bollobás' original applications of Theorem 1.5.1.

1.6 Intersection Saturation

In Section 1.3, we explored supersaturation problems where there are many instances of a forbidden intersection provided we have more than the extremal amount. In this section, we explore a related concept known as saturation: what is the minimum number of sets in a set system which avoids an intersection condition, but adding any edge necessarily creates a forbidden intersection. More formally, a set family $\mathcal{F} \subset \binom{[n]}{k}$ is

(n,k,t)-saturated if $|A \cap B| \neq t$ for all $A, B \in \mathcal{F}$ and for all $C \in \binom{[n]}{k}$ with $C \notin \mathcal{F}$, there exists $A \in \mathcal{F}$ so that $|A \cap C| = t$.

The motivation for intersection saturation is from the classical theory of saturation numbers for graphs. Given a forbidden graph F, an n-vertex graph $G \subset \binom{[n]}{2}$ is F-saturated if G does not contain any copies of F and any edge $e \in \binom{[n]}{2} \setminus G$ is so that $G \cup e$ contains a copy of F. The saturation number of F, denoted $\mathrm{sat}(n, F)$ is the minimum number of edges in an n-vertex F-saturated graph. In this language, the underlying object of study in this section is the hypergraph analog of the saturation number when F is the hypergraph consisting of two edges with t vertices in common.

While saturation problems for graphs have been extensively studied, there is not much known regarding hypergraph saturation. Perhaps somewhat surprisingly, we understand hypergraph saturation for cliques $\binom{[r]}{k}$ quite well:

Theorem 1.6.1 (Bollobás [Bol65])**.** *Fix $n \geq r \geq k$. Then*

$$sat\left(n, \binom{[r]}{k}\right) = \binom{n}{k} - \binom{n-r+k}{k}.$$

Proof. First, we consider the set system

$$\mathcal{F} = \{A \in \binom{[n]}{k} : A \cap [r-k] \neq \emptyset\}.$$

It follows that $\mathcal{F}$ does not contain a copy of $\binom{[r]}{k}$ as such a copy would necessarily contain at least k elements disjoint from $[r-k]$ and hence at least one set disjoint from $[r-k]$. To see that $\mathcal{F}$ is saturated, let $X \in \mathcal{F}^c$ so that $X \cap [r-k] = \emptyset$. Then $\mathcal{F} \cup \{X\}$ contains a copy of $\binom{[r]}{k}$ on the vertex set $X \cup [r-k]$. This shows the upper bound $\mathrm{sat}(n, \binom{[r]}{k}) \leq \binom{n}{k} - \binom{n-r+k}{k}$.

For the lower bound, let $\mathcal{H} \subset \binom{[n]}{k}$ be saturated with respect to $\binom{[r]}{k}$. Then let $\mathcal{A} = \binom{[n]}{k} \setminus \mathcal{H}$ and set $\mathcal{A} = \{A_1, \ldots, A_l\}$. For each A_i, it follows that $\mathcal{H} \cup \{A_i\}$ contains a copy of $\binom{[r]}{k}$. Let V_i denote the vertex set of this copy and set $B_i = [n] \setminus V_i$. It then clearly follows that $A_i \cap B_i = \emptyset$ for all $i \in [l]$. Noting that each V_i contains exactly one k-set from $\mathcal{A}$ (namely $A_i \subset V_i$), it follows that $A_j \cap B_i \neq \emptyset$ for all $j \neq i \in [l]$. As such, $(\mathcal{A}, \mathcal{B})$ is a Bollobás set pair with each set in $\mathcal{A}$ having size k and each set in $\mathcal{B}$ having size $n - r$. Using Theorem 1.5.1, we recover

$$|\mathcal{A}| = l \leq \binom{(n-r)+k}{k}.$$

Noting that $\mathcal{A} = \binom{[n]}{k} \setminus \mathcal{H}$, we obtain the desired lower bound that

$$|\mathcal{H}| \geq \binom{n}{k} - \binom{n-r+k}{k}. \qquad \square$$

It turns out that a good candidate from minimally (n, k, t)-saturated set systems are Steiner systems. For positive integers $t < k < n$, a *Steiner system* $\mathcal{S} = \mathcal{S}(n, k, t) \subset \binom{[n]}{k}$ is a set system where for all $T \in \binom{[n]}{t}$, there exists a unique $F \in \mathcal{S}$ so that $T \subset F$. The existence of Steiner systems, and more broadly designs, is a rich combinatorial area which has been extensively studied [Wil72, Kee18, GKLO20, JX83, Tie91] in the literature.

Given a Steiner system $\mathcal{S}(n, k, t)$, it is straightforward to see that given any distinct $S_1, S_2 \in \mathcal{S}(n, k, t)$, $|S_1 \cap S_2| \neq t$ as otherwise there would a t-set contained in *both* S_1 and S_2. Moreover, when we add a set $C \in \binom{[n]}{k}$ but $C \notin \mathcal{S}(n, k, t)$, it follows that we create an intersection of size *at least* t. Moreover, we actually create $\binom{k}{t}$ sets of size t which are contained two sets in our new system. It therefore turns out that for many integer pairs

(k,t), we necessarily have that all Steiner systems $\mathcal{S}(n,k,t)$ are (n,k,t)-saturated and as such we define

$$w(n,k,t) := \min\left\{ \frac{|\mathcal{F}| \cdot \binom{k}{t}}{\binom{n}{t}} : \mathcal{F} \subset \binom{[n]}{k} \quad \text{is an } (n,k,t)\text{-saturated set system} \right\}. \quad (1.13)$$

Up to re-scaling, $w(n,k,t)$ is the *saturation number* of the k-uniform hypergraph consisting of two edges intersecting in exactly t places. For more on saturation numbers of hypergraphs, see the survey of Faudree, Faudree, and Schmitt [FFS11].

A good lower bound on $w(n,k,t)$ comes from a hypergraph Turán problem. Let $K_k^{(t)}$ denote the t-uniform clique on k vertices and $\mathrm{ex}_t(n, K_k^{(t)})$ be the maximum number of edges of an $K_k^{(t)}$-free t-uniform hypergraph on n vertices. An averaging argument gives that $\pi(K_k^{(t)}) := \lim_{n\to\infty} \mathrm{ex}_t(n, K_k^{(t)})/\binom{n}{t}$ exists for all $k > t > 2$. Hypergraph Turán problems are often studied via the dual problem of determining the minimum size $\mathcal{F} \subset \binom{[n]}{t}$ so that every $A \in \binom{[n]}{k}$ contains an edge of $\mathcal{F}$. In this setting, the limiting density is denoted by $t(k,t)$ so that $t(k,t) + \pi(K_k^{(t)}) = 1$. Although the limiting value is not known for any $k > t > 2$, Turán [Tur41] conjectured the following for $t = 3$:

Conjecture 1.6.2 (Turán [Tur41]). *Let $k \geq 3$. Then $\pi(K_k^{(3)}) = 1 - 4/(k-1)^2$.*

Towards a lower bound on $w(n,k,t)$, let $\mathcal{F} \subset \binom{[n]}{k}$ be an (n,k,t)-saturated set system and $\partial_t(\mathcal{F}) = \{T \in \binom{[n]}{t} : \exists F \in \mathcal{F}, T \subset F\}$. Then $\binom{[n]}{t} \setminus \partial_t(\mathcal{F})$ is $K_k^{(t)}$-free as otherwise we may add that k-set to $\mathcal{F}$ without creating an intersection of size exactly t; contradicting the fact that $\mathcal{F}$ is (n,k,t)-saturated. Thus, $|\binom{[n]}{t} \setminus \partial_t(\mathcal{F})| \leq \pi(K_k^{(t)})\binom{n}{t}$.

Noting that $|\partial_t(\mathcal{F})| \leq |\mathcal{F}| \cdot \binom{k}{t}$, and dividing through by $\binom{n}{t}$, we recover:

$$w(n,k,t) \geq 1 - \pi(K_k^{(t)}) = t(k,t). \tag{1.14}$$

There has been substantial research towards hypergraph Turán numbers for cliques $K_k^{(t)}$. However, in the case where $t > 3$, there are not even conjectured limiting values for $\pi(K_k^{(t)})$ for many integers pairs (k,t) to match that of Conjecture 1.6.2. This is perhaps somewhat surprising as Theorem 1.6.1 determines the corresponding saturation number exactly. For more on the vast history of hypergraph Turán problems, see the surveys written by Keevash [Kee11] and Sidorenko [Sid95].

Using a result of Füredi[F1̈5] on the stability of extremal K_k-free graphs, we are able to prove that the bound in (1.14) is not tight when $k > t$ and $t = 2$.

Theorem 1.6.3. *Let n be sufficiently large, then $w(n,k,2) \geq (1 + k^{-2}) \cdot t(k,2)$.*

A Steiner system $\mathcal{S}(n,k,t)$ is (n,k,t)-saturated (and hence $w(n,k,t) \leq 1$) if each clique partition of $\binom{[k]}{t}$ into smaller cliques contains at least one t-set. This need not be true in general as the Fano plane exhibits a clique partition of $\binom{[7]}{2}$ into cliques of size three. As a result, we construct a (n,k,t)-saturated set system with $\Theta(n^t)$ edges:

Theorem 1.6.4. *For all $1 \leq t < k$ and $q \in \mathbb{R}$ such that $k = q^2 + q + t - 1$,*

$$w(n,k,t) \leq \binom{k + \lfloor q-1 \rfloor}{\lfloor q-1 \rfloor}. \tag{1.15}$$

The bound in (1.15) is unlikely to be tight and perhaps there even exists Steiner systems which are $o(n^t)$ sets away from being (n,k,t)-saturated. Moreover, there are also cases when $w(n,k,t) < 1$. Pikhurko [Pik01] proved that $w(n,4,3) \leq 4/9$ which is best possible if Conjecture 1.6.2 is true. We are able to extend the work of Pikhurko [Pik01] when $(k,t) = (5,4)$:

Theorem 1.6.5. *For n sufficiently large, $w(n,5,4) \leq 5/8$.*

Using the Rödl Nibble, we obtain a construction of a $(n,k,k-2)$-saturated set system for $k \in \{6,7\}$:

Theorem 1.6.6. *Let n be sufficiently large. Then $w(n,7,5) \leq 1/2$ and $w(n,6,4) \leq 3/4$.*

A similar construction as in Theorem 1.6.6 gives $w(n,11,8) \leq 67/128$ and it seems likely that this can be extended to other integer pairs (k,t). However, all integer pairs for which we were able to show $w(n,k,t) < 1$ are so that $2t > k$ and it would interesting to find a construction which works for $t \leq 2k$:

Problem 1.6.7. *Find an integer pair (k,t) with $2t \leq r$ for which $w(n,k,t) < 1$.*

It would also be interesting to study intersection saturation in the more general setting where a set family has a list of permissible intersection sizes. To this end, let $L \subset [0,k-1]$. Then $\mathcal{F} \subset \binom{[n]}{k}$ is an (n,k,L)-*saturated set system* if for each $A, B \in \mathcal{F}$, $|A \cap B| \in L$ and for each $C \in \mathcal{F}^c$, there exists $A \in \mathcal{F}$ with $|A \cap C| \notin L$. Deza, Erdős, and Frankl [DEF78] studied the problem of extremal, i.e. largest, (n,k,L)-saturated set systems and a good survey of the problem may be found in [FT18]. Let $s(n,k,L)$ denote the size of the smallest (n,r,L)-saturated system. In this paper, we considered the case of $L = [0, k-1] \setminus \{t\}$ and showed that $s(n, k, [0, r-1] \setminus \{t\}) = \Theta(n^t)$. Moreover, letting $2 \le t < s \le r-1$ be integers and setting $L = [0, t-1] \cup [s, k-1]$, it is not too hard to show that $\mathcal{F} = \partial_k(\mathcal{S}(n, 2k-s, t))$ is an (n, k, L)-saturated set system which together with an argument similar to (1.14) gives $s(n, k, [0, r-1] \setminus \{t\}) = \Theta(n^t)$.

When $L = [0, t-1]$, it is straightforward to see Steiner systems $\mathcal{S}(n, k, t)$, if they exist, are (n, k, L)-saturated set systems of maximal size. Moreover, the same argument as in (1.14) gives:

$$s(n, k, [0, t-1]) \cdot \binom{k}{t} \ge t(k, t) \cdot \binom{n}{t} \tag{1.16}$$

As Theorem 1.6.3 shows that (1.14) is not sharp for $k \ge 3$ and $L = [0, k-1] \setminus \{2\}$, we similarly believe that (1.16) is not always sharp.

Problem 1.6.8. *Find an integer pair (k, t) for which (1.16) is not sharp. That is, show there exists an $\epsilon = \epsilon(k, t) > 0$ where $s(n, k, [0, t-1]) \cdot \binom{k}{t} \ge (1+\epsilon) \cdot t(k, t) \cdot \binom{n}{t}$.*

A good candidate for Problem 1.6.8 is $(k, t) = (5, 4)$. Sidorenko [Sid81] conjectured that $\pi(K_5^{(4)}) = 11/16$ which if true would imply that $w(n, 5, 4) \ge 5/16$ while Theorem 1.6.5 gives that $w(n, 5, 4) \le 5/8$. Another good candidate for Problem 1.6.8 is $(k, t) = (7, 4)$ where taking two vertex disjoint $\mathcal{S}(n/2, 7, 4)$ is a $(n, 7, \{0, 1, 2, 3\})$-saturated set system which gives $s(n, 7, \{0, 1, 2, 3\}) \cdot \binom{7}{4} \le 1/8 \cdot \binom{n}{4}$ whereas Sidorenko [Sid21] recently showed $t(7, 4) \le .0866$.

For a subset $L \subset [0, k-1]$, let $m(L)$ be the smallest integer from $[0, k-1]$ not in L. It is straightforward to see that $s(n, k, L) = \Omega(n^{m(L)})$ by a similar argument as in (1.14) when $0 \in L$ and this vacuously holds otherwise. We conjecture that this lower bound is best possible:

Conjecture 1.6.9. *Let $k \geq 1$ and $L \subset [0, k-1]$. Then $s(n, k, L) = \Theta(n^{m(L)})$.*

In order to prove Conjecture 1.6.9, it suffices to, for each $k \geq 1$ and $L \subset [0, k-1]$, give a construction of an (n, k, L)-saturated set system of size $\Theta(n^{m(L)})$. Although we were able to verify Conjecture 1.6.9 for all $k \leq 4$ and when $L = [0, t-1]$ and $L = [0, t-1] \cup [s, k]$, Conjecture 1.6.9 might be too optimistic as the order of magnitude of extremal (n, k, L)-saturated set systems is wide open. See [DEF78, FT18, Fra13] for more on extremal (n, k, L)-saturated set systems.

Chapter 2

Non-trivial d-wise intersecting families

There have been many recent directions [BD71, HK17, Kup19, FHHZ18, FS04, GS02] in the classical Hilton-Milner theory generalizing Theorem 1.2.2. In this chapter, we prove Theorem 1.2.4, which resolved the conjecture of Hilton and Milner [HM67]. Our proof actually gives a bit more and in order to state our theorem, we need the following additional non-trivial d-wise intersecting family $\mathcal{B}(k,d) = \mathcal{B}_1(k,d) \cup \mathcal{B}_2(k,d)$ where

$$\begin{aligned} \mathcal{B}_1(k,d) &= \{B \in \tbinom{[n]}{k} : |B \cap [d-1]| = d-2, [d,k] \subseteq B\} \\ \mathcal{B}_2(k,d) &= \{B \in \tbinom{[n]}{k} : [d-1] \subseteq B, B \cap [d,k] \neq \emptyset\}. \end{aligned}$$

The role of this family is in the stability for non-trivial d-wise intersecting families of k-element sets when $2d < k$, in which case $|\mathcal{A}(k,d)| \leq |\mathcal{B}(k,d)| \leq |\mathcal{HM}(k,d)|$ with equalities if and only if $k = 3$ and $d = 2$ and are all isomorphic when $d = k$.

Theorem 2.0.1. *Let k, d be integers with $2 \leq d < k$. Then there exists $n_0(k,d)$ such that for $n \geq n_0(k,d)$, if $\mathcal{F}$ is a non-trivial, d-wise intersecting family of k-element subsets of $[n]$, then*

$$|\mathcal{F}| \leq \max\{|\mathcal{HM}(k,d)|, |\mathcal{A}(k,d)|\}.$$

Furthermore, if $d \geq 3$ and $2d \geq k$ and $|\mathcal{F}| > \min\{|\mathcal{HM}(k,d)|, |\mathcal{A}(k,d)|\}$, then $\mathcal{F}$ is isomorphic to a subfamily of $\mathcal{HM}(k,d)$ or $\mathcal{A}(k,d)$. If $d \geq 3$, $2d < k$, and $|\mathcal{F}| > |\mathcal{B}(k,d)|$, then $\mathcal{F}$ is isomorphic to a subfamily of $\mathcal{HM}(k,d)$.

