Title
QUANTUM FLUCTUATIONS AND THE EFFECTIVE ENERGY FUNCTIONAL IN THE SKYEME MODEL

Author
Ingermanson, R.

Publication Date
1985-03-01

Lawrence Berkeley Laboratory
UNIVERSITY OF CALIFORNIA

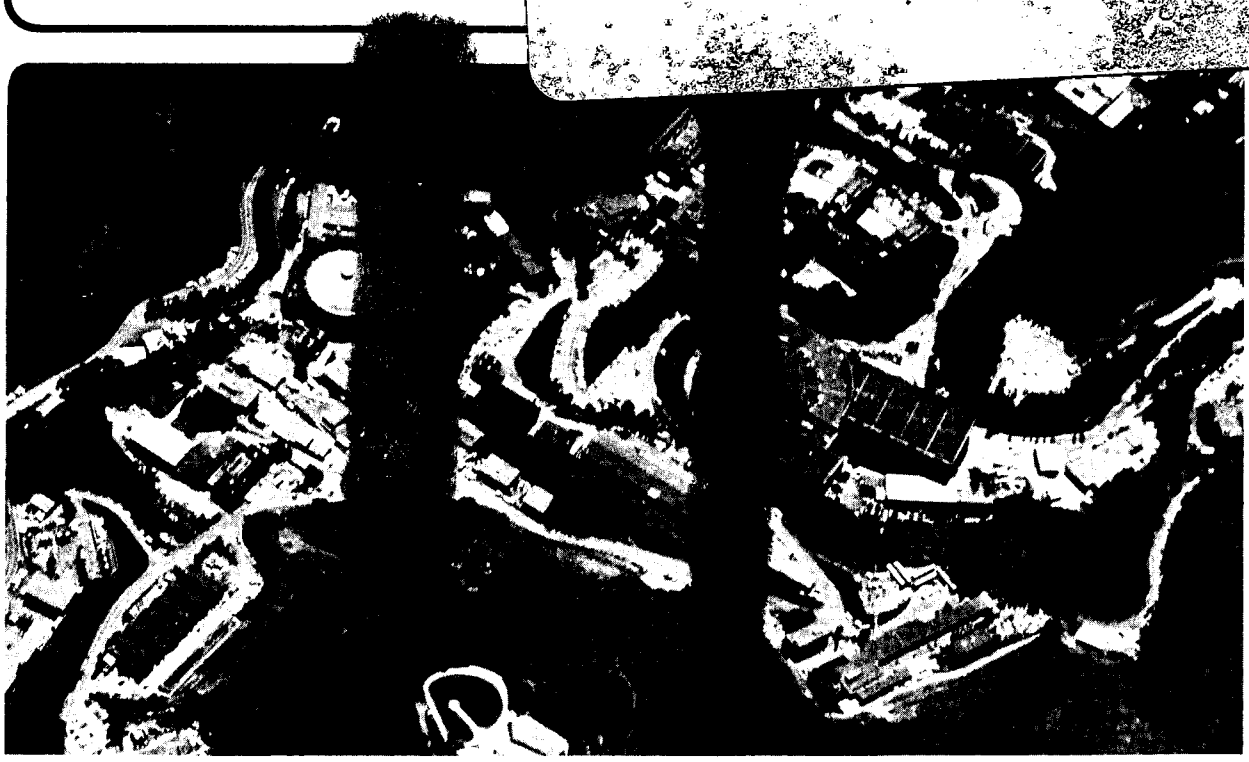
Physics Division

Submitted for publication

QUANTUM FLUCTUATIONS AND THE EFFECTIVE ENERGY FUNCTIONAL IN THE SKYRME MODEL

R. Ingermanson

March 1985



Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

DISCLAIMER

March 1, 1985 LBL-19122

QUANTUM FLUCTUATIONS AND THE EFFECTIVE ENERGY FUNCTIONAL IN THE SKYRME MODEL[1]

Randall Ingermanson

Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720, U.S.A.

ABSTRACT

We review the motivation for computing quantum corrections in the Skyrme model. We explicitly compute an approximate non-perturbative quantum energy functional for the Skyrmion. The result is quite similar in structure to the semi-classical formula of Adkins, Nappi and Witten. We show that certain weaknesses of the standard Skyrme model may be resolved by quantum corrections.

[1]This work was supported by the Director, Office of Energy Research, Office of High Energy Physics and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

1 Introduction

Recently, the old Skyrme model [1] has enjoyed a dramatic revival in popularity. The elegant paper by Adkins, Nappi and Witten (ANW) [2] still reflects very well the current status of the model. By this statement, we mean that the strengths of ANW's model are strengths of current versions of the model; conversely, weaknesses of ANW's model generally recur as weaknesses in the more recent work.

Since the strengths of ANW's model are well-known, we will focus here on the weaknesses, with an eye toward improving them:

1. The value of the axial coupling constant g_A is the most obvious defect of the Skyrme model. Most calculations are far too low. The existence of chiral anomalies in field theories suggests that an honest quantum calculation is needed here, rather than the semi-classical estimates that have been made to date.

2. Generally, the Skyrmion is too heavy. ANW skirted this problem by lowering F_π (*i.e.*, the mass scale) by some 30 per cent. This problem is typical of soliton physics; see, *e.g.*, Coleman's lectures [3] for a demonstration that the classical mass is an upper bound to the exact mass in a wide class of two-dimensional models. Quantum mass corrections can be quite substantial, as the exact results of the Sine-Gordon model show [4].

3. The Skyrme model has an infinite sequence of spin states with spin $= 1/2, 3/2 \ldots$ Experimentally, this sequence ends at $N_c/2$, where $N_c = 3$ is the number of colors in the underlying QCD theory.

   The Skyrme Lagrangian is supposed to represent the low energy part of the leading order in the $1/N_c$ expansion of QCD. Thus, it is not surprising that the sequence of spin states continues past $N_c/2$: in a tree-level calculation, N_c is invisible. (More accurately, the only N_c-dependent quantity, F_π, scales out of the equations.) In a quantum calculation, higher orders in $1/N_c$ would appear [5]. It is to be hoped that these extra terms would conspire to cut off the spin sequence.

4. Recently, there has been some discussion about the apparent inadequacy of the semi-classical energy functional in ANW's model [6,7].

This particular problem provides the main motivation for this paper. To discuss the problem, we first need to briefly review the model.

Consider the Lagrangian of the nonlinear σ-model

$$\mathcal{L}_0 = \tfrac{1}{4}F_\pi^2 \mathrm{Tr}[(\partial_\mu U)(\partial^\mu U^\dagger)] + \tfrac{1}{2}F_\pi^2 m_\pi^2 [(\mathrm{Tr}U) - 2].$$

$U(\vec{r},t)$ is an $SU(2)$ matrix-valued field. $U = \exp(2i\Pi^j\lambda^j/F_\pi)$, where Π^j are the pion fields, $F_\pi \approx 93$MeV, and $\mathrm{Tr}(\lambda^i\lambda^j) = \frac{1}{2}\delta^{ij}$. We assume the existence of a rotating classical solution

$$U(\vec{r},t) = A(t)U_0(\vec{r})A^\dagger(t),$$

and take Skyrme's ansatz

$$U_0 = \exp[iF(r)\vec{\tau}\cdot\hat{r}],$$

where $F(0) = \pi$ and $F(\infty) = 0$. $F(r)$ is to be chosen to minimize the energy of the soliton. $A(t)$ is parametrized by

$$A = a_0 + i\vec{a}\cdot\vec{\tau}$$

with the constraint $a_0^2 + \vec{a}\cdot\vec{a} = 1$.

Quantizing the "collective coordinates" $a^\mu(t)$ in the same way as ANW did, we eventually obtain the solitonic energy, as a functional of $F(r)$.

$$E^\ell[F] = M[F] + \frac{\ell(\ell+2)}{8\Lambda[F]}, \tag{1.1}$$

where $\ell/2 =$ spin $=$ isospin, and

$$\begin{aligned} M[F] &= 4\pi F_\pi^2 \int_0^\infty dr\, r^2 \left[\frac{1}{2}\left((F')^2 + \frac{2\sin^2 F}{r^2}\right) + m_\pi^2(1-\cos F)\right] \\ \Lambda[F] &= \frac{8\pi F_\pi^2}{3}\int_0^\infty dr\, r^2 \sin^2 F. \end{aligned} \tag{1.2}$$

($M[F]$ is the classical mass of a static soliton; $\Lambda[F]$ is the corresponding moment of inertia.)

