Packing Items from a Triangular Distribution

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Abstract. We consider the problem of packing n items which are drawn according to a probability distribution whose density function is triangular in shape. For triangles which represent density functions whose expectation is $1/p$ for $p = 3, 4, 5, \ldots$, we give a packing strategy for which the ratio of the number of bins used in the packing to the expected total size of the items asymptotically approaches 1.

Keywords: Bin packing, probabilistic analysis

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PACKING ITEMS FROM A TRIANGULAR DISTRIBUTION+

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Introduction

We are interested in the optimum packing of triangular distributions. We call a probability density function, $f(x)$, forward triangular if $f(x) = m(x-a)$, $x \in [a,b]$ and is 0 elsewhere. For a forward triangular function, the equations $\int_a^b f(x)dx = 1$, and $f(a) = 0$, require $f(x) = 2x/(b-a)^2 - 2a/(b-a)^2$, $x \in [a,b]$, yielding an expectation of $a + 2(b-a)/3$, that is two-thirds of the way from a to b. Backward triangular functions of the form $f(b) = 0$, $f(x) = m(b-x)$ have the corresponding property of having an expectation one-third of the way from a to b. We give results only for forward triangles. However proofs for backward triangles are almost identical. We define the optimum packing ratio of a distribution as the limit as the number of items drawn approaches infinity of the expected number of bins required to optimally pack the items divided by the expected total size of the items.

Because bin packing is NP-complete [3], much work has been done on algorithms which are not guaranteed to give the exact optimum. Here we mention a few of the papers most related to our problem, namely determining when the optimum packing ratio is one. Frederickson [2] analyzed the First-Fit-Decreasing Algorithm for distributions uniform over $[0,1]$ and showed that it wastes only an average of $O(n^{2/3})$ space. Knödel [5] and Lueker [7] designed algorithms for such functions that waste $O(\sqrt{n})$ space. Lueker [7] also showed that the wasted space in an optimal packing of such a function is, in fact, $\Theta(\sqrt{n})$. Furthermore, he showed that the First-Fit-Decreasing and Best-Fit-Decreasing Algorithms waste only this amount of space. Karmarkar [4] analyzed the Next-Fit Algorithm under a distribution uniform over the interval $[0,a]$ and explained some empirical results. Knödel [5] and Loulou [6] showed that for uniform density functions symmetric about $1/p$, where p is an even integer, the optimum packing ratio is one. Lueker [8] showed that for uniform density functions symmetric about $1/p$, where p is an odd integer, the optimum packing ratio is one. Karmarkar [4] attributes to Karp the problem of determining for what a and b the uniform distribution over $[a,b]$ allows an optimum packing ratio of one. Lueker [8]

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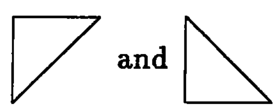
determined the packing ratio for a large subclass of these intrevals. Shor [11] has been able to provide constructive packing strategies which show that, for many other classes of intervals, the packing ratio is one. Using nonconstructive methods, Rhee and Talagrand [10] have shown that the optimum packing ratio is one for all of the intervals not covered by [8].

There has also been some work on non-uniform distributions. Knödel [5], Karmarkar [4], and Loulou [6] have established that decreasing density functions on the unit interval have optimum packing ratios of one. Rhee [9] has provided a characterization of the class of distributions which allow a packing ratio of one. This characterization does not, however, appear to be easy to apply. Rhee and Talagrand [10] also provide a sufficient condition, which is easily applied, which shows that, for a large class of nonuniform distributions, the packing ratio is one. In particular, the triangular densities which we discuss here meet this condition. Their proof, however, is not constructive: unlike the proof we provide below, it does not yield a concrete strategy for the packing.

