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Strongly Hilbert Modules

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Strongly Hilbert Modules

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ABSTRACT OF THE DISSERTATION

Strongly Hilbert Modules

by

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We will provide some results on Hilbert modules, namely an equivalent condition for faithful Noetherian modules to be Hilbert. Then, we will generalize the notion of a Hilbert rings and modules to create the concept of C-Hilbert rings and modules. Finally, to provide more examples of C-Hilbert modules, we will take the notion of strongly Hilbert rings and extend them to strongly Hilbert modules.

Contents

Chapter 1

Introduction

Hilbert rings were created by Oscar Goldman in 1951 to generalize the proof of Hilbert's Nullstellensatz Theorem. The Nullstellensatz Theorem is an important basis of Algebraic Geometry which relates algebraic structures of ideals to geometric figures. Goldman used Hilbert rings to reframe Nullstellensatz under purely algebraic notions. Since then, Algebraic Geometry has become more focused on schemes and varieties. However, Hilbert rings are still being studied by Commuative Algebraists who have created multiple generalizations of Hilbert rings.

Prime submodules were created in 1983 by McCasland in his doctoral thesis as a module theory analog of prime ideals. Every prime submodule can be associated with a prime ideal; however, it may not uniquely associated. Later, McCasland and Moore specified a largest such prime submodule, and called it $\mathfrak{p}$-maximal. In 2009, Naghipour created a new notion of being prime, which they named strongly prime submodules. This new notion of being prime involved two elements of the module, instead of one element from the ring and module, respectively.

In 2012, David Rush showed that for a submodule, being $\mathfrak{p}$-maximal or strongly prime is actually the same property. This allowed him to create a notion of G-submodules and Hilbert modules.

Noetherian rings can be viewed as generalizations of principal ideal rings, since every ideal in a Noetherian ring is finitely generated. This makes the ideals in Noetherian rings particularly nice to work with. Noetherian rings tend to have additional and simpler characterizations for properties. We will extend existing characterizations of subclasses of Noetherian rings to corresponding Noetherian modules.

In this work, we will go over the theory needed to construct Hilbert modules and demonstrate some associated new properties. Then, we will generalize the notion and theory of Hilbert rings and modules to create a new class of mathematical objects, of which Hilbert modules are an example. We will use this new process to create strongly Hilbert modules based on the work of Karamzadeh and Moslemi, and give some classification of the rings which give rise to strongly Hilbert modules.

In the second chapter we will highlight classical ring theory of this field, up to the definitions of Hilbert and Jacobson rings. Of special note is the additional characterization for Noetherian Hilbert rings, and the equivalence between Hilbert and Jacobson rings. These characterizations are the inspiration for most of the following work.

In the third chapter, we will briefly detail the theory surrounding prime submodules, then discuss the existing theory surrounding Hilbert and Jacobson modules. Of note is the equivalence between Hilbert and Jacobson modules, which has not been recognized until now. This will give us our first result.

Theorem. *Let M be a finitely generated R-module. Then M is a Hilbert module if and only if $R/\operatorname{Ann}_R(M)$ is a Hilbert ring.*

We will immediately use this theorem to extend the additional characterization for Noetherian Hilbert rings to modules.

Theorem. *Let M be a Noetherian, faithful, finitely generated R-module. Then M is a Hilbert module if and only if for every prime ideal $\mathfrak{p}$ in R such that $\dim(R/\mathfrak{p}) = 1$, there must exist infinitely many maximal ideals containing $\mathfrak{p}$.*

We will then note that Rush's construction of Hilbert modules can be used to create a new more general class of modules. We will start by defining C-Hilbert rings and modules, and then prove the following main result.

Theorem. *Suppose M is a faithful, finitely generated R-module. Then M is a C-Hilbert R-module, if and only if R is a C-Hilbert ring.*

In the fourth chapter, we spend time going over the work of Karamzadeh and Moslemi on creating strongly Hilbert rings. We note that those rings are examples of C-Hilbert rings, and thus can be used to define a new type of module we will name strongly Hilbert modules. We close with the following characterization for Noetherian strongly Hilbert modules.

Theorem. *Let M be a Noetherian, faithful, finitely generated R-module. Then M is a strongly Hilbert module if and only if for each prime ideal $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) \geq 1$ there exists an uncountable number of non-zero minimal prime ideals in $R/\mathfrak{p}$.*

We then close with some final thoughts and future directions for our work.

Chapter 2

Preliminaries

It is necessary to start with some basic definitions from commutative ring theory. Then, we will introduce the idea of a G-domain and use that to build G-ideals and Hilbert rings. We will close this section with an introduction to Jacobson rings and their relationship to Hilbert rings, which will be used to great effect in the next chapter.

2.1 Noetherian Rings

The ideals of an arbitrary ring can be difficult to describe or work with. Ideals generated by a single element are very easy to understand, so they do not provide particularly rich examples. Finitely generated ideals are only slightly more complicated to work with, but allow for many interesting examples. Thus, we will want to discuss rings where every ideal is finitely generated. First, we start with some useful notation for prime ideals.

Definition 2.1.1. The *spectrum* of a commutative ring R, denoted by $\operatorname{Spec} R$, is the set of all prime ideals of R. The spectrum of a ring can be equipped the Zariski topology, in which the closed sets are the sets defined as $V(I) = \{P \in \operatorname{Spec}(A) \mid I \subseteq P\}$, where I is an ideal. We will use $\operatorname{Spec} R$ as shorthand for the set of prime ideals of R.

The following chain conditions will allow us to isolate sets of ideals that are easier to work with.

Definition 2.1.2. Given a set of ideals A, A is said to satisfy *descending chain condition* if given any chain of ideals $I_1 \supseteq \cdots \supseteq I_{k-1} \supseteq I_k \supseteq I_{k+1} \supseteq \cdots$ where $I_j \in A$ for all j, then there exists an n such that: $I_n = I_{n+1} = \cdots$. Similarly, A is said to satisfy the *ascending chain condition* if the same condition holds for any chain of ideals $I_1 \subseteq \cdots \subseteq I_{k-1} \subseteq I_k \subseteq I_{k+1} \subseteq \cdots$.

Remark 2.1.3. If A is the set of all the ideals of R, then we say R is an *Artinian ring.*

Our primary interest is the case when $A = \operatorname{Spec} R$. After defining chain conditions, we can finally properly define Noetherian rings.

Definition 2.1.4. Let R be a commutative ring and A be the collection of all ideals of R. R is *Noetherian* if A satisfies the ascending chain condition.

Remark 2.1.5. Noetherian rings are named after Emmy Noether, a preeminent female mathematician from the turn of the 20th century. In addition to her extensive work in Abstract Algebra, she also created Noether's theorem, which many consider to be a basis for much of modern physics.

Noetherian rings are in general easier to work with than non-Noetherian rings. This idea is demonstrated in the following classic result.

Proposition 2.1.6. *A commutative ring R is Noetherian if and only if all of its ideals are finitely generated.*

This proposition demonstrates that Noetherian rings are generalizations of principle ideal rings.

Since every chain of descending prime ideals eventually terminates in a Noetherian ring, it makes sense to talk about the finite length of these chains. These lengths will be used to defined a way of measuring the size of rings, which will be called the dimension of a ring. These can be defined for arbitrary rings, but they are most natural in Noetherian rings; we will focus on using dimension in the Noetherian context.

First we will define the height of a prime ideal.

Definition 2.1.7. Given a prime $\mathfrak{p} \in R$, we define the *height of* $\mathfrak{p}$ to be the supremum of the lengths of all chains of prime ideals contained in $\mathfrak{p}$. Namely, the height n of $\mathfrak{p}$ is the length of longest chain of ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{n-1} \subsetneq \mathfrak{p}_n = \mathfrak{p}$.

Remark 2.1.8. Note we are counting the number of containment relations, so the height would be the number of prime ideals in the chain minus one.

Definition 2.1.9. The *Krull dimension* of a commutative ring R, denoted $\dim R$, is the supremum of the heights all prime ideals in R. Namely, $\dim R$ is the supremum of the lengths of all chains of prime ideals. Note that if $\dim R = \infty$, then for every positive integer k there exists a prime ideal $\mathfrak{p}$ of height k.

Unless otherwise stated, all rings will be assumed to be commutative with unity. We will be using the following notation throughout this paper:

- The symbol D is reserved for integral domains.

- The symbol K is reserved for the quotient field of D.

Most of the material in sections 2.2-2.4 is included in most commutative ring theory textbooks. We will be using Irving Kaplansky's *Commutative Rings* [4] as our primary source for this section.

2.2 G-domains

In commutative algebra, G-domains were introduced by Oscar Goldman and Wolfgang Krull as part of the effort to prove Hilbert's Nullstellensatz. Since there was already a class of rings with Krull's name attached to it, the honor went to Goldman.

Definition 2.2.1. An integral domain D is called a *G-domain* if its quotient field K is such that

$$K = D\left[\frac{a_1}{b_1}, \frac{a_2}{b_2} \ldots, \frac{a_i}{b_i}, \ldots, \frac{a_n}{b_n}\right].$$

In other words, K is a finitely generated ring over D.

Remark 2.2.2. D is a G-domain if and only if K can be generated as ring by a single element, since

$$K = D\left[\frac{a_1}{b_1}, \frac{a_2}{b_2} \ldots, \frac{a_i}{b_i}, \ldots, \frac{a_n}{b_n}\right] = D\left[\frac{1}{b_1 b_2 \ldots b_i \ldots b_n}\right].$$

Trivially, any field is G-domain, since zero additional elements need to be added to make it a field.

