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A SIMPLE CLASS OF BEAM TRANSPORT SYSTEMS
WITH OPTICALLY AXISYMMETRIC TRANSFER PROPERTIES*

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Abstract

It is demonstrated that for quadrupole-beam transport systems that have a specific symmetry, two of the three conditions that have to be satisfied in optically axisymmetric systems are automatically fulfilled. These systems therefore need to have only one free parameter to make them optically axisymmetric, and the resulting design formulas are presented and discussed. As a generalization, it is shown how dipole magnets, solenoids, or electrostatic elements can be included in the optical train. A system with lower symmetry, requiring adjustment of two parameters to achieve optical axisymmetry, is also described.

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1. Introduction

It happens occasionally that one has to design a beam transport system that has axisymmetric transfer properties. We consider, at least for most of this paper, arrays of magnetic quadrupoles with their longitudinal axes on the axis of a Cartesian coordinate system, and aligned so that their fields are perpendicular to the x-z and y-z planes. The requirement of axisymmetric beam transfer through the system then means that the transfer matrix M , establishing the relationship between the vectors $U_x = \begin{pmatrix} x \\ dx/dz \end{pmatrix}$ at both ends of the system, has to be identical to the matrix N for the y direction. Since the determinants of M and N must automatically equal one, the condition $M = N$ requires that three conditions are met. Unless extraordinary procedures are used to satisfy these conditions, both the design of such a system as well as its adjustment after it has been built will be quite difficult. In section 2, we show that for systems with a specific symmetry, two of those conditions are automatically satisfied, so that one has to adjust only one parameter to make the system optically axisymmetric. While some systems that have these symmetry properties may have been used in the past, it seems that the general "recipe" formulated here has not been stated before. In section 3, some specific formulas for these symmetric systems are given and discussed, in section 5 a generalization is described, and in section 6 a system is briefly mentioned that has a different symmetry and requires adjustment of two parameters to make it optically axisymmetric.

Throughout this paper, matrices are represented by capital letters, and matrix elements by the same subscripted lower case letters.

2. Description of a system with a specific symmetry, and its optical consequences.

Fig. 1 depicts schematically the symmetrical system we want to discuss. The system connecting point 1 to point 2 is arbitrary within the bounds set forth in section 1. The system connecting point 2 to point 3 is obtained by first forming the mirror image of system 1-2 relative to the plane perpendicular to the axis and intersecting the axis at point 2, and then rotating that whole system about the optical axis by 90° (the direction of rotation is immaterial). Instead of rotating system 2-3, one can also reverse the polarity of excitation of every element in system 2-3.

This means that when going from point 1 to point 3 in the y-z plane, one encounters the same elements as when going from point 1 to point 3 in the x-z plane, but their order is reversed. While it is well known (1) how a reversal of the order of elements affects a beam transport matrix, we want to derive the relevant relationships explicitly because of the somewhat unusual circumstances of our specific problem. To do so, we write down the first order trajectory equations

$$x''(z) + K_x(z) \cdot x(z) = 0, \quad (1a)$$

$$y''(z) + K_y(z) \cdot y(z) = 0. \quad (1b)$$

If we normalize z at point 1 to be zero ($z_1 = 0$), then $K_x(z_3 - z) = K_x(z)$ and $K_y(z_3 - z) = K_y(z)$ before system 2-3 has been rotated. After the rotation has been performed, i.e., for the system under consideration,

$$K_y(z) = K_x(z_3 - z) \quad (2)$$

holds.

If we represent the solutions of interest by

$$U_x(z_3) = MU_x(z_1) \quad (3a)$$

$$U_y(z_3) = NU_y(z_1), \quad (3b)$$

and if we use equ. (2) in equ. (1b), and relate that to solution (3a) of equ. (1a), we get

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U_y(z_1) = M \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U_y(z_3).$$

Comparing this to equ. (3b) yields

$$N = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)$$

Since the determinant of M equals one, the right side of this equation equals $\bar{M}$, but with its diagonal elements exchanged. Indicating that operation by a bar over the symbol representing the matrix, we obtain the result that for systems that possess the above defined symmetry,

$$N = \bar{M}. \quad (5)$$

This equation expresses the fact that 1.) independent (within the bounds stated in section 1) of the properties of the system connecting point 1 to point 2, the off-diagonal elements of N are the same as those of M; and that, as a consequence, 2.) to make the transfer properties of the system connecting points 1 and 3 axisymmetric (i.e., to make $N = M$), it is necessary and sufficient to satisfy

$$m_{11} = m_{22}, \quad (6)$$

since the matrix condition for axisymmetry,

$$M = \overline{M}, \quad (7)$$

is then satisfied.

