Title
Fibered biset functors

Authors
Boltje, Robert
Coşkun, Olcay

Publication Date
2018-12-01

DOI
10.1016/j.aim.2018.09.034

Fibered Biset Functors*

Robert Boltje†
Department of Mathematics
University of California
Santa Cruz, CA 95064
U.S.A.
boltje@math.ucsc.edu

Olcay Coşkun‡
Department of Mathematics
Boğaziçi University
Bebek, 34342 Istanbul
Türkiye
olcay.coskun@boun.edu.tr

December 4, 2016

Abstract

The theory of biset functors, introduced by Serge Bouc, gives a unified treatment of operations in representation theory that are induced by permutation bimodules. In this paper, by considering fibered bisets, we introduce and describe the basic theory of fibered biset functors which is a natural framework for operations induced by monomial bimodules. The main result of this paper is the classification of simple fibered biset functors.

Introduction

In [Bou96] and [Bou10b] Serge Bouc introduced and developed the theory of *biset functors*. This notion provides a framework for situations where structural maps that behave like *restriction*, *induction*, *inflation*, and *deflation*, or a subset of them, are present. Typical examples of biset functors are the various representation rings, as for instance the Burnside ring, the character ring, the Green ring, and the trivial source ring of a finite group G. The Brauer character ring of G and cohomology groups in fixed degree of G are other examples (no deflation in these cases). The theory of biset functors proved to be the right framework to prove striking results: The determination of the Dade group of endo-permutation modules of a p-group (see [BT00] for the final result, or [T07] for an overview) and the determination of the unit group of the Burnside ring of a p-group (see [Bou10a] or [Bou10b]). Both the Dade group and the unit group of the Burnside ring provide further examples of biset functors.

*MR Subject Classification 19A22, 20C15, 20C20

†Research supported by the NSF, DMS-0200592

‡Research supported partially by Tübitak-1001-113F240.

Biset functors are additive, abelian group valued functors on the biset category, whose objects are finite groups and whose morphism sets are given by the Grothendieck groups $B(G,H)$ of finite (G,H)-bisets. The composition is induced by a construction that imitates the tensor product of bimodules. Restriction, induction, inflation and deflation can be seen as particular transitive bisets.

Recently, a category analogous to the biset category, whose objects are finite sets and morphisms are correspondences between them, together with functors on this category have been considered by Bouc and Thévenaz, see [BT15]. Again surprising connections to other areas, for instance to lattice theory, came to light.

In this article we strive to systematically develop a similar theory of *fibered biset functors*. The motivation comes from the fact that representation rings of finite groups carry more structure, by considering multiplication with one-dimensional representations as structural maps. Monomial Burnside rings have been introduced by Dress in [D71] in great generality and utilized by the first author in the formalism of canonical induction formulas, see [Bol89] and [Bol98a]. See also [Ba04] and [R13] for some basic properties. Let A be a multiplicatively written abelian group. The A-fibered Burnside ring $B^A(G)$ is the Grothendieck group of A-fibered G-sets, i.e., $G \times A$-sets with finitely many orbits which are free as A-sets. A $\mathbb{Z}$-basis of $B^A(G)$ is parametrized by G-conjugacy classes of pairs (H,ϕ), where H is a subgroup of G and $\phi \in \mathrm{Hom}(H,A)$. Similarly one defines A-fibered (G,H)-bisets and their Grothendieck groups $B^A(G,H) = B^A(G\times H)$. Also fibered bisets allow a tensor product construction which gives rise to the A-fibered biset category over a commutative ring k, denoted by $\mathcal{C}_k^A$. Its objects are again finite groups and its morphism sets are given by $B_k^A(G,H) := k \otimes B^A(G,H)$. The k-linear functors from $\mathcal{C}_k^A$ to ${}_k\mathsf{Mod}$ together with natural transformations form the abelian category $\mathcal{F}_k^A$ of *A-fibered biset functors over k*. If $k=\mathbb{Z}$, we usually suppress the index k. In the case that A is the unit group of a field K, one obtains a natural linearization map from $B^A(G)$ to various representation rings of KG-modules, by mapping the class of (H,ϕ) to $\mathrm{Ind}_H^G(K_\phi)$, where K_ϕ denotes the one-dimensional KH-module associated with the homomorphism $\phi\colon H \to K^\times$. Using the linearization map one can interpret these representation rings as additive functors on the A-fibered biset category with values in the category of abelian groups, i.e., as *A-fibered biset functors*.

One has a natural embedding $B(G) \to B^A(G)$ and also a natural splitting map $B^A(G) \to B(G)$ of this embedding in the category of rings. The embedding allows to view the biset category as a subcategory of the A-fibered biset category. To view an A-fibered biset functor F via restriction as a biset functor means forgetting some of its structure. Thus, if a biset functor comes via restriction from an A-fibered biset functor it is worth trying to understand its richer structure as an A-fibered biset functor.

The main goal of this paper is to determine the simple A-fibered biset functors. For this we first need to study the tensor product of A-fibered bisets and how transitive A-fibered bisets can be factored through smaller groups. This is done in Sections 1 and 2. The situation for bisets is as follows: It is shown in [Bou10b] that a transitive (G,H)-biset $(G\times H)/U$, with $U \leqslant G \times H$, is equal to the product of five canonical bisets. More precisely, one has

$$\left(\frac{G\times H}{U}\right) = \ \mathrm{Ind}_P^G \times_P \mathrm{Inf}_{P/K}^P \times_{P/K} \mathrm{c}_{P/K,Q/L}^{\eta} \times_{Q/L} \mathrm{Def}_{Q/L}^Q \times_Q \mathrm{Res}_Q^H. \tag{1}$$

Here, $P = p_1(U)$ and $Q = p_2(U)$ are the first and the second projections of the subgroup $U \leqslant G \times H$ and $K = p_1(U \cap (P \times 1))$ and $L = p_2(U \cap (1 \times Q))$. In this case, the groups P/K and Q/L are isomorphic and a canonical isomorphism η is determined by U. Moreover, $\times_?$ denotes the product of bisets. See 2.7 for the description of the factors appearing in (1). We usually write a transitive A-*fibered* (G, H)-biset X in the form $\left(\frac{G\times H}{U,\phi}\right)$, where $U \leqslant G \times H$ is the stabilizer of the A-orbit of a chosen element $x \in X$ and $\phi\colon U \to A$ is the homomorphism arising from the action of U on this A-orbit. Different choices of x lead to $G \times H$-conjugates of (U, ϕ). In Corollary 2.5 we derive a Mackey formula for the tensor product of two transitive A-fibered bisets and use it to show an analogue of the decomposition in (1). As above, U determines a quintuple (P, K, η, L, Q). Further, the homomorphism $\phi\colon U \to A$ determines the following data: Let $\phi_1 \times \phi_2^{-1}$ be the decomposition of the restriction of ϕ to $K \times L$ and let $\hat{K}$ (resp. $\hat{L}$) be the kernel of ϕ_1 (resp. ϕ_2^{-1}). We write $l(U, \phi) = (P, K, \phi_1)$ and $r(U, \phi) = (Q, L, \phi_2)$ and call them the left and the right invariants of (U, ϕ). In Proposition 2.8 we prove the following decomposition

$$\left(\frac{G\times H}{U,\phi}\right) = \mathrm{Ind}_P^G \otimes_{AP} \mathrm{Inf}_{P/\hat{K}}^P \otimes_{AP/\hat{K}} \mathrm{Y} \otimes_{AQ/\hat{L}} \mathrm{Def}_{Q/\hat{L}}^Q \otimes_{AQ} \mathrm{Res}_Q^H. \tag{2}$$

Here, Y is a transitive A-fibered $(P/\hat{K}, Q/\hat{L})$-biset with stabilizing pair $(\bar{U}, \bar{\phi})$ such that the first and the second projections of $\bar{U}$ are surjective, and the homomorphisms $\bar{\phi}_1$ and $\bar{\phi}_2$ are faithful. In contrast to the theory of bisets is that in general the A-fibered biset Y can be factored through groups of smaller order than $P/\hat{K}$ and $Q/\hat{L}$. The problem remains to decompose Y further by factoring through a smallest possible group. It is solved in Section 10 under additional assumptions on A, see Hypothesis 10.1, which hold for instance if A is the group of units of an algebraically closed field. Under this assumption, we decompose X as follows. For simplicity, assume $P = G, Q = H, \hat{K} = 1$ and $\hat{L} = 1$. Set $\tilde{K} = G' \cap K$ and $\tilde{L} = H' \cap L$, where G' (resp. H') denotes the derived subgroup of G (resp. H). We construct a triple $(\tilde{G}, M, \mu)$, where $\tilde{G}$ is a central extension of G/K by $\tilde{K}$, M is a subgroup of $G \times \tilde{G}$, and $\mu \in \mathrm{Hom}(M, A)$, with additional properties as described in Section 10, and also a triple $(\tilde{H}, N, \psi)$ with analogous properties related to H, L, and $\tilde{L}$, such that

$$Y \cong \mathrm{Ins}_{\tilde{G}}^G \otimes_{A\tilde{G}} X \otimes_{A\tilde{H}} \mathrm{Del}_{\tilde{H}}^H\,,$$

where X is an A-fibered $(\tilde{G}, \tilde{H})$-biset which is reduced, that is, cannot be factored further through a group of order smaller than $|\tilde{G}| = |\tilde{H}|$. Here the *insertion* fibered biset $\mathrm{Ins}_{\tilde{G}}^G$ *inserts* the section $K/\tilde{K}$ into the group $\tilde{G}$ and the *deletion* fibered biset $\mathrm{Del}_{\tilde{H}}^H$ *deletes* the section $L/\tilde{L}$ from the group H. Unfortunately, the construction of $\mathrm{Ins}_{\tilde{G}}^G$ and $\mathrm{Del}_{\tilde{H}}^H$ uses choices which make them not canonical.

In Section 3 we introduce A-fibered biset functors and recall basic properties of such functor categories from [Bou96]. In order to determine the simple A-fibered biset functors over a commutative ring k one needs to find the simple modules of the endomorphism algebra $E_k^A(G) = B_k^A(G,G)$ of a finite group G which are annihilated by the ideal $I_k^A(G)$ of endomorphisms that factor through groups of smaller order. In order to determine the simple $E_k^A(G)/I_k^A(G)$-modules we consider the subalgebra $E_k^{A,c}(G)$ of $E_k^A(G)$ spanned over k by the isomorphism classes of *covering* transitive A-fibered (G,G)-bisets $\left(\frac{G\times G}{U,\phi}\right)$, i.e., satisfying $p_1(U) = G = p_2(U)$. This subalgebra of $E_k^A(G)$ covers $E_k^A(G)/I_k^A(G)$ and it is isomorphic to a product of matrix rings over group algebras $k\Gamma_{(G,K,\kappa)}$. Here, $\Gamma_{(G,K,\kappa)}$ is the group of isomorphism classes of covering transitive A-fibered (G,G)-bisets $\left(\frac{G\times G}{U,\phi}\right)$ with $l(U,\phi) = (G,K,\kappa) = r(U,\phi)$. This result is proved in the course of Sections 4, 5, and 6. In Section 4 we introduce the central idempotents of E_G^c which split the algebra E_G^c into matrix rings. In Section 5 we determine an equivalence relation on the pairs (K,κ) whose equivalence classes will parametrize the various matrix components. And in Section 6 we prove that $E_k^{A,c}(G)$ has the announced structure. In Section 7, the groups $\Gamma_{(G,K,\kappa)}$ are identified as extensions of a subgroup of the outer automorphism group $\mathrm{Out}(G/K)$ (determined by κ) and the group $\mathrm{Hom}(G/K,A)$. Section 8 is devoted to understanding which simple modules of E_G^c are annihilated by $E_G \cap I_G$. For a given matrix component, indexed by (K,κ), this only depends on the pair (K,κ). Pairs that will lead to simple E_G/I_G-modules are called *reduced* pairs. Unfortunately, for general A we do not have a handy criterion when (K,κ) is reduced. Proposition 8.6 gives a necessary and a (different) sufficient condition for (K,κ) to be reduced.

In Section 9 we show that the simple A-fibered biset functors are parametrized by equivalence classes of quadruples $(G,K,\kappa,[V])$, where G is a finite group, (K,κ) is a reduced pair, and $[V]$ is the isomorphism class of a simple $k\Gamma_{(G,K,\kappa)}$-module. This equivalence relation involves the notion of *linkage* of two tripes (G,K,κ) and (H,L,λ) by a pair (U,ϕ) with $U \leqslant G\times H$ and $\phi \in \mathrm{Hom}(U,A)$ such that $l(U,\phi) = (G,K,\kappa)$ and $r(U,\phi) = (H,L,\lambda)$. In Proposition 5.3 a group cohomology criterion is given to determine if two triples are linked. There exist examples of such linked triples with G and H not isomorphic. They lead to a negative answer of a question asked by Bouc, see Remark 9.8, which has been observed independently by Romero in [R12].

