Title

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Publication Date

2024-08-16

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August 15, 2024

Abstract

We analyze both finite and infinite systems of Riccati equations derived from stochastic differential games on infinite networks. We discuss a connection to the Catalan numbers and the convergence of the Catalan functions by Fourier transforms.

Keywords: Catalan functions, Riccati equation for periodic network, Stochastic differential games for infinitely many players

1 Introduction

The Catalan numbers C_n, $n \ge 0$ appear as a sequence of natural numbers defined by

$$C_n := \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix} = \frac{(2n)!}{n!\,(n+1)!}, \quad n \ge 0. \tag{1.1}$$

For example, $C_0 = 1$, $C_1 = 1$, $C_2 = 2$ and so on. This increasing sequence satisfies the recurrence relations

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0 = \sum_{j=1}^{n} C_{j-1} C_{n-j}, \quad n \ge 1 \tag{1.2}$$

and grows like $4^n n^{-3/2} / \sqrt{\pi}$, as $n \to \infty$. The Catalan numbers appear in many combinatorial counting problems, for example, counting of non-crossing partitions, the number of the Dyck words, the number of standard Young tableaux (see the monographs [5], [6], [7] by Stanley).

In this paper we shall discuss the Catalan numbers and more generally Catalan functions in the context of the stochastic differential games on infinite network introduced in the recent papers (Feng, Fouque and Ichiba [1] and [2], see also the referenced papers therein for the related mean-field games, some topics of stochastic differential games and their applications), where the Catalan functions are defined by the solution to the system of the infinite Riccati equations. Note that the system of the infinite Riccati equations determines the Nash equilibrium of the stochastic differential game for infinitely many players. Then we prove the convergence of the solution of the finite Riccati equation corresponding to a stochastic differential game for finitely many players (say N players) on a periodic network, as $N \to \infty$, to the solution of a system of infinite Riccati equations.

Following Feng, Fouque and Ichiba [1], let us recall the following Riccati equation for the countably many continuous functions φ_t^i, $i \in \mathbb{N}_0$, $0 \le t \le T$, given by the system

$$\dot{\varphi}_t^i = \frac{\mathrm{d}\varphi_t^i}{\mathrm{d}t} = \sum_{j=0}^{i} \varphi_t^j \, \varphi_t^{i-j} - \varepsilon^i; \quad i \in \mathbb{N}_0, \tag{1.3}$$

where ε^i are given by some real constants $\varepsilon^0 := \varepsilon$, $\varepsilon^1 := -\varepsilon$, $\varepsilon^i = 0$ for $i \ne 0, 1$, and the terminal conditions are $\varphi_T^0 := c$, $\varphi_T^1 := -c$, $\varphi_T^i := 0$ for $i \ne 0, 1$. Here, " $\dot{}$ " denotes the differentiation with respect to t, and the superscript i is not the power of function ϕ but the index $i \in \mathbb{N}_0$. Given $\varepsilon > 0$ and $c \ge 0$, the solution $\{\varphi_t^i, i \in \mathbb{N}, 0 \le t \le T\}$ of (1.3) exists and is unique (Lemma 1 of [1]). We call such sequence of functions the *Catalan functions*.

The solution φ_t^i, $0 \le t \le T$, $i \in \mathbb{N}_0$ depends on ε and T. Particularly, we take $\varepsilon = 1 = \varepsilon^0 = -\varepsilon^1$, and consider the stationary solution by letting the time derivative zero, that is, $\dot{\varphi}_t^i \equiv 0$, $i \in \mathbb{N}_0$, $t \ge 0$. Then the stationary solution $\{\varphi^i\}_{i \in \mathbb{N}_0}$ of (1.3) satisfies

$$\varphi^0 = 1 , \quad \varphi^1 = -\frac{1}{2} , \quad \text{and} \quad \varphi^i = -\frac{1}{2} \sum_{j=1}^{i-1} \varphi^j \varphi^{i-j} ; \quad i \ge 2 .$$

Thus, the relation between the stationary solution $\{\varphi^i\}_{i \ge 1}$ of (1.3) and the Catalan numbers $\{C_i\}_{i \in \mathbb{N}_0}$ in (1.1) is

$$\varphi^i = -\frac{2C_{i-1}}{4^i} ; \quad i \ge 1 . \tag{1.4}$$

Let us also recall the Riccati equation for N continuous functions ϕ_t^i, $i = 0, 1, \ldots, N-1$, $0 \le t \le T$, given by the following system

$$\dot{\phi}_t^i := \frac{\mathrm{d}\phi_t^i}{\mathrm{d}t} = \sum_{j=0}^{N-1} \phi_t^j \phi_t^{N+i-j} - \varepsilon^i ; \quad t \ge 0 \tag{1.5}$$

of ordinary differential equations for $i = 0, 1, \ldots, N-1$ and $0 \le t \le T$ with the given terminal values $\phi_T^0 := c =: -\phi_T^1 > 0$, $\phi_T^i := 0$, $i = 2, \ldots, N-1$ and real constants $\varepsilon^0 := \varepsilon =: -\varepsilon^1 > 0$ and $\varepsilon^i := 0$ for $i = 2, \ldots, N-1$. We impose the periodic condition $\phi_\cdot^i = \phi_\cdot^{i+N}$ for every $i \in \mathbb{Z}$. The solution $\{\phi_t^i, i = 0, 1, \ldots, N-1, 0 \le t \le T\}$ of (1.5) exists uniquely and depends on N.

