Title

The Borel complexity of the class of models of first-order theories

Journal

Proceedings of the American Mathematical Society, 153(09)

ISSN

0002-9939

Authors

Andrews, Uri
Gonzalez, David
Lempp, Steffen
et al.

Publication Date

2025

DOI

10.1090/proc/17308

THE BOREL COMPLEXITY OF THE CLASS OF MODELS OF FIRST-ORDER THEORIES

URI ANDREWS, DAVID GONZALEZ, STEFFEN LEMPP, DINO ROSSEGGER, AND HONGYU ZHU

Abstract. We investigate the descriptive complexity of the set of models of first-order theories. Using classical results of Knight and Solovay, we give a sharp condition for complete theories to have a $\mathbf{\Pi}^0_\omega$-complete set of models. In particular, any sequential theory (a class of foundational theories defined by Pudlák, Enayat, and Visser) has a $\mathbf{\Pi}^0_\omega$-complete set of models. We also give sharp conditions for theories to have a $\mathbf{\Pi}^0_n$-complete set of models.

1. Introduction

We characterize the possible Borel complexities of the set of models of a first-order theory. For a single formula φ, Wadge [Wa83, I.F.3 and I.F.4], using a result by Keisler [Ke65], showed that if φ is an $\exists_n$-formula which is not equivalent to a $\forall_n$-formula, then the set of models of φ is a $\mathbf{\Sigma}^0_n$-complete set under Wadge reduction. We extend this result to considering (possibly incomplete) first-order theories T and giving conditions on T determining the complexity of $\mathrm{Mod}(T)$, the set of models of T.

We show that a complete theory T has no $\forall_n$-axiomatization for any finite n if and only if $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_\omega$-complete. Prior to this result, showing that $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_\omega$-complete was difficult even for familiar theories, e.g., Rossegger [Ro20] asked this for the theory TA of true arithmetic. We also show that for any finite n, a (possibly incomplete) theory T has a $\forall_n$-axiomatization if and only if $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_n$. If T does not have a $\forall_n$-axiomatization, then $\mathrm{Mod}(T)$ is $\mathbf{\Sigma}^0_n$-hard.

By Vaught's proof [Va74] of the Lopez-Escobar theorem, showing that the set of models of T is $\mathbf{\Sigma}^0_n$ (or $\mathbf{\Pi}^0_n$) is equivalent to showing that T is equivalent to a Σ^{in}_n-formula (or Π^{in}_n-formula, respectively). Also, Wadge's lemma shows that if the set of models of T is $\mathbf{\Sigma}^0_n$ and not $\mathbf{\Pi}^0_n$, then it must be $\mathbf{\Sigma}^0_n$-complete. Thus, an equivalent way to present our main results is in terms of when a first-order theory T

2020 *Mathematics Subject Classification.* 03C62,03C52,03E15.

Key words and phrases. models of arithmetic, Borel complexity, countable models, descriptive set theory.

The first author acknowledges support from the National Science Foundation under Grant No. DMS-2348792. The third author's research was partially supported by AMS-Simons Foundation Collaboration Grant 626304. The fourth author's work was supported by the European Union's Horizon 2020 Research and Innovation Programme under the Marie Skłodowska-Curie grant agreement No. 101026834 ACOSE and the Austrian Science Fund (FWF) 10.55776/P36781. Part of this research was carried out while the third author was visiting Technical University of Vienna as a guest professor, and he would like to express his gratitude to this university. – The authors would like to express their gratitude to Roman Kossak for connecting them with Ali Enayat and Albert Visser, who were able to establish complementary results to ours on sequential theories.

is equivalent to a formula in $L_{\omega_1\omega}$. For example, it follows that a first-order theory is equivalent to a Π^{in}_n-sentence if and only if it has a $\forall_n$-axiomatization.

This is related to Keisler's result [Ke65, Corollary 3.4] that was recently reproved by Harrison-Trainor and Kretschmer [HK23]. If a formula in the infinitary logic $L_{\infty\omega}$ is Π^{in}_n and equivalent to a finitary formula, then it is equivalent to a finitary $\forall_n$-formula. In fact, our result, when applied to a single formula, implies the restricted version of this result for $L_{\omega_1\omega}$ formulas via an easy application of compactness. Keisler's result can be interpreted as showing that, though infinitary logic can express much more than finitary logic, it cannot express things more efficiently, i.e., in fewer quantifier alternations, than finitary logic.

Interestingly, all three proofs are quite different. Keisler used games and saturated models, Harrison-Trainor and Kretschmer used arithmetical forcing, and we use iterated priority constructions. One advantage of our technique is that, while combinatorially quite complicated, the metamathematics involved is quite tame. This suggests that our results are a consequence of, if not equivalent to, compactness. By contrast, Wadge's result that $\mathrm{Mod}(\varphi)$ is $\boldsymbol{\Sigma}^0_n$-complete if φ is $\exists_n$ and not equivalent to a $\forall_n$-formula relies upon Borel Wadge determinacy. Louveau and Saint-Raymond [LS87] showed that Borel Wadge determinacy can be proven in second-order arithmetic. However, all proofs known to date need strong fragments [DGHTa]. The second and major advantage is that we consider theories, not simply formulas.

2. Preliminaries

Given a Polish space X, the Borel hierarchy on X gives us a way to stratify subsets of X in terms of their descriptive complexity. A natural space is the space of countably infinite structures in a countable relational vocabulary τ, which we can view as a closed subspace of 2^ω as follows. Fix an enumeration of the atomic τ-formulas $(\varphi_i(x_0,\dots,x_i))_{i\in\omega}$; then given a τ-structure $\mathcal{A}$ with universe ω, define its *atomic diagram* by

$$D(\mathcal{A})(i) = \begin{cases} 1 & \mathcal{A} \models \varphi_i[x_j \mapsto j : j \leq i], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathrm{Mod}(\tau) \subseteq 2^\omega$ be the set of atomic diagrams of τ-structures with universe ω. Then it is easy to see that $\mathrm{Mod}(\tau)$ is a closed subset of Cantor space and thus a Polish space via the subspace topology.