Example 2.2.3. Consider $\mathbb{Z}_{2\mathbb{Z}} = \left\{\frac{a}{n} | a \in \mathbb{Z},\ n \text{ odd}\right\}$, the rationals with only odd denominators. This is the same as $\mathbb{Z}$ localized at the prime ideal $2\mathbb{Z}$. Given $a \in \mathbb{Z}_{2\mathbb{Z}}$, there exists a $b \in \mathbb{Z}_{2\mathbb{Z}}$ such that $ab = 2^k$ for some k. Thus, $\mathbb{Z}_{2\mathbb{Z}}\left[\frac{1}{2}\right]$ is a field, and by definition $\mathbb{Z}_{2\mathbb{Z}}$ is a G-domain.

There are other simple G-domains. For example, the formal power series ring over the rationals $\mathbb{Q}[[x]]$. $\mathbb{Q}[[x]]$ is a G-domain, since after adjoining the element $\frac{1}{x}$ the domain becomes the field $\mathbb{Q}\left((x)\right)$. However, it is more illustrative to mention a domain that is *not* a G-domain.

Example 2.2.4. $\mathbb{Z}$ is not a G-domain, since $\mathbb{Z}$ has an infinite number of prime elements. Therefore no finite list of rational numbers will allow you to generate all of their inverses.

This non-example demonstrates an important distinction between G-domains and other domains. A G-domain is a finite number of elements away from being a field, while a domain that is not a G-domain needs to have an infinite number of elements added to make it a field. The set of integers also suggests a connection between prime elements and G-domains. The following result solidifies this idea.

Proposition 2.2.5. *Let D be a unique factorization domain. Then D is a G-domain if and only if it has a finite number of non-associated prime elements.*

Proof. Assume D has a finite number of non-associated prime elements. Namely, D has a finite number of prime elements up to multiplication by a unit. Denote these prime elements as $p_1, p_2, ..., p_n$. Since D is a unique factorization domain, for any $a \in D$ we can write $a = up_1^{k_1}p_2^{k_2}...p_n^{k_n}$ where u is a unit of D and the k_i are non-negative integers. If K is the field of fractions of D, then

$$\begin{aligned} K &= \left\{\frac{a}{b} | a, b \in D, b \neq 0\right\} \\ &= \left\{\frac{a}{p_1^{k_1} p_2^{k_2} ... p_n^{k_n}} | a \in D, k_i \text{ are non-negative integers}\right\} \\ &= D\left[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\right]. \end{aligned}$$

Therefore D is a G-domain.

Next assume that D is a G-domain. Then, if K is the field of fractions of D, $K = D\left[\frac{1}{b}\right]$ where b is a non-zero element in D. Since D is an unique factorization domain we can write $b = up_1^{k_1} p_2^{k_2} ... p_n^{k_n}$ where u is a unit of D and the k_i are non negative integers. Therefore $K = D\left[\frac{1}{p_1 p_2 \cdots p_n}\right]$. Assume for contradiction that D has an infinite number of non-associated prime elements. Then there exists a prime element $q \in D$ such that $q \neq up_i$ for all $1 \leq i \leq n$ and u unit in D. But then $\frac{1}{q} \in K$ and $\frac{1}{q} \notin D\left[\frac{1}{p_1 p_2 \cdots p_n}\right]$. This is a contradiction to $K = D\left[\frac{1}{p_1 p_2 \cdots p_n}\right]$. Therefore D must have a finite number of non-associated prime elements. $\square$

Remark 2.2.6. If D is an unique factorization domain and has a finite number of primes, up to units, then it is a principal ideal domain. So there is no G-domain that is an unique factorization domain but not a principal ideal domain.

Even if we are not working with a unique factorization domain, there are several equivalent conditions to being a G-Domain [4].

Proposition 2.2.7. *Let D be an integral domain with quotient field K. For any non-zero element u in D, the following are equivalent:*

1. *Any non-zero prime ideal contains u.*

2. *Any non-zero ideal contains a power of u.*

3. $K = D[u^{-1}]$.

Remark 2.2.8. Note that condition 3 is the definition G-domain.

Proof. Assume every non-zero prime ideal in D contains a non-zero element u. Suppose for contradiction that I is a non-zero ideal that contains no power of u. Then I can be expanded to a prime ideal $\mathfrak{p}$ disjoint from $\{u^n\}$, which contradicts that every prime ideal contains u (so 1 implies 2).

Assume every non-zero ideal in D contains a power of a non-zero element u. Take any non-zero $b \in D$; The ideal $\langle b \rangle$ contains some power of u, say $u^n = bc$. This then implies $b^{-1} = cu^{-n} \in D[u^{-1}]$. But b was an arbitrary non-zero element in D; thus $K = D[u^{-1}]$ (so 2 implies 3).

Assume the quotient field of D is $K = D[u^{-1}]$ for some non-zero element u in D. If we take a non-zero prime ideal $\mathfrak{p}$ and any non-zero element b in $\mathfrak{p}$, then $b^{-1} = cu^{-n}$ for some $c \in D$ and $n \in \mathbb{Z}^+$, since $K = D[u^{-1}]$. Then $u^n = bc \in \mathfrak{p}$, and thus $u \in \mathfrak{p}$. (3 implies 1). □

There is an additional equivalent condition in the context of Noetherian rings, though we need the following lemma to prove it.

Lemma 2.2.9. *Let D be an integral domain having only a finite number of prime ideals. Then D is a G-domain.*

Proof. Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_n\}$ be all the non-zero prime ideals of an integral domain D. Then for each $\mathfrak{p}_i$, choose a non-zero element $a_i \in \mathfrak{p}_i$. Let $u = a_1 a_2 \dots a_n$ and note $u \in \mathfrak{p}_i$ for all i. Then by Theorem 2.2.7, $K = D[u^{-1}]$, which means D is a G-domain. □

We can generalize this idea as follows.

Proposition 2.2.10. *An integral domain D with the descending chain condition on finite intersections of prime ideals is a G-domain.*

Proof. Assume D is an integral domain with the descending chain condition on finite intersections of prime ideals. Let A be minimal among the ideals which are finite intersections of non-zero prime ideals. Thus $A \neq \{0\}$ and $A = \bigcap_{\mathfrak{p} \in \operatorname{Spec} D} \mathfrak{p}$. Let $u \in A$, then for all $\mathfrak{p} \in \operatorname{Spec}(D)$ $u \in \mathfrak{p}$. By Proposition 2.2.7, $K = D[u^{-1}]$, and D is a G-domain. □

We can use Lemma 2.2.9 to prove an additional characterization for Noetherian G-Domains [4].

Theorem 2.2.11. *A Noetherian integral domain D is a G-domain if and only if $\dim(D) \leq 1$ and D has only a finite number of maximal ideals (or, equivalently, prime ideals).*

Proof. Assume the Noetherian integral domain D has has only a finite number of maximal ideals and that $\dim(D) \leq 1$. Then, since $\dim(D) \leq 1$, every non-zero prime ideal in D is maximal. Thus, D has only a finite number of prime ideals, and by Lemma 2.2.9 D is a G-domain.

If we instead assume D is a Noetherian G-domain, by Proposition 2.2.7 the intersection of the non-zero prime ideals in D is non-zero. By Theorem 145 in [4], D has only a finite number of minimal prime ideals (prime ideals of height 1). If $\dim(D) > 1$, then there exist a chain of prime ideals $\{0\} \subset \mathfrak{p} \subset \mathfrak{q}$ where, without loss of generality, we may assume $\mathfrak{p}$ is of height 1 and $\mathfrak{q}$ is of height 2. Since there exists one prime ideal properly between $\{0\}$ and $\mathfrak{q}$ (namely $\mathfrak{p}$), by Theorem 144 also in [4], there must exist infinite many primes between them. But then D would have an infinite number of minimal prime ideals, which is a contradiction. So $\dim(D) \leq 1$, and D has a finite number of maximal ideals (since all non-zero ideals are of height 1, thus both minimal and maximal). □

It's a common theme that algebraic structures have a nice characterization in the Noetherian setting, and this theorem is this first of many in this paper. To get to our next Noetherian characterization we need to introduce our next majors objects: G-ideals and Hilbert rings.

2.3 G-ideals

With the notion of G-domains, we can move onto G-ideals, the main substructure we'll be dealing with.

Definition 2.3.1. A prime ideal $\mathfrak{p}$ of a commutative ring R is a *G-ideal* if $R/\mathfrak{p}$ is a G-domain.

An intuitive sense of the structure of a G-ideal is that they kill off all but a finite amount prime elements. Note that any maximal ideal is a G-ideal, since a field is trivially a G-domain.

Example 2.3.2. The ideal $\langle\, x \,\rangle$ in $\mathbb{Z}_{2\mathbb{Z}}\,[x]$ is a non-maximal G-ideal, since $\langle\, x \,\rangle \subset \langle\, x, 2 \,\rangle$ and $\mathbb{Z}_{2\mathbb{Z}}\,[x] \,/ \langle\, x \,\rangle \cong \mathbb{Z}_{2\mathbb{Z}}$, which is a G-domain.

Proposition 2.3.3. *Let a ring R have descending chain condition on finite intersections of prime ideals. Then each prime ideal $\mathfrak{p}$ of R is a G-ideal.*

Proof. Assume R has descending chain condition on finite intersections of prime ideals. Then for any prime ideal $\mathfrak{p}$ of R, $R/\mathfrak{p}$ will have descending chain condition on finite intersections of prime ideals. So by Proposition 2.2.10 $R/\mathfrak{p}$ is a G-domain and $\mathfrak{p}$ is a G-ideal. $\square$

We also have an interesting connection to nilradicals. Recall the following definition:

Definition 2.3.4. Given a ring R, the *nilradical*, N, of R is the set of all nilpotent elements of R. Namely: $N = \{x \in R \,|\, x^n = 0 \text{ for some positve integer } n\}$.

The nilradical can also be described as the intersection of all prime ideals in the ring.

The following result from [4] demonstrates an important connection between nilradicals and G-Domains in commutative rings.

Proposition 2.3.5. *The nilradical N of any commutative ring R is the intersection of all G-ideals in R.*

Proof. The nilradical is the intersection of all prime ideals of R, and thus needs to be a subset of the intersection of all G-ideals in R.

