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DEPARTMENT OF ECONOMICS

OWNER-OCCUPIED HOUSING AND THE COMPOSITION OF THE
HOUSEHOLD PORTFOLIO OVER THE LIFE CYCLE

BY

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Owner-Occupied Housing

and

the Composition of the Household Portfolio Over the Life Cycle

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Owner-Occupied Housing and

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ABSTRACT

The paper studies the impact of the portfolio constraint imposed by the consumption demand for housing (the "housing constraint") on the household's optimal holdings of financial assets. Since the ratio of housing to net worth declines as the household accumulates wealth, the housing constraint induces a life-cycle pattern in the portfolio shares of stocks and bonds. For reasonable degrees of risk aversion, the changes in portfolio composition over the life-cycle can be dramatic. For example, for a coefficient of relative risk aversion of 3, the ratio of stocks to net worth in the optimal portfolio is .09 for the youngest households (ages 18-30) and .60 for the oldest (age 70 and over).

Using data from the PSID on home values to construct household level panel data on the real after-tax return to owner-occupied housing, as well as data on the returns to financial assets, the paper estimates the vector of expected returns and the covariance matrix for the set of assets consisting of housing, mortgages, stocks, Treasury bonds, and T-bills. Numerical methods are used to calculate the mean-variance efficient frontier, conditional on different values of the housing constraint, and the optimal portfolios associated with different levels of relative risk aversion.

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The paper uses a mean-variance efficiency framework to examine the household's optimal portfolio problem when owner-occupied housing is included in the list of available assets. While it is straightforward to calculate the risk and return to housing, housing differs from stocks and bonds in a crucial way: since the household's ownership of residential real estate determines the level of its consumption of housing services, the household's demand for real estate is 'over-determined' in the sense that the level of real estate ownership, which is optimal from the point of view of the consumption of housing services, may differ from the optimal level of housing stocks from a portfolio point of view. In the absence of tax distortions and transactions costs, rental markets for housing would permit the household to separate its level of consumption of housing services from its investment in housing as an asset. We assume, instead, that the preferential tax treatment of owner occupied housing, and the transactions costs and agency costs involved in the rental market for housing create frictions large enough to effectively constrain households to include in their asset portfolio the level of housing consistent with their consumption demand for housing. For example, a young family will generally hold a portfolio consisting of a house worth several times their net worth, a large mortgage, and small amounts of financial assets; that is, their consumption demand for housing services causes the household to hold a larger position in real estate than they would hold on the basis of the portfolio demand for real estate.

Using data from the PSID on home values to construct household level panel data on the real after-tax return to owner-occupied housing, as well as data on the returns to financial assets, the paper estimates the vector of expected returns and the covariance matrix for the set of assets consisting of housing, mortgages, stocks, Treasury bonds, and T-bills. Considering purely as an asset, the inclusion of owner-occupied housing dramatically improves the efficient frontier because the return to housing is essentially uncorrelated with the return to stocks.

The paper focuses on the impact of the portfolio constraint imposed by the consumption demand for housing (the "housing constraint") on the household's optimal holdings of financial assets. Since the ratio of housing to net worth declines as the household accumulates wealth, the housing constraint induces a life-cycle pattern in the portfolio shares of stocks and bonds. Young households, which typically have large holdings of real estate relative to their net worth, are highly leveraged and therefore forced into a situation of high portfolio risk. As a result, these young households have a strong incentive to reduce the risk of their portfolio by using their net worth to either pay down their mortgage or buy bonds instead of buying stocks. In contrast, the optimal portfolio share of stocks is larger for older households with lower ratios of housing to net worth. For reasonable degrees of risk aversion, the changes in portfolio composition over the life cycle can be dramatic. For example, for a coefficient of relative risk aversion of 3, ratio of stocks to net worth in the optimal portfolio is .09 for the youngest households (18-30) and .60 for the oldest (70 and over).

Based on the estimated covariance matrix and an assumed value of the coefficient of relative risk aversion, we can calculate the optimal stock to net worth ratio as a function of the household's house value to net worth ratio. The empirical section of the paper compares a scatter plot of the stock to net worth ratio against the house value to net worth ratio for data from the 1989 wave of the PSID to the model's predictions for coefficients of relative risk aversion ranging from one to four. Since wealth data were collected for two waves of the PSID we also construct data on the change in the housing to net worth ratio and the change in the stock to net worth ratio over a five year interval and ask whether the cross-sectional relationship between the changes in the stock and housing ratios is consistent with the model.

I.   Risk Characteristics of Housing and the Efficient Frontier

In this section, we consider owner-occupied housing and mortgage instruments as assets, and characterize their risk characteristics by estimating their expected returns and covariance with standard financial assets, i.e. short-term Treasury bills, long (20-year) Treasury bonds, and stocks (as measured by the S&P 500). Of the vast literature on efficient portfolios, only a few papers incorporate real estate as an asset. Goetzmann and Ibbotson (1990) and Goetzmann (1993) used regression estimates of real estate price appreciation, and Ross and Zisler (1991) calculated returns from real estate investment trust funds, to characterize the risk and return to real estate investment. While regionally diversified real estate funds have recently become available, the vast majority of households invest in real estate by purchasing a particular house, rather than by purchasing shares of a diversified real estate fund. Further, the returns to an investment in a real estate fund and the returns to an investment in one's personal residence receive very different tax treatment. In selecting a small number of important assets to derive an efficient frontier, the specification of the real estate asset will depend on the nature of the investor. In analyzing the efficient frontier facing a large institutional investor such as TIAA-CREF, Ross and Zisler have appropriately used diversified real estate funds to characterize the risk characteristics of real estate.

Our focus, in contrast, is on the portfolio problem faced by the typical household. We assume that households hold real estate in the form of a specific house (rather than investing in a diversified fund and renting to satisfy their demand for housing services) due to tax distortions and transactions or agency costs associated with renting, but do no explicitly model the renting vs. owning decision.

To estimate returns on housing assets, we use data from the 1968-1992 waves of the Panel Study of Income Dynamics. Every year, the PSID asks homeowners how much their

house would sell for if the house were put on the market on the date of the interview, enabling us to calculate the return to owner-occupied housing at the household level[1]. The return to housing depends on appreciation of the value of the house, the value of the flow of housing services, and costs of ownership and maintenance. Lacking direct observations on the rental value of the house and the maintenance costs, these components are modeled as follows:

$$D_t = (r + d)P_{t-1} + PropertyTax_t \qquad (1)$$

$$COM_t = dP_{t-1} + (1 - t)PropertyTax_t \qquad (2)$$

where r is the real interest rate, d is the depreciation rate, and τ is the marginal income tax rate. The imputed annual rental value (analogous to the dividend on a stock), denoted D_t, reflects the assumption that landlords fully shift property taxes onto renters. In the absence of expenditures on maintenance and repairs, physical depreciation (at rate d) would be reflected in the real value of the house (P_t). However, we assume that both landlords and homeowners spend on maintenance and repairs an amount equal to the annual depreciation of the house so that the physical condition of the house is constant. In addition to maintenance and repairs, the cost of ownership and maintenance (COM_t) includes the net property tax payment (i.e., net of the deduction against income taxes). Denoting the real return on housing as R_t[2]:

$$R_t = \frac{P_t + D_t - COM_t - P_{t-1}}{P_{t-1}} = \frac{P_t + rP_{t-1} + tPropertyTax_t - P_{t-1}}{P_{t-1}}. \qquad (3)$$

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[1] As a rough measure of the accuracy of such subjective housing value measures, Skinner (1994) compares annual rate of self-reported price changes with the objective Commerce Department measures and finds the two series are quite close in mapping house price changes over the 1970s and 1980s.

[2] A possible bias arises from the fact that the house price is not the actual transaction price but the owner's estimate. If there is a systematic bias in the owner's estimate of the house price, then the return calculation may also be biased. Using the panel aspect of the 1985 and 1987 waves of the American Housing Survey, Goodman and Ittner (1993) compare owners' estimates with subsequent sales prices and find that the mean error of the home owner's estimate is 8.3%, and that the errors are not correlated with characteristics of the house or demographic characteristics of the owners. Given Goodman and Ittner's estimate of the homeowners' bias, it is straightforward to show, using equation (3), that the mean overestimation of 8.3% of the real house price would give us a downward bias in the calculated rate of return of around 0.06%, assuming a 33% marginal tax rate and a 2.5% property tax rate.

In computing the real return to housing, the nominal house value and nominal property tax payments as reported by the respondent are converted to real terms using the CPI-U deflator to obtain P_t and Property Tax$_t$. The short term interest rate, r, is assumed to be a fixed 5%. In general, the marginal income tax rate, τ, will vary both cross-sectionally and across time. For most waves of the PSID, it is possible to impute the household's marginal tax rate; thus one could construct a series of real returns to housing using a household- and time- specific value of the marginal tax rate. However, since our purpose is to construct an estimate of the (time invariant) covariance matrix faced by a representative household, we instead compute the real return to housing using an assumed marginal income tax rate of 33% (28% Federal and 5% state).

In order to incorporate tax effects, all asset returns are stated in after-tax, real terms. In the case of the return to housing, the imputed rent portion of the return is not taxed. While capital gains on housing are taxable in some circumstances, we assume that the bulk of realized capital gains escape taxation due to rollover provisions, the one-time $125,000 exclusion for homeowners over 55 years, and the fact that inherited real estate receives a stepped-up cost basis when transferred. We therefore assume that neither the rental value nor the capital gains portion of the return is taxed, and interpret R_t as the after-tax real return to housing.

In calculating real after-tax mortgage rates, we assume that the household takes out a fixed rate mortgage in the year in which the house is purchased and does not refinance as long as they remain in the same house. Taking into account the fact that the nominal rather than real mortgage interest payments are deductible, households pay a real after-tax interest rate of:

$$\text{Mortgage}_t = \frac{1 + (1 - t)\,\text{Nominal Mortgage}_s}{1 + \text{Inflation}_t} - 1. \tag{4}$$

In equation (4), Nominal Mortgage$_s$ denotes the fixed rate at which the mortgage was issued when the home was purchased in year s and Mortgage$_t$ denotes the after-tax real interest rate on the mortgage in a subsequent year t.   In implementing equation (4), Nominal Mortgage$_s$ is measured by the annual average of conventional home mortgage rates charged by major lenders in year s, τ is the hypothetical marginal tax rate of 33%, and the inflation rate is measured by the CPI-U.

The return on Treasury bills is measured by constructing the holding period return generated by rolling over short term Treasury bills for a period of one year, assuming that the holding period return is taxed at a marginal rate (Federal only) of 28%, then converting the after-tax nominal rate into real terms. Similarly, the return to Treasury bonds is measured by deflating the after-tax (28%) holding period yield on 20 year Treasury bonds.

Data on the S&P500 is used to measure stock returns. Using separate data series on dividend income and capital gains, we assume that the dividend income is taxed at the rate of 33%, while the capital gains portion of the return escapes taxation completely. The after-tax nominal return is then converted to a real return. Note that by assuming that capital gains are not taxed, our measure of the after-tax return to stocks probably overstates the actual return.

Real after-tax mortgage rates and returns to housing are calculated for homeowners in the PSID from 1968 to 1992. Excluding the poverty subsample, all households that owned a house over at least a two-year period between 1968 and 1992 were included (1,817 households). For each household, the return to housing R_t was calculated using equation (3) if the household was living in the same house during year t and year t-1. If, in year t, the household moved to a new house or became a renter, the value of R_t was set to missing.

The calculation of returns to housing relies on the change in the reported value of the house between interviews, which are usually conducted between February and April. We

Table 1: Expected Returns and Covariance Matrix

	T-Bills	Bonds	Stocks	House	Mortgage
Mean Return (arithmetic)	-.0038	.0060	.0824	.0659	.0000
Standard Deviation	.0435	.0840	.2415	.1424	.0336
Covariance Matrix					
T-Bills	.0018920				
T-Bonds	.0025050	.0070613			
Stocks	.0002008	.0040381	.0583292		
House	-.000119	-.000067	-.000178	.020284	
Mortgage	.0007087	.0023854	.0025400	-.0000057	.0011274
Correlation Matrix					
T- Bills	1.0000				
T- Bonds	.68533	1.0000			
Stocks	.01912	.19897	1.0000		
House	-.03339	-.004506	-.000771	1.0000	
Mortgage	.84119	.680286	.467954	-.001192	1.0000

interpret the housing return series as measuring the return over the March-to-March interval, and construct the returns on financial assets for the same time intervals. While the data on housing returns and mortgage rates are limited to the 1968-92 period, data for a longer period (1926-92) are used for T-bills, T-bonds, and the S&P500. A detailed description of the data and calculation of rates of return is provided in Appendix 1.

The mean returns, standard deviations, and covariance matrix of the five assets are reported in Table 1. In calculating the sample statistics of mean returns on housing assets and their covariance with other financial assets, we assume that all returns are drawn from the same distribution regardless of region and year. As a robustness measure, we trimmed the top and bottom 2% of housing returns on the grounds that the tails contain some extreme values likely to reflect gross measurement errors.