Our packing strategies involve assigning items of differing sizes to classes of bins in a probabilistic manner. For example, an item of size x may be assigned to a bin of class 1 with 50% probability, to a bin of class 2 with 30% probability and to a bin of class 3 with 20% probability. Our packing strategies can be represented by partitioning the original probability density function into regions. Each region represents the conditional probability density function for the assignment of an item into a bin of a certain class. In general, more than one region may be associated with a given class, in which case we will attempt to fill bins of that class by assigning them one item from each region. In the case of the triangular density function, it is sufficient that we consider only regions of triangular shape. Since only the density matters, the regions



represent the same conditional probability density function.

This paper is organized as follows. First we will give a packing strategy for a triangular function whose expectation is 1/3 that achieves an optimum packing ratio of 1. Then we show the same ratio for triangular functions whose expectations are 1/4 and 1/5. Finally we will use these packing strategies to show that any triangular function with an expectation of $1/p$, p integer, and $p \geq 3$ has an optimum packing ratio of 1.

Triangles with expectation of 1/3

Lemma 1: For a triangular density function with expectation c, and $k \geq 1$, we can divide it into 3^{2k} congruent subtriangles which can be partitioned into 3^{2k-1} sets of three triangles each such that the sum of the expectations of the subtriangles in each set is $3c$.

Proof: We will show the lemma by using induction on k. To facilitate the induction, we add to the inductive hypothesis the assertion that all three subtriangles in each set have the same orientation, i.e. either all are forward triangles or all are backward triangles.

First we will show that for $k = 1$, we can divide the triangle into $3^2 = 9$ smaller ones, such that there exist three sets of three subtriangles with the desired result.

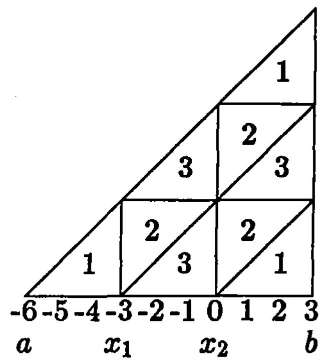


Figure 1.

Let w_0 be the length of the base of the original triangle, so $w_0 = b - a$. Divide the base into thirds, where the intervals $[a, x_1]$, $[x_1, x_2]$, and $[x_2, b]$ have equal lengths. Divide the triangle into nine congruent triangles, as in Figure 1. Let w_1 be the length of the base of one of the smaller triangles, so $w_1 = w_0/3$.

The expectation of the original triangle is c, two-thirds of the way between a and b, that is at x_2. We can rescale and translate the triangle, so that without loss of generality, we will assume $x_2 = 0$ and $w_1 = 3$. Under these conditions, the desired sum of the expectations in each set is 0. There are nine subtriangles with five different expectations as summarized in Table 1.

expectation	number of subtriangles
-4	1
-2	1
-1	2
1	2
2	3

Table 1.

These triangles may be partitioned into three sets of three triangles as follows. As Set 1, take the triangles whose expectations are $\{-4, 2, 2\}$ (labeled 1 in Figure 1). Set 2 is $\{-2, 1, 1\}$ (labeled 2). Set 3 is $\{-1, -1, 2\}$ (labeled 3). The sum of the expectations in each set is 0, each subtriangle in a set has the same orientation, and we have used every subtriangle.

For any k greater than 1 we assume we have a subdivision for $k - 1$ which satisfies the hypothesis. For any given set of three forward trangles at $k - 1$ level, k is formed by partitioning each of the three triangles into 9 small triangles in the same manner as was illustrated for $k = 1$ and forming these 27 small triangles into 9 sets $S_1, \ldots, S_9$ of 3 subtriangles as shown in Figure 2; there S_i can be found by selecting the three subtriangles that are labled i.