In Section 10, we show that under a further hypothesis on the group A, one can show that a pair (K,κ) is reduced if and only if $K \leqslant Z(G)\cap G'$ and κ is injective. This uses lengthy computations in the cohomology groups $H^2(G/K,A)$ and related cohomology groups. Finally, in Section 11 we realize some representation rings as simple fibered biset functors over appropriate fields of coefficients.

Notation Throughout the paper, we will adopt the following conventions:

If G is a group, $H \leqslant G$ is a subgroup and $x \in G$, then ${}^xH := xHx^{-1}$. By $H < G$ we indicate that H is a proper subgroup of G, i.e., $H \neq G$. If k is a commutative ring we denote by kG or $k[G]$ the group algebra of G over k. If $H \leqslant G$ and M is a left $k[H]$-module then ${}^xM$ is the $k[{}^xH]$-module with underlying k-module equal to M and ${}^xH$-action given by $xhx^{-1}\cdot m := hm$, for $h \in H$ and $m \in M$. If A is an abelian group, $\phi\colon H \to A$ is a group homomorphism and $x \in G$, then we define the homomorphism ${}^x\phi\colon {}^xH \to A$ by $xhx^{-1} \mapsto \phi(h)$ for $h \in H$.

For any ring R, the isomorphism class of an irreducible left R-module V is denoted by $[V]$ and the set of isomorphism classes of simple left R-modules is denoted by $\mathrm{Irr}(R)$.

Acknowledgement Work on this paper began 2007 during a visit of the second author at UC Santa Cruz. Since then both authors have enjoyed the support of the following institutions while working on this projects: UC Santa Cruz, MSRI, Boğaziçi University, Nesin Mathematics Village, Centre Interfacultaire Bernoulli. We would like to express our gratitude to these institutions.

1 A-Fibered Bisets

Throughout this article, we fix a multiplicatively written abelian group A. For every finite group G, we set

$$G^* := \mathrm{Hom}(G, A)$$

and view G^* as an abelian group with point-wise multiplication.

1.1 The categories ${}_G\mathsf{set}^A$ and ${}_G\mathsf{set}^A_H$. Let G be a finite group. An *A-fibered G-set* is a left $A \times G$-set X which is free as an A-set and has finitely many A-orbits. A morphism between two A-fibered G-sets is just an $A \times G$-equivariant map. The A-fibered G-sets and their morphisms form a category which we denote by ${}_G\mathsf{set}^A$.

We often consider a left G-set X also as a right G-set via

$$xg = g^{-1}x$$

for any $x \in X$ and $g \in G$ and vice-versa. However, when we view a left A-set as a right A-set we always do this via

$$xa = ax \tag{3}$$

and vice-versa. Elements of A play a similar role as the elements of a commutative ring k, when switching sides for modules over the group algebra kG.

If also H is a finite group we denote by the category ${}_G\mathsf{set}^A_H$ of *A-fibered (G, H)-bisets* as the category ${}_{G\times H}\mathsf{set}^A$. By the above convention, we can view an object $X \in {}_{G\times H}\mathsf{set}^A$ as equipped with a left G-action, a right H-action and a two-sided A-action through (3), all three commuting with each other.

1.2 The set $\mathcal{M}_G = \mathcal{M}^A(G)$. For a finite group G, we denote by $\mathcal{M}^A(G)$ the set of all pairs (K, κ) where K is a subgroup of G and $\kappa \in K^*$. Often we will just write $\mathcal{M}_G$ for $\mathcal{M}^A(G)$. The set $\mathcal{M}_G$ has a poset structure given by $(L, \lambda) \leqslant (K, \kappa)$ if $L \leqslant K$ and $\lambda = \kappa|_L$. Moreover, G acts on $\mathcal{M}_G$ by conjugation via ${}^g(K, \kappa) := ({}^gK, {}^g\kappa)$ for $g \in G$. Note that conjugation respects the poset structure. We denote the G-orbit of (K, κ) by $[K, \kappa]_G$.

Let also H be a finite group and let $(U, \phi) \in \mathcal{M}_{G\times H}$. We denote by

$$p_1 \colon G \times H \to G \quad \text{and} \quad p_2 \colon G \times H \to H$$

the projection maps and set

$$k_1(U) := \{g \in G \mid (g,1) \in U\} \quad \text{and} \quad k_2(U) := \{h \in H \mid (1,h) \in U\}\,.$$

We also set

$$k(U) := k_1(U) \times k_2(U) \leqslant U\,,$$

the largest 'rectangular' subgroup of U. Furthermore, for $i = 1, 2$, we define homomorphisms $\phi_i \in k_i(U)^*$ by

$$\phi|_{k(U)} = \phi_1 \times \phi_2^{-1}\,.$$

The reason we use ϕ_2^{-1} in the above equation is that later formulas will look nicer. Finally, we associate to (U, ϕ) its *left invariants* and *right invariants*

$$l(U,\phi) := (p_1(U), k_1(U), \phi_1) \quad \text{and} \quad r(U,\phi) := (p_2(U), k_2(U), \phi_2)\,.$$

Sometimes we will only be interested in part of these invariants and also define

$$l_0(U,\phi) := (k_1(U), \phi_1) \quad \text{and} \quad r_0(U,\phi) := (k_2(U), \phi_2)\,.$$

1.3 Proposition *Let $(U, \phi) \in \mathcal{M}_{G\times H}$ and set*

$$(P, K, \kappa) := l(U,\phi) \quad \text{and} \quad (Q, L, \lambda) := r(U,\phi)\,.$$

Moreover, set

$$\hat{K} := \ker(\kappa)\,, \quad \hat{L} := \ker(\lambda)\,, \quad \tilde{K} := \hat{K}P' \cap K \quad \text{and} \quad \tilde{L} := \hat{L}Q' \cap L\,.$$

Then, clearly,

$$1 \leqslant \hat{K} \leqslant \tilde{K} \leqslant K \leqslant P \leqslant G \quad \text{and} \quad 1 \leqslant \hat{L} \leqslant \tilde{L} \leqslant L \leqslant Q \leqslant H\,.$$

Moreover:

(a) *$\hat{K}$, $\tilde{K}$ and K are normal in P, and $\hat{L}$, $\tilde{L}$ and L are normal in Q.*

(b) *$K/\hat{K}$ is central in $P/\hat{K}$ and $L/\hat{L}$ is central in $Q/\hat{L}$.*

(c) *One has group isomorphisms*

$$P/K \cong U/(K \times L) \cong Q/L$$

which are induced by the projection maps $U \to P$ and $U \to Q$. The resulting isomorphism

$$\eta = \eta_U \colon Q/L \to P/K$$

is characterized by

$$\eta(qL) = pK \iff (p,q) \in U\,,$$

for $q \in Q$ and $p \in P$.

(d) *One has group isomorphisms*

$$\tilde{K}/\hat{K} \cong \ker(\phi|_{\tilde{K}\times\tilde{L}})/(\hat{K}\times\hat{L}) \cong \tilde{L}/\hat{L}$$

induced by the projection maps $\tilde{K}\times\tilde{L}\to\tilde{K}$ *and* $\tilde{K}\times\tilde{L}\to\tilde{L}$. *The resulting isomorphism*

$$\zeta = \zeta_{U,\phi}\colon \tilde{L}/\hat{L} \to \tilde{K}/\hat{K}$$

is characterized by

$$\zeta(\tilde{l}\hat{L}) = \tilde{k}\hat{K} \iff \kappa(\tilde{k}) = \lambda(\tilde{l})\,,$$

for $\tilde{l}\in\tilde{L}$ *and* $\tilde{k}\in\tilde{K}$.

(e) *If* $\hat{K}=1$, $\tilde{K}=K$ *and* $P=G$ *then* $|G|\leqslant|H|$.

Proof Parts (a), (b) and (c) are easy verifications and are left to the reader. Some parts of these statements are also known as Goursat's theorem.

(d) We only show that the projection

$$\bar{p}_1\colon \ker(\phi|_{\tilde{K}\times\tilde{L}})/(\hat{K}\times\hat{L}) \to \tilde{K}/\hat{K}\,,\quad (\tilde{k},\tilde{l})(\hat{K}\times\hat{L})\mapsto\tilde{k}\hat{K}\,,$$

is an isomorphism. (The other isomorphism is proved with the same arguments and the last statement follows easily.) Clearly, $\bar{p}_1$ is a well-defined group homomorphism, and it is easy to see that $\bar{p}_1$ is injective. To show that $\bar{p}_1$ is surjective, let $\tilde{k}\in\tilde{K}$. After multiplying $\tilde{k}$ with an element from $\hat{K}$ we may assume that $\tilde{k}\in P'$. Then, there exist elements $g_1,g_1',\ldots,g_n,g_n'\in P$ such that

$$\tilde{k} = [g_1,g_1']\cdots[g_n,g_n']\,.$$

By the definition of P there exist elements $h_1,h_1',\ldots,h_n,h_n' \in Q$ such that $(g_i,h_i),(g_i',h_i')\in U$ for $i=1,\ldots,n$. Set

$$\tilde{l} := [h_1,h_1']\cdots[h_n,h_n']\in Q'\,.$$

It follows that $(\tilde{k},\tilde{l})\in U'$, which implies that $\phi(\tilde{k},\tilde{l})=1$. Since $\tilde{K}\leqslant K$ we have $\tilde{l}\in L$. Since $\tilde{l}$ belongs also to Q', we have $\tilde{l}\in\tilde{L}$ and $\bar{p}_1(\tilde{k},\tilde{l})=\tilde{k}\hat{K}$.

(e) This follows immediately from parts (c) and (d). □

1.4 Stabilizing pairs. Let X be an A-fibered (G,H)-biset. We will denote the A-orbit of the element $x\in X$ by $[x]$. Note that $G\times H$ acts on the set of A-orbits of X by $(g,h)[x]:=[(g,h)x]$. For $x\in X$, let $S_x\leqslant G\times H$ denote the stabilizer of $[x]$.

We define a map

$$X\to\mathcal{M}_{G\times H}\,,\quad x\mapsto(S_x,\phi_x)\,, \tag{4}$$

with

$$\phi_x\colon S_x\to A$$

determined by the equation

$$(g,h)x = \phi_x(g,h)x$$

for $(g,h) \in S_x$. We call (S_x, ϕ_x) the *stabilizing pair* of x. With the notation introduced in the previous paragraph, we also obtain group homomorphisms

$$\phi_{x,1}\colon k_1(S_x) \to A \quad \text{and} \quad \phi_{x,2}\colon k_2(S_x) \to A$$

determined by the equations

$$\phi_{x,1}(g)x = gx \quad \text{and} \quad \phi_{x,2}(h)x = xh\,,$$

for $g \in k_1(S_x)$ and $h \in k_2(S_x)$. Since the action of $G \times H$ and A on X commute, the definitions of ϕ_x, $\phi_{x,1}$ and $\phi_{x,2}$ do not depend on the choice of x in its A-orbit. Note also that for $(g,h) \in G \times H$, we have

$$(S_{(g,h)x}, \phi_{(g,h)x}) = {}^{(g,h)}(S_x, \phi_x)\,, \quad \phi_{(g,h)x,1} = {}^{g}\phi_{x,1}\,, \quad \phi_{(g,h)x,2} = {}^{h}\phi_{x,2}\,.$$

Thus, the map defined in (4) is constant on the A-orbits of X, and considered as a map on the A-orbits of X, it is $G \times H$-equivariant.

1.5 Transitive A-fibered (G,H)-bisets. Let X be an A-fibered (G,H)-biset. It is clear that the $A \times G \times H$-action on X is transitive if and only if the $G \times H$-action on the set of A-orbits of X is transitive. In this case we call X a *transitive* A-fibered (G,H)-biset. There exists a bijective correspondence between the isomorphism classes of transitive A-fibered (G,H)-bisets and the $G \times H$-conjugacy classes of $\mathcal{M}_{G\times H}$. We describe this correspondence. If X is a transitive A-fibered (G,H)-biset, choose an element $x \in X$ and associate to X the class $[S_x, \phi_x]_{G\times H}$ as defined in (4). Conversely, given a pair (U,ϕ) we construct an A-fibered (G,H)-biset X by $X := (A \times G \times H)/U_\phi$, where U_ϕ is the subgroup of $G \times H \times A$ consisting of all elements $(\phi(u^{-1}), u)$, with $u \in U$. If we start with the conjugate pair ${}^{(g,h)}(U,\phi)$ instead of (U,ϕ), we obtain the conjugate subgroup ${}^{(1,g,h)}U_\phi$ of $G \times H \times A$ and therefore, we obtain isomorphic A-fibered (G,H)-bisets. It is easy to see that these two constructions are mutually inverse.