The finite system (1.5) leads us to the Nash equilibrium for the N-player, linear-quadratic stochastic differential game on the finite directed chain periodic network, while the infinite system (1.3) leads us to the Nash equilibrium for the infinitely many player, linear-quadratic stochastic differential game on the infinite directed chain network. In [1] and [2] the question of the convergence of the Nash equilibrium for the N-player game to the Nash equilibrium for the infinitely many player game was left as an open question in the periodic case considered here. In this paper we solve this open question positively.

The main results of this paper are the following propositions of convergence.

Proposition 1. *For any fixed $j \in \mathbb{N}_0$ and $t \in [0, T]$, the solution ϕ_t^j of the finite system* (1.5) *converges to φ_t^j of the infinite system* (1.3)*, as $N \to \infty$. That is,*

$$\lim_{N \to \infty} \phi_t^j = \varphi_t^j ; \quad j \in \mathbb{N}_0 , t \in [0, T] . \tag{1.6}$$

Proposition 2. *For any fixed $i \in \mathbb{N}_0$ and $t \in [0, T]$, we have the convergence results*

$$\lim_{N \to \infty} \sum_{j=0}^{N-1} \phi_t^j \phi_t^{N+i-j} = \sum_{j=0}^{i} \varphi_t^j \varphi_t^{i-j} , \quad \text{and} \quad \lim_{N \to \infty} \sum_{j=i+1}^{N-1} \phi_t^j \phi_t^{N+i-j} = 0 . \tag{1.7}$$

Proposition 3. *For any $K \in \mathbb{N}_0$, $T > 0$, the solution $\{\phi_t^i, i = 0, 1, \ldots, N-1, 0 \le t \le T\}$ of* (1.5) *and the solution $\{\varphi_t^i, i \in \mathbb{N}, 0 \le t \le T\}$ of* (1.3) *satisfy*

$$\lim_{N \to \infty} \sup_{0 \le i \le K} \sup_{0 \le t \le T} |\phi_t^i - \varphi_t^i| = 0 . \tag{1.8}$$

That is, the first K elements of the solution of (1.5) *converges uniformly to the first K elements of the solution of* (1.3)*, as $N \to \infty$.*

These results are proved in the following sections by Fourier transforms. The key observations are the representations (2.11) and (2.13) of the solutions $\{\phi_t^j\}$ and $\{\varphi_t^j\}$ of the Riccati equations (1.5) and (1.3) in terms of the solution $\{f_t(x)\}$ in (2.8) of an auxiliary Riccati equation (2.5) below.

After this manuscript was prepared, the recent papers [3] and [4] by Miana and Romero were brought up to our attention. In these papers a slightly general quadratic equation for Catalan generating functions, its spectrum and resolvent operator are studied from the point of view of functional analysis. In contrast to [3] and [4], the results here on the convergence of the solutions are more concrete, because of the specific form (1.3) of quadratic equation and because of the Fourier transforms. The generalization of the results in the current paper will be a theme of another paper.

2 Fourier transforms and Riccati equations

Let us define the discrete Fourier transform $\{\widehat{\phi}_t^k, k = 0, 1, \ldots, N-1\}$, $0 \le t \le T$ of the solution $\{\phi_t^i, i = 0, 1, \ldots, N-1, 0 \le t \le T\}$ of the Riccati equation (1.5) by

$$\widehat{\phi}_t^k := \sum_{j=0}^{N-1} \phi_t^j \exp\Big(-\frac{2\pi\sqrt{-1}\,jk}{N}\Big); \quad k = 0, 1, \ldots, N-1, 0 \le t \le T. \tag{2.1}$$

Here, the superscript k for $\widehat{\phi}_\cdot$ is not the power but the index. $\sqrt{-1}$ is the complex square root of -1. Inverting the discrete Fourier transform, we obtain

$$\phi_t^j = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{\phi}_t^k \exp\Big(\frac{2\pi\sqrt{-1}\,jk}{N}\Big); \quad j = 0, 1, \ldots, N-1, \tag{2.2}$$

and in particular,

$$\phi_t^0 = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{\phi}_t^k; \quad 0 \le t \le T. \tag{2.3}$$

Since the discrete Fourier transform of the convolution of two sequences is the product of their discrete Fourier transforms, it follows from the Riccati equation (1.5) that $\widehat{\phi}_t^k$ in (2.1) satisfies the one-dimensional Riccati equation

$$\dot{\widehat{\phi}}_t^k = (\widehat{\phi}_t^k)^2 - (1 - e^{-2\pi\sqrt{-1}k/N})\varepsilon; \quad 0 \le t \le T \tag{2.4}$$

with the terminal condition $\widehat{\phi}_T^k = (1 - e^{-2\pi\sqrt{-1}k/N})c$ for $k = 0, 1, \ldots, N-1$.

