Title
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Publication Date
2012-02-01

University of California, Berkeley
Department of Agricultural &
Resource Economics

CUDARE Working Papers

Year 2012 *Paper 1121*

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What's the Rate?
Disentangling the Weitzman and the Gollier Effect*

February 2012

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Abstract: The uncertainty of future economic development affects the term structure of discount rates and, thus, the intertemporal weights that are to be used in cost benefit analysis. The U.K. and France have recently adopted a falling term structure to incorporate uncertainty and the U.S. is considering a similar step. A series of publications discusses the following concern: A seemingly analogous argument used to justify falling discount rates can also be used to justify increasing discount rates. We show that increasing and decreasing discount rates mean different things, can coexist, are created by different channels through which risk affects evaluation, and have the same qualitative effect of making long-term payoffs more attractive.

JEL Codes: D61, D81, H43, Q54

Keywords: benefit cost analysis, discounting, term structure, uncertainty, Weitzman-Gollier puzzle

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*This paper would not exist if it was not for the interesting and insightful discussions on the puzzle with Larry Karp and Geir Asheim. I owe thanks to Christian Gollier for his comments on an early draft of the paper.

1 Introduction

Future economic development is uncertain. This uncertainty affects the discount rate and, thus, the intertemporal weights in cost benefit analysis. Climate change economics has been a prominent example of the dramatic effects that the discount rate has on optimal policy (Nordhaus 2007). A reasoning based on Weitzman (1998, 2001) has led the U.K. and France to adopt falling discount rates for long-term project evaluation, and the U.S. Environmental Protection Agency is currently preparing a similar proposal to the Office of Management and Budget. Such falling discount rates under uncertainty significantly increase the attention given to future benefits of long-term investments, including climate change mitigation and other public goods. A series of publications discussed the following concern: A seemingly analogous argument to the one that Weitzman used to argue for falling rates can be used to justify increasing discount rates under uncertainty (Gollier 2004). This finding is known as the Weitzman-Gollier puzzle.

Gollier & Weitzman (2010) recently proposed a resolution that won the prestigious Erik Kempe award of the European Association of Environmental and Resource Economists. The paper shows that an appropriate modification of the probabilities governing the uncertain productivity in the economy can yield formulas akin to both discount schemes: using present marginal utility to risk-adjust probabilities implies a rate similar to Weitzman's (1998, 2001) original formula, while using future marginal utility to risk-adjust probabilities implies a formula similar to Gollier (2004). However, a time-dependent probability adjustment can support almost any discounting formula. While Gollier & Weitzman (2010) offer interesting insights on the important role of marginal utilities, we find that the suggested solution to the reformulated puzzle does not fully embrace the core of the baffling nature of the original paradox. We show that both, increasing and decreasing discount rates, have a valid foundation. We do not attempt to resolve the puzzle by using time dependent adjustments in probability to convert a falling rate into one that appears to rise. Instead, we disentangle two channels through which uncertainty affects the value of a project payoff. One channel gives rise to a falling term structure (Weitzman effect), the other to an increasing term structure (Gollier effect).

Since Gollier (2004) demonstrated the puzzle, a series of publications have contributed to the issue. Summarizing the highlights in the discussion, Hepburn & Groom (2007) interpret the puzzle as a dependence of project value on evaluation time. They show that while discount rates decrease 'in the passage of time' they increase in the evaluation date. The authors point out that their finding leads to time inconsistent planning. Gollier (2010) shows that the dependence on evaluation time disappears in a more complete model where agents optimize consumption intertemporally. Gollier (2009) and Gollier & Weitzman (2010) find that Weitzman's formula is correct for logarithmic utility, and Freeman (2010) presents a model where Weitzman's formula is always correct.

In contrast to earlier papers, we carefully distinguish two effects. First, uncertain productivity affects the growth of baseline consumption in the economy. Second, uncertain productivity has an immediate effect on the payoff from an uncertain investment project. We show that it is the implied uncertainty over baseline consumption that makes Weitzman's formula of a decreasing certainty equivalent discount rate correct. In contrast, it is uncertainty with respect to the project's payoff that implies Gollier's increasing discount rate. We show that for general project evaluation both effects coexist. The Gollier effect vanishes only in the case of a project that is perfectly correlated with the uncertain market interest. Moreover, we show that both Weitzman's increasing and Gollier's decreasing rate make a project's long-term payoffs relatively more valuable, compared to a certain world with constant discount rates.

Section 2 introduces the setting of the paper and defines a project's annual surplus rate. Section 3 analyzes how to discount project payoffs under uncertainty about consumption growth and under uncertainty about project payoff. Section 4 analyzes the approaches of Weitzman and Gollier when uncertainty is generated by the market rate of interest. Section 5 concludes.

2 The Setting

We seek a general model that is able to disentangle the two different contributions of uncertainty that give rise to the Weitzman and the Gollier effect. For this purpose, we analyze a slightly more sophisticated intertemporal trade-off than the one usually employed in the literature on social discounting.