For $(U,\phi) \in \mathcal{M}_{G\times H}$ we denote the corresponding transitive A-fibered (G,H)-biset and its isomorphism class by

$$\left(\frac{G\times H}{U,\phi}\right) \quad \text{and} \quad \left[\frac{G\times H}{U,\phi}\right],$$

respectively.

1.6 Opposite of an A-fibered biset. Let X be an A-fibered (G,H)-biset. We define the *opposite* $X^{\mathrm{op}} \in {}_{H}\mathsf{set}_{G}^{A}$ of X as the A-fibered (H,G)-biset which has X as underlying set, but endowed with the left $A \times H \times G$-action given by

$$(a,h,g)x := (a^{-1}, g, h)x\,.$$

In particular, if $X = \left(\frac{G\times H}{U,\phi}\right)$ then $X^{\mathrm{op}} \cong \left(\frac{H\times G}{U^{\mathrm{op}},\phi^{\mathrm{op}}}\right)$ with

$$U^{\mathrm{op}} := \{(h,g) \in H \times G \mid (g,h) \in U\} \quad \text{and} \quad \phi^{\mathrm{op}}(h,g) := \phi(g,h)^{-1}\,.$$

Thus, $k_1(U^{\mathrm{op}}) = k_2(U)$, $k_2(U^{\mathrm{op}}) = k_1(U)$, $(\phi^{\mathrm{op}})_1 = \phi_2$, and $(\phi^{\mathrm{op}})_2 = \phi_1$.

1.7 The Grothendieck group. We denote the isomorphism class of an A-fibered (G,H)-biset X by $[X]$. Let X and Y be two A-fibered (G,H)-bisets. The disjoint union $X \coprod Y$ of X and Y is the categorical coproduct. On the set of isomorphism classes it induces the structure of a monoid $[X]+[Y] := [X \coprod Y]$. The corresponding Grothendieck group is called the *Burnside group* of A-fibered (G,H)-bisets and is denoted by $B^A(G,H)$. Every A-fibered (G,H)-biset X is represented by a class $[X]$ in $B^A(G,H)$ and the elements $\left[\frac{G\times H}{U,\phi}\right]$, with $[U,\phi]_{G\times H} \in G \times H\backslash\mathcal{M}_{G\times H}$, form a $\mathbb{Z}$-basis of the abelian group $B^A(G,H)$. We will denote again by $-^{\mathrm{op}}\colon B^A(G,H) \to B^A(H,G)$ the isomorphism induced by taking opposite fibered bisets. With this we have a canonical isomorphism $B^A(G,H) \cong B^A(H,G)$ of abelian groups.

As a special case, when H is the trivial group we obtain the Grothendieck group $B^A(G)$ of the category ${}_G\mathsf{set}^A$ with respect to disjoint unions.

1.8 Remark (a) The group $B^A(G)$ can be interpreted as the result of the $-_+$ construction, see [Bol98a], applied to the group rings $\mathbb{Z}G^* = \mathbb{Z}\mathrm{Hom}(G,A)$.

(b) Assume that $A = k^\times$, the unit group of an integral domain k. Then $B^A(G)$ can also be interpreted as the Grothendieck group of the k-linear additive category ${}_{kG}\mathsf{mon}$ of finite G-equivariant line bundles over k, which was introduced in [Bol01].

2 Tensor product of fibered bisets

In this section, we introduce the tensor product of fibered bisets. The usual product construction of bisets does not work in the fibered case, since under this product, the A-action may not remain free. The remedy is to consider the subset of free A-orbits in the product. We will call this construction the tensor product. After establishing associativity of the tensor product, we prove an explicit formula (see Corollary 2.5), called the Mackey formula, for the tensor product of two transitive A-fibered bisets, and a first level decomposition of a transitive fibered biset into a product of standard (fibered) bisets (see Proposition 2.8). The latter we call the standard decomposition. The middle factor of this decomposition will be decomposed further in Section 10. Both the Mackey formula and the standard decomposition are completely analogous to formulas in the biset category, see [Bou10b, Section 2.3].

2.1 The tensor product. Given finite groups G, H, K and objects $X \in {}_G\mathsf{set}^A_H$ and $Y \in {}_H\mathsf{set}^A_K$, we will define their *tensor product* $X \otimes_{AH} Y \in {}_G\mathsf{set}^A_K$. First recall the definition of the product $X \times_{AH} Y$ of the right $A \times H$-set X with the left $A \times H$-set Y: It is the set of $A \times H$-orbits of $X \times Y$ under the action

$$^{(a,h)}(x, y) = (x(a^{-1}, h^{-1}), (a, h)y)$$

for $(a, h) \in A \times H$ and $(x, y) \in X \times Y$. We will denote the $A \times H$-orbit of (x, y) by $[x, y]_{AH}$. Note that we have $[xa, y]_{AH} = [x, ay]_{AH}$ and $[xh, y]_{AH} = [x, hy]_{AH}$ for $a \in A$ and $h \in H$.

The set $X \times_{AH} Y$ is an A-set via $a[x, y]_{AH} := [ax, y]_{AH} = [x, ay]_{AH}$ and it is a (G, K)-biset via $(g, k)[x, y]_{AH} := [gx, yk^{-1}]_{AH}$. These two actions commute so that $X \times_{AH} Y$ is a left $A \times G \times K$-set. However, in general the A-action is not free. Note that the action of $G \times K$ permutes the free A-orbits of $X \times_{AH} Y$. This allows us to define

$$X \otimes_{AH} Y \in {}_G\mathsf{set}_K^A$$

as the union of the free A-orbits of $X \times_{AH} Y$. If the stabilizer of $[x, y]_{AH}$ in A is trivial, i.e., if $[x, y]_{AH} \in X \otimes_{AH} Y$, we will write $x \otimes_{AH} y$ instead of $[x, y]_{AH}$. Note that the construction $X \otimes_{AH} Y$ is functorial in X and Y. It is clear from the definitions that the tensor product respects disjoint unions:

$$(X \coprod X') \otimes_{AH} Y \cong (X \otimes_{AH} Y) \coprod (X' \otimes_{AH} Y) \,,$$
$$X \otimes_{AH} (Y \coprod Y') \cong (X \otimes_{AH} Y) \coprod (X \otimes_{AH} Y') \,,$$

for $X, X' \in {}_G\mathsf{set}_H^A$ and $Y, Y' \in {}_H\mathsf{set}_K^A$. It is also straightforward to verify that

$$(X \otimes_{AH} Y)^{\mathrm{op}} \cong Y^{\mathrm{op}} \otimes_{AH} X^{\mathrm{op}} \,,$$

as A-fibred (K, G)-bisets, under the map $[x, y] \mapsto [y, x]$.

2.2 Associativity. Let G, H, K, L be finite groups, let $X \in {}_G\mathsf{set}_H^A$, $Y \in {}_H\mathsf{set}_K^A$ and $Z \in {}_K\mathsf{set}_L^A$ be A-fibered bisets. We will show that there exists an isomorphism

$$(X \otimes_{AH} Y) \otimes_{AK} Z \cong X \otimes_{AH} (Y \otimes_{AK} Z)$$

which maps $(x \otimes_{AH} y) \otimes_{AK} z$ to $x \otimes_{AH} (y \otimes_{AK} z)$ for $x \in X$, $y \in Y$ and $z \in Z$. It is well-known that the map

$$\begin{aligned} (X \times_{AH} Y) \times_{AK} Z &\to X \times_{AH} (Y \times_{AK} Z) \,, \\ \Big[[x, y]_{AH}, z\Big]_{AK} &\mapsto \Big[x, [y, z]_{AK}\Big]_{AH} \,, \end{aligned} \tag{5}$$

is a bijection. It is also clear that it is an isomorphism of $A \times G \times L$-sets. Since $(X \otimes_{AH} Y) \otimes_{AK} Z$ is a subset of the left hand side and $X \otimes_{AH} (Y \otimes_{AK} Z)$ is a subset of the right hand side, it suffices to show that the above isomorphism restricts to these subsets. Note that $[[x, y]_{AH}, z]_{AK} \in (X \otimes_{AH} Y) \otimes_{AK} Z$ if and only if the stabilizer S_1 of $[x, y]_{AH}$ in A is trivial and the stabilizer S_2 of $[[x, y]_{AH}, z]_{AK}$ in A is trivial. Since $S_1 \leqslant S_2$, this is equivalent to the statement that $S_2 = 1$. Similarly, $[x, [y, z]_{AK}]_{AH} \in X \otimes_{AH} (Y \otimes_{AK} Z)$ if and only if the stabilizer T_2 of $[x, [y, z]_{AK}]_{AH}$ in A is trivial. But since the map in (5) is an A-equivariant isomorphism, $S_2 = T_2$, and the proof is complete.

2.3 Our next aim is to find an explicit formula for the tensor product of two transitive A-fibered bisets. For this purpose, we will need the following notation. Let G, H and K be finite groups, let $U \leqslant G \times H$ and $V \in H \times K$ be subgroups, and let $\phi \in U^*$ and $\psi \in V^*$ be homomorphisms satisfying

$$\phi_2|_{k_2(U)\cap k_1(V)} = \psi_1|_{k_2(U)\cap k_1(V)}. \tag{6}$$

Following [Bou96], we set

$$U * V := \{(g,k) \in G \times K \mid \text{there exists } h \in H \text{ with } (g,h) \in U \text{ and } (h,k) \in V\}.$$

Moreover, we define $\phi * \psi \in (U * V)^*$ by

$$(\phi * \psi)(g,k) := \phi(g,h)\psi(h,k),$$

where $h \in H$ is chosen such that $(g,h) \in U$ and $(h,k) \in V$, a construction that has also been used in [Bou10c]. Note that, by the condition in (6), this does not depend on the choice of $h \in H$.

For the following proposition, we fix transitive A-fibered bisets $X \in {}_G\mathsf{set}_H^A$ and $Y \in {}_H\mathsf{set}_K^A$, and also elements $x \in X$ and $y \in Y$. Let $(S_x, \phi_x) \in \mathcal{M}_{G\times H}$ and $(S_y, \phi_y) \in \mathcal{M}_{H\times K}$ denote their stabilizing pairs. Also, set $H_x := p_2(S_x)$ and $H_y := p_1(S_y)$. Note that every $A \times G \times K$-orbit of $X \times_{AH} Y$ has an element of the form $[x, hy]_{AH}$, with $h \in H$.

2.4 Proposition *Assume the above notation.*

(a) *Let $t, t' \in H$. The elements $[x, ty]_{AH}$ and $[x, t'y]_{AH}$ belong to the same $A \times G \times K$-orbit of $X \times_{AH} Y$ if and only if $H_x t H_y = H_x t' H_y$.*

(b) *Let $t \in H$. The stabilizer of $[x, ty]_{AH}$ in A is trivial if and only if*

$$\phi_{x,2}|_{H_t} = {}^t\phi_{y,1}|_{H_t},$$

where $H_t := k_2(S_x) \cap {}^tk_1(S_y)$.