For the other containment suppose $u \notin N$, and construct a G-ideal excluding u. Take the zero ideal, which is disjoint from $\{u^n\}$, and expand it to an ideal $\mathfrak{p}$ that is maximal with respect to being disjoint from $\{u^n\}$. $\mathfrak{p}$ is prime, since it is maximal with respect to excluding a multiplicatively closed set. In the domain $R/\mathfrak{p}$, let u^* denote the image of u. The maximality of $\mathfrak{p}$ ensures that every non-zero prime ideal in $R/\mathfrak{p}$ contains u^*. By Proposition 2.2.7, $R/\mathfrak{p}$ is a G-domain, and thus $\mathfrak{p}$ is a G-ideal. $\square$

Definition 2.3.6. The *radical* of an ideal I in a commutative ring R, denoted by $\operatorname{Rad}(I)$, is the set of elements whose power is in the ideal I. Namely,

$$\mathrm{Rad}\,(I) = \{r \in R \mid r^n \in I \text{ for some positive integer } n\}.$$

These two facts, along with Proposition 2.3.5, give us the following two Corollaries:

The radical of an ideal is the preimage of the nilradical in the quotient ring R/I. Also, the radical of any prime ideal $\mathfrak{p}$ is itself, namely , $\mathrm{Rad}\,(\mathfrak{p}) = \mathfrak{p}$.

Corollary 2.3.7. *Let I be any ideal in a commutative ring R. Then the radical of I is the intersection of all G-ideals containing I.*

Corollary 2.3.8. *Let $\mathfrak{p}$ be a prime ideal in a commutative ring R. Then $\mathfrak{p}$ is equal to the intersection of all G-ideals containing $\mathfrak{p}$.*

The two previous results imply that G-ideals lie somewhere between prime ideals and maximal ideals. We will explore this idea further.

2.4 Hilbert Rings

After building G-domains and G-ideals, we can now define a Hilbert ring.

Definition 2.4.1. A *Hilbert ring* is a ring R such that every G-ideal is a maximal ideal.

Example 2.4.2. $\mathbb{Z}$ is a Hilbert ring.

Proof. All prime ideals in $\mathbb{Z}$ are of the form $\{0\}$ or $p\mathbb{Z}$ where p is a prime element.

i. For $\{0\}$, we find $\mathbb{Z}/\{0\} \cong \mathbb{Z}$ and $\mathbb{Z}$ is not a G-domain; thus, $\{0\}$ is not a G-ideal.

ii. Any other prime ideal in $\mathbb{Z}$ will be of the form p$\mathbb{Z}$, and $\mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}_p$, which is a field.

Thus, any non-zero prime ideal is maximal. But note that any G-ideal in $\mathbb{Z}$ is a non-zero prime ideal, and thus $\mathbb{Z}$ is a Hilbert ring. $\square$

Example 2.4.3. $\mathbb{Q}[x]$ is also a Hilbert ring, since any non-zero prime ideal is maximal, and $\mathbb{Q}[x]/\{\mathbf{0}\} \cong \mathbb{Q}[x]$ is not a G-domain.

Remark 2.4.4. However, $\mathbb{Z}_{2\mathbb{Z}}[x]$ is not a Hilbert ring, since $\langle x \rangle$ is a G-ideal that is not maximal as it is contained in $\langle 2, x \rangle$.

Since maximal ideals and G-ideals are the same in a Hilbert ring, Corollary 2.3.7 gives us the following:

Proposition 2.4.5. *In a Hilbert ring, the radical of any ideal I is the intersection of the maximal ideals containing I.*

We will find the a more specific form of the Proposition 2.4.5 useful in our later proofs.

Corollary 2.4.6. *If R is a Hilbert ring, every prime ideal $\mathfrak{p}$ in R is equal to the the intersection of the maximal ideals containing $\mathfrak{p}$.*

To give us a better idea of how Hilbert rings work, let's go through a short exercise from [4].

Lemma 2.4.7. *Let R be a Hilbert ring having only a finite number of maximal ideals. Then these maximal ideals are the only prime ideals of R.*

Proof. By Corollary 2.4.6, every prime ideal of R is an intersection of some of the maximal ideals. Since there is only a finite number of maximal ideals of R, there can only be a finite number of prime ideals of R. Let $\mathfrak{q}$ be an arbitrary prime ideal of R. Since $R/\mathfrak{q}$ is a homomorphic image of R, it has only a finite number of prime ideals. By Lemma 2.2.9, $R/\mathfrak{q}$ is a G-domain, thus $\mathfrak{q}$ is a G-ideal. Since R is a Hilbert ring, $\mathfrak{q}$ must be a maximal ideal, showing that every prime ideal in R is maximal. □

The second of the Noetherian equivalent conditions is for Hilbert rings [4].

Theorem 2.4.8. *A Noetherian ring R is a Hilbert ring if and only if for every prime ideal $\mathfrak{p}$ in R such that* $\dim(R/\mathfrak{p}) = 1$, *there must exist infinitely many maximal ideals containing $\mathfrak{p}$.*

Proof. Assume that for every prime ideal $\mathfrak{p}$ such that $\dim(R/\mathfrak{p}) = 1$, there must exist infinitely many maximal ideals containing $\mathfrak{p}$. Let $\mathfrak{q}$ be a G-ideal, so $R/\mathfrak{q}$ is a G-domain. By Theorem 2.2.11, the dimension of $R/\mathfrak{q}$ is at most 1, and has only a finite number of maximal ideals. In the case where, $\dim(R/\mathfrak{q}) = 1$, then R has both finitely and infinitely many ideals containing $\mathfrak{q}$ which is not possible. Thus the $\dim(R/\mathfrak{q}) = 0$ and $\mathfrak{q}$ is a maximal ideal. Since an arbitrary G-ideal in R is maximal then R is a Hilbert ring.

Now, assume R is a Hilbert Ring. Let $\mathfrak{q}$ be a prime ideal in R such that $\dim(R/\mathfrak{q}) = 1$. Assume for contradiction that R has only finitely many maximal ideals containing $\mathfrak{q}$; then $R/\mathfrak{q}$ is a domain with only a finite number of maximal ideals. By Theorem 2.2.11, $R/\mathfrak{q}$ is a G-domain, so $\mathfrak{q}$ is a G-ideal. This is a contradiction to $\mathfrak{q}$ not being a maximal ideal, and therefore R must have infinitely many maximal ideals containing $\mathfrak{q}$. □

Example 2.4.9. $\mathbb{Q}[x_1, x_2, x_3, \ldots, x_n, \ldots]$ is a non-Noetherian Hilbert ring.

2.5 Jacobson Rings

Definition 2.5.1. A ring for which every prime ideal is an intersection of maximal ideals is called a *Jacobson ring.*

Remark 2.5.2. Jacobson rings can be defined for non-commutative rings, which is not the case for Hilbert rings.

Example 2.5.3. $\mathbb{Z}$ is a Jacobson ring. The only non-maximal prime ideal it contains is the zero ideal, and

$$\bigcap_{p \text{ prime}} p\mathbb{Z} = \{0\}.$$

The following theorem is the inspiration for much of our work in the next chapter. This equivalence has been known for half a century, though we'll produce a proof here of our own construction.

Theorem 2.5.4. *If R is a commutative ring, then R is Hilbert if and only if it is Jacobson.*

Proof. Suppose R is a Hilbert commutative ring. By 2.4.6, any prime ideal $\mathfrak{p}$ of R is an intersection of G-ideals. In a Hilbert ring, all G-ideals are maximal, so the prime ideal is the intersection of maximal ideals, and R is Jacobson.

On the other hand, if R is a Jacobson commutative ring, choose a nonzero G-ideal $\mathfrak{g}$. Then $D = R/\mathfrak{g}$ is a G-domain. Assume for contradiction D is not a field. Then by Proposition 2.2.7, there exists a $u \in D$ such that u is contained in every nonzero prime ideal of D. Thus, the intersection of all maximal ideals in D is not the zero ideal. Since R is Jacobson

$$\mathfrak{g} = \bigcap_{\mathfrak{g} \subset \mathfrak{m}} \mathfrak{m},$$

where all $\mathfrak{m}$ are maximal. But the maximal ideals in D are the maximal ideals of R containing $\mathfrak{g}$, so their intersection in $D = R/\mathfrak{g}$ should be $\mathbf{0}$, a contradiction. Hence, D must be a field, and R is a Hilbert ring. $\square$

This completes our study of Hilbert and Jacobson rings. In Chapter 3, we will discuss how these rings were used to construct Hilbert and Jacobson modules. Before that, we want to over some ideas from Karamzadeh's and Moslemi's work [5], which will address a newer type of ring called a strongly Hilbert ring. We will use this ring to construct our main results in Chapter 4.

2.6 *G*-type Domain

G-domains were defined using the difference between a finite set and an infinite set. We can generalize that idea to talk about the difference of larger cardinalities.

Definition 2.6.1. D is a *G-type domain* if there exists a countable multiplicatively closed set S in D with $K = D\left[S^{-1}\right]$ where we define $S^{-1} = \left\{\frac{1}{s} | s \in S\right\}$.

Observe that if A is countable subset of a ring R, then the multiplicatively closed set generated by A is still countable.

All G-domains are G-type domains, since any finite set is countable. For an example of a G-type domain that is not a G-domain we have our favorite domain: the integers.

Example 2.6.2. $\mathbb{Z}$ is a G-type domain, since if $S = \mathbb{Z}\backslash\{0\}$, then

$$\mathbb{Q} = \mathbb{Z}[S^{-1}] = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots].$$

In fact, any countable domain is a G-type domain. Take $S = D \backslash \{0\}$, and since D is countable, so is S and $K = D\left[S^{-1}\right]$. Thus D is a G-type domain.

We also get a version of Proposition 2.2.5 for G-type factorization domains.