For a first-order theory T, $\mathrm{Mod}(T) = \{D(\mathcal{A}) : \mathcal{A} \models T\}$ is a canonical subset of $\mathrm{Mod}(\tau)$, and it is natural to ask how complex $\mathrm{Mod}(T)$ is in terms of its Borel complexity. It is not hard to see that $\mathrm{Mod}(T)$ can be at most $\boldsymbol{\Pi}^0_\omega$: It follows from Vaught's proof [Va74] of the Lopez-Escobar theorem [Lo65] that an isomorphism-invariant subset of $\mathrm{Mod}(\tau)$ is $\boldsymbol{\Pi}^0_\alpha$ if and only if it is definable by a Π^{in}_α-formula in the infinitary logic $L_{\omega_1\omega}$ for $\alpha < \omega_1$. Note that we use the notation Π^{in}_α to refer to formulas in $L_{\omega_1\omega}$ at that complexity, and we use $\forall_n$ to refer to $L_{\omega\omega}$-formulas with n alternating quantifier blocks beginning with $\forall$. Since $\mathrm{Mod}(T)$ is the set of models of the infinitary formula $\bigwedge\!\!\bigwedge_{\varphi\in T}\varphi$, and every φ is $\exists_n$ for some $n \in \omega$, we get that $\mathrm{Mod}(T)$ is at most $\boldsymbol{\Pi}^0_\omega$.

However, for a fixed theory T, it turns out to be quite difficult to establish that $\mathrm{Mod}(T)$ is not simpler. The main theorem of this paper establishes a complete characterization of first-order theories T such that $\mathrm{Mod}(T)$ is $\boldsymbol{\Pi}^0_\omega$-complete. To establish this notion of completeness, we use Wadge reducibility; a subset X_1 of a Polish space Y_1 is *Wadge reducible* a subset X_2 of a Polish space Y_2 (denoted as $X_1 \leq_W X_2$) if there is a continuous function $f : Y_1 \to Y_2$ such that for all $y \in Y_1$, $y \in X_1$ if and only if $f(y) \in X_2$. A set $X \subseteq Y$ is Γ-*hard* for a pointclass Γ if for every $Z \in \Gamma$, $Z \leq_W X$, and X is Γ-*complete* if $X \in \Gamma$ and X is Γ-hard. Wadge [Wa83] showed that there is only one *Wadge degree* (equivalence class induced by $\leq_W$) of a $\mathbf{\Sigma}^0_\alpha$-set which is not $\mathbf{\Pi}^0_\alpha$ and vice versa. We refer the reader to Kechris [Ke95] for more on Wadge reducibility and a thorough introduction to descriptive set theory.

At last, we mention that we may assume that theories are in the language of graphs. It is well-known that every structure is interpretable in a graph, preserving most model-theoretic properties. For example, in [Ro22, Section 3.2], for a given relational vocabulary τ, a continuous function $g : \mathrm{Mod}(\tau) \to \mathrm{Mod}(\mathrm{Graphs})$ was given such that for all τ-structures, $\mathcal{A}$ embeds into $\mathcal{B}$ if and only if $g(\mathcal{A})$ elementarily embeds into $g(\mathcal{B})$. Easy modifications to the proofs there show that $\mathcal{A} \equiv \mathcal{B}$ if and only if $g(\mathcal{A}) \equiv g(\mathcal{B})$, and thus for any τ-structure $\mathcal{A}$, $\mathrm{Mod}(\mathrm{Th}(\mathcal{A})) \equiv_W \mathrm{Mod}(\mathrm{Th}(g(\mathcal{A}))$. The same reduction was given in a different context in [AM15, Proposition 2], the proof there also shows that $\mathrm{Mod}(\mathrm{Th}(\mathcal{A})) \equiv_W \mathrm{Mod}(\mathrm{Th}(g(\mathcal{A})))$.

3. Theories without bounded axiomatization

Definition 3.1. A first-order theory T is *boundedly axiomatizable* if there is some n such that T has a $\forall_n$-axiomatization.

Our main result for theories that are not boundedly axiomatizable is the following

Theorem 3.2. *A complete first-order theory T has a $\mathbf{\Pi}^0_\omega$-complete set of models if and only if T is not boundedly axiomatizable.*

In fact, Theorem 3.2 follows directly from the following more technical fact.

Theorem 3.3. *Let T be any complete first-order theory for which there is a collection of complete theories $\{T_n\}_{n\in\omega}$ such that for all $n \in \omega$, $T \neq T_n$ but $T \cap \exists_n = T_n \cap \exists_n$. Then, the collection of models of T is $\mathbf{\Pi}^0_\omega$-complete. Indeed, for each $\mathbf{\Pi}^0_\omega$-set P, there is a continuous function mapping any $p \in P$ to a model of T, and any $p \notin P$ to a model satisfying T_n for some n.*

We show in the next example the necessity of the assumption of completeness for the theory T in Theorem 3.2.

Example 3.4. Let $\mathcal{L}_k$ be disjoint relational languages, and for each k, let φ_k be an $\mathcal{L}_k$-sentence which is $\exists_k$ and not equivalent to any $\forall_k$-sentence. Let $\mathcal{L} = \bigcup_k \mathcal{L}_k \cup \{R_i \mid i \in \omega\}$ where each R_i is a new unary relation. Let T say that the set of realizations of each R_i is disjoint, and at most one is non-empty. Furthermore, let T say that any relation from $\mathcal{L}_k$ can only hold on tuples from the set of realizations of R_k. Let T further say that if R_k is non-empty, then φ_k holds. It is direct to see that T has no $\forall_n$-axiomatization for any n, and yet $\mathrm{Mod}(T)$ is $\mathbf{\Sigma}^0_\omega$.

We defer the proof of Theorem 3.3 to Section 5. Here we state some corollaries of Theorem 3.2:

Corollary 3.5. *For any completion T of Peano arithmetic* PA*, in particular for true arithmetic, the set of models of T is $\mathbf{\Pi}^0_\omega$-complete.* □

This follows from an observation of Rabin [Ra61] (which he suspects to have been known before) that no consistent extension of PA can be boundedly axiomatizable.

While Example 3.4 shows that Theorem 3.2 cannot be generalized to hold for incomplete theories, for many incomplete theories, one can use Theorem 3.3 to get a similar result. One simply has to find a suitable completion T and suitable theories T_n. One example of such a theory is PA.

Corollary 3.6. *Peano arithmetic has a $\mathbf{\Pi}^0_\omega$-complete set of models.*

Proof. Let $T = \mathrm{TA}$, and let T_n be a consistent completion of $\mathrm{TA} \cap \exists_n$, where $\mathsf{B}\Sigma^0_n$, the bounding principle for Σ^0_n-formulas, fails. That such T_n exists for every n follows from a result by Parsons [Pa70], see also [Ka91, Theorem 10.4]. Using Theorem 3.3 with this T and $(T_n)_{n\in\omega}$, we get that Mod(PA) is $\mathbf{\Pi}^0_\omega$-complete. □

Answering our question about other theories of arithmetic, Enayat and Visser [EVta] showed that no complete sequential theory can be boundedly axiomatizable. The sequential theories were first defined by Pudlák [Pu83] and rephrased by Pakhomov and Visser [PV22] as follows:

Definition 3.7. Given a theory T, we denote by $\mathrm{AS}(T)$ (*adjunctive set theory*) the extension of T by a new binary relation symbol $\in$ and the axioms

- AS1: $\exists x\, \forall y\, (y \notin x)$, and
- AS2: $\forall x\, \forall y\, \exists z\, \forall u\, (u \in z \leftrightarrow (u \in x \vee u = y))$.