As expected, stocks have the highest after-tax real rate of return at 8.2%, and Treasury bills the lowest, with negative 0.4%. The real return to housing at 6.6% is comparable to that of

stocks, while the return to housing has a substantially smaller standard deviation than the stock return. The after-tax real interest rate on a fixed mortgage is on average zero over this period. One striking characteristic of the covariance matrix is the extremely low and negative covariance of housing with other assets, including stocks, making owner-occupied housing an attractive asset for hedging fluctuations in financial assets. The finding that housing returns are essentially uncorrelated with stock returns is consistent with the findings of Goetzmann (1993) using repeated sales regression estimates. Summers (1981) provides a theoretical explanation for the negative correlation between returns on stocks and owner-occupied housing that arises from different tax treatment for gains from corporate capital income and owner-occupied housing.

Viewed purely as an asset, housing is an important component of the household's portfolio because it has an expected return and standard deviation comparable to (although smaller than) stocks, and because its return is essentially uncorrelated with the returns of the financial assets. Two different mean-variance efficient frontiers, drawn under the following assumptions, are plotted in Figure 1.

(i) The household can invest only in financial assets, i.e. T-bills, Treasury bonds, and stocks. Non-negativity constraints are imposed on the portfolio weights; thus households can lend, but not borrow, at the Treasury rates, and are not allowed to leverage their stock portfolios.

(ii) The household can invest in owner-occupied real estate and borrow with a fixed rate mortgage; the value of the mortgage cannot exceed the value of the house. The household can also invest in financial assets with the same non-negativity constraints as in the assumption (i).

The addition of housing and mortgages to the menu of available assets dramatically improves the efficient frontier. For example, a portfolio with standard deviation of 15% is

Figure 1: Efficient Frontier with and without Housing Assets
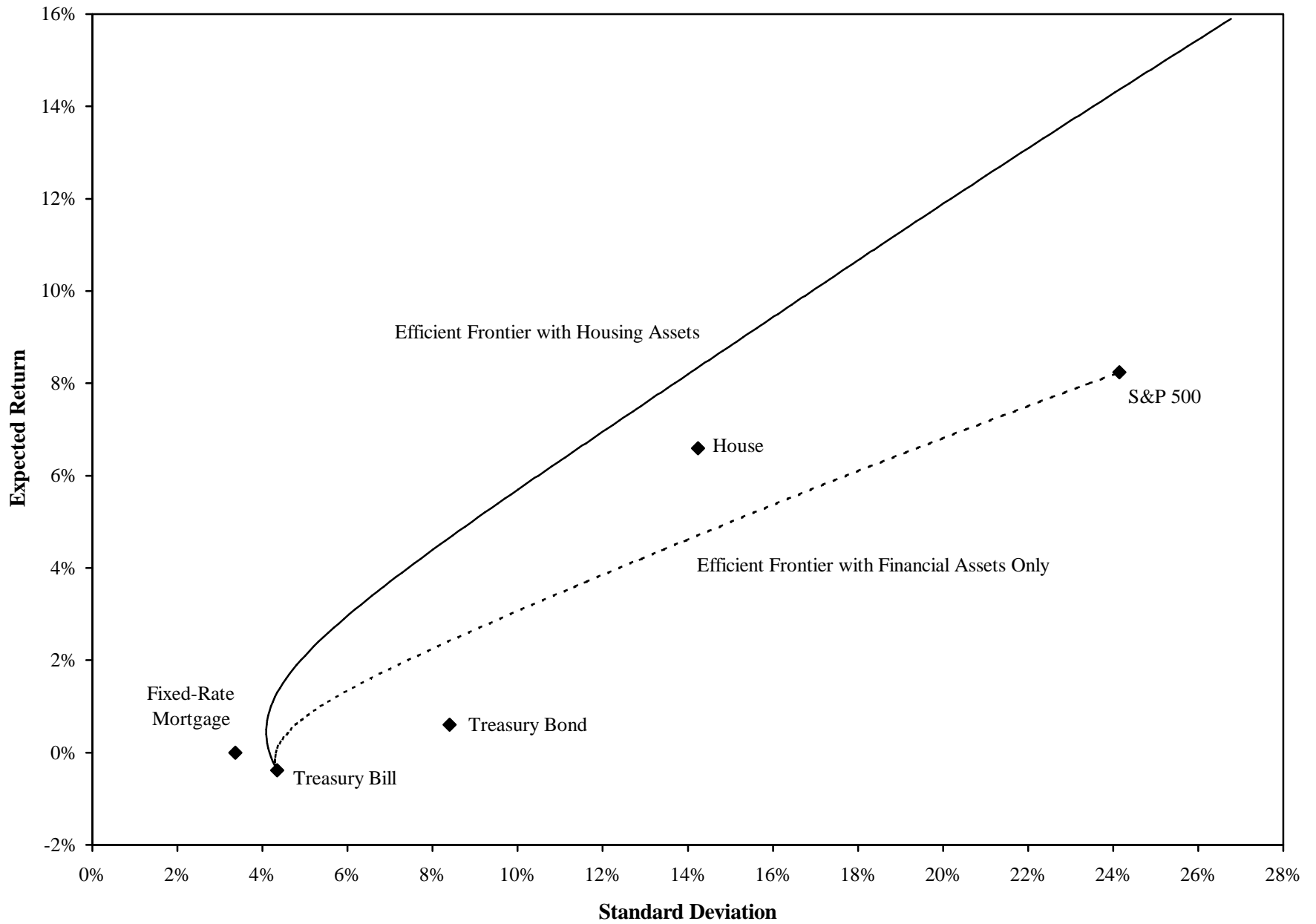
associated with an expected return of 5% if assets are restricted to financial assets only, but an expected return of 8.8% can be achieved with the same standard deviation when housing and mortgages are included. Figure 1 generally confirms the popular notion that home ownership is a good investment. Examination of portfolios on the frontier reveals that except for very low rates of expected return and risk, housing accounts for a considerable proportion of the optimal portfolio.

II.   The Life Cycle in Portfolio Composition

In the sense used by Modigliani, life cycle saving refers to the systematic accumulation of assets over the working years in conjunction with the potential decumulation of assets during retirement. If the household's portfolio consists purely of financial assets, under constant relative risk aversion, the composition of the optimal portfolio would not change as the household ages. If, however, other assets such as human capital and residential real estate are significant components of the household's wealth, changes in the risk and return characteristics of these assets over the life cycle would imply that the optimal proportions invested in financial assets could change over the life cycle. For example, as a young worker commits to an occupation, establishes a career and works toward retirement, labor income uncertainty tends to be resolved and the household may adjust its portfolio of financial assets by accepting more risk. Conversely, a retired worker who no longer has the option of varying his labor supply in response to adverse financial returns may want to hold a safer portfolio for precautionary purposes[3].

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[3]   Bodie, Merton and Samuelson (1992) model the investor's portfolio choice over the life cycle by incorporating labor supply flexibility. They show in a numerical simulation that the ability to vary labor supply ex post tends to induce the individual to assume greater risks in his investment portfolio ex ante. In other words, labor supply flexibility creates a kind of insurance against adverse investment outcomes, and the young with greater labor supply flexibility may take significantly greater investment risks than the old.

With rental markets for housing, a household can, in principle, divorce the size of its holdings of real estate assets from the level of housing services it consumes. However, due to tax distortions and transactions costs, rental housing is by no means a perfect substitute for owner-occupied housing. If the existence of tax distortions and market frictions implies that the household's homeownership is at least in part determined by its consumption demand for housing services, a life cycle pattern in homeownership (and mortgage liability) will induce a systematic life cycle pattern to holdings of financial assets[4].

In essence, the household's optimal level of housing is over-determined: the amount of housing it would hold purely on the basis of the consumption demand for housing services will seldom coincide with the amount of housing which is optimal for portfolio considerations. In a recent paper, Brueckner (1997) considers the interaction between the consumption demand and the investment demand for housing in a mean-variance portfolio model. He shows that when the amount of housing held for consumption purposes exceeds the amount which would be optimal for investment purposes, the household's overall portfolio is mean-variance inefficient in the sense that, in the absence of the constraint on housing, the household could achieve a higher

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[4]   Most of the previous studies of the life-cycle model have focused on age-wealth profiled and have been more concerned with the total size of a household's wealth over the life cycle than with its composition. Studies that have tested for changed in the composition of household portfolios over the life cycle have found conflicting evidence of significant age effects. Using data from 1978 Survey of Consumer Financial Decision, King and Leape (1987) find that the number of assets held by a household increases with age. Further, they find that the probability of owning stocks, corporate bonds, municipal bonds and savings bonds increases with age, while the probability of owning Treasury bonds declines with age. In order to explain this age effect, they reason that information on investment opportunities arrives randomly over time, and hence the probablility of owning 'information intensive' assets such as stocks will increase with age. Ioannides (1992), however, using the 1983 and 1986 Surveys of Consumer Finances, finds that coefficients on age are not significant at a 5% level in probit regressions of ownership of most financial assets, except for IRAs, housing equity and other debt. Ioannides controls for self-reported access to information about investment opportunities and behavioral characteristics related to investment behavior that are not available in King and Leape study, and finds that those variables do not contribute significantly to increasing probabilities of asset holdings, and therefore concludes that transaction costs rather than availability of information on investment opportunities should account for the low degree of portfolio diversification. Bertaut (1996), using the 1983 and 1989 Surveys of Consumer Finances, finds that the probability of owning stocks increases with age. Using the 1983, 1989, and 1992 Surveys of Consumer Finances, Poterba and Samwick (1997) find strong age effects on both the probability of ownership and the portfolio shares of stocks (directly held as well as all taxable equity including brokerage accounts and equity mutual funds) for the cohorts younger than 44 in 1983.

expected return without any increase in risk. That is, the consumption demand for housing constrains and distorts the portfolio decision.

Brueckner also considers the distortion to the housing consumption decision created by the portfolio motive. When the consumption motive for holding housing creates a portfolio distortion, the household treats the portfolio distortion as an additional cost and responds by reducing its consumption of housing services. We assume that the household takes into account the portfolio distortion created by its choice of housing consumption, but do not model the housing consumption decision explicitly. Instead, we take the household's level of housing consumption as given, and characterize the constrained optimal portfolios conditional on the level of housing consumption.

Our paper is very similar in spirit to Brueckner (1997) in that both papers use the mean-variance efficient portfolio framework to analyze the constraint imposed by the consumption demand for housing on the household's portfolio problem. However, the two papers are quite different in approach. Brueckner considers a general covariance matrix and mean vector of returns and a general utility function, and derives analytical results. Our approach is quantitative rather than analytical: we estimate the covariance matrix and vector of expected returns for housing and financial assets, we assume constant relative risk aversion utility and specific values of the relative risk aversion, and solve for the constrained optimal portfolios numerically.

To document the importance of housing assets in household's overall portfolio, Table 2 reports the mean asset-to-net worth ratio, by age group, of homeowners in 1989. Also reported in Table 2 is the expected return, μ, and standard deviation, σ, of each cohort's mean portfolio (calculated using the estimated expected returns and covariance matrix from Section I).

In Table 2 "Cash" refers to any money held in checking and savings accounts, money market account or Treasury bills and "Bonds" refers to any other savings including bond funds

Table 2: Life-Cycle Pattern of Asset Holdings – 1989 PSID Wealth Data
(Mean Ratio of Asset to Net Worth – Homeowners Only)

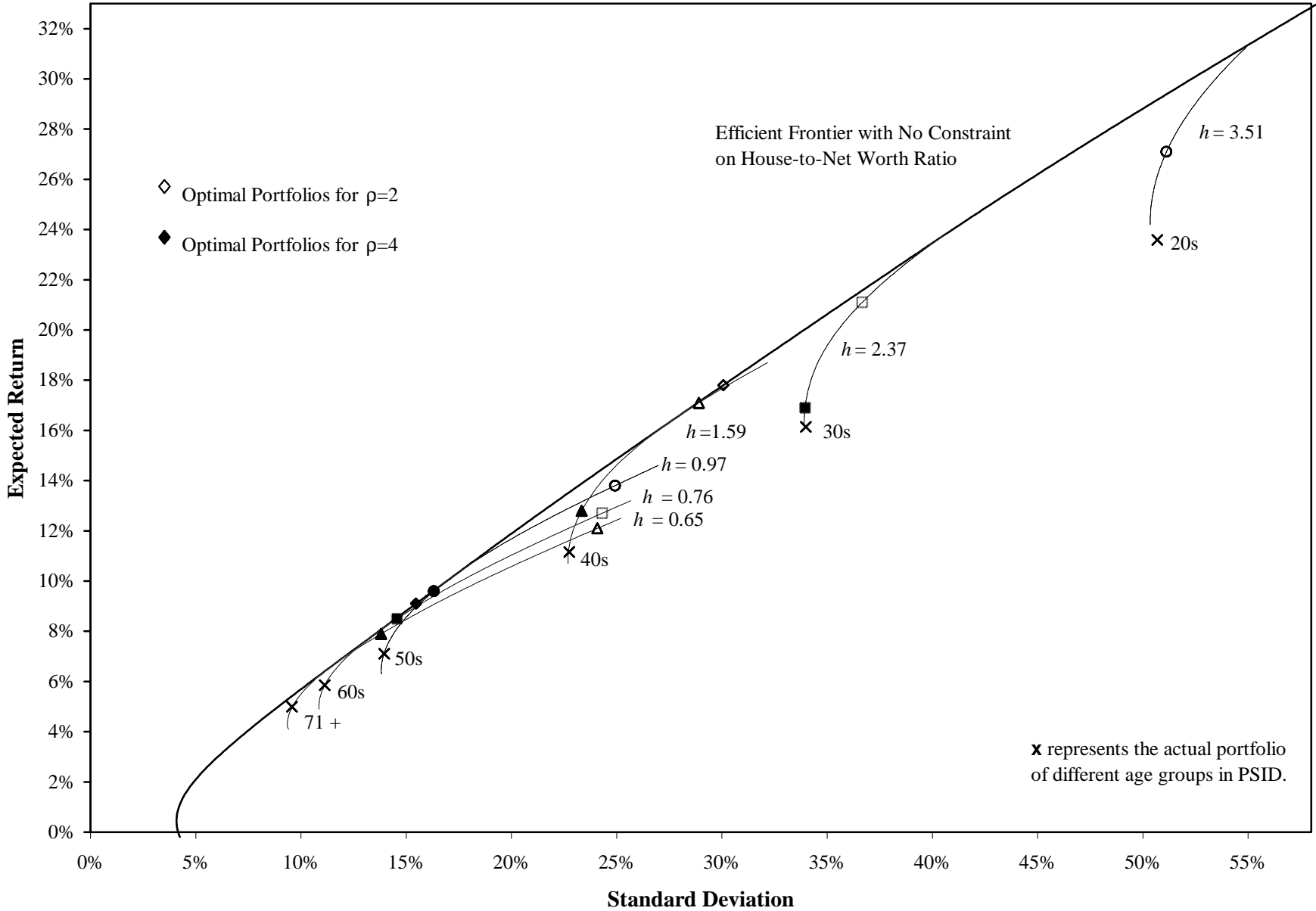| Age of Head	Cash	Bonds	Stocks	House	Mortgage	m	s
18 - 30	0.193	0.072	0.056	3.511	-2.833	0.236	0.507
31 - 40	0.169	0.067	0.068	2.366	-1.671	0.161	0.340
41 - 50	0.148	0.060	0.085	1.588	-0.882	0.112	0.227
51 - 60	0.200	0.058	0.092	0.969	-0.319	0.071	0.140
61 -70	0.254	0.048	0.113	0.757	-0.171	0.059	0.111
71 +	0.264	0.029	0.098	0.648	-0.038	0.050	0.097

