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Shan Zhou

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Aspects of quantum field theory

To mysteries and miracles.

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Publications

Liang Kong, Yin Tian, and Shan Zhou, "The center of monoidal 2-categories in $3+1$D Dijkgraaf-Witten Theory", Adv.Math. 360 (2020) 106928

James T. Liu, Leopoldo A. Pando Zayas, and Shan Zhou, "Subleading Microstate Counting in the Dual of Massive Type IIA", arXiv: 1808.10445

James T. Liu, Leopoldo A. Pando Zayas, and Shan Zhou, "Comments on Higher Rank Wilson Loops in $\mathcal{N} = 2^*$ ", JHEP 01(2018)047

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Abstract

Aspects of quantum field theory

by

Shan Zhou

Quantum field theory plays a central role in the study of theoretical physics, where a satisfying understanding of general field theories is usually absent. In this thesis, the path integral approach will be revisited, and it will be shown how to organize the computation of path integral into the framework of (fully) extended field theory. Several aspects of quantum field theory will be explored with this new viewpoint, including symmetries and anomalies.

Contents

Chapter 1

Introduction

1.1 Physics With Hindsight

Physics is powerful because it makes valid predictions of the future. Experiments are the most powerful tools we have for ruling out "unphysical" theories[1]. However, when no experiment is available yet, it is still possible to do physics.

A physical theory, after verified by multiple interconnected experiments, can usually be polished to obtain deep and beautiful mathematical structures that are completely beyond the imagination of its inventor (e.g. Newton must have no idea about symplectic geometry). On the other hand, it would be tempting to explain why the theory has to be of that form. If a theory is so successful, then there must be a reason behind it[2], and in many cases it seems to be possible to "prove" why the only possible theory is of the form shown on textbooks, without looking at sophisticated experiments[3].

[1]Being unphysical does not mean the theory is not useful. Many unphysical theories play important roles in mathematics.

[2]This is not a purely philosophical wish. From the viewpoint of renormalization group and effective theory, if a theory works for one model, it will eventually describe a universal class of phenomena, and ideally if we fully understand the universal class, we will be able to explain why irrelevant degrees of freedom are suppressed by the renormalization group and why the remaining degrees of freedom are essential. This is the meaning of "reason behind it".

[3]But we do have to refer to a few fundamental experiments, like the invariance of the speed of light

For example, let's try to "derive" special relativity: the speed of light is a constant, which suggests that the directions of time and space are comparable and should be described by a uniform geometry, called spacetime. We should be able to measure the distance, and Sylvester's law of inertia tells us that we only need to know the signature. Time, after all, is not the same as space, so the signature has to be $(-,+,+,+)$ (we will explain why we prefer the east coast sign convention in the following section). Now we have Minkowski spacetime and everything else of special relativity follows from its geometric and topological properties. Similar but much more technical argument exists for general relativity. Lovelock proved that under certain reasonable assumptions Einstein's equation of the spacetime metric is our only choice[1, 2].

Many classical textbooks are of that form and follow a logical instead of historical line of thinking. This type of argument for classical mechanics and relativity can be found in Landau's course of theoretical physics[3, 4]. The logic behind quantum mechanics, established by Birkhoff and von Neumann[5], is much more subtle[4] and is hence less known at these days, but if taking it as granted, then Weinberg were able to show that the quantum theory of relativistic dynamics has to be a field theory[6][5].

However, quantum field theory, unless all the previous theories, is not a well-defined theory. Its vanilla form involves infinitely many degrees of freedoms and divergences come from both the UR and IR side, and is hence more of less a formal notation for

and the principle of relativity.

[4]Consider the logical operations on results of experiments, which form a boolean lattice, then Stone representation theorem implies that the state is completely characterized by a set theoretic point in some phase space, which has to be classical mechanics. Discarding most of the logical identities, like $\neg\neg A = A$, results in pathological theories. However, abandoning distributivity $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ can lead to the (algebraic) theory of quantum mechanics. For example if the experiment is the measurement of a single particle, then A, B, C can be $p \in [-1,1], x \in [-1,0], x \in [0,1]$: knowing that $x \in [-1,1]$ only implies that the wavefunction $\psi(x)$ is supported on it and does not implies that the support is either $[-1,0]$ or $[0,1]$. The new logical system is referred as quantum logic.

[5]Given the strong preference of the logical development over the historical approach, it is no wonder that the title of Weinberg's masterpiece *The Quantum Theory of Fields* is a quantized version of Landau's *The Classical Theory of Fields*.

a universal class instead of some concrete theory. Though perturbative computation of weakly coupled theories and lattice simulation of strongly coupled theories are successful and match well with experiments, it is not clear how to go around many computational tricks and understand the physical meaning of many operations in field theories. On the other hand, many important theories are built and studied without explicit connection with experiments in real world. Many of them are of novel forms and traditional toolbox in quantum field theory does not work very well. There are attempts of developing quantum field theory from a logical line of thinking, like what was successfully done for classical physics, but none of them provides satisfying answer to many questions and it is still challenging to deal with quantum field theory in a rigorous way while keeping the physical meaning clear.

All of them suggest that we need to revisit existing concepts and tools in quantum field theory, so that we could have better understanding of existence constructions (though from hindsight) and develop a framework of field theories that provides new insights. This would be the main theme of this thesis.

1.2 Outline

The main topic of this thesis is about studying familiar concepts in quantum field theory with new viewpoint.

In Chapter 2, we will discuss a few rigorous ways of interpreting quantum field theory and explain why path integral is a preferred approach. We will then explain in what sense Wick rotation makes sense. After Wick rotation we get a Euclidean theory and we will explain how to organize path integral into the framework of extended quantum field theory. We will show that the extended structure is universal and every quantum field theory is extended. In particular, we will explain how to work with lattice regularized field theories where path integral is well defined.

In Chapter 3, we will discuss a few general properties and constructions in field theories, together with the symmetry topological field construction with separates the calculus of symmetry with its representation in a concrete quantum field theory.

In Chapter 4, we will explain different approaches to 't Hooft anomalies and gauge theories, and will show that the previous construction provides a unified viewpoint.

Chapter 2

Path Integral

Historically, quantum theories are obtained from quantization of classical theories. Although believed to describe the same physics, different quantization scheme of different formalism of classical mechanics could look very different and in practice one of them might dominate the other ones.

In the classical world, there do exist sophisticated constructions on top of the Lagrangian mechanics. It clarifies the precise meaning of the variational calculus and its cohomological structure in terms of the variational bicomplex[7], and has important applications in the study of general relativity[8, 9] and holography[10]. However, the Hamiltonian formalism is still considered as the one that has richer geometric structures[11].

In some sense the situation in the quantum world is similar. The modern form of quantum mechanics is obtained from quantization of the Hamiltonian dynamics and even in the early days of quantum mechanics there are already extensive studies on the mathematical structure behind the matrix formalism[12]. It still dominates the study of quantum mechanics after the Lagrangian version, path integral, is invented. One of the reasons is that it is very difficult to evaluate the propagator of non-free particles, like the hydrogen atom, though not impossible[13]. Another reason is that the mathematics behind the intuition of summation over path, i.e. the Feynman-Kac construction[14], is much more subtle than those lovely Hilbert spaces and operators acting on them[1].

In quantum field theory, both formalisms are important from the viewpoint of physicists. It is easier to prove the unitarity of a theory in the operator formalism while the symmetries are more transparent in the path integral formalism; besides, the computation of Feynman rules are usually easier in the path integral formalism while many non-perturbative result are obtained from operator-state based arguments[6]. For strict constructions, algebraic approaches seem to be more developed than the construction of path integral measures. Path integral is only well-understood in some special cases, including topological field theories[15], conformal field theories[16], and some supersymmetric field theories[17, 18, 19].

In this chapter, we are going to review the algebraic approach first, then explain why the path integral formalism is preferred in this thesis. After that, we will explore several novel aspects of quantum field theory with the help of path integral, and then explain the mathematical structure behind its definition. More structures and properties will be covered in the following chapters.

2.1 Local Field Theory

2.1.1 Wightman-style axioms

The first attempt of a strict description of quantum field theories is those Wightman-style axioms[20, 21]. Instead of providing a full list, we will only review a few important features.

There exists a Hilbert space describing the states of the theory. Physical states are

[1]In principle, functional analysis is less familiar for physicists, but the intuitions from finite dimensional Hilbert spaces work in most cases.

generated from the vacuum vector $|0\rangle$ by acting smeared fields on it

$$\phi_1(f_1)\dots\phi_n(f_n)\,|0\rangle \tag{2.1}$$

Fields in the theory should be understood as operator-valued distributions

$$\phi(f) := \int d^n x\,\phi(x)f(x) \tag{2.2}$$

and observables are n-point (generalized) functions

$$W(x_1,\dots,x_n) := \langle 0|\,\phi_1(x_1)\dots\phi_n(x_n)\,|0\rangle \tag{2.3}$$

It describes the relativistic structure: space-like fields should behave well under translations and Lorentz transformations; besides, they should also commute with each other

$$[\phi(x),\psi(y)] = 0 \tag{2.4}$$

where x, y are spacelike separated.

There should be a Euclidean version of field theories which is more or less equivalent to Wigntman's via Wick rotation.

However, the above set of axioms does not match very well with the need of physicists.

1. Operator-valued distributions are difficult while rarely used
2. Quantum fields are only tools of constructing observables like S-matrices, which implies that
   (a) we should be able to select another set of fundamental field variables
   (b) instead of the commutation relation of spacelike fields, only the cluster-decomposition

property is required to build physical theories

3. Extended operators, like Wilson loops, are missing in the above axioms
4. The theory has to be UV-complete and is defined at all scales
5. Intuition from free fields in Minkowski space, e.g. Hilbert space, vacuum state, particle spectrum, becomes skeptical for interacting theories in more complicated spacetime[22]

2.1.2 Local algebras

Given that the Wightman-style axioms are not satisfying, a more preferred while still rigorous method is developed, which is the local algebra approach.

Instead of specifying the field variables and states, it assigns an (von Neumann[23]) algebra of operators for each spacetime region

$$R \mapsto A(R) \tag{2.5}$$

which consists of observables supported in R. There are many equivalent definition of von Neumann algebra but the most convenient one is based on the commutant

$$S' := \{a \in \mathbb{B}(H) : [a, S] = 0\} \tag{2.6}$$

and von Neumann algebra is the one coincides with its double commutant

$$A = A'' \tag{2.7}$$

In particular, the von Neumann algebra $\vee(S)$ generated from S is its double commutant

$$\vee(S) := S'' \tag{2.8}$$

In physics, the commutant A' is related to observables spatially separated from the support of A

$$A(R) \subseteq (A(R'))' \tag{2.9}$$

where R' is the causal complement of R. This is a refined version of the Wightman axiom $[\phi(x), \psi(y)] = 0$.

The above algebraic construction could combine with the locality principle in quantum field theory. The algebra of locally generated / additive observables can be defined as

$$A(R) = \bigvee_{\cup B = R} A(B) \tag{2.10}$$

where

$$A_1 \vee A_2 := (A_1 \cup A_2)'' \tag{2.11}$$

This construction is additive

$$A(R_1 \cup R_2) = A(R_1) \vee A(R_2) \tag{2.12}$$

and isotonic

$$A(R_1) \subseteq A(R_2),\ R_1 \subseteq R_2 \tag{2.13}$$

The maximally allowable algebra of observables, restricted by the causality, is

$$A_m(R) := (A(R'))' \tag{2.14}$$

and the gap between it and the locally generated observables measures the nonlocal observables

$$A_m(R) = A(R) \vee \{b_{\text{nonlocal}}\} \tag{2.15}$$

In this sense, the local algebra approach can describe nonlocal operators[24] and is also independent of the existence of Poincare symmetry. It also leads to interesting results in field theory in a general curved spacetime, including the timelike tube theorem and the Reeh-Schlieder theorem[25, 26, 27]. However, these results also suggests that quantum field theory is defined at all scales (otherwise it seems to be unlikely to obtain the isomorphic of observables supported in spacetime regions of different size) and hence is likely to be a free theory (though there are concepts like renormalized trace for von Neumann algebras themselves[23], its connection with Wilsonian renormalization group is still not clear).

Exact constructions seems to be available only for free theories and it already involves complicated microlocal analysis. This does not mean the local algebra approach is useless: in the study of quantum gravity, one can derive many powerful results from free theories in curved spacetime, including Unruh effect[28] and Hawking radiation[29].

2.1.3 Path integral

Another issue of the local algebra approach is that, although it describes what a reasonable field theory should contain, it does not tell us how to compute correlation function of observables but leave it as some abstract information encoded in the trace function. If a field theory is local, it should at least in principle be able to compute observables from local data, like what calculus does for classical physics.

This is the starting point behind Feynman's path integral: cut the timeline into small pieces and gluing them together by inserting resolution of identity. In field theories where perturbative computation is no longer an option, it seems that it is easier to construct computable theories in the path integral formalism, at least after Wick rotation. Therefore in the following part of the thesis, we will explore a few aspects of quantum field theory from the viewpoint of path integrals.

2.2 Complex Spacetime

In the following of this thesis, we will focus on Euclidean field theories. However, instead of simply claiming that a relativistic quantum field theory is just a Wick-rotated version of a corresponding Euclidean field theory, we will first explain what a Wick rotation is, because the traditional idea of Wick rotation

$$t \mapsto \pm i\tau \tag{2.16}$$

does not provide a clear physical interpretation. For example, people often claim that Wick rotation is a type of analytic continuation, but that should be something for a function of complex variables z defined in a subset with limiting points, instead of a remote connection between the function values at z_0 and iz_0. We need to find a continuous path connecting Euclidean theories and Minkowski theories.

2.2.1 Wick rotation

Before diving into path integral, let's start with a simple warm-up: consider the Gaussian integral

$$\int_{-\infty}^{\infty} dx e^{iax^2} = \left(\frac{2\pi i}{a}\right)^{1/2} \tag{2.17}$$

Strictly speaking, the left hand side is only a formal expression and does not converge, unless we redefine the notion of integration. To regularize a divergent integral, we can introduce a finite cutoff $[-\Lambda, \Lambda]$ and take the limit $\Lambda \to \infty$ only at the end of the computation, or re-interpret the expression as a distribution that only makes sense after acting on test functions (smeared). All of them are prototypes of how we fight with infinity in quantum field theory. From this viewpoint, integrals that converge conditionally should be considered as something depending on the renormalization scheme, or in other words depending on UV details, which are hence better not to be viewed as a good theory of integration.