(c) *The elements $[x, ty]_{AH} \in X \times_{AH} Y$, where t runs through a set of representatives of the double cosets $H_x\backslash H/H_y$ such that $\phi_{x,2}|_{H_t} = {}^t\phi_{y,1}|_{H_t}$, form a set of representatives for the $A \times G \times K$-orbits of $X \otimes_{AH} Y$. For $t \in H$ with $\phi_{x,2}|_{H_t} = {}^t\phi_{y,1}|_{H_t}$, the stabilizing pair of $[x, ty]_{AH}$ is equal to*

$$(S_x * {}^{(t,1)}S_y,\ \phi_x * {}^{(t,1)}\phi_y).$$

Proof (a) This follows from the following chain of equivalences:

$$
\begin{aligned}
& [x,ty]_{AH} \text{ and } [x,t'y]_{AH} \text{ belong to the same } A\times G\times K\text{-orbit} \\
\iff & \exists g\in G, k\in K, a\in A\colon [x,t'y]_{AH} = [agx, tyk]_{AH} \\
\iff & \exists g\in G, k\in K, a,b\in A, h\in H\colon (x,t'y) = (agxh^{-1}b^{-1}, hbtyk) \\
\iff & \exists g\in G, k\in K, a,b\in A, h\in H\colon (g,h)\in S_x, ({t'}^{-1}ht, k^{-1})\in S_y, \\
& \qquad \phi_x(g,h) = a^{-1}b, \phi_y({t'}^{-1}ht, k^{-1}) = b^{-1} \\
\iff & \exists g\in G, h\in H, k\in K\colon (g,h)\in S_x, ({t'}^{-1}ht, k^{-1})\in S_y \\
\iff & \exists h\in H\colon h\in H_x, {t'}^{-1}ht\in H_y \\
\iff & \exists h\in H_x\colon t\in ht'H_y \\
\iff & t\in H_x t' H_y\,.
\end{aligned}
$$

(b) Let $a\in A$. Then we have the following chain of equivalences:

$$
\begin{aligned}
& [x,ty]_{AH} = [x,aty]_{AH} \\
\iff & \exists h\in H, b\in A\colon (x,ty) = (xh^{-1}b^{-1}, bhaty) \\
\iff & \exists h\in H, b\in A\colon h\in k_2(S_x), t^{-1}ht\in k_1(S_y) \\
& \qquad \phi_{x,2}(h^{-1}) = b, \phi_{y,1}(t^{-1}ht) = a^{-1}b^{-1} \\
\iff & \exists h\in k_2(S_x)\cap {}^{t}k_1(S_y)\colon {}^{t}\phi_{y,1}(h) = a^{-1}\phi_{x,2}(h) \\
\iff & a\in \operatorname{im}\Big(\phi_{x,2}|_{H_t}\cdot {}^{t}\phi_{y,1}^{-1}|_{H_t}\Big)\,.
\end{aligned}
$$

Thus, the stabilizer of $[x,ty]_{AH}$ in A is trivial if and only if $\phi_{x,2}|_{H_t} = {}^{t}\phi_{y,1}|_{H_t}$.

(c) With the results from (a) and (b), we only have to verify the statement about the stabilizing pair of $[x,ty]_{AH}$. Let $(W,\lambda)\in\mathcal{M}_{G\times K}$ denote the stabilizing pair of $[x,ty]_{AH}$, and let $(g,k)\in G\times K$ and $a\in A$. Then we have the chain of equivalences

$$
\begin{aligned}
& (g,k)\in W \text{ and } \lambda(g,k) = a \\
\iff & [gx, tyk^{-1}]_{AH} = [x,aty]_{AH} \\
\iff & \exists b\in A, h\in H\colon (gxh^{-1}b^{-1}, hbtyk^{-1}) = (x,aty) \\
\iff & \exists b\in A, h\in H\colon (g,h)x = bx, (t^{-1}ht,k)y = ab^{-1}y \\
\iff & (g,k)\in S_x * {}^{(t,1)}S_y \text{ and there exists } h\in H \text{ such that} \\
& \qquad (g,h)\in S_x,\ (h,k)\in {}^{(t,1)}S_y \text{ and } {}^{(t,1)}\phi_y(h,k) = a\phi_x(g,h)^{-1} \\
\iff & (g,k)\in S_x * {}^{(t,1)}S_y \text{ and } (\phi_x * {}^{(t,1)}\phi_y)(g,k) = a.
\end{aligned}
$$

Thus, $W = S_x * {}^{(t,1)}S_y$ and $\lambda = \phi_x * {}^{(t,1)}\phi_y$. □

The following Mackey formula is completely analogous to the formula for bisets, see [Bou10b, Lemma 2.3.24].

2.5 Corollary (Mackey formula) *Let* $(U,\phi) \in \mathcal{M}_{G\times H}$ *and* $(V,\psi) \in \mathcal{M}_{H\times K}$. *There exists an isomorphism of A-fibered* (G,K)*-bisets,*

$$\left(\frac{G\times H}{U,\phi}\right) \otimes_{AH} \left(\frac{H\times K}{V,\psi}\right) \cong \coprod_{\substack{t\in p_2(U)\backslash H/p_1(V)\\ \phi_2|_{H_t}={}^t\psi_1|_{H_t}}} \left(\frac{G\times K}{U * {}^{(t,1)}V, \phi * {}^{(t,1)}\psi}\right),$$

where $H_t := k_2(U) \cap {}^tk_1(V)$. *The above isomorphism maps the trivial coset of the* t*-component of the right hand side to the element* $U_\phi \otimes_{AH} (t,1,1)V_\psi$.

Proof This is an immediate consequence of Proposition 2.4(c) by choosing $x = U_\phi$ and $y = U_\psi$ to be the trivial cosets of $(G\times H\times A)/U_\phi$ and $(H\times K\times A)/U_\psi$, respectively. □

The proof of the following proposition is straightforward and is left to the reader, see also [Bou10b, Lemma 2.3.22] for Parts (a) and (c).

2.6 Proposition *Let* $(U,\phi) \in \mathcal{M}_{G\times H}$ *and* $(V,\psi) \in \mathcal{M}_{H\times K}$.

(a) *One has*

$$k_1(U) \leqslant k_1(U*V) \leqslant p_1(U*V) \leqslant p_1(U) \quad \text{and} \quad k_2(V) \leqslant k_2(U*V) \leqslant p_2(U*V) \leqslant p_2(V)$$

(b) *Assume that the restrictions of* ϕ_2 *and* ψ_1 *to* $k_2(U)\cap k_1(V)$ *coincide. Then*

$$l_0(U,\phi) \leqslant l_0(U*V,\phi*\psi) \quad \text{and} \quad r_0(V,\psi) \leqslant r_0(U*V,\phi*\psi).$$

(c) *Let* $\eta_U\colon p_2(U)/k_2(U) \to p_1(U)/k_1(U)$ *be the isomorphism from Proposition* 1.3(c). *Then one has*

$$k_1(U*V)/k_1(U) \cong \eta_U\Big((p_2(U)\cap k_1(V)k_2(U))/k_2(U)\Big).$$

(d) *If* $r(U,\phi) = l(V,\psi)$ *then* $l(U*V,\phi*\psi) = l(U,\phi)$ *and* $r(U*V,\phi*\psi) = r(V,\psi)$.

Next we want to show that every transitive A-fibered (G,H)-biset X is the tensor product of the canonical operations restriction, deflation, inflation, induction (which themselves are bisets) and a transitive A-fibered $(\bar{G},\bar{H})$-biset Y for certain sections $\bar{G}$ and $\bar{H}$ of G and H, respectively. First we need the following notations from [Bou96].

2.7 Notation (a) Let G and H be finite groups. For any subgroups $H_2 \leqslant H_1 \leqslant H$ and any group homomorphism $f\colon H_1 \to G$, we set

$${}_f\Delta(H_2) := \{(f(h),h) \mid h \in H_2\} \leqslant G\times H.$$

For any subgroups $G_2 \leqslant G_1 \leqslant G$ and any group homomorphism $f\colon G_1 \to H$, we set

$$\Delta_f(G_2) := \{(g,f(g)) \mid g \in G_2\} \leqslant G\times H.$$

Moreover, if f is the inclusion map of a subgroup, we only write $\Delta(H_2)$, resp. $\Delta(G_2)$.

(b) For a finite group G, a subgroup H of G and a normal subgroup N of G one defines the objects

$$\mathrm{Ind}_H^G := \left(\frac{G \times H}{\Delta(H), 1}\right) \in {}_G\mathsf{set}_H^A\,, \quad \mathrm{Res}_H^G := \left(\frac{H \times G}{\Delta(H), 1}\right) \in {}_H\mathsf{set}_G^A\,,$$
$$\mathrm{Inf}_{G/N}^G := \left(\frac{G \times G/N}{\Delta_\pi(G), 1}\right) \in {}_G\mathsf{set}_{G/N}^A\,, \quad \mathrm{Def}_{G/N}^G := \left(\frac{G/N \times G}{{}_\pi\Delta(G), 1}\right) \in {}_{G/N}\mathsf{set}_G^A\,,$$

where $\pi\colon G \to G/N$ denotes the canonical epimorphism. These objects are called *induction, restriction, inflation* and *deflation*.

The following proposition is a straightforward verification, using the explicit formula in Corollary 2.5. It is completely analogous to the proof of Lemma 3 in [Bou96, Section 3] or [Bou10b, Lemma 2.3.26].

2.8 Proposition *Let G and H be finite groups and let $(U, \phi) \in \mathcal{M}_{G\times H}$. Then, with the notation from Proposition 1.3, one has the decomposition*

$$\left(\frac{G \times H}{U, \phi}\right) \cong \mathrm{Ind}_P^G \otimes_{AP} \mathrm{Inf}_{P/\hat{K}}^P \otimes_{A(P/\hat{K})} X \otimes_{A(Q/\hat{L})} \mathrm{Def}_{Q/\hat{L}}^Q \otimes_{AQ} \mathrm{Res}_Q^H\,,$$

where

$$X = \left(\frac{P/\hat{K} \times Q/\hat{L}}{U/(\hat{K} \times \hat{L}), \bar{\phi}}\right)\,.$$

In the above proposition the homomorphism $\bar{\phi}\colon U/(\hat{K} \times \hat{L}) \to A$ is induced by ϕ and the group $U/(\hat{K} \times \hat{L})$ is viewed as a subgroup of $(P/\hat{K}) \times (Q/\hat{L})$ via the canonical isomorphism

$$\mathrm{can}\colon (P \times Q)/(\hat{K} \times \hat{L}) \to (P/\hat{K}) \times (Q/\hat{L})\,.$$

Note that setting $\bar{G} := P/\hat{K}$, $\bar{H} := Q/\hat{L}$, $\bar{U} := \mathrm{can}(U/(\hat{K} \times \hat{L})) \leqslant \bar{G} \times \bar{H}$, and $\bar{\phi}\colon \bar{U} \to A$ (using the above isomorphism), we have

$$p_1(\bar{U}) = \bar{G}\,, \quad \ker(\bar{\phi}_1) = \{1\}\,, \quad p_2(\bar{U}) = \bar{H}\,, \quad \ker(\bar{\phi}_2) = \{1\}\,.$$

3 A-fibered Biset Functors

Throughout this section, let k denote a commutative ring. In this section we recall and use the general results in [Bou96, Sections 2 and 4] on functor categories, and we specialize the approach in [Bou96, Section 3] to our situation.

3.1 Definition (a) By $\mathcal{C} = \mathcal{C}_k^A$ we denote the following category. Its objects are the finite groups. For finite groups G and H, we set

$$\mathrm{Hom}_{\mathcal{C}}(G, H) := B_k^A(H, G) := k \otimes B^A(H, G)\,.$$

If also K is a finite group then composition in $\mathcal{C}$ is defined by

$$-\underset{H}{\cdot}-\colon B_k^A(K,H)\times B_k^A(H,G)\to B_k^A(K,G)\,,\quad (y,x)\mapsto y\underset{H}{\cdot}x\,,$$

the k-linear extension of the bilinear map induced by taking the tensor product of A-fibered bisets. The identity element of G is the element $\left[\frac{G\times G}{\Delta(G),1}\right]$.

(b) An *A-fibered biset functor* over k is a k-linear functor from the k-linear category $\mathcal{C}_k^A$ to the k-linear category ${}_k\mathsf{Mod}$ of left k-modules. By $\mathcal{F}=\mathcal{F}_k^A$ we denote the category of A-fibered biset functors over k. Their morphisms are the natural transformations. Since the category ${}_k\mathsf{Mod}$ is abelian, also the category $\mathcal{F}_k^A$ is abelian, with point-wise constructions of kernels, cokernels, etc. As explained in Section 2 of [Bou96], this allows to define subfunctors, quotient functors, simple functors, projective functors, etc. If $M\in\mathcal{F}$, $m\in M(H)$ and $b\in B_k^A(G,H)$, we will usually write bm instead of $(\mathcal{F}(b))(m)$.

3.2 Remark (a) Two finite groups G and H are isomorphic as groups if and only if they are isomorphic in $\mathcal{C}$. In fact, if $\phi\colon G\to H$ is a group isomorphism then $\left[\frac{G\times H}{\Delta_\phi(G),1}\right]$ is an isomorphism between G and H in $\mathcal{C}$. Conversely, assume $a\in B_k^A(G,H)$ and $b\in B_k^A(H,G)$ satisfy $ab=1$ and $ba=1$. Then, the equation $ab=1$ and the Mackey formula imply that there exist standard basis elements $\left[\frac{G\times H}{U,\phi}\right]$ and $\left[\frac{H\times G}{V,\psi}\right]$ such that $U*V=\Delta(G)$. Proposition 2.6(a) then implies that $k_1(U)=\{1\}$ and $p_1(U)=G$. Now Proposition 1.3(c) implies that G is isomorphic to a section of H. Similarly, $ba=1$ implies that H is isomorphic to a section of G. Thus, G and H are isomorphic groups.