2.1 The Trade-Off

Our agent is given the opportunity to either invest a marginal fraction Δc^{inv} of an investment good or to consume a marginal fraction Δc^{cons} of a consumption good. The numbers Δc^{inv} and Δc^{cons} can differ. If he chooses consumption he will receive an immediate flow of utility. If he decides to invest he will receive the consumable payoff $\Delta c^{inv} \exp[rt]$ in period t. The agent's welfare is characterized by the usual discounted expected utility model with the constant rate of pure time preference δ. We employ discrete time only to avoid the discussion of utility flows of measure zero. The agent is indifferent between immediate consumption and investment if

$$u\big(c_0 + \Delta c^{cons}\big) - u(c_0) = \mathbf{E}\Big[u\big(c_t + \Delta c^{inv} \exp[rt]\big) - u\big(c_t\big)\Big] \exp(-\delta t). \quad (1)$$

The left hand side of equation (1) equals the utility gain from consuming Δc^{cons} in the present and the right hand side equals the utility gain from investing Δc^{inv} in the present and consuming $\Delta c^{inv} \exp[rt]$ in period t. In general, future baseline consumption c_t as well as the rate of return of the project r can be uncertain. We assume that future baseline consumption is governed by the potentially uncertain growth rate g: $c_t = c_0 \exp[gt]$.

We obtain the ratio of consumption units Δc^{cons} and investment units Δc^{inv} that leave the agent indifferent between either consuming or investing by analyzing the marginal trade-off

$$u'(c_0)\ \Delta c^{cons} = \mathbf{E}u'(c_0 \exp[gt]) \exp[rt] \exp(-\delta t)\ \Delta c^{inv}\ , \tag{2}$$

which is the total differential of equation (1), where c_t is expressed in terms of consumption growth. Equation (2) defines a relation between the variables Δc^{cons}, Δc^{inv}, g, and r. Initial consumption c_0 and pure time preference δ will always be exogenous. Equation (2) gives rise to different economically interesting relations that show how evaluation evolves in response to the payoff time t of the investment project. In particular, the next subsection derives a measure for the surplus of an investment project from the ratio ${}^{\Delta c^{cons}}/_{\Delta c^{inv}}$ satisfying equation (2).

We assume that capital and consumption are perfect substitutes so that we can compare Δc^{cons} and Δc^{inv} in the same measurement units of a single capital-consumption good. The assumption is crucial only in the case of the following equilibrium reasoning. If we analyze the payoff of a private project that can be scaled to an arbitrary level, then we expect that the last invested unit yields the same utility payoff as the marginal consumption unit. By construction of the trade-off that is modeled in equation (2), this equilibrium condition translates into $\Delta c^{cons} = \Delta c^{inv}$. Then, equation (2) states a relation between the growth rate g and the productivity r of such an equilibrium project. More precisely, equation (2) gives us this relation for a particular payoff time t. In general, the rates g and r satisfying this condition will depend on the project's payoff time t. Then, we use a time subscript, where g_t and r_t denote the *average* yearly rates over the time horizon t.

The condition $\Delta c^{cons} = \Delta c^{inv}$ is a frequent point of departure for calculating the social discount rate and, in the case of certainty, implies the Ramsey (1928) equation. In a market environment, where the project payoffs are tradable shares, an efficient investment has to satisfy $\Delta c^{cons} = \Delta c^{inv}$. In such a market environment we can interpret the payoff time of the project as the time to maturity of such a security. Keeping with the settings of Weitzman, Gollier, and the literature on the puzzle, we refer to r, the productivity of such an efficient equilibrium project, as the market interest rate.[1] It proves fruitful, however, to treat the general case where a marginal investment unit is not required to have the same welfare payoff as marginal consumption. We will only revert to the assumption $\Delta c^{inv} = \Delta c^{cons}$ in section 4 after we have analyzed the more general setting. The next subsection discusses a richer relation deriving from equation (2) that holds more generally and provides more information on actual project value.

2.2 Project Surplus and Market Failure

We employ equation (2) to introduce a measure for the surplus of the investment project over immediate consumption, as a function of the payoff time t. For the moment, we take baseline growth g and the rate of return on investment r as given random variables. We define the average yearly surplus rate of investing rather than consuming

$$\gamma_t = \frac{1}{t} \ln \frac{\Delta c^{cons}}{\Delta c^{inv}} ,$$

where the ratio ${}^{\Delta c^{cons}}/_{\Delta c^{inv}}$ is defined by equation (2). There are two different interpretations to the ratio ${}^{\Delta c^{cons}}/_{\Delta c^{inv}}$. If we think about consumption and investment for a moment as imperfect substitutes, then the ratio equals the marginal value of a unit of the investment good in units of the consumption good. If the ratio is larger than unity, a unit of the investment good is more valuable than a unit of the consumption good. Returning to the case of perfect substitutes, the ratio ${}^{\Delta c^{cons}}/_{\Delta c^{inv}}$ measures the surplus of the investment project in current consumption units. The rate γ_t expresses this surplus as a yearly average surplus rate.

The rate γ_t serves several purposes. First, we use it as a convenient tool for calculating equilibrium rates and certainty equivalent discount rates in

[1] In such an equilibrium, the project will generally be part of the market portfolio. Here, we follow the literature on the Weitzman-Gollier portfolio making it simply *the* market portfolio. The simplified setting suffices to make our point.

different settings. In particular, note that we obtain the condition $\Delta c^{cons} = \Delta c^{inv}$ on an investment paying at the market rate by setting $\gamma_t = 0$. Second, it helps us to avoid the unwarranted straight-jacket of restricting attention to projects whose productivity is perfectly correlated with the market rate of interest. As Gollier (2009) points out "There is a huge literature on the term structure of interest rates [...] But most results in this literature rely on arbitrage, a technique that is mostly useless when considering distant time horizons." The U.K. and France adopted falling discount rates precisely for time horizons lacking forward markets that could pin down market interest. In fact, we rely on social discount rates in the first place because of market limitations. First, markets are incomplete when it comes to the far future as e.g. affected by climate change, large infrastructure projects, or investment in basic research. Second, market failure limits the provision of public goods. Given this second failure, a dollar invested by the public hand is likely to result in a higher social welfare increase than the same dollar being consumed in the present, implying $\gamma_t > 0$ for many if not most governmental projects.