It follows from the considerations that lead to equ. (5) that the matrix M is obtained from the matrices A and B , describing the beam transfer properties from point 1 to point 2 in the x - z and y - z plane, through

$$M = \overline{BA}, \quad (8)$$

and equs. (6) and (7) are satisfied when

$$\overline{BA} = \overline{AB}. \quad (9)$$

It is interesting to discuss also the phase space that can be transmitted by systems that have the symmetry properties described at the beginning of this section. We assume that the apertures in the system connecting point 1 to point 2 are symmetric to both the x - z and y - z plane. We also assume that the apertures in the system connecting point 2 to point 3 are, as the optical elements, mirror images of the apertures in the 1-2 system that are rotated about the optical axis by 90° .

By "looking from point 2" at the possible particle trajectories coming from point 1 and going to point 3, it is easy to see that the stated symmetry conditions have the consequence that the x - x' diagram of transmittable phase space is the same as that diagram in y - y' space. Transforming these diagrams to point 1 or 3, it then follows that the phase space diagrams there have the same areas in the x - x' and y - y' planes, but that they will, in general, be of different shape. It should be noted that these statements depend only on the general system symmetry, and are true whether or not equ. (7) is satisfied.

3. Explicit system design formulae

It is useful to express equ. (7) in terms of the elements of the matrices A and B connecting point 1 to point 2 in the x and y direction, and then to express the elements of the total system matrix, M, by the elements of A and B also. We do this in two different ways:

1.) In literally the way just described, implying that the matching is accomplished by modifying the properties of A, B by making internal adjustments, using parameters such as excitation of some or all of the elements, spacing between elements, etc.

2.) In some cases, it may be convenient not to change the relationship between the elements of the arrays on either side of the symmetry point, but to use the spacing d between the two arrays as the parameter to satisfy equ. (7) (see fig. 2). As in 1, the matrices A, B describe the transfer from point 1 to point 2; only in this case, point 2 is not the symmetry point, but separated from it by the distance d/2. Although some of it is a little tedious, the derivation of these formulas is straightforward and need not be reproduced here.

The expressions become both transparent and compact if one introduces the operators E and O, defined by the following equations:

$$E(g) = (g + g_-)/2 \quad (10a)$$

$$O(g) = (g - g_-)/2 \quad (10b)$$

The argument g that the operators act on are functions of the elements a_{ij} of A and/or b_{kl} of B, and the function g_- is the same function, but with the symbols a and b exchanged. To use equ. (13) as an example, the

right side of that equation, written explicitly, reads $a_{22}b_{12} + b_{22}a_{12}$.

3.1 Matching formulas and M for a system according to fig. 1

The matching condition, equ. (7), is satisfied when

$$0(a_{22}b_{11} + a_{12}b_{21}) = 0, \quad (11)$$

and the elements of M are then given by

$$m_{11} = m_{22} = E(a_{22}b_{11} + a_{12}b_{21}) \quad (12)$$

$$m_{12} = 2 \cdot E(a_{22}b_{12}) \quad (13)$$

$$m_{21} = 2 \cdot E(a_{21}b_{11}) \quad (14)$$

3.2 Matching formulas and M of a system according to fig. 2

The matching condition, equ. (7), is satisfied when

$$d = \frac{0(a_{22}b_{11} + a_{12}b_{21})}{0(a_{21}b_{22})}, \quad (15)$$

and the elements of M are then given by

$$m_{11} = m_{22} = \frac{0(a_{22}a_{21})}{0(a_{21}b_{22})} \quad (16)$$

$$m_{12} = \frac{0(a_{22}^2)}{0(a_{21}b_{22})} \quad (17)$$

$$m_{21} = \frac{0(a_{21}^2)}{0(a_{21}b_{22})}. \quad (18)$$

3.3 Comments on the mathematical structure of equs. (11)-(18)

For a preliminary study of an optical system of the kind discussed here, it is convenient, and usually quite adequate, to represent an

optical system by a product of thin lens matrices $\begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$ and pure drift matrices $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$. Then the matrix B is obtained from A by reversing in A the signs of all k. To obtain the explicit expressions for the E- and O-functions appearing in equs. (11)-(18), it is not practical to write down explicitly the complete argument, since g_- is obtained from g by reversing the signs of all k's in g. Therefore an E-function in equs. (11)-(18) consists of only those terms of the argument that are products of an even (including zero) number of k's, and O-functions similarly contain only terms that consist of products of an odd number of k's. These properties of the E- and O-functions make them considerably more simple than their arguments and therefore permit ready understanding of their meaning.

4. Discussion of some system properties

Since the polarity of the elements on one side of the symmetry plane has to be opposite to the polarity of the elements on the other side, the symmetry plane cannot be straddled by a quadrupole. Consequently, the total system always has an even number of elements. While there exists a mathematical solution for a system consisting of only two quadrupoles, their lengths have to be so large that, not surprisingly, such a system is not practical. However, a system with a doublet on either side of the symmetry plane is quite practical, provided the quadrupoles close to the symmetry plane are 15 to 30% stronger than the outer elements. A more detailed discussion of this most primitive system, or a pair of triplets, is not difficult with the use of the formulas given above, but is outside the scope of this paper.