Proposition 2.6.3. *A unique factorization domain is a G-type domain if and only if it has only a countable number of prime elements (up to units).*

Thus, if K is a countable field, then $K[x]$ is a G-type domain that is not a G-domain.

Even if we are not working with a unique factorization domain, there are several equivalent conditions to being a G-type domain via an analogue of Proposition 2.2.7 for G-type domains.

Proposition 2.6.4. *Let D be a a domain with quotient field K, and let S be a multiplicatively closed set in D. The following are equivalent:*

1.Each non-zero prime ideal of D intersects S.

2.Each non-zero ideal of D intersects S.

3. $K = D[S^{-1}]$

Remark 2.6.5. Condition 3 is equivalent to D being a G-type domain.

Proof. Assume every non-zero prime ideal in D intersects S. Suppose for contradiction that I is a non-zero ideal that is disjoint from S. Then I can be expanded to a prime ideal $\mathfrak{p}$ disjoint from S, which contradicts that every non-zero prime ideal of D intersects S (so 1 implies 2).

Assume every non-zero ideal in D intersects S. Take any non-zero $b \in D$; the ideal $\langle b \rangle$ contains some element of S, say $s = bc$. Then $b^{-1} = cs^{-1} \in D[S^{-1}]$. But b was an arbitrary non-zero element in D, and thus $K = D[u^{-1}]$ (so 2 implies 3).

Assume the quotient field of D is $K = D[S^{-1}]$. If we take a non-zero prime ideal $\mathfrak{p}$ and any non-zero element b in $\mathfrak{p}$, then $b^{-1} = cs^{-1}$ for some $c \in D$, since $K = D[S^{-1}]$. Then $s = bc$ and thus $\mathfrak{p}$ intersects S. (3 implies 1). $\square$

These equivalent conditions for being a G-type domain will be immediately useful in the proof of Proposition 2.6.3 for general integral domains.

Lemma 2.6.6. *If a domain D has a countable number of prime ideals, then D is a G-type domain.*

Proof. The set Spec $D \backslash \{0\}$ is countable, so we may index them by $\mathbb{N}$ and consider $\{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, ...\}$ as all the non-zero prime ideals of D. Choose a non zero $a_i \in \mathfrak{p}_i$ for each $i \in \mathbb{N}$, and let S be the multiplicatively closed set generated by $\{a_1, a_2, a_3, ...\}$. Note that D is a domain, and S is a countable multiplicatively closed set such that for every non-zero prime ideal $\mathfrak{p}$ in D, $\mathfrak{p} \cap S \neq \emptyset$. Then by Proposition 2.6.4, the quotient field of D is equal to $D[S^{-1}]$. Thus D is a G-type domain. $\square$

The following Proposition is included in a proof in [5], but we will state and prove it by itself.

Proposition 2.6.7. *Let D be an integral domain. If D has the descending chain condition on prime ideals and only a countable number of nonzero minimal prime ideals then D is a G-type domain.*

Proof. Let $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3, \ldots, \mathfrak{p}_n, ...$ be the non-zero minimal prime ideals of D and note that any prime ideal $\mathfrak{q}$ contains one of the $\mathfrak{p}_n$'s. For each n, take $0 \neq a_n \in \mathfrak{p}_n$ and let S be the multiplicatively closed set generated by $\{a_1, a_2, a_3, \ldots, a_n, ...\}$. But $\mathfrak{q} \cap S \neq \emptyset$ for all prime ideals $\mathfrak{q}$. Thus by Proposition 2.6.4, $K = D[S^{-1}]$. Thus D is a G-type domain. $\square$

Now we have the tools necessary to prove an equivalent condition for Noetherian G-type domains.

Theorem 2.6.8. *Let D be a Noetherian domain. Then D is a G-type domain if and only if D has only a countable number of non-zero minimal prime ideals.*

Proof. Suppose D has only a countable number of non-zero prime ideals. Since D is Noetherian, it has descending chain condition on prime ideals by the height function. Then by the proof of Proposition 2.6.7, D is a G-type domain.

Conversely, if D is a G-type domain let $S = \{s_1, s_2, s_3, \ldots\}$ be a countable multiplicatively closed set such that $S \cap \mathfrak{p} \neq \emptyset$ for all non-zero prime ideals $\mathfrak{p}$. Assume for contradiction the set of non-zero minimal prime ideals is uncountable. Since S is countable and $S \cap \mathfrak{p} \neq \emptyset$ for all prime $\mathfrak{p}$, there must exist an element $s \in S$ such that s belongs to an uncountable number of non-zero minimal prime ideals. Let $\{\mathfrak{p}_\alpha\}$ be the subset of minimal non-zero prime ideals that contain s, namely $s \in \mathfrak{p}_\alpha$ for all α. Considering the ideal $\langle\, s \,\rangle$, all the $\mathfrak{p}_\alpha$'s are minimal over $\langle\, s \,\rangle$. But since D is Noetherian, there can only be a finite number of minimal prime ideals over any given ideal. This is a contradiction, and therefore D has a countable number of minimal prime ideals. $\square$

Example 2.6.9. $D = \mathbb{Z}\left[2^{\frac{k}{2^n}}\right]$ where $n \in \mathbb{N}$ and $1 \leq k < n$ is a countable non-Noetherian domain, and thus a G-type domain. D is countable, since it is a subring of the algebraic closure of $\mathbb{Z}$, which is countable. It is non-Noetherian, since $\left\langle \sqrt{2} \right\rangle \subset \left\langle \sqrt[4]{2} \right\rangle \subset \left\langle \sqrt[8]{2} \right\rangle \subset \ldots$ is an infinite chain of ascending ideals in D.

2.7 G-type Ideal

Definition 2.7.1. A prime ideal $\mathfrak{p}$ of a ring R is called a *G-type ideal* if $R/\mathfrak{p}$ is a G-type domain.

Remark 2.7.2. A way of intuiting what these G-ideals are is that they kill off all but a countable amount prime elements.

Example 2.7.3. Any G-ideal is a G-type ideal.

Example 2.7.4. In the ring $\mathbb{Z}[x]$, the prime ideal $\langle\, x \,\rangle = \{f(x) \in \mathbb{Z}[x] \,|\, f(0) = 0\}$ is a G-type ideal, since $\mathbb{Z}[x]/\langle\, x \,\rangle \cong \mathbb{Z}$ and $\mathbb{Z}$ is a G-type ideal. However, $\langle\, x \,\rangle$ is not a G-ideal, since $\mathbb{Z}$ is not a G-domain.

Example 2.7.5. In $\mathbb{Z}\times\mathbb{R}$, the prime ideal $\{0\}\times\mathbb{R}$ is a G-type ideal, since $(\mathbb{Z}\times\mathbb{R})/(\{0\}\times\mathbb{R}) \cong \mathbb{Z}$ is a G-type domain.

Proposition 2.7.6. *Let a ring R have descending chain condition on prime ideals, and $R/\mathfrak{p}$ has only a countable number of non-zero minimal prime ideals for each prime ideal $\mathfrak{p}$ in R. Then each prime ideal $\mathfrak{p}$ of R is a G-type ideal.*

Proof. Assume R has the descending chain condition on prime ideals, and $R/\mathfrak{p}$ has only a countable number of non-zero minimal prime ideals for each prime ideal $\mathfrak{p}$ in R. Given any prime ideal $\mathfrak{p}$ in R, then $R/\mathfrak{p}$ will have the descending chain condition on prime ideals and only a countable number of nonzero minimal prime ideals. Thus, by Proposition 2.6.7 $R/\mathfrak{p}$ will be a G-type domain, which makes $\mathfrak{p}$ a G-type ideal. $\square$

2.8 Strongly Hilbert Rings

Definition 2.8.1. A ring R is a *strongly Hilbert ring* if each G-type ideal in R is maximal.

Remark 2.8.2. Any strongly Hilbert ring is a Hilbert ring, since a G-ideal is a G-type ideal.

However, $\mathbb{Z}$ is not a strongly Hilbert ring, since $\{0\}$ is a G-type ideal that is not maximal.

Example 2.8.3. $R = \mathbb{C}[x]/\langle x^2 \rangle$ is a strongly Hilbert ring. (Note that it is not a domain, since $\overline{x} * \overline{x} = 0$). R is the image of a principal ideal ring, thus is one itself. In fact, all ideals in R can be described $g(x)R$, where $g(x) \in \mathbb{C}[x]$. Prime ideals in $\mathbb{C}[x]$ are $\{0\}$, or of the form $\langle x - a \rangle$ where $a \in \mathbb{C}$. In R, $0R$ is not prime since $\overline{x} * \overline{x} = 0$. However, $(x-a)R$ is prime, with $R/(x-a)R \cong \mathbb{C}$ Thus $(x-a)R$ is in fact maximal, and therefore any G-type ideal in R is maximal. Hence, R is a strongly Hilbert ring.

In general, $\dim R = 0$ implies R is a strongly Hilbert ring (since every prime ideal is already maximal).

Example 2.8.4. Let K be an uncountable field. Then $K[x_1, x_2, \ldots, x_n]$ and $K[x_1, x_2, \ldots, x_n, \ldots]$ are strongly Hilbert rings. Note that $K[x_1, x_2, ..., x_n, ...]$ is non-Noetherian.

If K is a countable field, $K[x]$ is never a strongly Hilbert ring. Since $\{0\}$ is prime and $K[x]$ is a G-type domain, $\{0\}$ is a G-type ideal that is not maximal.

Next, we have an equivalent condition for Noetherian Hilbert rings.