The third reason why we use γ_t is that the more general analysis is key to disentangling the Weitzman and the Gollier effect, i.e. identifying the channel increasing and the channel decreasing discount rates over time. Restricting attention to projects with private payoffs that are efficiently traded in an existing market melts the two channels, which is a major reason why the puzzle has been so puzzling. Finally, the rate γ_t will be the key to seeing that both uncertainty channels generally increase the value of long-term project payoffs, compared to a certain world with constant discounting. We will present and interpret our results in two different ways. First, we calculate the surplus rate γ_t. In a discussion on discounting, we can also interpret the negative of the annual surplus rate $-\gamma_t$ as a generalized discount rate. The standard consumption or social discount rate measures the value decline due to a shift of a consumption unit into the future. The generalized rate $-\gamma_t$ measures the value decline of consumption that is shifted into a *productive* investment for the future (hereafter a 'productive consumption shift'). Second, we explicitly calculate the equilibrium discount rate, i.e. the relation between the average annual rates r_t and g_t obtained from setting $\gamma_t = 0$.

3 Discounting under Uncertainty

This section fleshes out the different consequences of uncertainty over baseline consumption growth and uncertainty over the payoff of an investment project. The first implies decreasing and the second increasing discount rates. The two resulting discount rates have slightly different meanings. After discussing the two rates separately, we analyze the joint framework and show that the two seemingly opposed term structures have the same qualitative effect of giving more value to long-run project payoffs.

3.1 Certain Future

In the case of **certainty** we find using equation (2) that the annual surplus rate is

$$\gamma_t = r + \frac{1}{t} \ln \left[\frac{u'\big(c_0 \exp[gt]\big)}{u'(c_0)} \right] - \delta \ .$$

Employing an isoelastic utility function $u(c) = \frac{c^{1-\eta}}{1-\eta}$ the expression simplifies to

$$\gamma = r - \eta g - \delta \ . \tag{3}$$

The surplus rate of the project increases in its productivity and decreases in pure time preference. Moreover, consumption growth decreases the marginal utility from an additional unit of future consumption and, thus, further decreases the (present value) surplus. In an equilibrium, where project sizes and payoffs are continuous and carried out at the optimal level, γ_t has to equal zero. Then, equation (3) returns the well known Ramsey (1928) equation:

$$r = \delta + \eta g \ . \tag{4}$$

Focusing on discounting, $-\gamma$ measures, in terms of an average yearly rate, the value loss of a consumption-capital unit that is invested into the future t. This generalized discount rate for a productive consumption shift into the future follows the usual Ramsey formula, modified by the productivity of investment. Observe the difference between the information contained in the standard social or consumption discount rate and the generalization $-\gamma$. The standard discount rate describes the devaluation of consumption under a mere shift of a consumption-capital unit from the present into the future t. In contrast, the rate $-\gamma$ measures the devaluation of a consumption unit that is not merely shifted into the future, but invested into a productive project with payoff time t. If the project is unproductive we are back to the mere consumption shift and also $-\gamma$ reduces to $\delta + \eta g$.

3.2 Uncertain Growth

In the case of **uncertainty over the growth rate** we find

$$\gamma_t = r + \frac{1}{t} \ln \left[\frac{\mathrm{E} u'(c_0 \exp[gt])}{u'(c_0)} \right] - \delta \ .$$

Employing once more an isoelastic utility function $u(c) = \frac{c^{1-\eta}}{1-\eta}$ the expression simplifies to

$$\boldsymbol{\gamma_t =} r - \left[-\frac{1}{t} \ln \left[\mathrm{E} \exp[-\eta g t] \right] \right] - \delta = \boldsymbol{r - \mathcal{M}_t^-(\eta g) - \delta} \ . \tag{5}$$

The expression $\mathcal{M}_t^-(z) = -\frac{1}{t} \ln \left[\mathrm{E} \exp[-tz] \right]$ is a generalized mean and decreases in t (Hardy, Littlewood & Polya 1964). Thus, under growth uncertainty, the consumption growth related reduction of the project's annual surplus decreases in time. As time goes to infinity the mean approaches the minimal positively weighted value of ηg and the annual surplus approaches the corresponding maximum.

The current scenario is not, yet, the model of Weitzman (1998, 2001). However, the above reasoning is the essence of what makes the certainty equivalent discount rate in Weitzman's approach fall over time. When the overall growth of the economy is uncertain, a project with a certain payoff in the future becomes more valuable. Note that the serial correlation of uncertainty assumed in the setting is crucial for this result. The generalized discount rate $-\gamma_t = \delta + \mathcal{M}_t^-(\eta g) - r$ characterizes the devaluation of a capital-consumption good shifted into the future t by means of a productive investment project. This generalized discount rate falls over time as the mean $\mathcal{M}_t^-(\eta g)$ approaches the (elasticity weighted) minimal growth rate carrying positive weight.

In an equilibrium where project sizes and payoffs are continuous and carried out at the optimal level we know that $\gamma_t = 0$. In order to satisfy this condition, either r or g or both have to pick up a time dependence. In general we find[2]

$$r_t = \delta + \mathcal{M}_t^-(\eta g_t) \ . \tag{6}$$

If we keep the assumption of constant or, more precisely, fully serially correlated consumption growth $g_t = g$ we find a falling term structure of the market rate. The agents or agency in the economy implement long-term projects with an average yearly productivity that is lower than for short-term projects.