ANW computed $F(r)$ by minimising $M[F]$:

$$\frac{\delta M}{\delta F(r)} = 0.$$

In this approach, $F(r)$ is unstable to collapse. Following Skyrme, ANW added an extra piece, (the "Skyrme term"), to $\mathcal{L}_0$, having four

derivatives. This changed the scaling properties of $M[F]$, stabilizing $F(r)$. ANW then went on to compute many baryonic parameters semi-classically.

This approach has been criticized [6], on the grounds that one should minimize $E^\ell[F]$, rather than $M[F]$. One then finds that $F(r)$ is stabilized by the scaling properties of the term $\ell(\ell+2)/8\Lambda[F]$, so that the Skyrme term is unneeded (for the physical case $\ell \neq 0$.) This procedure seems less artificial than ANW's approach. It is plausible that this method would solve the problem of the infinite sequence of spin states. ℓ enters explicitly into the equation of motion; for large enough ℓ, it might transpire that no solutions exist.

Unfortunately, this new scheme has a major drawback [6]. Linearizing the differential equation

$$\frac{\delta E^\ell}{\delta F(r)} = 0, \tag{1.3}$$

in the limit of $r \to \infty$, we find

$$\frac{d}{dr}(r^2F') = k^2r^2F, \tag{1.4}$$

where k^2 is to be determined self-consistently by

$$k^2[F] \equiv m_\pi^2 - \frac{\ell(\ell+2)}{6\Lambda^2[F]}. \tag{1.5}$$

Equation (1.5) is the source of the problem to which we alluded earlier. A priori, k^2 can be positive, negative or zero. We examine these three possibilities in the light of (1.5):

Suppose $k^2 < 0$. Then the solution to (1.4) is $F \sim \sin(|k|\,r + \theta)/r$, where θ is some phase. One then finds that $\Lambda[F]$ is divergent, which means $k^2[F] = m_\pi^2 > 0$. This contradiction implies that k^2 can never be negative.

If $k^2 = 0$, then the asymptotic equation of motion is, instead of (1.4),

$$\frac{d}{dr}(r^2F') = 2F.$$

Then $F \sim 1/r^2$ and $\Lambda[F]$ is finite. However, (1.5) is hard to satisfy.

In particular, when $m_\pi = 0$, (1.5) has no solution at all (for $\ell > 0$). This means that (1.3) is inconsistent, for non-zero ℓ; our new scheme has caused the sequence of odd half-spin states to vanish altogether.

The situation is a little better, when $m_\pi \neq 0$. In this case, however, it is unnatural to expect that $k^2[F]$ will vanish in (1.5).

Suppose, therefore, that $k^2 > 0$. Then $F(r) \sim e^{-kr}/r$ with $0 < k \leq m_\pi$. $\Lambda[F]$ is finite and rough numerical work shows that $k \sim \frac{1}{2}m_\pi$.

We see that (1.3) is at least consistent, when $m_\pi \neq 0$. However, we now face phenomenological difficulties; the model cannot be made to fit the data. Braaten and Ralston [6,7] have pointed out several places at which the model fails. These are of two main types: 1) To get the right nucleon-nucleon interaction, we need to have $F \sim e^{-m_\pi r}/r$. Our exponent k is substantially less than m_π. 2) Our model satisfies several inequalities based on

$$\Lambda^2[F] \geq \frac{\ell(\ell+2)}{6m_\pi^2}, \tag{1.6}$$

which we deduce from (1.5). One example of such an inequality is $M_\Delta - M_N \leq 3(\sqrt{5}-1)m_\pi/\sqrt{8} = 1.31m_\pi$; this inequality fails by some 40 per cent.

These problems become more acute if we allow m_π to approach zero. The model is not smooth in the chiral limit.

These difficulties arise from (1.5). They indicate that our energy functional (1.1) needs modification to improve the large-distance behaviour of $F(r)$. This is an infrared problem. It resists all ultraviolet-type cures, such as adding a Skyrme term to $\mathcal{L}_0$, or coupling the omega boson to the system [8].

Recently, Carlson [9] has studied quantum corrections arising from the dilatational mode. He showed that the above infrared problem is relaxed, but not eliminated. More precisely, he proved that $F(r)$ gets an $r^{-3/2}$ tail, when $m_\pi = 0$. Thus, the equation of motion has a consistent solution. However, $\Lambda[F]$ diverges (albeit rather mildly). The method of proof does not work for non-zero m_π, so we cannot check whether Braaten and Ralston's inequalities remain (or become less sensitive to the chiral limit). However, it seems likely that adding more vibrational modes to the picture should further improve the large-distance behaviour of $F(r)$.

Motivated by the above four weaknesses in the usual Skyrme model, we believe it is worthwhile to attempt a serious quantum calculation, involving

all of the vibrational modes. Note that we do *not* have to rewrite all of Skyrmion physics. We expect that quantum fluctuations will play an important role in only two quantities: the energy and g_A. It is fair to expect ANW's semi-classical method to continue to give good estimates for the other baryonic parameters.

This paper will be concerned only with the energy functional $E^\ell[F]$. The main result of this paper is the (approximate) formula for $E^\ell[F]$ given in eq. (4.10). Given this, $F[r]$ can, in principle, be determined numerically. Following ANW, one could then estimate the other parameters semi-classically to decide which ones need a fuller quantum treatment.

This paper is organized as follows. In Section 2, we sketch the strategy to be used in calculating quantum effects. Section 3 is devoted to renormalization of the model and calculation of the energy of a static Skyrmion. In Section 4, we introduce collective coordinates and use Dirac's method of constrained quantization to compute the effective energy functional of a rotating Skyrmion. In Section 5, we discuss how our result may solve some of the above weaknesses of the usual Skyrme model.

2 A Quantum Strategy

The following four-step procedure would yield the full quantum effective energy exactly. (Unfortunately, this procedure is wildly impractical.)

1. Obtain the Hamiltonian $H[U]$ from the Lagrangian density $\mathcal{L}_0$.

2. For an arbitrary normalized quantum state (denoted by the Schrödinger wave-funtional $\Psi[U]$), calculate the effective energy functional

$$\begin{aligned} E[\Psi] &= \langle\Psi| H |\Psi\rangle \\ &= \int [dU]\Psi^*[U]H[U]\Psi[U]. \end{aligned}$$

3. Denote by $\mathcal{H}$ the set of all normalized wave-functionals. Compute the vacuum energy

$$E_{vac} = \min_{\Psi\in\mathcal{H}} E[\Psi].$$

Renormalize the parameters of the model in some appropriate scheme.

4. Denote by S the subset of $\mathcal{H}$ corresponding to rotating Skyrmions. Compute the soliton energy

$$E_{sol} = \min_{\Psi \in S} E[\Psi].$$

This strategy is difficult to follow, for two reasons. First, the path integrals involved cannot be done exactly. Second, the nonlinear σ-model is non-renormalizable. As a compromise with reality, we revise the strategy as follows.

1. Rewrite the Lagrangian as a linear σ-model (the $O(4)$ vector model, with field variables $\phi^\mu(x)$, $\mu = 0,1,2,3$.) Obtain the corresponding Hamiltonian $H[\phi]$.

2. Let $\mathcal{G}$ denote that subset of $\mathcal{H}$ consisting of *Gaussian* wave-functionals

$$\Psi[\phi;\Phi,f] = N_f \exp\left[-\tfrac{1}{2}\int_{x,y}(\phi^\mu_x - \Phi^\mu_x) f^{\mu\nu}_{xy}(\phi^\nu_y - \Phi^\nu_y)\right]. \qquad (2.1)$$

N_f is a normalization constant. We will usually write Ψ in a schematic functional notation

$$\Psi = N \exp\left[-\tfrac{1}{2}(\phi-\Phi) f (\phi-\Phi)\right].$$

Calculate the Gaussian effective energy functional

$$E[\Psi] = \langle \Psi | H | \Psi \rangle .$$

3. Approximate

$$E_{vac} = \min_{\Psi \in \mathcal{G}} E[\Psi],$$

and renormalize the model.

4. Approximate

$$E_{sol} = \min_{\Psi \in S \cap \mathcal{G}} E[\Psi].$$

The idea of computing the effective energy via Gaussian wave-functionals has been used by several authors. A detailed examination of this procedure in quantum mechanical systems was made by Stevenson [10], who included many references. Barnes and Ghandour [11] showed that field theories could be renormalized in this scheme, using the effective potential methods developed about a decade ago [12]. We can hardly expect this method to yield exact results. However, it is reasonable to expect qualitative correctness.