Theorem 2.8.5. *Let R be a Noetherian ring. Then R is a strongly Hilbert ring if and only if for each prime ideal $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) \geq 1$, there exists an uncountable number of non-zero minimal prime ideals in $R/\mathfrak{p}$.*

Proof. Assume for any prime ideal $\mathfrak{p}$ in R with $\dim(R/\mathfrak{p}) \geq 1$, there exists an uncountable number of non-zero minimal prime ideals in $R/\mathfrak{p}$. Let $\mathfrak{q}$ be an arbitrary G-type ideal, so $R/\mathfrak{q}$ is a Noetherian G-type domain. By Theorem 2.6.8, $R/\mathfrak{q}$ has only a countable number of non-zero minimal prime ideals. This implies $\dim R/\mathfrak{q} = 0$, since having $\dim R/\mathfrak{q} \geq 1$ would lead to contradiction on the cardinality of minimal prime ideals of $R/\mathfrak{q}$. But $\dim R/\mathfrak{q} = 0$ implies $\mathfrak{q}$ is a maximal ideal. Thus any arbitrary G-type ideal is maximal, and R is a strongly Hilbert ring.

Conversely, let R be a strongly Hilbert ring. If $\mathfrak{p}$ is a prime ideal with $\dim R/\mathfrak{p} \geq 1$. Then $\mathfrak{p}$ is not a maximal ideal, thus $\mathfrak{p}$ is not a G-type ideal. Then $R/\mathfrak{p}$ is not a G-type domain, so by Theorem 2.6.8 $R/\mathfrak{p}$ has an uncountable number of non-zero minimal prime ideals. □

We can get the forward direction without the assumption of being Noetherian via Lemma 2.6.6.

Lemma 2.8.6. *If R is a strongly Hilbert ring, then for each prime ideal $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) \geq 1$ there exists an uncountable number of prime ideals in $R/\mathfrak{p}$.*

Proof. Let R be a strongly Hilbert ring, and $\mathfrak{p} \in \operatorname{Spec} R$ such that $\dim(R/\mathfrak{p}) \geq 1$. Since $\dim(R/\mathfrak{p}) \geq 1$, $\mathfrak{p}$ is not a maximal ideal. Assume for contradiction that $R/\mathfrak{p}$ has a countable number of prime ideals. By Lemma 2.6.6, $R/\mathfrak{p}$ is a G-type domain. Thus by definition, $\mathfrak{p}$ is a G-type ideal. However, $\mathfrak{p}$ is not a maximal ideal, a contradiction to R being a strongly Hilbert ring. Therefore $R/\mathfrak{p}$ has an uncountable number of prime ideals. □

After finishing all of the necessary material from ring theory, we can finally discuss Hilbert and Jacobson modules in the next chapter.

Chapter 3

Hilbert and Jacobson Modules

3.1 Modules

Modules are a generalization of vector spaces, where the scalars come from an arbitrary ring instead of a field. For the rest of this work, we will assume all modules are unitary left modules over a commutative ring R and refer to them as R-modules.

Ideals are used to study the structure of rings in commutative ring theory. We would like to be able to use the tools and knowledge we have about ideals and apply them to modules. To do this, we will need a way to associate ideals with submodules.

Definition 3.1.1. Given two submodules L and N of an R-module M, we define

$$(N :_R L) := \{r \in R \,|\, rL \subseteq N\}.$$

We call this the *colon operator*.

The significance of this operator is not apparent until we introduce another fact: $(N :_R L)$ is an ideal. With this, we can construct a correspondence between submodules and ideals. Let L be the entire module M. Then given any submodule N, we have an associated ideal of R, $(N :_R M)$. We illustrate this with the following example.

Example 3.1.2. The following are some simple examples of the colon operator in $\mathbb{Z} \oplus \mathbb{Z}$:

- $((2\mathbb{Z} \oplus \mathbb{Z}) :_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})) = 2\mathbb{Z}$
- $((2\mathbb{Z} \oplus \{0\}) :_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})) = \{0\}$
- $((2\mathbb{Z} \oplus 3\mathbb{Z}) :_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})) = 6\mathbb{Z}$

One of the recurring themes of Chapter 2 was having additional characterizations for Noetherian rings. We would like to be able to keep that additional structure as we begin to work with modules. To do so, we need to define Noetherian modules.

Definition 3.1.3. An R-module is called *Noetherian* if it satisfies the ascending chain condition on its submodules.

The following classical theorem allows us to construct explicit examples of Noetherian submodules.

Proposition 3.1.4. *Let R be a Noetherian ring. Then any R-module is Noetherian if and only if it is finitely generated.*

Thus, any finitely generated $\mathbb{Z}$-module, such as $\mathbb{Z} \oplus \mathbb{Z}$, will be Noetherian. We will come back to this particular module when we talk about prime submodules. Before moving on, we want to note that for an R-module M to be non-Noetherian, the ring R has to be be non-Noetherian. However, the converse is not true. For a simple counterexample, look at $\mathbb{Z}$ as a $\mathbb{Z}[x_1, x_2, x_3, ...]$-module, with multiplication defined by

$$f(x_1, x_2, x_3, \dots, x_n) \cdot k := f(0, 0, 0, \dots, 0) \cdot k.$$

The module is Noetherian, but the ring $\mathbb{Z}[x_1, x_2, x_3, ...]$ is not. The reason the counterexample worked is because the annihilator of M killed off the possible infinitely ascending chains of submodules. However, when we look at modules where the annihilator is the zero ideal – namely faithful modules – we get the following result.

Lemma 3.1.5. *If M is a faithful Noetherian R-module, then the ring R is Noetherian.*

We will use this Lemma later to prove our new results for Noetherian modules.

3.2 Prime Submodules

Much of commutative ring theory is based around the study of prime ideals. Recently, McCasland and Moore [9] created the following notion of a prime submodule.

Definition 3.2.1. A proper submodule N of an R-module M is said to be *prime* if for all $a \in R$ and $x \in M$, if $ax \in N$ then either $x \in N$ or $a \in (N :_R M)$.

Since McCasland and Moore introduced this definition, there has been significant research activity generalizing ring-theoretic results regarding prime ideals to modules. The rest of the section will discuss these generalizations and the connection between prime submodules and prime ideals. The following proposition shows that the ideal associated to a prime submodule is prime.

Proposition 3.2.2. *If N is a prime submodule of an R-module M, then $(N :_R M)$ is a prime ideal of R.*

Example 3.2.3. Consider the submodule $2\mathbb{Z} \oplus 2\mathbb{Z}$ in the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$. If the product $r(a, b)$ is an element of $2\mathbb{Z} \oplus 2\mathbb{Z}$, then $ra, rb \in 2\mathbb{Z}$. Since $2\mathbb{Z}$ is a prime ideal, either $r \in ((2\mathbb{Z} \oplus 2\mathbb{Z}) :_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})) = 2\mathbb{Z}$ or $(a, b) \in 2\mathbb{Z} \oplus 2\mathbb{Z}$. Therefore, $2\mathbb{Z} \oplus 2\mathbb{Z}$ is a prime submodule.

Using the definition of a prime submodule, we can prove the following useful result about maximal submodules in finitely generated modules [6].

Proposition 3.2.4. *Let M be a finitely generated R-module. If N is a maximal submodule of M, then the ideal $(N :_R M)$ is a maximal ideal of R.*

Proof. Let N be a maximal submodule of a finitely generated R-module M. Assume that $(N :_R M)$ is not a maximal ideal. Let $\mathfrak{p}$ be a prime ideal properly containing $(N :_R M)$. Then $(N + \mathfrak{p}M)_{(\mathfrak{p})}$ is a $\mathfrak{p}$-prime submodule of M properly containing N. This is a contradiction to N being maximal. Therefore, $(N :_R M)$ must be a maximal ideal. $\square$

Remark 3.2.5. Note that the converse of this Proposition is not true. Going back to our example of $2\mathbb{Z} \oplus 2\mathbb{Z}$ in the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}$, $((2\mathbb{Z} \oplus 2\mathbb{Z}) :_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})) = 2\mathbb{Z}$, which is a maximal ideal in $\mathbb{Z}$. However, $2\mathbb{Z} \oplus 2\mathbb{Z}$ is not a maximal submodule, since $(2\mathbb{Z} \oplus 2\mathbb{Z}) \subseteq (2\mathbb{Z} \oplus \mathbb{Z})$.

After creating prime submodules, McCasland and Moore introduced the notion of the radical of a submodule in [9].

Definition 3.2.6. Given a submodule N of an R-module M, the *radical* of N is the intersection of all prime submodules of M containing N.

A concern with the ideal $(N :_R M)$ is that it is not unique with respect to the submodule N. We would like to be able to talk about a "unique" submodule associated with a prime ideal $\mathfrak{p}$. Since we are dealing with finitely generated modules, we can look for the largest module associated to an ideal. This leads us to the definition of I-maximal [9].

Definition 3.2.7. A proper submodule N of a finitely generated R-module M with $I = (N :_R M)$ is said to *I-maximal* if it is maximal among submodules K of M with $I = (K :_R M)$.

Example 3.2.8. The submodule $2\mathbb{Z} \oplus 2\mathbb{Z}$ is not $2\mathbb{Z}$-maximal, since $(2\mathbb{Z} \oplus 2\mathbb{Z}) \subsetneq (2\mathbb{Z} \oplus \mathbb{Z})$ and $((2\mathbb{Z} \oplus \mathbb{Z}) :_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z})) = 2\mathbb{Z}$. However, $2\mathbb{Z} \oplus \mathbb{Z}$ is $2\mathbb{Z}$-maximal, since $2\mathbb{Z} \oplus \mathbb{Z}$ is a maximal submodule of $\mathbb{Z} \oplus \mathbb{Z}$.

We will only be concerned with the case when I is a prime ideal. The following Proposition, from [9], will be useful in moving from prime ideals to prime submodules.

Proposition 3.2.9. *Let M be a finitely generated R-module. Given any prime ideal $\mathfrak{p}$ in R such that $\operatorname{Ann}(M) \subseteq \mathfrak{p}$, there exists a prime submodule N that is $\mathfrak{p}$-maximal.*

We are going to want to work with prime submodules that are $\mathfrak{p}$-maximal with respect to their ideal $\mathfrak{p}$. These can be hard to identify in general. In [12], Rush was able to demonstrate that a prime submodule being $\mathfrak{p}$-maximal is equivalent to two other conditions. We will start with defining those conditions, and then properly state Rush's theorem.