3.3 Uncertain Project Payoff

In the case of **uncertainty over the project payoff** we find

$$\gamma_t = \frac{1}{t} \ln \left[\mathrm{E} \exp[rt] \right] + \frac{1}{t} \ln \left[\frac{u'(c_0 \exp[gt])}{u'(c_0)} \right] - \delta \ .$$

[2]For an individual project we assumed a constant productivity r when calculating γ_t. Here, we turn the relation around and solve e.g. for the constant r that satisfies the equilibrium condition $\gamma_t = 0$ for every point in time t. Thus, r_t is the constant yearly average interest rate of a certain investment with payoff time t.

Employing once more an isoelastic utility function $u(c) = \frac{c^{1-\eta}}{1-\eta}$ simplifies the expression to

$$\boldsymbol{\gamma_t =} \frac{1}{t} \ln \left[\mathrm{E} \exp[rt] \right] - \eta g - \delta = \boldsymbol{\mathcal{M}_t^+(r) - \eta g - \delta} \ . \tag{7}$$

This time, the generalized mean $\mathcal{M}_t^+(z) = \frac{1}{t} \ln \left[\mathrm{E} \exp[tz] \right]$ is increasing in t. The mean grows over time to the largest value of r that receives positive weight. Thus, the project's average annual surplus grows over time under payoff uncertainty.

A first step in relating this reasoning to Gollier (2004) is to calculate the certain productivity that yields the same expected annual surplus as the uncertain project characterized by equation (7). We obtain this certainty equivalent productivity rate r_t^G by setting the annual surplus captured by equation (7) equal to that under certainty $\gamma_t^{cert} = r_t^G - \eta g - \delta$ (see equation 3), which results in the certainty equivalent productivity rate:

$$r_t^G = \mathcal{M}_t^+(r) \ .$$

The certainty equivalent productivity increases over time. The current scenario is not, yet, the model of Gollier (2004). However, the above reasoning is the essence of what makes the certainty equivalent interest rate in Gollier's approach increase over time. At first sight, it might seem contradictory that uncertainty makes the certainty equivalent interest rate fall in the first and increase in the second scenario. Let us call this finding the herald of the Weitzman-Gollier puzzle, emphasizing that it is not, yet, the actual puzzle. Our clear distinction between uncertainty over consumption growth and uncertainty over the investment payoff clarifies that we are dealing with two different settings. The actual Weitzman-Gollier puzzle arises when we merge the two different uncertainties and we discuss the puzzle in the next section. For now, we point out the difference between these two certain and certainty equivalent discount rates.

In our earlier finding, we analyzed the average annual surplus of a project with certain payoff in an uncertain world. This surplus increases over time.

The further we push the certain payoff into a future that is characterized by serially correlated growth uncertainty, the more the agent appreciates the certain unit (relative to the same shift in a certain world). In contrast, in the current scenario, we describe a certain project that is required to exhibit the same term structure as a project with (fully serially correlated) productivity uncertainty. We find that such a certainty equivalent project has to exhibit increasing productivity over time in order to be competitive. The intuition for this finding lies in the fully serially correlated exponential payoff growth. Over time, a high realization of the productivity rate yields exponentially increasing higher payoffs than a low realization. Then, the (linear) expected value operator shifts the resulting mean value more and more towards the value of those states with a high interest realization. Intuitively, the serial correlation implies that there is increasingly more to win relative to what there is to loose.

Finally, it would be misleading to conclude that the two types of uncertainty have opposite effects on how we value investments. The equations characterizing the project's annual surplus under growth and under payoff uncertainty (5) and (7) both imply that uncertainty increases the average annual surplus of an investment project. The same finding holds if we use the rate $-\gamma_t = \delta + \eta g - \mathcal{M}_t^+(r)$ to characterize how the economic agent discounts a shift of present consumption into the future by means of an uncertain investment project. As in the scenario with uncertainty over consumption growth, the devaluation rate falls over time: here, because the mean $\mathcal{M}_t^+(r)$ enters negatively and approaches the maximal positively weighted interest rate.

3.4 Joint Uncertainty

In the case of **joint uncertainty over growth and the project payoff** we find

$$\gamma_t = \frac{1}{t} \ln \left[\frac{\mathrm{E} u'(c_0 \exp[gt]) \exp[rt]}{u'(c_0)} \right] - \delta \; .$$

Employing once more an isoelastic utility function $u(c) = \frac{c^{1-\eta}}{1-\eta}$ the expression simplifies to

$$\boldsymbol{\gamma_t =} \frac{1}{t} \ln \Big[\mathrm{E} \exp[(r - \eta g)t] \Big] - \delta = \boldsymbol{\mathcal{M}_t^+(r - \eta g) - \delta} \ . \tag{8}$$

Thus, the project's annual surplus rate also increases over time under general uncertainty. The devaluation rate $-\gamma_t$ for consumption that we shift into investment decreases accordingly.

An interesting special case obtains when consumption growth uncertainty is independent of project payoff uncertainty. Then, the uncertainty contributions that we derived independently in the previous scenarios simply merge into the joint formula

$$\begin{aligned} \gamma_t &= \frac{1}{t} \ln \Big[\mathrm{E} \exp[rt] \Big] - \Big[- \frac{1}{t} \ln \big[\mathrm{E} \exp[-\eta g t] \big] \Big] - \delta \\ &= \boldsymbol{\mathcal{M}_t^+(r) - \mathcal{M}_t^-(\eta g) - \delta} \ . \end{aligned} \tag{9}$$

The project's annual surplus rate increases over time because (i) the mean of the uncertain payoff increases and (ii) consumption growth that is uncorrelated with project payoff increases the value of shifting a unit of the capital-consumption good into the future. The intuition for both of these time changes derives from the power of the exponential. As payoffs grow exponentially over time at a rate proportional to productivity, the high productivity rates and payoffs take over the average payoff. In contrast, consumption growth reduces the value of a future consumption unit. Here, the lowest rates take over the mean as consumption value declines exponentially. In terms of the generalized discount rate for consumption shifted productively into the future we have $-\gamma = \delta + \mathcal{M}_t^-(\eta g) - \mathcal{M}_t^+(r)$. This discount rate falls over time, as the small growth rates dominate the valuation of marginal consumption and, simultaneously, the large productivity rates dominate the payoff expectations.