3 The Vacuum State and Renormalization

Our first task is to transform $\mathcal{L}_0$ to the $O(4)$ vector model. We write

$$U(\vec{r},t) = u_0 + i\vec{u}\cdot\vec{\tau}$$

with the constraint $u_0^2 + \vec{u}\cdot\vec{u} = 1$. Defining

$$\phi^\mu \equiv F_\pi u_\mu,$$

the constraint becomes $\phi^\mu\phi^\mu = F_\pi^2$. $\mathcal{L}_0$ is then formally equivalent to the $\lambda \to \infty$ limit of

$$\mathcal{L} = \tfrac{1}{2}(\partial_\nu\phi^\mu)^2 + F_\pi m_\pi^2(\phi^0 - F_\pi) - \lambda(\phi^\mu\phi^\mu - \nu^2)^2$$

where

$$\nu^2 \equiv F_\pi^2 - \frac{m_\pi^2}{4\lambda}.$$

Identifying ϕ^0 with the sigma and ϕ^i with the pion triplet, we find that the pion has mass m_π and the sigma has mass $m_\sigma^2 = m_\pi^2 + 8\lambda F_\pi^2$. For finite λ the model is renormalizable (see, *e.g.*, Abers and Lee [12]), and is called the linear σ-model.

Classically, the system has a minimum energy when $\phi^0 = F_\pi$ and $\phi^i = 0$. Before computing the quantum energy, we introduce bare parameters into the action, anticipating the need for renormalization. The bare Lagrangian is therefore

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2}(\dot{\phi}^\mu)^2 - V_B(\phi) \qquad (3.1)\\ V_B(\phi) &= \tfrac{1}{2}(\partial_i\phi^\mu)^2 - F_B m_B^2(\phi^0 - F_B) + \lambda(\phi^\mu\phi^\mu - \nu_B^2)^2 \end{aligned}$$

where

$$\nu_B^2 \equiv F_B^2 - \frac{m_B^2}{4\lambda}.$$

λ will be understood to be bare. Since no experimental value exists for λ, it is pointless to renormalize it. We will consider λ to be a free parameter; it can be adjusted to fit the numerical Skyrmion parameters to the experimental data.

We do not include a wave-function renormalization $\phi^\mu \to \sqrt{Z}\phi^\mu$. (By the Ward identity [12], Z would be the same for all four components of ϕ^μ.) The reason is that Z turns out to be 1, in our scheme.

From (3.1), we easily find the Hamiltonian

$$H = \int_x [\tfrac{1}{2}\Pi^\mu\Pi^\mu + V_B(\phi)]$$

where

$$\Pi^\mu_x \equiv -i\frac{\delta}{\delta\phi^\mu_x} \quad .$$

Now let $\Phi^\mu_x \equiv \Phi^\mu(\vec{x})$ be any classical static field configuration. Then we will be interested in Gaussian wave-functionals of the form (2.1). We can assume that $f^{\mu\nu}_{xy}$ is symmetric, *i.e.*, that

$$f^{\mu\nu}_{xy} = f^{\nu\mu}_{yx}.$$

Note that $\langle\Psi|\,\phi^\mu_x\,|\Psi\rangle = \Phi^\mu_x$.

Our problem is to find the best possible $f^{\mu\nu}_{xy}$, the one which minimizes the energy

$$E[\Phi; f] \equiv \langle\Psi|\,H\,|\Psi\rangle\,.$$

Naturally, the best choice of $f^{\mu\nu}_{xy}$ depends on Φ^μ_x, so we will regard Φ^μ_x as some fixed function until further notice.

For convenience in doing the path integral, we shift ϕ^μ_x by Φ^μ_x:

$$\xi^\mu_x \equiv \phi^\mu_x - \Phi^\mu_x.$$

Note that $[d\xi^\mu] = [d\phi^\mu]$ and that

$$\frac{\delta}{\delta\xi^\mu_x} = \frac{\delta}{\delta\phi^\mu_x}.$$

Schematically, we write

$$\Psi[\xi; f] = Ne^{-\xi f\xi/2}.$$

It is easy to work out

$$\tfrac{1}{2}\int_x \Pi^\mu\Pi^\mu\Psi = T[\xi]\Psi,$$

where

$$T[\xi] \equiv \int_x \tfrac{1}{2}\left[f^{\mu\mu}_{xx} - \left(\int_y f^{\mu\nu}_{xy}\xi^\nu_y\right)^2\right].$$

Defining the "effective Hamiltonian"

$$H_{eff}[\xi] \equiv T[\xi] + \int_x V_B[\Phi + \xi],$$

we find that

$$\begin{aligned} E[\Phi; f] &= N^2\int [d\xi]H_{eff}[\xi]e^{-\xi f\xi} \\ &= H_{eff}[\delta/\delta J]N^2\int [d\xi]e^{-\xi f\xi}e^{J\xi}\Big|_{J=0} \\ &= H_{eff}[\delta/\delta J]e^{Jf^{-1}J/4}\Big|_{J=0} \end{aligned}$$

where we have used standard path integral tricks.

The result of this simple calculation can be split into a "classical energy" E_c, plus a "quantum energy" E_q:

$$E[\Phi; f] = E_c[\Phi] + E_q[\Phi; f]$$

where

$$\begin{aligned} E_c[\Phi] &\equiv \int_x V_B(\Phi) \qquad (3.2) \\ E_q[\Phi; f] &\equiv \int_x\Big[\tfrac{1}{4}f^{\mu\mu}_{xx} - \tfrac{1}{4}\int_y \delta_{xy}\nabla^2_x(f^{-1})^{\mu\mu}_{xy} \\ &\quad + \tfrac{1}{4}\lambda(f^{-1})^{\mu\nu}_{xx}\left([(f^{-1})^{\rho\rho}_{xx} + 4(\Phi^2 - \nu_B^2)]\delta^{\mu\nu} + 2(f^{-1})^{\mu\nu}_{xx} + 8\Phi^\mu\Phi^\nu\right)\Big]. \end{aligned}$$

(In this paper, Φ^2 is always to be understood as $\Phi^\mu\Phi^\mu$, not as the 2nd component of Φ^μ.)

We want to choose $f^{\mu\nu}_{xy}$ so as to minimize E_q in (3.2). In practice, it is convenient to vary E_q with respect to $(f^{-1})^{\mu\nu}_{xy}$. The variational equation, explicitly, is

$$\begin{aligned} 0 &= \frac{\delta E_q}{\delta(f^{-1})^{\mu\nu}_{xy}} \qquad (3.3) \\ &= -\tfrac{1}{4}\int_z (f^{\mu\rho}_{xz}f^{\rho\nu}_{zy}) - \tfrac{1}{4}\nabla^2_x\delta_{xy}\delta^{\mu\nu} \\ &\quad + \tfrac{1}{4}\lambda\delta_{xy}\left([2(f^{-1})^{\rho\rho}_{xx} + 4(\Phi^2_x - \nu_B^2)]\delta^{\mu\nu} + 4(f^{-1})^{\mu\nu}_{xx} + 8\Phi^\mu_x\Phi^\nu_x\right). \end{aligned}$$

This equation is exactly soluble if Φ^μ_x happens to be a constant function; for slowly varying Φ^μ_x, it is sometimes approximately soluble. Before deriving these solutions, we obtain an extremely valuable identity.

Multiplying the variational equation (3.3) by $(f^{-1})^{\nu\alpha}_{yw}$ and integrating over $\vec{y}$, we find

$$\begin{aligned} -\nabla^2_x(f^{-1})^{\mu\alpha}_{xw} = f^{\mu\alpha}_{xw} - \lambda(f^{-1})^{\nu\alpha}_{xw}\Big(&[2(f^{-1})^{\rho\rho}_{xx} + 4(\Phi^2_x - \nu_B^2)]\delta^{\mu\nu} \\ &+ 4(f^{-1})^{\mu\nu}_{xx} + 8\Phi^\mu_x\Phi^\nu_x\Big). \qquad (3.4) \end{aligned}$$

This identity is valid for the exact solution $f^{\mu\nu}_{xy}$ of our variational equation. Without knowing anything about $f^{\mu\nu}_{xy}$, this "Magic Identity" enables us to simplify (3.2) drastically:

$$E_q = \int_x \left[\tfrac{1}{2} f^{\mu\mu}_{xx} - \tfrac{1}{4}\lambda\left(|(f^{-1})^{\mu\mu}_{xx}|^2 + 2|(f^{-1})^{\mu\nu}_{xx}|^2\right)\right]. \tag{3.5}$$

Now we can look for solutions $f^{\mu\nu}_{xy}$. The variational equation (3.3) is rather imposing, so we look first to a simpler field theory for insight. Barnes and Ghandour [11] considered the well-known $\lambda\phi^4$ scalar field theory, in four dimensions. Setting $\Phi = 0$, they chose a wave-functional

$$\Psi = Ne^{-\phi f \phi/2}.$$

Their exact solution for the corresponding variational equation took the form

$$f_{xy} = \int \frac{d^3\vec{p}}{(2\pi)^3}\cos[\vec{p}\cdot(\vec{x}-\vec{y})]\omega(\vec{p}),$$

where

$$\omega(\vec{p}) = \sqrt{\vec{p}^2 + \mathcal{M}^2}$$

and $\mathcal{M}$ was some constant to be determined self-consistently.