Definition 3.2.10. A proper submodule N of a R-module M is said to be *strongly prime* if $((N + Rx) :_R M)\, y \subseteq N$ implies either $x \in N$ or $y \in N$ for $x, y \in M$.

Remark 3.2.11. Every strongly prime submodule is a prime submodule.

Definition 3.2.12. If D is an integral domain with quotient field K and M is a D-module, the *rank* of M is defined as the dimensions of $K \otimes_D M$ as a vector space over K.

The following theorem of Rush from the same publication gives equivalent conditions for a submodule N to be $\mathfrak{p}$-maximal.

Theorem 3.2.13. *Let N be a submodule of a finitely generated R-module M, and $\mathfrak{p}$ be a prime ideal of R. The following are equivalent:*

(1) N is a strongly prime submodule with $(N :_R M) = \mathfrak{p}$;

(2) N is $\mathfrak{p}$-maximal;

(3) N is prime submodule with $(N :_R M) = \mathfrak{p}$ and M/N is an $(R/\mathfrak{p})$-module of rank 1.

This theorem is powerful tool that will allow us identify and prove results on $\mathfrak{p}$-maximal submodules. The following proposition, which is a partial inverse to 3.2.4, quickly follows.

Proposition 3.2.14. *Let M be a finitely generated R-module. Suppose N is a submodule of M such that $(N :_R M) = \mathfrak{p}$ is a maximal ideal and N is $\mathfrak{p}$-maximal. Then N is a maximal submodule of M.*

Proof. Let N be a submodule of M such that $(N :_R M) = \mathfrak{p}$ is a maximal ideal and N is $\mathfrak{p}$-maximal. Since N is $\mathfrak{p}$-maximal, by Theorem 3.2.13, M/N is an $(R/\mathfrak{p})$-module of rank 1. But $\mathfrak{p}$ is maximal, so $R/\mathfrak{p}$ is a field. Thus, N is a maximal submodule of M. $\square$

3.3 Hilbert Modules

Using the notion of $\mathfrak{p}$-maximal prime submodules, Rush defined the notion of a G-submodule in [12] by uniquely associating a prime submodule with a G-ideal in R.

Definition 3.3.1. A submodule N of a finitely generated R-module M is a G-*submodule* if $\mathfrak{p} = (N :_R M)$ is a G-ideal of R and N is $\mathfrak{p}$-maximal.

Remark 3.3.2. Every maximal submodule is a G-submodule, since by Proposition 3.2.4 $(N :_R M)$ will be a maximal ideal, and thus a G-ideal.

Example 3.3.3. Let M be the $\mathbb{Z}_{2\mathbb{Z}}[x]$-module $\mathbb{Z}_{2\mathbb{Z}}[x] \oplus \mathbb{Z}_{2\mathbb{Z}}[x]$. We claim that the submodule $N = x\mathbb{Z}_{2\mathbb{Z}}[x] \oplus \mathbb{Z}_{2\mathbb{Z}}[x]$ is a G-submodule. Note that $(N :_{\mathbb{Z}_{2\mathbb{Z}}[x]} M) = x\mathbb{Z}_{2\mathbb{Z}}[x]$. Recall that $\mathbb{Z}_{2\mathbb{Z}}[x] / (x\mathbb{Z}_{2\mathbb{Z}}[x]) \cong \mathbb{Z}_{2\mathbb{Z}}$ which is a G-domain, thus $x\mathbb{Z}_{2\mathbb{Z}}[x]$ is a G-ideal. N is $x\mathbb{Z}_{2\mathbb{Z}}$-maximal, since the only proper submodule of M that contains N is $(2, x)\,\mathbb{Z}_{2\mathbb{Z}}[x] \oplus \mathbb{Z}_{2\mathbb{Z}}[x]$, and

$$\big((2, x)\,\mathbb{Z}_{2\mathbb{Z}}[x] \oplus \mathbb{Z}_{2\mathbb{Z}}[x] :_{\mathbb{Z}_{2\mathbb{Z}}[x]} M\big) \;=\; (2, x)\,\mathbb{Z}_{2\mathbb{Z}}[x].$$

Thus, N is $\mathfrak{p}$-maximal for a G-ideal and is a G-submodule. Note that N is not a maximal submodule.

Remark 3.3.4. A G-submodule for a given G-ideal is not unique. For example, by the same reasoning as above, $\mathbb{Z}_{2\mathbb{Z}}[x] \oplus (x)\mathbb{Z}_{2\mathbb{Z}}[x]$ is also a G-submodule associated with the G-ideal $x\mathbb{Z}_{2\mathbb{Z}}[x]$.

After defining G-submodules we get the following new corollary of 2.7.6.

Corollary 3.3.5. *Let R have the descending chain condition on finite intersections of prime ideals, and M be a finitely generated R-module. If a submodule N of M is strongly prime, then N is a G-submodule.*

Proof. Let N be a strongly prime submodule and $\mathfrak{p} = (N :_R M)$. Then by Theorem 3.2.13 N is $\mathfrak{p}$-maximal. But by Theorem 2.3.3 every prime ideal of R is a G-ideal. Thus, N is a G-submodule. □

Returning to material the from [12], recall from Corollary 2.3.7 that the radical of an ideal is the intersection of all G-ideals containing it. We have a similar result for modules.

Proposition 3.3.6. *If N is a submodule of a finitely generated R-module M, then the radical of N is equal to the intersection of all of the G-submodule of M containing N.*

Similar to the ring theory version, we will find the following corollary concerning prime submodules particularly useful.

Corollary 3.3.7. *If N is a prime submodule of a finitely generated R-module M, then N is equal to the intersection of all of the G-submodule of M containing N.*

Now, we can finally define Hilbert modules.

Definition 3.3.8. A finitely generated R-module is said to be a *Hilbert module* if each G-submodule of M is a maximal submodule of M.

Remark 3.3.9. Rush originally named these modules *Jacobson modules* in an attempt to avoid confusion to Hilbert C^*-modules from Hilbert spaces. We will refer to them as Hilbert modules to highlight the connection of this definition with definition 2.4.1. This will also allow us to save the name Jacobson modules for another module we will define soon.

The following major result from [12] shows the relationship between Hilbert rings and modules.

Theorem 3.3.10. *If R is a Hilbert ring, then every finitely generated R-module M is a Hilbert module.*

Proof. If R is a Hilbert ring and N is a G-submodule of the finitely generated R-module M, then by definition $\mathfrak{p} = (N :_R M)$ is a G-ideal of R, and N is $\mathfrak{p}$-maximal. Thus $\mathfrak{p}$ is a maximal ideal of R, since R is Hilbert. Since $\mathfrak{p}$ is a maximal ideal and N is $\mathfrak{p}$-maximal then by Proposition 3.2.14 N is a maximal submodule of M. Therefore M is a Hilbert module. $\square$

Example 3.3.11. This theorem give us a multiple example of Hilbert modules. For instance, $\mathbb{Z} \oplus \mathbb{Z}$ over $\mathbb{Z}$ is a Hilbert module since $\mathbb{Z}$ is a Hilbert ring.

Now consider the reverse direction of 3.3.10. If an R-module M is a Hilbert module does that mean R is a Hilbert ring? Unfortunately that is not the case.

Remark 3.3.12. Recall from 2.4.4 that $\mathbb{Z}_{2\mathbb{Z}}[x]$ is not a Hilbert ring. Look at the simple module $\mathbb{Z}_{2\mathbb{Z}}[x] / \langle 2, x \rangle$ over $\mathbb{Z}_{2\mathbb{Z}}[x]$. This module only has one submodule, namely the zero submodule. Since every submodule is maximal then every G-submodule is maximal. Therefore $\mathbb{Z}_{2\mathbb{Z}}[x] / \langle 2, x \rangle$ is a Hilbert module.

However, with some minor alterations to remove such cases we can obtain an equivalence relationship between Hilbert rings and modules. To do so we need to introduce the notion of a Jacobson module.

3.4 Jacobson Modules

In [7] Mani Shirazi and Sharif used the notion of a Jacobson rings to define a new characterization of a particular type of module.

Definition 3.4.1. A R-module M is said to be a *Jacobson module* if every prime submodule is an intersection of maximal submodules.

In [7], these modules were called Hilbert modules. We will refer to them as Jacobson modules to highlight the connection of this definition with definition 2.5.1. We ended the last chapter by showing that Hilbert rings are the same as Jacobson rings in the commutative setting. We have a similar result for modules [12].

Theorem 3.4.2. *A finitely generated R-module M is a Hilbert module if and only if it is a Jacobson module.*

Proof. Assume M is a Hilbert module. Since every G-submodule is maximal then by Corollary 3.3.7 every prime submodule N of M is equal to the intersection of all maximal submodules containing N. Thus, M is a Jacobson module.

Assume M is a Jacobson module. Let N be a G-submodule of M. Then since N is prime $N = \bigcap\limits_{i \in I} M_i$ where M_i are maximal submodules of M. So by Proposition 3.2.4 if M_i is maximal then each $\mathfrak{m}_i = (M_i :_R M)$ is a maximal ideal. Thus,

$$\mathfrak{p} \;=\; (N :_R M) \;=\; \left(\bigcap_{i \in I} M_i :_R M\right) \;=\; \bigcap_{i \in I} (M_i :_R M) \;=\; \bigcap_{i \in I} \mathfrak{m}_i.$$

But by the definition of G-submodule, $\mathfrak{p}$ is a G-ideal of R and $D = R/\mathfrak{g}$ is a G-domain. Assume for contradiction D is not a field. Then by Proposition 2.2.7, there exists a $u \in D$ such that u is contained in every nonzero prime ideal of D. Thus, the intersection of all maximal ideals in D is not the zero ideal. However $\mathfrak{p} = \bigcap\limits_{i \in I} \mathfrak{m}_i$, where all $\mathfrak{m}_i \in R$ are maximal. But the maximal ideals in D are precisely the maximal ideals of R containing $\mathfrak{g}$, so their intersection in $D = R/\mathfrak{g}$ should be 0, a contradiction. Hence, D must be a field, and $\mathfrak{p}$

is a maximal ideal. N is $\mathfrak{p}$-maximal where $\mathfrak{p}$ is a maximal ideal, so by Proposition 3.2.14 N is a maximal submodule. Since every G-submodule in M is maximal, M is a Hilbert module. $\square$

Hilbert and Jacobson modules were created independently from each other. In fact, the original statement of Theorem 3.4.2 is that M is a Hilbert module if and only if every prime submodule is an intersection of maximal submodules. It is only in this work that these two modules are recognized to be the same. This is powerful new idea, and it allows us to use theorems about Jacobson modules to prove results on Hilbert modules. We are going to focus on the following theorem from [7].