A theory is *sequential* if it allows a definitional extension to $\mathrm{AS}(T)$.

Note here that Adjunctive Set Theory does not even require Extensionality. Pudlák's original definition was in terms of being able to define Gödel's β-function, which then allows for a weak coding of sequences. (Note that any extension of a sequential theory is again sequential.) Examples of sequential theories include PA^- (by Jeřábek [Je12]) and essentially all versions of set theory, but not Robinson's Q (by Visser [Vi17]). Thus, Enayat and Visser's result [EVta] yields that $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_\omega$-complete for essentially any "foundational" complete theory, in particular, any completion of PA^-.

Finally, note that our reduction in Theorem 3.3 produces models of different theories T_n for different x in the $\mathbf{\Sigma}^0_\omega$-outcome, depending on how we witness that $x \notin P$. We note that it is necessary to use infinitely many theories in the $\mathbf{\Sigma}^0_\omega$-outcome, since the union of $\mathrm{Mod}(T_i)$ for finitely many theories T_i is always $\mathbf{\Pi}^0_\omega$, and the complement of P could be $\mathbf{\Sigma}^0_\omega$-hard.

4. Theories with bounded axiomatization

In this section, we will present results on the Wadge degrees of models of first-order theories with bounded axiomatization via the quantifier complexity of their axiomatizations. Our proofs will rely on the following lemma that will rely on theorems of Knight and Solovay. We delay its proof to Section 5.

Lemma 4.1. *Suppose $n \geq 1$ and T^+ and T^- are distinct complete theories such that $T^- \cap \exists_n \subseteq T^+ \cap \exists_n$. Then, for any $P \in \mathbf{\Sigma}^0_n$, there is a Wadge reduction f such that $f(p) \in \mathrm{Mod}(T^+)$ if $p \in P$, and $f(p) \in \mathrm{Mod}(T^-)$ otherwise. In particular, $\mathrm{Mod}(T^+)$ is $\mathbf{\Sigma}^0_n$-hard, and $\mathrm{Mod}(T^-)$ is $\mathbf{\Pi}^0_n$-hard.*

To apply Lemma 4.1 to incomplete theories, we use the following Lemma which allows us to find completions satisfying the hypotheses of Lemma 4.1.

Definition 4.2. A *level-sentence set* for $\mathcal{L}$ is either the set of $\exists_n$- or the set of $\forall_n$-sentences in $\mathcal{L}$ for some n.

For a level-sentence set Λ, we let $\neg\Lambda$ be the set of sentences equivalent to the negation of a sentence in Λ.

Definition 4.3. For a (possibly incomplete) theory T, and a level-sentence set Λ, we let $\mathrm{Th}_\Lambda(T)$ denote $\Lambda \cap \overline{T}$, where $\overline{T}$ is the deductive closure of T.

Lemma 4.4. *Let Λ be a level-sentence set for $\mathcal{L}$. Let A be a set of finitary sentences and φ a finitary sentence such that $A \not\vdash \varphi \leftrightarrow \psi$ for any $\psi \in \neg\Lambda$. Then there are complete consistent theories $T^+ \supseteq A \cup \{\varphi\}$ and $T^- \supseteq A \cup \{\neg\varphi\}$ such that $\Lambda \cap T^- \subseteq \Lambda \cap T^+$. Furthermore, if T is any theory consistent with $A \cup \{\varphi\} \cup \mathrm{Th}_\Lambda(A \cup \{\neg\varphi\})$, then T^+ can be chosen to contain T.*

Proof. The lemma follows from the following two claims that allow us to choose such T^+ and T^-.

Claim 4.5. The theory $A \cup \{\varphi\} \cup \mathrm{Th}_\Lambda(A \cup \{\neg\varphi\})$ is consistent.

Proof. Suppose that $A \cup \{\varphi\} \cup \mathrm{Th}_\Lambda(A \cup \{\neg\varphi\})$ is inconsistent. By compactness, there is $\psi \in \mathrm{Th}_\Lambda(A \cup \{\neg\varphi\})$ such that $A \cup \{\varphi\} \vdash \neg\psi$. But then $A \vdash \varphi \leftrightarrow \neg\psi$ as well as $\neg\psi \in \neg\Lambda$, a contradiction. □

Now, choose T^+ to be any complete consistent extension of $A \cup \{\varphi\} \cup \mathrm{Th}_\Lambda(A \cup \{\neg\varphi\})$. Observe that if T is any theory consistent with $A \cup \{\varphi\} \cup \mathrm{Th}_\Lambda(A \cup \{\neg\varphi\})$, then T^+ can be chosen to contain T.

Claim 4.6. The theory $(\neg\Lambda \cap T^+) \cup A \cup \{\neg\varphi\}$ is consistent.

Proof. Suppose not, then by compactness, there is $\psi \in \neg\Lambda \cap T^+$ such that $A \cup \{\neg\varphi\} \vdash \neg\psi$. But then $\neg\psi \in \Lambda$ and $A \cup \{\neg\varphi\} \vdash \neg\psi$, so $\neg\psi \in T^+$, contradicting that T^+ is consistent. □

Let T^- be a completion of $(\neg\Lambda \cap T^+) \cup A \cup \{\neg\varphi\}$. Observe that T^- and T^+ satisfy the lemma. □

Corollary 4.7. *Let Λ be a level-sentence set, and let T be a theory which is not Λ-axiomatizable (i.e., $\mathrm{Th}_\Lambda(T)$ does not imply all of T). Then there are complete theories T_0, T_1 such that $T \subseteq T_0$, T is inconsistent with T_1, and $\Lambda \cap T_0 \subseteq \Lambda \cap T_1$.*

Proof. Let $A = \mathrm{Th}_\Lambda(T)$, and let $\varphi \in T$ be such that $A \not\vdash \varphi$. Observe that $A \not\vdash \varphi \leftrightarrow \psi$ for any $\psi \in \Lambda$, since otherwise ψ would be in $\mathrm{Th}_\Lambda(T) = A$, contradicting $A \not\vdash \varphi$.