Let us assume that our own variational equation has an analogous solution. (For the moment, we also assume that $\Phi^\mu_x = $ constant.)

We choose the ansatz

$$f^{\mu\nu}_{xy} = R^{\mu\rho} g^{\rho\sigma}_{xy} (R^T)^{\sigma\nu}, \tag{3.6}$$

where $g^{\rho\sigma}_{xy}$ is diagonal and R is some orthogonal matrix. By analogy to the simpler case, we expect

$$g^{\rho\sigma}_{xy} = \delta^{\rho\sigma}\int \frac{d^3\vec{p}}{(2\pi)^3}\cos[\vec{p}\cdot(\vec{x}-\vec{y})]\omega_\sigma(\vec{p}) \tag{3.7}$$

where

$$\omega_\sigma(\vec{p}) \equiv \sqrt{\vec{p}^2 + \mathcal{M}^2_\sigma}.$$

$\mathcal{M}_\sigma$ is to be determined by substituting this ansatz into the variational equation. By isospin invariance, we expect that $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3$.

The requirement that $g^{\rho\sigma}_{xy}$ be diagonal is a strong constraint. Substituting the $f^{\mu\nu}_{xy}$ ansatz (3.6) into the variational equation (3.3), we find that $g^{\rho\sigma}_{xy}$ is diagonal (and isospin invariant) only if $(R^T)^{\mu\nu}\Phi^\nu = |\Phi|\,\delta^{\mu 0}$. In words, R

rotates the north pole to Φ^μ. This leaves a lot of freedom in our choice of R. A convenient choice is

$$R^{\mu\nu} = \frac{1}{\Phi}\begin{bmatrix} \Phi^0 & -\Phi^n \\ \Phi^m & [\Phi\delta^{mn} - \frac{\Phi^m\Phi^n}{\Phi+\Phi^0}] \end{bmatrix} \tag{3.8}$$

where, from now on, we write the norm of Φ^μ as

$$\Phi \equiv |\Phi^\mu|.$$

The variational equation (3.3) now imposes a consistency condition on $\mathcal{M}_\mu$:

$$\mathcal{M}^2_\mu = 4\lambda\Big[\Phi^2 - \nu^2_B + 2\Phi^2\delta^{\mu 0} + \tfrac{1}{2}\sum_\rho (g^{-1})^{\rho\rho}_{xx} + (g^{-1})^{\mu\mu}_{xx}\Big] \tag{3.9}$$

where no sum on μ is intended.

Note that

$$(g^{-1})^{\rho\sigma}_{xy} = \delta^{\rho\sigma}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\cos[\vec{p}\cdot(\vec{x}-\vec{y})]}{\omega_\rho(\vec{p})} \quad ,$$

which is also diagonal.

It is convenient to define some (divergent) integrals:

$$I_\mu \equiv \int \frac{d^3\vec{p}}{(2\pi)^3}\omega_\mu(\vec{p})$$

$$J_\mu \equiv \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\omega_\mu(\vec{p})}$$

$$K_\mu \equiv \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\omega_\mu(\vec{p})^3} \quad .$$

We will regulate these by a cutoff Λ_c, which is a free parameter in the theory (presumably several hundred MeV).

The consistency condition (3.9) now simplifies to

$$\mathcal{M}^2_\mu = 4\lambda\Big[\Phi^2 - \nu^2_B + 2\Phi^2\delta^{\mu 0} + J_\mu + \tfrac{1}{2}\sum_\nu J_\nu\Big] \tag{3.10}.$$

(Note that we can consistently satisfy the isospin invariance requirement $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}_3$. This implies that $f^{\mu\nu}_{xy}$ is invariant under the isospin transformation defined by $R \to RM$, where M is any orthogonal matrix that leaves the north pole fixed.)

The preceding analysis proves that our ansatz (3.6) exactly satisfies the variational equation (3.3) when Φ^μ_x is constant. The situation is much different when Φ^μ_x is not a constant function. Generally, the above type of solution is unworkable. The reason is that $\mathcal{M}_\mu$ is supposed to be independent of $\vec{x}$.

But the consistency condition (3.10) on $\mathcal{M}_\mu$ shows that this is impossible, except when Φ_x^2 happens to be independent of $\vec{x}$.

Fortunately for us, Skyrmions correspond to precisely this special condition. In our current notation, a static Skyrmion is represented by

$$\begin{aligned} \Phi_x^0 &= F_\pi c \\ \Phi_x^i &= F_\pi s \hat{x}_i \\ c &\equiv \cos F \\ s &\equiv \sin F. \end{aligned} \tag{3.11}$$

Thus, $\Phi_x^2 = F_\pi^2$, and so we have some hope of extending our previous solution (3.6) to the Skyrmion problem. A natural generalization of (3.6) is

$$\begin{aligned} f_{xy}^{\mu\nu} &= R_x^{\mu\rho} g_{xy}^{\rho\sigma} (R^T)_y^{\sigma\nu} \\ (f^{-1})_{xy}^{\mu\nu} &= R_x^{\mu\rho} (g^{-1})_{xy}^{\rho\sigma} (R^T)_y^{\sigma\nu} \end{aligned} \tag{3.12}$$

where R_x is defined by Φ_x^μ exactly as R was previously defined by Φ^μ in (3.8).

This choice for $f_{xy}^{\mu\nu}$ fails to satisfy the variational equation (3.3) exactly. The failure stems from two terms involving derivatives of R_x. If Φ_x^μ is a slowly varying function, then (3.12) is an approximate solution.

[A technical comment: if $\Phi_x^\mu \to$ constant for $|\vec{x}| \to \infty$, then the "restricted variational equations"

$$\frac{\partial E}{\partial(\mathcal{M}_\mu^2)} = 0$$

yield our consistency conditions (3.10) for $\mathcal{M}_\mu$.]

Substituting (3.12) into (3.5) we find that (3.2) becomes

$$\begin{aligned} E[\Phi^\mu] = \int_x \Big[& \tfrac{1}{2}(\partial_i \Phi^\mu)^2 - F_B m_B^2 (\Phi^0 - F_B) + \lambda(\Phi^2 - \nu_B^2)^2 \\ & + \tfrac{1}{2}\sum_\mu I_\mu - \tfrac{1}{4}\lambda\Big(\sum_\mu J_\mu\Big)^2 - \tfrac{1}{2}\lambda \sum_\mu (J_\mu^2)\Big]. \end{aligned} \tag{3.13}$$

It is obvious now that no wave-function renormalization is needed, as we claimed earlier. This is due to our approximations. A better calculation would (presumably) add corrections involving derivatives of Φ_x^μ.

Let $\mathcal{E}(\Phi^\mu)$ denote the energy density of a constant field Φ^μ. We introduce some new notation here. A bar over any quantity means to evaluate that quantity at the field value $\Phi^0 = F_\pi$, $\Phi^i = 0$. In this notation, we can write a renormalization prescription [11,12] very simply:

$$\overline{\frac{\partial \mathcal{E}}{\partial \Phi^0}} = 0 \quad ; \quad \overline{\frac{\partial^2 \mathcal{E}}{\partial \Phi^1 \partial \Phi^1}} = m_\pi^2 \quad ; \quad \overline{\frac{\partial^2 \mathcal{E}}{\partial \Phi^0 \partial \Phi^0}} = m_\sigma^2 \quad .$$

These define F_π, m_π and m_σ, in that order. Note that $\mathcal{E}$ depends on Φ^0 and Φ^2 (and on $\mathcal{M}_\mu$ through I_μ and J_μ). Also, $\mathcal{M}_\mu$ depends on Φ^2 and on the other $\mathcal{M}_\nu$'s through the consistency condition (3.10). We calculate

$$\begin{aligned} \frac{\partial \mathcal{E}}{\partial \Phi^\mu} &= -F_B m_B^2 \delta^{\mu 0} + 2\Phi^\mu \frac{\partial \mathcal{E}}{\partial(\Phi^2)} \\ \frac{\partial^2 \mathcal{E}}{\partial \Phi^\mu \partial \Phi^\nu} &= 2\left[\delta^{\mu\nu} + 2\Phi^\mu \Phi^\nu \frac{\partial}{\partial(\Phi^2)}\right] \frac{\partial \mathcal{E}}{\partial(\Phi^2)}. \end{aligned}$$

Applying these to the renormalization prescription, we find instantly that

$$\begin{aligned} 0 &= \overline{\frac{\partial \mathcal{E}}{\partial \Phi^0}} = -F_B m_B^2 + 2F_\pi \overline{\frac{\partial \mathcal{E}}{\partial(\Phi^2)}} \\ m_\pi^2 &= \overline{\frac{\partial^2 \mathcal{E}}{\partial \Phi^1 \partial \Phi^1}} = 2\overline{\frac{\partial \mathcal{E}}{\partial(\Phi^2)}} \quad . \end{aligned}$$

Combining these, we find the important result

$$F_B m_B^2 = F_\pi m_\pi^2.$$

(This is the analog of the QCD result that the quantity $m_q \langle \bar{q} q \rangle$ is unchanged by renormalization [13].)