Source: Authors' tabulation from the PSID. See Appendix 1 for details.

and other assets such as rights in a trust and cash values of insurance policies. Thus these two categories correspond only very loosely to the Treasury bills or 20-year Treasury bonds used in the estimation of the risk characteristics of financial assets. Details on the PSID wealth data are reported in Appendix 1.

In the portfolio analysis which follows, we condition on the mean ratio of house value to net worth, denoted h (e.g. h=3.51 for 18-30 year olds, h=2.37 for 31-40 year olds), and use quadratic programming to calculate the constrained efficient frontier associated with each fixed level of h. The optimization is subject to the following constraints: households can borrow only in the form of a mortgage, the value of the mortgage cannot exceed the value of the house, and the portfolio weights on T-bills, T-bonds, and stocks must be nonnegative.

Figure 2, which plots the unconstrained efficient frontier as well as six age-specific constrained efficient frontiers, shows that the housing constraint has an enormous effect on the risk and return tradeoff available to the household. The youngest homeowners, with h=3.51, are limited to the constrained efficient frontier at the upper right. For this group, the minimum variance portfolio has an expected return of .24 and a standard deviation of over .50. As the household accumulates wealth and the value of h falls, the constrained frontier moves to the left.

Figure 2: Efficient Frontiers with Fixed Housing Investment

Efficient Frontier with No Constraint
on House-to-Net Worth Ratio

◇ Optimal Portfolios for ρ=2

◆ Optimal Portfolios for ρ=4

$h = 3.51$

✕ 20s

$h = 2.37$

$h = 1.59$

$h = 0.97$

$h = 0.76$

$h = 0.65$

✕ 30s

✕ 40s

✕ 50s

✕ 60s

71 +

Expected Return

Standard Deviation

✕ represents the actual portfolio
of different age groups in PSID.

It is not until the value of *h* falls to levels typical of households in their 50's, 60's, and 70's that the constrained efficient frontier becomes a reasonable approximation of the unconstrained frontier. Figure 2 also plots the constrained optimal portfolios for $\rho \approx 2$ and $\rho \approx 4$[5]. On a given constrained frontier, of course, the optimal portfolio for $\rho \approx 4$ has lower expected return and risk than the optimal portfolio for $\rho \approx 2$. Due to its highly leveraged position in housing, however, a young household with *h*=3.51 and $\rho \approx 4$ holds a much riskier portfolio than a fifty-year old household with $\rho \approx 2$. Using the mean portfolio shares reported in the PSID for the different age cohorts (Table 2), Figure 2 also plots the actual portfolio for each age group. The actual portfolios are reasonably well predicted by the model under the assumption that $\rho \approx 4$[6].

Note that in contrast to Brueckner's analysis, the constrained efficient frontier is not necessarily tangent to the unconstrained frontier. Brueckner derives the tangency property by considering a constraint on the value of *h*, but assuming that the weights on all other assets have interior solutions. In our model, the household faces a mortgage constraint which says that the value of the mortgage cannot exceed the value of the house, and this constraint is often binding. For example, consider the junction of the constrained efficient frontier for *h*=3.51. Along the entire length of the constrained frontier, the household holds a mortgage equal in value to its house and holds an amount equal to its net worth in a combination of stocks and bonds. As risk aversion declines, and the household moves along the constrained frontier toward riskier portfolios, this movement is achieved by reducing the bond share and increasing the stock share while house and mortgage remain at their constrained values. Once the portfolio consists of

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[5] Appendix 2 explains the method for deriving the approximate values of the coefficient of relative risk aversion in the mean-variance efficient frontier context.

[6] The value of ρ equal to 4 is consistent with other studies based on CAPM and financial data. Friend and Blume (1975), using movements of stock market price indices, find ρ to be between 1.5 and 12. Grossman and Shiller (1981) conclude ρ to be around 4 in order to produce the historical pattern of stock price movements, and

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h=3.51, mortgage=-3.51, stocks=1, bonds=0, and T-bills=0, the fact that the mortgage constraint is binding implies that the household has obtained the highest return portfolio which is consistent with *h*=3.51. In order to move along the unconstrained efficient frontier, the household would maintain the shares of stocks=1, bonds=0, T-bills=0, and increase the value of *h* while remaining fully leveraged. Thus at the intersection of the constrained and unconstrained frontiers, the slope of the constrained frontier reflects the risk and return tradeoff involved in trading bonds for stocks, while the slope of the unconstrained frontier reflects the risk and return tradeoff involved in acquiring a larger, fully mortgaged house while maintaining a stock share of unity.

Table 3 reports the optimal portfolio weights for different values of ρ when the efficient frontier is estimated with (i) financial assets only, (ii) the value of the housing constraint, *h*, fixed at various levels corresponding to different stages of the life cycle, and (iii) no constraint imposed on *h*. At a low level of risk aversion ($\rho \approx 1$), the efficient portfolio includes only stocks and a house, which is leveraged with a 100% mortgage, regardless of the ratio of housing assets to net worth. For higher degrees of risk aversion, the optimal portfolio weights change dramatically and exhibit a clear pattern of changes in financial asset composition in response to changes in the value of *h*. For example, for a coefficient of relative risk aversion of 3, the optimal portfolio for a young household with *h*=3.51 has a bond share of .91 and a stock share of .09. Twenty years later, with a typical housing ratio of 1.59, the optimal portfolio has a bond share of .57 and a stock share of .43. In retirement, when the housing ratio is .65, the optimal portfolio has a bond share of .40 and a stock share of .60.

Fratantoni (1996) studies the effects of homeownership on risky asset holding by modeling the commitment to a fixed mortgage payment combined with uncertain labor income

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Pindyck (1988) estimates ρ to be between 1.6 and 3.5 in order for the response of equity premium to changes in the volatility of returns to be consistent with stock price movements.

Table 3: Portfolio Weights for Different Constraints on *h*

Housing-to-NW Ratio	Assets in Portfolio	Coefficient of Relative Risk Aversion (r)			
		$r \gg 1$	$r \gg 2$	$r \gg 3$	$r \gg 4$
Financial Assets Only	**Treasury Bills**	0	0	0.2425	0.4618
	Treasury Bonds	0	0.2669	0.2681	0.1580
	Stocks	1	0.7331	0.4894	0.3802
3.51	**Treasury Bills**	0	0	0	
	Treasury Bonds	0	0.5605	0.9143	Not Attainable*
	Stocks	1	0.4395	0.0857	
	House	3.51	3.51	3.51	
	Mortgage	-3.51	-3.51	-3.51	
2.37	**Treasury Bills**	0	0	0	0
	Treasury Bonds	0	0.3622	0.7026	0.6190
	Stocks	1	0.6378	0.2974	0.1093
	House	2.37	2.37	2.37	2.37
	Mortgage	-2.37	-2.37	-2.37	-2.0983
1.59	**Treasury Bills**	0	0	0	0
	Treasury Bonds	0	0.2125	0.5660	0.4914
	Stocks	1	0.7875	0.4340	0.2452
	House	1.59	1.59	1.59	1.59
	Mortgage	-1.59	-1.59	-1.59	-1.3269
0.97	**Treasury Bills**	0	0	0	0
	Treasury Bonds	0	0.1093	0.4628	0.3978
	Stocks	1	0.8907	0.5372	0.3599
	House	0.97	0.97	0.97	0.97
	Mortgage	-0.97	-0.97	-0.97	-0.7277
0.76	**Treasury Bills**	0	0	0	0
	Treasury Bonds	0	0.0720	0.4255	0.3640
	Stocks	1	0.9280	0.5745	0.3969
	House	0.76	0.76	0.76	0.76
	Mortgage	-0.76	-0.76	-0.76	-0.4070
0.65	**Treasury Bills**	0	0	0	0
	Treasury Bonds	0	0.0556	0.3960	0.3434
	Stocks	1	0.9444	0.6040	0.4136
	House	0.65	0.65	0.65	0.65
	Mortgage	-0.65	-0.65	-0.65	-0.4070
No Constraint on *h*	**Treasury Bills**	0	0	0	0
	Treasury Bonds	0	0.2431	0.4933	0.3825
	Stocks	1	0.7569	0.5067	0.3768
	House	3.6490	1.7317	1.1570	0.8744
	Mortgage	-3.6490	-1.7317	-1.1570	-0.6337

* Note that for the portfolio with the constraint *h*=3.51, the implied value of ρ is approximately 3.1 at the minimum variance portfolio. Since no investor would hold a portfolio on the downward sloping part of the efficient frontier, we consider that the mean-variance efficient portfolio is not attainable under the assumption of ρ≈4.

in a simulation model. In his calibration, however, a higher degree of risk aversion ($\rho = 10$) is required to produce a share of risky assets in total wealth below 10%, and the share of risky assets in total wealth is higher for young households who have just bought a house than for older households nearing retirement.

Given the estimated vector of expected returns and covariance matrix, the model predicts that households generally hold both a mortgage and bonds. Since mortgages and bonds are similar assets, an alternative approach would be to consider a mortgage and a bond perfect substitutes; i.e., assume that there is a single financial instrument which is called a mortgage if one takes a negative position and called a bond if one takes a positive position, in addition to stocks and housing. If the optimal portfolios were calculated for this simplified menu of assets, the share of the new hybrid instrument would be roughly equal to the reported sum of mortgage and bond shares and the share of stocks would be largely unaffected. That is, if we consider mortgages and bonds to be perfect substitutes, the model can give predictions about the sum of the holdings of mortgage and bonds, but cannot say anything about the shares of mortgage and bonds separately.

The basic message for the life cycle of portfolio composition is that young households with high values of h are constrained to hold a highly risky portfolio and therefore use their net worth to reduce portfolio risk rather than attempting to further increase their expected return. To reduce portfolio risk, the young households hold small shares of stocks (about .08 for $\rho \approx 3$), and reduce their leverage, either explicitly by paying down the mortgage, or implicitly by keeping the 100% mortgage and simultaneously holding bonds. The mean-variance analysis explains why some households would hold very low levels of stocks, but does not explain why households would hold bonds rather than paying down their mortgages. However, if the household is indifferent between paying down the mortgage and holding bonds from the point of view of

mean-variance efficiency, the actual portfolio decision may depend on other considerations such as transactions costs. If the brokerage fees incurred by selling bonds are less than the transactions costs in refinancing a mortgage, it may indeed by optimal for the household to reduce its leverage by simultaneously holding a mortgage and bonds rather than explicitly paying down the mortgage.

III.   The Predictive Performance of the Model

Using the wealth data from the 1989 wave of the PSID, the six panels of Figure 3 plot, for homeowners of different age cohorts, the household's ratio of stocks to net worth as a function of the ratio of house value to net worth. The ratio of mortgage value to net worth is also plotted. To characterize the stock and mortgage holdings predicted by the model, the relationship between the housing ratio, h, and the optimal portfolio shares was calculated for additional values of h. When plots of the optimal stock share as a function of h appeared to be essentially linear, the predicted value of the stock to net worth ratio, s, was regressed on h. The results of the OLS regressions are reported in Table 4.

For a coefficient of relative risk aversion of unity, the optimal portfolio consists of a

Table 4: Regression of Portfolio Weights of Stocks in Optimal Portfolio

|	Coefficient of Relative Risk Aversion		
	$\rho \approx 2$	$\rho \approx 3$	$\rho \approx 4$
Constant	1.0653	0.7151	0.5383
	(.00215)	(.00178)	(.00176)
h	-0.1780	-0.1782	-0.1812
	(.00098)	(.00118)	(.00121)
Sample Size	14	14	14
R^2	.9996	.9994	.9994

Note: standard errors in parenthesis

mortgage equal in value to the house, and a portfolio share of unity on stocks, for all values of h.