An intuition learnt from the Gaussian type integral is that integration of exponential functions are usually well-defined when the exponent has a positive-definite real part. In principal, we can introduce an arbitrary regulator of the form $e^{-\epsilon|x|}$ or $e^{-\epsilon x^2}$ and the result should be independent of the choice of regulator. However, one of them has a better interpretation from a physicist's viewpoint - complex spacetime.

A quantum theory is believed to be described by a path integral[2]

$$\int_{\phi,A} [\mathcal{D}\phi\mathcal{D}A] e^{iS[g,\phi,A]} \tag{2.18}$$

where the action S is a local functional of the spacetime metric g and field configuration of ϕ, A[3]. The above discussion suggests that the exponent should have a negative definite real part to make the integral convergent, at least formally.

The idea can be shown more in a more concrete way[30, 31]. Let M be the background spacetime and D be its dimension. Consider a p-form gauge field with $p+1$-form field strength $F = dA$, then a natural condition for allowable spacetimes is that the usual

[2]Let's postpone the discussion of the meaning of path integral to the following sections of this chapter. In this section, path integrals are understood as some formal notation.

[3]A general field can be defined as a sheaf over the spacetime M, valued in higher categories. Here we will only consider a simple case where physics are described by gauge fields and scalar fields.

action

$$I_{p+1} = \int_M d^D x \sqrt{\det g} F_{i_1 \dots i_{p+1}} F^{i_1 \dots i_{p+1}} \tag{2.19}$$

together with the one without gauge fields

$$I_0 := m^2 \int_M d^D x \sqrt{\det g} \phi^2 \tag{2.20}$$

should have a positive real part, or in a pointwise expression

$$\mathrm{Re} \left(\sqrt{\det g} F_{i_1 \dots i_{p+1}} F^{i_1 \dots i_{p+1}} \right) > 0 \tag{2.21}$$

We can diagonalize the gravatational field

$$g_{ij} = \lambda_i \delta_{ij} \tag{2.22}$$

and then the positivity condition says that for any subset S of $\{1, 2, \dots, D\}$

$$\mathrm{Re} \left(\sqrt{\det g} \prod_{i \in S} \lambda_i \right) > 0 \tag{2.23}$$

For I_0, the condition is

$$\mathrm{Re} \left(\sqrt{\det g} \right) > 0 \tag{2.24}$$

then it can be shown[30, 31] that the requirement for allowable spacetimes can be expressed as

$$\sum |\arg \lambda_i| < \pi \tag{2.25}$$

and as a special case, the Minkowski metric

$$g = \mathrm{diag}(-1, 1, 1, 1) \tag{2.26}$$

is on the boundary but does not belong to the set of allowable metrics. That's the reason why many Feynman diagrams in relativistic field theory seem to be divergent even at tree level. In other words, the relativistic spacetime is a notion of classical physics, while in quantum physics it should be replaced by the complex spacetime

$$g_c = \mathrm{diag}(e^{i(\pi-\epsilon)}, 1, 1, 1) = \mathrm{diag}(-1 + i\epsilon, 1, 1, 1) \tag{2.27}$$

2.2.2 Feynman's $i\epsilon$-prescription

In this thesis, the main purpose of introducing complex spacetime is to justify our choice of spacetime signature: the relativistic case is now really an analytic continuation.

Before studying general properties of Euclidean field theories, we would like to add a brief comment about the ϵ term in the metric.

Let us derive the Feynman propagator of scalar fields with the complex spacetime. The action is

$$S[\phi] = \int d^4x \left(g_c^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right) = \int d^4x \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + i\epsilon (\partial_t \phi)^2 \right) \tag{2.28}$$

notice that $\partial_t \phi^2$ is positive definite, the epsilon term is exactly the one that leads us to the Feynman's $i\epsilon$ term.

Similar results can be derived for spin-1/2 and spin-1 fields.

In this sense, the so called Wick rotation is a well-defined operation and the Lorentzian theory and the Euclidean theory are unified as a single theory for complex spacetime. In the following part of this thesis, we will work with Euclidean spacetime and the above arguments justify why the result should be generic and also valid for theories with (almost) real time.

2.3 Topological Operator

Now we will consider symmetries of a quantum field theory. We will do it twice: in this section, we will derive everything from the traditional usage of path integral, which is roughly some formal notation that needs further clarification but is believed to characterize some universal properties of quantum field theories. The goal is to get familiar with extended operators in quantum field theory and it also motivates the categorical methods in physics. Notice that we have not explain in what sense we will define a quantum field theory yet. After explaining it in the remaining part of this chapter, we will revisit symmetries in field theory in the next chapter and 't Hooft anomalies associated to them in the following one.

In general, symmetries of a path integral could be different from symmetric transformation of the Lagrangian. The path integral consists of both the integrand and the measure in the moduli space of fields. Besides, just like the integrand is one of many equivalent representations in a usual integral, because of the change of variable formula (choice of local coordinate chart), operations on the integrand should have some intrinsic meaning while it might have different forms in different coordinate systems. For path integral, although we usually stick to a convenient set of field variables, concepts of physical meaning should be independent of the choice of fundamental fields.

At this moment, however, we will consider a quantum symmetry as an operation on a fixed set of fundamental fields which does not change the path integral

$$\delta_\epsilon(e^{-S}) = 0 \tag{2.29}$$

and the usual argument in quantum field theory suggests the existence of a current operator

$$\delta_\epsilon S[\phi] = -\int d^D x \epsilon(x) \partial_\mu j^\mu(x) \tag{2.30}$$

Here an operator is interpreted as an insertion into the path integral, instead of some linear operator acting on the physical Hilbert space. Products of operators are interpreted as a relation in the sense of expected value. For example, the operator equation

$$O_1 O_2 = O_3 \tag{2.31}$$

means

$$\int [\mathcal{D}\phi] e^{-S} O_1 O_2 \cdots = \int [\mathcal{D}\phi] e^{-S} O_3 \ldots \tag{2.32}$$

Given the current operator j^μ, we can derive the Ward identity for correlation functions, which should also be invariant under the variation

$$0 = \int [\mathcal{D}\phi] \delta_\epsilon (O_1 \ldots O_n e^{-S}) \tag{2.33}$$

where O_i is again some insertion into the path integral. The intuition is that the variation of fields is just a substitution of variables and physical quantities should not depend on how we choose the fundamental field variables.

The product rule of differential gives

$$0 = \sum_i \langle \ldots \delta_\epsilon O_i(x_i) \ldots \rangle + \left\langle O_1 \ldots O_n \int d^D x \epsilon(x) d * j \right\rangle \tag{2.34}$$

Integrating the current operator over a spatial slice will get the charge operator

$$Q = \int d^{D-1} x j^0 \tag{2.35}$$

However, a natural question is why we have to integrate it over a fixed surface. Ward identity actually tells us the differential of current operator vanishes as long as the integration domain does not contain any insertion. Then integrating both sides of the Ward identity over a codimension-1 hypersurface, we conclude that the following operator

$$Q(\Sigma) := \int_{\Sigma} *j \tag{2.36}$$

is a topological operator

$$Q(\Sigma) = Q(\Sigma + \partial N) \tag{2.37}$$

as long as

$$x_i \notin N \tag{2.38}$$

When Σ is a double copy of the spatial slice

$$\Sigma = \partial[-\delta, \delta] \times \mathbb{R}^{D-1} \tag{2.39}$$

the integration of the Ward identity tells that charge operator are generators of symmetry transformation and after passing from Lie algebra to Lie group

$$U(\Sigma)O(x) = O^g(x^g) \tag{2.40}$$

when x is bounded by Σ and the symmetry could act on both the operator itself and spacetime. Now simply by choosing a different integration domain, we find that quantum symmetry is not necessary a representation of the symmetry group

$$U : G \to \mathrm{Aut}(H) \tag{2.41}$$

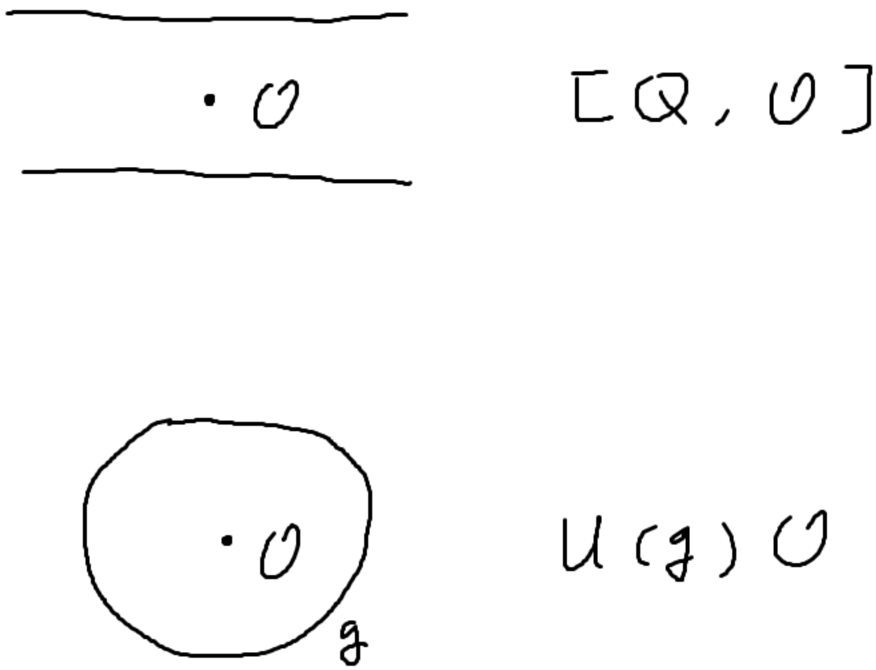

Figure 2.1: Action of topological operators

A more general form is a collection of topological operators, charged by some group element, that acts on point-like operators, and satisfy the fusion rule

$$U(\Sigma_1, g)U(\Sigma_2, h) = U(\Sigma_3, gh) \tag{2.42}$$

where Σ_1 can be continuously deformed into Σ_2 and Σ_3. The absence of insertion can be viewed as a special topological operator

$$U(\Sigma, 1) = 1 \tag{2.43}$$

Recall that the above derivation depends on a certain choice of fundamental fields and coordinate systems, while the calculus of topological operators does not. It is then suggested that topological operators are more intrinsic representation of symmetries in quantum field theory, although they are still representations. On the other hands, as long as we recognize a set of topological operators that are closed under mutual fusion, we will find a certain type of symmetry in the field theory. The fusion of topological operators would still work even if their charge g is not invertible, which corresponds to the representation of an algebra A instead of a group G. There are a lot to say about those topological operators, but before that, we need to pause a little bit and explain the algebraic structures emergent from physics.

2.4 Interlude: Higher Algebra from Physics

In the previous section, we briefly talked about the "calculus" of topological operators while the result seems to be some purely algebraic structure. This is a common phenomena in both mathematics and physics: when the object we are manipulating is topological (does not depend on geometric details), a structure originally defined by some version of calculus has often an algebraic form which allows us to derive many structural results. The algebras are so clean and powerful that people eventually forget their geometric origin. If we view the symmetries in quantum field theory from hindsight (in the sense of what we discussed in the first chapter), this might be what happens: we start with Ward identity and topological operators, then find that they are a simple algebraic structure, group, and then we can study representation theory of groups from the group algebra itself and the whole theory of Lie groups and Lie algebras does not contain anything about topological operators.

However, in this section, we are going to show that it is beneficial to remember the geometric origin, and at the same time review some physical ideas behind category theory, which will be used in the following sections. In physics, operational symmetries are not just a group element: we cannot do something as CPT-transformation. The symmetry transformations we can perform are usually the ones continuously connected to the identity element, like a rotation.

In physics, this kind of "path" connecting two operations could encode important information; for example, a particle can accumulate different phase when moved along different paths. It suggests that the algebraic structure extracted from physics should also describe these higher operations (path connecting two operations).

A traditional geometric representation looks like that: a space is represented by a point; structure-preserving maps between spaces are represented by an interval or arrow connecting the two point; maps between maps are represented by a 2-cell between two arrows. One example could be homotopy between maps. It is possible to formalize the above idea and define an even higher algebraic structure as something equivalent to a special simplicial set: being able to compose two maps means that given an ordered triangle $[0,1,2]$ with edge $(0,2)$ missing, we can extend it to a full triangle by assigning $(0,2)$ the composition of $f_{(0,1)}$ and $f_{(1,2)}$, and similarly for any n-simplex with one face missing, we should be able to extend it to a full simplex. Simplicial sets satisfying this (weak Kan) extension condition is a good model for $(\infty, 1)$-categories, where ∞ means there are maps at arbitrarily higher order (though they can be identity, the trivial one) and 1 means higher maps are invertible up to equivalence.

The above construction is fruitful[32, 33, 34] but does not quite fit the requirement of physics, where higher maps are not necessarily (weakly) invertible. A dual viewpoint seems to be more natural and useful.