Restricting attention now to fields Φ_x^μ such that the norm $\Phi = F_\pi$, we obtain

$$\begin{aligned} E[\Phi^\mu] &= E_{vac} + \int_x \left[\tfrac{1}{2}(\partial_i \Phi^\mu)^2 - F_\pi m_\pi^2 (\Phi^0 - F_\pi)\right] \\ E_{vac} &= \int_x \Big[F_B^2 m_B^2 - F_\pi^2 m_\pi^2 + \lambda(F_\pi^2 - \nu_B^2)^2 \\ &\qquad + \tfrac{1}{2}\sum_\mu \overline{I_\mu} - \tfrac{1}{4}\lambda\Big(\sum_\mu \overline{J_\mu}\Big)^2 - \tfrac{1}{2}\lambda \sum_\mu \overline{J_\mu^2}\Big] \quad . \end{aligned} \tag{3.14}$$

E_{vac} is a meaningless infinite quantity, which we ignore from now on.

When Φ_x^μ represents the static Skyrmion, the energy can be computed:

$$E[F] = E_{vac} + M[F]. \tag{3.15}$$

This is the classical static energy. Note the agreement with the observation we made earlier, that the classical energy is generally an upper bound to the true energy. (An upper bound, because we have done a Rayleigh-Ritz variational calculation).

Naively, this is all we need to know about renormalization. For future reference, we note some useful facts. One can explicitly compute

$$\frac{\partial \mathcal{E}}{\partial(\Phi^2)} = 2\lambda(\Phi^2 - \nu_B^2) + 2\lambda J_0 + \lambda \sum_\mu J_\mu \quad .$$

Combining this with (3.10) and our equation for m_π^2, we find

$$\begin{aligned}\overline{M_0^2} &= m_\pi^2 + 8\lambda F_\pi^2 \\ \overline{M_1^2} &= m_\pi^2 + 4\lambda(\overline{J_1} - \overline{J_0}).\end{aligned} \tag{3.16}$$

Given λ and a cutoff Λ_c, these two equations determine $\overline{M_0}$ and $\overline{M_1}$.

A very intricate calculation is required to show that

$$m_\sigma^2 = m_\pi^2 + 8\lambda F_\pi^2 \left[\frac{(1+2\lambda\overline{K_1}) - 6\lambda\overline{K_0}(1+4\lambda\overline{K_1})}{(1+5\lambda\overline{K_1}) + 3\lambda\overline{K_0}(1+4\lambda\overline{K_1})}\right], \tag{3.17}$$

which should be compared to the tree-level result. (Note that the $\overline{K_\mu}$ are log-divergent.) This equation puts a constraint on the allowed range of λ and Λ_c, because m_σ is supposed to be much larger than m_π.

4 Quantum Energy of Rotating Solitons

So far, we have invested a lot of work to obtain very meager results. The only thing we have *really* proved is that the true mass of the static Skyrmion is no greater than the classical mass $M[F]$, a fact which we already suspected. In this section, we improve our trial wave-functional by introducing ANW's collective coordinates $a^\mu(t)$. The new wave-functional will be Gaussian in all directions in function space orthogonal to rigid rotations of the Skyrmion. The rotations will be quantized following ANW.

(The result of following this program is the effective energy functional (4.10) given at the end of this Section. This result uses only notation introduced in Sections 1–3; it can be understood without reading the laborious calculations of the current Section.)

In executing our program, we face a well-known technical problem: after adding the collective coordinates, we will have four more coordinates than degrees of freedom. We solve this problem by using Dirac's method of constrained quantization [14]. The four collective coordinates will be added at the expense of introducing four constraints on the fluctuations of ϕ^μ.

An extra subtlety arises from the fact that the four collective coordinates are constrained to lie on a 3-sphere. Thus, the constraints on ϕ^μ are themselves constrained! This is not a deep problem; we will point out the potential pitfalls along the way.

Mathematically, the problem of one soliton with four collective coordinates is very similar to the problem of four solitons with one collective coordinate apiece. The problem of N solitons has been treated very clearly by Tomboulis and Woo (TW) [15]. The following is a condensed and adapted version of TW's derivation of the Hamiltonian, canonical coordinates and Dirac-Poisson brackets.

Recall that the bare Lagrangian is

$$\mathcal{L} = \tfrac{1}{2}(\dot\phi^\mu)^2 - V_B(\phi).$$

We would like to consider quantum fluctuations about a rotating classical soliton

$$\Phi^\mu(\vec x, t) = M^{\mu\nu}(t)\Phi_S^\nu(\vec x)$$

where Φ_S^ν is the static Skyrmion field given by (3.11), and $M^{\mu\nu}(t)$ is a rotation matrix. In terms of the collective coordinates defined in Section 1, we compute

$$M^{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & [(a_0^2 - \vec a\cdot\vec a)\delta_{mn} + 2a_m a_n + 2a_0 a_\ell \epsilon_{\ell mn}] \end{bmatrix} \tag{4.1}.$$

(Because of the constraint $a^\mu a^\mu = 1$, this formula for $M^{\mu\nu}$ is not unique.)

We again shift ϕ^μ by Φ^μ:

$$\phi^\mu(\vec x, t) = \Phi^\mu(\vec x, t) + \xi^\mu(\vec x, t).$$

Then $\dot\phi_x^\mu = \dot a^\nu C_x^{\nu\mu} + \dot\xi_x^\mu$, where the coefficients $C_x^{\nu\mu}$ are chosen to satisfy

$$\dot\Phi_x^\mu = \dot a^\nu C_x^{\nu\mu}.$$

Substituting $\phi = \Phi + \xi$ into $\mathcal{L}$, we find

$$\begin{aligned} L(t) &= \int_x \mathcal{L}(\vec x, t) \\ &= \tfrac{1}{2}\dot a^\nu \alpha^{\nu\rho}\dot a^\rho + \int_x [\dot a^\nu C^{\nu\mu}\dot\xi^\mu + \tfrac{1}{2}(\dot\xi^\mu)^2 - V_B] \end{aligned}$$

where we defined

$$\alpha^{\nu\rho} \equiv \int_x C_x^{\nu\mu} C_x^{\rho\mu}.$$

The momenta conjugate to a^μ and ξ_x^μ are

$$\begin{aligned} P^\mu &\equiv \frac{\partial L}{\partial \dot a^\mu} = \dot a^\nu \alpha^{\nu\mu} + \int_x C_x^{\mu\nu}\dot\xi_x^\nu \\ \Pi_x^\mu &\equiv \frac{\delta L}{\delta \dot\xi_x^\mu} = \dot a^\nu C_x^{\nu\mu} + \dot\xi_x^\mu \end{aligned}$$

These momenta satisfy the constraints

$$\psi^\mu \equiv P^\mu - \int_x C_x^{\mu\nu}\Pi_x^\nu = 0. \tag{4.2}$$

Note that the ambiguities inherent in $M^{\mu\nu}$ are also present in $C_x^{\nu\mu}$, $\alpha^{\nu\rho}$ and P^μ, but *not* in ψ^μ. All this ambiguity is certainly irrelevant to the physics. We are free to choose $M^{\mu\nu}$ so as to simplify our computations. The best choice seems to be the one displayed in (4.1); it results in a "covariant" looking form for $\alpha^{\mu\nu}$:

$$\alpha^{\mu\nu} = 4\Lambda[F](\delta^{\mu\nu} + \tfrac{1}{2}a^\mu a^\nu)$$
$$(\alpha^{-1})^{\mu\nu} = \frac{1}{4\Lambda[F]}(\delta^{\mu\nu} - \tfrac{1}{3}a^\mu a^\nu),$$

where $\Lambda[F]$ is the classical moment of inertia (1.2). For future reference, we write an intermediate result required to verify the above formulae:

$$\begin{aligned} C_x^{\nu 0} &= 0 \\ C_x^{0m} &= 2F_x s_x \hat{x}_i(a_0\delta_{im} + a_j\epsilon_{ijm}) \\ C_x^{nm} &= 2F_x s_x \hat{x}_i(\delta_{nm}a_i + \delta_{in}a_m - \delta_{im}a_n + a_0\epsilon_{inm}) \quad . \end{aligned} \tag{4.3}$$