Theorem 3.4.3. *Let M be a faithful, finitely generated R-module. Then M is a Jacobson module if and only if R is a Jacobson ring.*

Remark 3.4.4. Recall, a module is said to be faithful if the annihilator of the module is the zero ideal. Namely, a R-module M is faithful if and only if $(\{0\} :_R M) = \{0\}$.

Proof. Assume that R is a Jacobson ring. Then by Theorem 2.5.4, R is a Hilbert ring. Since R is Hilbert and M is a finitely generated R-module, by Theorem 3.3.10 M is a Hilbert module. Finally, use Theorem 3.4.2 to find that M is a Jacobson module.

Now assume that the finitely generated R-module M is a Jacobson module. Let $\mathfrak{p}$ be a prime ideal of R. Then by Proposition 3.2.9, since $\operatorname{Ann}(M) = \{0\} \subseteq \mathfrak{p}$ there exists a prime submodule K of M with $\mathfrak{p} = (K :_R M)$. Since K is a prime submodule in a Jacobson module $K = \bigcap_{i \in I} M_i$, where M_i are maximal submodules. By Proposition 3.2.4, for each submodule M_i the ideal $(M_i :_R M)$ is a maximal ideal of R. We will denote this maximal ideal $\mathfrak{m}_i$.

Thus,

$$\mathfrak{p} \;=\; (K :_R M) \;=\; \left(\bigcap_{i\in I} M_i :_R M\right) \;=\; \bigcap_{i\in I}(M_i :_R M) \;=\; \bigcap_{i\in I}\mathfrak{m}_i.$$

We have shown that any arbitrary prime ideal in R is an intersection of maximal ideals, and thus R is a Jacobson ring. $\square$

Remark 3.4.5. Note that we use the fact that Hilbert and Jacobson modules are equivalent to prove this theorem. This was not how it was originally proved in [7], since they did not have our notion of a Hilbert module. We used our new proof to highlight this new connection.

The following result from [7] give a complete characterization of the relationship between Jacobson rings and modules.

Corollary 3.4.6. *Let M be a finitely generated R-module. Then M is a Jacobson module if and only if $R/\mathrm{Ann}_R(M)$ is a Jacobson ring.*

Proof. Let M be a non-zero finitely generated R-module. Then M is a Jacobson R-module if and only if M is a Jacobson $(R/\mathrm{Ann}_R(M))$-module. By Theorem 3.4.3, this is equivalent to $R/\mathrm{Ann}_R(M)$ being a Jacobson ring. $\square$

As we mentioned at the end of the discussion on Theorem 3.3.10 we wanted when M being a Hilbert Modules implies that R is a Hilbert ring. Using Theorems 3.4.2 and 3.4.3, we have our first major new result.

Theorem 3.4.7. *Let M be a faithful, finitely generated R-module. If M is a Hilbert module then R is a Hilbert ring.*

Corollary 3.4.8. *Let M be a finitely generated R-module. Then M is a Hilbert module if and only if $R/\operatorname{Ann}_R(M)$ is a Hilbert ring.*

We now have have a full classification of Hilbert modules. It firmly solidifies the notion that being Hilbert is really a property of the ring; under certain conditions this can be inherited properly to the module. One issue is that we had to use the theory of Jacobson modules in order to get this result. Since it is possible to prove this same result for Jacobson modules without mentioning Hilbert modules, the converse should be doable. That will be one of our goals for Chapter 4. Another thing we note is the process of defining G-submodules was very flexible, and we could replace other types of ideals for G-ideals and still get a meaningful definition. Instead of doing all of those one by one, we will consolidate them in Chapter 4.

Before moving on, we want to prove one more new result using this Hilbert module and ring equivalence. Recall that from Theorem 2.4.8, a Noetherian ring R is a Hilbert ring if and only if for every prime ideal $\mathfrak{p}$ such that $\dim(R/\mathfrak{p}) = 1$, there must exist infinitely many maximal ideals containing $\mathfrak{p}$. We would like to have a similar classification of Noetherian Hilbert modules. Like Theorem 3.4.3, we require our modules to be faithful.

Theorem 3.4.9. *Let M be a Noetherian, faithful, finitely generated R-module. Then M is a Hilbert module if and only if for every prime ideal $\mathfrak{p}$ in R such that $\dim(R/\mathfrak{p}) = 1$, there must exist infinitely many maximal ideals containing $\mathfrak{p}$.*

Proof. Let M be a Noetherian, faithful, finitely generated R-module. Since M is a faithful, Noetherian R-module, by Lemma 3.1.5 the ring R is Noetherian.

Assume that M is a Hilbert module. By Corollary 3.4.8, M is a Hilbert Module if and only if R is a Hilbert ring. By Theorem 2.4.8, R is Hilbert ring if and only if for every prime ideal $\mathfrak{p}$ such that $\dim(R/\mathfrak{p}) = 1$, there must exist infinitely many maximal ideals containing $\mathfrak{p}$. □

This theorem gives a classification for Noetherian Hilbert modules based only on the maximal ideals in the ring R. This powerful result sadly only holds true for faithful modules since otherwise there is no way to guarantee that ring R will be Noetherian.

This concludes our study of Jacobson and Hilbert modules together. There is still more results that can be obtained by studying their interdependence but we will be moving on to generalizing and directly proving the theorems on Hilbert modules. Then we will introduce a new module to justify this work which we will name strongly Hilbert modules.

Chapter 4

Strongly Hilbert Modules

4.1 Generalized Hilbert Modules

We want to generalize the notion of a G-submodule and Hilbert modules. Since most of the structure of these Hilbert modules were inherited from ring theory, if we can define this process more generally it would allow us to create new types of modules, and then state theorems about them en masse. To do this we will first need to generalize the notion of a G-ideal and Hilbert ring.

Definition 4.1.1. Let R be a commutative ring with identity. Given a collection C of prime ideals in R, we define an ideal to be a C-*ideal* if it is contained in that collection C.

We define a ring R to be a C-*Hilbert* if every C-ideal is a maximal ideal.

Example 4.1.2. If we choose our collection of ideals C to be G-ideals, then the C-Hilbert ring is a Hilbert ring.

Example 4.1.3. If we choose our collection of ideals C to be G-type ideals, then the C-Hilbert ring is a strongly Hilbert ring.

Example 4.1.4. If we choose our collection of ideals to be all prime ideals, then a C-Hilbert ring is a ring of dimension zero.

Thus, the notion of a C-Hilbert rings allows us to talk about Hilbert and strongly Hilbert rings at the same time. As seen in the example of a ring with dimension zero, it also allows us to prove results on other structures in addition to those two known structures. Moving on, we can define C-Hilbert modules in the same way Rush defined Hilbert modules in [12].

Definition 4.1.5. Suppose M is a finitely generated R-module. We define a submodule N of M to be a *C-submodule* if $(N :_R M) = \mathfrak{p}$ is a C-ideal and N is $\mathfrak{p}$-maximal.

We define a finitely generated R-module M to be a *C-Hilbert module* if every C-submodule of M is a maximal submodule.

We now get to one of our two major results relating C-Hilbert rings and modules. It is a version of Theorem 3.3.10, with C-Hilbert modules instead of Hilbert modules. Note that proof itself is derived from [12] , where we essentially replace G-ideals and G-submodules with our newly defined C-ideals and C-submodules.

Theorem 4.1.6. *If R is a C-Hilbert ring, then every finitely generated R-module M is C-Hilbert module.*

Proof. Suppose R is a C-Hilbert ring, and N is a C-submodule of the finitely generated R-module M. We need to prove that $\boldsymbol{N}$ is maximal. By definition, $\mathfrak{p} = (N :_R M)$ is a C-ideal of R, and N is $\mathfrak{p}$-maximal. Since R is an C-Hilbert ring, $\mathfrak{p}$ is a maximal ideal of

R. N is a $\mathfrak{p}$-maximal submodule and $\mathfrak{p}$ is a maximal ideal, so by Proposition 3.2.14 N is a maximal submodule of M. Therefore, every C-submodule of M is maximal and M is a C-Hilbert module. $\square$

With this new Theorem, Theorem 3.3.10 can be considered a corollary of this more general result. Our second major result relating C-Hilbert rings and modules is a new version of Theorem 3.4.7. The proof is new and is independent of Jacobson modules.

Theorem 4.1.7. *Suppose M is a faithful, finitely generated R-module. If M is a C-Hilbert module, then R is a C-Hilbert ring.*

Proof. Let $\mathfrak{p}$ be a C-ideal of R. Then since $\mathfrak{p}$ is a prime ideal and $\mathrm{Ann}(M) = \{0\} \subseteq \mathfrak{p}$ by Theorem 3.2.9 there exists a prime submodule N of M that is maximal with respect to having $(N :_R M) = \mathfrak{p}$. So N is a $\mathfrak{p}$-maximal submodule where $\mathfrak{p}$ is C-ideal. Thus, N is a C-submodule in a C-Hilbert module, and therefore maximal. But if N is a maximal submodule, then by Proposition 3.2.4 $(N :_R M) = \mathfrak{p}$ must be a maximal ideal of R. Since every C-ideal is maximal, by definition R is a C-Hilbert ring. $\square$

Note that Theorem 3.4.7 becomes a corollary of this more general result. Combining these two theorems, we also derive the following corollary.