4 The Weitzman-Gollier Puzzle Revisited

This section employs the analysis laid out in the previous section to clarify the Weitzman-Gollier puzzle. We assume isoelastic utility throughout the discussion. Moreover, we maintain the crucial assumption of the literature on the Weitzman-Gollier puzzle that the productivity of the uncertain project (or market interest) is fully serially correlated.

4.1 The "Weitzman Effect"

4.1.1 The "Pure Weitman Effect"

Weitzman (1998, 2001) argues for a social discount rate that falls over time. His stepping stone is the insight that one should average discount factors rather than rates in order to derive a meaningful certainty equivalent cost benefit analysis. He points out that a certainty equivalent discount factor gives rise to a certainty equivalent discount rate of the form

$$r_t^W \equiv \mathcal{M}_t^-(r^{market}) \ ,$$

where r^{market} is the uncertain productivity rate determining economic growth. While Weitzman derives the certainty equivalent discount rate formally from the corresponding factor, he does not derive the underlying discount factor and his intuitive argument leaves some room for interpretation. His discussion emphasizes uncertainty over economic growth. We argue that Weitzman is concerned with growth uncertainty changing the marginal value of a dollar because of its impact on the resulting wealth and consumption levels. There are other interpretations of his reasoning.[3] We show below that our interpretation of Weitzman's intuitive approach is the one validating his formula (or, generally, a close relative). This insight is implicit also in the work of Gollier (2010), Gollier & Weitzman (2010), and Freeman (2010).

[3] In particular, Freeman & Groom (2010) show that Weitzman's (2001) formula does not withstand the perspective of aggregating forecasts (or opinions) over experts.

We know from equation (6) that the interest rate of an equilibrium project with certain payoff in an uncertain world is $r_t^{cert} = \delta + \mathcal{M}_t^-(\eta g_t)$. Weitzman (1998, 2001) does not explicitly spell out utility or consumption. Instead, he argues exclusively using the market interest rate in the economy. We have to check whether his market interest based argument gives rise to the correct formula for r_t^{cert}. For this purpose, we have to relate the market interest r^{market} to consumption growth. The assumption of full serial correlation implies that the interest rate is known with certainty from period 1 on and, from then on, the market equilibrium satisfies the Ramsey equation (4). Denoting the *annual* growth rate from period τ to period $\tau + 1$ by g^τ we therefore find

$$r^{market} = \delta + \eta g^\tau \text{ for all } \tau \geq 1 , \tag{10}$$

where equation (10) has to hold for every realization of the market interest and, thus, also for the random variable itself. From period 1 on we therefore find a constant annual growth rate $g^* \equiv g^1 = ... = g^{t-1}$ that is perfectly correlated to the market rate of interest r^{market}.

Let us assume for a moment that equation (10) is also satisfied for $\tau = 0$. We discuss and relax this assumption in section 4.1.2. It is satisfied e.g. in a Lucas (1978) tree economy. Then, we have $r^{market} = \delta + \eta g^*$ and

$$r_t^W = \mathcal{M}_t^-(r^{market}) = \mathcal{M}_t^-(\delta + \eta g^*) = \delta + \mathcal{M}_t^-(\eta g^*) = r_t^{cert} , \tag{11}$$

confirming that Weitzman's r_t^W correctly describes the certainty equivalent discount rate. Observe that the full serial correlation of market interest is key for equality (11) to hold. It implies that the agents observe the market rate during the first period and adjust their consumption paths according to equation (10). This way, the growth rate of consumption becomes perfectly correlated also with the market rate of interest. Therefore, Weitzman's formula holds. We call the effect observed in equation (11) the pure Weitzman effect.

4.1.2 The "Generalized Weitzman Effect"

In general, the assumption that equation (10) is satisfied for $\tau = 0$ may not hold. In the following, we discuss this assumption and show how r_t^W deviates from the correct certainty equivalent discount rate r_t^{cert} when the assumption does not hold. While capital productivity from period 0 to period 1 is subject to the particular realization of r^{market}, consumption growth does not necessarily follow the growth of invested capital. In an economy with productive capital, interest determines not just growth, but also the share of the capital-consumption good that is consumed. This share shifts when r^{market} is realized and, for that period, consumption growth and capital growth diverge (as opposed to being governed by equation 10). A (brute force) example excluding such consumption shifts is the Lucas (1978) tree economy, an endowment economy with perishable consumption.[4] Freeman (2010) shows that, indeed, Weitzman's formula is always correct in the Lucas tree economy. He also gives an argument close to the one above, why this result is not expected to carry over to the AK production economy used in Gollier (2009), Gollier (2010), and Gollier & Weitzman (2010). In the following, we state the more general relation between Weitzman's r_t^W and the correct certainty equivalent discount rate.