The Hamiltonian is easily computed:

$$\begin{aligned} H &= P^\mu\dot{a}^\mu + \int_x \Pi_x^\mu\dot{\xi}_x^\mu - L \\ &= \int_x \left[\tfrac{1}{2}(\Pi^\mu)^2 + V_B\right] \quad . \end{aligned}$$

Following TW, we decompose Π_x^μ as

$$\Pi_x^\mu = \overline{\Pi}_x^\mu + P^\sigma(\alpha^{-1})^{\sigma\nu}C_x^{\nu\mu}.$$

In terms of these new variables, the constraints (4.2) take the "linearized" form

$$\overline{\psi}^\mu \equiv \int_x C_x^{\mu\nu}\overline{\Pi}_x^\nu = 0 \tag{4.4}$$

and the Hamiltonian becomes

$$H = H_{rot} + H_{vib}$$

where

$$\begin{aligned} H_{rot} &\equiv \tfrac{1}{2}P^\mu(\alpha^{-1})^{\mu\nu}P^\nu \\ H_{vib} &\equiv \int_x \left[\tfrac{1}{2}(\overline{\Pi}^\mu)^2 + V_B\right]. \end{aligned} \tag{4.5}$$

As discussed by TW, we need to impose "gauge conditions" on ξ_x^μ. A suitable set of "gauge conditions" for our model is (analogous to (4.4))

$$\chi^\mu \equiv \int_x C_x^{\mu\nu}\xi_x^\nu = 0. \tag{4.6}$$

Differentiating this with respect to time yields

$$\int_x C_x^{\mu\nu}\dot{\xi}_x^\nu = -\beta^{\mu\rho}\dot{a}^\rho$$

where we define

$$\beta^{\mu\rho}[\xi] \equiv \int_x \frac{\partial C_x^{\mu\nu}}{\partial a^\rho}\xi_x^\nu \quad .$$

This allows us to prove the important result

$$P^\mu\dot{a}^\mu + \int_x \Pi_x^\mu\dot{\xi}_x^\mu = \overline{P}^\mu\dot{a}^\mu + \int_x \overline{\Pi}_x^\mu\dot{\xi}_x^\mu$$

where

$$\overline{P}^\mu \equiv P^\sigma(\delta^{\sigma\mu} - [\alpha^{-1}\beta]^{\sigma\mu}).$$

Hence, $\overline{P}^\mu$, a^μ, $\overline{\Pi}_x^\mu$ and ξ_x^μ form a canonical set. In matrix form,

$$P^T = \overline{P}^T(I - \alpha^{-1}\beta)^{-1} = \overline{P}^T\sum_{m=0}^{\infty}(\alpha^{-1}\beta)^m.$$

Noting that α^{-1} and β are symmetric matrices, we can rewrite (4.5) purely in terms of the canonical variables:

$$H_{rot} = \tfrac{1}{2}\overline{P}^T\sum_{k=0}^{\infty}(k+1)(\alpha^{-1}\beta)^k\alpha^{-1}\overline{P}. \tag{4.7}$$

Now we need to study the Poisson brackets of the system. The "naive" brackets are

$$\{\xi_x^\mu, \overline{\Pi}_y^\nu\} = \delta^{\mu\nu}\delta_{xy}$$
$$\{a^\mu, \overline{P}^\nu\} = \delta^{\mu\nu} - a^\mu a^\nu.$$

All others vanish. The second bracket looks rather abnormal; it reflects the constraint $a^\mu a^\mu = 1$.

It is easy to compute

$$\{\overline{\psi}^\mu, \overline{\psi}^\nu\} = 0 = \{\chi^\mu, \chi^\nu\} \quad ; \quad \{\overline{\psi}^\mu, \chi^\nu\} = -\alpha^{\mu\nu} \quad .$$

Under these conditions, TW showed that Dirac's "star"-brackets can be simplified to

$$\{f,g\}^* \equiv \{f,g\} - [\{\overline{\psi}^\mu, \chi^\nu\}]^{-1}[\{f,\chi^\mu\}\{\overline{\psi}^\nu, g\} - \{f,\overline{\psi}^\mu\}\{\chi^\nu, g\}].$$

With this definition of brackets, the constraints can be imposed as operator equations in the quantum theory (if no ordering problems arise).

Because $\{a^\mu, \overline{\psi}^\nu\} = 0 = \{a^\mu, \chi^\nu\}$, we find at once that

$$\{a^\mu, g\}^* = \{a^\mu, g\}.$$

Similarly, $\{\xi^\mu_x, \chi^\nu\} = 0 = \{\overline{\Pi}^\mu_x, \psi^\nu\}$ implies that

$$\{\xi^\mu_x, \xi^\nu_y\}^* = 0$$
$$\{\overline{\Pi}^\mu_x, \overline{\Pi}^\nu_y\}^* = 0.$$

A short computation is needed to show that

$$\{\xi^\mu_x, \overline{\Pi}^\nu_y\}^* = L^{\mu\nu}_{xy},$$

where

$$L^{\mu\nu}_{xy} \equiv \delta^{\mu\nu}\delta_{xy} - (\alpha^{-1})^{\rho\sigma} C^{\rho\mu}_x C^{\sigma\nu}_y. \tag{4.8}$$

The star-brackets $\{\overline{P}^\mu, g\}^*$ are somewhat more complicated, but we really don't need them in this paper.

Now we are ready to quantize the model. The recipe we will follow is based on the picture of a slowly rotating classical soliton. First, we compute the effects of quantum fluctuations in ξ^μ_x, regarding the coordinates $a^\mu(t)$ as classical. Second, we quantize the a^μ in the same way as ANW did.

This recipe depends on the approximation that fluctuations in a^μ are much slower than fluctuations in ξ^μ_x. The effect of this approximation is to make the commutators $[\overline{P}^\mu, \xi^\nu_x] = 0 = [\overline{P}^\mu, \overline{\Pi}^\nu_x]$ in the quantized theory. This simplifies calculations involving H_{rot} somewhat, and has no effect on those involving H_{vib}.

[Note: How good is this approximation? From ANW, we estimate the rotational frequency $\omega \approx \sqrt{\ell(\ell+2)}(M_\Delta - M_N)/3$. This is roughly 170 MeV for the nucleon and 380 MeV for the delta. This is to be compared to the vibrational frequencies which range roughly between m_π and Λ_c. Thus, our approximation is rather crude. Our motivation for using this approximation stems from a desire to remain close to ANW's semi-classical picture. If our results ultimately prove unsatisfactory, then we can always improve them by using Dirac's procedure to compute the correct commutators $[\overline{P}^\mu, \xi^\nu_x]$ and $[\overline{P}^\mu, \overline{\Pi}^\nu_x]$. This will lead to some extra terms in H_{rot}.]

The only non-vanishing commutators are therefore taken to be:

$$[\xi^\mu_x, \overline{\Pi}^\nu_y] = iL^{\mu\nu}_{xy}$$
$$[a^\mu, \overline{P}^\nu] = i(\delta^{\mu\nu} - a^\mu a^\nu).$$

For simplicity, we will truncate the series (4.7) after the first term ($k = 0$). (The second, fourth, ... vanish in the path integral anyway, for reasons to be seen shortly). H_{rot} is then independent of ξ^μ_x, so the path integral is trivial, and we only need to be concerned with the wave-function of the collective coordinates. Note that $a^\mu \overline{P}^\mu = 0$ holds as an operator equation when acting on such a wave-function. Therefore, we can replace H_{rot} by

$$H'_{rot} = \frac{\overline{P}^\mu \overline{P}^\mu}{8\Lambda[F]}$$

which is precisely ANW's expression for the rotational Hamiltonian. Taking over ANW's wave-functions, we have the rotational energy

$$E^\ell_{rot} = \frac{\ell(\ell+2)}{8\Lambda[F]}.$$

All that remains is to compute the expectation value of H_{vib} in a soliton state. First, we need to compute the vibrational energy about the rotated configuration $\Phi^\mu(\vec{x}) = M^{\mu\nu}(A)\Phi^\nu_S(\vec{x})$. This will yield the effective potential for the collective coordinates. By isospin invariance, this effective potential must be independent of a^μ; we can choose any convenient value of a^μ for the calculation. (The most convenient choice is $a_0 = 1$, $a_i = 0$. This choice should be understood to be in effect throughout the remainder of this paper.) Because of the isospin invariance, the second stage of computation (integration over the collective coordinates) can be skipped entirely.