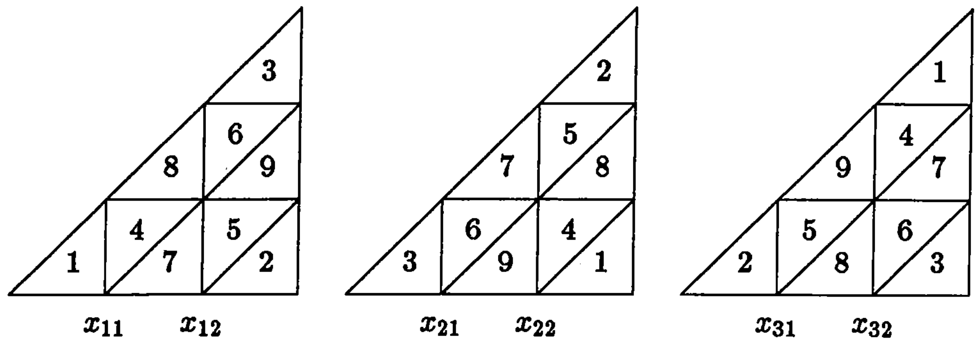


Figure 2.

The sum of the expectations of a set of these smaller triangles is $x_{12} + o_1 + x_{22} + o_2 + x_{32} + o_3$ where x_{12}, x_{22}, and x_{32} are the expectations of the larger triangles and o_1, o_2, and o_3 are the displacements of the expectations of the smaller triangles from the expectations of the larger triangles to which they belong. For S_1 (assuming the same normalization as in the case of $k = 1$), $o_1 = -4$, $o_2 = 2$, and $o_3 = 2$. The sum of the expectations of this set is $x_{12} - 4 + x_{22} + 2 + x_{32} + 2 = x_{12} + x_{22} + x_{32}$, which, since the triangles are congruent, is three times the expectation of the large triangles. This implies the expectation of S_1 is the same as the expectation of the large triangles. The sum of the expectations of any of the sets is, in general, the sum of the expectations of the larger triangles plus the sum of the offsets from the different expectations. Therefore, by choosing each of the sets of smaller triangles such that $o_1 + o_2 + o_3 = 0$, we can form level k. The nine subsets shown in Figure 2 are chosen so that this is the case.

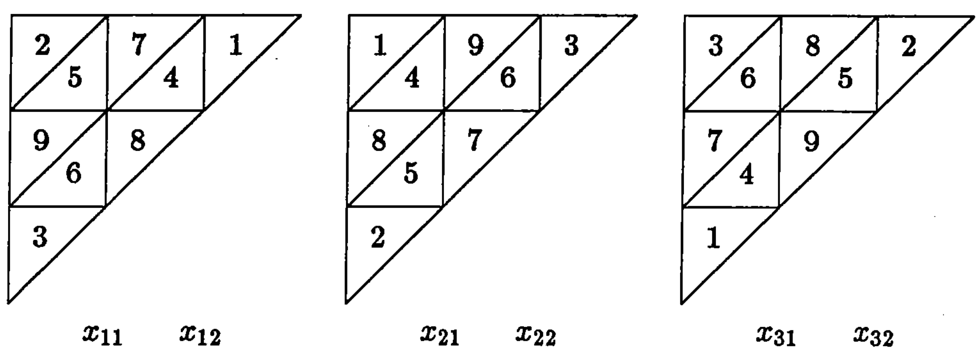


Figure 3.

This construction can also be used if the three triangles are backward, as in Set 2 of Figure 1. To form the new sets, we flip the triangles about their means and change the signs of the offsets and form the 9 sets as shown in Figure 3. Again, the sum of the offsets for each of the sets is zero so the expectation of each set is the same as the expectation of the original three triangles.

Observe that each set contains either three forward or three backward triangles. Thus, we have constructed a subdivision for k which satisfies the inductive hypothesis (and the assumption that each set contains triangles of the same orientation). Finally we note that the width of the base of each of the 27 smaller triangles is 1/3 of the three triangles from which they were formed, so $w_k = w_{k-1}/3$, that is, $w_k = w_0/3^k$.