It is worth noting that the sizes of these families are given by

$$\begin{aligned} |\mathcal{A}(k,d)| &= (d+1)\binom{n-d-1}{k-d} + \binom{n-d-1}{k-d-1} \sim (d+1)\binom{n}{k-d} \\ |\mathcal{HM}(k,d)| &= \binom{n-d+1}{k-d+1} - \binom{n-k-1}{k-d+1} + d - 1 \sim (k-d+2)\binom{n}{k-d}. \end{aligned}$$

This is particularly interesting as for large values of n, $|\mathcal{HM}(k,d)|$ and $|\mathcal{A}(k,d)|$ are both substantially smaller than the trivial d-wise intersecting family of size $\binom{n-1}{k-1}$. We use the Delta system method to prove Theorem 2.0.1 in the case where $k \geq d \geq 3$. When $d = 2$, stability versions of Theorem 1.2.2 appear in Han and Kohayakawa [HK17] and Kostochka and Mubayi [KM17]. The Delta system method is a powerful tool in extremal combinatorics that initially appeared in Deza, Erdős and Frankl's [DEF78] study of (n,k,L)-systems. It has also been used by Frankl and Füredi [FF87] in Chvátal's problem of avoiding d-simplicies as well as by Füredi [Für14, FJS14] on the problem of embedding expansions of forests in r-graphs for $r \geq 4$. The application of the Delta system method here gives $n_0(k,d) = d + e(k^2 2^k)^{2^k}(k-d)$.

2.1 Preliminaries

In this section, we will prove some basic facts and structural results pertaining to non-trivial d-wise intersecting families. We will first show, as was initially done by Hilton and Milner in [HM67], that there cannot be a k-uniform non-trivial d-wise intersecting family for $d > k$.

Lemma 2.1.1. [HM67] *Let $n \geq d > k$, then there does not exist a d-wise intersecting non-trivial $\mathcal{F} \subseteq \binom{[n]}{k}$.*

Proof. Fix $A \in \mathcal{F}$. For each $a \in A$, there exists $X_a \in \mathcal{F}$ so that $a \notin X_a$ by the definition of non-trivial. Then $A \cap \bigcap_{a \in A} X_a = \emptyset$ which is a contradiction. $\square$

A similar argument as in Lemma 2.1.1 also gives an upper bound on the m-wise intersections from a non-trivial d-wise intersecting family.

Lemma 2.1.2. *Let $m \geq 1$ be a positive integer. Given a non-trivial d-wise intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ and $A_1, \ldots, A_m \in \mathcal{F}$,*

$$|\bigcap_{i=1}^{m} A_i| \geq d - (m-1).$$

Proof. Suppose not, and that there exists $A_1, \ldots, A_m \in \mathcal{F}$ with $A = \cap_{i=1}^{m} A_i$ and $|A| < d - (m-1)$. For each $a \in A$, by the non-triviality of $\mathcal{F}$, we may find $X_a \in \mathcal{F}$ so that $a \notin X_a$ and hence violate the d-wise intersecting property of $\mathcal{F}$. $\square$

In the case where $d = k$, Hilton and Milner [HM67] noted that $\binom{[k+1]}{k}$ is a non-trivial k-wise intersecting family and proved this is the only such example up to isomorphism.

Proposition 2.1.3. [HM67] *If $n \geq k+1$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is a k-wise intersecting non-trivial family, then $\mathcal{F}$ is isomorphic to $\binom{[k+1]}{k} = \mathcal{A}(k,k)$.*

Proof. Let $A \in \mathcal{F}$, then without loss of generality we may assume that $A = [k]$. Observe that there exists $A_1 \in \mathcal{F}$ so that $1 \notin A_1$ and by Lemma 2.1.2, $|A \cap A_1| = k-1$. Without loss of generality, let $A_1 \setminus A = \{k+1\}$. Then for each $i \in [2,k]$, there exists $A_i \in \mathcal{F}$ so that $i \notin A_i$. Next, Lemma 2.1.2 yields that $|A \cap A_i| = k-1$ and $|A_1 \cap A_i| = k-1$ and as a result $A_i \cap [k+2, n] = \emptyset$. Putting these all together we get that $\binom{[k+1]}{k} \subseteq \mathcal{F}$ and then noting that no other k-sets may be added to $\binom{[k+1]}{k}$ while preserving the k-wise intersection property yields the desired result. $\square$

2.2 Structure of d-wise intersecting families

We start by defining some terminology regarding Delta systems that may be found in the survey written by Mubayi and Verstraete [MV16]. A *Delta system* is a hypergraph Δ such that for all distinct $X_1, X_2 \in \Delta$, $X_1 \cap X_2 = \cap_{E\in\Delta} E$. We let $\Delta_{k,s}$ be a k-uniform Delta system with s edges and define $\mathrm{core}(\Delta) := \cap_{E\in\Delta} E$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ and $X \subseteq [n]$, then define the *core degree* of X in $\mathcal{F}$ to be the size of the largest Delta system contained in $\mathcal{F}$ which has core equal to X. That is, let

$$d^\star_{\mathcal{F}}(X) := \max\{s : \exists\ \Delta_{k,s} \subseteq \mathcal{F} \text{ so that } \mathrm{core}(\Delta_{k,s}) = X\}.$$

In this section, we will examine the collection of d-sets with large core degree with respect to a non-trivial d-wise intersecting family. We will show that this collection of d-sets is necessarily isomorphic to a subfamily of one of the corresponding collections of d-sets in the extremal examples $\mathcal{HM}(k, d)$ and $\mathcal{A}(k, d)$. Moreover, given enough d-sets with large core degree, we show that $|\mathcal{F}|$ is less than or equal to $\max\{|\mathcal{A}(k, d)|, |\mathcal{HM}(k, d)|\}$.

By Lemma 2.1.2, $|A\cap B| \geq d-1$ for all $A, B \in \mathcal{F}$ in a non-trivial d-wise intersecting family $\mathcal{F}$, and hence $d^\star_{\mathcal{F}}(X) \leq 1$ whenever $|X| < d-1$. Moreover, since $|X| < d-1 < k$, it follows that X cannot be the core of a delta system $\Delta_{k,1}$ and thus $d^\star_{\mathcal{F}}(X) = 0$. We will now show that $(d-1)$-sets cannot have large core degree in non-trivial d-wise intersecting families.

Lemma 2.2.1. *Let $k \geq d \geq 3$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be non-trivial d-wise intersecting and $X \in \binom{[n]}{d-1}$. Then $d^\star_{\mathcal{F}}(X) < k$.*

Proof. Without loss of generality, suppose that $X = [d-1]$ is so that $d^{\star}_{\mathcal{F}}(X) \geq k$. Thus, there exists $\Delta_{k,k} = \{F_1, \ldots, F_k\} \subseteq \mathcal{F}$ so that $\mathrm{core}(\Delta_{k,k}) = [d-1]$. Next, by the nontriviality of $\mathcal{F}$, for each $j \in [d-1]$, there exists $X_j \in \mathcal{F}$ so that $j \notin X_j$.

Now, when $d \geq 4$, since $F_1 \cap F_2 = [d-1]$ and $\mathcal{F}$ is d-wise intersecting, for $3 \leq m \leq d-1$,

$$F_1 \cap F_2 \cap \Big(\bigcap_{j \in [d-1] \setminus \{m\}} X_j \Big) = \{m\}.$$

As a result, $|X_m \cap [d-1]| = d-2$ and hence $|X_m \cap (F_i \setminus [d-1])| \geq 1$ for all $i \in [k]$ by Lemma 2.1.2. This yields a contradiction as

$$|X_m| \geq |X_m \cap [d-1]| + \sum_{j=1}^{k} |X_m \cap (F_i \setminus [d-1])| > k.$$

When $d = 3$, the result follows similarly by considering the cases where $2 \in X_1$ and $2 \notin X_1$. □

We are interested in d-sets which have large core degree since they intersect elements of our family $\mathcal{F}$ in many places. To this end, we say $D \in \binom{[n]}{d}$ has *large core degree* if $d^{\star}_{\mathcal{F}}(D) \geq k$.

Lemma 2.2.2. *Let $k \geq d \geq 3$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be non-trivial d-wise intersecting and $D \in \binom{[n]}{d}$ have large core degree. Then $|A \cap D| \geq d-1$ for all $A \in \mathcal{F}$.*

Proof. Seeking a contradiction, suppose there exists such a $D \subseteq [n]$ and $A \in \mathcal{F}$ so that $|A \cap D| < d-1$. By definition of large core degree, there exists a Delta system $\Delta_{k,k} = \{F_1, \ldots, F_k\} \subseteq \mathcal{F}$ so that $\mathrm{core}(\Delta_{k,k}) = D$. By Lemma 2.1.2, $|A \cap (F_j \setminus D)| \geq (d-1) - |A \cap D|$

for all $j \in [k]$. This gives a contradiction as

$$|A| \geq |A \cap D| + \sum_{j=1}^{k} |A \cap (F_j \setminus D)| > k. \qquad \square$$

Given a family $\mathcal{F}$, we let $\mathcal{S}_d(\mathcal{F})$ be the possibly empty collection of d-sets with large core degree in $\mathcal{F}$. We first show that $\mathcal{S}_d(\mathcal{F})$ is a $(d-1)$-intersecting family; i.e. the intersection of any pair of sets in the family has size at least $(d-1)$.

Lemma 2.2.3. *Let $k \geq d \geq 3$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be a non-trivial d-wise intersecting family. Then $\mathcal{S}_d(\mathcal{F}) \subseteq \binom{[n]}{d}$ is a $(d-1)$-intersecting family.*

Proof. Let $d \geq 3$. Suppose there exists $D_1, D_2 \in \mathcal{S}_d(\mathcal{F})$ so that $|D_1 \cap D_2| \leq d-2$. By definition, there exists $\Delta^1_{k,k} = \{F_1, \ldots, F_k\} \subseteq \mathcal{F}$ and $\Delta^2_{k,k} = \{G_1, \ldots, G_k\} \subseteq \mathcal{F}$ so that $\mathrm{core}(\Delta^i_{k,k}) = D_i$ for $i = 1, 2$. Note that there necessarily exists $F_i \in \Delta^1_{k,k}$ so that $|(F_i \setminus D_1) \cap D_2| = 0$. By Lemma 2.1.2, $|(F_i \setminus D_1) \cap (G_j \setminus D_2)| \geq (d-1) - |D_1 \cap D_2|$ for any $1 \leq j \leq k$. This gives a contradiction since

$$|F_i| \geq |D_1 \cap D_2| + \sum_{j=1}^{k} |(F_i \setminus D_1) \cap (G_j \setminus D_2)| > k. \qquad \square$$

We now note that for $n \geq k(k-d)+d$, the two extremal families $\mathcal{A}(k,d)$ and $\mathcal{HM}(k,d)$ are so that:

$$\begin{aligned} \mathcal{S}_d(\mathcal{A}(k,d)) &= \binom{[d+1]}{d} \\ \mathcal{S}_d(\mathcal{HM}(k,d)) &= \{A \in \binom{[k+1]}{d} : [d-1] \subseteq A\}. \end{aligned}$$

As a result of Lemma 2.2.3, we are interested in $\mathcal{S} \subseteq \binom{[n]}{d}$ which are $(d-1)$-intersecting. In the proof of Theorem 2.0.1, we will iteratively find d-sets with large core degree. The following structural type result yields that the collection of these sets is necessarily isomorphic to a subfamily of the collection of d-sets with large core degree in the extremal families $\mathcal{A}(k,d)$ and $\mathcal{HM}(k,d)$.

Lemma 2.2.4. *Let $k \geq d \geq 3$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be a non-trivial d-wise intersecting family. Then $\mathcal{S}_d(\mathcal{F})$ is isomorphic to a subfamily of $\binom{[d+1]}{d}$ or $\{D \in \binom{[n]}{d} : [d-1] \subseteq D\}$. In the latter case, it also follows that $\mathcal{S}_d(\mathcal{F})$ is isomorphic to a subfamily of $\{D \in \binom{[k+1]}{d} : [d-1] \subseteq D\}$.*

Proof. Given distinct $F_1, F_2 \in \mathcal{S}_d(\mathcal{F})$, we have $|F_1 \cap F_2| = d-1$ and hence without loss of generality, we may assume that $F_1 = [d]$ and $F_2 = [d-1] \cup \{d+1\}$. Now, we let

$$\mathcal{S}_1 := \{F \in \mathcal{S}_d(\mathcal{F}) \setminus \{F_1, F_2\} : |F \cap [d-1]| = d-2\}$$

and note that if $F \in \mathcal{S}_1$, then $\{d, d+1\} \subseteq F$ as $\mathcal{S}$ is $(d-1)$-intersecting. We then let

$$\mathcal{S}_2 := \{F \in \mathcal{S}_d(\mathcal{F}) \setminus \{F_1, F_2\} : |F \cap [d-1]| = d-1\}$$

so that $\mathcal{S}_d(\mathcal{F}) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \{F_1, F_2\}$. For all $F \in \mathcal{S}_1$ and for all $G \in \mathcal{S}_2$, $|F \cap G| = d-2$ and thus if $\mathcal{S}_2 \neq \emptyset$, then $\mathcal{S}_1 = \emptyset$. This completes the proof of the first statement.

For the second statement, seeking a contradiction, and without loss of generality, suppose that $\{D \in \binom{[k+2]}{d} : [d-1] \subseteq D\} \subseteq \mathcal{S}_d(\mathcal{F})$. Then, as $\mathcal{F}$ is non-trivial, there exists $X_1 \in \mathcal{F}$ so that $1 \notin X_1$. By Lemma 2.2.2, it follows that $[d, k+2] \subseteq X_1$ and hence $[2, k+2] \subseteq X_1$ which is a contradiction. $\square$

As a result of Lemma 2.2.4, we think of $\mathcal{S}_d(\mathcal{F}) \subseteq \binom{[k+1]}{d}$. We will now show that if a non-trivial d-wise intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ has a particular structure of d-sets with large core degree, then $|F| \leq |\mathcal{HM}(k,d)|$.

Lemma 2.2.5. *Let $k \geq d \geq 3$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be a non-trivial d-wise intersecting family. If $\{D \in \binom{[k]}{d} : [d-1] \subseteq D\} \subseteq \mathcal{S}_d(\mathcal{F})$, then $|\mathcal{F}| \leq |\mathcal{HM}(k,d)|$. Moreover, if $|\mathcal{F}| > |\mathcal{B}(k,d)|$, then $\mathcal{F}$ is necessarily isomorphic to some subfamily of $\mathcal{HM}(k,d)$.*

Proof. We have that $D_x := [d-1] \cup \{x\} \in \mathcal{S}_d(\mathcal{F})$ for all $x \in [d,k]$. As a result of Lemma 2.2.2, for all $A \in \mathcal{F}$, $|A \cap [d-1]| \geq d-2$. We let $\mathcal{F}_1 := \{A \in \mathcal{F} : |A \cap [d-1]| = d-2\}$ and $\mathcal{F}_2 := \{A \in \mathcal{F} : |A \cap [d-1]| = d-1\}$. For each $i \in [d-1]$ and each $X_i \in \mathcal{F}$ so that $i \notin X_i$, Lemma 2.2.2 gives that $[k] \setminus \{i\} \subseteq X_i$ since $D_x := [d-1] \cup \{x\} \in \mathcal{S}_d(\mathcal{F})$ for all $x \in [d,k]$.

We now have two cases based on the collection over $i \in [d-1]$ of non-empty 1-uniform link graphs of $[k] \setminus \{i\}$. First, we consider the case where these 1-uniform link graphs do not all have size one consisting of the same vertex. That is, we may find $\{X_1, \ldots, X_{d-1}\} \subseteq \mathcal{F}$ so that $i \notin X_i$ for each $i \in [d-1]$ and so that

$$\bigcap_{i=1}^{d-1} X_i = [d,k].$$

Let $A \in \mathcal{F}_1$ so that $|A \cap [d-1]| = d-2$, then by Lemma 2.2.2 it follows that $[d,k] \subseteq A$. Next, for $A \in \mathcal{F}_2$, if $A \cap [d,k] = \emptyset$, then $A \cap (X_1 \cap \cdots \cap X_{d-1}) = \emptyset$; a contradiction. Thus $A \cap [d,k] \neq \emptyset$. Putting these together, it follows that:

$$\begin{aligned}\mathcal{F}_1 &\subseteq \{A \in \binom{[n]}{k} : |A \cap [d-1]| = d-2, [d,k] \subseteq A\}\\ \mathcal{F}_2 &\subseteq \{A \in \binom{[n]}{k} : [d-1] \subseteq A, A \cap [d,k] \neq \emptyset\}.\end{aligned}$$

Thus $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| \leq |\mathcal{B}(k,d)| \leq |\mathcal{HM}(k,d)|$.