Observe also that $T \cup \mathrm{Th}_{\neg\Lambda}(A \cup \{\neg\varphi\})$ is consistent. Otherwise, there would be a formula $\psi \in \Lambda$ such that $T \vdash \psi$, thus $\psi \in A$, and $A \cup \{\neg\varphi\} \models \neg\psi$. But then $A \vdash \varphi$, which is a contradiction. So, we can apply Lemma 4.4 to the triple $\neg\Lambda, A, \varphi$ to get two complete theories $T^- \supseteq A \cup \{\neg\varphi\}$ and $T^+ \supseteq T$ with $\neg\Lambda \cap T^- \subseteq \neg\Lambda \cap T^+$. Finally, let $T_0 = T^+$ and $T_1 = T^-$. □

Lemma 4.8. *Let T be a theory without a $\forall_n$-axiomatization. Then* $\mathrm{Mod}(T)$ *is* $\mathbf{\Sigma}^0_n$*-hard.*

Proof. By Corollary 4.7, we have complete theories $T_0 \supseteq T$ and T_1 inconsistent with T such that $\forall_n \cap T_0 \subseteq \forall_n \cap T_1$. Thus $\exists_n \cap T_1 \subseteq \exists_n \cap T_0$, and applying Lemma 4.1 shows that $\mathrm{Mod}(T)$ is $\mathbf{\Sigma}^0_n$-hard. □

Lemma 4.9. *Let T be a theory without an $\exists_n$-axiomatization. Then* $\mathrm{Mod}(T)$ *is* $\mathbf{\Pi}^0_n$*-hard.*

Proof. By Corollary 4.7, we have complete theories $T_0 \supseteq T$ and T_1 inconsistent with T such that $\exists_n \cap T_0 \subseteq \exists_n \cap T_1$, and applying Lemma 4.1 shows that $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_n$-hard. □

Theorem 4.10. *Let T be a theory and $n \in \omega$. Then* $\mathrm{Mod}(T) \in \mathbf{\Pi}^0_n$ *if and only if T is $\forall_n$-axiomatizable.*

Proof. If $\mathrm{Mod}(T) \in \mathbf{\Pi}^0_n$, then it is not $\mathbf{\Sigma}^0_n$-hard. So, by Lemma 4.8, it must have a $\forall_n$-axiomatization. On the other hand, if T is $\forall_n$-axiomatizable, then $\mathrm{Mod}(T) \in \mathbf{\Pi}^0_n$, as the infinitary conjunction over all sentences in the axiomatization is Π^{in}_n. □

For $\exists_n$-axiomatizable theories, the situation is not as simple as the one for $\forall_n$-axiomatizable theories seen in Theorem 4.10. If A is a $\exists_n$-axiomatization of T, then $\mathrm{Mod}(T) = \mathrm{Mod}(\psi)$, where $\psi = \bigwedge\!\!\!\bigwedge_{\varphi \in A} \varphi$. However, if A is not a finite axiomatization, then ψ is not $\Sigma^{\mathrm{in}}_\alpha$, but rather $\Pi^{\mathrm{in}}_{\alpha+1}$. Combining this with the contrapositive of Lemma 4.9, we obtain the following

Proposition 4.11. *Let T be a theory and $n \in \omega$. If* $\mathrm{Mod}(T) \in \mathbf{\Sigma}^0_n$*, then T is $\exists_n$-axiomatizable. On the other hand, if T is $\exists_n$-axiomatizable, then* $\mathrm{Mod}(T) \in \mathbf{\Pi}^0_{n+1}$. □

We now give examples of $\exists_n$-axiomatizable theories of different Wadge degrees showing that the bounds in Proposition 4.11 cannot be improved.

Example 4.12. For $k \leq 2$, there are $\exists_k$-axiomatizable $\aleph_0$-categorical theories with a $\mathbf{\Pi}^0_{k+1}$-complete set of models.

Proof. For $k = 1$, let τ be the signature consisting of one unary relation symbol P, and let T be the theory saying that P is infinite and coinfinite. T is easily seen to be $\exists_1$-axiomatizable, $\aleph_0$-categorical, and $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_2$-complete.

For $k = 2$, let τ be the signature consisting of a single binary relation symbol R. Let T say that R is irreflexive, symmetric and for every x, there is at most one y such that $R(x, y)$. Finally, let T say that there are infinitely many x satisfying $\exists y R(x, y)$ and infinitely many x satisfying $\neg\exists y R(x, y)$. Then T is $\aleph_0$-categorical and $\exists_2$-axiomatizable. To show that $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_3$-complete we will give a Wadge reduction from the $\mathbf{\Pi}^0_3$-complete subset P of $2^{\omega\times\omega}$ consisting of all elements having infinitely many empty columns, i.e., $P = \{x \in 2^{\omega\times\omega} : \exists^\infty m \forall n\, x(m, n) = 0\}$ [Ke95, Exercise 23.2]. Given $x \in 2^{\omega\times\omega}$, we produce a structure $\mathcal{M}$ with domain the set $M = \{a_i \mid i \in \omega\} \cup \{b_{\langle j,k\rangle} \mid k \text{ is least such that } x(j, k) = 1\} \cup \{c_i \mid i \in \omega\} \cup \{d_i \mid i \in \omega\}$. We then set $R(c_i, d_i)$ and $R(a_i, b_{\langle i,k\rangle})$ for any i and any i, k such that $b_{\langle i,k\rangle}$ exists. Note that the domain of M as defined is a computable set, so there is an effective bijection with ω, and thus we may consider this to be a map from $2^{\omega\times\omega}$ to $\mathrm{Mod}(\tau)$. It is straightforward to check that $\mathcal{M}$ is a model of T if and only if $x \in P$. □

Example 4.13. There is a finitely $\exists_3$-axiomatizable $\aleph_0$-categorical theory with a $\mathbf{\Sigma}^0_3$-complete set of models.

Proof. Consider the theory of the linear ordering $2 \cdot \mathbb{Q} + 1 + \mathbb{Q}$ together with its successor relation S. This theory is $\aleph_0$-categorical and is axiomatizable by the axioms for linear orderings, the definition of the successor relation, the statement that there is neither a least nor greatest element, and the following $\exists_3$-formula:

$$(\star) \quad \begin{gathered}\exists x\,[(\forall y < x)\exists z\,[y < z < x] \wedge (\forall y > x)(\exists z\,[x < z < y] \wedge (\forall u > x)\neg S(y,u)\,) \wedge \\ (\forall y < x)\,(\exists z(S(y,z) \vee S(z,y)) \wedge (\exists u\, S(y,u) \to \forall v\, \neg S(v,y)))]\end{gathered}$$