The matching condition, equ. (11), contains obviously the momentum of the particle. It is easy to show that for a pair of thin lens

doublers, it is not possible to satisfy equ. (11), as well as making the first derivative of that equation, with respect to the momentum, vanish. Whether this achromaticity can be achieved for more complicated systems has not yet been determined.

While the ability to make a system optically axisymmetric by adjusting only one parameter can be extremely advantageous, it is important to realize also that the condition $m_{11} = m_{22}$ represents a limitation of the system that can make the accomplishment of some tasks with only one such system impossible. To illustrate this point, we assume that we have a touching pair of identical, but oppositely excited, solenoids that are sufficiently weak so that their optical effect can be described by a thin axisymmetric lens. We want to compensate this thin lens effect in such a way that the transfer matrix describing the effect of the solenoids and the compensating system equals a pure drift matrix with a drift length equaling the physical length of the total system. It is easy to show that one system of the kind described here cannot perform this function if it is required that the compensating system has elements only on one side of the solenoid system.

5. Generalizations

We can allow dipole and combined dipole-quadrupole magnets as elements of the beam transport systems discussed in this paper if we amend and modify sections 1 and 2 in the following way.

When designing system 1-2, and the first element has been put into its desired location, the following elements have to be emplaced according to the following rules: 1.) The desired central trajectory of the new element has to join without discontinuity in location or direction the

central trajectory emerging from the previous element. 2.) The elements must have one (dipole) or two (quadrupole) symmetry planes that do not contain any tangential field components, and such a symmetry plane must be either parallel or perpendicular to that kind of symmetry plane of the preceding element.

To form the system connecting point 2 to point 3, we proceed analogously to the prescription given in section 2: we first form a mirror image of system 1-2 relative to a plane that is perpendicular to the central trajectory emerging from the last element of system 1-2, and then rotate the preliminary system 2-3 like a rigid body by 90° about the central trajectory emerging from the last element of system 1-2. The alternate prescription given in section 2 (reversal of excitation of all elements in system 2-3) obviously cannot be applied here.

If equs. (1a) and (1b) are now interpreted as differential equations describing the deviation of trajectories from the central trajectory, equ. (2) is obviously again valid, and all subsequent conclusions and equations in sections 2-4 can be applied to the more general systems discussed in this section. However, since the systems discussed now involve bending, we also have dispersion that does not occur in pure quadrupole systems. To make matters worse, the bending takes place in at least two planes, so that systems involving dipoles are not likely to be used frequently.

To be complete, it is worth stating that the electrostatic equivalents of the elements discussed so far can also be used without any need to modify the theory. In addition, solenoids can be used if the rotation of the x-z and y-z reference planes about the solenoid axis is

properly taken into account. A solenoid is also the exception to the rule stated in section 4 that the symmetry point 2 cannot be straddled by an element.

6. A system with different symmetry, requiring two adjustments to achieve optical axisymmetry

Because we found that the symmetry properties of the systems discussed so far may limit their usefulness under some circumstances, we want to describe a system with a different symmetry that does not produce a transfer matrix with identical diagonal elements. As a drawback, the system described now requires the adjustment of two parameters to achieve optical axisymmetry.

We start again with a system connecting points 1 and 2. We then duplicate that system and connect its entrance end (corresponding to point 1) to the exit end of system 1-2, with the reference trajectories and the symmetry planes appropriately aligned. As the last step, we again rotate system 2-3 by 90° about the central orbit as it exists at point 2. Using the same notation used above, optical axisymmetry of the transfer properties from point 1 to point 3 is then achieved when

$$BA = AB \quad (19)$$

is satisfied, and this is the case when

$$O(a_{12}b_{21}) = 0, \quad (20)$$

$$O(a_{11}b_{12} + a_{12}b_{22}) = 0. \quad (21)$$

The conditions expressed by equ. (19) for the diagonal elements of AB and BA are simultaneously satisfied when equ. (20) is fulfilled, requiring the satisfaction of only one more condition. We have chosen here to express that explicitly for the 12-element of equ. (19), giving equ. (21). We don't discuss systems of this symmetry type in more detail because these systems are not likely to be of nearly the same interest as those discussed in sections 1-5.

7. Acknowledgements

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References

1. See for instance S. Penner, Rev. Sc. Instr. 32 150 (1961).

Figure Captions

Fig. 1 Schematic representation of symmetric system.

Fig. 2 Schematic representation of system with adjustable spacing between its two parts.

0 0 0 0 4 5 0 4 4 1 4

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XBL 764-1270

Fig. 1

XBL 764-1271

Fig. 2

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