Next, suppose all of the 1-uniform link graphs all have size one consisting of the same vertex (say $k+1$). Thus for all $i \in [d-1]$, there exists a unique set $X_i \in \mathcal{F}$ so that $i \notin X_i$ and that $X_i = [k+1] \setminus \{i\}$. Hence $\mathcal{F}_1 = \{[k+1] \setminus \{i\} : i \in [d-1]\}$. Now, let $A \in \mathcal{F}_2$ be so that $[d-1] \subseteq A$ and suppose that $A \cap [d, k+1] = \emptyset$. Then, $A \cap X_1 \cap \cdots \cap X_{d-1} = \emptyset$; a contradiction. Thus

$$\mathcal{F}_2 \subseteq \{A \in \binom{[n]}{k} : [d-1] \subseteq A, A \cap [d, k+1] \neq \emptyset\}$$

and hence $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| \leq |\mathcal{HM}(k,d)|$. The second part of the statement holds since if $|\mathcal{F}| > |\mathcal{B}(k,d)|$, we cannot be in the first case and are necessarily in the second case for which $\mathcal{F}$ is isomorphic to a subfamily of $\mathcal{HM}(k,d)$. □

We now will prove the analog of Lemma 2.2.5 when $\mathcal{S}_d(\mathcal{F})$ is isomorphic to a subfamily of $\binom{[d+1]}{d}$.

Lemma 2.2.6. *Let $k \geq d \geq 3$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be a non-trivial d-wise intersecting family. Given that $|\mathcal{S}_d(\mathcal{F})| \geq 3$ and $\mathcal{S}_d(\mathcal{F}) \subseteq \binom{[d+1]}{d}$, then $\mathcal{F} \subseteq \mathcal{A}(k,d)$.*

Proof. Let $D_1, D_2, D_3 \in \mathcal{S}_d(\mathcal{F})$ be distinct d-sets with large core degree. By Lemma 2.2.2 and Lemma 2.2.4, we may assume $D_i = [d+1] \setminus \{i\}$ and that $|A \cap D_i| \geq d-1$ for all

$A \in \mathcal{F}$ for $i = 1, 2, 3$. This then implies that $|A \cap [d+1]| \geq d$ for all $A \in \mathcal{F}$ and thus the result follows. $\square$

2.3 The Delta System Method

Given a k-partite hypergraph $\mathcal{H} \subseteq \binom{[n]}{k}$ with parts $X_1, \ldots, X_k$, we let the *projection* of $E \subseteq [n]$ be

$$\text{proj}(E) = \{i : E \cap X_i \neq \emptyset\}.$$

Let $\mathcal{H}_E = \{X \cap E : X \in \mathcal{H}\}$ be the trace of E on $\mathcal{H}$ and let the *intersection pattern* of E on $\mathcal{H}$ be

$$I_{\mathcal{H}}(E) = \{\text{proj}(A) : A \in \mathcal{H}|_E\}.$$

We are now able to state Füredi's intersection semilattice lemma [Für83].

Lemma 2.3.1. [Für83] *For fixed $s, k \in \mathbb{N}$, there exists $c := c(s,k)$ so that for all $\mathcal{H} \subseteq \binom{[n]}{k}$ there exists a k-partite $\mathcal{H}^\star \subseteq \mathcal{H}$ with $|\mathcal{H}^\star| > c|\mathcal{H}|$ and a $\mathcal{J} \subseteq 2^{[k]}$ so that the following hold:*

1. *For all $A_1, A_2 \in \mathcal{J}$, $A_1 \cap A_2 \in \mathcal{J}$ (i.e., $\mathcal{J}$ is intersection closed).*

2. *For all $E \in \mathcal{H}^\star$ that $I_{\mathcal{H}^\star}(E) = \mathcal{J}$.*

3. *For all $X \cap E \in \mathcal{H}^\star|_E$, $d^\star_{\mathcal{H}^\star}(X \cap E) \geq s$.*

Given $\mathcal{H} \subseteq \binom{[n]}{k}$ so that it satisfies the conclusions of Lemma 2.3.1, we say that $\mathcal{H}$ is $(s, \mathcal{J})$-homogeneous and if there exists an $s \in \mathbb{N}$ so that $\mathcal{H}$ is $(s, \mathcal{J})$-homogeneous, we say that $\mathcal{H}$ is $\mathcal{J}$-homogeneous. Let $\mathcal{J} \subseteq 2^{[k]}$, then the *rank* of $\mathcal{J}$, denoted $\rho(J)$ is defined as

$$\rho(\mathcal{J}) := \min\{|E| : E \subseteq [k] : d_{\mathcal{J}}(E) = 0\}$$

where $d_{\mathcal{J}}(E)$ is the *degree* of the set E in $\mathcal{J}$.

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be non-trivial and d-wise intersecting. Applying Lemma 2.3.1 with $s = k$ to large subfamilies $\mathcal{H} \subseteq \mathcal{F}$ when n sufficiently large gives a particular intersection structure $\mathcal{J}$. To this end, let $n_0(k, d) := d + e(k^2 2^k)^{2^k}(k - d)$ and we let $c_k := c(k, k) > (k^2 2^k)^{-2^k}$.

Lemma 2.3.2. *Let $\mathcal{H} \subseteq \mathcal{F}$ be $\mathcal{J}$-homogeneous and so that $|\mathcal{H}| \geq \binom{n-d}{k-d}$ where $n > n_0(k, d)$. Then*

$$\mathcal{J} \subseteq \bigcup_{l=d}^{k-1} \binom{[k]}{l}.$$

Moreover, $|\mathcal{J} \cap \binom{[k]}{d}| = 1$ so that there exists $D \in \mathcal{S}_d(\mathcal{F})$ as $d^\star_{\mathcal{F}}(D) \geq d^\star_{\mathcal{H}}(D) \geq k$.

Proof. Lemma 2.1.2 gives that $|A \cap B| \geq d - 1$ for all $A, B \in \mathcal{F}$ which yields that

$$\mathcal{J} \subseteq \bigcup_{l=d-1}^{k-1} \binom{[k]}{l}.$$

Lemma 2.2.1 yields that $\mathcal{J} \cap \binom{[k]}{d-1} = \emptyset$ and hence

$$\mathcal{J} \subseteq \bigcup_{l=d}^{k-1} \binom{[k]}{l}.$$

Now, since $\mathcal{J}$ is intersection closed, $|\mathcal{J} \cap \binom{[k]}{d}| \leq 1$. If $\mathcal{J} \cap \binom{[k]}{d} = \emptyset$, then without loss of generality, suppose that $[d+1]$ is the inclusion minimal element of $\mathcal{J}$. Suppose there exists $X \in \mathcal{J}$ so that $[d+2, k] \subseteq X$, then there is an $i \in [d+1]$ so that $i \notin X$ as

$[k] \notin \mathcal{J}$. Now, $X \cap [d+1] \in \mathcal{J}$, but $|X \cap [d+1]| < (d+1)$ so we necessarily have that $d_{\mathcal{J}}([d+2,k]) = 0$. As a result, $\rho(\mathcal{J}) \leq k-d-1$. Next, if $\mathcal{H}^\star \subseteq \binom{[n]}{k}$ is $\mathcal{J}$-homogeneous with $\rho(\mathcal{J}) = r$, then $|\mathcal{H}^\star| \leq \binom{n}{r}$ (see [MV16]). Thus, $|\mathcal{H}^\star| \leq \binom{n}{k-d-1}$. However, for $n > n_0(k,d)$,

$$|\mathcal{H}^\star| > c_k|\mathcal{H}| \geq c_k\binom{n-d}{k-d} > \binom{n}{k-d-1}.$$

In the case where the minimal element has size strictly larger than $d+1$, the proof is similar as $\binom{n}{k-d-1} > \binom{n}{k-a}$ for all $a > d+1$ when $n \geq n_0(k,d)$. $\square$

2.4 Proof of Theorem 2.0.1

In this section, we will prove Theorem 2.0.1 by repeated applications of Lemma 2.3.1 and Lemma 2.3.2 and the structural results from Section 2.2.

Proof of Theorem 2.0.1. As a result of Lemma 2.2.5 and Lemma 2.2.6, it suffices to show that $|\mathcal{S}_d(\mathcal{F})| \geq 3$ and that we either have $\mathcal{S}_d(\mathcal{F})$ contains $\{A \in \binom{[k]}{d} : [d-1] \subseteq A\}$ with $|\mathcal{S}_d(\mathcal{F})| = k-d+1$ or $\mathcal{S}_d(\mathcal{F})$ is isomorphic to a subfamily of $\binom{[d+1]}{d}$. An application of Lemma 2.3.1 and Lemma 2.3.2 yields a d-set D_1 which has large core degree. We now consider

$$\mathcal{H}_1 := \{A \in \mathcal{F} : D_1 \nsubseteq A\}$$

and again applying Lemma 2.3.1 and Lemma 2.3.2 yields a d-set D_2 which has large core degree and $D_1 \neq D_2$. We can iteratively apply Lemma 2.3.1 and Lemma 2.3.2 s times to get $\{D_1, \ldots, D_s\} \in \mathcal{S}_d(\mathcal{F})$ where the particular value of s depends on $|\mathcal{F}|$. That is, we need s to be small enough so that the number of sets contained in at least one of $\{D_1, \ldots, D_s\}$ is less than $|\mathcal{F}|$.

In the case where $2d < k$, we may suppose that $|\mathcal{F}| > |\mathcal{B}(k,d)| > |\mathcal{A}(k,d)|$ and we take $s = k - d + 1$. Noting that $s > d + 1$ then yields that $\mathcal{S}_d(\mathcal{F})$ is not isomorphic to a subfamily of $\binom{[d+1]}{d}$. Lemma 2.2.4 and Lemma 2.2.5 then yield that $\mathcal{F}$ is isomorphic to a subfamily of $\mathcal{HM}(k,d)$.

In the case where $2d = k$, we may suppose that $|\mathcal{F}| > |\mathcal{A}(k,d)| > |\mathcal{B}(k,d)|$ and also take $s = k - d + 1$ where we note $k - d + 1 = d + 1$. Noting that $|\mathcal{F}| > |\mathcal{A}(k,d)|$ then yields that $\mathcal{S}_d(\mathcal{F})$ is not isomorphic to $\binom{[d+1]}{d}$. Lemma 2.2.4 and Lemma 2.2.5 then yield that $\mathcal{F}$ is isomorphic to a subfamily of $\mathcal{HM}(k,d)$.

In the case where $2d \geq k + 1$, we may suppose that $|\mathcal{F}| > |\mathcal{HM}(k,d)| > |\mathcal{B}(k,d)|$. When $d < k - 1$, we take $s = k - d + 1 \geq 3$ and when $d = k - 1$, and an Inclusion-Exclusion argument yields that we may take $s = 3$. In both of these cases, noting that $|\mathcal{F}| > |\mathcal{HM}(k,d)|$ then yields that $\mathcal{S}_d(\mathcal{F})$ is not isomorphic to a subfamily of $\{A \subseteq [k] : [d-1] \subseteq A\}$ by Lemma 2.2.5. Thus we may use Lemma 2.2.4 and Lemma 2.2.6 to get that $\mathcal{F}$ is isomorphic to a subfamily of $\mathcal{A}(k,d)$. $\square$

This Chapter contains material from: J. O'Neill, J. Verstraëte, "Non-trivial d-wise intersecting families", J. Combin. Theory Ser. A, 178, 2021. The dissertation author was one of the primary investigators and authors of this paper.

Chapter 3

Supersaturation for Oddtown and Eventown

3.1 Proof of Theorem 1.3.6

It suffices to consider $\mathcal{A} = \{A_1, \ldots, A_{n+1}\}$ with $|A_i|$ odd for all $i \in [n+1]$ and to prove that we cannot have $\mathrm{op}(\mathcal{A}) \leq 2$. Let $v_i := v_{A_i} \in \mathbb{F}_2^n$ for each $i \in [n+1]$ and consider the vector space

$$V = \{(\epsilon_1, \ldots, \epsilon_{n+1}) \in \mathbb{F}_2^{n+1} : \epsilon_1 v_1 + \cdots + \epsilon_{n+1} v_{n+1} = 0\}. \tag{3.1}$$

Without loss of generality, it suffices to consider three cases:

Case 1: There exists one such odd intersection pair which we may assume is $\{A_1, A_2\}$. Let $(\epsilon_1, \ldots, \epsilon_{n+1}) \in V$ be nonzero. Taking inner products with each v_i and $\epsilon_1 v_1 + \cdots + \epsilon_{n+1} v_{n+1}$, we obtain $\epsilon_3 = \epsilon_4 = \cdots = \epsilon_{n+1} = 0$ and $\epsilon_1 = \epsilon_2$. If $\epsilon_1 = \epsilon_2 = 1$, then $0 = v_1 + v_2$ which implies $A_1 = A_2$; a contradiction.

Case 2: There exists two such odd intersection pairs which we may assume are $\{A_1, A_2\}$ and $\{A_1, A_3\}$. It then follows that $\{v_2, \ldots, v_{n+1}\}$ is linearly independent. Consider the nonzero vector $(\epsilon_1, \ldots, \epsilon_{n+1}) \in V$. If $\epsilon_1 = 0$, then $0 = \epsilon_2 v_2 + \cdots + \epsilon_{n+1} v_{n+1}$ and hence $\epsilon_2 = \epsilon_3 = \cdots = \epsilon_{n+1} = 0$; a contradiction. Thus, we must have $\epsilon_1 = 1$. Taking inner products with each v_i and $\epsilon_1 v_1 + \cdots + \epsilon_{n+1} v_{n+1}$, we obtain $\epsilon_1 + \epsilon_2 = 0$ and $\epsilon_1 + \epsilon_3 = 0$ and $\epsilon_4 = \epsilon_5 = \cdots = \epsilon_{n+1} = 0$. Using that $\epsilon_1 = 1$, it follows that $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. As such, $v_1 + v_2 + v_3 = 0$ which implies that each element in $A_1 \cup A_2 \cup A_3$ is in exactly two of $\{A_1, A_2, A_3\}$. Thus, for each $x \in A_1$, it follows that $x \in A_2$ or $x \in A_3$ but not both and hence we reach a contradiction as

$$|A_1| = |A_1 \cap A_2| + |A_1 \cap A_3| = 1 + 1 = 0 \pmod 2.$$

Case 3a: There exists two such odd intersection pairs which we may assume are $\{A_1, A_2\}$ and $\{A_3, A_4\}$ and $\dim(V) = 2$. By the rank-nullity theorem,

$$\dim(\text{span}\{v_1, \dots, v_{n+1}\}) = n - 1$$

and observe that $\{v_1, v_3, v_5, v_6, \dots, v_{n+1}\}$ is an orthogonal basis. Let $(\epsilon_1, \dots, \epsilon_{n+1}) \in V$ be nonzero. For all $i \geq 5$, taking inner products with v_i and $\epsilon_1 v_1 + \cdots + \epsilon_{n+1} v_{n+1}$, we obtain $\epsilon_i = 0$. For $i \leq 4$, taking inner products with v_i and $\epsilon_1 v_1 + \cdots + \epsilon_{n+1} v_{n+1}$ we recover that $0 = \epsilon_1 + \epsilon_2$ and $0 = \epsilon_3 + \epsilon_4$. Hence $\epsilon_1 = \epsilon_2 \in \{0, 1\}$ and $\epsilon_3 = \epsilon_4 \in \{0, 1\}$. Suppose $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = \epsilon_4 = 0$. Then $0 = \epsilon_1 v_1 + \cdots + \epsilon_{n+1} v_{n+1} = v_1 + v_2$ and we reach a similar contradiction as in Case 1. Similarly, we cannot have $\epsilon_1 = \epsilon_2 = 0$ and $\epsilon_3 = \epsilon_4 = 1$. As such, the only nonzero vector in V is $(1, 1, 1, 1, 0, \dots, 0)$ which contradicts the dimension of V.