One can easily verify that $L \cong 2 \cdot \mathbb{Q} + 1 + \mathbb{Q}$ for any countable linear ordering L satisfying $(\star)$, thus this is a finite $\exists_3$-axiomatization for an $\aleph_0$-categorical theory. As the axiomatization is finite, we get an axiomatization by a single $\exists_3$-formula and thus its Wadge degree is at most $\mathbf{\Sigma}^0_3$ by the Lopez-Escobar theorem. To see hardness, note that it was shown in [GRta, Theorem 3.3] that the isomorphism class of $2 \cdot \mathbb{Q} + 1 + \mathbb{Q}$ is $\mathbf{\Sigma}^0_4$-complete in the space of linear orderings (without successor relation). Now, towards a contradiction, assume it is not $\mathbf{\Sigma}^0_3$-hard in $\mathrm{Mod}(\{\leq, S\})$. Then Lemma 4.8 gives a $\Pi^{\mathrm{in}}_3(\leq, S)$-sentence φ such that $\mathrm{Mod}(\varphi) = \mathrm{Iso}(2\cdot\mathbb{Q}+1+\mathbb{Q}, \leq, S)$. But clearly φ translates into a $\Pi^{\mathrm{in}}_4(\leq)$-formula, contradicting that $\mathrm{Iso}(2 \cdot \mathbb{Q} + 1 + \mathbb{Q}, \leq)$ is $\mathbf{\Sigma}^0_4$-complete in $\mathrm{Mod}(\{\leq\})$. □

We next show that 3 is minimal possible in Example 4.13. This is similar to a result of Arnold Miller [Mi83] that states that no countable structure can have a $\mathbf{\Sigma}^0_2$-isomorphism class.

Proposition 4.14. *Let φ be a consistent Σ^{in}_2-sentence. Then φ has a finitely generated model. In particular, if T is a complete relational theory and* $\mathrm{Mod}(T) \in \mathbf{\Sigma}^0_2$*, then* $\mathrm{Mod}(T) = \emptyset$.

Proof. Suppose φ is Σ^{in}_2, i.e., of the form $\bigvee\!\!\!\bigvee_{i\in\omega} \exists \overline{x}\theta_i(\overline{x})$, where θ_i are conjunctions of $\forall_1$-sentences. Assume without loss of generality that $(\mathcal{A}, \overline{a}) \models \theta_i(\overline{a})$ for some i, then every substructure of $(\mathcal{A}, \overline{a})$ satisfies $\theta_i(\overline{a})$ and thus the substructure of $\mathcal{A}$ generated by $\overline{a}$ satisfies φ.

Now assume that T is a complete relational theory and $\mathrm{Mod}(T) \in \mathbf{\Sigma}^0_2$. Then by Lopez-Escobar, $\mathrm{Mod}(T) = \mathrm{Mod}(\varphi)$ for a Σ^{in}_2-formula φ. It then follows from the above argument that T has a finite model $\mathcal{A}$. Hence, $\mathrm{Mod}(T) = \mathrm{Iso}(\mathcal{A})$ by the completeness of T. So, T does not have a countably infinite model, and $\mathrm{Mod}(T)$ is empty. □

Next we show that Examples 4.12 and 4.13 can be generalized to higher quantifier levels.

Lemma 4.15. *Let $n \geq 2$. Let T be an $\exists_n$-axiomatizable theory such that* $\mathrm{Mod}(T)$ *is $\mathbf{\Sigma}^0_n$-complete (or $\mathbf{\Pi}^0_{n+1}$-complete, respectively). Let T' be the Δ^0_2-Marker extension of T (see* [AM15, Lemma 2.8]*). Then T' is an $\exists_{n+1}$-axiomatizable theory such that* $\mathrm{Mod}(T)$ *is $\mathbf{\Sigma}^0_{n+1}$-complete (or $\mathbf{\Pi}^0_{n+2}$-complete, respectively).*

Proof. We focus on the case where $\mathrm{Mod}(T)$ is $\mathbf{\Sigma}^0_n$-complete, with the $\mathbf{\Pi}^0_{n+1}$-complete case being similar. Note that $\mathrm{Mod}(T')$ is $\mathbf{\Sigma}^0_{n+1}$, since from a structure B, it is $\Delta^0_3(B)$ to check that it is a Δ^0_2-Marker extension of a structure $\hat{B}$, with $\hat{B}$ being uniformly $\Delta^0_2(B)$. Finally, B is a model of T' if and only if $\hat{B}$ is a model of T, which is $\Sigma^0_n(B')$. Putting all together, we get that $\mathrm{Mod}(T')$ is $\mathbf{\Sigma}^0_{n+1}$.

Since $\mathrm{Mod}(T)$ is $\mathbf{\Sigma}^0_n$-complete, there is a Wadge reduction of $P_n = \{k^\frown p \mid k \in p^{(n)}, p \in 2^\omega\}$ to $\mathrm{Mod}(T)$. We will convert this into a Wadge reduction of $P_{n+1} =$

$\{k^\frown p \mid k \in p^{(n+1)}, p \in 2^\omega\}$ to $\mathrm{Mod}(T')$. Since P_{n+1} is a $\mathbf{\Sigma}^0_{n+1}$-complete subset of ω^ω, this shows $\mathrm{Mod}(T')$ is $\mathbf{\Sigma}^0_{n+1}$-complete.[1]

Let g be the continuous map reducing P_n to $\mathrm{Mod}(T)$, and let D be an oracle which computes g. Given $k^\frown p$, $g(k^\frown p')$ is uniformly computable from $D \oplus p'$. Then $D \oplus p$ can uniformly compute a copy of the Δ^0_2-Marker extension of $g(k^\frown p')$. This yields the $\mathbf{\Sigma}^0_{n+1}$-hardness of $\mathrm{Mod}(T')$.

It is straightforward to check that if T is $\exists_n$-axiomatizable for $n \geq 2$, then T' is $\exists_{n+1}$-axiomatizable. □

Since Marker extensions preserve $\aleph_0$-categoricity and finite axiomatizability, we can generalize Examples 4.12 and 4.13 to higher quantifier levels.

Example 4.16. For every $n \geq 1$, there is an $\exists_n$-axiomatizable $\aleph_0$-categorical theory T such that $\mathrm{Mod}(T)$ is $\mathbf{\Pi}^0_{n+1}$-complete.

For every $n \geq 3$, there is a finitely $\exists_n$-axiomatizable $\aleph_0$-categorical theory T such that $\mathrm{Mod}(T)$ is $\mathbf{\Sigma}^0_n$-complete.

Example 4.17. For any $n \geq 3$, there is an $\exists_n$-axiomatizable $\aleph_0$-categorical theory T such that $\mathrm{Mod}(T)$ is a properly $\mathbf{\Delta}^0_{n+1}$-set.