Recall the calculus of topological operators discussed in the previous section: for each codim-1 hypersurfaces Σ, a label or charge or brane-type is associated to it, and let the set of them be denoted by X

$$X := \{U(\Sigma, g)\} \tag{2.44}$$

The fusion of two topological operator is again a topological operator, which defines a map

$$X \times X \to X \tag{2.45}$$

and the fused topological operator is denoted as

$$U(\Sigma, gh) := U(\Sigma, g)U(\Sigma, h) \tag{2.46}$$

Let us temporarily forget the issue of projective representation, then the result of fusion does not depend on the order

$$(U(\Sigma, g)U(\Sigma, h))U(\Sigma, k) = U(\Sigma, g)(U(\Sigma, h)U(\Sigma, k)) \tag{2.47}$$

Inserting nothing can be viewed as a special topological operator $e \in X$, whose fusion

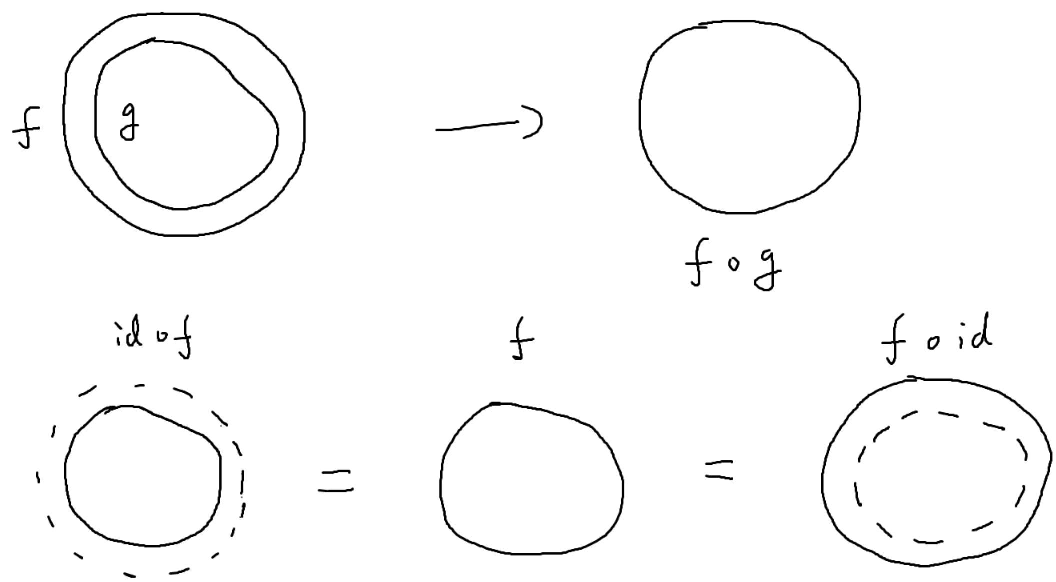

Figure 2.2: Topological operators form a category

with every other operator does not change anything

$$U(\Sigma, e)U(\Sigma, g) = U(\Sigma, g)U(\Sigma, e) = U(\Sigma, g) \tag{2.48}$$

We allow non-invertible operators, so the existence of anti-operator is not listed here.

Topological operators acts on point-like defects

$$U(\Sigma, g)O(x) = O^g(x) \tag{2.49}$$

Let A be the space of point-like defects at x, then the action defines a map

$$X \times A \to A \tag{2.50}$$

One might conclude that those topological operators form an algebra and the top

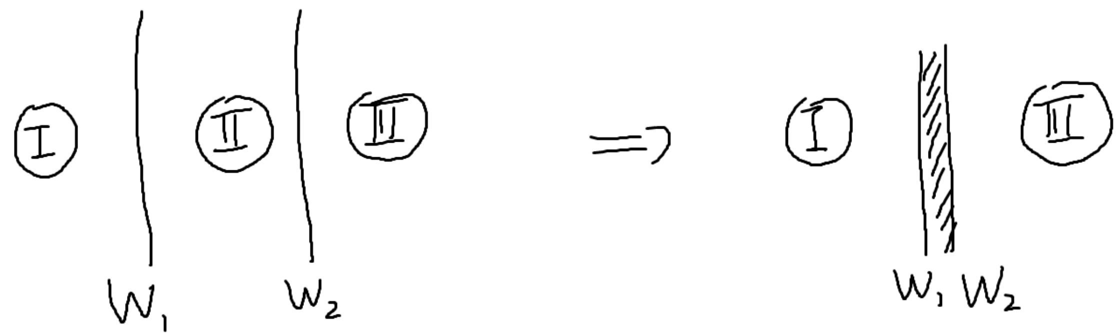

Figure 2.3: Domain walls

dimensional regions separated by those topological operators define representations of the algebra. However, we only discussed situations with fixed Σ.

Consider the following example: topological operators are now domain walls between theories and after passing through the domain wall, a particle in one theory becomes another particle in another theory, so that they are not necessarily belong to the same set A and a domain wall defines a map $A_1 \to A_2$. Notice that the map is in abstract sense: in set theory, the domain and codomain of functions should be well distinguished, but for a domain wall, which one is on the left and which one is on the right is just a problem of convention. This suggest that one can always reverse the order of domain and codomain, as long as we swap all the pairs simultaneously.

The composition of domain wall is again the fusion of them and the fusion is associative. There always exists a trivial domain wall. Then, calculus of domain walls (or in general codim-1 objects) is exactly the same as the usual category theory, where objects are physical theories and morphisms are domain walls.

Now it is easier to define higher categories: apart from topological operators $U(\Sigma, f)$

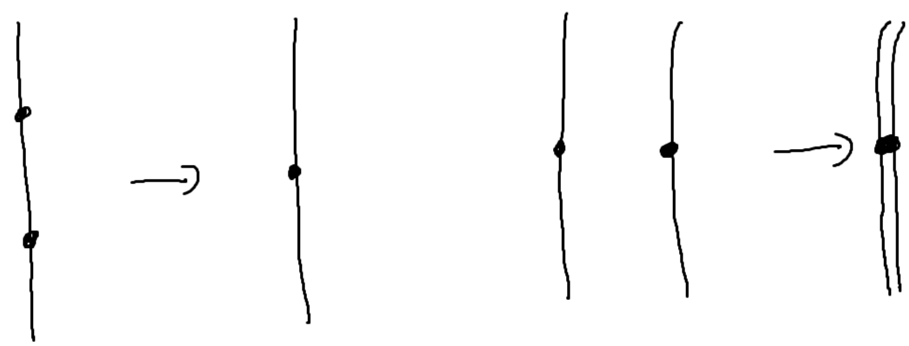

Figure 2.4: Vertical and horizontal fusions

supported on hypersurfaces, one could also consider higher codimensional topological operators

$$U(\Gamma, \alpha),\ \Gamma \subseteq \Sigma,\ \mathrm{codim}\Gamma = 2 \tag{2.51}$$

Now there are two ways of fusing these new operators:

1. Fix Σ and fuse Γ along Σ,
2. Fuse Γ supported on different Σs by fusing those codim-1 operators

The result should not depend on the order of fusing: given $\alpha_{i,j}$ supported on f_i where $i, j = 1, 2$

$$(\alpha_{1,1}\alpha_{1,2})(\alpha_{2,1}\alpha_{2,2}) = (\alpha_{1,1}\alpha_{2,1})(\alpha_{1,2}\alpha_{2,2}) \tag{2.52}$$

In fact, the above construction defines a 2-category. One might be skeptical whether it is necessary to define a categorical structure from geometric manipulations. First, the correspondence between k-morphism and codim-k surfaces is the standard string diagram

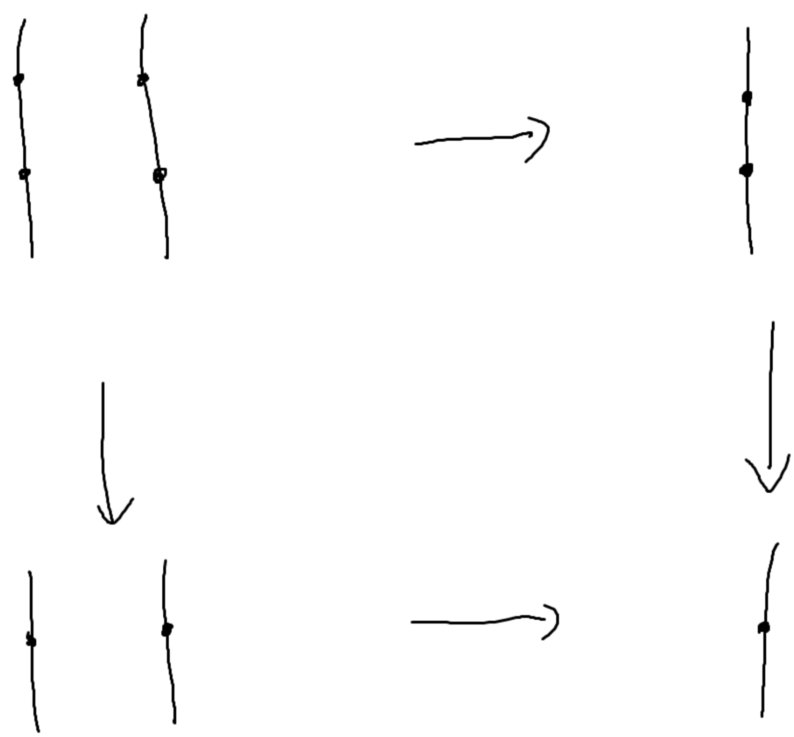

Figure 2.5: Consistency condition for 2-morphisms

technique[35], which also simplifies some constructions and arguments in group theory[36] and general relativity[37, 38]. More importantly the construction provides an operational picture for higher morphisms, which is very convenience for its usage in quantum field theory. The calculus of topological operators are one example.

Unlike $(\infty, 1)$-categories, there are no general theory for higher categories, but a natural conjecture is that in n-category, k-morphism in the simplicial set picture is a k-simplex can be viewed as the dual geometry of a codimension-k membrane in the above construction, and in this sense they describe two dual description of the same algebraic structure.

The final concept we would need is functor. Given two categories, we want to map k-morphisms to k-morphisms while preserving their structures, that's a functor. One might be wondering why other categorical constructions, like limits or adjunctions, are not discussed here. Typically, categorical constructions are performed by testing the relation of an object with all other objects in the category (universal properties), which requires a full knowledge of the whole category. However, in the study of non-topological quantum field theory, a precise definition of higher category is usually missing, which prevents us from doing categorical computations.

Now, we have enough preliminaries and could work with real path integrals, and explain in what sense "calculus" of topological operators, i.e. (higher-)category theories, are in fact some generalization of differential calculus.

2.5 Path Integral as A Functor

It is used to be expected that path integral is just a more complicated Lebesgue integration and all we need to do is to define the measure of path integral rigorously, at least for Euclidean field theories

$$\int [\mathcal{D}\phi] e^{-S_E[\phi]} \tag{2.53}$$

For quantum mechanics, the goal is achieved by the formula of Feynman and Kac[39, 40], but a general construction for higher dimensions is absent.

It should not be a surprise from the viewpoint of physics: quantum mechanics is a $0+1$ dimensional quantum field theory which is defined for all the scales. On the other hand, a general quantum field theory is an effective theory that describe the long range degree of freedoms, or in other words low energy physics under certain cutoff. This is an essential difference: unlike usual integrals where all roads lead to the same continuum limit, the continuum of spacetime or integration domain is more like an illusion and the formula of path integral should be better viewed as some formal notation. The formal notation is useful: in the previous sections we showed how nontrivial results can be derived from it; therefore a natural question is: if path integral, at least in its current form, is not a well-defined construction that has all the properties we want, what might be the most important ones?

2.5.1 Locality

We start with the following idea: as long as we can compute the path integral, we should in principle know everything about the physics.

It would be good to have a Lagrangian density, but not all the theories are constructed from a Lagrangian, not to say a Lagrangian density. On the other hand, we can prove results for field theories, though more or less heuristically, without assuming the existence of a Lagrangian, like what we did in the previous section for topological operators. In this sense, a theory of path integral should not depends on the existence of Lagrangian.

Without assuming anything related to Lagrangian, let's consider what we could do if we are able to compute path integral approximately/numerically. First given a closed manifold M, we should be able to compute the path integral on that, which is the partition function

$$\mathcal{Z} = \int [\mathcal{D}\phi] e^{-S_E[\phi]} \tag{2.54}$$

Strictly speaking, partition function is defined up to a normalization constant, which implies that path integral is valued in an $\mathbb{C}^\times$-torsor. However, on the one hand, introducing the whole theory of (possibly higher) torsors brings unnecessary complexities into the theory; on the other hand, just like physicists still use state vectors and keep tracking of phase factors while quantum states are in fact points in a projective space (complex lines in the Hilbert space), it might be beneficial to keep the plain form of partition function. In the following we still view the result of path integral as a number in $\mathbb{C}$.

Evaluating partition function itself as a black box is not satisfying. Though it is possible to couple the theory with a background current J and compute a generating functional,

$$W[J] := \int [\mathcal{D}\phi] e^{-S_E[\phi] - \int J\phi} \tag{2.55}$$

and its Legendre transformation

$$\Gamma[\phi] := -\int \phi J + W[J],\ \frac{\delta W[J]}{\delta J} = \phi \tag{2.56}$$

not every observable in a quantum field can be reduced to them. The expected value of extended operators, like Wilson loops, is one example. Another example is theories with nontrivial boundaries.

It is hence suggested that the recipe of a path integral should at least contain the following information: given the boundary condition b on ∂M, a complex number is computed as output

$$I : b \mapsto \int_{\phi|_{\partial M}=b} [\mathcal{D}\phi] e^{-S_E[\phi]} \tag{2.57}$$

The boundary condition can be very general data, but it would be helpful to take a look at the case where they are really boundary condition of field configurations.