For values of ρ of 2, 3, or 4, the OLS regressions of s on h each had R^2 approaching unity. Furthermore, the estimates of the slope coefficients were numerically very similar at -.18, implying that a 10% reduction in h would induce a 1.8% increase in s. The relationship between h and s is linear only for the interval in which both h and s are nonzero; for values of h above a critical value, the optimal share s is zero, and for h equal to zero the optimal value of s lies below the line associated with nonzero values of h.

For comparison with the PSID data, the predicted relationship between s and h is plotted in Figure 3 for values of ρ equal to 2, 3, and 4; for $\rho \approx 1$, the relationship between s and h is a horizontal line at $s=1$. In Figure 3a, which plots the wealth data for homeowners in the 18-30 cohort, most (68%) hold no stocks. If $\rho \approx 4$, the model predicts that for any household with a ratio of house value to net worth greater than 2.97, zero is indeed the optimal share for stocks. If $\rho < 4$, the model predicts that homeowners with h less than about 2.97 hold a positive share in stocks; thus the preponderance of households with h less than 2.97 and no reported stock holdings is inconsistent with the model. However, if we restrict our attention to households with nonzero stock holdings, most of these households have values of s within the predicted band for $\rho \approx 2$ to $\rho \approx 4$. Only one household has a portfolio consistent with $\rho \approx 1$[7].

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[7]   Using the Health and Retirement Study, Barsky, Juster, Kimball and Shapiro (1997) show how the coefficient of relative risk aversion may be inferred from a respondent's answer to survey questions of the form, "If you are given the opportunity to take a new job with a 50% chance it will double your family income and a 50% chance it will cut your family income by a third, would you take the new job?" Respondents who decline the first gamble are then asked if they would accept a gamble of 50% chance of doubling and 50% chance of a smaller cut (e.g. 20% or 10%); respondents who accept the initial gamble are asked if they would also accept a gamble of 50% chance of doubling and 50% chance of a larger cut (e.g. 50% or 75%). Luoh and Stafford (1997) apply the approach to data from the 1996 wave of the PSID, which contained questions of the same form. Based on the responses, Luoh and Stafford find the following distribution of values of relative risk aversion among the employed PSID households: for 6.5%, $0 < \rho < .31$; for 13.5%, $.31 < \rho < 1$; for 14.8%, $1 < \rho < 2$; for 15.3%, $2 < \rho < 3.84$; for 18%, $3.84 < \rho < 7.52$; for 31%, $\rho > 7.52$. In terms of the scatters presented in Figure 3, the Luoh and Stafford results (which are generally comparable to the results of Barsky, et. al. from the HRS) suggest that about a half of households have coefficients of relative risk aversion between zero and 3.8, and the remaining half have relative risk aversion greater than 3.8.

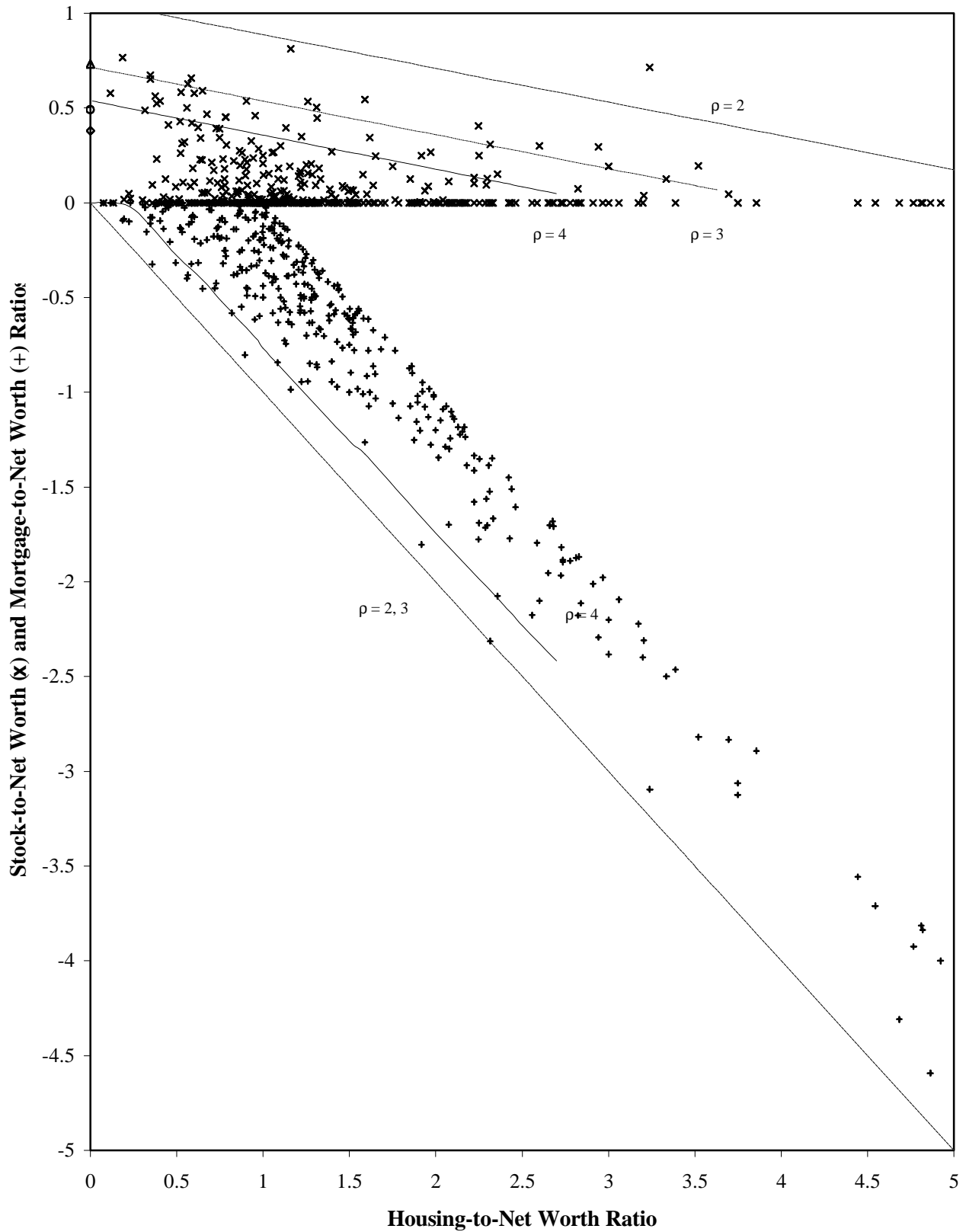# Figure 3a: Age in 1989 = 18 to 30 (N=228)
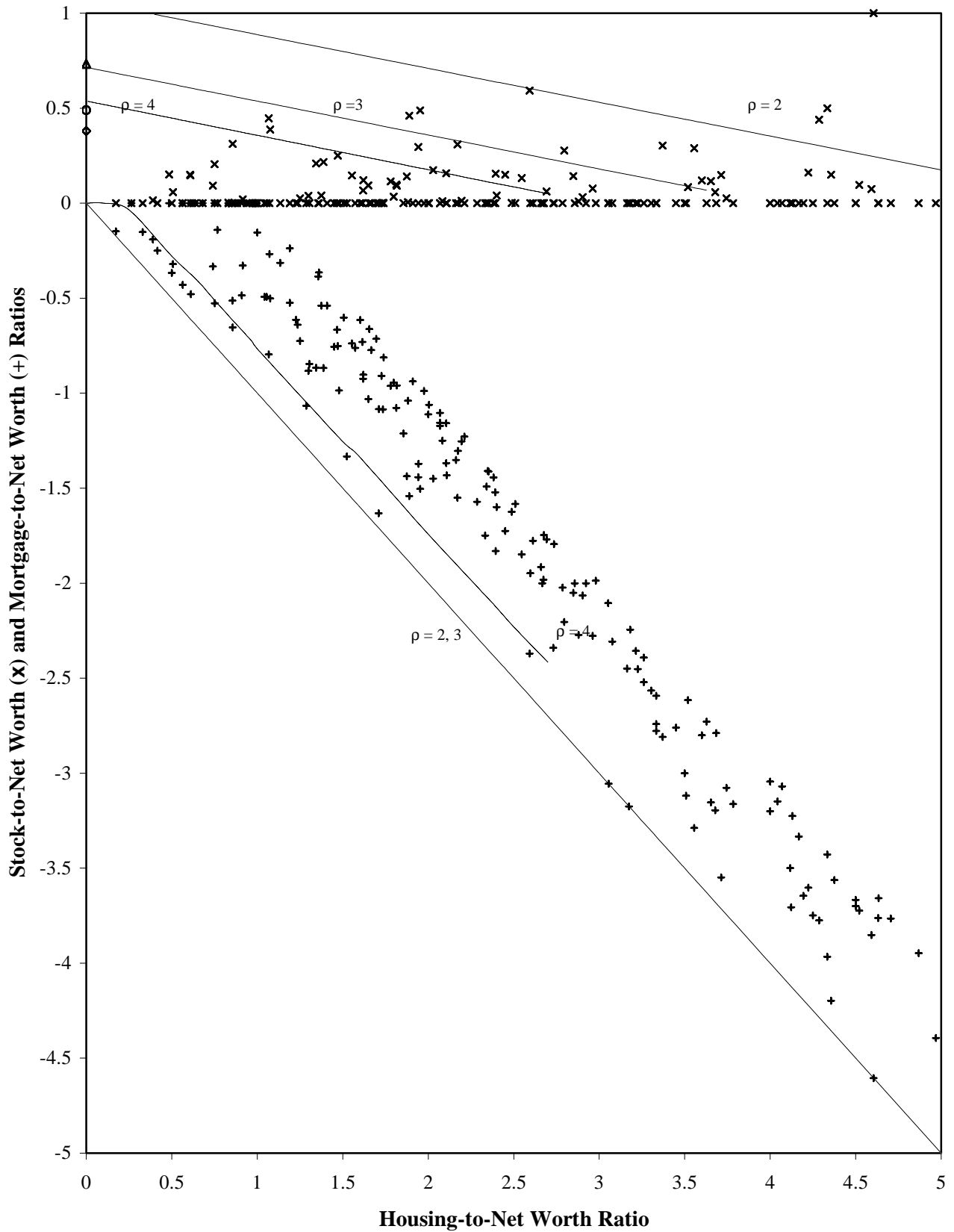
Note: In this sample, 67.5% of households hold no stocks and 8.3% do not hold mortgages.
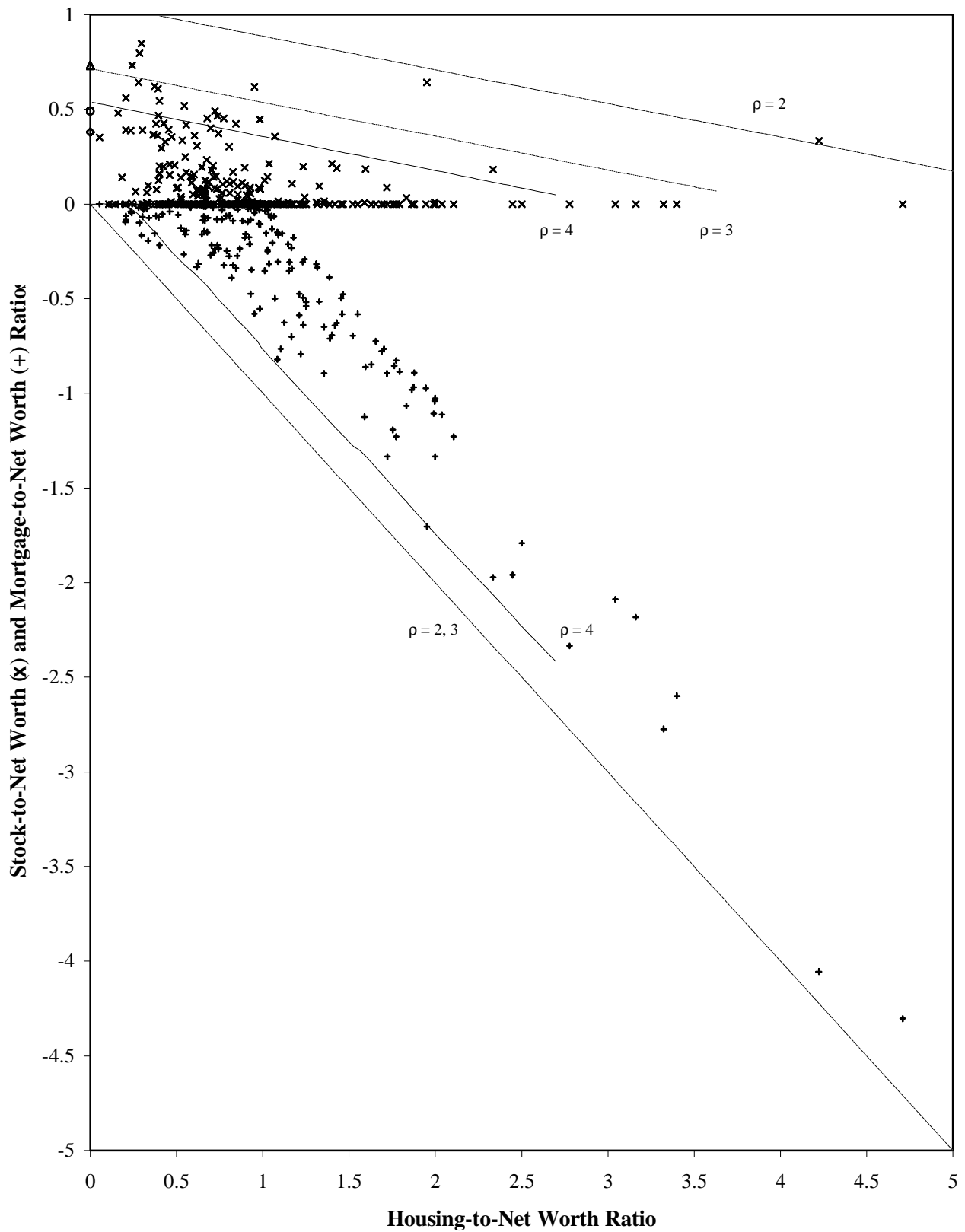
Figure 3b: Age in 1989 = 31 to 40 (N=620)
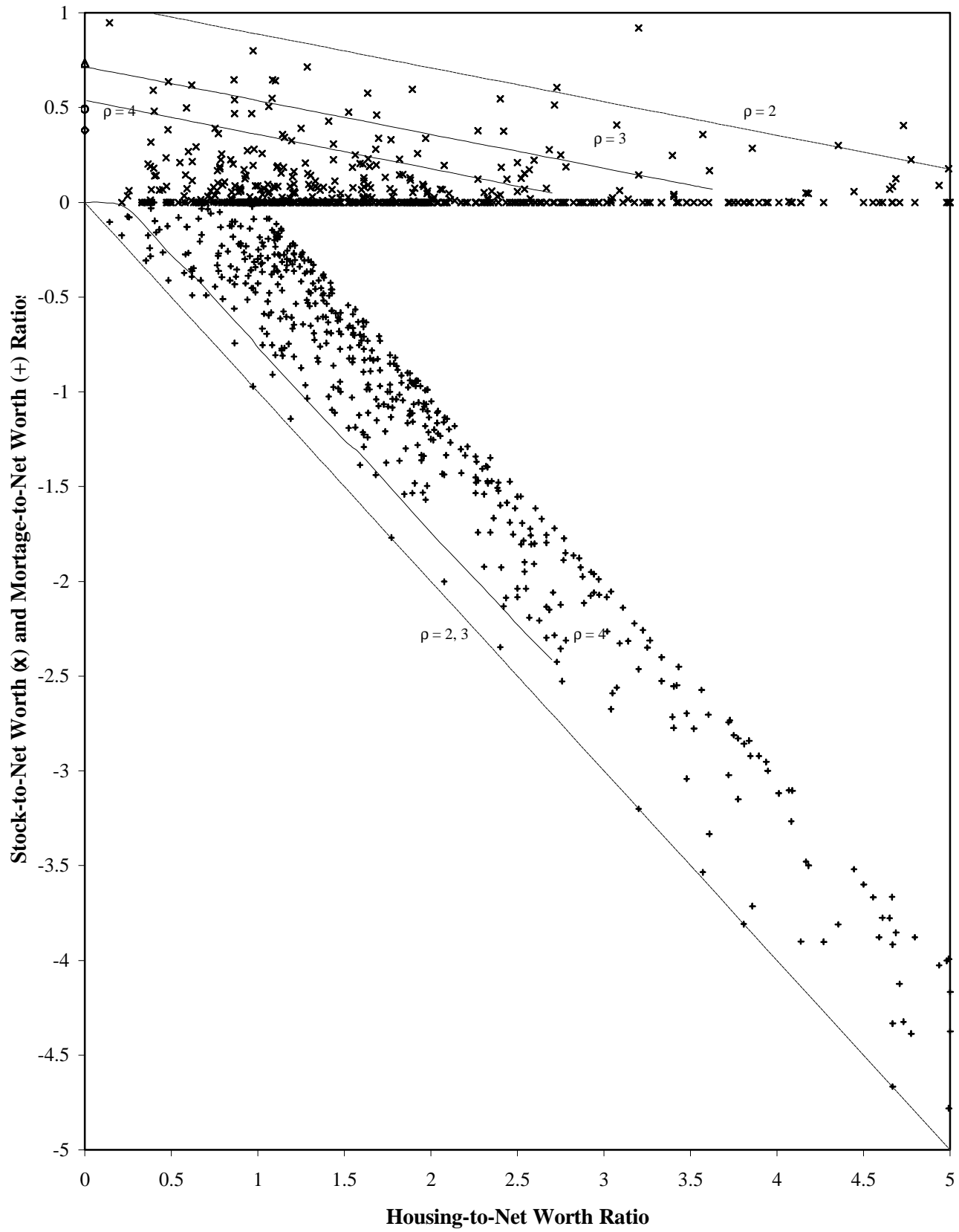
Note: In this sample, 58.4% of households hold no stocks and 9.7% do not hold mortgages.