In a similar manner, replacing k/N by x in (2.4), let us consider the following, one-dimensional, auxiliary Riccati equation for $\mathbb{C}$-valued function $\{f_t(x), 0 \le t \le T, x \in [0,1]\}$ defined by

$$\dot{f}_t(x) = (f_t(x))^2 - (1 - e^{-2\pi\sqrt{-1}x})\varepsilon; \quad 0 \le t \le T, \quad x \in [0,1] \tag{2.5}$$

with the terminal condition $f_T(x) = (1 - e^{-2\pi\sqrt{-1}x})c$, $x \in [0,1]$.

Since both Riccati equations (2.4) and (2.5) are scalar-valued ordinary differential equations, we solve them explicitly by the standard method of solving the general Riccati equation of the form

$$\dot{y}_t = a_t + b_t y_t + c_t (y_t)^2; \quad 0 \le t \le T \tag{2.6}$$

with some (smooth) functions $a_\cdot, b_\cdot, c_\cdot$. That is, solving a second-order ordinary differential equation

$$\ddot{u}_t - \Big(b_t + \frac{\dot{c}_t}{c_t}\Big)\dot{u}_t + a_t c_t u_t = 0 \tag{2.7}$$

for $\{u_t\}$, we obtain the solution $y_t = -\dot{u}_t/(c_t u_t)$, $0 \le t \le T$ for the general Riccati equation. The solutions to our Riccati equations (2.4) and (2.5) are given by the following proposition.

Proposition 4. *The solution of the auxiliary Riccati equation* (2.5) *is given by*

$$f_t(x) = \sqrt{\varepsilon}\,\mathfrak{r}(x)\,e^{\sqrt{-1}\boldsymbol{\theta}(x)} \cdot \frac{\mathfrak{a}^+(x)\mathfrak{e}_t^+(x) - \mathfrak{a}^-(x)\mathfrak{e}_t^-(x)}{\mathfrak{a}^+(x)\mathfrak{e}_t^+(x) + \mathfrak{a}^-(x)\mathfrak{e}_t^-(x)}, \tag{2.8}$$

where $\mathfrak{a}^{\pm}(x)$ *and* $\mathfrak{e}_t^{\pm}(x)$ *are* $\mathbb{C}$*-valued functions defined by*

$$\mathfrak{a}^{\pm}(x) := \sqrt{\varepsilon} \pm c\,\mathfrak{r}(x)\,e^{\sqrt{-1}\boldsymbol{\theta}(x)}, \quad \mathfrak{e}_t^{\pm}(x) := \exp\Big(\pm\sqrt{\varepsilon}\,\mathfrak{r}(x)e^{\sqrt{-1}\boldsymbol{\theta}(x)}\,(T-t)\Big); \quad 0 \le t \le T \tag{2.9}$$

with

$$\mathfrak{r}(x) := [2(1-\cos(2\pi x))]^{1/4}, \quad \boldsymbol{\theta}(x) := \frac{1}{2}\arctan\Big(\frac{\sin(2\pi x)}{1-\cos(2\pi x)}\Big) \in [0,\pi) \tag{2.10}$$

for fixed $x \in [0,1]$.

Proof. For each fixed $x \in [0,1]$, we shall solve the Riccati equation (2.5) for $\{f_t(x)\}$, as the special case of the general Riccati equation (2.6) with $a_t := -(1-e^{-2\pi\sqrt{-1}x})\varepsilon$, $b_t := 0$, $c_t = 1$, $0 \le t \le T$. By the transformation from $y_\cdot$ in (2.6) to $u_\cdot$ in (2.7), it amounts to solving the second-order differential equation

$$\ddot{u}_t + (1-e^{-2\pi\sqrt{-1}x})\varepsilon u_t = 0; \quad 0 \le t \le T.$$

With the definitions (2.10) of $\mathfrak{r}(x)$ and $\boldsymbol{\theta}(x)$, the square roots of $-(1-e^{-2\pi\sqrt{-1}x})$ is given by $\pm\sqrt{-1}\mathfrak{r}(x)e^{\sqrt{-1}\boldsymbol{\theta}(x)}$. Hence, the solution $u_\cdot$ to the second-order differential equation is given by

$$u_t(x) = \mathfrak{c}_1(x)\cdot e^{\sqrt{-1}\mathfrak{r}(x)e^{\sqrt{-1}\boldsymbol{\theta}(x)}t} + \mathfrak{c}_2(x)\cdot e^{-\sqrt{-1}\mathfrak{r}(x)e^{\sqrt{-1}\boldsymbol{\theta}(x)}t}; \quad 0 \le t \le T$$

for some $\mathfrak{c}_i(x)$, $i = 1,2$ which are determined by the terminal condition $f_T(x) = -\dot{u}_T(x)\,/\,u_T(x)$, and $f_t(x) = -\dot{u}_t(x)\,/\,u_t(x)$ is given by (2.8) for $x \in [0,1]$, $t \in [0,T]$. □