Consider the following quantization scheme for path integral: let M be a manifold whose boundary ∂M has two component N_1, N_2. The Hilbert space associated to each one are generated by formal linear combination of field configurations

$$B_i := \mathrm{Span}(\mathrm{Map}(N_i, \mathbb{C})) \tag{2.58}$$

Given two basis boundary condition

$$b_i \in \mathrm{Map}(N_i, \mathbb{C}) \subseteq B_i \tag{2.59}$$

we can compute the path integral

$$I[b_1, b_2] = \int_{\phi|_{N_i}=b_i} [\mathcal{D}\phi] e^{-S_E[\phi]} \tag{2.60}$$

by formally integrating out degree of freedoms supported in $M - \partial M$, which is interpreted as the matrix element between b_1 and b_2

$$\langle b_2 | \, e^{-\tau H} \, | b_1 \rangle = I[b_1, b_2] \tag{2.61}$$

and hence the path integral defines a linear transformation

$$I : B_1 \to B_2 \tag{2.62}$$

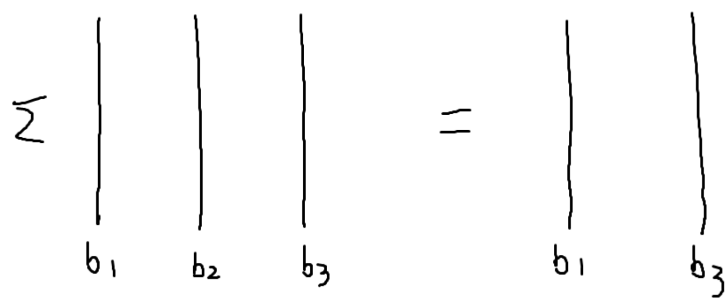

Figure 2.6: Glue two path integrals

Ordinary integrals can be computed by decomposing the integration domain into small intervals and approximate the integral by the value of the function on that interval (weighted by the measure of the interval). This is also the key observation that makes path integral computable is: we can decompose the spacetime into two pieces M_1, M_2 which intersect at a one codimenional higher submanifold, and the boundary condition is decomposed into b_1, b_2, b_{12}, b_{21}, where $b_{12} = \bar{b}_{21}$ is supported in $M_1 \cap M_2$, then compute

$$I_1[M_1, b_1, b_{12}], I_2[M_1, b_1, b_{21}] \tag{2.63}$$

and the composition

$$I[M, b] = \int [\mathcal{D}b_{12}] I_1[M_1, b_1, b_{12}] I_2[M_1, b_1, b_{21}] \tag{2.64}$$

Repeat the procedure above, we will be able to reduce the computation of path integral to a summation of boundary conditions on codimension-1 surfaces. In theories where there is a simple relation between those boundary conditions and classical field configurations, the summation is equivalent to the folklore "path integral is a sum over all field

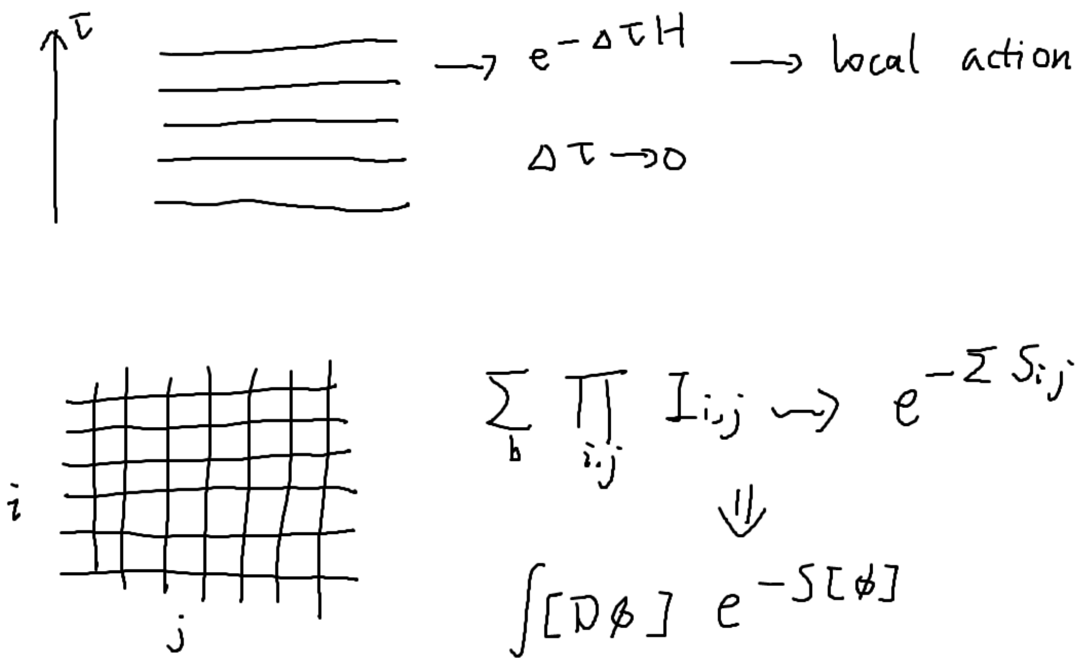

Figure 2.7: Domain of path integral

configurations".

Since we are working in the Euclidean signature, there does not exist a canonical direction of time, and instead of a linear transformation, a path integral would better be viewed as a linear functional

$$I[M, -] : B \to \mathbb{C} \tag{2.65}$$

and the composition rule above should be viewed as contraction of tensors

$$I^{M_1}_{b_1,b_{12}} I^{M_2}_{b_{12},b_2} = I^{M}_{b_1,b_2} \tag{2.66}$$

where b_k here might better be viewed as abstract indices.

Let us consider a trivial case where the boundary condition is a singleton set, then the path integral is simply a number

$$I : M \mapsto I[M] \in \mathbb{C} \tag{2.67}$$

and the composition rule becomes

$$I[M_1 \cup M_2] = I[M_1]I[M_2] \tag{2.68}$$

Taking the logarithm on both sides, we recover the additivity of measures. In this sense, the Lebesgue integral is a special case of path integral, which justifies our previous statement that path integral is a generalized calculus.

When the path integral on a small spacetime patch can be written as a local functional (the minus logarithm of it is called action S_E), we recover the formal path integral

$$\int [\mathcal{D}\phi] e^{-S_E[\phi]} \tag{2.69}$$

where the subscript E should now be understood as effective instead of Euclidean.

This is the intuition behind the Atiyah-Segal-Kontsevitch definition of quantum field theory:

1. There is a geometric category of n-dimensional bordisms Bord_n and a suitable category of vector spaces Vect
2. Objects in Bord_n are $(n-1)$-dimensional manifolds
3. Morphisms in Bord_n are cobordism connecting them
4. A quantum field theory is a functor F that maps $n-1$ dimensional manifolds to vector spaces, which are interpreted as the Hilbert space of the system

5. n dimensional cobordisms are mapped to linear transformations, which are interpreted as the result of path integral

6. Disjoint union of manifolds are mapped to tensor product of vector spaces

In fact, the above list is just an over-simplified description. There are many technical details involved. For the domain of the functor, boundary conditions are typically defined on germs of manifolds, while for the codomain of the functor, we need to choose a suitable category of topological vector spaces so that the computation converges. However, the path integral picture should always work. In the next section we will explain how to define path integral for discrete geometry but before that we need to talk about an important generalization of the above framework.

2.5.2 Full locality

Looking at the functorial definition of quantum field theory, a natural question is why we stop at codimension 1. This kind of consideration leads to extended quantum field theories where we have data for higher codimensional geometry.

However, there is a huge gap between this mathematical consideration and physics. While the path integral interpretation of Atiyah-style definitions is more or less well-known, the physical interpretation of extended theories, which involves higher categorical computation, are less known. Besides, it is often believed that the extended field theories are mainly for topological field theories.

In the remaining part of this section, we are going to show that extended structures are in fact quite natural, and every quantum field theory is extended.

The key observation is again the principle of locality: if the partition function can be computed by cutting the spacetime into small cells and gluing them together by multiplying the subintegrals and add them together, we should be able to do similar things for the space of states.

Consider an $n-1$ dimensional slice N_1 with boundary $N_2 := \partial N_1$. The Hilbert space of the theory is constructed from field configurations

$$\mathrm{Map}(N_1, \mathbb{C}) \tag{2.70}$$

but now N_1 itself has boundaries, which could restrict the possible field configurations.

Let b_2 be the constraint of field configuration on N_2, for example the Dirichlet boundary condition

$$\phi|_{N_2}(z) = b_2(z),\ z \in N_2 \tag{2.71}$$

then the Hilbert space for N_1 is no longer

$$\mathrm{Span}_{\mathbb{C}}(\mathrm{Map}(N_1, \mathbb{C})) \tag{2.72}$$

but a subspace of that. In this sense, given a boundary condition N_2 of boundary condition N_1, we get a vector space

$$b_2 \mapsto V_{b_2} \in \mathrm{Vect} \tag{2.73}$$

Recall that the map

$$b_1 \mapsto I[b_1] \in \mathbb{C} \tag{2.74}$$

can be interpreted as a linear map or a linear functional, and we can use it to cut and paste the path integrals on smaller regions, we should be able to do similar things for the above construction.

In $\mathbb{C}$, one can do addition and multiplication and the identity element exists for both operations. In Vect, the concept of summation is the direct sum

$$\mathrm{Span}(B_1) \oplus \mathrm{Span}(B_2) \simeq \mathrm{Span}(B_1 \cup B_2) \tag{2.75}$$

while the multiplication is given by the tensor product

$$\mathrm{Span}(B_1) \otimes \mathrm{Span}(B_2) \simeq \mathrm{Span}(B_1 \times B_2) \tag{2.76}$$

The multiplication identity is $\mathbb{C}$ and the addition identity is 0. This is the structure of a 2-vector space, where linear combinations have vector spaces instead of complex numbers as their coefficients. The corresponding category is written as 2Vect.

Given two boundary condition $b_{2,1}, b_{2,2}$, if the vector space determined by them is $V(b_{2,1}, b_{2,2})$, then we have the following gluing rule

$$V(b_{2,1}, b_{2,3}) = \bigoplus_{b_{2,2} \in B_2} \left(V(b_{2,1}, b_{2,2}) \otimes V(b_{2,2}, b_{2,3})\right) \tag{2.77}$$

this is what we could use to glue Hilbert spaces.

Let's consider a simple situation where B_2 is a singleton. The physical meaning is that the common boundary make no restriction on both sides, in other words, we have two non-interacting subsystems. The above equation becomes

$$V(b_{2,1}, b_{2,3}) = V(b_{2,1}, *) \otimes V(*, b_{2,3}) \tag{2.78}$$

which simply says composition of quantum systems is performed by tensor product. In this sense, the requirement of field theory functor being monoidal, i.e. maps disjoint unions to tensor product, is just a special case of the calculus of codimension-2 data.

We just showed that every quantum field theory is extended as long as it is defined

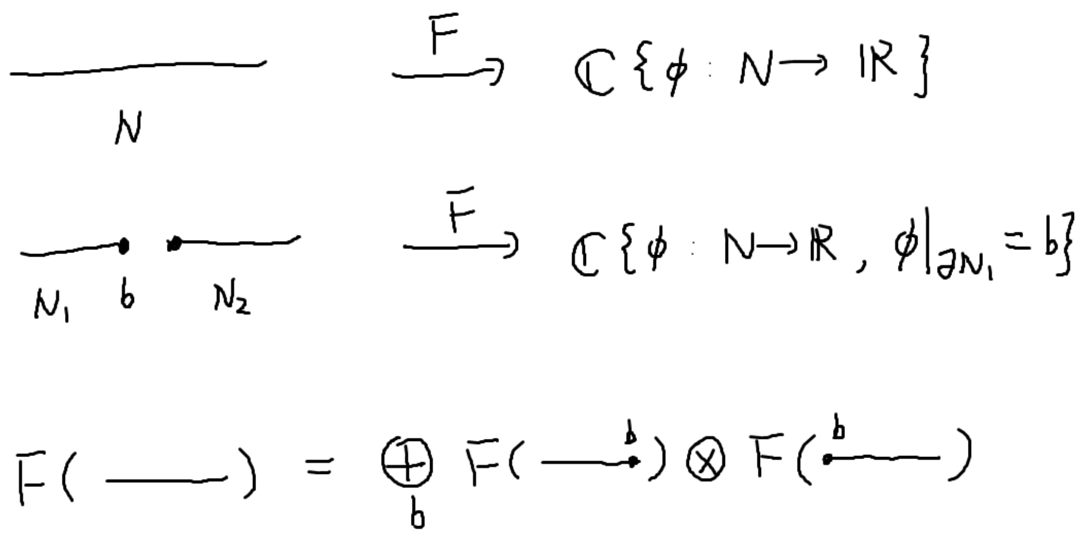

Figure 2.8: Glue codim-2 surfaces

by a reasonable path integral:

1. Given boundary condition, we can evaluate path integrals on the n-dimensional bulk, which is valued in $\mathbb{C}$
2. Boundary conditions are hence $\mathbb{C}$-valued linear combination, which is valued in Vect
3. Given boundary condition of boundary condition, we can construct the space of states on the $n-1$ dimensional boundary, which is valued in Vect
4. Boundary conditions of boundary conditions are hence Vect-valued linear combination, which is valued in 2Vect

We could continue the above construction to all codimensions, which is the full locality of a field theory. However, in the following part of this thesis, we will only consider one-codimensional-higher extended theories described above.

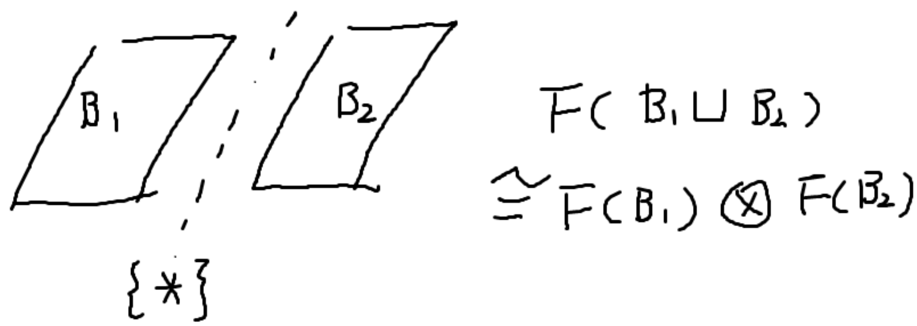

Figure 2.9: Monoidality

As a simple corollary, when the boundary condition is empty, i.e. there are no degree of freedoms on the boundary and the two subsystems are isolated, the extended path integral shows that the total system is the tensor product of the two subsystems, which used to be part of the Atiyah-style axioms but now a provable property in our framework.

2.6 Lattice Models

In this section we will explain in what sense the above formal definition of path integral would also work for lattice models, and hence we have a well-defined path integral for lattice-regularized field theories.

The canonical way of doing geometry is based on the concept of manifolds, which is a two-step construction: first define differential geometry with the usual tools of calculus in $\mathbb{R}^n$, then define global geometry by choosing a open covering of the underlying space and gluing them together with partition of unity.