We obtain the equilibrium condition that has to hold between period 0 and period 1 from section 3.4 on joint uncertainty. Setting $\gamma = 0$ and $t = 1$ in equation (8) delivers the desired relation between consumption growth and the interest of the uncertain equilibrium market project: $\mathcal{M}_1^+(r^{market} - \eta g^0) = \delta$, which by equation (10) implies the condition $\mathcal{M}_1^+(\eta(g^* - g^0)) = 0$. Without further assumptions on the structure of the economy, such as an AK production model, the distribution of r^{market} does not fix the distribution of g^0. In general, the distribution of g^0 depends on the model structure, the distribution of r^{market}, η, and δ. The yearly average consumption growth rate over the time horizon of the project now depends on the time horizon

[4] Here, the project under evaluation would be the planting of a new tree carrying fruits only in a single period.

and is composed of $g_t = \sum_{\tau=0}^{t-1} g^\tau / t = g^* + (g^0 - g^*)/t$. Then, the correct certainty equivalent discount rate r^{cert} becomes

$$r_t^{cert} = \delta + \mathcal{M}_t^-(\eta g_t) = \delta + \mathcal{M}_t^-\left(\eta g^* + \eta \frac{g^0 - g^*}{t}\right) \equiv r_t^{\widehat{W}} \ , \tag{12}$$

while Weitzman's original market rate based discounting formula still delivers

$$r_t^W = \mathcal{M}_t^-(r^{market}) = \mathcal{M}_t^-(\delta + \eta g^*) = \delta + \mathcal{M}_t^-(\eta g^*) \ . \tag{13}$$

Weitzman's r^W picks up the relevant consumption growth change after uncertainty resolves, but misses the consumption growth change $g^0 - g^*$ during the resolution of uncertainty that captures shifts in the consumption share.[5] A comparison of equations (12) and (13) teaches two lessons. First, in the long run Weitzman's r^W converges to the correct formula for the certainty equivalent discount rate.[6] Second, while Weitzman's r_t^W starts out at the mean rate and then falls to its minimal value as t increases,[7] the correct formula can initially differ, potentially significantly, from expected market interest. Again, the underlying reason is that agents will generally adjust their consumption-investment behavior to the interest rate. We call the certainty equivalent rate $r_t^{\widehat{W}}$ in equation (12) the generalized Weitzman formula. It exhibits the following generalized Weitzman effect. Like the original Weitz-

[5]Let us elaborate this point. The stock variable capital is fixed at a given point in time and productivity takes on a constant realization r_i^{market} from period 0 on. Therefore, a change in consumption share must drive a wedge between g_0 and g^* and, similarly, a difference between g_0 and g^* must result from a change in consumption share. An increase in the consumption share will increase g^0 over g^* making the contribution positive. Observe that the term is a random variable.

[6]The consumption shift $g^0 - g^*$ is independent of the project's time horizon and, thus, its influence dies off as t grows large. Observe that we do not simply state that the certainty equivalent discount *rate* converges to the same value, but that Weitzman's discounting *formula* converges to the correct formula, which is a stronger statement.

[7]Observe that $\lim_{t\to 0} r_t^W = \lim_{t\to 0} \mathcal{M}_t^-(r^{market}) = \mathbf{E}\, r^{market}$. Given our discrete time model the limit is only indicative that the rate starts out closely below the mean and continues to fall.

man formula, the discount rate $r_t^{\widehat{W}}$ falls monotonically in t.[8] Moreover, the generalized discounting formula converges to the original Weitzman formula analyzed in section 4.1.1 for long-term evaluation.

4.1.3 Relation to Gollier & Weitzman (2010)

We briefly relate equation (12) to the certainty equivalent discount rate derived by Gollier & Weitzman (2010). The authors piece together an AK growth model under certainty with a prepended cost benefit model of investment under uncertainty. Their setting does not explicitly model consumption before uncertainty resolves. They assume that the agent only commits to the investment project before uncertainty resolves, but then invests after uncertainty resolution. This simplification implies that the agent invests at a time where the project is, in general, no longer optimal. However, the timing simplification allows the authors to omit the explicit reasoning about consumption growth while uncertainty resolves: their model does not show the consumption shift due to uncertainty resolution and the authors assume equation (10) to hold for all times. Nevertheless, their prepended cost benefit reasoning under uncertainty correctly picks up the modification of the certainty equivalent discount rate leading to $r_t^{\widehat{W}}$ rather than r_t^W. Gollier and Weitzman obtain the correct formula by adjusting the probability distribution underlying the market interest by the ratio of marginal utility post uncertainty resolution and expected marginal utilities preceding uncertainty resolution. This adjustment is elegant and implies the correct discounting formula, however, it makes it hard to see the economic intuition underlying

[8]This insight closely relates to the finding stated in Gollier & Weitzman (2010). Our timing, however, slightly differs in that investment actual takes place at the time of commitment before uncertainty resolves. Thus we give a short proof of the statement for our setting. The condition $\mathcal{M}_1^+(\eta(g^* - g^0)) = 0$ derived from $\gamma_1 = 0$ implies that $\sum_i p_i \exp[\eta(g^* - g^0)] = 1$. Therefore $r_t^{\widehat{W}} = \delta - \frac{1}{t} \ln \left[\sum_i p_i \exp[-\eta(g_i^* - g_i^0)] \exp[-\eta g_i^* t] \right] = \delta - \frac{1}{t} \ln \left[\sum_i {}^{p_i \exp[-\eta(g_i^* - g_i^0)]} / {}_{\sum_i p_i \exp[-\eta(g_i^* - g_i^0)]} \exp[-\eta g_i^* t] \right] \equiv \delta - \frac{1}{t} \ln \left[\sum_i q_i \exp[-\eta g_i^* t] \right]$. The redefined probability weights q_i turn the expression once more into the form $\delta + \mathcal{M}_t^-(\eta g_i^*)$ embracing the generalized mean, which decreases in time.

the adjustment.