Thus for a translated triangle, we can produce 3^{2k-1} sets of three triangles such that the sum of the expectations in each set is 0. Reversing the translation yields a sum of $3c$. ■

The previous lemma gives a method for packing a triangle with expectation 1/3 into bins of size $1 + 2w_k$; we just blindly pick one item from each member of any set formed and pack the items together into one bin. We will call this a packing strategy. However, the bin size is more than what we are allowing, namely unit capacity. The next lemma shows how to get around this.

Lemma 2: The items drawn from a triangular distribution with expectation $1/p$, $p = 3$, have an optimum packing ratio of one.

Proof: We will use "p" instead of "3" in the following argument because most of it will generalize for other p. For a fixed k, using the recursion shown in the previous lemma, the sum of items drawn one from each member of a set where the sum of the expectations is $cp = 1$ cannot exceed the sum of the expectations plus p times 2/3 of the width of the subtriangles (w_k). For $p = 3$, this is equal to $1 + 2w_k$. However for a packing to be legal, we must reduce this sum to one. This can be done as follows. Instead of packing using the strategy of the original triangle R with an expectation of $1/p$ with base $[a, b]$, pack using the strategy defined by the new triangle L which is a translation of R by $2w_k/3$ to the left, as shown in Figure 4.

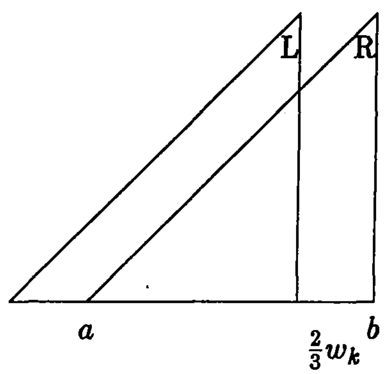


Figure 4.

The expectation of triangle L is $1/p - 2w_k/3$. By the previous lemma and reasoning analogous to the above, the maximum sum of one item from each triangle in any set formed is $p(1/p - 2w_k/3) + 2w_k = 1$ (since $p = 3$).

Suppose we have n items independently drawn according to our triangular distribution. For arbitrarily small ϵ and arbitrarily large k, the value of the maximum number of items in any subtriangle is bounded above by $(1 + \epsilon)n/p^{2k}$, except with exponentially small probability [1]. Let A_0 be the event that the maximum number of items obeys this bound, and A_1 be the event that it does not.

First assume that A_0 holds. To pack the items in L∩R, choose one (arbitrary) item from each of the p triangles that form a set according to the strategy of packing L together into one bin. Since the items are generated by R, but we are using the packing strategy of L, the subtriangles along the hypotenuse of L will have fewer items expected. If we run out of items in a particular subtriangle before we have finished packing all the items in that set, either because the expected number of items was less for that subtriangle than for the others, or because the number of items in that subtriangle doesn't correspond to the expected number of items, we just continue packing but with less items per bin. Since we have at most $(1+\epsilon)n/p^{2k}$ items per triangle, we need at most $(1+\epsilon)n/p^{2k}$ bins per set. We have p^{2k-1} sets, so to pack $L \cap R$, we need at most $(1+\epsilon)n/p$ bins.

Now consider $\bar{L} \cap R$, that is the area of R that is not in L. This strip is a subset of the union of the triangles in the rightmost portion of R, i.e. over $[b-w_k, b]$. We can overcount the number of items in $\bar{L} \cap R$ by counting the number of items in this column of triangles. We have $p^k + p^k - 1 = 2p^k - 1$ triangles in the column. Pack all the items found in this column individually in bins. This uses a maximum of $(1+\epsilon)n(2p^k-1)/p^{2k}$ bins. Hence, when event A_0 holds, the total number of bins is bounded by $(1+\epsilon)n(1/p+(2p^k-1)/p^{2k})$.