It is well known that the entire theory of general relativity is built upon that. However,

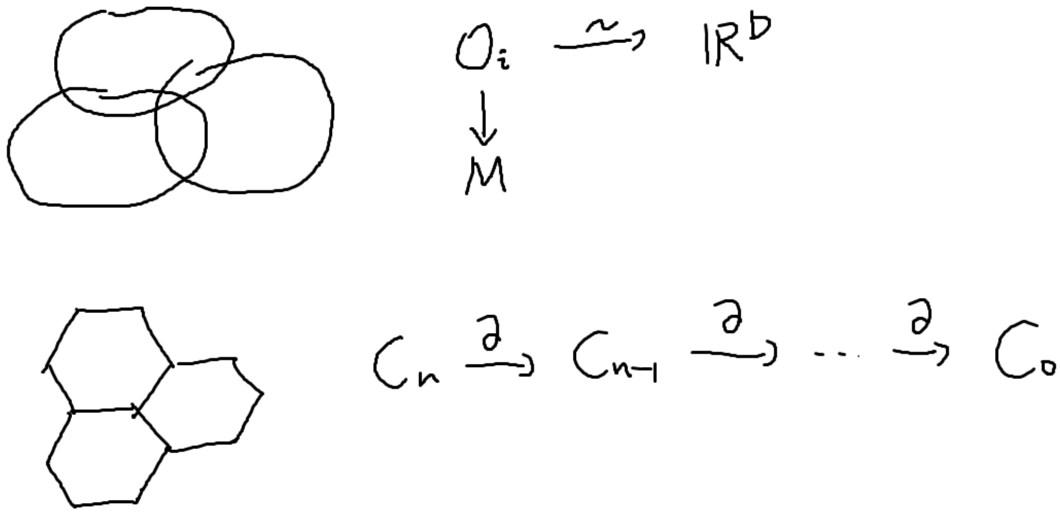

Figure 2.10: Cells vs manifolds

general relativity is too geometric in the sense that most of the information is already encoded into the (pseudo)-Riemannian manifold. It might be more illustrating when considering the local algebra approach to quantum field theory: for each open set R in the spacetime M, a von Neumann algebra $\mathcal{A}_R$ is defined which consists of the operators supported in R, and we have shown how to use this construction to extract non-local operators from the causal structure of a field theory.

As we have explained, the local algebra approach is not satisfying for what we are interested in, which would better be described by path integrals.

Path integrals are built upon a different geometric structure: a cell-decomposition of spacetime. In the above analysis of formal path integrals, we allow an arbitrarily small spacetime region; however, non-topological/conformal interacting field theories are usually defined under certain energy scale and require regularization, which means we are only allowed to probe certain subset of spacetime cells.

Besides, in topological quantum field theory, people are interested in the value of field functor on all possible manifolds, while in the study of non-topological field theories, it is usually enough to study the theory on a fixed background geometry and it is actually difficult to define a field theory on a general manifold. Given these considerations, we propose a modified definition of lattice regularized quantum field theory:

1. A quantum field theory is a functor from a discrete bordism category Bord to the category of higher vector spaces

$$F : \mathrm{Bord}(M) \to n\mathrm{Vect} \tag{2.79}$$

2. The discrete bordism category Bord(M) depends on the background geometry M, where objects are a subset of points, 1-morphisms are lines connecting points, 2-morphisms are surfaces bounded by 1-morphisms, ...

3. For each cell C_n, given the boundary condition b_1 on its codim-1 boundary ∂C_n, the field theory functor F assigns a complex number, which is the building block of the path integral, whose minus logarithm is called (effective) action

$$e^{-S_E[C_n,b_1]} = F(C_n)(b_1) \tag{2.80}$$

4. Cells are glued by contracting boundary condition b'' on their common boundaries

$$F(C \cup C')(b,b') = \sum_{b''} F(C)(b,b'')F(C')(b'',b') \tag{2.81}$$

5. For each $n-1$-cell C_{n-1}, given the boundary condition b_2 on its codim-2 boundary ∂C_{n-1}, the field theory functor F assigns a vector space, which is the building block

of the space of states

$$F(C_{n-1}) = \bigoplus_{b_2} F(b_2) \tag{2.82}$$

6. $n-1$-cells are glued by higher version of contraction

$$F(C_{n-1,1} \cup C_{n-1,2}) = \bigoplus_{b_2} (F(C_{n-1,1}, b_2) \otimes F(C_{n-1,2}, b_2)) \tag{2.83}$$

7. In principle, the construction should be done for all codimensions

Let us illustrate the above construction on a cubic lattice with spins on sites and interactions between nearest pairs of spins. Then the Hilbert space is the tensor product of single spins. This can be derived from the calculus of our extende quantum field theories: the $n-2$ dimensional boundary conditions are the spin configurations on that. Consider a boundary consisting of a single spin, if it separates two subsystem $\mathcal{H}_{1,2}$, then the gluing rule gives

$$\mathcal{H} = \bigoplus_{s\in\{\pm\}} (\mathcal{H}_1 \otimes \mathcal{H}_2) \simeq \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathbb{C}^2 \tag{2.84}$$

Breaking the hypersurface into minimal cells and repeatedly use the above relation, we can show that the total Hilbert space is just the tensor product of each spin.

If we choose the boundary as the one separating two nearest pair of spins, then the boundary contains no data and the total system is the tensor product of the two subsystem. Repeat this computation we will find that the total Hilbert space is empty. The reason is that in this case there is no interaction between the nearest pair and we are effectively introducing a net of defects into the system.

Now let's show a more concrete and nontrivial example: a system with up to first order derivatives in its effective action. The matter field can be represented by

$$(\partial\phi)^2 + V(\phi) \to e^{-\sum_{\langle i,j\rangle} J_{ij} s_i s_j - \sum_i V(s_i)} \tag{2.85}$$

where

$$J_{ij} = 0 \text{ if } |i-j| > 1 \tag{2.86}$$

An intuition is that the boundary condition should encode information of spins on the opposite side of the boundary. There are multiple methods to achieve the same goal. One of them is to define a pair of field variables on the boundary and glue them together by introducing a delta function (which is a multiplication of the number of surfaces minus one delta functions identifying a pair of surfaces) Another method, which might show

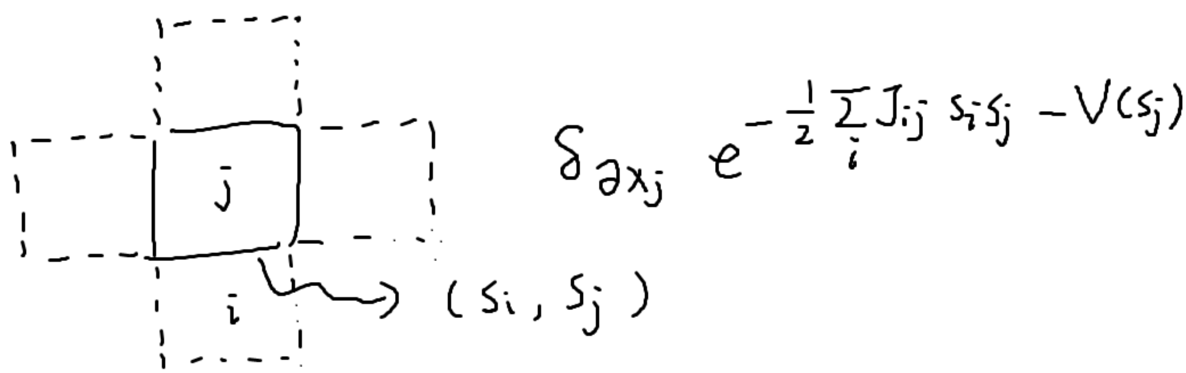

Figure 2.11: Path integral for cubic lattice

more intuition than the above construction, is to split a single field variable into the number of faces copies of that, then introducing a delta function identifying all of its faces and a potential function for each cell whose faces are created from the same field variable, and the interacting term for other cells.

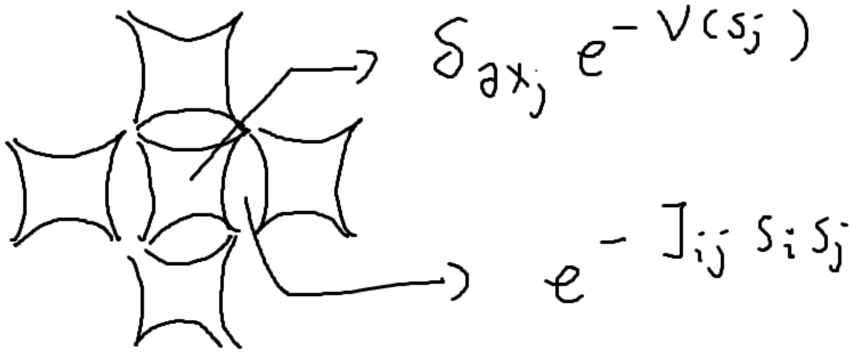

Figure 2.12: Glue spins

In both of the above schemes, if we follow the traditional path integral quantization intuition, we will find that we actually create an enlarged Hilbert space, and the delta functions in the effective action serve as a gauge fixing term selecting the physical Hilbert subspaces.

Chapter 3

Symmetry and Defect

In the previous chapter, we showed the effectiveness of path integral formalism in quantum field theory and how algebraic structures emerge from it. In this chapter, we are going to continue the exploration of structures in quantum field theory.

3.1 Generalized Symmetries

As we have shown in the previous chapter, symmetries in quantum field theory can be represented by topological operators of codimension 1.

Nothing prevents us from considering higher codimensional topological operators. These are called higher form symmetries.

In general, a p-form symmetry is a codimension $p+1$ operator $U(\Sigma_{p+1})$ charged by some algebra A. When fusing higher codimensional operators, since they can be fused with any order by continuously swapping their location, we have

$$U(\Sigma_{p+1}, a_1)U(\Sigma_{p+1}, a_2) = U(\Sigma_{p+1}, a_2)U(\Sigma_{p+1}, a_1) \tag{3.1}$$

which implies that A is always commutative for $p > 0$.

Higher form symmetries are not something artificial. In fact it appears in free Maxwell theory. Recall that operators for 0-form symmetries are topological because of

$$d\,j = 0 \tag{3.2}$$

in regions without defects, we can construct 1-form symmetries from the following Maxwell equation

$$dF = 0,\; d * F = 0 \tag{3.3}$$

The corresponding topological operators are

$$U^m(\Sigma_2, g) = \exp\left(i\theta \int_{\Sigma_2} F\right),\; g = e^{i\theta} \tag{3.4}$$

$$U^e(\Sigma_2, g) = \exp\left(i\theta \int_{\Sigma_2} *F\right),\; g = e^{i\theta} \tag{3.5}$$

and the charged objects are Wilson lines.

In some sense, the higher form symmetries can be reconstructed from charged operators. For extended operators L_1, L_2 connected by a one dimensional less domain wall, we can detect its charge by surrounding it with the topological operator. The topological operator can be continuously deformed from L_1 to L_2 and hence L_1 and L_2 has the same charge, so the equivalent classes of charged objects are those that cannot be connected by a domain wall. In Maxwell theory, Wilson lines are classified by their corresponding representations, which is labelled by an integer, and different types of Wilson lines cannot be connected together, so the symmetries are the Pontryagin dual to $\mathbb{Z}$ ($\hat{G}$, the Pontryagin dual of an abelian group G, is the group homomorphism $G \to U(1)$), which is $U(1)$.

In the previous chapter, we already showed that every quantum field theory is extended and it is necessary to study the higher structure. Here, we showed it in another way that extended operators are quite common and the study of higher structures will be required even for free Maxwell theory.

A more general construction of theories with higher form symmetries are the topological ones. In a topological theory, all the excitations are topological and symmetries and defects are indistinguishable.

One way of obtaining topological theories from an action, instead of some more abstract construction, is the Dijkgraaf-Witten theory

$$S = 2\pi \int a_p \cup_\eta \delta b_{d-p-1} \tag{3.6}$$

where

1. a_p is a G-valued p-cochain, which assigns a group element g to each p-simplex
2. b_{d-p-1} is a $\hat{G}$-valued $d-p-1$-cochain
3. δ maps a cochain b to a one-degree higher one, whose value is the signed summation of the value of b on its boundary
4. η is the natural pairing $G \times \hat{G} \to \mathbb{R}/\mathbb{Z}$
5. $\cup_\eta$ is the cup product maps p-cochain and q-cochain to $p+q$-cochain, the value is given by evaluating these two cochains separately and then use the pairing η

The discrete gauge theory will play an important role in the study of symmetries, although we will take a slightly different approach.

3.2 Structures of Quantum Field Theory

Now we will continue the discussion of quantum field theory defined by path integrals. In the previous discussion of topological operators, we use some vague language about operators / defects / insertions of path integral. We would like to make it slightly more clear here.

First, the action of our path integral is by definition local

$$\int \prod_i db_{1,i} e^{-\sum_j S_E[C_{n,j}]} \tag{3.7}$$

where $b_{1,i}$ are codimention-1 boundary conditions and $C_{n,j}$ is a cell decomposition of the spacetime.

A defect, in general, is a modification of the path integral. Consider a path integral defined by locally defined action, a modification of the path integral is another local action S' differing by a few local terms, then

$$e^{-S'_E} = e^{-S_E + (S_E - S'_E)} = e^{-S_E} e^{S_E - S'_E} \tag{3.8}$$

shows that defect introduced by modifying the action is equivalent to adding insertions into the path integral.

For example, in the last example of the previous chapter, we showed how to build the extended path integral for the dynamics of matter fields. We can introduce a coupling with background gauge fields by replacing the derivatives with their covariant versions, or in the lattice version by modifying the interacting cells.

$$(\partial\phi)^2 \mapsto (D\phi)^2 \to \phi_2^\dagger U \phi_1 + \phi_1^\dagger U^\dagger \phi_2 \tag{3.9}$$

This is equivalent to introducing top-dimensional defects in all interacting cells, or equivalently after reducing to the previous quantization scheme, codim-1 defects on all boundaries (s_1, s_2).