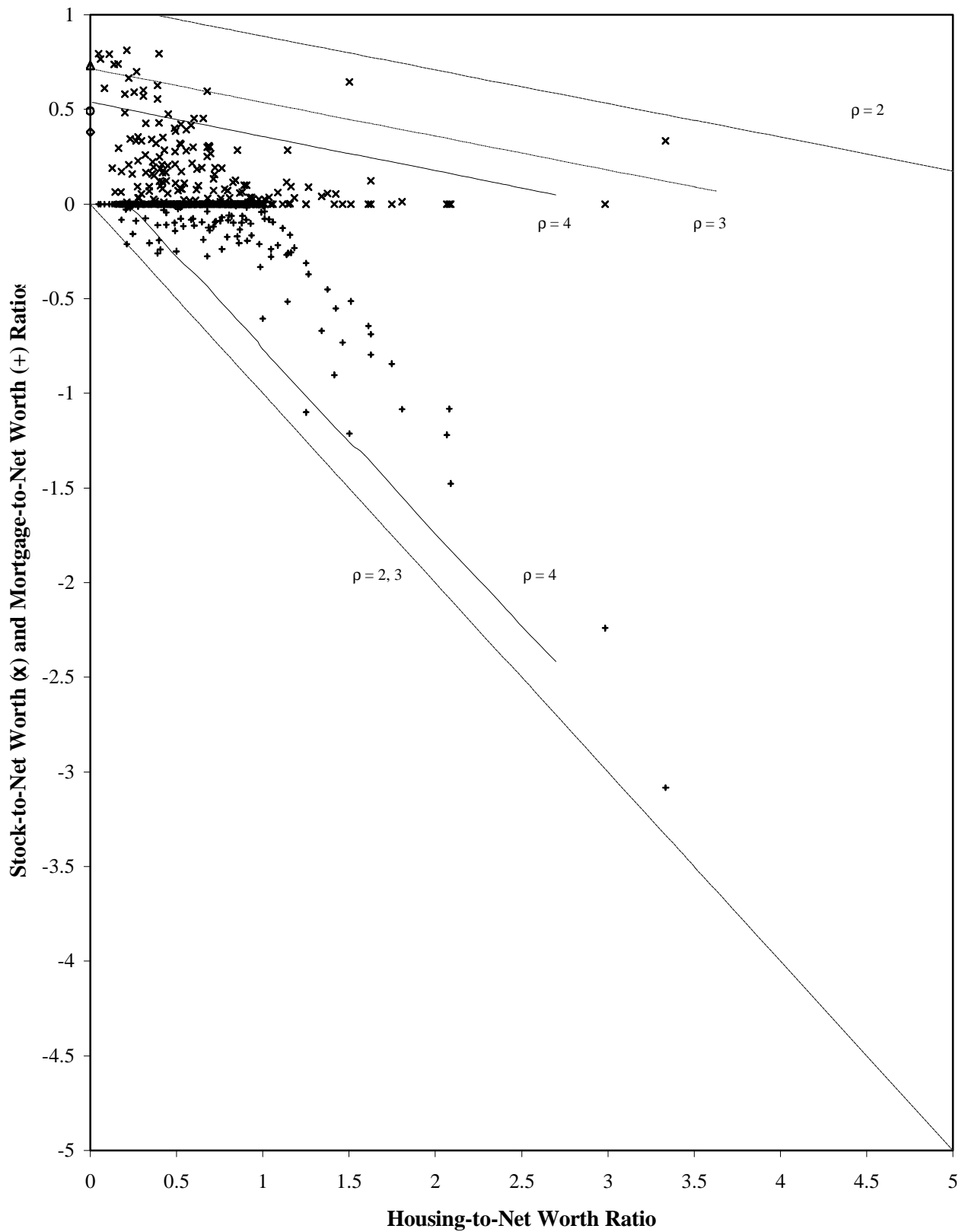**Figure 3c: Age in 1989 = 41 to 50 (N=428)**

Note: In this sample, 53.5% of households hold no stocks and 15.7% do not hold mortgages.

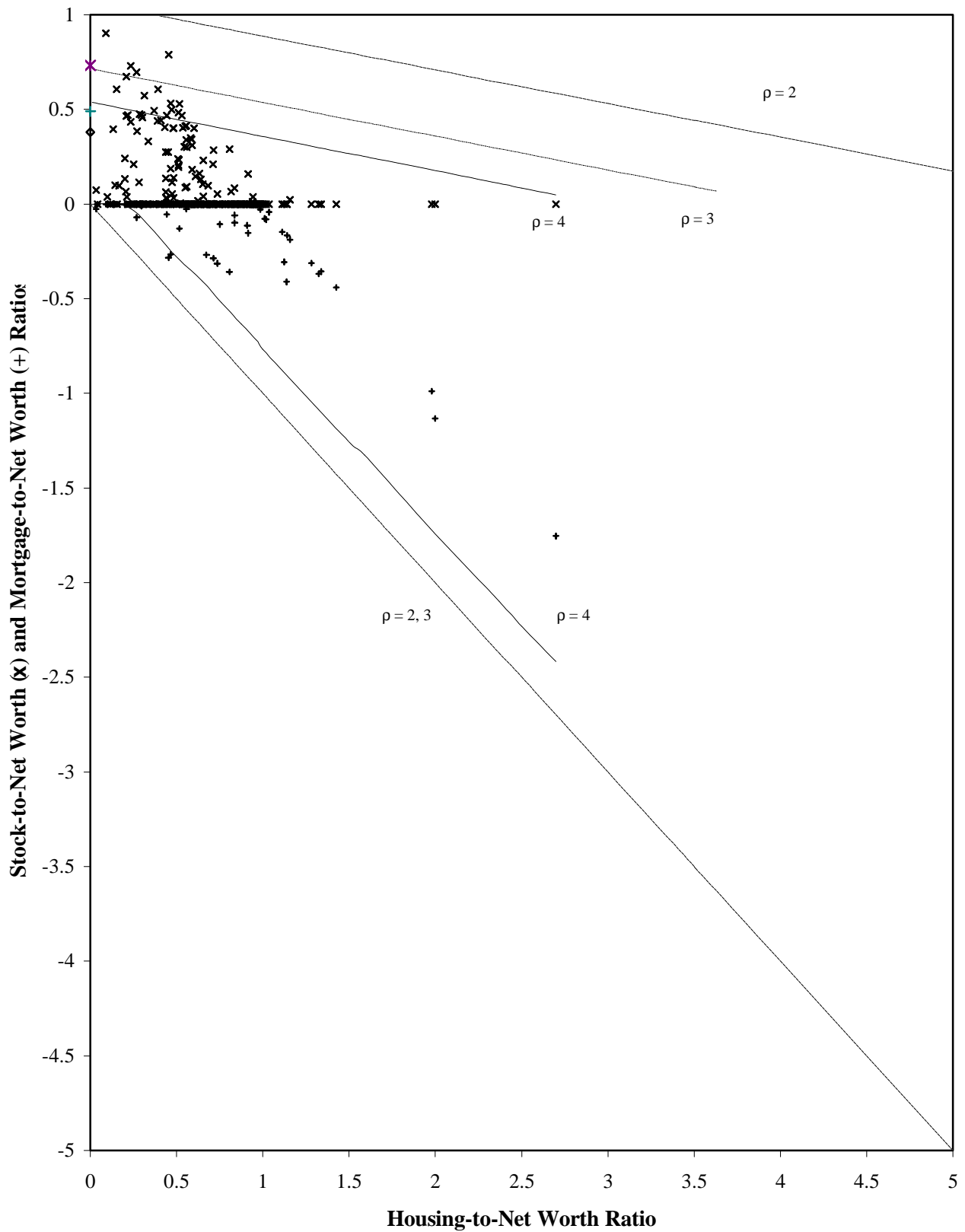**Figure 3d: Age in 1989 = 51 to 60 (N=312)**

Note: In this sample, 52.7% of households hold no stocks and 41.3% do not hold mortgages.

Figure 3e: Age in 1989 = 61 to 70 (N=289)

Note: In this sample, 51.6% of households hold no stocks and 66.1% do not hold mortgages.

Figure 3f: Age in 1989 = 71 and over (N=250)

Note: In this sample, 65.2% of households hold no stocks and 86.8% do not hold mortgages.