(b) By Part (a), one may replace $\mathcal{C}$ with a full subcategory as long as every isomorphism type of finite groups is represented and one obtains a functor category that is equivalent to $\mathcal{F}$. Thus, we may assume that $\mathrm{Ob}(\mathcal{C})$ is a set, without changing the equivalence class of $\mathcal{F}$.

The following remark shows that $\mathcal{F}$ also has some functorial properties and rigidity when changing the abelian group A.

3.3 Remark (a) If $f\colon A\to A'$ is a homomorphism of abelian groups then one obtains an induced k-linear homomorphism $B_k^A(G,H)\to B_k^{A'}(G,H)$ for any two finite groups G and H. Moreover, these homomorphisms induce a k-linear functor $\mathcal{C}_k^A\to\mathcal{C}_k^{A'}$, and by restriction along this functor a k-linear functor $\mathcal{F}_k^{A'}\to\mathcal{F}_k^A$ between the associated functor categories. If f is an isomorphism then all these induced k-linear homomorphisms and functors are isomorphisms.

(b) The inclusion $\mathrm{tor}A\subseteq A$ induces a k-linear isomorphism $B_k^{\mathrm{tor}A}(G,H)\xrightarrow{\sim}B_k^A(G,H)$ for any two finite groups G and H, and further k-linear isomorphisms $\mathcal{C}_k^{\mathrm{tor}A}\to\mathcal{C}_k^A$ and $\mathcal{F}_k^A\xrightarrow{\sim}\mathcal{F}_k^{\mathrm{tor}A}$.

3.4 We recall several constructions from [Bou96, Section 2].
For a finite group G let $E_G=E_k^A(G)$ denote the endomorphism algebra of G in $\mathcal{C}_k^A$, i.e.,

$$E_G=E_k^A(G):=\mathrm{End}_{\mathcal{C}}(G)=B_k^A(G,G)\,.$$

Clearly, E_G is a k-algebra, and for any fibered biset functor F, the k-module $F(G)$ has the structure of a left E_G-module. This way one obtains a functor, given by evaluation at G,

$$\mathcal{E}_G\colon \mathcal{F}_k^A \to {}_{E_G}\mathsf{Mod}$$

which maps a functor $F \in \mathcal{F}_k^A$ to $F(G)$ and a natural transformation $\eta\colon F \to F'$ to η_G. The evaluation functor $\mathcal{E}_G$ has a left adjoint which we now describe. To a left E_G-module V one associates the functor $L_{G,V} \in \mathcal{F}_k^A$, which is defined on an object H of $\mathcal{C}$ by

$$L_{G,V}(H) := \mathrm{Hom}_{\mathcal{C}}(G,H) \otimes_{E_G} V = B_k^A(H,G) \otimes_{E_G} V\,,$$

where $\mathrm{Hom}_{\mathcal{C}}(G,H)$ is considered as a right E_G-module via composition of morphisms. If also K is an object of $\mathcal{C}$ and if $\phi \in \mathrm{Hom}_{\mathcal{C}}(K,H)$ then $L_{G,V}(\phi)$ is given by composition with ϕ in the first factor of the tensor product. This way one obtains a functor

$$L_{G,-}\colon {}_{E_G}\mathsf{Mod} \to \mathcal{F}_k^A$$

which is left adjoint to $\mathcal{E}_G$.

Lemma 1 in [Bou96] specializes in our situation to the following Lemma.

3.5 Lemma *Let V be a simple left E_G-module. Then the A-fibered biset functor $L_{G,V}$ has a unique maximal subfunctor $J_{G,V}$. Its evaluation at a finite group H is given by*

$$J_{G,V}(H) = \left\{ \sum_i x_i \otimes v_i \in B_k^A(H,G) \otimes_{E_G} V \;\Big|\; \forall y \in B_k^A(G,H)\colon\ \sum_i (y \mathop{\cdot}_{H} x_i)(v_i) = 0 \right\}.$$

Moreover, the simple head $S_{G,V}$ of $L_{G,V}$ satisfies $S_{G,V}(G) \cong V$.

3.6 The essential algebra. For a finite group G we set

$$I_G = I_k^A(G) := \sum_{|H|<|G|} B_k^A(G,H) \mathop{\cdot}_{H} B_k^A(H,G) \subseteq B_k^A(G,G) = E_G\,.$$

The sum runs over all finite groups H of order smaller than $|G|$. Obviously, I_G is an ideal of E_G and we denote by $\bar{E}_G := E_G/I_G$ the factor k-algebra. It is called the *essential algebra* of G.

3.7 Proposition (a) *Let $S \in \mathcal{F}$ be a simple functor and let G be a finite group such that $V := S(G) \neq \{0\}$. Then V is a simple E_G-module and $S \cong S_{G,V}$ in $\mathcal{F}$.*

(b) *Let $S \in \mathcal{F}$ be a simple functor and let G be a finite group of smallest order satisfying $S(G) \neq \{0\}$. Then the simple E_G-module $S(G)$ is annihilated by I_G. In particular, every simple functor $S \in \mathcal{F}$ is isomorphic to $S_{G,V}$ for some finite group G and some simple E_G-module V which is annihilated by I_G.*

(c) *Let G be a finite group and let V be a simple E_G-module which is annihilated by I_G. Then G is* a minimal group *for $S_{G,V}$,* i.e., *$S_{G,V}(G) \neq \{0\}$ and for every finite group H with $S_{G,V}(H) \neq \{0\}$ one has $|G| \leqslant |H|$.*

(d) *For finite groups G, H and simple modules $V \in {}_{E_G}\mathsf{Mod}$ and $W \in {}_{E_H}\mathsf{Mod}$ one has $S_{G,V} \cong S_{H,W}$ if and only if $S_{G,V}(H) \cong W$ as E_H-modules.*

Proof Part (a) follows from a short argument given at the beginning of Section 4 in [Bou96]. Part (b) is immediate from the definitions and Part (a). To prove Part (c) assume that $|H| < |G|$. Then the description of $J_{G,V}(H)$ in Lemma 3.5 implies that $L_{G,V}(H) = J_{G,V}(H)$. Thus, $S_{G,V}(H) = \{0\}$. Finally, Part (d) follows immediately from Part (a) and the last statement in Lemma 3.5. $\square$

4 Idempotents in E_G

In this section, we introduce idempotents in E_G that will play an important role later.

Recall that G acts on $\mathcal{M}_G$ by conjugation. We will denote the G-fixed point set by $\mathcal{M}_G^G$. A pair $(K,\kappa) \in \mathcal{M}_G$ is G-fixed if and only if K is a normal subgroup of G and κ is a G-stable homomorphism. Note that if also H is a finite group, $(U,\phi) \in \mathcal{M}_{G\times H}$ and $p_1(U) = G$ then, by Proposition 1.3(a) and (b), one has $l_0(U,\phi) \in \mathcal{M}_G^G$.

4.1 Definition Let G be a finite group and let $(K,\kappa) \in \mathcal{M}_G^G$. We define the A-fibered (G,G)-biset $E_{(K,\kappa)}$ as

$$E_{(K,\kappa)} := \left(\frac{G\times G}{\Delta_K(G),\phi_\kappa}\right),$$

where

$$\Delta_K(G) := \{(g_1,g_2) \in G\times G \mid g_1K = g_2K\} = (K\times\{1\})\Delta(G) = (\{1\}\times K)\Delta(G)$$
$$\text{and} \quad \phi_\kappa(g_1,g_2) = \kappa(g_2^{-1}g_1) = \kappa(g_1g_2^{-1}).$$

Note that $\Delta_K(G)$ is a subgroup of $G\times G$, since K is normal in G, and that ϕ_κ is a homomorphism, since κ is G-stable. Note also that $E_{(K,\kappa)}^{\mathrm{op}} \cong E_{(K,\kappa)}$.

4.2 Proposition *Let G and H be finite groups and let $(K,\kappa),(K',\kappa') \in \mathcal{M}_G^G$. Moreover, let $(U,\phi) \in \mathcal{M}_{G\times H}$ with $l(U,\phi) = (G,K',\kappa')$.*

(a) *One has $l(\Delta_K(G),\phi_\kappa) = (G,K,\kappa) = r(\Delta_K(G),\phi_\kappa)$.*

(b) *One has*

$$E_{(K,\kappa)} \otimes_{AG} \left(\frac{G\times H}{U,\phi}\right) \cong \begin{cases} \emptyset, & \text{if } \kappa|_{K\cap K'} \neq \kappa'|_{K\cap K'},\\ \left(\frac{G\times H}{(K\times 1)U,\kappa\cdot\phi}\right), & \text{if } \kappa|_{K\cap K'} = \kappa'|_{K\cap K'},\end{cases}$$

where $(\kappa\cdot\phi)((k,1)u) := \kappa(k)\phi(u)$ for $k\in K$ and $u \in U$. Moreover, in the second case, one has

$$l((K\times 1)U,\kappa\cdot\phi) = (G,KK',\kappa\cdot\kappa').$$

In particular one has

$$E_{(K,\kappa)} \otimes_{AG} E_{(K',\kappa')} \cong \begin{cases} \emptyset, & \text{if } \kappa|_{K\cap K'} \neq \kappa'|_{K\cap K'},\\ E_{(KK',\kappa\cdot\kappa')}, & \text{if } \kappa|_{K\cap K'} = \kappa'|_{K\cap K'}.\end{cases}$$

(c) *Assume that* $(K,\kappa) \leqslant (K',\kappa')$. *Then*

$$E_{(K,\kappa)} \otimes_{AG} \left(\frac{G\times H}{U,\phi}\right) \cong \left(\frac{G\times H}{U,\phi}\right).$$

In particular, one has $E_{(K,\kappa)} \otimes_{AG} E_{(K',\kappa')} \cong E_{(K',\kappa')}$ *and* $E_{(K,\kappa)} \otimes_{AG} E_{(K,\kappa)} \cong E_{(K,\kappa)}$.

(d) *Assume that* $p_2(U,\phi) = H$. *Then*

$$\left(\frac{G\times H}{U,\phi}\right) \otimes_{AH} \left(\frac{G\times H}{U,\phi}\right)^{\mathrm{op}} \cong E_{(K',\kappa')}$$

in ${}_G\mathsf{set}_G^A$.

Proof Part (a) is an easy verification. Parts (b) and (d) follow immediately from the explicit Mackey formula in Corollary 2.5. Part (c) is a special case of part (b). □

4.3 For $(K,\kappa) \in \mathcal{M}_G^G$ we set

$$e_{(K,\kappa)} := [E_{(K,\kappa)}] \in B_k^A(G,G) = E_G\,.$$

Note that $e_{(1,1)} = 1 \in E_G$. By Proposition 4.2, we have

$$e_{(K,\kappa)} \cdot e_{(L,\lambda)} = \begin{cases} e_{(KL,\kappa\lambda)}, & \text{if } \kappa|_{K\cap L} = \lambda|_{K\cap L}, \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

for all $(K,\kappa), (L,\lambda) \in \mathcal{M}_G^G$. This implies that, we obtain a commutative subalgebra

$$\bigoplus_{(K,\kappa)\in\mathcal{M}_G^G} k e_{(K,\kappa)} \tag{8}$$

of E_G. For $(K,\kappa), (L,\lambda) \in \mathcal{M}_G^G$, let $\mu^{\triangleleft}_{(K,\kappa),(L,\lambda)}$ denote the Möbius coefficient with respect to the poset $\mathcal{M}_G^G$. Since κ is determined by λ if $(K,\kappa) \leqslant (L,\lambda)$, this coefficient is equal to the Möbius coefficient $\mu^{\triangleleft}_{K,L}$ of K and L with respect to the poset of normal subgroups of G. For $(K,\kappa) \in \mathcal{M}_G^G$, we set

$$f_{(K,\kappa)} := \sum_{(K,\kappa)\leqslant(L,\lambda)\in\mathcal{M}_G^G} \mu^{\triangleleft}_{K,L} e_{(L,\lambda)} \tag{9}$$

and obtain by Möbius inversion that

$$e_{(K,\kappa)} = \sum_{(K,\kappa)\leqslant(L,\lambda)\in\mathcal{M}_G^G} f_{(L,\lambda)}\,. \tag{10}$$

It follows that also the elements $f_{(K,\kappa)}$, $(K,\kappa) \in \mathcal{M}_G^G$, form a k-basis of the algebra in (8). Moreover, note that

$$\sum_{(K,\kappa)\in\mathcal{M}_G^G} f_{(K,\kappa)} = e_{(1,1)} = 1 \in E_G\,. \tag{11}$$

The next proposition shows that these basis elements are mutually orthogonal idempotents.