Let us turn to the fusion of defects. For non-topological theories, if we want to fuse defects that does not overlap with each other, then the only thing we can do is to surround them by a hypersurface and evaluate the path integral of that. The gluing rule of path integral suggests that knowing the data on the hypersuface is equivalent to the data on the defects to fuse. This is a generalized version of state-operator correspondence, where defects correspond to states, but there is no canonical way. Taking some categorical limit might formally construct a canonical correspondence because of the universal properties, but it does not seem to provide additional information, at least in the discrete case.

When defects overlap with each other, they can be fused by considering the vacuum between them. For example, for codimensional 2 defects $b_{2,1}$ and $b_{2,2}$ with bulk $b_{1,1}$ and $b_{1,2}$, if there are no restriction on the vacuum between them

$$V_{b_{2,1},b_{2,2}} = \mathbb{C} \tag{3.10}$$

then the gluing rule is

$$\bigoplus_{b_{2,1}} \bigoplus_{b_{2,2}} V(b_{1,1}) \otimes \mathbb{C} \otimes V(b_{1,2}) = \bigoplus_{b_{2,1},b_{2,2}} V(b_{1,1}) \otimes V(b_{1,2}) \tag{3.11}$$

which is the stacking construction (put two theory together as if they do not interact with each other).

The data

$$V_{b_{2,1},b_{2,2}} = \mathbb{C} \tag{3.12}$$

can be interpreted as an insulated vacuum, or in other words, there is nothing in the spacetime.

A trivial theory is a functor that assigns 1 at top level, $\mathbb{C}$ at the next level, and singleton boundary condition that always maps to $\mathbb{C}$. The stacking of such a theory with any other theory will not change that theory. It is in this sense that stacking defines a product operation among all possible quantum field theories and trivial theory is its identity element.

Nontrival fusions

$$V_{b_{2,1},b_{2,2}} = 1(b_{2,1}, b_{2,2})\mathbb{C} \tag{3.13}$$

where $1(-,-)$ is a $0-1$ valued function, is also possible. An example could be a delta function selecting certain boundary conditions. This is equivalent to inserting a delta function into the path integral, while now be interpreted as a defect in the vacuum.

From the above examples, it seems to be reasonable to view a quantum field theory as something that is clouded by (possibly twisted) vacuum.

Another issue that we have not discussed yet is the background field. In the standard definition of functorial field theories the background field is defined as a sheaf over the category of open sets of the background spacetime, valued in category of simplicial sets. This basically says that background field is a local field. However, as we have seen in the previous section, the quantum field theory can be defined on cells, which represents a regularized theory. The manifold structure of the background spacetime is no longer needed. Then, it would be natural to directly define the background field on the cell structure. After all, classical fields should contain no more information than quantum fields (we need to make more accurate measurement in order to observe quantum phenomena). One possible strategy is to modify the target category from nVect to something more complicated. However, we will see that there are more natural way of dealing with background fields.

It would be interesting to consider maps between two quantum field theories. A general functor in higher categories could be difficult to define, so here we will only consider some special case.

The first one is compactification. If we have an $n+1$ dimensional theory F defined on $M \times S$, then we can define an n dimensional theory F' on M by mapping $F'(X)$ to $F(X \times S)$. This will be useful in the next section.

Another possible one is the projection map from one field theory to another, where the projection is defined for both the discrete bordism category and nVect. The physical interpretation should be the renormalization group, and the limit of a sequence of such kinds of projections describe an IR fixed point. Morphisms between functorial field the-

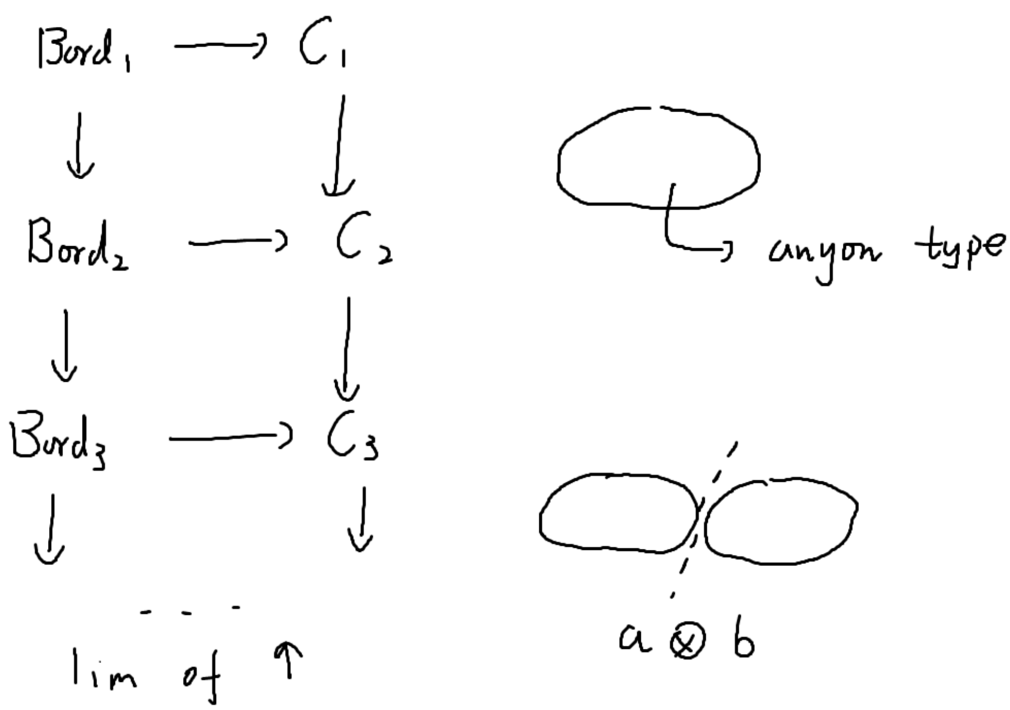

Figure 3.1: Renormalization group morphisms

ories should be a commutative diagram: maps between bordisms identify observables, while maps between target categories identify values of observables, and the commutativity should be witnessed by the information lost in the renormalization process. In the IR fixed point, there should be strong finiteness on the target category and the extended path integral essentially describe the topological dynamics of anyons. We do not have a concrete model for the above statement but that would be the intuition from the extended path integral framework.

We discussed some structures behind the path integral above. In the next section, we are going to talk about one of the most important data in field theories: symmetry.

3.3 Symmetry as A Topological Field Theory

We showed that (higher-form) symmetries are represented by topological operators. On the other hand, in a topological theory, topological operators and topological defects are indistinguishable and hence a natural conjecture is that there are some connection between topological field theories and symmetric field theories[41].

Instead of considering a single symmetry operator $U(\Sigma, g)$, let us consider a net of symmetry operators $U(\Sigma_1, g_1) \dots U(\Sigma_k, g_k)$. They should satisfy a consistency condition: if a defect moves along a loop and passes a sequence of symmetry operators, then the net effect should be zero. Of course, non-topological defect cannot move freely, so it should be understood as a passive movement, i.e. move the network of topological operators in opposite direction.

This defines a flat G-connection A. The same data can be computed from a topological theory, which assigns codimension 1 cells a group algebra $\mathbb{C}[G]$, and the fusion of codimension 2 cells (discussed in the previous section) guarantees that the valid states are linear combinations of flat G-connections. Pairing this topological theory with the original theory, we will get an external way of characterizing symmetries in a quantum field theory.

To be more exact, a quantum field theory F with symmetry group G, is a composition of a non-topological theory $\tilde{F}$, a topological theory discussed above ρ, and a bulk theory σ connecting them together, and the original quantum field theory is equivalent to the compactification of the above triplet

$$F \simeq \rho \otimes_{\sigma} \tilde{F} \tag{3.14}$$

The physical intuition is that bulk theory is topological so the length between the two boundaries does not make sense, and we should be able to shrink their distance until the extra dimension is invisible. This is essentially a dimensional reduction process, which

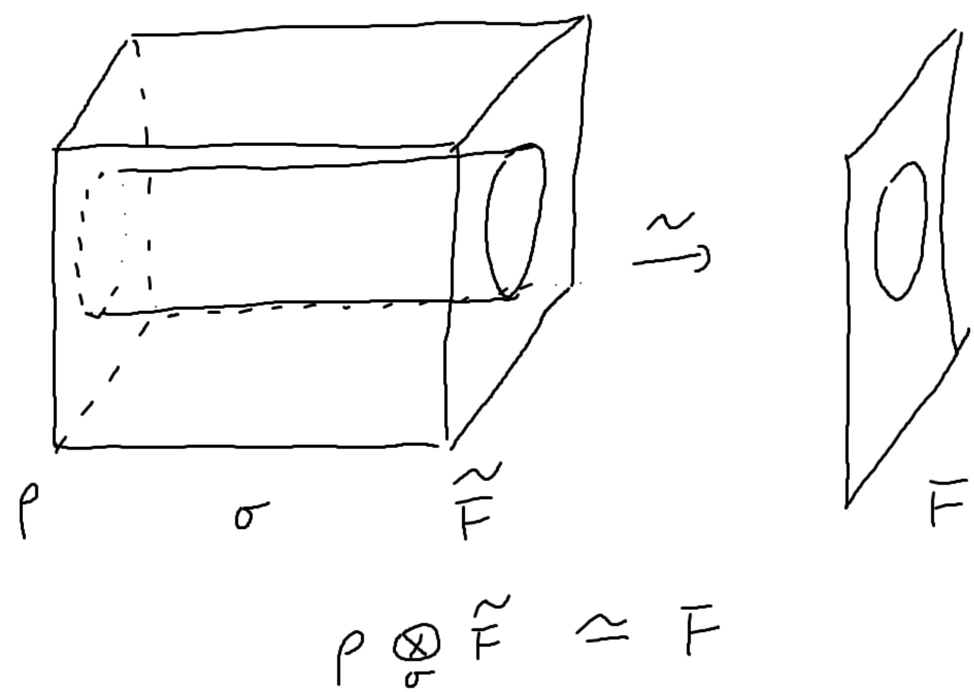

Figure 3.2: Symmetry topological field theory

is a special case of our previous intuition for renormalization group morphism. Here at bordism level we identify

$$X \mapsto X \times I \tag{3.15}$$

and at functor level we identify

$$F(X) = (\rho \otimes_\sigma \tilde{F})(X \times I) \tag{3.16}$$

This is an exact renormalization and the commutative diagram should be strict.

As an example, quantum mechanics with symmetry G can be described by a $1+1$ dimensional field theory with boundaries. For the non-topological boundary, a closed loop consists of an interval of distance τ on the boundary and another loop in the bulk, with point-like defects on it, labelled by $g \in G$. The value of the field theory on the top degree is

$$\langle h_o| \, e^{-\tau H} U(g) \, |h_i\rangle \tag{3.17}$$

The point-like defects act as a symmetry operator because it can approach either the in state or the out state from the bulk, which shall not change the result of the path integral because the bulk is a topological theory. The point-like excitations can be moved freely in the bulk and can be moved onto the topological boundary. Evaluating the field theory on the topological boundary gives a vector space spanned by flat G-connections. The compatification is equivalent to pairing a flat G-connection with the above amplitude.

We can make it more concrete by explicitly assigning every extended object a corresponding value in our extended path integral. We assign the symmetry group G for each point in the bulk, promote the Hilbert space to a $\mathbb{C}[G]$-module, and the bulk intervals are assigned $\mathbb{C}[G]$ as a bimodule. The extended structure is defined by the module tensor product rules.

Notice that we allow a projective representation of the symmetry group by introducing