Proposition 5. *With* $\{f_t(x)\}$ *defined in* (2.8)*, the solution of the Riccati equation* (2.4) *and the solution of the Riccati equation* (1.5) *are represented by*

$$\widehat{\phi}_t^k = f_t\Big(\frac{k}{N}\Big), \quad \text{and} \quad \phi_t^k = \frac{1}{N}\sum_{j=1}^{N} f_t\Big(\frac{k}{N}\Big)\exp\Big(2\pi\sqrt{-1}j\cdot\frac{k}{N}\Big) \tag{2.11}$$

for $k = 0,1,\dots,N-1$, $0 \le t \le T$. *Thus, there exists a constant* $c_T := \sup_{0\le t\le T}\sup_{x\in[0,1]}|f_t(x)| \in (0,\infty)$, *such that*

$$\sup_{N\ge 2}\sup_{0\le k\le N-1}\sup_{0\le t\le T}|\phi_t^k| \le \sup_{N\ge 2}\sup_{0\le k\le N-1}\sup_{0\le t\le T}|\widehat{\phi}_t^k| \le c_T. \tag{2.12}$$

Proof. For each fixed $k = 0,1,\dots,N-1$, we solve the Riccati equation (2.4) for the discrete Fourier transform $\widehat{\phi}_t^k$ and obtain $\widehat{\phi}_t^k = f_t(k/N)$ in a similar procedure, replacing k/N by x in the proof of Proposition 4. Substituting it to the inverse discrete Fourier transform (2.2), we obtain (2.11). The uniform bound (2.12) is obtained directly by the representations (2.11). □

In order to prove Proposition 1, we derive the following representation of the infinite Riccati solution $\{\varphi_t^k\}$ in terms of the auxiliary Riccati solution $\{f_t(x)\}$ in (2.8).

Proposition 6. *With the solution* $\{f_t(x)\}$ *in* (2.8) *of the auxiliary Riccati equation* (2.5)*, the solution* $\{\varphi_t^j\}$ *of the infinite Riccati equation* (1.3) *is represented as*

$$\varphi_t^j = \int_0^1 f_t(x)e^{2\pi\sqrt{-1}jx}\mathrm{d}x; \quad j \in \mathbb{N}_0, \ 0 \le t \le T. \tag{2.13}$$

Consequently, we have the upper bound

$$\sup_{j\in\mathbb{N}_0}\sup_{0\le t\le T}|\varphi_t^j| \le c_t = \sup_{0\le t\le T}\sup_{x\in[0,1]}|f_t(x)| \in (0,\infty). \tag{2.14}$$

Proof. Note that the family $\{e^{-2\pi\sqrt{-1}jx}, j \in \mathbb{N}_0\}$ of continuous functions on $[0,1]$ forms an orthonormal basis of the space $L^2([0,1])$, and the right hand of (2.13) is the j-th Fourier coefficient of f_t with respect to this orthonormal basis, that is,

$$f_t(x) = \sum_{j=0}^{\infty}\mathbf{c}_{j,t}e^{-2\pi\sqrt{-1}jx}, \quad \mathbf{c}_{j,t} := \int_0^1 f_t(y)e^{2\pi\sqrt{-1}jy}\mathrm{d}y; \quad x \in [0,1], t \in [0,T]. \tag{2.15}$$

To show (2.13), we shall verify that the Fourier coefficients $\{\mathbf{c}_{j,t}\}$ satisfy the infinite Riccati equation (1.3) and we apply its uniqueness of the solution. Since $\{f_t(x)\}$ satisfies the auxiliary Riccati equation (2.5), by the direct calculation we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 f_t(x)e^{2\pi\sqrt{-1}jx}\mathrm{d}x &= \int_0^1 \dot{f}_t(x)e^{2\pi\sqrt{-1}jx}\mathrm{d}x \\ &= \int_0^1 ((f_t(x))^2-(1-e^{-2\pi\sqrt{-1}x})\varepsilon)e^{2\pi\sqrt{-1}jx}\mathrm{d}x \\ &= \int_0^1 (f_t(x))^2 e^{2\pi\sqrt{-1}jx}\mathrm{d}x - \varepsilon\int_0^1 (1-e^{-2\pi\sqrt{-1}x})e^{2\pi\sqrt{-1}jx}\mathrm{d}x \\ &= \int_0^1 (f_t(x))^2 e^{2\pi\sqrt{-1}jx}\mathrm{d}x - \varepsilon^j\,, \quad j\in\mathbb{N}_0\,,\ t\in[0,T]\,,\end{aligned} \tag{2.16}$$

where $\{\varepsilon^j\}$ was defined as $\varepsilon^0 = \varepsilon = -\varepsilon^1 > 0$, and $\varepsilon^i = 0$, $i\geq 2$. For the first term of the right hand, it follows from (2.15) and the convolution of the Fourier series that