When event A_1 holds, we will use n bins and pack one item per bin. Since A_1 holds with exponentially small probability, for large n, a bound on the expectation of the total number of bins used over all events is $(1+2\epsilon)n(1/p+(2p^k-1)/p^{2k})$. Since ϵ can be chosen arbitrarily small, and k can be chosen arbitrarily large, we get the optimum packing ratio arbitrarily close to one by dividing by the expected total size of the items, that is, dividing by n/p. ■

Triangles with expectation of 1/4

Using reasoning similar to the case for $p = 3$, we can optimally pack a triangle with an expectation of $1/4$. A partition of $4^2 = 16$ subtriangles at each level of the recursion is used.

Lemma 3: For a triangular density function with expectation c, and $k \geq 1$, we can divide it into 4^{2k} subtriangles which can be partitioned into 4^{2k-1} sets of four triangles each such that the sum of the expectations of the subtriangles in each set is $4c$.

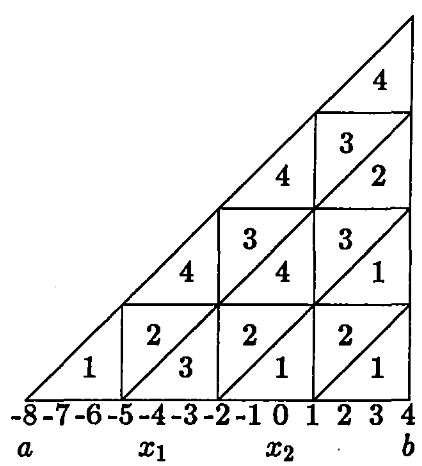


Figure 5.

Proof: Again without loss of generality, we will deal with the translated triangle, setting $x_2 = 0$ and $w_1 = 3$. Under this translation, we desire the sum of the expectations of the subtriangles to be zero. The $k = 1$ level of the recursion is illustrated in Figure 5. The subtriangles with their expectations are shown in Table 2. The sets whose expectations sum to 0 are: $\{-6, 0, 3, 3\}$ (Set 1), $\{-4, -1, 2, 3\}$ (Set 2), $\{-3, -1, 2, 2\}$ (Set 3), and $\{-3, 0, 0, 3\}$ (Set 4). Again, as in Lemma 1, we have used every triangle.

expectation	number of subtriangles
−6	1
−4	1
−3	2
−1	2
0	3
2	3
3	4

Table 2.

The recursive step works in a fashion analogous to Lemma 1, provided that all four triangles have the same orientation, i.e. either all are forward triangles, or all are backward triangles. Notice that we may have three backward triangles and one forward triangle (Sets 2 and 3), or vice versa, or two forward and two backward triangles at deeper recursion levels. However this method of packing still works, provided we can partition the 64 subtriangles into sets of four with each set having an expectation of 0.

To form a set we must choose one subtriangle from each of the four triangles, so that the sum of the expectations is 0. For the partition to work, we must use each subtriangle in exactly one of the sixteen sets. Table 3 summarizes the shifts in expectation of the subtriangles.

expectation	subtriangles of forward triangle	subtriangles of each backward triangle
−6	1	
−4	1	
−3	2	4
−2		3
−1	2	
0	3	3
1		2
2	3	
3	4	2
4		1
6		1

Table 3.

A correct partition for one forward and three backward triangles is: $\{-6,-3,3,6\}$, $\{-4,3,-3,4\}$, $\{0,6,-3,-3\}$, $\{0,-3,6,-3\}$, $\{2,4,-3,-3\}$, $\{2,-3,4,-3\}$, $\{-1,1,-3,3\}$, $\{-1,-3,1,3\}$, $\{3,-2,-2,1\}$ twice, $\{3,-2,1,-2\}$, $\{3,1,-2,-2\}$, $\{-3,3,0,0\}$, $\{-3,0,3,0\}$, $\{0,0,0,0\}$, and $\{2,0,0,-2\}$. In the above notation, the first member of the set is the expectation of the subtriangle chosen from the first triangle and so forth. The first member of the partition is from the forward triangle and the last three members are each from the three different backward triangles. Note that each subtriangle from each triangle is used exactly once and the sum of the expectations of each set is zero.