Proof. Fix T_0 to be an $\exists_{n-1}$-axiomatizable $\aleph_0$-categorical theory such that $\mathrm{Mod}(T_0)$ is $\mathbf{\Pi}^0_n$-complete. Fix T_1 to be an $\exists_n$-axiomatizable $\aleph_0$-categorical theory such that $\mathrm{Mod}(T_1)$ is $\mathbf{\Sigma}^0_n$-complete. Let T have a unary predicate U and say that the set of elements realizing U is a model of T_0 and the set of elements realizing $\neg U$ is a model of T_1. Then T is $\exists_n$-axiomatizable and $\mathrm{Mod}(T)$ is $D_2(\mathbf{\Sigma}^0_n)$-complete. □

We observe that a special case of Theorem 4.10 implies a case of a theorem of Keisler [Ke65, Corollary 3.4], recently reproved by Harrison-Trainor and Kretschmer [HK23].

Theorem 4.18. *If a finitary first-order formula φ is equivalent to $\psi \in \Pi^{\mathrm{in}}_n$, then there is a $\forall_n$-formula θ such that $\varphi \equiv \theta$.*

Proof. By adding constants, we may assume that φ is a sentence. Since φ is equivalent to ψ, we get that $\mathrm{Mod}(\{\varphi\}) \in \mathbf{\Pi}^0_n$. Thus, Theorem 4.10 shows that φ has a $\forall_n$-axiomatization. Compactness implies that φ is equivalent to a single $\forall_n$-sentence. □

Combining our results from this section, we obtain the following characterization.

Theorem 4.19. *Let T be a theory and $n \in \omega$. Then the following are equivalent.*

1. *T has a $\forall_n$-axiomatization but no $\forall_{n-1}$-axiomatization.*
2. *The Wadge degree of $\mathrm{Mod}(T)$ is in $[\mathbf{\Sigma}^0_{n-1}, \mathbf{\Pi}^0_n]$.*

Note that the intervals $[\mathbf{\Sigma}^0_{n-1}, \mathbf{\Pi}^0_n]$ contain $\aleph_1$-many different $\mathbf{\Delta}^0_n$-Wadge degrees.

Question 4.20. Which $\mathbf{\Delta}^0_n$-Wadge degrees are the degree of $\mathrm{Mod}(T)$ for some (complete) finitary first-order theory?

[1] To see that P_{n+1} is $\mathbf{\Sigma}^0_{n+1}$-complete, first note that it is $\mathbf{\Sigma}^0_{n+1}$. If there was D such that P_{n+1} is $\Pi^0_{n+1}(D)$, then P_{n+1} would be $\Delta^0_{n+1}(D)$. Hence, we would get that for any C computing D that $k \in C^{(n+1)}$ if and only if $k^\frown \chi_C \in P_{n+1}$ and this would be $\Delta^0_{n+1}(C)$, hence computable from $C^{(n)}$. But this would contradict that the Turing jump is proper. So, by Wadge's lemma, P_{n+1} is $\mathbf{\Sigma}^0_{n+1}$-complete.

5. Proofs of the two technical results

In the present section, we will prove Theorem 3.3 and Lemma 4.1. We will first prove Theorem 3.3 and then introduce some minor modifications to the proof to prove Lemma 4.1.

The proof of Theorem 3.3 relies on a theorem of Knight that initially appeared in [Kn87] where it is proved using a worker argument, a technique for (possibly infinite) iterated priority constructions developed by Harrington. In [Kn99], a new proof of this result can be found based on a version of Ash and Knight's α-systems [AK00]. Let us introduce all the definitions necessary to state this theorem.

A *Scott set* $\mathcal{S}$ is a subset of 2^ω that is closed under Turing reducibility, join, and satisfies Weak König's Lemma, i.e., if $T \in \mathcal{S}$ codes an infinite binary tree, then there is a path f through T such that $f \in \mathcal{S}$. An *enumeration* of a countable Scott set $\mathcal{S}$ is a set $r \in 2^\omega$ satisfying $\mathcal{S} = \{r^{[n]} \mid n \in \omega\}$, i.e., $\mathcal{S}$ equals the set of columns of r. If $A = r^{[i]}$, then we say that i is an r-index for A.

We are now ready to state Knight's theorem.

Theorem 5.1 ([Kn99, Theorem 2.5])**.** *Let T be a complete theory. Suppose $r \leq_T x$ is an enumeration of a Scott set $\mathcal{S}$, with functions t_n which are $\Delta^0_n(x)$ uniformly in n, such that for each n, $\lim_s t_n(s)$ is an r-index for $T \cap \exists_n$, and for all s, $t_n(s)$ is an R-index for a subset of $T \cap \exists_n$. Then T has a model $\mathcal{B}$ with $\mathcal{B} \leq_T x$.*

We will need the following uniform version of Theorem 5.1. To prove this, one could simply note that the techniques involved in the proof of Theorem 5.1—α-systems and standard finite-injury constructions—are uniform. However, as the proofs in [Kn99, Theorem 2.5] are combinatorially quite difficult, we provide some guidance for the interested reader in the form of a proof sketch.

Theorem 5.2. *Let r be an enumeration of a Scott set $\mathcal{S}$. Then there exists a Turing operator $\Gamma : 2^\omega \times \omega \to 2^\omega$ which satisfies the following:*

Let x be any set and Φ_e be such that $\lim_s \Phi_e^{x^{(n-1)}}(s)$ is an r-index for $T \cap \exists_n$, and for every s, $\Phi_e^{x^{(n-1)}}(s)$ is an r-index for a subset of $T \cap \exists_n$. Then $\Gamma^{x \oplus r}(e) \in \mathrm{Mod}(T)$.

Proof sketch. First, let us point out a notational difference. In Theorem 5.1, the functions t_n are required to be $\Delta^0_n(x)$ uniformly in n, while in this theorem, the corresponding functions are $\Phi_e^{x^{(n-1)}}$, where Φ_e is a Turing operator that requires the $(n-1)^{\text{th}}$ jump of x to produce the right output. This discrepancy comes from the definition of the arithmetic hierarchy. The set $x^{(n)}$ is the canonical complete $\Delta^0_{n+1}(x)$-set and is not $\Delta^0_n(x)$.

We will now point out which parts of the proof of Theorem 5.1 guarantee the required uniformity. In [Kn99], Theorem 2.5 (i.e., our Theorem 5.1) is obtained as a direct consequence of Theorems 2.1 and 2.3. The proof of Theorem 2.1 is a finite injury construction and, as should be clear to readers familiar with such constructions, is uniform. The only crucial part is that it needs an effective enumeration of the Scott set $\mathcal{S}$, i.e., an enumeration of $\mathcal{S}$ with additional computable functions on the indices that witness that $\mathcal{S}$ is a Scott set. Marker [MM84] showed that one can pass from an enumeration of a Scott set to an effective enumeration, and thus this is not a hindrance.