$$\begin{aligned}\int_0^1 (f_t(x))^2 e^{2\pi\sqrt{-1}jx}\mathrm{d}x &= \int_0^1 \Big(\sum_{\ell=0}^{\infty}\mathbf{c}_{\ell,t}e^{-2\pi\sqrt{-1}\ell x}\sum_{k=0}^{\infty}\mathbf{c}_{k,t}e^{-2\pi\sqrt{-1}kx}\Big)e^{2\pi\sqrt{-1}jx}\mathrm{d}x \\ &= \int_0^1 \Big(\sum_{k=0}^{\infty}\mathbf{b}_{k,t}e^{-2\pi\sqrt{-1}kx}\Big)e^{2\pi\sqrt{-1}jx}\mathrm{d}x = \mathbf{b}_{j,t} := \sum_{k=0}^{j}\mathbf{c}_{k,t}\mathbf{c}_{j-k,t} \\ &= \sum_{k=0}^{j}\Big(\int_0^1 f_t(x)e^{2\pi\sqrt{-1}kx}\mathrm{d}x\Big)\Big(\int_0^1 f_t(x)e^{2\pi\sqrt{-1}(j-k)x}\mathrm{d}x\Big)\,.\end{aligned} \tag{2.17}$$

Substituting this expression in (2.16), and because of (2.15), we obtain the infinite Riccati equation

$$\begin{aligned}\dot{\mathbf{c}}_{j,t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 f_t(x)e^{2\pi\sqrt{-1}jx}\mathrm{d}x \\ &= \sum_{k=0}^{j}\Big(\int_0^1 f_t(x)e^{2\pi\sqrt{-1}kx}\mathrm{d}x\Big)\Big(\int_0^1 f_t(x)e^{2\pi\sqrt{-1}(j-k)x}\mathrm{d}x\Big) - \varepsilon^j \\ &= \sum_{k=0}^{j}\mathbf{c}_{k,t}\mathbf{c}_{j-k,t} - \varepsilon^j\,; \quad j\in\mathbb{N}_0\,, 0\leq t\leq T\,,\end{aligned} \tag{2.18}$$

equivalent to (1.3). Also, the terminal condition is satisfied

$$\mathbf{c}_{T,j} = \int_0^1 f_T(x)e^{2\pi\sqrt{-1}jx}\mathrm{d}x = \int_0^1 c(1-e^{-2\pi\sqrt{-1}x})e^{2\pi\sqrt{-1}jx}\mathrm{d}x = c^j\,,$$

where $\{c^j\}$ was defined as $c^0 = c = -c^1 > 0$ and $c^i = 0$, $i\geq 2$. Thus, by the uniqueness of the solution to the infinite Riccati equation (1.3), we identify $\mathbf{c}_{j,t} = \varphi_t^j$, $j\in\mathbb{N}_0$, $t\in[0,T]$ as in (2.13). □

2.1 Proof of Proposition 1

Now we shall prove Proposition 1. Substituting (2.11) into the inverse discrete Fourier transform (2.2), we obtain the Riemann sum

$$\phi_t^j = \frac{1}{N}\sum_{k=0}^{N-1}\widehat{\phi}_t^k\exp\Big(\frac{2\pi\sqrt{-1}jk}{N}\Big) = \frac{1}{N}\sum_{k=0}^{N-1}f_t\Big(\frac{k}{N}\Big)\exp\Big(2\pi\sqrt{-1}j\cdot\frac{k}{N}\Big)$$

for $j = 0,1,\ldots,N-1$, $0\leq t\leq T$. Since $f_t(x)e^{2\pi\sqrt{-1}kx}$ is a continuous function of x for every fixed j and t, taking the limit as $N\to\infty$, we obtain the limit of ϕ_t^j,

$$\lim_{N\to\infty}\phi_t^j = \lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}f_t\Big(\frac{k}{N}\Big)\exp\Big(2\pi\sqrt{-1}j\cdot\frac{k}{N}\Big) = \int_0^1 f_t(x)e^{2\pi\sqrt{-1}jx}\mathrm{d}x = \varphi_t^j \tag{2.19}$$

for each fixed $j \in \mathbb{N}_0$ and $t \in [0, T]$, thanks to the identification in Proposition 6. $\square$

2.2 Proof of Proposition 2

The first part of the convergence results (1.7) is obtained in a similar manner as in the proof of Proposition 1. Indeed, using (2.2) and (2.11), we rewrite the sum as a Riemann sum, and then we take the limit, as $N \to \infty$,