The case where three triangles are forward and one triangle is backward is symmetric. We just flip the signs of the offsets in the list for combining triangles.

The case where two triangles are forward and two triangles are backward is simple. Pair a forward and a backward triangle. Match each subtriangle of one triangle with expectation e with a subtriangle of the other triangle with expectation $-e$. Put any two such pairs together into a set. ■

Lemma 4: The items drawn from a triangular distribution with expectation $1/p$, $p = 4$, have an optimum packing ratio of one.

Proof: We proceed as in Lemma 2. The maximum sum of items drawn one from each member of a set where the sum of the expectations of the set is one is $cp + 4(2w_k/3) = 1 + 8w_k/3$.

Shift Triangle L over to the left $2w_k/3$ units. The maximum sum of one item from each triangle in any set formed is $p(1/p - 2w_k/3) + 8w_k/3 = 1$. The rest of the argument for $p = 4$ is exactly the same as for $p = 3$. ■

Triangles with expectation of 1/5

Lemma 5: For a triangular density function with expectation c and $k \geq 1$, we can divide it into 5^{2k} subtriangles such which can be partitioned into 5^{2k-1} sets of five triangles each such that the sum of the expectations of the subtriangles in each set is $5c$.

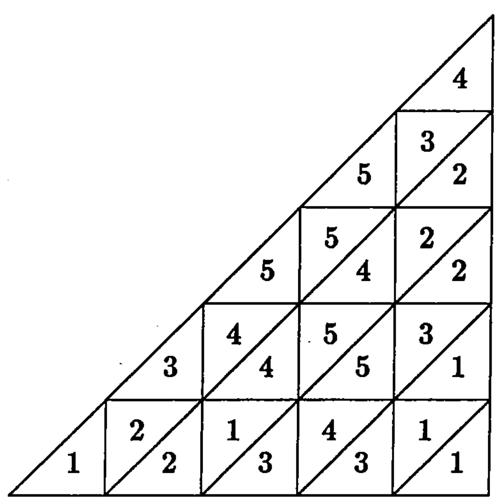


Figure 6.

Proof: Again without loss of generality, we will deal with the translated triangle, setting $x_2 = 0$ and $w_1 = 3$. Under this translation, we again desire the sum of the expectations of the subtriangles to be zero. The $k = 1$ level of the recursion is illustrated in Figure 6. The subtriangles with their shifts in expectation are shown in first two columns of Table 4 and the sets whose expectations sum to zero are: $\{-8, -3, 3, 4, 4\}$ (Set 1), $\{-6, -5, 3, 4, 4\}$ (Set 2), $\{-5, -2, 1, 3, 3\}$ (Set 3), $\{-3, -2, 0, 1, 4\}$ (Set 4), and $\{-2, 0, 0, 1, 1\}$ (Set 5).

expectation	subtriangles of forward triangles	subtriangles of backward triangles
−8	1	
−6	1	
−5	2	
−4		5
−3	2	4
−2	3	
−1		4
0	3	3
1	4	
2		3
3	4	2
4	5	
5		2
6		1
8		1

Table 4.

The recursive step works analogously to that in Lemma 1, provided again that all five triangles have the same orientation, i.e. either all are forward triangles, or all are backward triangles.