4.4 Proposition *For all* $(K,\kappa),(L,\lambda)\in\mathcal{M}_G^G$ *one has*

$$e_{(K,\kappa)}f_{(L,\lambda)} = f_{(L,\lambda)}e_{(K,\kappa)} = \begin{cases} f_{(L,\lambda)}, & \text{if } (K,\kappa)\leqslant(L,\lambda),\\ 0, & \text{otherwise,}\end{cases}$$

and

$$f_{(K,\kappa)}f_{(L,\lambda)} = \begin{cases} f_{(K,\kappa)}, & \text{if } (K,\kappa)=(L,\lambda),\\ 0, & \text{otherwise.}\end{cases}$$

Proof First note that equations (7) and (9) imply immediately that $e_{(K,\kappa)}f_{(L,\lambda)} = f_{(L,\lambda)}$ when $(K,\kappa)\leqslant(L,\lambda)$. Recall also that $e_{(K,\kappa)}$ and $f_{(L,\lambda)}$ commute.

Next we prove the remaining statements of the proposition by induction on $d = d(K,\kappa)+d(L,\lambda)$, where $d(K,\kappa)$ is defined as the largest $n\in\mathbb{N}_0$ such that there exists a chain $(K,\kappa)=(K_0,\kappa_0)<\cdots<(K_n,\kappa_n)$ in $\mathcal{M}_G^G$.

If $d=0$ then (K,κ) and (L,λ) are maximal in $\mathcal{M}_G^G$. Thus, $f_{(K,\kappa)} = e_{(K,\kappa)}$ and $f_{(L,\lambda)} = e_{(L,\lambda)}$. We only have to show that $e_{(K,\kappa)}e_{(L,\lambda)} = 0$ when $(K,\kappa)\neq(L,\lambda)$. So assume that $e_{(K,\kappa)}e_{(L,\lambda)}\neq 0$. Then equation (7) implies that $\kappa|_{K\cap L} = \lambda|_{K\cap L}$ and that there exists a pair $(KL,\kappa\lambda)\in\mathcal{M}_G^G$ with $(K,\kappa)\leqslant(KL,\kappa\lambda)\geqslant(L,\lambda)$. Since (K,κ) and (L,λ) are maximal in $\mathcal{M}_G^G$, we obtain $(K,\kappa)=(KL,\kappa\lambda)=(L,\lambda)$.

Now assume that $d>0$ and that the proposition holds for smaller values of d. We first show that if $f_{(K,\kappa)}f_{(L,\lambda)}\neq 0$ then $(K,\kappa)=(L,\lambda)$. In fact $f_{(K,\kappa)}f_{(L,\lambda)}\neq 0$ implies, after expanding $f_{(K,\kappa)}$ and $f_{(L,\lambda)}$ according to equation (9) and using equation (7), that $\kappa|_{K\cap L}=\lambda|_{K\cap L}$ and that

$$f_{(K,\kappa)}f_{(L,\lambda)} \in \bigoplus_{(KL,\kappa\lambda)\leqslant(M,\mu)\in\mathcal{M}_G^G} ke_{(M,\mu)}\,.$$

This implies that $e_{(KL,\kappa\lambda)}f_{(K,\kappa)}f_{(L,\lambda)} = f_{(K,\kappa)}f_{(L,\lambda)}$ by Equation (7). Assume that $(K,\kappa)\neq(L,\lambda)$. Then $(K,\kappa)<(KL,\kappa\lambda)$ or $(L,\lambda)<(KL,\kappa\lambda)$, and by induction we obtain $e_{(KL,\kappa\lambda)}f_{(K,\kappa)}=0$ or $e_{(KL,\kappa\lambda)}f_{(L,\lambda)}=0$. In either case we obtain $f_{(K,\kappa)}f_{(L,\lambda)} = e_{(KL,\kappa\lambda)}f_{(K,\kappa)}f_{(L,\lambda)}=0$, a contradiction.

Next we assume $(K,\kappa)=(L,\lambda)$. By the result of the previous paragraph and by induction we have

$$e_{(K,\kappa)} = e_{(K,\kappa)}^2 = \Big(\sum_{(K,\kappa)\leqslant(K',\kappa')\in\mathcal{M}_G^G} f_{(K',\kappa')}\Big)^2 = f_{(K,\kappa)}^2 + \sum_{(K,\kappa)<(K',\kappa')\in\mathcal{M}_G^G} f_{(K',\kappa')}\,.$$

Comparing this with Equation (10) we obtain $f_{(K,\kappa)}^2 = f_{(K,\kappa)}$.

Finally, Equation (10) implies that

$$e_{(K,\kappa)}f_{(L,\lambda)} = \sum_{(K,\kappa)\leqslant(K',\kappa')\in\mathcal{M}_G^G} f_{(K',\kappa')}f_{(L,\lambda)}\,.$$

By induction and by what we already proved, the latter sum is equal to $f_{(L,\lambda)}$ if $(K,\kappa)\leqslant(L,\lambda)$ and equal to 0 otherwise. □

In Section 9 we will need the following lemma.

4.5 Lemma *Let G and H be finite groups and let $(U,\phi)\in\mathcal{M}_{G\times H}$ with $p_1(U)=G$ and $p_2(U)=H$. Set $(K,\kappa):=l_0(U,\phi)$ and $(L,\lambda):=r_0(U,\phi)$. Then*

$$\left[\frac{G\times H}{U,\phi}\right]\underset{H}{\cdot}f_{(L,\lambda)}=f_{(K,\kappa)}\underset{G}{\cdot}\left[\frac{G\times H}{U,\phi}\right]\underset{H}{\cdot}f_{(L,\lambda)}=f_{(K,\kappa)}\underset{G}{\cdot}\left[\frac{G\times H}{U,\phi}\right]$$

Proof We only prove the first equation. The second one follows by a similar argument or by applying $-^{\mathrm{op}}$. Set $x:=\left[\frac{G\times H}{U,\phi}\right]$. Note that $e_{(K,\kappa)}x=x$ (see Proposition 4.2(c)). Using the definition of $f_{(K,\kappa)}$, it suffices to show that $e_{(K',\kappa')}xf_{(L,\lambda)}=0$ for all $(K',\kappa')\in\mathcal{M}_G^G$ with $(K',\kappa')>(K,\kappa)$. By Proposition 4.2(b), we have $e_{(K',\kappa')}x=\left[\frac{G\times H}{V,\psi}\right]$ for some $(V,\psi)\in\mathcal{M}_{G\times H}$ satisfying $l(V,\psi)=(G,K',\kappa')$ and $r(V,\psi)=(H,L',\lambda')$ for some $(L',\lambda')\in\mathcal{M}_H^H$ with $(L,\lambda)\leqslant(L',\lambda')$. By Proposition 1.3(c), we have $G/K\cong H/L$ and also $G/K'\cong H/L'$. Since $K<K'$, this implies $L<L'$. Thus, $e_{(K',\kappa')}xf_{(L,\lambda)}=e_{(K',\kappa')}xe_{(L',\lambda')}f_{(L,\lambda)}=0$ by Proposition 4.4, and the proof is complete. □

5 Linkage

In this section, we introduce an equivalence relation, that we call linkage, on the set $\mathcal{M}_G^G$ of G-fixed elements in $\mathcal{M}_G$. The linkage classes will be used in later sections, especially in the determination of the parametrizing set for the simple fibered biset functors.

5.1 Definition (a) Let G and H be finite groups and let $(K,\kappa)\in\mathcal{M}_G^G$ and $(L,\lambda)\in\mathcal{M}_H^H$. We say that (G,K,κ) and (H,L,λ) are *linked* if there exists $(U,\phi)\in\mathcal{M}_{G\times H}$ with $l(U,\phi)=(G,K,\kappa)$ and $r(U,\phi)=(H,L,\lambda)$. In this case we also write $(G,K,\kappa)\underset{(U,\phi)}{\sim}(H,L,\lambda)$ or just $(G,K,\kappa)\sim(H,L,\lambda)$. Note that if also I is a finite group and $(M,\mu)\in\mathcal{M}_I^I$ and if $(V,\psi)\in\mathcal{M}_{H\times I}$ is such that $(H,L,\lambda)\underset{(V,\psi)}{\sim}(I,M,\mu)$ then $(G,K,\kappa)\underset{(U*V,\phi*\psi)}{\sim}(I,M,\mu)$. Therefore, the relation $\sim$ is an equivalence relation.

(b) As a special case we also say that elements (K,κ) and (K',κ') of $\mathcal{M}_G^G$ are G-*linked* if $(G,K,\kappa)\sim(G,K',\kappa')$. We use again the notation $(K,\kappa)\sim_G(K',\kappa')$ or just $(K,\kappa)\sim(K',\kappa')$. Note that if $(K,\kappa)\sim_G(K',\kappa')$ then $G/K\cong G/K'$ by Proposition 1.3 and therefore, $|K|=|K'|$. We write $\{K,\kappa\}_G$ for the G-linkage class of (K,κ).

5.2 Let G be a finite group and let $K\leqslant Z(G)$. Let $\sigma\colon G/K\to G$ be a section of the canonical epimorphism $\pi\colon G\to G/K$ (i.e., $\pi\circ\sigma=\mathrm{id}_{G/K}$). Then σ defines a 2-cocycle $\alpha\in Z^2(G/K,K)$ by the equation

$$\sigma(x)\sigma(y)=\alpha(x,y)\sigma(xy)\,,\quad \text{for } x,y\in G/K. \tag{12}$$

When σ runs through all possible sections of π then α runs through a full cohomology class $[\alpha]$ in $H^2(G/K,K)$. We say that α *describes* the central extension $1\to K\to G\to G/K\to 1$.

5.3 Proposition *Let G and H be finite groups and let $(K,\kappa)\in\mathcal{M}_G^G$ and $(L,\lambda)\in\mathcal{M}_H^H$. Assume that κ and λ are faithful and let $\alpha\in Z^2(G/K,K)$ and $\beta\in Z^2(H/L,L)$ be cocycles which describe the central extensions $1\to K\to G\to G/K\to 1$ and $1\to L\to H\to H/L\to 1$, respectively. Then the following are equivalent:*

(i) $(G,K,\kappa)\sim(H,L,\lambda)$.

(ii) *There exists an isomorphism $\eta\colon H/L\to G/K$ such that*

$$[\kappa\circ\alpha]=[\lambda\circ\beta\circ(\eta^{-1}\times\eta^{-1})]$$

as elements in $H^2(G/K,\mathrm{tor}A)$.

Proof Let $\sigma\colon G/K\to G$ and $\tau\colon H/L\to H$ be sections of the canonical epimorphisms such that

$$\sigma(x)\sigma(y)=\alpha(x,y)\sigma(xy)\quad\text{and}\quad\tau(w)\tau(z)=\beta(w,z)\tau(wz)$$

for all $x,y\in G/K$ and $w,z\in H/L$.

We first show that (i) implies (ii). Let $(U,\phi)\in\mathcal{M}_{G\times H}$ with $l(U,\phi)=(G,K,\kappa)$ and $r(U,\phi)=(H,L,\lambda)$. Then we obtain an isomorphism $\eta\colon H/L\to G/K$ from Proposition 1.3(c). We set $\tau':=\tau\circ\eta^{-1}\colon G/K\to H$ and $\beta':=\beta\circ(\eta^{-1}\times\eta^{-1})\in Z^2(G/K,L)$. For all $x,y\in G/K$ we obtain

$$\begin{aligned}&\kappa\Big(\alpha(x,y)\Big)\lambda\Big(\beta'(x,y)\Big)^{-1}\\=&\kappa\Big(\sigma(x)\sigma(y)\sigma(xy)^{-1}\Big)\lambda\Big(\tau'(x)\tau'(y)\tau'(xy)^{-1}\Big)^{-1}\\=&\phi\Big(\sigma(x)\sigma(y)\sigma(xy)^{-1},\tau'(x)\tau'(y)\tau'(xy)^{-1}\Big)\\=&\phi\Big(\sigma(x),\tau'(x)\Big)\phi\Big(\sigma(y),\tau'(y)\Big)\phi\Big(\sigma(xy),\tau'(xy)\Big)^{-1}\\=&\mu(x)\mu(y)\mu(xy)^{-1},\end{aligned}$$

where $\mu\colon G/K\to\mathrm{tor}A$ is defined by $\mu(x)=\phi(\sigma(x),\tau'(x))$. Note here that, by the definition of η and τ', we have $(\sigma(x),\tau'(x))\in U$ for every $x\in G/K$.