$$\begin{aligned}
\sum_{j=0}^{N-1} \phi_t^j \phi_t^{N+i-j} &= \sum_{j=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} \widehat{\phi}_t^k e^{2\pi\sqrt{-1}kj/N} \cdot \frac{1}{N} \sum_{\ell=0}^{N-1} \widehat{\phi}_t^\ell e^{2\pi\sqrt{-1}(N+i-j)\ell/N} \\
&= \frac{1}{N^2} \sum_{k,\ell=0}^{N-1} f_t\Big(\frac{k}{N}\Big) f_t\Big(\frac{\ell}{N}\Big) \sum_{j=0}^{N-1} e^{2\pi\sqrt{-1}(k-\ell)j/N} \cdot e^{2\pi\sqrt{-1}i\ell/N} \\
&= \frac{1}{N^2} \sum_{k,\ell=0}^{N-1} f_t\Big(\frac{k}{N}\Big) f_t\Big(\frac{\ell}{N}\Big) \cdot N \cdot \mathbf{1}_{\{k=\ell\}} \cdot e^{2\pi\sqrt{-1}i\ell/N} \qquad (2.20) \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \Big[f_t\Big(\frac{k}{N}\Big)\Big]^2 e^{2\pi\sqrt{-1}i\ell/N} \\
&\xrightarrow[N\to\infty]{} \int_0^1 (f_t(x))^2 e^{2\pi\sqrt{-1}ix} \mathrm{d}x = \sum_{j=0}^{i} \mathbf{c}_{j,t}\mathbf{c}_{i-j,t} = \sum_{j=0}^{i} \varphi_t^j \varphi_t^{i-j}
\end{aligned}$$

for every $t \in [0, T]$ and $i \geq 0$, because of (2.13) and (2.17). Here, $\mathbf{1}_{\{k=\ell\}}$ is the indicator function which takes 1 on the set $k = \ell$ and 0, otherwise, and $\mathbf{c}_{\cdot,t}$ was defined in (2.15). This proves the first part of the convergence results (1.7).

For the second part of the convergence results, combining the first part (2.20) with the convergence of $\{\phi_t^i\}$ in Proposition 1, we obtain

$$\sum_{j=i+1}^{N-1} \phi_t^j \phi_t^{N+i-j} = \sum_{j=0}^{N-1} \phi_t^j \phi_t^{N+i-j} - \sum_{j=0}^{i} \phi_t^j \phi_t^{N+i-j} \xrightarrow[N\to\infty]{} \sum_{j=0}^{i} \varphi_t^j \varphi_t^{i-j} - \sum_{j=0}^{i} \varphi_t^j \varphi_t^{i-j} = 0. \qquad (2.21)$$

Therefore, we conclude the proof of Proposition 2. $\square$

2.3 Proof of Proposition 3

We shall evaluate the difference $D_N(t) := \sup_{0\leq i\leq K} \sup_{0\leq s\leq t} |\phi_s^i - \varphi_s^i|$, $0 \leq t \leq T$. With the time-reversal $\overline{\phi}_t^i := \phi_{T-t}^i$, $\overline{\varphi}_t^i := \varphi_{T-t}^i$, $0 \leq t \leq T$, it follows from the Riccati equations that for $i = 0, 1, \ldots, N-2$, $0 \leq t \leq T$,

$$\begin{aligned}
-\dot{\overline{\phi}}_t^i + \dot{\overline{\varphi}}_t^i &= \dot{\phi}_t^i - \dot{\varphi}_t = \sum_{j=0}^{N-1} \phi_t^j \phi_t^{N+i-j} - \sum_{j=0}^{i} \varphi_t^j \varphi_t^{i-j} \\
&= \sum_{j=i+1}^{N-1} \phi_t^j \phi_t^{N+i-j} + \sum_{j=0}^{i} [(\phi_t^j - \varphi_t^j)\phi_t^{i-j} + \varphi_t^j(\phi_t^{i-j} - \varphi_t^{i-j})] \\
&= \sum_{j=i+1}^{N-1} \overline{\phi}_t^j \overline{\phi}_t^{N+i-j} + \sum_{j=0}^{i} [(\overline{\phi}_t^j - \overline{\varphi}_t^j)\overline{\phi}_t^{i-j} + \overline{\varphi}_t^j(\overline{\phi}_t^{i-j} - \overline{\varphi}_t^{i-j})].
\end{aligned}$$

Since we have $\overline{\phi}_0^i = \phi_T^i = \varphi_T^i = \overline{\varphi}_0^i$, integrating both sides over $[0, s](\subseteq [0, T])$, taking the absolute values and using the triangle inequality, we obtain

$$|\overline{\phi}_s^i - \overline{\varphi}_s^i| \leq \int_0^s \Big|\sum_{j=i+1}^{N-1} \overline{\phi}_u^j \overline{\phi}_u^{N+i-j}\Big| \mathrm{d}u + \int_0^s \sum_{j=0}^{i} [|\overline{\phi}_u^j - \overline{\varphi}_u^j| \cdot |\overline{\phi}_u^{i-j}| + |\overline{\varphi}_u^j| \cdot |\overline{\phi}_u^{i-j} - \overline{\varphi}_u^{i-j}|] \mathrm{d}u \qquad (2.22)$$