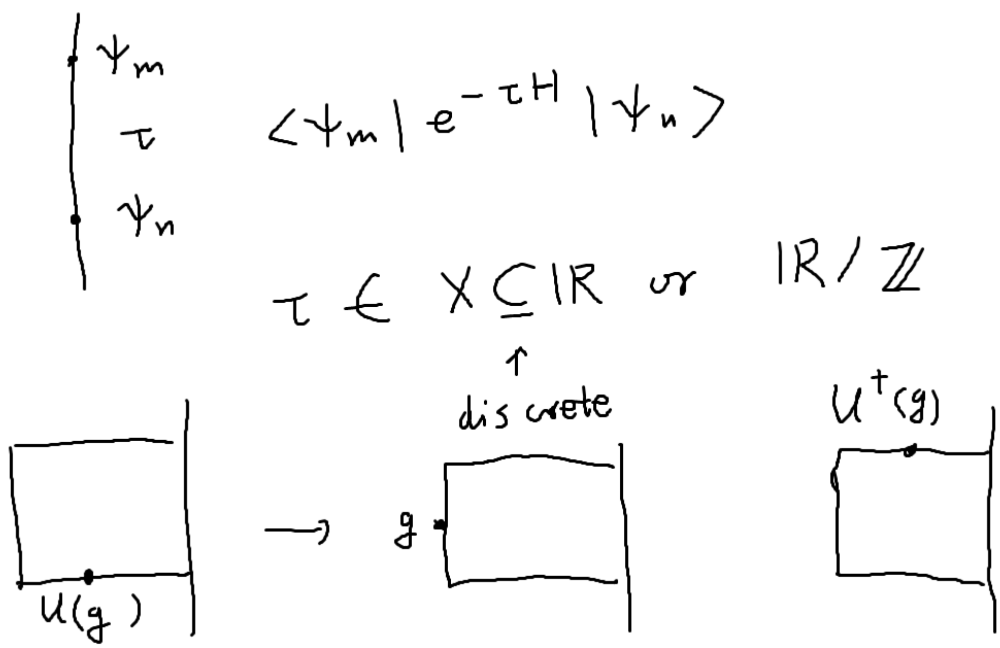

Figure 3.3: Symmetry in quantum mechanics

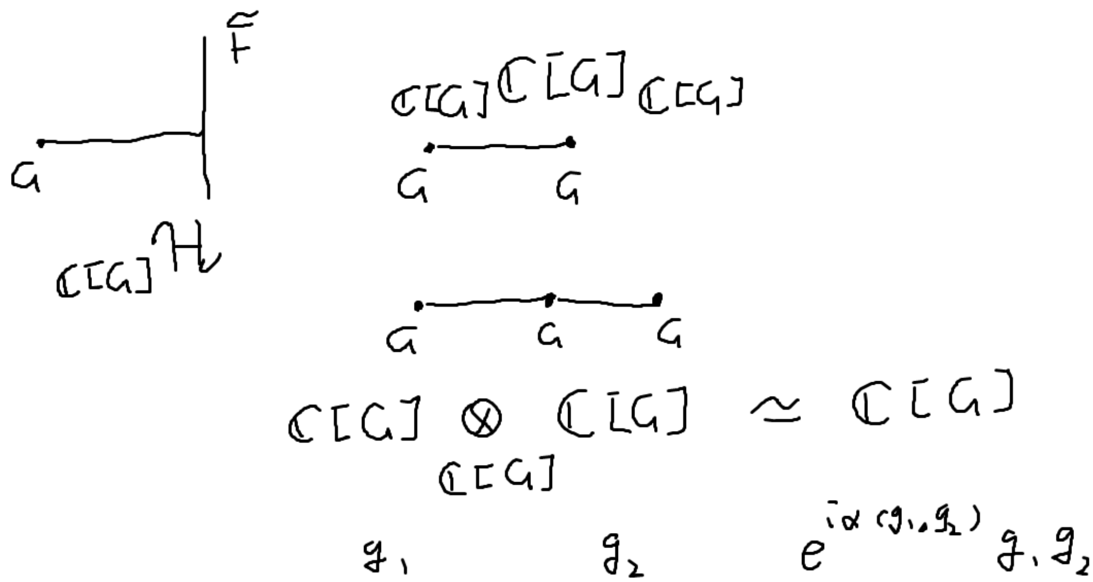

Figure 3.4: Representation of symmetry group

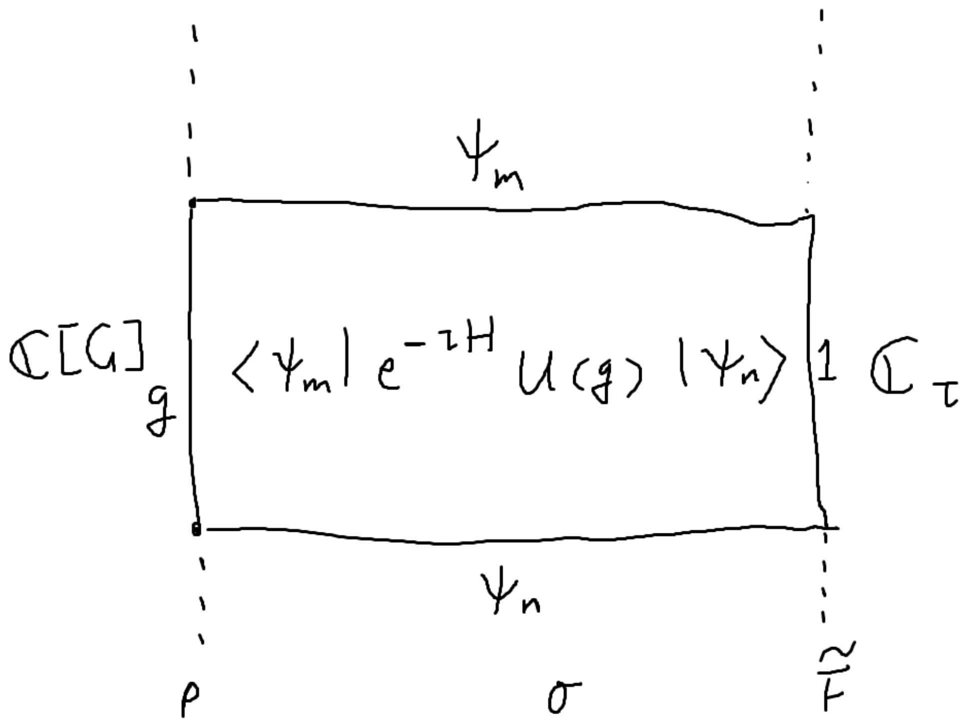

Figure 3.5: Bulk data

a twisted tensor product.

The bulk data is relatively easy to describe, and the only trick is that in category theory isomorphic but different objects can play important roles. We should assign a labelled complex line for the non-topological boundary and use it to select the correct amplitude.

Notice that every theory has a trivial symmetry group $G = 1$. In the above framework, it should be interpreted as the fact that every theory can be the boundary of one dimensional higher vacuum, which is a trivial field theory

$$F : \cdots \mapsto 1, \mathbb{C}, \{*\} \tag{3.18}$$

The top-dimensional data in the boundary should now be a complex line instead of a complex number, and the interpretation should be the ambiguity in defining the partition function.

Chapter 4

Gauge and Anomaly

Symmetries and anomalies are powerful tools in quantum field theory[42, 43]. They are preserved by the renormalization group flow, which connects properties on the IR side with those on the UV side. In the previous chapter, we showed that how to define symmetric quantum field theories in an external way which separates the calculus of topological operators with the physical degrees of freedoms. In this chapter, we will continue explore anomalies in quantum field theory.

4.1 ’t Hooft Anomaly

’t Hooft anomalies are not the only type of anomalies in quantum field theory. However, from our construction of extended field theory, the action is not necessarily related to a classical field theory with good symmetries and the path integral is not defined by trying to introduce a measure into the space of fields. Therefore the usual quantum anomaly caused by non-invariance of path integral measures is usually absent by definition, and the ’t Hooft anomaly seems to be more relevant and important to this framework.

There are multiple definitions of 't Hooft anomaly, which are believed to describe the same physics. This would be the topic of this section.

't Hooft anomaly[44] is defined as a property of a quantum theory when coupled with a background gauge theory: if a theory has symmetry G, then coupling[1] it with a background G-gauge theory A and evaluating the partition function, we can test the response of the theory with respect to a gauge transformation of the background gauge field

$$Z[A^{\lambda}] = e^{i\alpha(A,\lambda)} Z[A] \tag{4.1}$$

If the phase factor $\exp i\alpha(A,\lambda)$ cannot be absorbed by a redefinition of fields, then we cannot gauge the theory: nontrivial observables in gauge theory have to be gauge-invariant. In this case we say that the symmetry G has a 't Hooft anomaly.

The above description is technically correct but it does not tell us where the anomaly comes from. An answer to the question is that anomaly is the projectivity of a quantum theory. We will provide an abstract explanation together with a concrete one.

Let's start with the abstract argument[45]. In the previous chapters, we define a field theory as a functor valued in the category of 2-vector spaces. Strictly speaking, a quantum system is defined up to a phase factor and should hence be valued in the category of projective spaces.

In category theory, an important lesson is that when taking quotient, it is preferred to explicitly keep the isomorphisms. For example, consider the group action of G on a set X. The orbits are equivalent classes and we can construct a set theoretic quotient X/G. Another way of doing the quotient is to construct a groupoint where objects are $x \in X$ and isomorphisms are $g : x \to gx$, which defines a 1-category $X//G$, and the set X/G is its skeleton.

[1]The precise meaning of coupling will be discussed in the following sections, though it is almost done in the previous chapter.

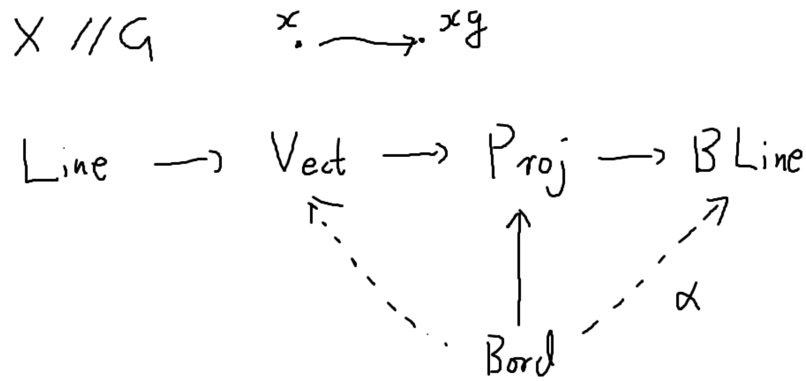

Figure 4.1: Abstract anomaly

Similarly, if we take a quotient of the category of vector spaces Vect with respect to the category of complex lines Line, then the result should be a 2-category Proj: objects are vector spaces, 1-morphisms are linear maps between vector spaces, and 2-morphisms are $\lambda \in \mathbb{C}^\times$ from T to λT. The obstruction of lifting the field theory to a functor valued in Vect is measured by a functor from the bordism category to the classifying category BLine, which can be defined as the category of $\mathbb{C}^\times$-gerbes.

The functor from bordisms to BLine can be considered as a quantum field theory, except that the codim-1 objects are mapped to boundary conditions and top dim objects are mapped to (1-dimentional) vector spaces. Compare with our previous discussion about boundaries, it implies that the theory defined by this functor is on the boundary of a bulk theory. This is the story of anomaly inflow: an anomaly can only be cancelled by coupling the theory with a bulk invertible topological theory.

Now let us consider a more concrete construction. We have seen that flat G-connections correspond to configuration of topological operators, and a gauge transformation is nothing more than a rearrangement of these defects, which involves fusion of topological operators. If the fusion is projective, i.e., assotiative up to a phase factor, then the partition function is not invariant under gauge transformations.

However, recall the following result: in quantum field theory, there is no nontrivial projective representation on the Hilbert space[46]. First consider the situation where the symmetry is unbroken and there is an invariant vacuum state $|0\rangle$. Symmetry operators can at most act as a phase factor

$$U(g)\,|0\rangle = e^{i\theta(g)}\,|0\rangle \tag{4.2}$$

Then for the projective representation

$$U(g_1)U(g_2) = e^{i\alpha(g_1,g_2)}U(g_1g_2) \tag{4.3}$$

acting on the vacuum we get (mod 2π)

$$e^{i\alpha(g_1,g_2)} = e^{i\theta(g_1)}e^{i\theta(g_2)}e^{-i\theta(g_1g_2)} \tag{4.4}$$

which implies that the projective representation lies in the trivial group cohomology, or in other words, it can be absorbed into the definition of symmetry operator

$$\bar{U}(g) := e^{-i\theta(g)}U(g) \tag{4.5}$$

which acts in the same way on states (complex lines in Hilbert space) but without projective representation ($\bar{U}(g_1)\bar{U}(g_2) = \bar{U}(g_1g_2)$).

Then consider field theories with spontaneous symmetry breaking, in the sense that the Hilbert space is equipped with a decomposition into superselection sectors

$$\mathcal{O}_1 \dots \mathcal{O}_n \left|b\right\rangle \tag{4.6}$$

and observables have zero matrix element between states in different superselection sectors. Here the vacuum space is normalized by

$$\langle b_1 | b_2 \rangle = \delta_{b_1,b_2} \tag{4.7}$$

even if they are parameterized by a continuous family of order parameters b. Symmetry operators act by

$$U(g)\mathcal{O}_1 \dots \mathcal{O}_n \left|b\right\rangle = \mathcal{O}_1 U(g)^{-1} U(g) \dots \mathcal{O}_n U(g)^{-1} U(g) \left|b\right\rangle = \mathcal{O}_1^g \dots \mathcal{O}_n^g U(g) \left|b\right\rangle \tag{4.8}$$

and because the symmetry is broken, $U(g)\left|b\right\rangle$ must belong to another sector

$$U(g)\left|b\right\rangle = e^{if(g,b)} \left|gb\right\rangle \tag{4.9}$$

Define a central operator B that detects superselection sectors

$$B\mathcal{O}_1 \dots \mathcal{O}_n \left|b\right\rangle = b\mathcal{O}_1 \dots \mathcal{O}_n \left|b\right\rangle \tag{4.10}$$

then a redefined symmetry operator

$$\tilde{U}(g) := e^{-if(g,B)} U(g) \tag{4.11}$$

has the same action on the algebra of observables but gives a non-projective representation of the symmetry on the Hilbert space.

At first look, there is now a paradox: while anomalies are associated with nontrivial projective representations of symmetry operators, there is no such kinds of symmetry operators in quantum field theory. The argument above is of course not perfect; it is based on a Wightman-styled representation of the state space and does not deal with higher form symmetries. However, we would like to argue that even if we accept the non-existence of nontrivial projective operators in quantum field theory, it is still consistent with the existence of 't Hooft anomalies.

First, the above argument depends on the existence of vacuum vector, which is a feature of quantum field theory in flat spacetime. So at least it is possible to build an anomalous theory in quantum mechanics.

For example, consider a $0+1$ dimensional field theory, a.k.a. quantum mechanical theory on S^1, then the partition function is given by $\operatorname{Tr} e^{-\beta H}$. A symmetry operator is a topological defect (because it commutes with the Hamiltonian; we showed more details at the end of the previous chapter). We cut the spacetime S^1 into intervals with endpoints $\{\tau_i\}$ and a compatible background gauge theory is equivalent to an assignment $i \mapsto g_i$. The coupled theory is defined by

$$Z[\{g_i\}] = \operatorname{Tr} e^{-\beta H} \prod_i U(g_i, \tau_i) \tag{4.12}$$

A gauge transformation is a fusion of nearby defects, for example

$$Z[g_1, g_2] = \operatorname{Tr} e^{-\beta H} U(g_1, \tau_1) U(g_2, \tau_2) \to \operatorname{Tr} e^{-\beta H} U(g_1 g_2, \tau) = Z[g_1 g_2] \tag{4.13}$$

If the symmetry has a nontrivial projective representation, then the partition function before and after the transformation differs by a phase factor

$$Z[g_1, g_2] = \exp i\alpha(g_1, g_2) Z[g_1 g_2] \tag{4.14}$$

and hence we cannot gauge the symmetry.

On the other hand, the argument is done within the Hamiltonian formalism, which implies a fixed space-time decomposition, and the symmetry operators have to be space-like. Therefore, if the operators are time-like or the spacetime we are interested in does not have a space-time decomposition that allows the existence of a vacuum vector, then it is still possible to build a field theory with anomalies.

4.2 Differential Cohomology

In the above discussion, although the 't Hooft anomaly is defined as the violation of gauge invariance after coupling the theory with a background gauge field, we intentionally avoid defining the coupling between background gauge theory and dynamic fields by introducing terms like

$$\exp\left(\int d^4x A_\mu j^\mu\right) \tag{4.15}$$

into the integrand. The reason is

1. not every theory is defined by a Lagrangian, as we have already shown how to define a path integral from local data where the action is an effective quantity derived from the properties of path integrals.