Next we show that (ii) implies (i). We define

$$U:=\{(g,h)\in G\times H\mid gK=\eta(hL)\}\,.$$

Then $k_1(U)=K$, $k_2(U)=L$, $p_1(U)=G$ and $p_2(U)=H$. We only need to show that $\kappa\times\lambda^{-1}\in(K\times L)^*$ can be extended to a homomorphism $\phi\in U^*$. Consider the central extension

$$1\to K\times L\to U\to U/(K\times L)\to 1$$

and note that $p_1\colon U\to G$ induces an isomorphism $\bar{p}_1\colon U/(K\times L)\to G/K$. Furthermore, note that, with

$$\tau':=\tau\circ\eta^{-1}\colon G/K\to H\quad\text{and}\quad\beta':=\beta\circ(\eta^{-1}\times\eta^{-1})\in Z^2(G/K,L)\,,$$

the function

$$\rho\colon G/K \to U\,, \quad x \mapsto \big(\sigma(x), \tau'(x)\big)\,,$$

is a section of the surjection $U \to U/(K\times L) \stackrel{\bar{p}_1}{\to} G/K$ and that

$$\gamma\colon G/K \times G/K \to K \times L\,, \quad (x,y) \mapsto \big(\alpha(x,y), \beta'(x,y)\big)\,,$$

defines a cocycle $\gamma \in Z^2(G/K, K\times L)$ such that

$$\rho(x)\rho(y) = \gamma(x,y)\rho(xy)$$

for all $x, y \in G/K$. This means that $\rho' := \rho\circ\bar{p}_1\colon U/(K\times L) \to U$ is a section of $U \to U/(K\times L)$ with corresponding cocycle $\gamma' := \gamma\circ(\bar{p}_1\times\bar{p}_1) \in Z^2(U/(K\times L), K\times L)$. Now, $\kappa\times\lambda^{-1} \in (K\times L)^*$ extends to a homomorphism $\phi\in U^*$ if and only if in the following part of the Hochschild-Serre five term exact sequence (see [K87, Theorem 1.5.1]),

$$\cdots \longrightarrow \mathrm{Hom}(U, \mathrm{tor}A) \longrightarrow \mathrm{Hom}(K\times L, \mathrm{tor}A) \stackrel{\delta}{\longrightarrow} H^2(U/(K\times L), \mathrm{tor}A) \longrightarrow \cdots\,,$$

the homomorphism $\kappa\times\lambda^{-1} \in \mathrm{Hom}(K\times L, \mathrm{tor}A)$ belongs to the kernel of the connecting homomorphism δ. However, by [K87, Theorem 1.5.1], one has $\delta(\kappa\times\lambda^{-1}) = [(\kappa\times\lambda^{-1})\circ\gamma']$. Therefore, it suffices to show that $[(\kappa\times\lambda^{-1})\circ\gamma] = 1 \in H^2(G/K, \mathrm{tor}A)$. But

$$\big((\kappa\times\lambda^{-1})\circ\gamma\big)(x,y) = \kappa\big(\alpha(x,y)\big)\cdot\lambda\big(\beta'(x,y)\big)^{-1} = \mu(x)\mu(y)\mu(xy)^{-1}$$

for some function $\mu\colon G/K \to \mathrm{tor}A$, by the hypothesis in (ii). □

5.4 Remark (a) Let G and H be finite groups and let $(K,\kappa) \in \mathcal{M}_G^G$ and $(L,\lambda) \in \mathcal{M}_H^H$. Let $\hat{K} := \ker(\kappa)$, $\hat{L} := \ker(\lambda)$ and set

$$\bar{G} := G/\hat{K}\,, \quad \bar{K} := K/\hat{K}\,, \quad \bar{H} := H/\hat{L}\,, \quad \bar{L} := L/\hat{L}\,.$$

Furthermore, let $\bar{\kappa} \in \bar{K}^*$ and $\bar{\lambda} \in \bar{L}^*$ denote the homomorphisms that inflate to κ and λ, respectively. It is easy to see that

$$(G,K,\kappa) \sim (H,L,\lambda) \iff (\bar{G},\bar{K},\bar{\kappa}) \sim (\bar{H},\bar{L},\bar{\lambda})\,.$$

Therefore, the question if (G,K,κ) is linked to (H,L,λ) can be reduced to the case where κ and λ are faithful and can be answered by the criterion in Proposition 5.3.

(b) Proposition 5.3 is still true if $H^2(G/K, \mathrm{tor}A)$ in (ii) is replaced by $H^2(G/K, A)$. The proof is exactly the same, mutatis mutandis. Note also that the natural map $H^2(G/K, \mathrm{tor}A) \to H^2(G/K, A)$ is injective by the long exact cohomology sequence, since $H^1(G/K, A/\mathrm{tor}A) \cong \mathrm{Hom}(G/K, A/\mathrm{tor}A)$ is trivial.

5.5 (a) Recall that, for $(K,\kappa)\in\mathcal{M}_G^G$, we denote by $\{K,\kappa\}_G$ the G-linkage class of (K,κ) in $\mathcal{M}_G^G$. The partial order on $\mathcal{M}_G^G$ induces a partial order on the linkage classes, defined for $(K,\kappa),(L,\lambda)\in\mathcal{M}_G^G$ by

$$\{K,\kappa\}_G\leqslant\{L,\lambda\}_G$$

if and only if there exists $(K',\kappa')\in\{K,\kappa\}_G,(L',\lambda')\in\{L,\lambda\}_G$ with $(K',\kappa')\leqslant(L',\lambda')$. In order to see that this relation is transitive it suffices to show that if $(K,\kappa)\leqslant(L,\lambda)$ in $\mathcal{M}_G^G$ and $(K',\kappa')\sim_G(K,\kappa)$ in $\mathcal{M}_G^G$ then there exists $(L',\lambda')\in\mathcal{M}_G^G$ with $(K',\kappa')\leqslant(L',\lambda')\sim_G(L,\lambda)$. The existence of (L',λ') is seen as follows. Since $(K,\kappa)\sim_G(K',\kappa')$, there exists $(U,\phi)\in\mathcal{M}_{G\times G}$ with $l(U,\phi)=(G,K,\kappa)$ and $r(U,\phi)=(G,K',\kappa')$. We can write

$$E_{(L,\lambda)}\otimes_{AG}\left(\frac{G\times G}{U,\phi}\right)\cong\left(\frac{G\times G}{V,\psi}\right)$$

for some $(V,\psi)\in\mathcal{M}_{G\times G}$. Using $(K,\kappa)\leqslant(L,\lambda)$, Proposition 4.2(b) implies $l(V,\psi)=(G,L,\lambda)$. Now Propositions 1.3(a) and 2.6(b) imply $r(V,\psi)=(G,L',\lambda')$ for some $(L',\lambda')\in\mathcal{M}_G^G$ with $(K',\kappa')\leqslant(L',\lambda')\sim_G(L,\lambda)$, as desired.

(b) For $(K,\kappa)\in\mathcal{M}_G^G$ we define the elements

$$e_{\{K,\kappa\}_G}:=\sum_{(K',\kappa')\in\{K,\kappa\}_G}e_{(K',\kappa')}\in B_k^A(G,G)$$

and

$$f_{\{K,\kappa\}_G}:=\sum_{(K',\kappa')\in\{K,\kappa\}_G}f_{(K',\kappa')}\in B_k^A(G,G)\,.$$

The following proposition now follows immediately from Proposition 4.4.

5.6 Proposition *Let $(K,\kappa),(L,\lambda)\in\mathcal{M}_G^G$. Then*

$$\begin{aligned} e_{\{K,\kappa\}_G}f_{\{L,\lambda\}_G}=f_{\{L,\lambda\}_G}e_{\{K,\kappa\}_G}&=0\quad\textit{unless } \{K,\kappa\}_G\leqslant\{L,\lambda\}_G,\\ e_{\{K,\kappa\}_G}f_{\{K,\kappa\}_G}=f_{\{K,\kappa\}_G}e_{\{K,\kappa\}_G}&=f_{\{K,\kappa\}_G}\quad\textit{and}\\ f_{\{K,\kappa\}_G}f_{\{L,\lambda\}_G}&=\begin{cases} f_{\{K,\kappa\}_G}\,, & \textit{if } \{K,\kappa\}_G=\{L,\lambda\}_G,\\ 0\,, & \textit{otherwise.}\end{cases}\end{aligned}$$

6 The algebra E_G^c and the group $\Gamma_{(G,K,\kappa)}$

This section is devoted to the study of the structure of the subalgebra E_G^c of E_G generated by the classes of transitive A-fibered (G,G)-bisets whose stabilizing pairs have full projections. In Section 8 it will be shown that this algebra covers the quotient $\bar{E}_G$. In this section we will show that it is isomorphic to a direct product of matrix algebras over certain group algebras $k\Gamma_{(G,K,\kappa)}$. These group algebras, together with the linkage classes, will be the main ingredients of the parametrization of the simple fibered biset functors in Section 9.

6.1 The algebra E_G^c and the group $\Gamma_{(G,K,\kappa)}$. (a) Let G and H be finite groups and let $(U,\phi) \in \mathcal{M}_{G\times H}$. We say that (U,ϕ) is *covering* if $p_1(U) = G$ and $p_2(U) = H$. We denote by $\mathcal{M}_{G\times H}^c$ the set of all covering pairs of $\mathcal{M}_{G\times H}$. Note that if (U,ϕ) as above is covering then $l_0(U,\phi) \in \mathcal{M}_G^G$ and $r_0(U,\phi) \in \mathcal{M}_H^H$. If also K is a finite group and also $(V,\psi) \in \mathcal{M}_{H\times K}^c$ then, by the Mackey formula,

$$\left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_{H} \left[\frac{H\times K}{V,\psi}\right] = \begin{cases} \left[\frac{G\times K}{U*V,\phi*\psi}\right], & \text{if } \phi_2 = \psi_1 \text{ on } k_2(U)\cap k_1(V),\\ 0 & \text{otherwise,}\end{cases}$$

with $(U*V,\phi*\psi) \in \mathcal{M}_{G\times K}^c$ in the first case. This implies that the k-span

$$E_G^c = E_k^{A,c}(G) = B_k^{A,c}(G,G)$$

of the canonical basis elements $\left[\frac{G\times G}{U,\phi}\right]$ of $E_G = B_k^A(G,G)$, with $(U,\phi) \in \mathcal{M}_{G\times G}^c$, is a k-subalgebra of $E_G = B_k^A(G,G)$. Note that $e_{(K,\kappa)}, f_{(K,\kappa)}, e_{\{K,\kappa\}_G}, f_{\{K,\kappa\}_G} \in E_G^c$ for all $(K,\kappa) \in \mathcal{M}_G^G$.

(b) Let G be a finite group and let $(K,\kappa) \in \mathcal{M}_G^G$. It follows from Propositions 2.6(d) and 4.2 that the standard basis elements $\left[\frac{G\times G}{U,\phi}\right]$ of E_G^c with $l(U,\phi) = (G,K,\kappa) = r(U,\phi)$ form a finite group $\Gamma_{(G,K,\kappa)}$ under multiplication, with identity element $e_{(K,\kappa)}$ and inverses induced by taking opposite fibered bisets.