We modify the setting of Gollier & Weitzman (2010) in order to explicitly analyze consumption at time 0^- before uncertainty resolves (project commitment time) and compare it to their modeled consumption at 0^+ after uncertainty resolves. We define a rate that measures consumption change between time 0^- and 0^+ as the random variable $g_{0^-}^{GW} = \ln\left(\frac{c_{0^+}^{GW}}{c_{0^-}^{GW}}\right)$. Note that the actual growth rate at 0 in the Gollier-Weitzman setting would be infinite and, therefore, we have defined $g_{0^-}^{GW}$ as if happening over the course of a year. A short calculation[9] shows that the resulting instantaneous growth jump satisfies $g_{0^-}^{GW} = g^0 - g^*$: in the Gollier-Weitzman setting, the consumption jump triggered by changes in the consumption-investment ratio happens instantaneously at time 0, between committing to the investment project and actually carrying out the consumption-investment decision. The probability bias in Gollier & Weitzman's (2010) version of the corrected Weitzman formula precisely picks up the jump in consumption growth due to shifts in the consumption-investment ratio. In their model, the shift happens instantaneously, right before the AK economy starts to operate, after committing to the project, but before investing into the project. In our setting, these consumption shifts happen during the first period after the agent has invested into the project.

4.2 The "Gollier Effect"

Gollier (2004) uses the same setting as above with an uncertain market interest r^{market}. He studies a certainty equivalent project yielding the same

[9] Starting point is equation (13) in Gollier & Weitzman (2010). We introduce consumption in period 0^- by observing that an expected utility maximizer must satisfy $u'(c_{0^-}^{GW}) = \mathbf{E}u'(c_{0^+}^{GW})$. This relation translates their probability bias $u'(c_{0^+}^{GW})/\mathbf{E}u'(c_{0^+}^{GW})$ into the term $u'(c_{0^+}^{GW})/u'(c_{0^-}^{GW}) = \exp(\eta g_{0^-}^{GW})$. This latter equality turns their formula (14) for the certainty equivalent discount rate into $\mathcal{M}_t^-(\eta g_{0^-}^{GW}/t + r)$. Using furthermore their equation (11), which essentially states the Ramsey formula under certainty, we obtain the certainty equivalent discount rate $\mathcal{M}_t^-(\eta g_{0^-}^{GW} + \eta g^*)$ whose comparison with our equation (12) results in the stated relation.

return as the market portfolio. Instead, we start out studying a certainty equivalent project yielding the same annual surplus as an arbitrary uncertain public project. We denote the annual surplus of the certainty equivalent project by γ_t^{cert} and the annual surplus of the uncertain public project by γ_t^{unc}. We want to know when the annual surplus of the certainty equivalent project $\gamma_t^{cert} = r_t^{cert} - \mathcal{M}_t^-(\eta g_t) - \delta$ (equation 5) equals that of the uncertain project $\gamma_t^{unc} = \mathcal{M}_t^+(r^{unc} - \eta g_t) - \delta$ (equation 8). We find[10]

$$\gamma_t^{cert} = \gamma_t^{unc} \quad \Rightarrow \quad r_t^{cert} = \mathcal{M}_t^+(r^{unc} - \eta g) + \mathcal{M}_t^-(\eta g) \ . \tag{14}$$

In the cases where the consumption growth rate g is either certain or independent of r^{unc} we find

$$r_t^{cert} = \mathcal{M}_t^+(r^{unc}) = r_t^G \ , \tag{15}$$

using equation (9) in the case of independence. Formula (15) is precisely the equation suggested in Gollier (2004) to calculate the certainty equivalent discount rate. It implies an increasing term structure.

Obviously, the original Weitzman formula (12) or the generalized Weitzman formula (11) and equation (15) cannot simultaneously define a certainty equivalent discount rate for an uncertain market project (except for degenerate cases). Equation (15) holds in the case of certain consumption growth or independence between consumption growth and the interest r^{unc}. Thus, the equation cannot hold for the market interest rate. As we have shown in section 4.1, the serially correlated market interest rate is perfectly correlated with consumption growth from period 1 on (equation 10). Instead of equation (15), for $r^{unc} = r^{market}$ equation (14) together with the equilibrium

[10] The certainty equivalent productivity rate picks up a time index because there exists a different certainty equivalent project for every payoff time t. As we observed in section 4.1.2 also the growth rate will generally vary over time once we make it endogenous by imposing equilibrium conditions.

condition $\gamma_t^{market} = 0$ yields once more

$$\begin{aligned} r_t^{cert} &= \mathcal{M}_t^+(r^{market} - \eta g_t) + \mathcal{M}_t^-(\eta g_t) = \delta + \mathcal{M}_t^-(\eta g_t) \\ &= \delta + \mathcal{M}_t^-\left(\eta g^* + \eta \frac{g_0 - g^*}{t}\right) . \end{aligned}$$

The fact that equation (15) does not hold for the market interest does not imply that we should dismiss the insight of Gollier (2004). It is still true that risky projects generally become increasingly more valuable if the payoffs are serially correlated. The certainty equivalent discount rate for a project that has to match the expected payoff of such an uncertain project has to exhibit an increasing term structure. In fact, equation (8) shows that under full serial correlation, the only case where this insight is not correct is the case where r is the market interest rate. It makes economic sense that the market rate of interest is the exception to the rule. Otherwise, a gamble at the market rate would always give rise to a surplus, if only the payoff time would be pushed far enough into the future. But then, the market rate of interest could not be an equilibrium rate. In the case of limited forward markets and public projects, however, this argument does not usually apply and the Gollier effect remains.