Case 3b: There exists two such odd intersection pairs which we may assume is $\{A_1, A_2\}$ and $\{A_3, A_4\}$ and $\dim(V) = 1$. The reasoning from Case 3a gives that the vector space $V = \{0, (1, 1, 1, 1, 0, \dots, 0)\}$. By the rank-nullity theorem, $\text{span}\{v_1, \dots, v_{n+1}\} = \mathbb{F}_2^n$. Clearly any basis for $\mathbb{F}_2^n$ consisting of n vectors from $\{v_1, \dots, v_{n+1}\}$ cannot contain all of the elements in $\{v_1, v_2, v_3, v_4\}$ as $v_1 + v_2 + v_3 + v_4 = 0$. Without loss of generality, assume that $\{v_2, \dots, v_{n+1}\}$ forms a basis for $\mathbb{F}_2^n$ and consider $U = \text{span}\{v_3, v_4\}$. It then follows that $\dim(U^\perp) = n - 2$ and $\{v_2, v_5, v_6, \dots, v_{n+1}\}$ forms a basis for $U^\perp$. As such, since $v_1 \in U^\perp$, we have $v_1 = \alpha_2 v_2 + \alpha_5 v_5 + \cdots + \alpha_{n+1} v_{n+1}$. However, this is a contradiction as $v_1 = v_2 + v_3 + v_4$ and $\{v_2, \dots, v_{n+1}\}$ is a basis for $\mathbb{F}_2^n$.

3.2 Proof of Theorem 1.3.4

3.2.1 The case of $s = 1$

Let $\mathcal{A} = \{A_1, \ldots, A_m\} \subseteq 2^{[n]}$ be a collection of even-sized subsets where $m = 2^{\lfloor n/2 \rfloor} + 1$ and suppose $\mathcal{A}' \subseteq \mathcal{A}$ is a maximal (with respect to $\mathcal{A}$) eventown subfamily of $\mathcal{A}$. If $|\mathcal{A}'| \leq 2^{\lfloor n/2 \rfloor - 1}$, then since for each $A \in \mathcal{A} \setminus \mathcal{A}'$, there exists $B \in \mathcal{A}'$ for which $|A \cap B|$ is odd,

$$\mathrm{op}(\mathcal{A}) \geq |\mathcal{A} \setminus \mathcal{A}'| \geq 2^{\lfloor n/2 \rfloor} + 1 - 2^{\lfloor n/2 \rfloor - 1} > 2^{\lfloor n/2 \rfloor - 1}.$$

If $|\mathcal{A}'| = t > 2^{\lfloor n/2 \rfloor - 1}$, then without loss of generality assume that $\mathcal{A}' = \{A_1, \ldots, A_t\}$. Setting $v_i = v_{A_i} \in \mathbb{F}_2^n$ as the characteristic vector of A_i, we can consider the vector space $W = \mathrm{span}\{v_1, \ldots, v_t\}$. Then, $W \subseteq W^{\perp}$ and since $\dim(\mathbb{F}_2^n) = \dim(W) + \dim(W^{\perp})$, it follows that $\dim(W) \leq \lfloor n/2 \rfloor$. Further, as $t > 2^{\lfloor n/2 \rfloor - 1}$, a straightforward counting argument gives that we must have that $\dim(W) = \lfloor n/2 \rfloor$.

For each A_i with $i > t$, we consider the linear functional $\chi_i : W \to \mathbb{F}_2$ given by $w \mapsto \langle w, v_i \rangle$. By the first isomorphism theorem, it follows that $W/\ker(\chi_i) \cong \mathbb{F}_2$ and $\dim(\ker(\chi_i)) = \lfloor n/2 \rfloor - 1$. As a result, $|\{v_1, \ldots, v_t\} \setminus \ker(\chi_i))| \geq t - 2^{\lfloor n/2 \rfloor - 1}$. Hence there are at least $t - 2^{\lfloor n/2 \rfloor - 1}$ sets in $\mathcal{A}'$ which have odd-sized intersection with A_i. Since we can do this for all $t < i \leq m$, it follows that

$$\mathrm{op}(\mathcal{A}) \geq \left(t - 2^{\lfloor n/2 \rfloor - 1}\right) \cdot (m - t). \tag{3.2}$$

The result then follows by an optimization problem and considering all positive integers $t : 2^{\lfloor n/2 \rfloor - 1} < t \leq 2^{\lfloor n/2 \rfloor}$.

3.2.2 The case of $s = 2$

Let $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq 2^{[n]}$ be a collection of even-sized subsets with $m = 2^{\lfloor n/2 \rfloor} + 2$. Let $\mathcal{A}' \subseteq \mathcal{A}$ be a maximal (with respect to $\mathcal{A}$) eventown subfamily of $\mathcal{A}$ of maximum size. If $|\mathcal{A}'| \leq 2^{\lfloor n/2 \rfloor - 2} + 1$, then we may partition $\mathcal{A}$ by $\mathcal{A} = \mathcal{A}_1' \sqcup \mathcal{A}_2' \sqcup \cdots \sqcup \mathcal{A}_l'$ with $l \geq 4$ so that for each $A_i \in \mathcal{A}_i$ and each $j < i$, there exists $B_j \in \mathcal{A}_j$ with $|A_i \cap B_j|$ odd and hence

$$\mathrm{op}(\mathcal{A}) \geq |\mathcal{A} \setminus \mathcal{A}_1'| + |\mathcal{A} \setminus (\mathcal{A}_1' \sqcup \mathcal{A}_2')| + \cdots + |\mathcal{A}_l'| \geq \frac{3}{4} \cdot 2^{\lfloor n/2 \rfloor} + \frac{1}{2} \cdot 2^{\lfloor n/2 \rfloor} > 2^{\lfloor n/2 \rfloor}.$$

If $|\mathcal{A}'| \in [2^{\lfloor n/2 \rfloor - 2} + 2, 2^{\lfloor n/2 \rfloor - 1} - 1]$ or $|\mathcal{A}'| \geq 2^{\lfloor n/2 \rfloor - 1} + 2$, then we are done by (3.2).

If $|\mathcal{A}'| = 2^{\lfloor n/2 \rfloor - 1} + 1 = t$, then without loss of generality assume that the maximal eventown subfamily is $\mathcal{A}' = \{A_1, \dots, A_t\}$. Setting $v_i = v_{A_i} \in \mathbb{F}_2^n$ as the characteristic vector of A_i, we let $W = \mathrm{span}\{v_1, \dots, v_t\}$. As in Section 3.2.1, we note that $\dim(W) = \lfloor n/2 \rfloor$ and will consider the linear functional $\chi_i : W \to \mathbb{F}_2$ given by $w \mapsto \langle w, v_i \rangle$ with $W_i := \ker(\chi_i)$. We then set $\mathcal{B}' = \mathcal{A} \setminus \mathcal{A}'$ and further consider $\mathcal{B}'' \subseteq \mathcal{B}'$ to be the sets $B \in \mathcal{B}'$ for which $\mathrm{op}(\mathcal{A}' \cup \{B\}) = 1$. We now consider a few cases based on $|\mathcal{B}''|$:

Case 1: $|\mathcal{B}''| \leq 1$

For each $B \in \mathcal{B}' \setminus B''$, there exists at least two sets in $\mathcal{A}'$ with odd-sized intersections with B, so

$$\mathrm{op}(\mathcal{A}) \geq |\mathcal{B}''| + 2 \cdot |\mathcal{B}' \setminus \mathcal{B}''| \geq 1 + 2(2^{\lfloor n/2 \rfloor - 1}) > 2^{\lfloor n/2 \rfloor}.$$

Case 2: $2 \leq |\mathcal{B}''| \leq 2^{\lfloor n/2 \rfloor - 2}$

As $\mathcal{B}'' \neq \emptyset$, there exists an $i > t$ where $W_i = \ker(\chi_i)$ is so that $|\{v_1, \dots, v_t\} \setminus W_i| = 1$. For each $A_j \in \mathcal{B}' \setminus \mathcal{B}''$, we know that $|\{v_1, \ldots, v_t\} \setminus W_j| \geq 2$ and in particular $W_i \neq W_j$. As W_i and W_j are vector subspaces of $\mathbb{F}_2^n$ of dimension $\lfloor n/2 \rfloor - 1$, it follows that $|W_i \cap W_j| \leq 2^{\lfloor n/2 \rfloor - 2}$ and hence $|\{v_1, \ldots, v_t\} \setminus W_j| \geq |W_i \setminus W_j| \geq 2^{\lfloor n/2 \rfloor - 2}$. As such, we get

$$\mathrm{op}(\mathcal{A}) \geq |\mathcal{B}''| + (2^{\lfloor n/2 \rfloor - 2}) \cdot |\mathcal{B}' \setminus \mathcal{B}''| > 2^{\lfloor n/2 \rfloor}. \tag{3.3}$$

Case 3: $|\mathcal{B}''| \geq 2^{\lfloor n/2 \rfloor - 2}$

Let $\mathcal{A}'' = \{A \in \mathcal{A}' : \exists\, B \in \mathcal{B}'' \text{ so that } |A \cap B| \text{ is odd}\}$. We first consider the case where $|\mathcal{A}''| \leq n$ and then will show that we actually necessarily have $|\mathcal{A}''| \leq n$. In this case, there exists $A_i \in \mathcal{A}''$ for which $\mathrm{op}(\mathcal{B}'' \cup \{A_i\}) \geq \frac{1}{n} \cdot 2^{\lfloor n/2 \rfloor - 2}$. Let $B_1'', B_2'' \in \mathcal{B}''$ be so that $|A_i \cap B_1''|$ and $|A_i \cap B_2''|$ are both odd. Then $|B_1'' \cap B_2''|$ is odd as otherwise $\mathcal{A}' \setminus \{A_i\} \cup \{B_1'', B_2''\}$ is a eventown subfamily of $\mathcal{A}$ of larger size. As a result,

$$\mathrm{op}(\mathcal{A}) \geq \binom{\frac{1}{n} \cdot 2^{\lfloor n/2 \rfloor - 2}}{2} > 2^{\lfloor n/2 \rfloor}. \tag{3.4}$$

Seeking a contradiction, suppose that $|\mathcal{A}''| \geq n+1$. As each element from $\mathcal{B}''$ has an odd number of elements in common with exactly one set from $\mathcal{A}''$, we may find distinct $X_1, \ldots, X_{n+1} \in \mathcal{A}''$ and $Y_1, \ldots, Y_{n+1} \in \mathcal{B}''$ so that $|X_i \cap Y_i|$ is odd for all $i \in [n+1]$ and $|X_i \cap Y_j|$ is even for all $i \neq j \in [n+1]$. As $\mathcal{A} \subseteq 2^{[n]}$, this violates Theorem 1.4.3 as there are $n+1$ such set pairs.

If $|\mathcal{A}'| = 2^{\lfloor n/2 \rfloor}$ and $\dim(W) = \lfloor n/2 \rfloor$, then we may argue in a similar fashion as above. If $\dim(W) < \lfloor n/2 \rfloor$, then the result follows by (3.2). Thus we have shown $\mathrm{op}(\mathcal{A}) \geq 2^{\lfloor n/2 \rfloor}$ in all cases.

This chapter 3 contains material from: J. O'Neill, "Towards supersaturation for oddtown and eventown", arxiv:2109.09925, 2021. The dissertation author was the sole author of this paper.

Chapter 4

On k-wise oddtown problems

Let $K_{n:k}$ denote the *Kneser graph* with $V(K_{n:k}) = \binom{[n]}{k}$ and $(A, B) \in E(K_{n:k})$ if and only if $A \cap B = \emptyset$. Given pairwise disjoint subsets $X_1, \ldots, X_k$ of $[n]$, let $\prod_{i=1}^{k} X_i$ denote the complete k-partite k-graph with edges consisting of exactly one vertex from each part $X_1, \ldots, X_k$ and $X_1 \sqcup \cdots \sqcup X_k$ denote their disjoint union. For $k = 2$, we refer to complete bipartite graphs as *bicliques*.

4.1 Hypergraph Covering Problems

In this section, we relate the problem at hand to a decomposition problem of a particular hypergraph. We first define the corresponding collection of set families.

Definition 1. *Let $\mathcal{A}_j = \{A_{j,i} : 1 \leq i \leq m\}$ for $j \in [k]$. Then $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ forms a Bollobás set (k,t)-tuple modulo 2 if*

$$\Big|\bigcap_{i \in [k]} A_{j,i_j}\Big| = 0 \pmod 2 \iff |\{i_1, \ldots, i_k\}| < t.$$

Let $b_{k,t}(n)$ be the size of the largest Bollobás set (k,t)-tuple modulo 2 with ground set $[n]$.

In the non-modular setting, Bollobás set (k,t)-tuples were first explored by the authors [OV21a] and are a generalization of the well-studied Bollobás set pairs [Bol65] when $k = t = 2$. Using the language of Definition 1, Theorems 1.4.4 and 1.4.5 (i.e. the "reverse" versions) say $b_{k,k}(n) = \Theta(n^{1/\lfloor k/2 \rfloor})$ and $b_{k,t}(n) = \Theta(n^{1/(t-1)})$ respectively. In order to determine the asymptotics of $b_{k,t}(n)$, we consider the following related hypergraph covering problem:

Definition 2. *Set $X_i = \{x_{i,j} : j \in [n]\}$ for $i \in [k]$. For $H \subseteq \prod_{i\in[k]} X_i$, a modulo 2 cover of H is a collection of complete k-partite subgraphs of $\prod_{i\in[k]} X_i$ which cover each edge of H an odd number of times and each non-edge an even number of times. We denote the minimum size of a modulo 2 cover of H by $f_k(H)$.*

Let $H_{k,t}(n)$ be the subhypergraph of $\prod_{i\in[k]} X_i$ defined by

$$H_{k,t}(n) = \{\{x_{1,i_1}, x_{2,i_2}, \ldots, x_{k,i_k}\} : |\{i_1, \ldots, i_k\}| \geq t\}.$$

Then, as in [OV21a, Section 1.2] there is a one to one correspondence between a modulo 2 cover of $H_{k,t}(n)$ with m complete k-partite k-graphs and a Bollobás set (k,t)-tuple modulo 2 consisting of subsets of $[m]$. Hence,

$$f_{k,t}(n) := f_k(H_{k,t}(n)) = \min\{m : b_{k,t}(m) \geq n\}. \tag{4.1}$$

The problem in Definition 2 is closely related to the hypergraph extension of the Graham-Pollak [GP72] problem. Given a hypergraph $H \subseteq \binom{[n]}{k}$, let $g_k(H)$ denote minimum number of complete k-partite subgraphs of H needed to cover each edge of H exactly once and let $g_k(n) = g_k(\binom{[n]}{k})$.

Graham and Pollak [GP72] proved for all $n \geq 1$, $g_2(n) = n-1$. Alon [Alo86] proved that $g_3(n) = n-2$ for $n \geq 2$ and proved that $g_k(n) = \Theta(n^{\lfloor k/2 \rfloor})$. Relating the problem to that of a minimal biclique cover of the Kneser graph $K_{n:k}$, Cioabă et al. [CKV09] improved the bounds on $g_k(n)$ to:

$$\frac{2\binom{n-1}{k}}{\binom{2k}{k}} \leq g_{2k}(n) \leq \binom{n-k}{k}. \tag{4.2}$$

Currently Leader, Milićević, and Tan [LMT18] have the best general bounds. A key ingredient in our proof of Theorem 1.4.4 is to similarly relate this hypergraph covering problem to a graph covering problem as in Cioabă et al. [CKV09]. To this end, define the *ordered* Kneser graph $OK_{n:k}$ as follows:

$$\begin{aligned} V(OK_{n:k}) &= \{(i_1, i_2, \ldots, i_k) : \{i_1, \ldots, i_k\} \in \binom{[n]}{k}\} \\ E(OK_{n:k}) &= \{((i_1, \ldots, i_k), (j_1, \ldots, j_k)) : \{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_k\} = \emptyset\}. \end{aligned}$$

The ordered Kneser graph is a $k!$ blowup of the Kneser graph $K_{n:k}$ and our interest in it is the following lemma which relates modulo 2 covers of $H_{2k,2k}(n)$ and biclique covers of $OK_{n:k}$.