The proof of Theorem 2.3 uses an $(n+1)$-approximation system relative to $\Delta^0_2(X)$. These $(n+1)$-approximation systems are a special version of α-systems developed by Ash and Knight. A standard reference is [AK00]. The proof in Knight [Kn99] just shows that the premises of Theorem 2.3 are sufficient to obtain an $(n+1)$-approximation system satisfying the premises of [Kn99, Corollary 4.4]. This corollary essentially says that there is a function E that takes paths π through a special tree and produces a $\Sigma^0_2(x)$-object $E(\pi)$. This E is easily seen to be continuous from its definition, see [Kn99, p268, third paragraph]. But one issue is that we cannot obtain π from x, rather π is the limit of approximations π^n which are $\Delta^0_n(x)$, uniformly in n. Thankfully, as is explained in the proof of [Kn99, Theorem 4.1] that gives rise to [Kn99, Corollary 4.4], $E(\pi) = E(\pi^1)$ and π^1 can be uniformly computed from x. As $E(\pi) = E(\pi^1)$ is what we are after, we get the desired continuity. □

5.1. **Proof of Theorem 3.3.** We fix a $\mathbf{\Pi}^0_\omega$-set $P \subseteq 2^\omega$, theories T and $\{T_n\}_{n\in\omega}$, and show how to obtain the Scott set $\mathcal{S}$ and functionals t_n such that, using an input $p \in 2^\omega$, we satisfy Theorem 5.2 and thus output a model of T if $p \in P$ and a model of some T_n otherwise.

Given a $\mathbf{\Pi}^0_\omega$-set P, we can fix a decreasing sequence of $\mathbf{\Pi}^0_n$-sets P_n such that $P = \bigcap_{n\geq 1} P_n$. Let c be strong enough so that each P_n is uniformly $\Pi^0_n(c)$.

Next, we fix an enumeration r of a Scott set $\mathcal{S}$ containing $T \cap \exists_k$ for each k and $\{T_n \cap \exists_k\}_{n,k\in\omega}$. Let $y = c \oplus r \oplus T \oplus \bigoplus_{n\in\omega} T_n$. We will describe a computation Φ_e satisfying the hypotheses of Theorem 5.2, namely that $\Phi_e^{(y\oplus p)^{(n-1)}}(s)$ will be constant in s and is an r-index for $T_p \cap \exists_n$ for a complete theory T_p. Furthermore, we will ensure that if $p \in P$ then $T_p = T$; and if $p \notin P$ then $T_p = T_n$ for some $n \in \omega$. Note that we are applying Theorem 5.2 with the oracle $x = y \oplus p$.

We describe the index e by giving a uniform method of computing an r-index for $T_p \cap \exists_n$ from $(y \oplus p)^{(n-1)}$. For $n = 1$, we output a fixed index for $T \cap \exists_1$. For $n \geq 2$, $\Phi_e^{(y\oplus p)^{(n-1)}}$ depends on whether $p \in P_{n-1}$. Since P_{n-1} is uniformly $\Pi^0_{n-1}(c)$, it is uniformly computable in $(c\oplus p)^{(n-1)} \leq_T (y\oplus p)^{(n-1)}$ to determine membership of p in P_{n-1}.

For $n \geq 2$, let Φ_e be the algorithm defined as follows. Given an oracle of the form $(y \oplus p)^{(n-1)}$, let k_0 be the least $k < n$ such that $p \notin P_k$ if such k exists, and n otherwise. Let $\Phi_e^{(y\oplus p)^{(n-1)}}(s)$ output the least r-index of $T_{k_0} \cap \exists_n$. Note that finding this index is not effective in r and T_{k_0} but is effective in $(r \oplus T_{k_0})' \leq_T y' \leq_T (y \oplus p)^{(n-1)}$ (needing one jump here is why we treat the case $n = 1$ differently).

We have just produced a uniform sequence of computations, thus we can find a single index e (note that y and p are used in the oracle, but not in identifying the index) such that if $p \in P$ then for every n, $\Phi_e^{(y\oplus p)^{(n-1)}}$ satisfies the conditions in Theorem 5.2 for T and if $p \notin P$, then the conditions in Theorem 5.2 are satisfied for some T_n. Thus, the Turing operator Γ in Theorem 5.2 gives a Wadge reduction from P to $\mathrm{Mod}(T)$, as $\Gamma^{y\oplus p} \in \mathrm{Mod}(T)$ if and only if $p \in P$.

5.2. **Proof of Lemma 4.1.** The proof is almost the same as the proof of Theorem 3.3, except that one of our approximation functions will not be constant. The difference is in how we obtain the index e to apply Theorem 5.2. We let $y = c \oplus r \oplus f \oplus g$, where P is $\Sigma^0_n(c)$, and r is an enumeration of a Scott set which contains both T^+ and T^-; now f and g are functions such that $f(n)$ is an r-index for $T^- \cap \exists_n$ and $g(n)$ is an r-index for $T^+ \cap \exists_n$. Let $P = \bigcup_{i\in\omega} P_i$, where the P_i are uniformly $\Delta^0_n(c)$.

Here, for $k < n$, we let $\Phi_e^{(y\oplus p)^{(k-1)}}(s)$ output $f(k)$, which is an r-index for $T^- \cap \exists_k = T^+ \cap \exists_k$. We now describe the algorithm to compute $\Phi_e^{(y\oplus p)^{(n-1)}}(s)$ for $k = n$. First, check whether $p \in \bigcup_{t<s} P_t$. Note that this is uniformly computable from $(c \oplus p)^{(n-1)}$. If $p \notin \bigcup_{t<s} P_t$, we output $f(n)$, which is an r-index for $T^- \cap \exists_n$. If $p \in \bigcup_{t<s} P_t$, we output $g(n)$, which is an r-index for $T^+ \cap \exists_n$. Note that the value of $\Phi_e^{(y\oplus p)^{(n-1)}}(s)$ may change at most once as s increases. For $k > n$, we let $\Phi_e^{(y\oplus p)^{(k-1)}}$ be the algorithm that checks whether $p \in P$ and outputs $g(k)$, an r-index for $T^+ \cap \exists_k$ if $p \in P$, and outputs $f(k)$, the index for $T^- \cap \exists_n$ otherwise. The fact that $T^- \cap \exists_n \subseteq T^+ \cap \exists_n$ guarantees that if $p \in P$, then Φ_e satisfies the conditions for Theorem 5.2 with $T = T^+$, and that if $p \notin P$, Φ_e satisfies the conditions for Theorem 5.2 with $T = T^-$. Thus, Γ gives a Wadge reduction from P to $\mathrm{Mod}(T^+)$.