4.3 Reconciling "Weitzman and Gollier Effects"

The original point of departure of both authors was the question how to evaluate and discount an investment project under uncertainty. Both authors have contributed an insightful piece to the puzzle. Serially correlated uncertainty over the market rate of interest always leads to a falling term structure. However, a public decision maker frequently faces the question whether to invest in a non-market project with potentially risky payoff. The risky project will often be discrete and not perfectly correlated with the market rate of interest.[11] We can compare the annual surplus of a certain and

[11]Note that a public decision maker generally faces a resource constraint preventing him from providing the optimal allocation of a public good even when the project can be scaled

an uncertain project by using equations (5) and (8). We can rewrite the first equation as

$$\boldsymbol{\gamma_t^{cert}} = r^{cert} - \mathcal{M}_t^-(\eta g_t) - \delta = \boldsymbol{r^{cert} - r_t^{\widehat{W}}} \ .$$

As $r_t^{\widehat{W}}$ falls over time, the average annual surplus of the certain project increases. The further we shift the certain payoff into the (serially correlated) uncertain future, the lower is the productivity required to make the investment worthwhile. The annual surplus of the uncertain project is $\gamma^{unc} = \mathcal{M}_t^+(r^{unc} - \eta g) - \delta$ or, in the case of independence between the market rate of interest and project risk,

$$\boldsymbol{\gamma_t^{unc}} = \mathcal{M}_t^+(r^{unc}) - \mathcal{M}_t^-(\eta g) - \delta = \boldsymbol{r_t^G - r_t^{\widehat{W}}} \ .$$

The surplus of the uncertain payoff increases in payoff time for both accounts. First, r_t^W falls over time because of consumption growth uncertainty. Second, r_t^G increases over time because the serial correlation implies that the high project payoff grows faster than average. In particular, indifference between the certain and the uncertain investment project holds if

$$\gamma_t^{cert} = \gamma_t^{unc} \quad \Leftrightarrow \quad r_t^{cert} = r_t^G \ , \tag{16}$$

i.e. if the certain project exhibits an increasing term structure. Note, however, that equation (16) is generally not an efficiency condition. We emphasize once more that these last equations assume independence between the uncertain project payoff and the market rate of interest, and that $r_t^G = \mathcal{M}_t^+(r^{unc})$ increases in the uncertainty of the investment project, not the uncertainty of the market rate of interest.

5 Conclusions

We laid out how consumption growth and payoff uncertainty affect the social or consumption discount rate in the serially correlated setting of the

to an arbitrary level.

Weitzman-Gollier puzzle. We showed that Weitzman's conclusion of falling discount rates derives from uncertainty over economic baseline growth reducing the consumption value of a capital-consumption good. We showed that a more generally applicable certainty equivalent discounting formula, also appearing in Gollier & Weitzman (2010), converges to Weitzman's original formula in the long run. Instead of utility adjusting probabilities, we generalized Weitzman's formula by explicitly incorporating the neglected consumption growth shifts. These shifts of the consumption-investment ratio happen in response to the resolution of uncertainty. In our setting these shifts happen explicitly in the first period. We showed that these shifts also happen implicitly and instantaneously right before the first consumption period in the setting of Gollier & Weitzman (2010).

In contrast to the Weitzman effect, Gollier's reasoning in favor of increasing discount rates derives from analyzing a project's payoff uncertainty, emphasizing the capital nature of the capital-consumption good. The increasing term structure of Gollier's certainty equivalent rates reflects that the relative advantage of a risky project, over a certain project, increases over time (given serial correlation). We have shown that uncertainty over both, baseline growth and project payoff, generally work in the same direction of increasing the value of investing in long-term projects rather than short term projects. The Gollier effect disappears only in the case where the project payoffs are certain or perfectly correlated with the market interest.

Instead of listing the broad field of applications where uncertainty in the intertemporal trade-off is of major economic importance, we close with an illustration contrasting the Gollier and the Weitzman effect. We assume two small projects, each of which pays a bundle of milk and honey. The first project has a certain and constant rate of productivity. The second project pays either nothing or a milk and honey volume that grows at a constant productivity rate dominating that of the certain project. When we compare the two projects and the three possible payoffs, the payoff in the high productivity state dominates all others exponentially over time. For a sufficiently long time horizon, an economic agent (with isoelastic utility) would therefore prefer the uncertain project. That is the "Gollier effect".

Let us now assume that the risky project can be scaled arbitrarily. The economic agents would invest more and more into the risky project (given a sufficiently long project horizon). At some investment level, the high productivity state literally leads to the land of milk and honey. Swimming in a river of milk (or anticipating doing so), the agents no longer care for the exponentially dominating payoff they obtain in the high productivity state by investing another unit. At this point (at the latest...), the project has become an equilibrium project and the productivity rates are market interest rates: they determine the overall milk and honey consumption in the economy. Now, the certain project has become relatively more attractive. It pays out less milk and honey in the state of the world where agents already are in the land of milk and honey, but pays out more in the state of the world in which the risky market project leaves the agents empty-handed. This certainty characteristic is valued relatively higher by the agents, the larger the spread of milk and honey over the risk states. Thus, the certain project is valued increasingly more over time. That is the "Weitzman effect".

Observe that we first compared two small projects, one certain and the other uncertain. Then, we compared the small certain project to the uncertain and large market project that determines overall consumption levels. Finally, observe that we are in a world of serially correlated exponential productivities. Thus, the smallness assumptions of a project will be challenged more easily the further its payoff lies in the future.

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