Lemma 4.1.1. *Let $\{\mathcal{C}_i : i \in [t]\}$ be a modulo 2 cover of $H_{2k,2k}(n)$. Then, there exists a modulo 2 biclique cover $\{\mathcal{C}'_i : i \in [t]\}$ of $OK_{n:k}$ and in particular $f_{2k,2k}(n) \geq f_2(OK_{n:k})$.*

Proof. For each complete $2k$-partite $2k$-graph $\mathcal{C}_i := \prod_{j=1}^{2k} X_{i,j}$ in a modulo 2 cover of $H_{2k,2k}(n)$, define the complete bipartite graph $\mathcal{C}'_i$ with parts $\prod_{j=1}^{k} X_{i,j} \cap V(OK_{n:k})$, and $\prod_{j=k+1}^{2k} X_{i,j} \cap V(OK_{n:k})$. Observe that $\{\mathcal{C}'_i\}$ is a modulo 2 biclique cover of $OK_{n:k}$. □

Our next lemma shows a connection between the k-graphs $H_{k,k}(n)$ and $\binom{[n]}{k}$.

Lemma 4.1.2. *Let $k \geq 2$. Then $g_k(H_{k,k}(n)) \leq k! g_k(n)$.*

Proof. Let $m = g_k(n)$ and $\{\mathcal{C}_i : i \in [m]\}$ be such a minimal k-partite, k-uniform cover of $\binom{[n]}{k}$. Then, given $\mathcal{C}_i = \prod_{i \in [k]} C_i$ and a bijection $\pi : [k] \to [k]$, define

$$\mathcal{C}_{i,\pi} = \prod_{i \in [k]} C_{\pi(i)} \subseteq H_{k,k}(n).$$

It follows that $\{\mathcal{C}_{i,\pi}\}$ forms a decomposition of $H_{k,k}(n)$ and implies the bound. $\square$

4.2 Proof of Theorem 1.4.4

4.2.1 Two families

In the setting where $t = k = 2$, using a straightforward argument, we are able to prove a "reverse" bipartite oddtown theorem:

Proposition 4.2.1. *For all $n \geq 1$,*

$$b_{2,2}(n) = \begin{cases} n & n \text{ is odd} \\ \\ n+1 & n \text{ is even.} \end{cases}$$

Proof. For n even, and $i \in [n]$ consider $A_i = \{i\}$ with $B_i = \{i\}^c$ and $B_{n+1} = A_{n+1} = [n]$. This is a Bollobás set pair modulo 2 of size $n+1$ and thus $b_{2,2}(n) = n+1$ for even n by the bounds in (1.2).

For n odd, let $m = b_{2,2}(n)$ and $(\mathcal{A}, \mathcal{B})$ be a Bollobás set pair modulo 2 of size m. If $\{v_{A_i} : A_i \in \mathcal{A}\}$ is linearly independent in $\mathbb{F}_2^n$, then $m \leq n$. Thus, we may assume that there exists a non-trivial solution to $\sum_{i=1}^m \epsilon_i v_{A_i} = 0$ in $\mathbb{F}_2^n$. For all $j \in [m]$, it follows that

$$0 = \left\langle \sum_{i=1}^{m} \epsilon_i v_{A_i}, v_{B_j} \right\rangle = \sum_{\epsilon_i = 1; i \neq j} \langle v_{A_i}, v_{B_j} \rangle = \sum_{\epsilon_i = 1; i \neq j} |A_i \cap B_j| = |\{i : \epsilon_i = 1, i \neq j\}|$$

over the field $\mathbb{F}_2$. Since this is true for all $j \in [m]$, the existence of one non-zero ϵ_i implies they are all equal to one. In particular, $(m-1)$ is even and hence $m \in \{n, n+1\}$ is odd and thus $m = n$. For n odd, taking $A_i = \{i\}$ with $B_i = \{i\}^c$ for all $i \in [n]$ shows this is best possible. $\square$

4.2.2 More than two families

In this section, for $k \geq 2$, we will prove that

$$\left(\frac{1}{2} + o(1)\right)\binom{n}{k} \leq f_{2k,2k}(n) \leq (2k)!\binom{n}{k} \tag{4.3}$$

$$\left(\frac{1}{2} + o(1)\right)\binom{n}{k-1} \leq f_{2k-1,2k-1}(n) \leq (2k-1)!\binom{n}{k-1} \tag{4.4}$$

which imply Theorem 1.4.4 by (4.1) and that the bound in the theorem is the correct order of magnitude. Using Lemma 4.1.2 with the upper bound in (4.2), we get the upper bounds in (4.3) and (4.4).

By considering the link of the vertex $x_{1,1}$ in $H_{2k,2k}(n)$ and in each complete $2k$-partite $2k$-graph in a minimal modulo 2 cover of $H_{2k,2k}(n)$, it follows that $f_{2k,2k}(n) \geq f_{2k-1,2k-1}(n-1)$. As a result, the lower bound in (4.4) follows by the corresponding lower bound in (4.3). As such, it suffices to prove the lower bound in (4.3). By Lemma 4.1.1, it then suffices to prove that $f_2(OK_{n:k}) \geq (1/2 + o(1))\binom{n}{k}$.

As the adjacency matrix of each biclique has rank at most 2, the subadditivity of matrix rank and taking the minor of $OK_{n:k}$ which corresponds to the Kneser graph $K_{n:k}$ give

$$f_2(OK_{n:k}) \geq \frac{1}{2}\mathrm{rank}(A(OK_{n:k})) \geq \frac{1}{2}\mathrm{rank}(A(K_{n:k})) \tag{4.5}$$

where the rank of the matrices is over $\mathbb{F}_2$. Let $M_{n,k,l} \in \{0,1\}^{\binom{n}{k}\times\binom{n}{l}}$ where the (K, L) entry for $K \in \binom{[n]}{k}$ and $L \in \binom{[n]}{l}$ is the indicator of $K \subseteq L$ and note that $M_{n,k,n-k} = A(K_{n:k})$.

Wilson [Wil90, Theorem 1] determined the rank for matrices $M_{n,k,l}$ over $\mathbb{F}_p$ for $k \leq \min\{l, n-l\}$:

$$\operatorname{rank}(M_{n,k,l}) = \sum_{i:\, p \nmid \binom{l-i}{k-i}} \binom{n}{i} - \binom{n}{i-1}. \tag{4.6}$$

Applying (4.6) when $p = 2$ and $l = n-k$ gives the following for the rank over $\mathbb{F}_2$:

$$\operatorname{rank}(A(K_{n:k})) \geq \binom{n}{k} - \binom{n}{k-1} = (1+o(1))\binom{n}{k}. \tag{4.7}$$

Hence, using (4.5) and (4.7) give

$$f_{2k,2k}(n) \geq f_2(OK_{n:k}) \geq \Big(\frac{1}{2} + o(1)\Big)\binom{n}{k}.$$

Taking the bounds from (4.3) and (4.4) and noting in (4.1) that $f_{k,k}(n) = \min\{m : b_{k,k}(m) \geq n\}$, we get the following bounds on $b_{2k,2k}(n)$ and $b_{2k-1,2k-1}(n)$:

$$\begin{aligned} \Big(\frac{1}{2k} + o(1)\Big)n^{\frac{1}{k}} &\leq b_{2k,2k}(n) \leq \Big(k2^{\frac{1}{k}}\Big)n^{\frac{1}{k}} \\ \Big(\frac{1}{2k^2} + o(1)\Big)n^{\frac{1}{k-1}} &\leq b_{2k-1,2k-1}(n) \leq \Big((k-1)2^{\frac{1}{k-1}} + o(1)\Big)n^{\frac{1}{k-1}}. \end{aligned}$$

4.3 Proof of Theorem 1.4.5

In this section, we will prove that if $2t-2 \leq k$, then

$$f_{k,t}(n) \leq \Big(\frac{t^k}{t!} + o(1)\Big)n^{t-1} \quad \text{and} \quad b_{k,t}(n) \leq \Big(t - 1 + o(1)\Big)n^{\frac{1}{t-1}}. \tag{4.8}$$

Since $f_{k,t}(n) = \min\{m : b_{k,t}(m) \geq n\}$, the upper bound on $f_{k,t}(n)$ in (4.8) implies a lower bound on $b_{k,t}(n)$ which together with the upper bound on $b_{k,t}(n)$ in (4.8) proves Theorem

1.4.5 as they imply

$$\Big(\Big(\frac{t!}{t^k}\Big)^{\frac{1}{t-1}} + o(1)\Big)n^{\frac{1}{t-1}} \leq b_{k,t}(n) \leq \Big(t-1+o(1)\Big)n^{\frac{1}{t-1}}.$$

4.3.1 Upper bound on $f_{k,t}(n)$

Let $\phi : [n]^k \to \{\pi : \pi \text{ set partition of } [k]\}$ be the correspondence of an edge of $[n]^k$ to a set partition of $[k]$ where each part consists of the coordinates with equal indices. For a set partition π of $[k]$, define $H_k^\pi(n) = \{e \in [n]^k : \phi(e) = \pi\}$ so that

$$H_{k,t}(n) = \bigsqcup_{|\pi| \geq t} H_k^\pi(n). \tag{4.9}$$

It follows that $f_k(H_k^\pi(n)) \leq n^{|\pi|}$ as the number of edges in $H_k^\pi(n)$ is at most $n^{|\pi|}$ and one may take a modulo 2 cover consisting of single edges. Moreover, adding the complete k-partite k-graph $[n]^k$ to a minimal cover, it follows that

$$f_{k,t}(n) \leq 1 + f_k([n]^k \setminus H_{k,t}(n)). \tag{4.10}$$

Since $f_k(\cdot)$ is subadditive with respect to disjoint unions, using (4.9) gives

$$f_k([n]^k \setminus H_{k,t}(n)) \leq \sum_{|\pi| \leq t-1} f_k(H_k^\pi(n)) \leq \sum_{|\pi| \leq t-1} n^{|\pi|} = (S(k,t-1) + o(1))n^{t-1} \tag{4.11}$$

where $S(k,t)$ is the Stirling number of the second kind. We then get the upper bound on $f_{k,t}(n)$ in (4.8) by using the bound $S(k,t-1) \leq S(k,t) \leq t^k/t!$ together with (4.10) and (4.11).

4.3.2 Upper bound on $b_{k,t}(n)$

We first prove the following lemma which relates $b_{k,t}(n)$ to $b_{2,2}(n)$.

Lemma 4.3.1. *Given $n \geq 1$, let $\alpha = \alpha(k,t) = 2t - k + 2$. Then letting $m = b_{k,t}(n)$,*

$$b_{2,2}(n) \geq \begin{cases} \binom{m}{t-1} & \text{if } 2t-2 \leq k \\ \\ \binom{m-\alpha}{k-t+1} & \text{if } 2t-2 > k \end{cases}$$

In particular, both quantities are less than or equal to $n+1$ by (1.2).

Proof. Let $m = b_{k,t}(n)$ for $2t-2 \leq k$ and let $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ be a Bollobás set (k,t)-tuple modulo 2 of size m where $\mathcal{A}_i = \{A_{j,i} \subseteq [n] : i \in [m]\}$ for all $j \in [k]$. For each $I = \{i_1 < \cdots < i_{t-1}\} \in \binom{[m]}{t-1}$, define the families $\mathcal{A} := \{A_I\}$ and $\mathcal{B} := \{B_I\}$ where:

$$\begin{aligned} A_I &:= A_{1,i_1} \cap \cdots \cap A_{t-1,i_{t-1}} \\ B_I &:= A_{t,i_1} \cap \cdots \cap A_{2t-2,i_{t-1}} \end{aligned}$$

Note that $(\mathcal{A}, \mathcal{B})$ is a Bollobás set pair modulo 2 and hence the result follows.

Similarly, for $2t-2 > k$, let $m = b_{k,t}(n)$ and let $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ be a Bollobás set (k,t)-tuple modulo 2 of size m where $\mathcal{A}_i := \{A_{j,i} \subseteq [n] : i \in [m]\}$ for all $j \in [k]$. Consider the index set

$$\mathcal{F} := \binom{[\alpha+1, m]}{k-t+1}.$$

For $I = \{i_1 < \cdots < i_{k-t+1}\} \in \mathcal{F}$, define the families $\mathcal{A} := \{A_I\}$ and $\mathcal{B} := \{B_I\}$ where

$$\begin{aligned} A_I &:= A_{1,1} \cap \cdots \cap A_{\alpha,\alpha} \cap A_{\alpha+1,i_1} \cap \cdots \cap A_{t-1,i_{k-t+1}} \\ B_I &:= A_{k-t+2,i_1} \cap \cdots \cap A_{k,i_{t-1}}. \end{aligned}$$

Note that $(\mathcal{A}, \mathcal{B})$ is a Bollobás set pair modulo 2 and hence the result follows. $\square$

We are now able to prove the upper bound on $b_{k,t}(n)$ in (4.8) when $2t-2 \leq k$. Letting $m = b_{k,t}(n)$, Lemma 4.3.1 yields that $\binom{m}{t-1} \leq n+1$. Using the bound $\binom{m}{t-1} \geq (m/(t-1))^{t-1}$, it follows that

$$m \leq \Big(t - 1 + o(1)\Big) n^{\frac{1}{t-1}}.$$

This chapter contains material from: J. O'Neill, J. Verstraëte, "A note on k-wise oddtown problems", arxiv:2011.09402, 2020. The dissertation author was one of the primary investigators and authors of this paper.

Chapter 5

A generalization of the Bollobás set pairs inequality

5.1 Proof of Theorem 1.5.2

Given a Bollobás set (k,t)-tuple $(\mathcal{A}_1,\dots,\mathcal{A}_k)$ with $\mathcal{A}_j=\{A_{j,i}:1\le i\le m\}$ and a surjection $\phi:[k]\to[t]$, consider $\mathcal{A}_\ell(\phi):1\le \ell\le t$ where $\mathcal{A}_\ell(\phi)=\{A_{\ell,i}(\phi):1\le i\le m\}$ and

$$A_{\ell,i}(\phi)=\bigcap_{h:\phi(h)=\ell} A_{h,i}.$$

It follows that $(\mathcal{A}_1(\phi),\dots,\mathcal{A}_t(\phi))$ is a Bollobás set (t,t)-tuple and hence it suffices to prove Theorem 1.5.2 in the case where $t=k$. In this setting, surjections $\phi:[k]\to[k]$ simply permute the k families and as such we suppress the notation of ϕ for the remainder of this section. The proof of Theorem 1.5.1, given a Bollobás set pair, defines a collection of permutations $\mathcal{C}_i$ for $i\in[m]$ and shows that these chains are necessarily disjoint. Similarly, given a Bollobás set (k,k)-tuple, we will define a collection of chains $\mathcal{C}_\sigma$ for every ordered collection σ of $(k-1)$ distinct indices of $[m]$ and show these chains are pairwise disjoint.

Let $(\mathcal{A}_1,\dots,\mathcal{A}_k)$ with $\mathcal{A}_j=\{A_{j,i}:1\le i\le m\}$ be a Bollobás set (k,k)-tuple, and set

$$X=\bigcup_{i=1}^{m}(A_{1,i}\cup A_{2,i}\cup\cdots\cup A_{k,i})$$

with $|X|=n$. For $\sigma\in[m]_{(k-1)}$, define a subset $\mathscr{C}_\sigma$ of permutations $\pi:X\to[n]$ by

$$\mathscr{C}_\sigma:=\left\{\pi:X\to[n]:\max_{x\in A_{1,\sigma}}\pi(x)<\min_{y\in A_{2,\sigma}}\pi(y)\le\max_{y\in A_{2,\sigma}}\pi(y)<\cdots<\min_{z\in A_{k,\sigma}}\pi(z)\right\}.$$

Letting $U_\sigma:=A_{1,\sigma}\cup\cdots\cup A_{k,\sigma}$, elementary counting methods give

$$|\mathscr{C}_\sigma|=\binom{n}{|U_\sigma|}|A_{1,\sigma}|!\cdots|A_{k,\sigma}|!(n-|U_\sigma|)!=n!\cdot\binom{|U_\sigma|}{|A_{1,\sigma}|\cdots|A_{k,\sigma}|}^{-1}. \tag{5.1}$$

We will now prove a lemma which states that $\{\mathscr{C}_\sigma\}_{\sigma\in[m]_{(k-1)}}$ forms a disjoint collection of a permutations. The general proof only works for $k \geq 4$, so we first consider $k = 3$.

Lemma 5.1.1. *If $\sigma_1, \sigma_2 \in [m]_{(2)}$ are distinct, then $\mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2} = \emptyset$.*

Proof. Seeking a contradiction, suppose there exists $\pi \in \mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2}$. After relabeling, it suffices to consider the following five cases.