Figures 3b-3f provide comparable plots for older cohorts. While the predicted relationship between *s* and *h* is independent of age, the scatter becomes more concentrated at lower values of *h* as the cohort ages. Based on the estimated covariance matrix and vector of expected returns, for coefficients of relative risk aversion of 2 or 3, the model predicts that even as the ratio of house value to net worth declines over the life-cycle, the household should nevertheless maintain a 100% mortgage in order to invest in higher return assets. This feature of the optimal portfolio is strongly contradicted by the PSID data; for the two oldest cohorts (61-70, and over 70), the majority of homeowners hold no mortgage. Because of the adding-up constraint on the portfolio shares, the fact that the model underpredicts mortgage liability for older households translates into an underprediction of stock holdings for those households.

The six panels in Figure 3 are cross-sectional snapshots of wealth holdings in 1989. Since the PSID contains wealth data for all survey households in both 1984 and 1989, we can use the panel dimension of the survey to study, at the household level, the change in portfolio shares over the five year interval[8]. According to the model, households optimally hold a stock share of zero if the housing constraint, *h*, exceeds some threshold, $\tilde{H}$, which depend on the value of ρ. If the household's values of *h* are below the threshold for both years (1984 and 1989), the model predicts that, conditional on the observed change in *h*, the change in the stock share is:

$$\Delta s_t = -0.18 \, \Delta h_t, \tag{5}$$

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[8] There have been a few studies that have used the same data set to examine the effects of housing capital appreciation on the saving behavior of home owners. Using the self-reported house price appreciation in the PSID, Skinner (1996) finds a reduction of 2.5 to 5.4 in saving per dollar increase in housing, and Engelhardt (1996) estimates marginal propensity to consume out of real housing capital gains is .14 for the mean saver household and .03 for the median saver household. On the other hand, Hoynes and McFadden (1994), augmenting the PSID wealth data with house price data from 112 metropolitan areas for the 1984-1989 period, find that an increase in the growth rate of real housing prices of ten percentage points will lead to an increase in the non-housing savings rate of 2.64 percentage points. Interesting findings of both Skinner and Engelhardt are that there is an asymmetry in the real saving response to real housing capital gains. The real savings offset comes from the households that experienced capital losses and households that experienced positive capital gains do not reduce their savings.

Table 5:   Implications for Regression Coefficients

	$h_{89} \leq \tilde{H}$	$h_{89} > \tilde{H}$
$h_{84} \leq \tilde{H}$	Constant = 0 Slope ≈ -0.18	Constant = positive Slope ≈ Negative, but smaller than -0.18 in absolute value.
$h_{84} > \tilde{H}$	Constant = positive Slope ≈ Negative, but smaller than -0.18 in absolute value.	Constant ≈ 0 Slope ≈ 0

where $\Delta s_t = s_t - s_{t-1}$, and $\Delta h_t = h_t - h_{t-1}$. If, on the other hand, the value of h is larger than $\tilde{H}$ in both years, the model predicts zero stock holdings in each year and $\Delta s_t = 0$. If the household moves across the threshold $\tilde{H}$ between 1984 and 1989 (i.e., $h \leq \tilde{H}$ in one year, and $h > \tilde{H}$ in the other), the model predicts that the change in the portfolio share of stock is:

$$\Delta s_t = \text{constant} + \beta \Delta h_t, \tag{6}$$

where the constant term is positive and the slope coefficient β is negative and smaller in absolute value than .18. The implications for regression coefficients are summarized in Table 5.

Table 6 presents some descriptive statistics for the 1,327 PSID households that were homeowners either in 1984 or 1989. The descriptive statistics are provided for three types of households; those that owned their home in each year, those that went from owning in 1984 to renting in 1989, and those that went from renting to owning. In presenting the scatter plot of Δs_t against Δh_t, Figure 4 uses distinct symbols for the three groups of households. Because the linear relationship between h and s is not predicted to hold at $h=0$, the empirical prediction of

Variables in:	Own-Own			Own-Rent			Rent-Own		
	Mean	**S.D.**	**Median**	**Mean**	**S.D.**	**Median**	**Mean**	**S.D.**	**Median**
1984									
Age	45.8	14.9	42	42.1	16.3	37	33.3	11.5	30
(% over 65)	(14.4%)			(6.9%)			(2.8%)		
Family Size	3.00	1.29	3	2.77	1.48	2	2.61	1.27	2
No. of Rooms	6.26	1.32	6	5.91	1.12	3	4.84	1.55	5
Head Labor Y	23,725	23,858	22,000	17,211	13,923	17,750	19,236	14,824	17,000
(% Y84=0)	(13.1%)			(9.1%)			(4.2%)		
Net Worth	100,204	396,389	56,250	44,033	70,784	30,000	12,783	26,096	3,300
Ptf Weight of:									
Cash & Bonds	.217	.199	.153	.209	.216	.150	.907	.233	1.000
Stocks	.066	.139	0	.052	.148	0	.093	.233	0
(% Stock=0)	(61.0%)			(77.3%)			(74.8%)		
House	1.698	5.558	1.057	5.571	16.177	1.819	0	0	0
Mortgage	-.982	5.576	-.270	-4.831	16.282	-.870	0	0	0
1989									
Family Size	2.99	1.33	3	2.77	1.54	2	3.03	1.389	3
No. of Rooms	6.73	1.88	7	5.35	1.82	5	6.28	1.70	6
Head Labor Y	30,188	41,241	26,000	20,836	19,747	16,996	29,692	23,049	20,000
(% Y89=0)	(21.2%)			(15.9%)			(6.3%)		
Net Worth	145,442	240,507	92,086	24,630	33,768	11,650	76,371	130,477	36,000
Ptf Weight of:									
Cash & Bonds	.244	.211	.183	.822	.282	1	.271	.253	.189
Stocks	.092	.168	0	.178	.282	0	.066	.136	0
(% Stock=0)	(54.4%)			(63.6%)			(61.5%)		
House	1.351	2.014	.953	0	0	0	3.150	4.246	2.018
Mortgage	-.688	2.007	-.174	0	0	0	-2.487	4.262	-1.229
5-Yr. Change									
Dh	-.347	5.745	-.114	-5.571	16.177	-1.819	3.150	4.246	2.018
Ds	.026	.144	0	.126	.319	0	-.028	.247	0
(% $Ds = 0$)	(45.0%)			(56.8%)			(53.1%)		
Dm	.294	5.764	.0177	4.831	16.282	.870	-2.487	4.262	-1.229
D(cash&bond)	.027	.229	.0121	.613	.381	.738	-.635	.340	-.754
No. of Obs.	1140 (85.9%)			44 (3.3%)			143 (10.8%)		

Source: Authors' tabulation from the PSID.

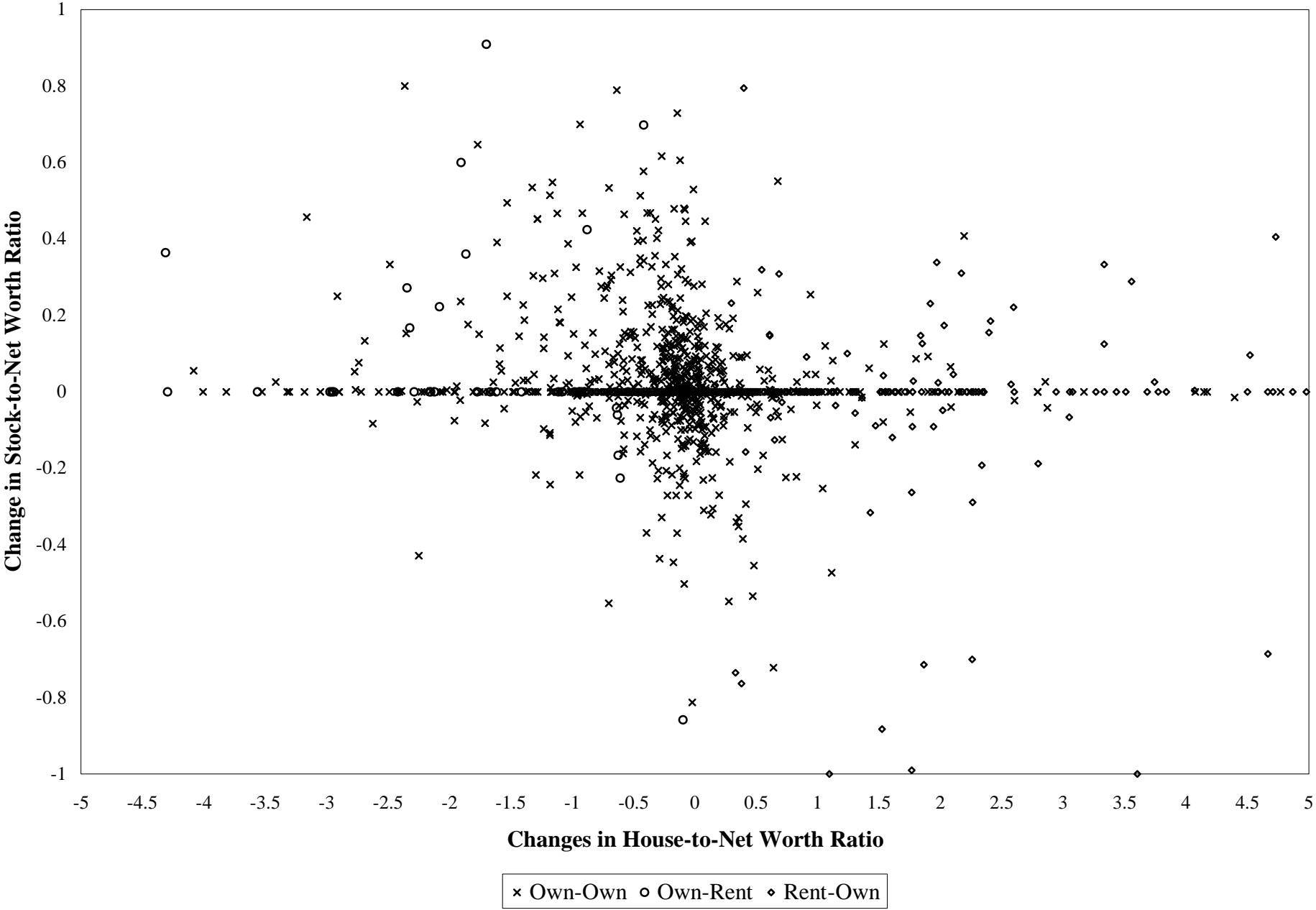**Figure 4: Changes in Asset Holdings between 1984 and 1989**

Table 7:   Transition of Portfolio Weights in Housing for the PSID Households between 1984 and 1989

	$h_{89} \leq 2.97$		$h_{89} > 2.97$		Column Total	
$h_{84} \leq 2.97$	1,000 (87.7%)	of which $\Delta s \neq 0$ 568 (56.8%)	39 (3.4%)	of which $\Delta s \neq 0$ 12 (30.8%)	1039 (91.1%)	580 (55.8%)
$h_{84} > 2.97$	69 (6.1%)	of which $\Delta s \neq 0$ 34 (49.4%)	32 (2.8%)	of which $\Delta s \neq 0$ 13 (40.6%)	101 (8.9%)	47 (46.5%)
Row Total	1069 (93.8%)	602 (56.3%)	71 (6.2%)	25 (35.2%)	1140 (100%)	627 (55.0%)

Source : Authors' tabulation from the PSID.

Table 5 will not hold for households which move between renting and owning. For this reason, the regression results are based only on households which were homeowners in both years.

The implications of the model are tested under the assumption that $\rho \approx 4$, which corresponds to the threshold level $\tilde{H} = 2.97$. Table 7 reports the transition matrix of housing portfolio weights for the 1,140 households that were homeowners in both 1984 and 1989. For this group, the majority (88%) have housing shares less than the threshold level of 2.97 in both years. According to the model (with the assumption of $\rho \approx 4$), the 88% of households, for which $h \leq 2.97$ in both years should hold strictly positive shares of stocks in both years, and Δs_t should be nonzero. Contradicting the model, 43% of these households have zero stock holdings in both years. While we acknowledge the fact that the model cannot explain the large number of households with zero stock holdings, we use only observations with nonzero values of s in at least one of the years in the regressions. For households for which $h > 2.97$ in both years, the model predicts that stock holdings in each year, and therefore Δs_t should be zero. In the data, Δs_t

Table 8: Regression Estimates for Changes in Portfolio Weights

Dependent Variable : $\Delta s_t = \dfrac{\text{Stocks}_t}{\text{NetWorth}_t} - \dfrac{\text{Stocks}_{t-1}}{\text{NetWorth}_{t-1}}$

	N (%)	(1)	(2)	(3)	(4)	$h_{84}>2.97$ or $h_{89}>2.97$ (5)	$h_{84}>2.97$ and $h_{89}>2.97$ (6)
			$h_{84}\leq 2.97$ and $h_{89}\leq 2.97$				
Constant		0.02322	0.01157	---	---	0.09081	.07449
		(.00821)	(.00933)			(.03007)	(.04626)
$\mathbf{D}_h$	568 (100%)	-0.10294 (.01492)		-0.11684 (.01473)		-0.00518 (.00508)	-0.00000 (.00829)
$\mathbf{D}_h \leq -1.5$	10 (1.8%)		-0.07264 (.02889)		-0.07933 (.02840)		
$-1.5 < \mathbf{D}_h \leq 0.5$	88 (15.7%)		-0.12768 (.02351)		-0.13997 (.02133)		
$-0.5 < \mathbf{D}_h < 0.5$	426 (72.6%)		-0.22047 (.04177)		-0.23828 (.03925)		
$0.5 \leq \mathbf{D}_h < 1.5$	31 (5.5%)		-0.07513 (.04009)		-0.06175 (.03864)		
$1.5 \leq \mathbf{D}_h$	4 (0.7%)		0.002894 (.05251)		0.00948 (.05226)		
No. of Obs.		568	568	568	568	46	13
R^2		0.0776	0.0997			.0230	0.0000
Adjust. R^2		0.0760	0.0917			.0008	-0.0909

Note: standard errors in parenthesis

is zero for 59% of this group. However, for consistency with the other regressions, only observations with nonzero values of *s* for at least one year are used for this group as well.