For the other combinations we will proceed as in Lemma 3. Table 4 gives the shift in expectation for each of the subtriangles. It suffices to choose sets of five subtriangles each such that each subtriangle is in exactly one set, each set contains one subtriangle from each of the five triangles, and the shifts in expectation for each set sum to zero. If two triangles are backward and three triangles are forward, e.g. Set 1, then we can combine the subtriangles in the following manner (Again, the first element is from the first triangle and so on. The first three members are from the forward triangles and the last two members are from the backward triangles.): $\{-8, -8, 0, 8, 8\}$, $\{0, -6, -6, 6, 6\}$, $\{-5, 0, -5, 5, 5\}$ twice, $\{-2, 4, -8, 3, 3\}$, $\{-6, -3, 3, 3, 3\}$, $\{4, 3, 1, -4, -4\}$ four times, $\{3, 1, 4, -4, -4\}$, $\{4, 4, -2, -3, -3\}$, $\{-3, -5, 4, 2, 2\}$ twice, $\{3, 4, -3, -3, -1\}$, $\{3, -2, 3, -1, -3\}$, $\{-2, 4, 4, -3, -3\}$ twice, $\{3, -2, -3, 0, 2\}$, $\{1, -3, 3, -1, 0\}$, $\{1, -2, 3, -1, -1\}$, $\{0, 1, -2, 2, -1\}$, $\{1, 1, -2, 0, 0\}$, $\{1, 1, 0, -1, -1\}$, and $\{0, 0, 0, 0, 0\}$. The case where two triangles are forward and three triangles are backward is done similarly; we just flip the signs of the offsets in Table 4 and in the sets derived from them.

The case where four triangles are forward and one triangle is backward is again done similarly. Combine the triangles in the following sets (Again, the first element is from the first triangle and so on. The first four members are from the forward triangles and the last member is the backward triangle.): $\{-8,-8,4,4,8\}$, $\{4,4,-8,-6,6\}$, $\{-5,4,4,-8,5\}$, $\{-5,3,-6,3,5\}$, $\{3,-6,3,4,-4\}$, $\{-6,3,3,4,-4\}$, $\{4,-5,4,1,-4\}$ twice, $\{4,4,1,-5,-4\}$, $\{1,4,-5,3,-3\}$ twice, $\{3,1,4,-5,-3\}$, $\{-2,-2,1,4,-1\}$ twice, $\{4,1,-2,-2,-1\}$, $\{0,3,3,-3,-3\}$, $\{3,-3,-3,0,3\}$ twice, $\{-3,3,0,-2,2\}$, $\{-3,0,3,-2,2\}$, $\{0,0,-2,3,-1\}$, $\{1,0,0,-3,2\}$, $\{-2,1,1,0,0\}$, $\{1,-2,0,1,0\}$, and $\{0,1,-2,1,0\}$. The case where four triangles are backward and one triangle is forward is done analogously, again by flipping the signs of the offsets and the signs inside the sets. The above partitions exhaust the cases which can occur in the recursive step. ■

Lemma 6: The items drawn from a triangular distribution with expectation $1/p$, $p = 5$, have an optimum packing ratio of one.

Proof: The maximum sum of items drawn one from each member of a set where the sum of the expectations of the set is one is $1 + 10w_k/3$.

Shift Triangle L over to the left $2w_k/3$ units. The maximum sum of one item from each triangle in any set formed is now 1. The rest of the argument for $p = 5$ is exactly the same as for $p = 3$ in Lemma 2. ■

Triangles with expectation $1/p$, $p \geq 3$, p integer

The following corollary is an extension of the above lemmas which will allow us some variation in the capacity of the bins.

Corollary: A triangle with expectation α can be packed in bins of size $p\alpha$, $p = 3,4,5$ with an optimum packing ratio of one.

Proof: Simply shift the coordinates for Lemmas 2,4, and 6. ■

Theorem: Any triangle with expectation $1/p$, with $p \geq 3$ and p integer, can be packed with an optimum packing ratio of one.

Proof: Any integer $p \geq 3$ can be expressed as $p = 3c_3 + 4c_4 + 5c_5$ with c_i a non-negative integer, $i = 3,4,5$. We will divide each bin into sub-bins and pack each sub-bin separately.

First we will divide the triangle T into three smaller triangles, T_3, T_4, and T_5, as shown in Figure 7. The division is such that the areas of T_3, T_4, and T_5 are $(c_3+c_4+c_5)^{-1}$ times, respectively, c_3, c_4, and c_5. Specifically, if T has the corners $\{(a,0),(b,0),(b,2/(b-a))\}$, then T_3 has corners $\{(a,0),(b,0),(b,kc_3)\}$, T_4 has corners $\{(a,0),(b,kc_3),(b,kc_3+kc_4)\}$, and T_5 has corners $\{(a,0),(b,kc_3+kc_4),(b,2/(b-a))\}$, where $k = 2(c_3+c_4+c_5)^{-1}(b-a)^{-1}$.