(1) $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 4\}$ (2) $\sigma_1 = \{1, 3\}$ and $\sigma_2 = \{2, 3\}$

(3) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{1, 3\}$ (4) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{2, 3\}$

(5) $\sigma_1 = \{1, 2\}$ and $\sigma_2 = \{3, 1\}$.

In case (1), without loss of generality, $\max\{\pi(x) : x \in A_{1,1}\} \leq \max\{\pi(x) : x \in A_{1,2}\}$ and thus $\pi \in \mathscr{C}_{\sigma_2}$ yields

$$\max_{x\in A_{1,1}} \pi(x) \leq \max_{x\in A_{1,2}} \pi(x) < \min_{y\in A_{2,4}\setminus A_{1,2}} \pi(y).$$

Then as $A_{1,1} \cap A_{2,4} \cap A_{3,2} \neq \emptyset$, there exists $w \in A_{1,1} \cap A_{2,4} \cap A_{3,2}$. It follows that $w \notin A_{1,2}$ since if $w \in A_{1,2}$, then $w \in A_{1,2} \cap A_{2,4} \cap A_{3,2} \neq \emptyset$; a contradiction. But this yields a contradiction as

$$\pi(w) \leq \max_{x\in A_{1,1}} \pi(x) \leq \max_{x\in A_{1,2}} \pi(x) < \min_{y\in A_{2,4}\setminus A_{1,2}} \pi(y) \leq \pi(w).$$

In case (2), without loss of generality, $\max\{\pi(x) : x \in A_{1,1}\} \leq \max\{\pi(x) : x \in A_{1,2}\}$ and we recover a similar contradiction as case (1) by noting that there exists $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ with $w \notin A_{1,2}$.

In case (3) we may assume $\max\{\pi(x) : x \in A_{2,2} \backslash A_{1,1}\} \leq \max\{\pi(x) : x \in A_{2,3} \backslash A_{1,1}\}$ and $\pi \in \mathscr{C}_{1,3}$ yields $\max\{\pi(x) : x \in A_{2,3} \setminus A_{1,1}\} < \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_{2,3})\}$. Thus

$$\max\{\pi(x) : x \in A_{2,2} \setminus A_{1,1}\} < \min\{\pi(x) : x \in A_{3,1} \setminus (A_{1,1} \cup A_{2,3})\}$$

and there exists $w \in A_{1,3} \cap A_{2,2} \cap A_{3,1}$ with $w \notin A_{1,1}$ and $w \notin A_{2,3}$. It follows that $\pi(w) < \pi(w)$, a contradiction.

In case (4), if $\max\{\pi(x) : x \in A_{1,1}\} \leq \max\{\pi(x) : x \in A_{1,2}\}$, then using $w \in A_{1,1} \cap A_{2,3} \cap A_{3,2}$ and noting $w \notin A_{1,2}$, we get a contradiction. Thus, we may assume otherwise and $\pi \in \mathcal{C}_{1,2}$ gives

$$\max_{x \in A_{1,2}} \pi(x) < \max_{x \in A_{1,1}} \pi(x) < \min_{z \in A_{3,1} \setminus (A_{1,1} \cup A_{2,2})} \pi(z).$$

This is a contradiction as there exists $w \in A_{1,2} \cap A_{2,3} \cap A_{3,1}$ with $w \notin A_{1,1}$ and $w \notin A_{2,2}$.

In case (5), if $\max\{\pi(x) : x \in A_{1,1}\} \leq \max\{\pi(x) : x \in A_{1,3}\}$, then we may proceed as in the latter part of case (4) using $w \in A_{1,1} \cap A_{2,2} \cap A_{3,3}$ and $w \notin A_{2,1}$ and $w \notin A_{1,3}$ to get a contradiction. Otherwise, proceeding as in case (1) and noting there exists $w \in A_{1,3} \cap A_{2,2} \cap A_{3,1}$, but $w \notin A_{1,1}$ yields a contradiction. $\square$

A similar argument yields the analog of Lemma 5.1.1 to the case where $k \geq 4$.

Lemma 5.1.2. *Let $k \geq 4$. If $\sigma_1, \sigma_2 \in [m]_{(k-1)}$ are distinct, then $\mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2} = \emptyset$.*

Proof. Since $\sigma_1, \sigma_2 \in [m]_{(k-1)}$ are distinct, there exists minimal $h \in [k-1]$ so that $\sigma_1(h) \neq \sigma_2(h)$. Seeking a contradiction, suppose there exists a $\pi \in \mathscr{C}_{\sigma_1} \cap \mathscr{C}_{\sigma_2}$. Without loss of generality,

$$\max\{\pi(x) : x \in A_{h,\sigma_1}\} \leq \max\{\pi(x) : x \in A_{h,\sigma_2}\} < \min\{\pi(z) : z \in A_{k,\sigma_2}\}.$$

Now, consider a bijection $\tau : [k-1] \setminus \{h\} \to [k-1] \setminus \{1\}$ which has no fixed points. As in Lemma 5.1.1, we want to show that there exists a $w \in A_{h,\sigma_1} \cap A_{k,\sigma_2}$ and consider two separate cases.

First, suppose that $\sigma_1(h) \notin \sigma_2([k-1])$. As $|\{\sigma_1(h), \sigma_2(1), \ldots, \sigma_2(k-1)\}| = k$, there exists

$$w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap \bigcap_{l \in [k-1] \setminus \{h\}} A_{l,\sigma_2(\tau(l))}. \tag{5.2}$$

Next, suppose that $\sigma_1(h) = \sigma_2(x)$ for some x. We now claim that $x \neq 1$. If $h = 1$, then this is trivial. If $h > 1$, then $\sigma_1(1) = \sigma_2(1)$, so $\sigma_1(h) \neq \sigma_2(1)$ since $\sigma_1(h) \neq \sigma_1(1)$. For τ as above, there exists $y \in [k-1] \setminus \{h\}$ so that $\tau(y) = x$. Taking γ distinct from $\{\sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\}$, $|\{\sigma_1(h), \gamma, \sigma_2(1), \ldots, \sigma_2(k-1)\} \setminus \{\sigma_2(x)\}| = k$ and hence there exists

$$w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)} \cap A_{y,\gamma} \cap \bigcap_{l \in [k-1] \setminus \{y,h\}} A_{l,\sigma_2(\tau(l))}. \tag{5.3}$$

By construction, $w \in A_{h,\sigma_1(h)} \cap A_{k,\sigma_2(1)}$. Suppose there exists a $t \in [k-1] \setminus \{h\}$ so that $w \in A_{t,\sigma_2(t)}$. As τ has no fixed points, replacing the set in the k-wise intersection corresponding to $\mathcal{A}_t$ with $A_{t,\sigma_2(t)}$ in either (5.2) or (5.3), w is an element of this new k-wise intersection with $(k-1)$ distinct indices; a contradiction. If $w \in A_{h,\sigma_2(h)}$, then we may similarly replace $A_{h,\sigma_1(h)}$ with $A_{h,\sigma_2(h)}$ in the k-wise intersection in either (5.2) or (5.3) to get a contradiction. Thus, $w \notin A_{1,\sigma_2(1)} \cup \cdots \cup A_{k-1,\sigma_2(k-1)}$ and hence $w \in A_{h,\sigma_1} \cap A_{k,\sigma_2}$ so that $\pi(w) < \pi(w)$; a contradiction. $\square$

Using Equation (5.1), Lemma 5.1.1, and Lemma 5.1.2, we are now able to prove Theorem 1.5.2 in the case where $t = k$. There are $n!$ total permutations, and Lemma 5.1.1 and Lemma 5.1.2 yield that each of which appears in at most one of the sets $\mathscr{C}_\sigma$ for $\sigma \in [m]_{(k-1)}$. Hence, using $|\mathscr{C}_\sigma|$ in Equation (5.1),

$$\sum_{\sigma \in [m]_{(k-1)}} |\mathscr{C}_\sigma| = \sum_{\sigma \in [m]_{(k-1)}} n! \cdot \binom{|A_{1,\sigma} \cup \cdots \cup A_{k,\sigma}|}{|A_{1,\sigma}| \cdots |A_{k,\sigma}|}^{-1} \leq n!$$

and thus the result follows by dividing through by $n!$.

Sharpness of Theorem 1.5.2:

We give a simple construction establishing the sharpness of Theorem 1.5.2 for $k \geq t = 2$. Let $n \geq 4k$ and using addition modulo n, define $A_{1,i} = \{i\}^c$, $A_{j,i} = \{i - (j-1), i + (j-1)\}^c$ for $j \in [2, k-1]$, and $A_{k,i} = \{i - k + 2, i - k + 3, \ldots, i + k - 2\}$. Letting $\mathcal{A}_j = \{A_{j,i}\}_{i \in [n]}$ for all $j \in [k]$, we will show $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ is a Bollobás $(k, 2)$-tuple. Since $|A_{1,i}| = n - 1$ and $|A_{2,i} \cap \cdots \cap A_{k,i}| = 1$, Theorem 1.5.2 with $t = 2$ and surjection $\phi : [k] \to [2]$ with $\phi(1) = 1$ and $\phi(i) = 2$ for $i \neq 1$ gives

$$1 \;\geq\; \sum_{i=1}^{n} \binom{|A_{1,i}| + |A_{2,i} \cap \cdots \cap A_{k,i}|}{|A_{1,i}|}^{-1} \;=\; \sum_{i=1}^{n} \frac{1}{n} \;=\; 1.$$

By construction, for all $i \in [n]$, $A_{1,i} \cap A_{2,i} \cap \cdots \cap A_{k,i} = \emptyset$. It thus suffices to show these are the only empty k-wise intersections. To this end, for $\boldsymbol{i} = (i_1, \ldots, i_{k-1})$, define

$$A(\boldsymbol{i}) := A_{1,i_1} \cap \cdots \cap A_{k-1,i_{k-1}}.$$

Lemma 5.1.3. *Let $\boldsymbol{i} = (i_1, \ldots, i_{k-1})$. If $A(\boldsymbol{i})^c = A_{k,i_k}$, then $i_1 = \cdots = i_k$.*

Proof. We proceed by induction on k where the result is trivial when $k = 2$. In the case where $k > 2$, $i_{k-1} - k + 2 = i_k + x$ for some x such that $-(k-2) \le x \le (k-2)$ and thus $i_{k-1} + (k-2) = i_{k-1} - (k-2) + (2k-4) = i_k + x + (2k-4)$.

Next, there is a y such that $-(k-2) \le y \le (k-2)$ with $i_{k-1} + (k-2) = i_k + y$, and since $n \ge 4k$, $x + 2k - 4 = y$ with equality over $\mathbb{Z}$ and moreover $i_{k-1} + (k-2) = i_k + (k-2)$ over $\mathbb{Z}$ and hence $i_k = i_{k-1}$. Removing these elements from each set, the result then follows by induction. $\square$

If $A_{1,i_1} \cap \cdots \cap A_{k,i_k} = \emptyset$, then as $A(\boldsymbol{i}) = A_{1,i_1} \cap A_{2,i_2} \cap \cdots A_{k-1,i_{k-1}}$,

$$\emptyset = A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k-1,i_{k-1}} \cap A_{k,i_k} = A(\boldsymbol{i}) \cap A_{k,i_k}.$$

The result follows by noting $|A(\boldsymbol{i})| \ge n - (2k-3)$, $|A_{k,i_k}| = 2k - 3$, and using Lemma 5.1.3.

An Explicit Construction:

Let $k \ge 3$. An explicit construction of a Bollobás $(k,2)$-tuple $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ where $|\mathcal{A}_i| = 2^n$ and each $\mathcal{A}_i$ consists of subsets of X for $|X| = kn$ may be described as follows. Let $I_j := \{x_{j,1}, x_{j,2}, \ldots, x_{j,k}\}$ and consider $X = I_1 \sqcup \cdots \sqcup I_n$. Now, for each $f : [n] \to [2]$ and $j \in [k]$, define

$$A_{j,f} := \{x_{1,f(1)+j-1}, \ \ldots \ , x_{n,f(n)+j-1}\}^c$$

where we work modulo k within the subscripts of I_j. It is straightforward to check that $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k)$ is a Bollobás $(k,2)$-tuple. This establishes the lower bound on $\beta_{k,2}(n)$ in Equation (1.6) and hence the upper bound on $f_{k,2}(n)$ in Equation (1.11).

5.2 Proof of Theorem 1.5.4

5.2.1 Upper bound on $f_{k,t}(n)$

We wish to find a covering of $H_{k,t}(n)$ with complete k-partite k-graphs and assume the parts of $H_{k,t}(n)$ are $X_1, X_2, \ldots, X_k$. For each subset T of $[k]$ of size t, consider the uniformly random coloring $\chi_T : [n] \to T$. Given such a χ_T, let $Y_i \subseteq X_i$ be the vertices of color i for $i \in T$; that is $Y_i := \{x_{ij} : \chi(j) = i\}$ and $Y_i = X_i$ for $i \notin T$. Denote by $H(T, \chi)$ the (random) complete k-partite hypergraph with parts $Y_1, Y_2, \ldots, Y_k$, and note that $H(T, \chi) \subseteq H_{k,t}(n)$. We place each $H(T, \chi)$ a total of N times independently and randomly where

$$N = \left\lfloor \frac{(t+1)t^t \log_2 n}{(k-t+1)\log_2 e} \right\rfloor$$

and produce $\binom{k}{t}N$ random subgraphs $H(T, \chi)$. For a set partition π of $[k]$, let $|\pi|$ denote the number of parts in the partition and index the parts by $[|\pi|]$. Given a set partition $\pi = (P_1, P_2, \ldots, P_s)$, let

$$f(\pi, t) = \sum_{T \in [s]^{(t)}} \prod_{i \in T} |P_i|.$$

If U is the number of edges of $H_{k,t}(n)$ not in any of these subgraphs, then

$$\mathbb{E}(U) \le \sum_{|\pi| \ge t} n^{|\pi|}(1 - t^{-t})^{Nf(\pi,t)} = \sum_{t \le s \le k} n^s \sum_{|\pi| = s} (1 - t^{-t})^{Nf(\pi,t)}. \tag{5.4}$$

For sufficiently large n, we claim that $\mathbb{E}(U) < 1$, which implies there exists a covering of $H_{k,t}(n)$ with at most $\binom{k}{t}N$ complete k-partite k-graphs, as required. The following technical lemma states that f is a decreasing function in the set partition lattice, and that $f(\pi, t)$ increases when we merge all but one element of a smaller part of π with a larger part of π:

Lemma 5.2.1. *Let $k \geq s \geq t \geq 2$, and let $\pi = (P_1, P_2, \ldots, P_s)$ be a partition of $[k]$.*

(i) *If π' is a refinement of π with $|\pi'| = s+1$, then $f(\pi, t) \leq f(\pi', t)$.*

(ii) *If $|P_1| \geq |P_2| \geq 2$ and $a \in P_2$, and π' is the partition $(P'_1, P'_2, \ldots, P'_s)$ of $[k]$ with $P'_1 = P_1 \cup P_2 \setminus \{a\}$ and $P'_2 = \{a\}$ and with $P'_i = P_i$ for $3 \leq i \leq s$, then $f(\pi', t) \leq f(\pi, t)$.*

Proof. We will only prove (i) as the proof of (ii) is similar. Let $\pi = P_1|P_2|\cdots|P_s$ and without loss of generality, $\pi' = P_x|P_y|P_2|\cdots|P_s$. Setting $\mathcal{T}(\overline{1}) = \{T \in [s]^{(t)} : 1 \notin T\}$ and $\mathcal{T}'(\overline{x}, \overline{y}) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x, y \notin T\}$, it follows that

$$\sum_{T \in \mathcal{T}(\overline{1})} \prod_{i \in T} |P_i| = \sum_{T \in \mathcal{T}'(\overline{x}, \overline{y})} \prod_{i \in T} |P_i|.$$

Now, letting $\mathcal{T}(1) = \{T \in [s]^{(t)} : 1 \in T\}$ and $\mathcal{T}'(x, \overline{y}) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \in T, y \notin T\}$ and $\mathcal{T}'(\overline{x}, y) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \notin T, y \in T\}$, we see that

$$\sum_{T \in \mathcal{T}(1)} \prod_{i \in T} |P_i| = \sum_{T \in \mathcal{T}'(\overline{x}, y)} \prod_{i \in T} |P_i| + \sum_{T \in \mathcal{T}'(x, \overline{y})} \prod_{i \in T} |P_i|$$

since $|P_1| = |P_x| + |P_y|$. Thus letting $\mathcal{T}'(x, y) = \{T \in \{x, y, 2, \ldots, s\}^{(t)} : x \in T, y \in T\}$,

$$f(\pi', t) - f(\pi, t) = \sum_{T \in \mathcal{T}'(x, y)} \prod_{i \in T} |P'_i|$$

and in particular $f(\pi, t) \leq f(\pi', t)$. $\square$

By Lemma 5.2.1, a set partition of $[k]$ into s parts which minimizes $f(\pi, t)$ consists of one part of size $k - s + 1$ and $s - 1$ singleton parts and hence

$$\min\{f(\pi, t) : |\pi| = s\} = (k - s + 1)\binom{s-1}{t-1} + \binom{s-1}{t}. \tag{5.5}$$