Corollary 4.1.8. *Suppose M is a faithful, finitely generated R-module. Then M is a C-Hilbert module if and only if R is a C-Hilbert ring.*

Unfortunately, these do not extend to the case when $\mathrm{Ann}(M) \neq \mathbf{0}$. Being a C-ideal, and thus a C-Hilbert ring, is not necessarily retained under homomorphic images. This is due to the way we defined C-ideals based of an arbitrary collection of prime ideals in R.

Thus, we can not immediately find an additional relationship between the C-ideals of R and $R/\mathrm{Ann}_R(M)$.

Next, we will use strongly Hilbert rings to make a new example of C-Hilbert modules.

4.2 Strongly Hilbert Modules

Using the tools developed in the construction of Hilbert modules, we can create a new object: *Strongly* Hilbert modules. We start with the definition of a G-type submodule.

Definition 4.2.1. A submodule N of a finitely generated R-module M is a *G-type submodule* if $\mathfrak{p} = (N :_R M)$ is a G-type ideal of R and N is $\mathfrak{p}$-maximal.

Remark 4.2.2. Any G-submodule is a G-type submodule, since any G-ideal is a G-type ideal.

Remark 4.2.3. A G-type submodule is an example of an C-submodule, with G-type ideals being our collection C.

It is helpful to consider an example of a G-type submodule that is not a G-submodule.

Example 4.2.4. Let $M = \mathbb{Z}[x] \oplus \mathbb{Z}[x]$ be a $\mathbb{Z}[x]$-module, and note that M is finitely generated by $\{(1,0),(0,1)\}$. Let

$$N \;=\; x\mathbb{Z}[x] \oplus \mathbb{Z}[x] \;=\; \{(x \cdot f(x), g(x)) \,|\, f(x), g(x) \in \mathbb{Z}[x]\}\,,$$

which is a submodule of M. $(N :_{\mathbb{Z}[x]} M) = \langle x \rangle$ is a G-type ideal, as $\mathbb{Z}[x]/\langle x \rangle \cong \mathbb{Z}$ is a G-type domain. N is $\langle x \rangle$-maximal, since the only proper, prime submodules of M that contain N are $(p,x)\,\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ where p is a prime integer. However, $\big((p,x)\mathbb{Z}[x] \oplus \mathbb{Z}[x] :_{\mathbb{Z}[x]} M\big) = (p,x)\,\mathbb{Z}[x] \oplus \mathbb{Z}[x]$. So N is $\mathfrak{p}$-maximal for a G-type ideal and thus is a G-type submodule.

After defining G-type submodules, we can prove the following corollary of Proposition 2.7.6.

Corollary 4.2.5. *Let a ring R have the descending chain condition on prime ideals, and let M be a finitely generated R-module. Additionally, suppose $R/\mathfrak{p}$ has only a countable number of non-zero minimal prime ideals for each prime ideal $\mathfrak{p}$ in R. If a submodule N of M is strongly prime, then N is a G-type submodule.*

Proof. Let N be a strongly prime submodule and $\mathfrak{p} = (N :_R M)$. Then by Theorem 3.2.13, N is $\mathfrak{p}$-maximal. But by Theorem 2.7.6, every prime ideal of R is a G-type ideal. Thus, N is a G-type submodule. $\square$

Now we can finally define strongly Hilbert modules.

Definition 4.2.6. A finitely generated R-module M is said to be a *strongly Hilbert module* if each G-type submodule in M is maximal submodule.

Remark 4.2.7. Any strongly Hilbert module is a Hilbert module since if every G-type submodule is maximal then so is every G-submodule.

Remark 4.2.8. A strongly Hilbert module is a example of a C-Hilbert module, with G-type ideals being our collection C.

Theorem 4.2.9. *If R is a strongly Hilbert ring, then each finitely generated R-module M is a strongly Hilbert module.*

Proof. This is an immediate result of Theorem 4.1.6, since strongly Hilbert is an example of C-Hilbert where we choose our collection of prime ideals to be the set of G-type ideals. $\square$

Example 4.2.10. Since $\mathbb{C}[x]$ is a strongly Hilbert ring, any finitely generated $\mathbb{C}[x]$-module will be a strongly Hilbert module. For example, $\mathbb{C}[x] \oplus \mathbb{C}[x]$ over the ring $\mathbb{C}[x]$ is a strongly Hilbert module.

Recall any ring R of dimension zero is a strongly Hilbert ring, since the ring would have no prime ideal that is not maximal. By Theorem 4.2.9, if $\dim R = 0$ any finitely generated R-module is a strongly Hilbert module. We can expand this notion by looking at a module with only maximal submodules.

Example 4.2.11. Any simple module M is a strongly Hilbert module, since $\{0\}$ is the only maximal submodule (and any maximal submodule is a G-type submodule).

We can actually take this idea a little further, by looking at modules where every strongly prime submodule is maximal.

Corollary 4.2.12. *Let M be a finitely generated R-module such that every strongly prime submodule is a maximal submodule. Then M is a strongly Hilbert module.*

Proof. Let M be a finitely generated R-module such that every strongly prime submodule is maximal. Let N be a G-type submodule of M. Then $\mathfrak{p} = (N :_R M)$ is a G-type ideal and N is $\mathfrak{p}$-maximal. Then by 3.2.13 since N is $\mathfrak{p}$-maximal it is a strongly prime submodule. But every strongly prime submodule of M is a maximal submodule. Thus N is a maximal submodule. Since every G-type submodule in M is a maximal then M is a strongly Hilbert module. □

This Corollary gives us the following example of a strongly Hilbert module.

Example 4.2.13. Let $M = \mathbb{Z}/4\mathbb{Z}$ be a $\mathbb{Z}$-module. We claim M is a strongly Hilbert module. M has only two proper submodules: $0M$ and $2M$.

1. $0M$ is not a G-type submodule, since $(0M :_{\mathbb{Z}} M) = 4\mathbb{Z}$ is not a prime ideal, and thus cannot be a G-type ideal.

2. $2M$ is a G-type submodule, since $(2M :_{\mathbb{Z}} M) = 2\mathbb{Z}$ is a maximal ideal, thus G-type. Also, $2M$ is $2\mathbb{Z}$-maximal, since it is a maximal submodule.

Thus, all G-type submodules of M are maximal submodules, so M is a strongly Hilbert module.

However, note that $\mathbb{Z}$ is not a strongly Hilbert ring. Thus, the converse of Theorem 4.2.9 is not true in general. If we restrict ourselves to faithful modules, the converse does hold.

Theorem 4.2.14. *Let M be a faithful, finitely generated R-module. If M is a strongly Hilbert module, then R is a strongly Hilbert ring.*

Proof. This is an immediate result of Theorem 4.1.7, since strongly Hilbert is an example of C-Hilbert, where we chose our collection C of prime ideals to be the set of G-type ideals. $\square$

Corollary 4.2.15. *Let M be a faithful, finitely generated R-module. Then M is a strongly Hilbert module if and only if R is a strongly Hilbert ring.*

Corollary 4.2.16. *Let M be a finitely generated R-module. Then M is a strongly Hilbert module if and only if $R/\mathrm{Ann}_R(M)$ is a strongly Hilbert module.*

Proof. Let M be a non-zero finitely generated R-module. Then M is a strongly Hilbert R-module if and only if M is a strongly Hilbert $(R/\mathrm{Ann}_R(M))$-module. By Corollary 4.2.15, this is equivalent to $R/\mathrm{Ann}_R(M)$ being a strongly Hilbert ring. $\square$

We close with our final equivalent condition in the Noetherian case, this time for strongly Hilbert modules.

Theorem 4.2.17. *Let M be a Noetherian, faithful, finitely generated R-module. Then M is a strongly Hilbert module if and only if for each prime ideal $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) \geq 1$, there exists an uncountable number of non-zero minimal prime ideals in $R/\mathfrak{p}$.*

Proof. Let M be a Noetherian, faithful, finitely generated R-module. Since M is a faithful, Noetherian R-module, the ring R is Noetherian.

Assume that M is a strongly Hilbert module. Since M is faithful, by Corollary 4.2.15 M is a strongly Hilbert Module if and only if R is a Hilbert ring. By Theorem 2.8.5, R is Hilbert ring if and only if for each prime ideal $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) \geq 1$ there exists an uncountable number of non-zero minimal prime ideals in $R/\mathfrak{p}$. □

Chapter 5

Future Thoughts

5.1 Improve C-Hilbert Rings and Modules

C-Hilbert rings are a useful abstraction of existing generalizations of Hilbert rings, but it still has room to be refined. One unfortunate issue is that the current definition, tied to the property of a ring or module being C-Hilbert, is not necessarily retained under homomorphic images. However, it is the case that being Hilbert or strongly Hilbert is retained under homomorphisms, so this is a meaningful weakness. Further refinement of how we defined our collection C could rectify this issue. Also, we should find a better name for these objects than C-Hilbert.

5.2 More Examples of C-Hilbert Rings and Modules

There are generalizations of Hilbert rings other than being strongly Hilbert that have been published over the roughly 50 years since Hilbert rings were defined. Through the lens of our C-Hilbert notion, we can revisit and describe these objects. Then, we can create new modules from those objects.

5.3 Infinitely Generated Hilbert Modules

An annoying problem is thus far is that we have been restrained to working with only finitely generated modules. We needed to restrict ourselves to ensure that the we can always find a submodule is $\mathfrak{p}$-maximal for a given prime ideal $\mathfrak{p}$. In the future, we hope to expand the notion of Hilbert modules to infinitely generated modules, as long we can always find a submodule that is maximal maximal with respect to given prime ideal. Naively, infinitely generated Noetherian modules seem like a good candidate.

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