The problem is now reduced to evaluation of

$$E_{vib} \equiv \langle\Psi| H_{vib} |\Psi\rangle = \frac{\int [d\xi] \prod_\mu \delta(\chi^\mu[\xi]) \Psi^*[\xi] H_{vib}[\xi] \Psi[\xi]}{\int [d\xi] \prod_\nu \delta(\chi^\nu[\xi]) \Psi^*[\xi] \Psi[\xi]}.$$

This calculation is only slightly more difficult than the path integral done in Section 3. We again define a kinetic functional $T[\xi]$ by

$$\tfrac{1}{2}\int_x \overline{\Pi}^\mu_x \overline{\Pi}^\mu_x \Psi = T[\xi]\Psi$$

where

$$T[\xi] \equiv \tfrac{1}{2}\int_x \left[\int_{y,z} (L^{\mu\nu}_{xy} f^{\nu\sigma}_{yz} L^{\sigma\mu}_{zx}) - \left(\int_{y,z} L^{\mu\nu}_{xy} f^{\nu\sigma}_{yz} \xi^\sigma_z\right)^2\right].$$

Thus, $H_{vib}\Psi = H_{eff}[\xi]\Psi$, where

$$H_{eff}[\xi] \equiv T[\xi] + \int_x V_B(\Phi + \xi).$$

We use the usual integral representations of the delta-functions

$$\begin{aligned}\prod_\mu \delta(\chi^\mu) &= \prod_\mu \int_{-\infty}^{\infty} \frac{d\rho_\mu}{2\pi} \exp(i\rho_\mu \chi^\mu) \\ &= \frac{1}{(2\pi)^4} \int d^4\vec{\rho} \exp(i\sum_\mu \rho_\mu \chi^\mu).\end{aligned}$$

Substituting our particular choice for $\chi^\mu[\xi]$, (4.6), we can rewrite

$$i\sum_\mu \rho_\mu \chi^\mu = \int_x \theta_x^\nu \xi_x^\nu$$

where we define

$$\theta_x^\nu \equiv i\sum_\mu \rho_\mu C_x^{\mu\nu}.$$

The formal computation of the path integral is now easy:

$$\begin{aligned}E_{vib} &= \frac{\int d^4\vec{\rho} \int [d\xi] H_{eff}[\xi] e^{-\xi f \xi} e^{\theta\xi}}{\int d^4\vec{\rho} \int [d\xi] e^{-\xi f \xi} e^{\theta\xi}} \\ &= \frac{\int d^4\vec{\rho}\, H_{eff}[\delta/\delta\theta] e^{\theta f^{-1}\theta/4}}{\int d^4\vec{\rho}\, e^{\theta f^{-1}\theta/4}}.\end{aligned}$$

Since θ^μ is linear in $\vec{\rho}$, the odd terms contained in H_{eff} will not contribute. The above formula is slightly peculiar; we don't set $\theta = 0$ after doing the functional derivatives.

The resulting expression is best organized as follows:

$$E_{vib} = E_0 + \frac{\int d^4\vec{\rho}\,[E_2(\vec{\rho}) + E_4(\vec{\rho})] e^{\theta f^{-1}\theta/4}}{\int d^4\vec{\rho}\, e^{\theta f^{-1}\theta/4}}$$

where

$$\begin{aligned}E_0 = \int_x \Big[&\tfrac{1}{4} \int_{y,z} (L_{xy}^{\mu\nu} f_{yz}^{\nu\sigma} L_{zx}^{\sigma\mu}) + V_B(\Phi) - \tfrac{1}{4} \int_y \delta_{xy} \nabla_x^2 (f^{-1})_{xy}^{\mu\mu} \\ &+ \tfrac{1}{4}\lambda (f^{-1})_{xx}^{\mu\nu} \left([(f^{-1})_{xx}^{\rho\rho} + 4(\Phi^2 - \nu_B^2)]\delta^{\mu\nu} + 2(f^{-1})_{xx}^{\mu\nu} + 8\Phi^\mu\Phi^\nu \right) \Big]\end{aligned}$$

$$\begin{aligned}E_2(\vec{\rho}) = -\tfrac{1}{8} \int_x \left(\int_y L_{xy}^{\mu\nu} \theta_y^\nu \right)^2 + \tfrac{1}{2} \int_{x,y,z} (f^{-1})_{xy}^{\mu\alpha} \theta_y^\alpha \theta_z^\beta \Big[&-\tfrac{1}{4} \nabla_x^2 (f^{-1})_{xz}^{\mu\beta} \\ &+ \tfrac{1}{4}\lambda (f^{-1})_{xz}^{\nu\beta} \left([2(f^{-1})_{xx}^{\rho\rho} + 4(\Phi_x^2 - \nu_B^2)]\delta^{\mu\nu} + 4(f^{-1})_{xx}^{\mu\nu} + 8\Phi_x^\mu \Phi_x^\nu \right) \Big]\end{aligned}$$

$$E_4(\vec{\rho}) = \int_x \tfrac{1}{16} [(f^{-1}\theta)_x^\mu (f^{-1}\theta)_x^\mu]^2.$$

This can be greatly simplified. We assume that $f_{xy}^{\mu\nu}$ is given by (3.12). Substituting (4.8) into E_0, and comparing to (3.2), (3.14) and (3.15), we find

$$E_0 = E_{vac} + M[F] - \tfrac{1}{4}(\alpha^{-1})^{\alpha\beta} \int_{x,y} C_x^{\alpha\mu} f_{xy}^{\mu\nu} C_y^{\beta\nu}.$$

Also, it is easy to check that

$$\int_y L_{xy}^{\mu\nu} \theta_y^\nu = 0.$$

This, along with the Magic Identity (3.4), allows us to simplify

$$\begin{aligned}E_2(\vec{\rho}) &= \tfrac{1}{8} \int_x (f^{-1}\theta)_x^\mu (f\theta)_x^\mu \\ &= \tfrac{1}{8} \int_x \theta_x^\mu \theta_x^\mu \\ &= -\tfrac{1}{8} \alpha^{\mu\nu} \rho_\mu \rho_\nu.\end{aligned}$$

We observe that $E_4(\vec{\rho})$ involves five integrals over position space and four integrals over momentum space. In addition, the $\vec{\rho}$-integral over $E_4(\vec{\rho})$ will bring in two more non-local factors. We feel that such a highly non-local term should be ignored in our approximate treatment.

Our vibrational energy therefore reduces to

$$E_{vib} = E_{vac} + M[F] - \tfrac{1}{4}(\alpha^{-1})^{\rho\sigma} \int_{x,y} C_x^{\rho\mu} f_{xy}^{\mu\nu} C_y^{\sigma\nu} - \tfrac{1}{8} \alpha^{\mu\nu} \frac{\int d^4\vec{\rho}\, \rho_\mu \rho_\nu e^{\theta f^{-1}\theta/4}}{\int d^4\vec{\rho}\, e^{\theta f^{-1}\theta/4}}. \quad (4.9)$$

Note that

$$\theta f^{-1} \theta \equiv \int_{x,y} \theta_x^\alpha (f^{-1})_{xy}^{\alpha\beta} \theta_y^\beta = -\rho_\mu \left(\int_{x,y} C_x^{\mu\alpha} (f^{-1})_{xy}^{\alpha\beta} C_y^{\nu\beta} \right) \rho_\nu.$$

Thus, the $\vec{\rho}$-integral in (4.9) is Gaussian. Before doing this integral, we consider the somewhat forbidding terms involving $\int_{x,y} C_x^{\mu\rho} f_{xy}^{\rho\sigma} C_y^{\nu\sigma}$ (or $f \to f^{-1}$). $C_x^{\mu\nu}$ was displayed in (4.3); it is not very complicated when $a_0 = 1$ and $a_i = 0$. Substituting in $C_x^{\mu\nu}$ and the explicit form for (3.8)

$$R_x^{\mu\nu} = \begin{bmatrix} c_x & -s_x \hat{x}_n \\ s_x \hat{x}_m & [\delta_{mn} - (1 - c_x)\hat{x}_m \hat{x}_n] \end{bmatrix},$$

we find that the angular integrations can be done first. The result vanishes unless $\mu = \nu$. This factorizes the $\vec{\rho}$-integral in (4.9) into a product of one-dimensional Gaussian integrals.