In what follows, we denote a set partition of $[k]$ into s parts which minimizes $f(\pi, t)$ by π_s. For n large enough, and all s where $t \le s \le k$, we will show

$$\frac{\sum_{|\pi|=t}(1 - t^{-t})^{Nf(\pi,t)}}{\sum_{|\pi|=s}(1 - t^{-t})^{Nf(\pi,t)}} \ge n^{s-t}.$$

Replacing the numerator with its largest term and each term in denominator with its largest term,

$$\frac{\sum_{|\pi|=t}(1 - t^{-t})^{Nf(\pi,t)}}{\sum_{|\pi|=s}(1 - t^{-t})^{Nf(\pi,t)}} \ge \frac{(1 - t^{-t})^{Nf(\pi_t,t)}}{S(k, s)(1 - t^{-t})^{Nf(\pi_s,t)}} = \frac{1}{S(k, s)}(1 - t^{-t})^{N(f(\pi_t,t)-f(\pi_s,t))}$$

where $S(k, s)$ is the Stirling number of the second kind. Taking $n \ge S(k, s)$, we will now show the following:

Claim 5.2.2.

$$\frac{1}{S(k, s)}(1 - t^{-t})^{N(f(\pi_t,t)-f(\pi_s,t))} \ge n^{s-t}. \tag{5.6}$$

Proof. First, we recall that

$$N = \left\lfloor \frac{(t + 1)t^t \log_2 n}{(k - t + 1) \log_2 e} \right\rfloor \quad \text{and} \quad f(\pi_s, t) = (k - s + 1)\binom{s-1}{t-1} + \binom{s-1}{t}.$$

As a result, when $t \le s < k$, a calculation yields that

$$f(\pi_{s+1}, t) - f(\pi_s, t) = (k - s)\binom{s-1}{t-2}. \tag{5.7}$$

Letting $n \geq S(k,t)$, after taking $\log_2(\cdot)$ on both sides of (5.6), it suffices to prove that

$$N \cdot \frac{f(\pi_s,t) - f(\pi_t,t)}{t^t}\left(- t^t \log_2(1 - t^{-t})\right) \geq (s-t+1)\log_2(n). \tag{5.8}$$

Using the fact that $(1-t^{-t})^{t^t} \leq e^{-1}$ and our choice of N, it suffices to show that

$$f(\pi_s,t) - f(\pi_t,t) \geq \frac{(s-t+1)(k-t+1)}{t+1}. \tag{5.9}$$

The inequality in (5.9) holds for all $k \geq s > t \geq 3$ by using (5.7). $\square$

Therefore, the index $s = t$ maximizes the right hand side of Equation (5.4), and hence

$$\mathbb{E}[U] \leq (k-t+1)(n^t)\sum_{|\pi|=t}(1-t^{-t})^{Nf(\pi,t)} < (k-t+1)n^t S(k,t)(1-t^{-t})^{N(k-t+1)} < 1$$

for our choice of N provided $n \geq kS(k,t)$. Thus,

$$f_{k,t}(n) \leq \binom{k}{t}\frac{(t+1)t^t \log_2 n}{(k-t+1)\log_2 e} = \frac{(t+1)t^{t-1}}{\log_2 e}\binom{k}{t-1}\log_2 n.$$

5.2.2 Lower bound on $f_{k,2}(n)$

In this section, we show

$$f_{k,2}(n) \geq \min\{m : \binom{m}{\lceil m/k \rceil} \geq n\}. \tag{5.10}$$

Let $\{H_1, H_2, \ldots, H_m\}$ be a covering of $H_{k,2}(n)$ with $m = f_{k,2}(n)$ complete k-partite k-graphs. We recall $H_{k,2}(n) = K_{n,n,\ldots,n}\backslash M$, where M is a perfect matching of $K_{n,n,\ldots,n}$. For $i \in [k]$ and $j \in [n]$, define $A_{i,j} = \{H_r : x_{ij} \in V(H_r)\}$ and $\mathcal{A}_i = \{A_{i,j} : 1 \leq j \leq n\}$.

As in (1.9), $(\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_k)$ is a Bollobás $(k,2)$-tuple of size n. For convenience, for each $i \in [k]$, let $\phi_i : [k] \to [2]$ be so that $\phi_i^{-1}(1) = \{i\}$. Taking the sum of inequality from Theorem 1.5.2 with $t = 2$ over all $i \in [k]$,

$$\sum_{i=1}^{k} \sum_{j=1}^{n} \binom{|A_{1,j}(\phi_i) \cup A_{2,j}(\phi_i)|}{|A_{1,j}(\phi_i)|}^{-1} \leq k. \tag{5.11}$$

We use this inequality to give a lower bound on $f_{k,2}(n) = m$. First we observe

$$\sum_{r=1}^{m} |V(H_r)| = \sum_{j=1}^{n} \sum_{i=1}^{k} |A_{i,j}| = \sum_{j=1}^{n} \sum_{i=1}^{k} |A_{1,j}(\phi_i)|. \tag{5.12}$$

Let ∂H denote the set of $(k-1)$-tuples of vertices contained in some edge of a hypergraph H. Then

$$\sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} |A_{2,j}(\phi_i)|. \tag{5.13}$$

Putting the above identities together,

$$\sum_{r=1}^{m} |V(H_r)| + \sum_{r=1}^{m} |\partial H_r \cap \partial M| = \sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|). \tag{5.14}$$

We note $|\partial H_r \cap \partial M| \leq |V(H_r)|/(k-1)$, and therefore

$$\sum_{r=1}^{m} |\partial H_r \cap \partial M| \leq \frac{1}{k-1} \sum_{r=1}^{m} |V(H_r)|. \tag{5.15}$$

It follows that

$$\sum_{j=1}^{n} \sum_{i=1}^{k} (|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) \leq \frac{k}{k-1} \sum_{r=1}^{m} |V(H_r)|. \tag{5.16}$$

Subject to the linear inequalities (5.12) and (5.16), the left side of (5.11) is minimized when $kn|A_{1,j}(\phi_i)| = \sum_{r=1}^{m} |V(H_r)|$ and $kn(|A_{1,j}(\phi_i)| + |A_{2,j}(\phi_i)|) = (k-1)|A_{1,j}(\phi_i)|$. Since $|V(H_r)| \leq (k-1)n$ for all $r \in [m]$, (5.11) implies $\binom{m}{\lceil m/k \rceil} \geq n$, which gives (5.10). $\square$

5.2.3 Lower bound on $f_{k,k}(n)$

Let $\mathcal{H} = \{H_1, H_2, \dots, H_m\}$ be a minimal covering of $H_{k,k}(n)$ with complete k-partite k-graphs, so $m = f(H_{k,k}(n))$. Given a k-partite k-graph H, consider its 2-shadow $\partial_2(H) = \{R \subseteq V(H) : |R| = k-2, R \subseteq e \text{ for some } e \in H\}$. Let $\partial_2(\mathcal{H}) = \bigcup_{i=1}^m \partial_2(H_i)$.

Given $R \in \partial_2(\mathcal{H})$ and $H_i \in \mathcal{H}$, let $H_i(R) := \{e \in \binom{V(H_i)}{2} : e \cup R \in H_i\}$ be the possibly empty link graph of the edge R in the hypergraph H_i and let $V(H_i(R))$ be the set of vertices in the link graph. Observe that double counting yields

$$\sum_{R \in \partial_2(\mathcal{H})} \left(\sum_{i=1}^m |V(H_i(R))| \right) = \sum_{i=1}^m \left(\sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right). \tag{5.17}$$

An optimization argument yields $|\partial_2(H_i)|$ is maximized when the parts of H_i are of equal or nearly equal maximal size. Since $|V(H_i(R))| \leq 2(n-k+2)$, the right hand side of Equation (5.17) is bounded above by

$$\sum_{i=1}^m \left(\sum_{R \in \partial_2(H_i)} |V(H_i(R))| \right) \leq m \cdot \binom{k}{2} \cdot \left(\frac{n}{k}\right)^{k-2} \cdot 2(n-k+2). \tag{5.18}$$

For a lower bound on the left hand side of Equation (5.17), fix $R \in \partial_2(\mathcal{H})$ and without loss of generality suppose that $R = \{x_{1,1}, \dots, x_{k-2,k-2}\}$. Let $Y = [k-1, n]$. Let $K_{Y,Y}$ be the complete bipartite graph with two distinct copies of Y and $\mathcal{M} = \{(x_{k-1,i}, x_{k,i} : i \in Y\}$ be a perfect matching in $K_{Y,Y}$. Then, $\{H_1(R), \dots, H_m(R)\}$ forms a biclique cover of $K_{Y,Y} \setminus \mathcal{M}$. Applying the convexity result of Tarjan [Tar75, Lemma 5],

$$\sum_{i=1}^m |V(H_i(R))| \geq (n-k+2)\log_2(n-k+2).$$

Noting that $|\partial_2(\mathcal{H})| = \binom{k}{2}(n)_{(k-2)}$, the left hand side of Equation (5.17) is bounded below by

$$\sum_{R\in\partial_2(\mathcal{H})}\left(\sum_{i=1}^{m}|V(H_i(R))|\right) \geq \binom{k}{2}(n)_{(k-2)}(n-k+2)\log_2(n-k+2). \tag{5.19}$$

Comparing the bounds from Equation (5.18) and Equation (5.19),

$$m \geq \frac{(n)_{(k-2)}\log_2(n-k+2)}{2\left(\frac{n}{k}\right)^{k-2}} \geq \frac{k^{k-2}}{2}\log_2 n$$

provided that n is large enough.

For $t \geq 3$ and $t < k$, the lower bound on $f_{k,t}(n)$ in Theorem 1.5.4 is obtained from the lower bounds on $f_{t-1,t-1}(n-1)$ as follows: Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a minimal covering of $H_{k,t}(n)$ with complete k-partite k-graphs, so $m = f(H_{k,t}(n))$. Given $T \in \binom{[k]}{k-t+1}$, define $H_T \subseteq H_{k,t}(n)$ by

$$H_T := \{\{x_{1,i_1}, \ldots, x_{k,i_k}\} \in H_{k,t}(n) : i_j = 1 \; \forall \; j \in T\}.$$

It follows that at least $f_{t-1,t-1}(n-1)$ of the complete k-partite k-graphs in $\mathcal{H}$ are needed to cover H_T. Moreover, for distinct $T, T' \in \binom{[k]}{k-t+1}$, the corresponding complete k-partite k-graphs from $\mathcal{H}$ are necessarily pairwise disjoint and hence

$$f_{k,t}(n) \geq \binom{k}{k-t+1} f_{t-1,t-1}(n-1) \geq \binom{k}{t-1}\frac{(t-1)^{t-3}}{2}\log_2 n$$

provided that n is large enough.

This chapter contains material from: J. O'Neill, J. Verstraëte, "A generalization of the Bollobás set pairs inequality", The Electronic Journal of Combinatorics, 28(3), 2021. The dissertation author was one of the primary investigators and authors of this paper.

Chapter 6

Intersection saturated set systems

6.1 Clique partitions

In this section, we explore clique partitions. Given integers $2 \leq t < k$, a *clique partition* of $\binom{[k]}{t}$ is a collection of proper subsets $X_1, \ldots, X_m \subset [k]$ such that $\bigcup_i \binom{X_i}{t}$ is a partition of $\binom{[k]}{t}$. Let $\mathrm{cp}(k,t)$ denote the minimum size of a clique partition of $\binom{[k]}{t}$.

When $t = 2$, the classical theorem of de Bruijn and Erdős [dBE48] states that $\mathrm{cp}(k,t) = k$. In the study of (n,k,t)-saturated set systems, we are interested in the minimum clique size in any clique partition and use the following result of Alon, Mellinger, Mubayi and Verstraëte [AMMV12]:

Theorem 6.1.1 (Alon et al.[AMMV12]). *Let $t \geq 2$ and $k \in \mathbb{N}$. Define $q \in \mathbb{R}$ by $k = q^2 + q + t - 1$. Then,*

$$cp(k,t) \geq \frac{\binom{k}{t}}{\binom{q+t-1}{t}}.$$

In particular, in any such clique partition there exists X_i with $|X_i| \leq \lfloor q + t - 1 \rfloor$.

6.2 Rödl Nibble

In this section, we mention a generalization of the Rödl Nibble by Frankl and Rödl [FR85]. For a hypergraph H, let $\nu(H)$ be the maximum number of pairwise disjoint edges in H and $t(H)$ to be the minimum t for which t edges in H cover all of $V(H)$.

Theorem 6.2.1 (Frankl-Rödl [FR85]). *Suppose $\epsilon > 0$ is arbitrary, H is a d-uniform hypergraph on X, $|X| = n$, and $a > 3$ is a real number. Then there exists positive real*

δ such that if for some D one has $(1-\delta)D < d_H(v) < (1+\delta)D$ for all $x \in X$ and $d_H(\{x,y\}) < D/(\log(n)^a)$ holds for all distinct $x, y \in X$. Then for all $n > n_0(\delta)$,

$$t(H) \leq \frac{n(1+\epsilon)}{d} \text{ and } \nu(H) \geq \frac{n}{d}(1-d\epsilon).$$

Lemma 6.2.2. *Let $\mathcal{S} = \mathcal{S}(n,7,5)$. Then there exists $\mathcal{S}_1 \subset \partial_6(\mathcal{S}(n,7,5))$ for which each 4-tuple $T \in \binom{n}{4}$ is contained in at most one edge in $\mathcal{S}_1$ and $|\mathcal{S}_1| = (1-o(1))\binom{n}{4}/\binom{6}{4}$.*

Proof. Let $V(H) = \binom{[n]}{4}$ with $d = \binom{6}{4}$ and $E(H) = \{\{A \in \binom{[n]}{4} : A \subset B\} : B \in \partial_6(\mathcal{S})\}$. For each $A \in \binom{[n]}{4}$, it follows that the link of A in $\mathcal{S}$ is a 3-uniform perfect matching for which each edge contributes three to the degree of A in H and hence $d_H(A) = 3 \cdot \frac{n-4}{3} = n-4$. Moreover, for $A_1, A_2 \in \binom{[n]}{4}$, it follows that $d_H(\{A_1, A_2\}) \leq 2$ and hence we satisfy the conditions of Theorem 6.2.1. As such $\nu(H) \geq (1-o(1))\frac{|V|}{d} = (1-o(1))\binom{n}{4}/\binom{6}{4}$ which corresponds to such an $\mathcal{S}_1$ as desired. $\square$

6.3 Large Sets of Designs

Given $n \geq k \geq t$, a *large set of designs* is a partition of $\binom{[n]}{k}$ into pairwise edge disjoint Steiner systems $\mathcal{S}(n,k,t)$. It is not difficult to see that we necessarily need $\binom{k-i}{t-i} | \binom{n-i}{t-i}$ and $\binom{k}{t}^{-1}\binom{n}{t} | \binom{n}{k}$ in order for a large set of designs to exist. Completing the work of Lu [JX83], Tierlinck [Tie91] proved that large sets of designs exists when $t = 2$ and $k = 3$. Keevash [Kee18] then proved that large sets of designs exist for $t \leq k$ and n sufficiently large:

Theorem 6.3.1 (Keevash [Kee18]). *Suppose $k \geq t$ is fixed and $n \geq n_0(k,t)$ is large and so that it satisfies the above divisibility conditions. Then there exists a large set of designs.*

6.4 Proof of Theorem 1.6.3

Let $T_{n,k-1}$ denote the largest such complete $(k-1)$-partite graph on $[n]$. We will use the following stability theorem from Füredi [F15]:

Theorem 6.4.1 (Füredi [F15]). *Suppose H is an n-vertex K_k-free graph with $|E(H)| \geq |T_{n,k-1}| - s$. Then there exists a $(k-1)$-chromatic G with $V(G) = V(H)$, $E(G) \subseteq E(H)$, $|E(G)| \geq |E(H)| - s$.*

Let $k > t = 2$ and suppose $\mathcal{F} \subset \binom{[n]}{k}$ is $(n,k,2)$-saturated with

$$\binom{k}{2} \cdot |\mathcal{F}| \leq (1+\epsilon) \cdot \frac{1}{k-1}\binom{n}{2}$$

where $\epsilon = \epsilon(k)$ is some constant to be chosen later. Observe that $\partial_2(\mathcal{F})^c$ is K_k-free, and

$$|\partial_2(\mathcal{F})^c| \geq \left(1 - \frac{1}{k-1}\right)\binom{n}{2} - \frac{\epsilon}{k-1}\binom{n}{2} \geq |T_{n,k-1}| - \frac{\epsilon}{k-1}\binom{n}{2}$$

since $|\partial_2(\mathcal{F})| \leq \binom{k}{2} \cdot |\mathcal{F}|$. As such, applying Theorem 6.4.1 to $\partial_2(\mathcal{F})^c$ with $s = \epsilon\binom{n}{2}/(k-1)$ gives pairwise disjoint $X_1, \ldots, X_{k-1} \subset [n]$ and $G \subseteq K(X_1, \ldots, X_{k-1})$ with $E(G) \subseteq \partial_2(\mathcal{F})^c$ and so that $|E(G)| \geq |\partial_2(\mathcal{F})^c| - \epsilon\binom{n}{2}/(k-1)$.