2. a general gauge field might be nontrivial and cannot be represented by a globally defined connection $A_\mu(x)$. As a special case, when G is discrete, there is simply no nonzero gauge connection A and all the information of the gauge field is encoded

into its global structure.

This is the starting point of Dijkgraaf-Witten theory[47], but we would like to show that it can already be seen in a much simpler and more familiar theory, just like the existence of 1-form symmetries in Maxwell theory.

Consider a particle in relativistic spacetime. The worldline, or 0-brane W (parameterized by $x^\mu(\tau)$), has an action

$$\int_W T ds,\ ds = d\tau\sqrt{-(dx/d\tau)^2} \tag{4.16}$$

and classically it moves along the geodesics. Now if the particle is electrically charged by q, then it should be coupled with a background Maxwell theory with field strength tensor $F_{\mu\nu}$ (notice that we make no assumption about the existence of A_μ and $F = dA$, though it is so in Maxwell theory in Minkowski spacetime). The coupling term in the action $S[W, F, q]$ is unknown, but it is possible to consider a variation of the worldline

$$W' = x^\mu(\tau) + \delta x^\mu(\tau) \tag{4.17}$$

and the variation defines a bulk region B parameterized by τ and σ $(0 \le \sigma \le \epsilon)$

$$(\tau, \sigma) \mapsto x^\mu(\tau) + \sigma\delta x^\mu(\tau) \tag{4.18}$$

The only 2-form we have is the field strength, so the minimal coupling is constructed as

$$S[W', F, q] - S[W, F, q] = \int_B qF = \epsilon \int d\tau q F_{\mu\nu} \frac{dx^\mu(\tau)}{d\tau} \delta x^\nu(\tau) \tag{4.19}$$

Variating the free brane action

$$\delta \int d\tau T \sqrt{-(dx/d\tau)^2} = \epsilon \int d\tau T \frac{(dx_\mu/d\tau)(d\delta^\mu/d\tau)}{\sqrt{-(dx/d\tau)^2}} \tag{4.20}$$

Integrating by part, we get the equation of motion

$$\frac{d}{d\tau}\left(T\frac{dx_\mu}{ds}\right) = qF_{\mu\nu}\frac{dx^\nu}{d\tau} \tag{4.21}$$

which is the covariant form of the Lorentz force.

Notice that when a globally defined gauge connection A exists, the following term (in path integral)

$$\exp\left(2\pi i q \int_W A\right) \tag{4.22}$$

defines the same coupling as the above construction and is the same as the Wilson line, that explains why we name the 0-brane by W.

The action in this formalism will automatically work for a generalized Maxwell theory, as long as we replace the 0-brane W with a general p-brane.

Now consider a quantum field theory defined by a path integral, then the coupling should be defined by a map from cocycles c to a unit complex number $e^{2\pi i\chi(c)}$ which is in general unknown. But we know how to compare a brane and its variation

$$\chi(\partial B) = \int_B F_\chi \mod \mathbb{Z} \tag{4.23}$$

where F_χ is the corresponding field strength. This defines the differential cohomology[48] and is what should be integrated in the path integral. It is called cohomology because it provides an interpolation between the singular and de Rham cohomology.

The discrete gauge theory (generalized Dijkgraaf-Witten theory) presented in last chapter is defined by such a differential cohomology. Besides, the anomaly in the following quantum mechanical system

$$S = \frac{m}{2}\int d\tau \dot{q}^2 - \frac{i}{2\pi}\int d\tau \theta \dot{q} \tag{4.24}$$

is also described by this construction (notice that the term containing odd number of time derivatives is always imaginary, in both relativistic and Euclidean signature).

However, instead of exploring anomaly from this perspective, we will prefer another construction based on the symmetry topological field theory in the previous chapter, where the anomaly and its inflow is transparent.

4.3 Discrete Gauge Theory

Finally we are going to return to the idea that anomaly comes from coupling the symmetry with background gauge theories.

4.3.1 On manifolds

The classical notion of gauge fields consists of a choice of trivialization (global data)

$$U_i \to M \tag{4.25}$$

$$g_{ij} : U_i \cap U_j \to G \tag{4.26}$$

together with field strengths (local data)

$$A_i^\mu : U_i \to T_e G \tag{4.27}$$

To couple such a gauge field to a dynamic theory, a heuristic recipe[46] is to first choose a good covering of spacetime so that the intersection of any two open set only consists of a single component and is shrinkable

$$U_i \cap U_j \simeq \Sigma_{ij} \times I \tag{4.28}$$

then twist the path integral with the help of the background gauge field

$$\int [\mathcal{D}\phi] e^{-\sum S_E[\phi, A_i]} \prod \tilde{U_{ij}} \tag{4.29}$$

where

$$S_E[\phi, A_i] = S_E[\phi] + \mathrm{Tr} \int \phi A_i \tag{4.30}$$

and

$$\tilde{U_{ij}} = U(\Sigma_{ij}, g_{ij}) e^{-\int_{\Sigma_{ij}} \cdots} \tag{4.31}$$

In the case where G is discrete, there is no local degree of freedoms and all the information of a gauge field is encoded into its global structure g_{ij}.

4.3.2 On cells

The above construction might seem a little bit weird, because shrinking open sets does not seem to be a standard operation in the theory of manifolds. However, if we consider it in the framework of our path integral over cells, then it seems to be a nature construction.

First let us show how to construct the extended path integral for a continuous gauge group G.

The ideal is similar to the construction of matter fields: we need to introduce infor-

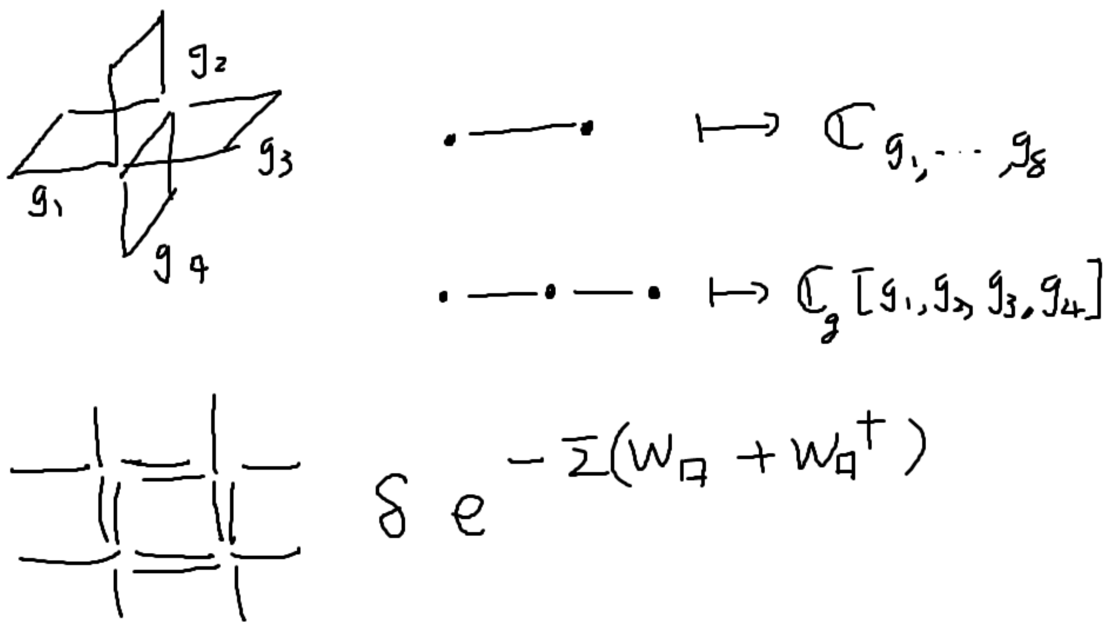

Figure 4.2: Lattice gauge theory

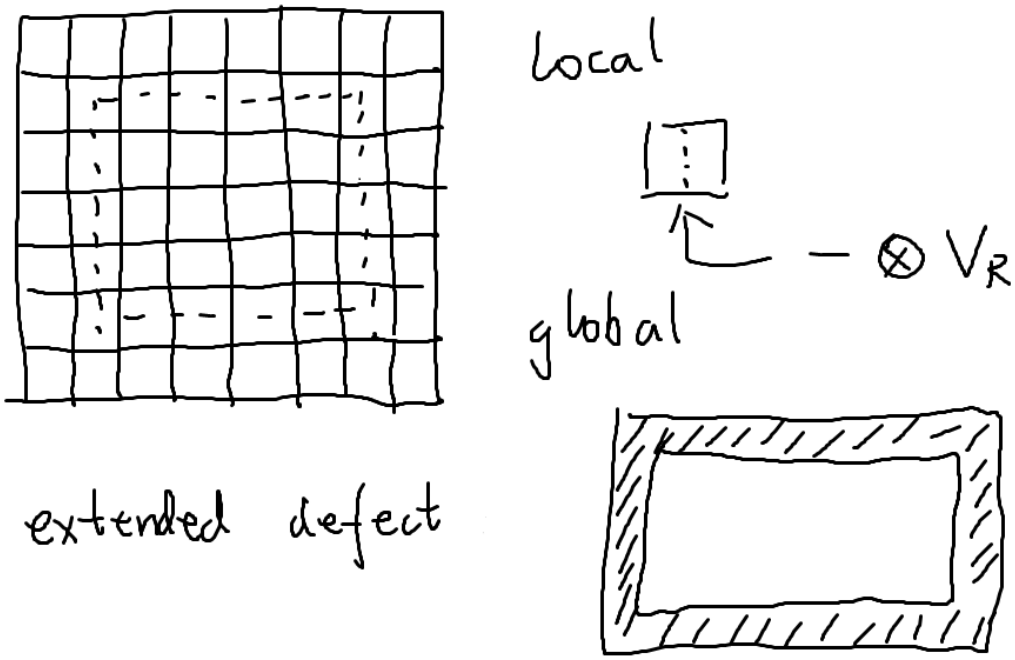

Figure 4.3: Wilson lines

mation around the boundary g_1, g_2, g_3, g_4 (notice that $S^0 = \{\mathrm{pt}^+, \mathrm{pt}^-\}$, the construction for matter fields is its codim-0 version) and then introducing a delta function to remove the redundancy.

Observables in gauge theory should be gauge invariant, which is an extended defect like Wilson lines. The global state-operator dual to this Wilson line is the hypersuface around it; however this is not local enough: we cannot compute it from extended path integral. To solve this issue, we need to introduce extra degree of freedoms along the Wilson line

$$F(b_2) \mapsto F(b_2) \otimes V_R \tag{4.32}$$

wile R is the representation of the Wilson line. In this sense, we go beyond the previous ideal of identifying defects with insertions into the path integral: the modification can be made for higher codimensional data and the twist is reflected in the extended effective action, instead of the top-degree one.

Now let's turn back to anomalies. Recall how we separate symmetries from evolution of physical states: a quantum field theory with symmetry G is a compactification of a topological boundary theory, whose states are flat G-connections, together with a non-topological theory that describes the other degree of freedoms. The discrete background gauge fields are represented by a defect network of topological operators on the topological boundary, while if moved onto the non-topological boundary with the topological invariance of the bulk theory, it becomes action of symmetry operators.

Let A run over nonequivalent flat G-connections, then the non-topological boundary is the following state after evaluating the path integral

$$\sum_{A} Z[A] \left|A\right\rangle \tag{4.33}$$

and the topological boundary will select a suitable value as the partition function of the full system. For example, if imposing Dirichlet boundary condition on it, then it selects the partition function matching with the boundary condition.

As we have argued above, if the fusion of topological operators is projective, then the system is anomalous. To absorb the phase factor, we need to couple the topological boundary with another bulk topological theory, where the boundary condition are the above Dirichlet boundary conditions, the vector spaces are always $\mathbb{C}$, and evaluating the path integral in the bulk gives the inverse of the phase factor coming from fusing topological operators represented by boundary conditions. This is also the same as the abstract construction above (functor valued in $B\mathrm{Line}$).

Now we have two bulk theories and two boundary theories. By fusing the topological boundary with the non-topological one (i.e. compactification), we get a physical theory

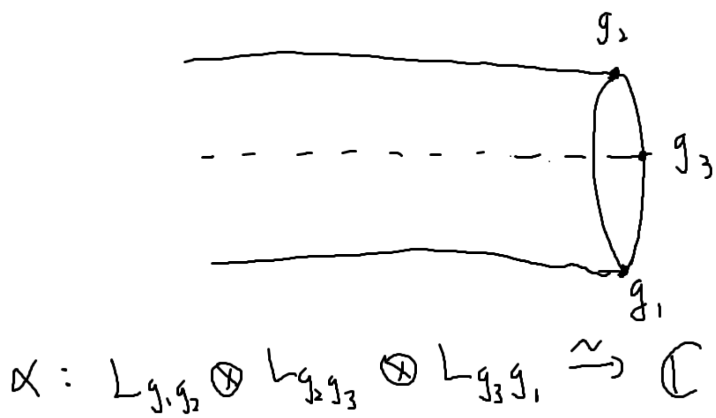

Figure 4.4: Anomaly inflow

living on the boundary of an invertible field theory, whose sole role is to absorb the phase factors from projective representation. We now have a unification of different descriptions of discrete gauge theories and 't Hooft anomaly.

Chapter 5

Conclusions

We discussed a few rigorous ways of interpreting quantum field theory and explained why path integral is a preferred approach.

We then discussed in what sense doing path integral in Euclidean signature is reasonable and after Wick rotation how to organize path integral into the framework of extended quantum field theory. We showed that the extended structure is universal and every quantum field theory is extended. In particular, we explained how to work with lattice regularized field theories where path integral is well defined.

With the new framework, we discussed a few general properties and constructions in field theories, together with the symmetry topological field construction with separates the calculus of symmetry with its representation in a concrete quantum field theory.

After that, we explained different approaches to 't Hooft anomalies and gauge theories, and showed that the previous construction provides a unified viewpoint.

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