Regression results are presented in Table 8. If the relationship between Δs_t and Δh_t, is constrained to be linear over all values of Δh_t, the slope coefficient is -.103, somewhat smaller than the predicted value of -.18 for the observations for which the house value is less than 2.97 in both years (column 1 and 3). If a piecewise linear specification is used (column 2 and 4), the regression coefficient for moderate changes in *h* $(-0.5<\Delta h< 0.5)$ is -.22, with a standard error of

.042. For observations which crossed the threshold $\tilde{H}$ (i.e. $h>2.97$ in one of the two years), the slope coefficient is negative but smaller in absolute value than .18 and statistically insignificant and the constant term is statistically significant and positive (column 5), consistent with the predictions of the model. For observations for which $h>2.97$ in both years (column 6), the slope coefficient is numerically and statistically insignificantly different from zero and the constant term is positive but statistically insignificant, consistent with the model.

IV.   Conclusions

Most quantitative analyses of optimal portfolios ignore wealth held in the form of owner-occupied real estate and concentrate solely on financial assets. Presumably the neglect of real estate as an asset is a consequence of data and measurement problems rather than the belief that real estate is an unimportant part of the household's portfolio. Based on data from the PSID, the paper finds that the inclusion of housing as an asset dramatically improves the unconstrained efficient frontier available to households, confirming the popular conception that the homeownership is a good investment. However, to the extent that the household's holding of real estate is determined at least in part by its consumption demand for housing services (as opposed to determined purely for its role in portfolio diversification), the consumption demand for housing may place a constraint on the household's portfolio problem.

The paper estimates the risk and return to financial assets (short term bonds, long term bonds, mortgages and stocks) and residential real estate, then uses standard mean-variance portfolio theory to calculate the household's optimal holdings of the financial assets, conditional on their holding of housing. Since the ratio of housing investment to net worth declines over the household's life-cycle, the constraint imposed on the portfolio problem by the consumption demand for housing services induces a dramatic life-cycle pattern in the portfolio shares of

stocks and bonds. For example, young households, which typically have large holdings of real estate relative to their net worth, are highly leveraged and therefore forced into a situation of high portfolio risk (and return). As a result, these young households respond to the housing constraint by using their net to either pay down their mortgage or buy bonds instead of buying stocks. In comparison, ownership of stocks is much more attractive to older households which have accumulated greater wealth and therefore reduced their ratio of housing to net worth. For older households with lower ratios of housing to net worth, the constrained efficient frontier is much better approximated by the unconstrained efficient frontier, and the constrained optimal portfolio more closely resembles the unconstrained optimal portfolio with substantial amounts of stocks.

The model shows that households which are ex ante identical, in the sense that they have identical preferences toward risk and identical perceptions of the risk and return to different assets, will nevertheless hold quite different portfolios of financial assets because each household is optimizing their portfolio subject to a constraint on housing, and this constraint varies across households. For example, for a moderate degree of relative risk aversion of three, a household with a house to net worth ratio of 3.5 (typical of those in their twenties) hold an optimal portfolio which contains 8.5% stocks, while a mature household with a house to net worth ratio of .76 (typical of those in their sixties) should hold a portfolio share of 57.5% in stocks.

Empirical prediction of asset holdings at the household level is notoriously difficult. Assuming plausible levels for the household's degree of risk aversion, the model does reasonably well in explaining household data on the change in stock holding, at the household level, across two waves of the PSID.

The analysis in the paper is entirely partial equilibrium, and assumes that the expected return vector and covariance matrix are time-invariant. Nevertheless, it is fun to speculate on the

consequences of the housing constraint in a dynamic, general equilibrium setting. If one accepts

the basic premise that the household's optimal portfolio is constrained by the housing ratio, and

this ratio tends to be related to age, then demographic bulges such as the baby boom, birth

dearth, and baby boom echo may have important effects on asset prices. The effect of the baby

boom generation on the expected return to housing has been demonstrated in the literature

(Manchester, 1989; Mankiw and Weil, 1989). Similarly, demographic effects have been invoked

as a possible explanation of the recent stock market boom.   However, the treatment of

demographic effects on the stock market in the popular press generally focuses on the idea that

the baby boomers, finally recognizing that retirement is on the horizon, have comparatively high

saving rates. In this view, the aggregate flow of saving is unusually high because the baby boom

cohort has a high saving rate and constitutes a large fraction of the overall population. Our

analysis focuses on portfolio shares rather than savings rates, and shows that as the housing

constraint is gradually relaxed over the life cycle, the aging baby boomers should shift their

portfolio composition away from bonds and toward stocks. Thus even in the absence of high

saving rates, the baby boom generation could have a systematic effect on asset prices.

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Appendix 1   A Description of Sample Selection and Variables

I.   MICRO DATA

Panel Study of Income Dynamics (PSID)

Twenty-five years or "waves" of the PSID data (1968 - 1992) were available at the time of this study. The surveys are usually conducted in March, but were sometimes done in late February or April, and most of the questions refer to the preceding calendar year. Because of ambiguity about the time that the questions refer to and differences in interview dates, we interpret the responses as referring to the first quarter of the year, or a certain date during the first quarter for some questions, and time our aggregate data (prices, rates of return) accordingly. We use the term "households" or "family unit (FU)" interchangeably to refer to the data for a particular family through survey periods and "observation" to the data for a particular family in a particular survey year.

A. Housing Return

a. Sample Selection

We use only the SRC (non-poverty) sample present in 1992. From the total of 6,771 households included in the survey as of 1992, we first exclude households that never owned home between 1968 and 1992. This procedure reduces our sample to 5,846. We then exclude households where values of house or remaining mortgage principal have ever been assigned by PSID staff between 1968 and 1992, which reduces our sample to 1,817.

The PSID reports data on split-off households as separate FUs retroactively. That is, if a new household is formed by a child of the original sample FU by split-off, then the same variable values for a split-off FU are recorded as its parents' FU as if it were an independent household

until the year of split-off. Because presence of such data distorts estimates of means and variance, we treat data on split-off FUs as missing while they cohabit with the original FUs.

b. Change of Variable Values

Several variables are top-coded and we treat the top-coded observation missing. These variables are:

(i) Homeowner's House Value - The variables are top-coded at $99,999 before 1974 and $999,999 after 1975. Seven FUs are affected by this operation.

(ii) Remaining Mortgage Principal - The variables are top-coded at $99,999 from 1968 to 1981, and at $999,999 after 1983. Thirteen FUs are affected by this operation.

(iii) Annual Property Tax - The variables are top-coded at $9,998 (9999 is reserved for 'don't know/not applicable' answer) before 1981, and at $99,999 after 1983. This affects 46 FUs.

There are a number of cases where the house value changes by more than tenfo ld (or by one-tenth) in a year even when a FU did not move. Close inspection of these observations led us to detect that these values are off by one or more digits. We conclude that these are coding errors, and thus change the house value if drastic changes of house values are obviously attributable to a coding error. There are 45 such observations, and we change the value of these observations by either dropping or adding zero at the end. The list of observations where variable value was changed is available from the authors upon request.

c. Return on Housing Equity

A household-specific rate of return on housing is constructed incorporating property tax payments and the household's marginal tax rate. The housing return, denoted R_t, is calculated as:

$$R_t = \frac{P_t + D_t - COM_t - P_{t-1}}{P_{t-1}}, \qquad\qquad (A1.1)$$

where P_t is the real price of house in year t, D_t is the imputed rent ('*dividend*'), and COM_t is the cost of ownership and maintenance. D_t and COM_t are defined as:

$$D_t = (r + d)P_{t-1} + PropertyTax_t \qquad\qquad (A1.2)$$

$$COM_t = dP_{t-1} + (1 - t)PropertyTax_t \qquad\qquad (A1.3)$$

where r is real interest rate and d is the depreciation rate, τ is the marginal income tax rate of the household. The imputed rental value, D_t is calculated assuming that landlords would fully shift property taxes onto renters. In computing the cost of ownership and maintenance, COM_t, we assume that the owner spends on maintenance and repair an amount equal to the current depreciation on the house and itemizes deductions, so that the net property tax bill is (1- τ) Property Tax$_t$. This gives:

$$R_t = \frac{P_t + rP_{t-1} + t\ PropertyTax_t - P_{t-1}}{P_{t-1}}. \qquad\qquad (A1.4)$$

The price of the house is a direct survey question of the PSID throughout. The question is asked if FU owns the house they live in, and it is worded as "Could you tell me what the present value of your house is -- I mean about how much would it bring if you sold it today." The property tax amount is also a direct survey question except waves 11, 21 and 22 (1978, 88, and 89, respectively). The Wave VI Codebook (Survey Research Center: 1973) describes the estimation method of the property tax, and we use the same method to calculate property tax for 1978. Property ax rates are assessed depending on where the family lives, as presented in Table A1.

Table A1: Property Tax Rate

Distance from Nearest City of 50,000 or more	New England States	All Other States
0 - 5 miles	2.5%	2.0%
5 - 49 miles	2.0%	1.5%
50 or more miles	1.5%	1.0%

Source : Survey Research Center (1973), p.140

For 1988 and 1989, we do not have data on distance from the nearest city of 50,000 or more. Instead, we used the size of the largest city in county as an indicator and applied the tax rate of 1.5% (2.0% for New England states) if the largest city in county is greater than 50,000, and 1.0% (1.5% for New England states) otherwise.

House price and property tax are then deflated by CPI-U. We use constant interest real rate of 5% as opportunity cost of capital (r), and the marginal tax rate (τ) of 33% comprising 28% of Federal income tax and 5% of state income tax. The return is then calculated using the equation (4).

d. Further Exclusion of Observations

We want to construct return variables as meaningful as possible. We further exclude observations if any of the following applies.

(i)   When a FU moves - When a household changes residence, two houses they live in are not the same one. We delete the calculated return variable in the year if a family unit has moved during the previous year.

(ii)   When a female head marries to a nonsample male - PS ID usually considers male/'husband' as the head of household. If a female head (who has become a head because of split-off or divorce from the original sample family) marries to a nonsample male and starts living with

him, then her 'husband' becomes the head of household in subsequent survey years. If the female head owned a house before the marriage but moved out to live with her new 'husband', who already owned a house, then the new household is living in a different house from the previous year, even though the 'head' interviewed may respond 'No' to the question whether he has moved since spring of the previous year. To exclude such cases, we treat return variable missing if Family Composition Change variable indicates there had been a change of head due to marriage of the female head to a nonsample male.

B.   Mortgage Rate

We take into account that different households face different effective mortgage rates arising from differences in timing of the house purchase.   We calculate effective after-tax mortgage rates, assuming that households take a fixed-rate mortgage in the year they buy a house and do not refinance as long as they remain in the same house. Since mortgage interest payments are deductible, households pay an effective real rate of:

$$\text{Mortgage}_t = \frac{1 + (1-t)\text{Nominal Mortgage}_s}{1 + \text{inflation}_t} - 1 \qquad (A1.5)$$

where τ is the marginal tax rate, which we assume to be 33%, Nominal Mortgage $_s$ is the annual average of conventional home mortgage rate charged by major lenders, and inflation $_t$ is measured as the March – March percentage change in CPI-U of year t.

In calculating sample statistics of mean returns on housing and its covariance with other financial assets, we assume that all returns are drawn from the same distribution regardless of region and year. After deleting observations for which there are missing values of the return to housing or mortgage, we obtain 23,398 observations of housing return and mortgage rate. As a robustness measure, we trim the top and bottom 2% of housing returns, on the grounds that the

tails contain some extreme values likely to reflect gross measurement errors. Mean returns are calculated from this sample of 22,462 observations and then we use these mean returns to calculate standard deviations and the covariance. Since the number of homeowners in the PSID changes every year, we need be careful in avoiding introducing biases by weighing some years with more observations heavier than others with fewer observations. We hence construct 24 vectors of returns and covariance matrices corresponding to each year in our sample period and then weigh them equally to arrive at the single vector of returns and covariance matrix for the sample period of 1968-92.

C.   Wealth Data

We use the 1989 PSID Wealth Supplement to construct mean asset-net worth ratios as reported in Table 2 of the text. As in Section A of this Appendix, we first select from the non-poverty sample households that are homeowners in 1989, and delete any households that would splitoff later in the survey. We also change the house price variable that are presumably entered erroneously, as discussed before, which affects four observations. To be consistent with nonnegativity constraints on financial asset holdings, we delete observations that report negative net holdings of shares of stocks of publicly held corporations, mutual funds, or investment trust, including stocks in IRAs (v17326), money in checking or saving accounts, money market bonds, or Treasury bills including IRAs (v17329), or other savings such as bond funds, cash value in a life insurance policy, a valuable collection for investment purposes, or rights in trust or estate (v17332). We also delete observations where amount of mortgage outstanding (v16326) is greater than the reported value of the house (v16323). In addition, we delete observations where asset values are topcoded (house value and mortgage outstanding at $999,999; other real estate

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and value of own farm/business at $9,999,999), or those where the sum of individual wealth data does not equal to the net worth variable (v17389) generated by PSID. Further, we delete observations reporting negative net worth (24 observations) or no financial assets (137 observations). This leaves us with 2,127 observations.