Definition. *An edge $A \in \mathcal{F}$ is called good if $A \subset X_i$ for some $i \in [k-1]$ and $d_{\mathcal{F}}(e) = 1$ for all $e \in \binom{A}{2}$. Otherwise, we refer to A as a bad edge. Set $\mathcal{B} \subset \mathcal{F}$ as the collection of bad edges.*

Seeking to show there are few bad edges in $\mathcal{F}$, we further consider $\mathcal{B}_1 = \{A \in \mathcal{B} : \exists\, i \neq j \text{ with } A \cap X_i \neq \emptyset, A \cap X_j \neq \emptyset\}$ and $\mathcal{B}_2 = \{A \in \mathcal{B} : \exists\, i \text{ with } A \subset X_i\}$.

Claim. $|\mathcal{B}| = |\mathcal{B}_1| + |\mathcal{B}_2| \leq \frac{4\epsilon}{3(k-1)}\binom{n}{2}$.

Proof. Let $\mathcal{F}_i = \mathcal{F}|_{X_i}$ be the trace of $\mathcal{F}$ on X_i so that $|\mathcal{B}_2| = |\mathcal{F}_1 \cap \mathcal{B}_2| + \cdots + |\mathcal{F}_{k-1} \cap \mathcal{B}_2|$. Then

$$\binom{k}{2} \cdot (|\mathcal{F}_1| + \cdots + |\mathcal{F}_{k-1}| + |\mathcal{B}_1|) = \binom{k}{2}|\mathcal{F}| \leq (1+\epsilon) \cdot \frac{1}{k-1}\binom{n}{2} \tag{6.1}$$

and for each $i \in [k-1]$ it follows that

$$\binom{k}{2}|\mathcal{F}_i| = \sum_{e \in \partial_2(\mathcal{F}_i)} d_{\mathcal{F}_i}(e) = |\partial_2(\mathcal{F}_i)| + \sum_{e \in \partial_2(\mathcal{F}_i)} (d_{\mathcal{F}_i}(e) - 1). \tag{6.2}$$

Since $\mathcal{F}$ is $(n,k,2)$-saturated, for each $A \in \mathcal{B}_2 \cap \mathcal{F}_i$ there exists $A \neq B \in \mathcal{B}_2 \cap \mathcal{F}_i$ which contribute at least three to the sum on the right hand side of (6.2) as $|A \cap B| \geq 3$. Therefore,

$$\sum_{e \in \partial_2(\mathcal{F}_i)} (d_{\mathcal{F}_i}(e) - 1) \geq \frac{3}{2} \cdot |\mathcal{F}_i \cap \mathcal{B}_2|. \tag{6.3}$$

Note that each edge $A \in \mathcal{B}_1$ intersects at most $\binom{k-1}{2}$ edges in $K(X_1, \ldots, X_{k-1})^c$ and moreover that $|\partial_2(\mathcal{F})^c \cap K(X_1, \ldots, X_{k-1})^c| \leq \epsilon\binom{n}{2}/(k-1)$ since each edge in the intersection needs to be removed from $\partial_2(\mathcal{F})^c$ in order to create G. It thus follows that

$$|\partial_2(\mathcal{F}_1)| + \cdots + |\partial_2(\mathcal{F}_{k-1})| + \binom{k-1}{2}|\mathcal{B}_1| + \frac{\epsilon}{k-1}\binom{n}{2} \geq |K(X_1, \ldots, X_{k-1})^c| \geq |T^c_{n,k-1}|. \tag{6.4}$$

As a result, using (6.2), (6.3) and (6.4), we recover

$$\binom{k}{2}\left(|\mathcal{F}_1| + \cdots + |\mathcal{F}_{k-1}| + |\mathcal{B}_1|\right) \geq \frac{1}{k-1}\binom{n}{2} - \frac{\epsilon}{k-1}\binom{n}{2} + (k-1)|\mathcal{B}_1| + \frac{3}{2}|\mathcal{B}_2|.$$

Hence, using (6.1) it follows that

$$(1+\epsilon)\cdot\frac{1}{k-1}\binom{n}{2} \geq \frac{1}{k-1}\binom{n}{2} - \frac{\epsilon}{k-1}\binom{n}{2} + (k-1)|\mathcal{B}_1| + \frac{3}{2}|\mathcal{B}_2|.$$

We then recover the desired bound by noting $k \geq 3$ and rearranging to get

$$\frac{2\epsilon}{k-1}\binom{n}{2} \geq (k-1)|\mathcal{B}_1| + \frac{3}{2}|\mathcal{B}_2| \geq \frac{3}{2}\left(|\mathcal{B}_1| + |\mathcal{B}_2|\right).$$

□

Lemma 6.4.2. *Let $A \in \mathcal{F}$ be a good edge and $A' \subset A \subset X_i$ with $|A'| = k-1$ and $v \in [n] \setminus X_i$. Then there exists $u \in A'$ with $\{u, v\} \in \partial_2(\mathcal{F})$.*

Proof. It follows that $A' \cup \{v\} \notin \mathcal{F}$ as A is a good edge. As such, there must be a set $B \in \mathcal{F}$ which intersects $A' \cup \{v\}$ in exactly two places. Using that A is good, we recover the desired claim. □

Let $e(X_i, X_j)$ be the number of edges in $\partial_2(\mathcal{F})$ with one vertex in X_i and the other vertex in X_j. Using Lemma 6.4.2, for each good edge $A \in \mathcal{F}$ and vertex v from another block, there are at least two edges between v and a vertex from A in $\partial_2(\mathcal{F})$. Note that $\mathcal{F}_i \cap (\mathcal{F} \setminus \mathcal{B})$ forms a partial Steiner system and hence vertices have degree at most $|X_i| - 1$. An optimization argument then gives

$$\sum_{\{i,j\} \in \binom{[k]}{2}} e(X_i, X_j) \geq \frac{(|\mathcal{F}| - |\mathcal{B}|)\cdot 2\frac{(k-2)n}{r-1}}{\frac{n}{k-1}} \geq 2(k-2)(|\mathcal{F}| - |\mathcal{B}|).$$

We reach a contradiction if $2(k-2)(|\mathcal{F}|-|\mathcal{B}|) > \epsilon\binom{n}{2}/(k-1)$ which is equivalent to

$$2(k-2)\left(\frac{2(1+\epsilon)}{k(k-1)} - \frac{4\epsilon}{3}\right) > \epsilon.$$

It then follows that taking $\epsilon = k^{-2}$ gives a contradiction.

6.5 Proof of Theorem 1.6.4

In this section, we show that at least one of the following set families is (n, k, t)-saturated.

Construction 1. *Let $n \geq k > t \geq 1$ be integers and consider $q \in \mathbb{R}$ where $k = q^2+q+t-1$. For $0 \leq r \leq \lfloor q-1 \rfloor$, consider $\partial_k(\mathcal{S}(n, k+r, t))$.*

Claim. *If $\partial_k(\mathcal{S}(n, k+r, t))$ is not (n, k, t)-saturated, then there exists a clique partition of $\binom{[k]}{t}$ into cliques of size at least $t+r+1$.*

Proof. Suppose $\partial_k(\mathcal{S}(n, k+r, t))$ is not (n, k, t)-saturated, then there exists an $R \in \binom{[n]}{k}$ which may be added without creating an intersection of size t. However, since each $T \in \binom{[n]}{t}$ is covered by an edge of $\mathcal{S}(n, k+r, t)$, it follows that the trace $\partial_k(\mathcal{S}(n, k+r, t))|_R$ is a clique partition of $\binom{R}{t}$. Suppose there exists a set X_i in the clique partition of $\binom{R}{t}$ with $|X_i| \leq t+r$. This means that there exists $A \in \mathcal{S}(n, k+r, t))$ with $|A \cap R| \leq r+t$. Then, by appropriately choosing a subset $A'' \subset A$ of size k, we recover an intersection of size t; a contradiction. $\square$

By Theorem 6.1.1, each clique partition of $\binom{[k]}{t}$ into proper subcliques has a clique of size at most $t + \lfloor q-1 \rfloor$ and hence the above claim gives that at least one of the set families in Construction 1 is (n,k,t)-saturated which then implies Theorem 1.6.4 by taking the largest such family.

6.6 Proof of Theorem 1.6.5

In this section, we construct a $(n,5,4)$-saturated set system which gives $w(n,5,4) \leq 5/8$:

Construction 2. *Let* $n = 0 \pmod{12}$ *and consider pairwise disjoint* X *and* Y *with* $|X| = n-3$ *and* $|Y| = n$ *and further let* $X = S_X \sqcup X'$ *and* $Y = S_Y \sqcup Y'$ *with* $|S_X| = 9$ *and* $|S_Y| = 15$. *For convenience, let* $S = S_X \sqcup S_Y$. *By Theorem 6.3.1, we can decompose* $\binom{X'}{4}$ *and* $\binom{Y}{4}$ *as follows:*

$$\binom{X'}{4} = \bigsqcup_{i \in [n-15]} \mathcal{S}_i(X',4,3) \quad \textit{and} \quad \binom{Y}{4} = \bigsqcup_{i \in [n-3]} \mathcal{S}_i(Y,4,3).$$

Letting $X = \{x_1, \ldots, x_{n-3}\}$ *and* $Y' = \{y_1, \ldots, y_{n-15}\}$, *we now define* $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ *where*

$$\begin{aligned}
\mathcal{F}_1 &= \{\{y_i\} \cup T : T \in \mathcal{S}_i(X',4,3)\ , i\ \in [n-15]\} \\
\mathcal{F}_2 &= \{\{x_i\} \cup T : T \in \mathcal{S}_i(Y,4,3)\ ,\ i \in [n-3]\}.
\end{aligned}$$

Claim. *There exists* $(2n-3,5,4)$*-saturated* $\mathcal{F}' \subset \binom{[2n-3]}{5}$ *with* $\mathcal{F} \subset \mathcal{F}'$ *and* $|\mathcal{F}'| = |\mathcal{F}| + O(n^3)$.

Proof. As $\mathcal{F}$ is not $(2n-3,5,4)$-saturated, we greedily add edges until our new set system $\mathcal{F}'$ is $(2n-3,5,4)$-saturated. It then suffices to show $|\mathcal{F}'| = |\mathcal{F}| + O(n^3)$. First, it follows that we may not add any A with $A \cap S = \emptyset$ by considering the cases of $|A \cap X'|$ and $|A \cap Y|$. Next, there are $\Theta(n^3)$ edges with $|A \cap S| \geq 2$, and so it suffices to consider the case where $|A \cap S| = 1$. It then follows that the only edges we may add are so that $|A \cap S| = 1$, $|A \cap X'| = 2$, and $|A \cap Y'| = 2$. For $s \in S$ and $e \in \binom{X'}{2}$, let $G_{s,e} = \{f \in \binom{Y'}{2} : \{s\} \cup e \cup f \in \mathcal{F}'\}$ be the link graph of $\{s\} \cup e$ in $\mathcal{F}'$. Since $\mathcal{F}'$ does not contain an intersection of size four, $G_{s,e}$ is necessarily a graph matching so

$$|\mathcal{F}' \setminus \mathcal{F}| = \sum_{s \in S} \sum_{e \in \binom{X'}{2}} |G_{s,e}| \leq |S| \cdot \binom{n}{2} \cdot \frac{n}{2} = \Theta(n^3). \qquad \square$$

When $n = i \pmod{12}$, letting $\mathcal{F} \subset \mathcal{F}' \subset \binom{[n-i]}{4}$ be as in Construction 2 and the above claim, and take $\mathcal{F}'' \subset \binom{[n]}{5}$ to be an $(n,5,4)$-saturated set system which contains $\mathcal{F}'$. Setting $S = [n-i+1, n]$, there are $O(n^3)$ edges in $\mathcal{F}''$ with $|A \cap S| \geq 2$ and for each $s \in S$, it follows that $|\mathcal{F}(\{s\})| = O(n^3)$ as it is $(n-i,4,3)$-saturated by the result of Frankl-Füredi [FF87].

6.7 Proof of Theorem 1.6.6

In this section, we construct a $(n,7,5)$-saturated set system which gives $w(n,7,5) \leq 1/2$:

Construction 3. *Let X_1 and X_2 be a balanced partition of $[n]$. Set $\mathcal{F}_0 = \mathcal{S}(X_1,7,5) \sqcup \mathcal{S}(X_2,7,5)$. Let $\mathcal{S}_1 \subset \partial_6(\mathcal{S}(X_1,7,5))$ and $\mathcal{S}_2 \subset \partial_6(\mathcal{S}(X_2,7,5))$ be as in Lemma 6.2.2.*

Then define $\mathcal{F}_1$ by

$$\mathcal{F}_1 := \{\{x\} \cup A : x \in X_1; B \in \mathcal{S}_2\} \sqcup \{\{x\} \cup A : x \in X_2; B \in \mathcal{S}_1\}.$$

Claim. *There exists a $(n,7,5)$-saturated $\mathcal{F} \supset \mathcal{F}_0 \cup \mathcal{F}_1$ with $|\mathcal{F}| \leq |\mathcal{F}_0| + |\mathcal{F}_1| + o(n^5)$.*

Proof. As $\mathcal{F}_0 \cup \mathcal{F}_1$ is not $(n,7,5)$-saturated, we greedily add edges until our new set system $\mathcal{F}$ is $(n,7,5)$-saturated. The only sets $A \in \binom{[n]}{7}$ which we could have added to $\mathcal{F}_0 \cup \mathcal{F}_1$ are so that $|A \cap X_1| \in \{1,3,4,6\}$. It suffices to show that $|\mathcal{F}| \leq |\mathcal{F}_0| + |\mathcal{F}_1| + o(n^5)$.

By symmetry, it suffices to consider the cases where $|A \cap X_1| = 1$ and $|A \cap X_1| = 4$. Fix $x \in X_1$, then $\mathcal{F} \setminus (\mathcal{F}_0 \cup \mathcal{F}_1)(x) = o(n^4)$ by the construction of $\mathcal{S}_2$. Hence, the number of edges with $|A \cap X_1| = 1$ we can add to $\mathcal{F}_0 \cup \mathcal{F}_1$ is $o(n^5)$. We now consider edges of the form $|A \cap X_1| = 4$. If $A \cap X_1 \in \partial_4(\mathcal{S}_1)$, then it is not hard to see that there exists a $B \in \mathcal{F}_1$ with $|A \cap B| = 5$. If $A \cap X_1 \notin \partial_4(\mathcal{S}_2)$, then consider the link of A in $\mathcal{F}$. This is a 3-uniform set system which does not contain an intersection of size one. By oddtown rules, $|\mathcal{F}(A)| \leq n$ and as there are $o(n^4)$ sets $A' \notin \partial_4(\mathcal{S}_1)$, it follows that there are at most $o(n^5)$ edges of the form $|A \cap X_1| = 4$ we can add to $\mathcal{F}_0 \cup \mathcal{F}_1$. $\square$

For other large values of n, we may argue similarly as in the end of Section 6.6 to get the desired upper bound for $w(n,7,5)$. This construction can also be adapted when $(k,t) = (6,4)$ which gives $w^\star(n,6,4) \leq 3/4$.

This chapter contains material from: J. O'Neill, J. Verstraëte, "A note on intersection saturation", which is currently being prepared for submission for publication. The dissertation author was one of the primary investigators and authors of this paper.

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