Then the difference $D_N(t)$ satisfies the inequality

$$\begin{aligned} D_N(t) \,&=\, \sup_{0\le i\le K}\sup_{0\le s\le t}|\phi^i_s-\varphi^i_s| \,=\, \sup_{0\le i\le K}\sup_{0\le s\le t}|\overline{\phi}^i_s-\overline{\varphi}^i_s| \\ &\le \int_0^t \sup_{0\le i\le K}\sup_{0\le u\le s}|\sum_{j=i+1}^{N-1}\overline{\phi}^j_u\overline{\phi}^{N+i-j}_u|\mathrm{d}u \\ &\quad+\int_0^t \sup_{0\le i\le K}\sup_{0\le u\le s}\max(|\overline{\phi}^i_u|\,,|\overline{\varphi}^i_u|)D_N(s)\mathrm{d}s \\ &\le c_{N,1}(t)+\int_0^t c_{N,2}(s)D_N(s)\mathrm{d}s\,, \end{aligned} \tag{2.23}$$

where we defined

$$c_{N,1}(t) \,:=\, t\cdot\sup_{0\le i\le K}\sup_{T-t\le u\le T}|\sum_{j=i+1}^{N-1}\phi^j_u\phi^{N+i-j}_u|\le c_{N,1}(T)\,,$$

$$c_{N,2}(t) \,:=\, K\cdot\sup_{0\le i\le K}\sup_{T-t\le u\le T}\max(|\phi^i_u|\,,|\varphi^i_u|)\le c_{N,2}(T)<\infty$$

for $0\le t\le T$. Note that by (2.12) and (2.14), we have $\sup_N c_{N,2}(T)<\infty$. Applying the Gronwall inequality, we obtain

$$D_N(T)\le c_{N,1}(T)\exp\Big(\int_0^T c_{N,2}(t)\mathrm{d}t\Big)\,. \tag{2.24}$$

Since the function $f_\cdot(\cdot)$ is bounded, we may refine the proof of Propositions 1-2. Particularly, the approximation of the Riemann sum in (2.20) is uniform over $i=0,1,\ldots,K$ and over $[0,T]$. Thus, we obtain

$$\lim_{N\to\infty}c_{N,1}(T) \,=\, \lim_{N\to\infty}T\cdot\sup_{0\le i\le K}\sup_{0\le u\le T}|\sum_{j=i+1}^{N-1}\phi^j_u\phi^{N+i-j}_u| \,=\, 0\,.$$

Therefore, combining this with (2.24), we conclude the proof of Proposition 3:

$$\lim_{N\to\infty}\sup_{0\le i\le K}\sup_{0\le t\le T}|\phi^i_t-\varphi^i_t| \,=\, \lim_{N\to\infty}D_N(T) \,\le\, \lim_{N\to\infty}c_{N,1}(T)\exp\Big(\int_0^T c_{N,2}(t)\mathrm{d}t\Big) \,=\, 0\,.$$

□

As a consequence of Proposition 3, we have the following corollary which resolves the open question left in [1].

Corollary 2.1. *The N-player Nash equilibrium of linear quadratic stochastic differential games on the directed chain periodic network in [1] converges to the infinitely many player Nash equilibrium of linear quadratic stochastic differential games on the infinite directed chain network in [1].*

Acknowledgement

Part of research was supported by National Science Foundation NSF DMS-2008427. T.I. would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Stochastic Systems for Anomalous Diffusions (SSD), supported by EPSRC grant no EP/K032208/1, where a part of work on this paper was undertaken.

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3 Appendix

3.1 Finite system solved by matrix Riccati equation

The above Riccati equation (1.5) can be written as a matrix Riccati equation

$$\dot{\Phi}(t) = \Phi(t)\Phi(t) - \mathbf{E}\,, \quad \Phi(T) := \mathbf{C}\,, \tag{3.1}$$

where $\Phi(\cdot)$ is the $N \times N$ matrix-valued function $\Phi(t) := (\Phi_{i,j}(t))_{0 \le i,j \le N-1}$, $0 \le t \le T$ with $\Phi_{i,j}(t) := \phi_t^{i-j}$ for $0 \le i, j \le N-1$ with the condition $\phi_\cdot^i = \phi_\cdot^{i+N}$ for every $i \in \mathbb{Z}$ and $\mathbf{E}$ is an $N \times N$ matrix given by

$$\Phi(t) := \begin{pmatrix} \phi_t^0 & \phi_t^{N-1} & \cdots & & \phi_t^1 \\ \phi_t^1 & \phi_t^0 & \ddots & & \phi_t^2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \phi_t^{N-1} \\ \phi_t^{N-1} & \cdots & & \phi_t^1 & \phi_t^0 \end{pmatrix}, \quad \mathbf{E} := \begin{pmatrix} \varepsilon & 0 & \cdots & 0 & -\varepsilon \\ -\varepsilon & \varepsilon & \ddots & \ddots & 0 \\ 0 & -\varepsilon & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\varepsilon & \varepsilon \end{pmatrix},$$

and the $N \times N$ matrix $\mathbf{C}$ determines the terminal condition

$$\mathbf{C} := \begin{pmatrix} c & 0 & \cdots & 0 & -c \\ -c & c & \ddots & \ddots & 0 \\ 0 & -c & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -c & c \end{pmatrix}.$$

Here, $\dot{\Phi}(t)$ stands for the element wise differentiation of $\Phi(t)$ with respect to t.