We then use the reported values of "Cash" (v17329), "Bonds" (v17332), "Stocks" (v17326), and the House Value in 1989 (v16323) and Mortgage Outstanding (v16326). The net worth is calculated by adding "Cash", "Bond", "Stocks", and "House" and then subtracting "Mortgage Outstanding". Here, "Cash" refers to any money held in checking and savings accounts, money market bonds or Treasury bills and "Bonds" refers to any other savings including bond funds and other assets such as rights in a trust and cash value of insurance policies. Thus these two categories correspond only very loosely to the Treasury bills or 20 year Treasury bonds used in the estimation of the risk characteristics of the financial assets

II.   MACRO DATA

Center for Research of Security Prices (CRSP)

Returns for stocks, bonds, bills are calculated from monthly data series on stocks, bonds, bills and inflation provided by Ibbotson and Associates for CRSP. The variables used are as follows (letters in parenthesis indicate the respective CRSP mnemonic codes). The description of data is taken from *CRSP Indices File Guide* (Center for Research in Security Prices: 1996).

Treasury Bills (USTRET) - US Treasury Bills total return. Each month a one-bill portfolio containing the shortest-term bill having not less than one month to maturity is constructed. The holding period return is the difference in prices between the month-end and the previous month-end, where prices are:

$$1 - y_t \frac{D}{360},$$

where y_t is yield at time t, and D is the number of days to maturity.

<u>Treasury Bond (GBTRET)</u> - U.S. Treasury Bonds total return. Each year, a one bond portfolio was constructed containing a bond with a term of approximately 20 years and a reasonably current coupon, and whose returns did not reflect potential tax benefits, impaired negotiability, or special redemption or call privileges. Where callable bonds had to be used, the term of the bond was assumed to be as simple average of the maturity and first call dates minus the current data. The bond was "held" for the calendar year and returns were recorded. Returns are the change in the flat price, where flat price is the average of the bond's bid and ask prices, plus the accrued coupon.

<u>Stocks Capital Appreciation (CSCRET)</u> - S&P 500 stock capital appreciation. The capital appreciation component of the common stock return is taken as the change in the S&P 500 as reported in the Wall Street Journal over 1977-92, and in Standard and Poor's Trade Securities Statistics from 1926-76.

<u>Stocks Income Return (CSIRET)</u> – S&P 500 stocks income return. For 1977-92, income is calculated as realized dividends. Dividends are accumulated over the month and then invested on the last trading day of the month in the S&P 500 at the day's closing level. For 1926-76, quarterly dividends are extracted from rolling yearly dividends reported quarterly in Standard and Poor's Trade and Securities Statistics, then allocated to months within each quarter, assuming payments were made o the same month and day as they were in 1974.

<u>CPI (CPIRET)</u> - Consumer Price Index for All Urban Consumers, not seasonally adjusted (CPI-U NSA).

Returns on different assets receive different tax treatments. As for the housing asset, the imputed rent portion of the return is not taxed. While capital gains on housing are taxable in some circumstances, we assume that the bulk of realized capital gains escape taxation due to rollover provisions, the one time $125,000 exclusion for homeowners over 55, and the fact that inherited real estate receives a stepped-up cost basis when transferred. We simplify the calculation of housing return by assuming that neither the rental value nor the capital gains portion of the return is taxed. As for Treasury bills and Treasury bonds, we assume annual return is subject to a federal income tax at a marginal rate of 28%. In respect to stocks, we assume income from dividends is taxed at a marginal rate of 33%, while capital appreciation escape taxation completely. Formally, returns on financial assets are calculated as follows:

$$r_t^{TreasuryBill} = \frac{(1-t)\prod_{i=1}^{12}(1+USTRET_i)}{1+inflation_t} - 1,$$

$$r_t^{TreasuryBond} = \frac{(1-t)\prod_{i=1}^{12}(1+GBTRET_i)}{1+inflation_t} - 1, \text{ and}$$

$$r_t^{Stock} = \frac{\prod_{i=1}^{12}(1+CSCRET_i+(1-t)CSIRET_i)}{1+inflation_t} - 1.$$

CITIBASE

Mortgage rates we used are taken from CITIBASE Monthly data tape. We use Conventional Home Mortgage Rate - Loans Closed (National Average for All Major Lenders: FYMCLE). We first take annual average (April to March) of mortgage rates, and convert it to after-tax effective rate using marginal tax rate of 33%, and then deflate by inflation rate. Detailed description is provided in Micro Data section under Mortgage Rate.

Appendix 2   Mathematics of Efficient Frontier and Coefficient of Risk Aversion

No Constraints on Asset Holdings

In the absence of a riskless asset, the minimum-variance portfolio with expected return μ is the solution $\mathbf{w}(\mu)$ to the following problem (see, for example, Ingersoll (1987)).

$$\text{Minimize} \qquad \mathbf{w'Vw} \tag{A2.1a}$$

$$\text{Subject to} \qquad \mathbf{i'w} = 1 \tag{A2.1b}$$
$$\mathbf{r'w} = \mu \tag{A2.1c}$$

where $\mathbf{w}$ is a vector of portfolio weights of n assets, $\mathbf{V}$ is the n x n covariance matrix of individual assets, $\mathbf{i}$ an n x 1 vector of ones, and $\mathbf{r}$ an n x 1 vector of expected returns of individual assets. Forming the Lagrangean, differentiating it with respect to $\mathbf{w}$, and solving the system of first order conditions, we obtain the solution set:

$$\mathbf{w^*}(\mu) = \lambda \mathbf{V^{-1}i} + \gamma \mathbf{V^{-1}r} \tag{A2.2}$$

Lagrange multipliers λ and γ can be determined from the budget constraint (1b) and (1c):

$$? = \frac{C - B\mu}{D}, \qquad\qquad ? = \frac{A\mu - B}{D},$$

where

$$A = \mathbf{i'V^{-1}i} \qquad\qquad B = \mathbf{i'V^{-1}r} = \mathbf{r'V^{-1}i} \tag{A2.3}$$
$$C = \mathbf{r'V^{-1}r} \qquad\qquad D = AC - B^2$$

The equation of the optimal portfolio variance is

$$\sigma^2(\mu) = \mathbf{w^{*'}V\,w^*} = \mathbf{w^{*'}V}(\lambda \mathbf{V^{-1}i} + \gamma \mathbf{V^{-1}r})$$
$$= \lambda + \gamma\mu. \tag{A2.4}$$

The slope of this efficient frontier measures the marginal rate of transformation (MRT) between risk and return:

$$MRT = \frac{d\mu}{ds} = \frac{Ds}{A\mu - B} \tag{A2.5}$$

Blake (1996) shows that if the utility function is of the form of constant relative risk aversion,

$$U(W) = \frac{1}{1-?} W^{1-?} \tag{A2.6}$$

and if portfolio returns are normally distributed with mean μ and variance σ^2 and if investors maximize expected utility, then indifference curves are represented by a local approximation (see also Levy and Markowitz, 1979; Sharpe, 1992) :

$$\mu = EU + \frac{1}{2} ?s^2 \tag{A2.7}$$

where EU is an index of expected utility, and ρ is the Arrow-Pratt coefficient of relative risk aversion[1]. The slope of indifference curves measures the marginal rate of substitution (MRS) between risk and return, or the price of risk reduction, that is given by:

$$MRS = \frac{d\mu}{ds} = ?s \tag{A2.8}$$

The optimal portfolio is that for which (5) and (8) are equal, which implies that the coefficient of risk aversion is given by:

$$? = \frac{D}{A\mu - B} \tag{A2.9}$$

Expected utility maximizing investors must select optimal portfolio consistent with (9) holding.

Nonnegativity Constraints on Asset Holdings and Mortgage Constraint

We have so far considered the case where no restrictions other than (1b) and (1c) are imposed. This does not impose any short sales constraints and hence elements of **w** can be

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[1]   If the utility function is constant absolute risk aversion of the form $U(w) = e^{-\rho w}$, then (7) holds exactly, given the assumption of normality of asset returns. See Hirshleifer and Riley (1992).

negative.   The cases that involve the short-sales constraint or other constraints are simple

extensions of the above. In the case of short-sales constraint and the mortgage constraint, where

the size of mortgage cannot be bigger than the value of the house, we modify the problem by

adding the following constraints:

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ -w_n \end{bmatrix} \geq \mathbf{0} \tag{A2.1d}$$

and

$$\mathbf{x}\cdot\mathbf{w} \geq 0, \tag{A2.1e}$$

where w_n is the portfolio weight of mortgage that cannot be positive, $\mathbf{x}$ is an nx1 vector of the

form:

$$\mathbf{x} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix},$$

where 1's in the $(n-1)^{th}$ and n^{th} rows of vector $\mathbf{x}$ corresponds to the portfolio weight of housing

asset and mortgage, respectively.

The solution to this problem carries an additional term of shadow prices of the constraints,

$$\mathbf{w}^*(\mu) = \lambda\mathbf{V^{-1}i} + \gamma\mathbf{V^{-1}r} + \kappa\mathbf{V^{-1}x} + \mathbf{V^{-1}w} \tag{A2.2$'$}$$

where $\mathbf{w}$ is an n x 1 vector of the Lagrange multipliers associated with the inequality constraints

such that the complementary slackness $\mathbf{w}\cdot\mathbf{w} = 0$ is satisfied, and κ the Lagrange multiplier

associated with mortgage constraint such that $\kappa\,\mathbf{x}\cdot\mathbf{w} = 0$.

With a new term in the system of first order conditions, the values of λ and γ now

become:

$$? = \frac{-B\mu + C - ?(CE - BF) - (C\pmb{\gamma}' - B\mathbf{r}')\mathbf{V}^{-1}\pmb{?}}{D},$$

$$(A2.3)'$$

$$? = \frac{A\mu - B - ?(AF - BE) - (A\mathbf{r}' - B\pmb{\gamma}')\mathbf{V}^{-1}\pmb{?}}{D}$$

where

$$E = \mathbf{i}'\mathbf{V}^{-1}\mathbf{x} = \mathbf{x}'\mathbf{V}^{-1}\mathbf{i} \qquad F = \mathbf{r}'\mathbf{V}^{-1}\mathbf{x} = \mathbf{x}'\mathbf{V}^{-1}\mathbf{r}.$$

The optimal portfolio variance is still given by (4) with the new values of λ and γ.

Exactly as before, the coefficient of relative risk aversion can be calculated by equating the slope of the efficient frontier to the MRS of the representative investor:

$$? = \frac{2D}{2A\mu - 2B - ?(AF - BE) - (A\mathbf{r}' - B\pmb{\gamma}')\mathbf{V}^{-1}\pmb{?}}$$

$$(A2.9)'$$

As for non-zero elements of $\mathbf{w}$ and the value of κ, the values of Lagrange multipliers obtained from the quadratic programming subroutine are used to calculate ρ when the inequality constraints and mortgage constraint are binding.

Housing Constraint

Households often finance the purchase of a house with a high proportion of mortgage, because of the lumpiness and indivisibility of housing investment and the inseparability of housing consumption and investment. In such situations, the optimization problem becomes slightly more complicated but can be handled in exactly the same way as before, by simply adding a new equality constraint. The leverage constraint can be written as:

$$\mathbf{z}'\mathbf{w} = h$$

$$(A2.1f)$$

where $\mathbf{x}$ is an n x 1 vector of the form:

$$\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}.$$

Here the $(n-1)^{th}$ element of $\mathbf{w}$ corresponds to the portfolio weight of housing asset, as before.

Solving the optimization problem (1a) with constraints (1b) to (1f) gives the optimal portfolio weight:

$$\mathbf{w}^* (\mu) = \lambda \mathbf{V^{-1}i} + \gamma \mathbf{V^{-1}r} + \kappa\, \mathbf{V^{-1}x} + \eta \mathbf{V^{-1}z} + \mathbf{V^{-1}w}$$

(A2.2)″

where η is the Lagrange multiplier associated with house-size constraint.   Treating κ and η as parameters and solving the system of equations for λ and γ, we obtain:

$$? = \frac{- B\mu + C - ?(CE - BF) + ?(CG - BH) - (C? - B\mathbf{r}')\mathbf{V^{-1}?}}{D},$$

(A2.3)″

$$? = \frac{A\mu - B - ?(AF - BE) - ?(AH - BG) - (A\mathbf{r}' - B?)\mathbf{V^{-1}?}}{D}$$

where

$$G = \mathbf{i}?\mathbf{V^{-1}z} = \mathbf{z}\,?\mathbf{V^{-1}i} \qquad H = \mathbf{r}\,?\mathbf{V^{-1}z} = \mathbf{z}\,?\mathbf{V^{-1}r}.$$

The optimal portfolio variance is now determined by

$$\sigma^2 (\mu) = \lambda + \gamma\mu + \eta h, \tag{A2.4′}$$

and the coefficient of relative risk aversion is given by:

$$? = \frac{2D}{2A\mu - 2B - ?(AF - BE) - ?(AH - BG) - (A\mathbf{r}' - B?)\mathbf{V^{-1}?}}.$$

(A2.9)″

Outputs from the quadratic programming subroutine for the values of η, κ, and $\mathbf{w}$ are used to calculate the values of ρ.