References

[AK13] Andrews, Uri and Knight, Julia F., *Spectra of atomic theories*, J. Symbolic Logic **78** (2013), 1189-1198.

[AM15] Andrews, Uri and Miller, Joseph S., *Spectra of theories and structures*, Proc. Amer. Math. Soc. 143 (2015), 1283-1298

[AK00] Ash, Christopher J. and Knight, Julia F., *Computable structures and the hyperarithmetical hierarchy*, Stud. Logic Found. Math., Vol. 144, North-Holland, Amsterdam, 2000.

[DGHTa] Day, Adam R.; Greenberg, Noam; Harrison-Trainor, Matthew A.; and Turetsky, Daniel D., *An effective classification of Borel Wadge classes*, preprint.

[EVta] Enayat, Ali and Visser, Albert, *Incompleteness of boundedly axiomatizable theories*, `https://arxiv.org/abs/2311.14025`, submitted for publication.

[GRta] Gonzalez, David and Rossegger, Dino, *Scott sentence complexities of linear orderings*, `https://arxiv.org/pdf/2305.07126.pdf`.

[HK23] Harrison-Trainor, Matthew A. and Kretschmer, Miles J., *Infinitary logic has no expressive efficiency over finitary logic*, preprint.

[Je12] Jeřábek, Emil, *Sequence encoding without induction*, Math. Logic Quaterly **58** (2012), 244–248.

[Ka91] Kaye, Richard W., *Models of Peano arithmetic*, Oxford Logic Guides 15, Oxford University Press, New York, 1991, x+292 pp.

[Ke95] Kechris, Alexander S., *Classical descriptive set theory*, Grad. Texts in Math. Vol. 156, Springer-Verlag, New York, 1995, xviii+402 pp.

[Ke65] Keisler, H. Jerome, *Finite approximations of infinitely long formulas*, in: “The theory of models. Proceedings of the 1963 International Symposium at Berkeley” (eds. J. W. Addison, Leon Henkin, Alfred Tarski), North-Holland Publishing Co., Amsterdam, 1965), pp. 158–169.

[Kn87] Knight, Julia F., *Degrees of Models with Prescribed Scott Set*, Classification theory (Chicago, IL, 1985), 182191. Lecture Notes in Math., 1292 Springer-Verlag, Berlin, 1987

[Kn99] Knight, Julia F., *True approximations and models of arithmetic*, in: “Models and computability (Leeds, 1997)”, London Math. Soc. Lecture Note Ser. Vol. 259, Cambridge University Press, Cambridge, 1999, 255-278.

[Lo65] Lopez-Escobar, Edgar G. K., *An interpolation theorem for denumerably long formulas*, Fund. Math. **57** (1965), 253–272.

[LS87] Louveau, Alain and Saint-Raymond Jean, *Borel Classes and Closed Games: Wadge-Type and Hurewicz-Type Results*, Trans. Amer. Math. Soc. **304** (1987), 431-67.

[MM84] Macintyre, Angus J. and Marker, David E., *Degrees of recursively saturated models*, Trans. Amer. Math. Soc. **282** (1984), 539-554.

[Ma89] Marker, David E., *Non Σ_n axiomatizable almost strongly minimal theories*, J. Symbolic Logic **54** (1989), 921–927.

[Mi83] Miller, Arnold W., *On the Borel classification of the isomorphism class of a countable model*, Notre Dame J. Formal Logic **24** (1983), 22–34.

[PV22] Pakhomov, Fedor N. and Visser, Albert, *Finitely axiomatized theories lack self-comprehension*, Bull. Lond. Math. Soc. **54** (2022), 2513–2531.

[Pa70] Parsons, Charles D., *On a number theoretic choice schema and its relation to induction*, Stud. Logic Found. Math. **60** (1970), 459–473.

[Pu83] Pudlák, Pavel P., *Some prime elements in the lattice of interpretability types*, Trans. Amer. Math. Soc. **280** (1983), 255–275.

[Ra61] Rabin, Michael O., *Non-standard models and independence of the induction axiom*, in: "Essays on the foundations of mathematics. Dedicated to A. A. Fraenkel on his seventieth anniversary" (eds. Y. Bar-Hillel, E. I. J. Poznanski, M. O. Rabin, and A. Robinson), Magnes Press, The Hebrew University, Jerusalem, 1961, pp. 287–299.

[Ro20] Rossegger, Dino, *Complexity of the set of models of* TA, question on MathOverFlow, `https://mathoverflow.net/questions/370114/complexity-of-the-set-of-models-of-ta`, 2020.

[Ro22] Rossegger, Dino, *Degree Spectra of Analytic Complete Equivalence Relations*, The Journal of Symbolic Logic 87 (4): 166376.

[Sc65] Scott, Dana S., *Logic with denumerably long formulas and finite strings of quantifiers*, in: "The Theory of Models" (Proc. 1963 Internat. Sympos. Berkeley), (eds. J. W. Addison, L. Henkin, and A. Tarski), North-Holland, Amsterdam, 1965, pp. 329–341.

[Sh64] Shepherdson, John C., *A non-standard model for a free variable fragment of number theory*, Bulletin de lAcadémie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques, **12** (1964), 79–86.

[So82] Solovay, Robert M., *Degrees of models of true arithmetic*, preprint circulated in 1982.

[Va07] Vanden Boom, Michael, *The effective Borel hierarchy*, Fund. Math. **195** (2007), 269–289.

[Va74] Vaught, Robert L., *Invariant sets in topology and logic*, Fund. Math. **82** (1974), 269–294.

[Vi17] Visser, Albert, *On* Q, Soft Comput. **21** (2017), 39–56.

[Wa83] Wadge, William W., *Reducibility and determinateness on the Baire space*, Ph.D. Thesis, University of California, Berkeley, 1983, 334 pp.

[Wi85] Wilmers, George M., *Bounded existential induction*, J. Symbolic Logic **50** (1985), 72–90.

(Andrews, Lempp, Zhu) Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706-1325, USA

Email address: `andrews@math.wisc.edu`
URL: `http://www.math.wisc.edu/~andrews`
Email address: `lempp@math.wisc.edu`
URL: `http://www.math.wisc.edu/~lempp`
Email address: `hongyu@math.wisc.edu`
URL: `https://sites.google.com/view/hongyu-zhu/`

(Gonzalez) Department of Mathematics, University of California, Berkeley, Berkeley, California 94720-1234, USA

Email address: `david_gonzalez@berkeley.edu`
URL: `https://www.davidgonzalezlogic.com/`

(Rossegger) Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8-10, 1040 Wien, AUSTRIA

Email address: `dino.rossegger@tuwien.ac.at`
URL: `https://drossegger.github.io/`