Let us consider the time reversal parametrized by $\tau := T - t$ and $\Psi(\tau) := \Phi(T - \tau)$, $0 \le t \le T$, $0 \le \tau \le T$. Then the matrix-valued Riccati equation is

$$\dot{\Psi}(\tau) = -\Psi(\tau)\Psi(\tau) + \mathbf{E} \tag{3.2}$$

for $0 \le \tau \le T$ with the initial value $\Psi(0) := \mathbf{C}$. Its solution is given by

$$\Psi(\tau) = (\mathbf{O}_{21}(\tau) + \mathbf{O}_{22}(\tau)\mathbf{C})(\mathbf{O}_{11}(\tau) + \mathbf{O}_{12}(\tau)\mathbf{C})^{-1}\,, \tag{3.3}$$

where $\mathbf{O}_{ij}(\cdot)$, $1 \le i, j \le 2$ are the $N \times N$ block matrix elements of $\mathbf{O}(\cdot)$ defined by

$$\mathbf{M} := \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{E} & \mathbf{0} \end{pmatrix}, \quad \mathbf{O}(\tau) := \begin{pmatrix} \mathbf{O}_{11}(\tau) & \mathbf{O}_{12}(\tau) \\ \mathbf{O}_{21}(\tau) & \mathbf{O}_{22}(\tau) \end{pmatrix} := \exp(\mathbf{M}\tau), \tag{3.4}$$

for $0 \le \tau \le T$. Here $\mathbf{0}$ is $N \times N$ zero matrix and $\mathbf{I}$ is $N \times N$ identity matrix. Thus, we obtain the solution to the Riccati equation (1.5) as the first column of $\Phi(t) = \Psi(T - t)$ for $0 \le t \le T$.

The characteristic polynomial of the $2N \times 2N$ matrix $\mathbf{M}$ in (3.4), in terms of $\lambda \in \mathbb{C}$, is simply given by

$$\det(\lambda \mathbf{I} - \mathbf{M}) \,=\, (\lambda^2 - \varepsilon)^N - (-\varepsilon)^N, \tag{3.5}$$

and hence the eigenvalues are

$$\lambda \,=\, \pm\sqrt{\varepsilon \cdot \Big(1 - \exp\Big(\sqrt{-1} \cdot \frac{2\pi\, k}{N}\Big)\Big)}\,; \quad k \,=\, 0, 1, \dots, N-1\,,$$

and $\lambda = 0$ has multiplicity of 2. Thus, the size of the eigenvalues is bounded by $\sqrt{2\varepsilon}$. For example, in the case of $N = 4$, the eight eigenvalues are

$$\{0, 0, \pm\sqrt{(1 + \sqrt{-1})\varepsilon}\,, \pm\sqrt{(1 - \sqrt{-1})\varepsilon}\,, \pm\sqrt{2\varepsilon}\,\}\,.$$

The direct numerical calculation of (3.3) is not stable for a large τ, because of multiple eigenvalues. It is often suggested (e.g., Vaughan [8]) to calculate iteratively

$$\Psi((k+1)\Delta\tau) = (\mathbf{O}_{21}(\Delta\tau) + \mathbf{O}_{22}(\Delta\tau)\Psi(k\Delta\tau))(\mathbf{O}_{11}(\Delta\tau) + \mathbf{O}_{12}(\Delta\tau)\Psi(k\Delta))^{-1}\,; \quad k \,=\, 0, 1, 2, \dots$$

with $\Psi(0) = \mathbf{C}$, where $\Delta\tau$ is set to be small.

3.2 Generating function for infinite Riccati equation

For the infinite system (1.3) let us recall the generating function $S_t(z) := \sum_{k=0}^{\infty} z^k \varphi_t^k$ for $\varphi_\cdot^k$, $k = 0, 1, 2, \dots$ satisfies the scaler Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t} S_t(z) \,=\, [S_t(z)]^2 - \varepsilon(1 - z)\,, \quad 0 \le t \le T\,, \quad S_T(z) \,=\, c(1 - z)$$

for $|z| < 1$. As in Proposition 4, the solution to this Riccati equation is given by

$$S_t(z) \,=\, \sqrt{\varepsilon(1 - z)} \cdot \frac{\overline{\mathfrak{a}}^{+}\overline{\mathfrak{e}}_t^{+} - \overline{\mathfrak{a}}^{-}\overline{\mathfrak{e}}_t^{-}}{\overline{\mathfrak{a}}^{+}\overline{\mathfrak{e}}_t^{+} + \overline{\mathfrak{a}}^{-}\overline{\mathfrak{e}}_t^{-}}\,,$$

where

$$\overline{\mathfrak{a}}^{\pm} \,:=\, \sqrt{\varepsilon(1 - z)} \pm c(1 - z)\,, \quad \overline{\mathfrak{e}}_t^{\pm} \,:=\, \exp\Big(\pm\sqrt{\varepsilon(1 - z)}(T - t)\Big)$$

for $0 \le t \le T$, $|z| < 1$.