Further subdivide each triangle T_i into c_i smaller triangles T_{ij}, $i = 3,4,5$, $j = 1,...,c_i$ as shown in Figure 8. Each of the triangles T_{ij} has expectation $1/p$. These can also be considered conditional density functions and packed according to the strategies of Lemmas 2,4, and 6. Pack each of the triangles T_{ij} into sub-bins of size i/p, $i = 3,4,5$. The optimum packing ratio is one for each sub-bin, by the corollary. To form a bin of size 1, combine c_3 sub-bins of size $3/p$, c_4 sub-bins of size $4/p$, and c_5 sub-bins of size $5/p$.

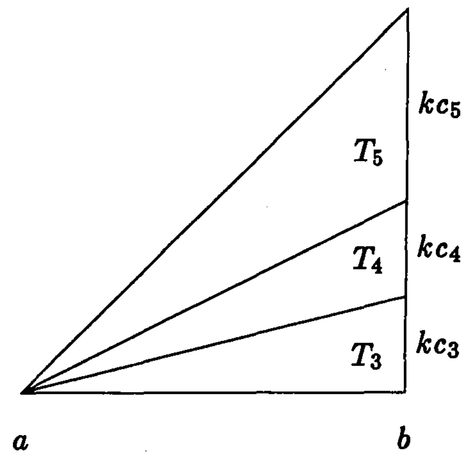
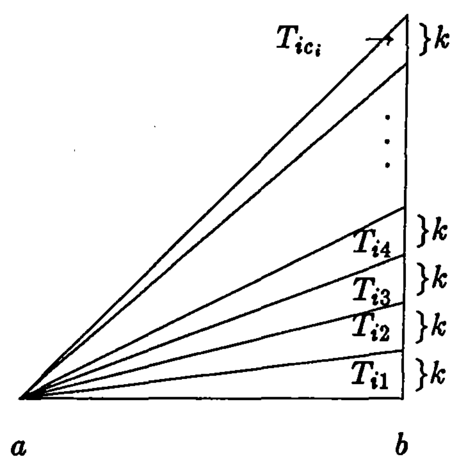


Figure 7.



Triangle T_i

Figure 8.

This implies that the optimum packing ratio for the entire triangle into bins of size 1 is one. ∎

Triangles with expectation of 1/2

It is easily seen that triangles with expectation of 1/2 have an optimum packing ratio of more than one. Note that a triangle may be either forward or backward. Observe that a forward triangle with expectation 1/2 has 5/9 of its area to the right of the line $x = 1/2$. This means that 5/9 of the items are bigger than 1/2, and thus no two of them can be placed in the same bin. Assuming that we can use the smaller items to fill the partially full bins without wasted space, we would still have at least 1/9 of the items packed in individual bins. This means that the optimum packing ratio for a forward triangle with expectation one is at least 10/9. A backward triangle with expectation 1/2 has 1/9 of its area to the right of the line $x = 1 - a$. This means that 1/9 of the items are so big that they cannot be combined with any other items and still fit into one bin, so they must be placed into individual bins. Even if the other items could be packed without wasted space, these items packed individually are enough to make the optimum packing ratio more than one.

Conclusion

We have shown strategies which achieve the optimum packing ratio of one for items drawn from triangular density functions with expectation $1/p$ for any integer $p \geq 3$.

Open Questions

How can we use this theorem to construct optimum packing strategies for triangular functions with other expectations? Initial investigation suggests that this can form a basis for the determination of optimum packing ratios for classes of intervals, $[a, b]$, for triangular functions in general, much like Lueker and Shor did for the uniform density case.

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