A lengthy calculation is required to actually carry out the paragraph above. The details are straightforward, however, and we find that ANW's results are modified schematically to

$$E^\ell[F] = E_{vac} + M[F] + \frac{\ell(\ell+2) - P[F]}{8\Lambda[F]} - N[F]\Lambda[F] \qquad (4.10)$$

where

$$P[F] \equiv \tfrac{32}{9} F_\pi^2 \int_0^{\Lambda_c} dp\, p^2 \Big(\omega_0 h_1^2(p) + \omega_1 [h_2^2(p) + 3h_3^2(p)] \Big)$$

and

$$N[F] \equiv \frac{3}{64F_\pi^2} \left(\frac{1}{\int_0^{\Lambda_c} dp\, p^2 \left[\frac{h_1^2(p)}{\omega_0} + \frac{h_2^2(p)}{\omega_1} \right]} + \frac{3}{\int_0^{\Lambda_c} dp\, p^2 \frac{h_3^2(p)}{\omega_1}} \right).$$

The $h_i(p)$ are integral transforms of the Skyrme function:

$$h_1(p) = \int_0^\infty dx\, x^2 s_x^2 j_0(px)$$
$$h_2(p) = \int_0^\infty dx\, x^2 s_x c_x j_1(px)$$
$$h_3(p) = \int_0^\infty dx\, x^2 s_x j_1(px)$$

where j_0 and j_1 are spherical Bessel functions

$$j_0(\alpha) = \frac{\sin\alpha}{\alpha}$$
$$j_1(\alpha) = \frac{\sin\alpha}{\alpha^2} - \frac{\cos\alpha}{\alpha}.$$

We have dropped the bar-notation of Section 3. It should be understood that ω_μ is defined in terms of $\overline{M}_\mu$.

5 Conclusion

It is time to ask whether we have made any progress. A quantitative answer must wait for a numerical solution to the complicated equation of motion

$$\frac{\delta E^\ell}{\delta F(r)} = 0.$$

Qualitatively, we can check whether we have moved in the right direction, relative to the problems discussed in the introduction.

The clearest point is that the Skyrmion is now lighter. $E^\ell[F]$ as given in (4.10) is just the semi-classical result (1.1), minus a positive quantity.

We might now fear that the energy could be unbounded from below. As a crude test of this possibility, we should examine the scaling properties of $E^\ell[F]$. We would also like to understand the asymptotic behavior of $F(r)$.

An exact analysis of these two issues is very difficult. As a first step, we approximate $\omega_\mu(\vec{p}) \approx M_\mu$ in the expressions for $P[F]$ and $N[F]$ following (4.10). Sending $\Lambda_c \to \infty$, we can use the orthogonality relations

$$\int_0^\infty dp\, p^2 j_n(px) j_n(py) = \frac{\pi}{2x^2}\delta(x-y).$$

In this way, $P[F]$ and $N[F]$ simplify to

$$P[F] = \tfrac{32}{9} F_\pi^2 [M_0 Q_1 + M_1 (Q_2 + 3Q_3)]$$
$$N[F] = \frac{3}{64F_\pi^2} \left[\left(\frac{Q_1}{M_0} + \frac{Q_2}{M_1} \right)^{-1} + 3 \left(\frac{Q_3}{M_1} \right)^{-1} \right]$$

where

$$Q_1[F] = \tfrac{1}{2}\pi \int_0^\infty dr\, r^2 \sin^4 F$$
$$Q_2[F] = \tfrac{1}{2}\pi \int_0^\infty dr\, r^2 \sin^2 F \cos^2 F$$
$$Q_3[F] = \tfrac{1}{2}\pi \int_0^\infty dr\, r^2 \sin^2 F.$$

Note that, in this approximation, (4.9) reduces to (4.10) quite easily.

Scaling analysis is now very simple. Rescale distances by a dilatation parameter α . Then the $\alpha \to 0$ limit of $E^\ell[F]$ is just

$$E^\ell[F(\alpha r)] \to \frac{(+\text{const})}{\alpha^3}.$$

Also, the $\alpha \to \infty$ limit of $E^\ell[F]$ is

$$E^\ell[F(\alpha r)] \to (+\text{const})\alpha^3.$$

Therefore, $E^\ell[F(\alpha r)]$ has a minimum for some intermediate value of α.

The asymptotic form of $F(r)$ is also straightforward to study. As in Section 1, we find

$$\frac{d}{dr}(r^2 F') = k^2 r^2 F$$

$$k^2[F] \equiv m_\pi^2 + \frac{P[F] - \ell(\ell+2)}{6\Lambda^2[F]} - \frac{4N[F]}{3} - \frac{7M_1}{6\Lambda[F]} + \frac{3\Lambda[F]}{512F_\pi^4 M_1} \left[\left(\frac{Q_1[F]}{M_0} + \frac{Q_2[F]}{M_1} \right)^{-2} + 6 \left(\frac{Q_3[F]}{M_1} \right)^{-2} \right]. \qquad (5.1)$$

If $k^2 < 0$, then $F \sim \sin(|k|r + \theta)/r$, and one finds that Λ, Q_2 and Q_3 diverge, while Q_1 remains finite. (5.1) then reduces to $k^2 = m_\pi^2$, a contradiction.

Just as in Section 1, it is unnatural to expect that k^2 will vanish exactly, because of the complicated consistency condition (5.1).

We therefore assume that $k^2 > 0$, and so $F \sim e^{-kr}/r$. In this case, Λ and all the Q_n's are finite. The situation is now very different from that in Section 1. We expect that (5.1) can be satisfied, even when $m_\pi = 0$. Thus, the inconsistencies we previously found for the massless case have vanished.

For the physical value of m_π, it remains to be seen whether the model is still numerically inaccurate:

1) To get the right nucleon-nucleon interaction, we would like to have $k \approx m_\pi$. Some preliminary numerical work indicates that $k^2[F]$ is quite sensitive to F. It is possible to tune k over quite a large range by replacing $F(r)$ by $F(\alpha r)$, where $\frac{1}{2} < \alpha < 2$. The best we can say at present is that k is near m_π for functions F near the one found by ANW.

2) It is easy to show that (1.6) no longer automatically holds. This means that the inequalities found by Braaten and Ralston are no longer valid. Thus, we have some hope of matching the parameters of the model to the data.

At the moment, we can say nothing about the problem of the infinite sequence of spin states in the model. This question must wait until numerical solutions are available.

The author intends to make a detailed numerical investigation of the equation of motion in the near future.

While this paper was being typed, Carlson completed a paper extending his previous work [16]. In addition to quantizing rotations and dilatations, he quantized the *radial* vibrational modes in a Gaussian approximation. He reports fairly good values for the mass of the nucleon (and excited states), somewhat high values for the mass of the delta (and excited states) and an excellent value for g_A.

Acknowledgements

I thank O. Alvarez and J. W. Carlson for extensive discussions on this topic. I also thank K. Bardakci, Y. C. Kao and M. Suzuki for helpful comments and questions. This work was supported by the Director, Office of Energy Research, Office of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

References

[1] T. H. R. Skyrme, Proc. Roy. Soc. A260 (1961) 127.

[2] G. S. Adkins, C. R. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552.

[3] S. Coleman, in New Phenomena in Subnuclear Physics, ed. A. Zichichi, part A, (Plenum Press, New York, 1977) p. 297.

[4] R. F. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D11 (1975) 3424.

[5] E. Witten, Nucl. Phys. B160 (1979) 57.

[6] M. Bander and F. Hayot, Phys. Rev. D30 (1984) 1837; E. Braaten and J. P. Ralston, Argonne preprint ANL-HEP-PR-84-12, February 1984; E. Braaten and J. P. Ralston, Phys. Rev. D31 (1985) 598.

[7] J. P. Ralston, Argonne Preprint ANL-HEP-CP-84-45, May 1984.

[8] G. S. Adkins and C. R. Nappi, Phys. Lett. 137B (1984) 251.

[9] J. W. Carlson, Nucl. Phys. B253(1985) 149.

[10] P. M. Stevenson, Phys. Rev. D30 (1984) 1712.

[11] T. Barnes and G. I. Ghandour, Phys. Rev. D22 (1980) 924.

[12] E. S. Abers and B. W. Lee, Phys. Rep. 9C (1973) 1; S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888.

[13] S. Weinberg, Phys. Rev. D8 (1973) 3497.

[14] P. A. M. Dirac, Can. J. Math. 2 (1950), 129; Proc. Roy. Soc. London A246 (1958) 326; Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University New York, 1964.

[15] E. Tomboulis and G. Woo, Annals of Physics 98 (1976) 1.

[16] J. W. Carlson, Berkeley preprint UCB-PTH-85/5, February 1985.

This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.

TECHNICAL INFORMATION DEPARTMENT
LAWRENCE BERKELEY LABORATORY
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720