(c) More generally, assume that $(K,\kappa) \in \mathcal{M}_G^G$ and $(L,\lambda) \in \mathcal{M}_H^H$ and set

$${}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)} := \left\{ \left[\frac{G\times H}{U,\phi}\right] \mid l(U,\phi) = (G,K,\kappa),\ r(U,\phi) = (H,L,\lambda) \right\}.$$

This set is non-empty if and only if (G,K,κ) and (H,L,λ) are linked. Assume from now on that this is the case. Then it is a $(\Gamma_{(G,K,\kappa)},\Gamma_{(H,L,\lambda)})$-biset and each of the groups $\Gamma_{(G,K,\kappa)}$ and $\Gamma_{(H,L,\lambda)}$ acts transitively and freely on ${}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$. This is easily verified by tensoring with opposites of standard basis elements and using Proposition 2.6(d). Therefore, each element $\left[\frac{G\times H}{U,\phi}\right] \in {}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$ induces an isomorphism

$$\gamma\colon \Gamma_{(H,L,\lambda)} \xrightarrow{\sim} \Gamma_{(G,K,\kappa)}, \quad y \mapsto \left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}_{H} y \mathop{\cdot}_{H} \left[\frac{G\times H}{U,\phi}\right]^{\mathrm{op}},$$

and if also $\left[\frac{G\times H}{U',\phi'}\right] \in {}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$ then the resulting isomorphism γ' satisfies $\gamma' = c_z \circ \gamma$, where $z = \left[\frac{G\times H}{U',\phi'}\right] \mathop{\cdot}_{G} \left[\frac{G\times H}{U,\phi}\right]^{\mathrm{op}} \in \Gamma_{(H,L,\lambda)}$. As $(\Gamma_{(G,K,\kappa)},\Gamma_{(H,L,\lambda)})$-biset we have ${}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)} \cong \left[\frac{\Gamma_{(G,K,\kappa)}\times\Gamma_{(H,L,\lambda)}}{\Delta_\gamma}\right]$, where $\Delta_\gamma := \{(\gamma(y),y) \mid y \in \Gamma_{(H,L,\lambda)}\}$. As a consequence, one obtains a canonical bijection

$$\mathrm{Irr}(k\Gamma_{(H,L,\lambda)}) \xrightarrow{\sim} \mathrm{Irr}(k\Gamma_{(G,K,\kappa)}) \tag{13}$$

induced by the category equivalence $k[{}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}] \otimes_{k\Gamma_{(H,L,\lambda)}} -$ between the category of left $k\Gamma_{(H,L,\lambda)}$-modules and the category of left $k\Gamma_{(G,K,\kappa)}$-modules. Its inverse is given by tensoring with the bimodule $k[{}_{(H,L,\lambda)}\Gamma_{(G,K,\kappa)}] = k[{}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}^{\mathrm{op}}]$. The bijection in (13) coincides with the one given by transport of structure via the isomorphism γ.

The goal of this section is the following theorem describing the structure of the k-algebra $E_G^c = B_k^{A,c}(G,G)$.

6.2 Theorem *There exists a k-algebra isomorphism*

$$\bigoplus_{\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim} \mathrm{Mat}_{|\{K,\kappa\}_G|}(k\Gamma_{(G,K,\kappa)}) \xrightarrow{\sim} E_G^c$$

with the following property: For every $(K,\kappa) \in \mathcal{M}_G^G$, writing $\{K,\kappa\}_G = \{(K_1,\kappa_1),\ldots,(K_n,\kappa_n)\}$, the element $f_{(K_i,\kappa_i)} \in E_G^c$ is mapped to the diagonal idempotent matrices $e_i = \mathrm{diag}(0,\ldots,0,1,0,\ldots,0) \in \mathrm{Mat}_{|\{K,\kappa\}_G|}(k\Gamma_{(G,K,\kappa)})$, $i = 1,\ldots,|\{K,\kappa\}_G|$, in the $\{K,\kappa\}_G$-component.

Before we can prove the theorem we need a few auxiliary results.

6.3 Lemma *Let $(U,\phi) \in \mathcal{M}_{G\times G}^c$ and let $(K,\kappa),(L,\lambda) \in \mathcal{M}_G^G$. If $f_{\{K,\kappa\}_G}\left[\frac{G\times G}{U,\phi}\right] f_{\{L,\lambda\}_G} \neq 0$ then $\{K,\kappa\}_G = \{L,\lambda\}_G$.*

Proof If the above expression is non-zero then there exist $(K',\kappa') \in \{K,\kappa\}_G$ and $(L',\lambda') \in \{L,\lambda\}_G$ such that $f_{(K',\kappa')}\left[\frac{G\times G}{U,\phi}\right] f_{(L',\lambda')} \neq 0$ in E_G^c. Expanding $f_{(L',\lambda')}$ as in Equation (9), we see that there exists $(L'',\lambda'') \in \mathcal{M}_G^G$ with $(L',\lambda') \leqslant (L'',\lambda'')$ and $f_{(K',\kappa')}\left[\frac{G\times G}{U,\phi}\right] e_{(L'',\lambda'')} \neq 0$. So also $\left[\frac{G\times G}{U,\phi}\right] e_{(L'',\lambda'')}$ is non-zero and of the form $\left[\frac{G\times G}{V,\psi}\right]$ for some $(V,\psi) \in \mathcal{M}_{G\times G}^c$ satisfying $r_0(V,\psi) \geqslant (L'',\lambda'')$ and $(K'',\kappa'') := l_0(V,\psi) \geqslant l_0(U,\phi)$ by Proposition 2.6(b). By Proposition 4.2, we now have

$$0 \neq f_{(K',\kappa')}\left[\frac{G\times G}{U,\phi}\right] e_{(L'',\lambda'')} = f_{(K',\kappa')} e_{(K'',\kappa'')}\left[\frac{G\times G}{U,\phi}\right] e_{(L'',\lambda'')}$$

which implies $(K'',\kappa'') \leqslant (K',\kappa')$ by Proposition 4.4. Altogether, we obtain

$$\begin{aligned}\{L,\lambda\}_G = \{L',\lambda'\}_G &\leqslant \{L'',\lambda''\}_G \leqslant \{r_0(V,\psi)\}_G = \{l_0(V,\psi)\}_G \\ &= \{K'',\kappa''\}_G \leqslant \{K',\kappa'\}_G = \{K,\kappa\}_G\,.\end{aligned}$$

Similarly, we can prove $\{K,\kappa\}_G \leqslant \{L,\lambda\}_G$. □

6.4 Corollary *Let G be a finite group. The elements $f_{\{K,\kappa\}_G}$, $\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim$, are mutually orthogonal central idempotents of $E_G^c = B_k^{A,c}(G,G)$ and their sum is equal to 1. In particular, one has a decomposition*

$$E_G^c = \bigoplus_{\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim} f_{\{K,\kappa\}_G} E_G^c \tag{14}$$

of E_G^c into two-sided ideals.

Proof We already know from Proposition 5.6 that the elements $f_{\{K,\kappa\}_G}$, $\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim$, are mutually orthogonal idempotents of E_G^c and we know from Equation (11) that their sum is equal to 1. From this and from Lemma 6.3, we obtain the three decompositions

$$E_G^c = \bigoplus f_{\{K,\kappa\}_G} E_G^c = \bigoplus f_{\{K,\kappa\}_G} E_G^c f_{\{K,\kappa\}_G} = \bigoplus E_G^c f_{\{K,\kappa\}_G}$$

where each of the three sums runs over $\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim$. Since $f_{\{K,\kappa\}_G} E_G^c f_{\{K,\kappa\}_G} \subseteq f_{\{K,\kappa\}_G} E_G^c$ and $f_{\{K,\kappa\}_G} E_G^c f_{\{K,\kappa\}_G} \subseteq E_G^c f_{\{K,\kappa\}_G}$, we obtain equality

$$f_{\{K,\kappa\}_G} E_G^c = f_{\{K,\kappa\}_G} E_G^c f_{\{K,\kappa\}_G} = E_G^c f_{\{K,\kappa\}_G}$$

for every $\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim$. Now let $b \in E_G^c$ be arbitrary. Then

$$\sum_{\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim} f_{\{K,\kappa\}_G} b = b = \sum_{\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim} b f_{\{K,\kappa\}_G}$$

and we obtain $f_{\{K,\kappa\}_G} b = b f_{\{K,\kappa\}_G}$ for every $\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim$ by the equality of the above decompositions. This completes the proof. □

6.5 Besides the decomposition (14) of E_G^c into ideals we have a second natural decomposition of E_G^c, this time into k-submodules, again indexed by $\mathcal{M}_G^G/\sim$. For $\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim$, we denote by $E_G^{c,\{K,\kappa\}}$ the k-span of all standard basis elements $\left[\frac{G\times G}{U,\phi}\right]$ with $(U,\phi) \in \mathcal{M}_{G\times G}^c$ satisfying $l_0(U,\phi) \in \{K,\kappa\}_G$. Note that this last condition is equivalent to requiring $r_0(U,\phi) \in \{K,\kappa\}_G$. We have the obvious decomposition

$$E_G = \bigoplus_{\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim} E_G^{c,\{K,\kappa\}} \tag{15}$$

into k-submodules. The next lemma provides a connection between the latter decomposition and the one in (14).

6.6 Lemma *Let G be a finite group and let $(K,\kappa) \in \mathcal{M}_G^G$.*

(a) *Let $(U,\phi) \in \mathcal{M}_{G\times G}^c$ with $r_0(U,\phi) = (K,\kappa)$ and let $(L,\lambda) \in \mathcal{M}_G^G$ with $(K,\kappa) \not\leqslant (L,\lambda)$. Then $\left[\frac{G\times G}{U,\phi}\right] f_{(L,\lambda)} = 0$.*

(b) *One has*

$$\bigoplus_{\{K,\kappa\}_G \leqslant \{L,\lambda\}_G \in \mathcal{M}_G^G/\sim} E_G^{c,\{L,\lambda\}} = \bigoplus_{\{K,\kappa\}_G \leqslant \{L,\lambda\}_G \in \mathcal{M}_G^G/\sim} E_G^c f_{\{L,\lambda\}_G} .$$

(c) *The projection map*

$$\omega\colon E_G^{c,\{K,\kappa\}} \to E_G^c f_{\{K,\kappa\}_G}, \quad b \mapsto b f_{\{K,\kappa\}_G} ,$$

with respect to the decomposition (14) is an isomorphism of k-modules whose inverse is the projection map with respect to the decomposition (15).

(d) *Enumerating the elements of* $\{K,\kappa\}_G$ *as* $(K_1,\kappa_1),\ldots,(K_n,\kappa_n)$*, the isomorphism* ω *is the direct sum of the* k*-module isomorphisms*

$$\omega_{ij}\colon k[{}_{(G,K_i,\kappa_i)}\Gamma_{(G,K_j,\kappa_j)}] \to f_{(K_i,\kappa_i)}E_G^c f_{(K_j,\kappa_j)}\,, \quad b_{ij}\mapsto f_{(K_i,\kappa_i)}b_{ij}f_{(K_j,\kappa_j)}\,,$$

for $i,j\in\{1,\ldots,n\}$.

(e) *Let* $i,j,l,m\in\{1,\ldots,n\}$, $b_{ij}\in k[{}_{(G,K_i,\kappa_i)}\Gamma_{(G,K_j,\kappa_j)}]$ *and* $b_{lm}\in k[{}_{(G,K_l,\kappa_l)}\Gamma_{(G,K_m,\kappa_m)}]$. *If* $j=l$ *then* $b_{ij}b_{lm}\in k[{}_{(G,K_i,\kappa_i)}\Gamma_{(G,K_m,\kappa_m)}]$. *In any case one has*

$$\omega_{ij}(b_{ij})\omega_{lm}(b_{lm}) = \begin{cases}\omega_{il}(b_{ij}b_{lm}), & \text{if } j=l,\\ 0, & \text{if } j\neq l.\end{cases} \tag{16}$$

(f) *The map*

$$k[\Gamma_{(G,K,\kappa)}]\to f_{(K,\kappa)}E_G^c f_{(K,\kappa)}\,, \quad a\mapsto f_{(K,\kappa)}af_{(K,\kappa)}\,,$$

is a k*-algebra isomorphism.*

Proof (a) One has

$$\left[\frac{G\times G}{U,\phi}\right] f_{(L,\lambda)} = \left[\frac{G\times G}{U,\phi}\right] e_{(K,\kappa)}f_{(L,\lambda)} = 0$$

by Propositions 4.2(c) and 4.4.

(b) The left hand side of the equation is an ideal of E_G^c by Proposition 2.6(b), and it contains $f_{\{L,\lambda\}_G}$ for every $\{L,\lambda\}_G\geqslant\{K,\kappa\}_G$. This shows that the left hand side contains the right hand side. Conversely, let $(M,\mu)\in\mathcal{M}_G^G$ with $\{M,\mu\}_G\not\geqslant\{K,\kappa\}_G$. Then, by Part (a), the left hand side annihilates $f_{\{M,\mu\}_G}$. But the right hand side equals the annihilator of the set of all $f_{\{M,\mu\}_G}$ with $\{M,\mu\}_G\not\geqslant\{K,\kappa\}_G$, by Corollary 6.4. This shows that the left hand side is contained in the right hand side.

(c) Note that the subsums of both sides in the equation in Part (b), that are indexed by $\{L,\lambda\}_G>\{K,\kappa\}_G$, are also equal. In fact, this follows by applying (b) to the elements $\{L,\lambda\}_G$. Therefore, $E_G^{c,\{K,\kappa\}}$ and $E_G^c f_{\{K,\kappa\}_G}$ are both complements to the same submodule of the direct sum in (b). The assertion is now immediate.

(d) By definition we have a direct sum decomposition into k-submodules

$$E_G^{c,\{K,\kappa\}} = \bigoplus_{i,j=1}^n k[{}_{(G,K_i,\kappa_i)}\Gamma_{(G,K_j,\kappa_j)}]\,. \tag{17}$$

Since $f_{\{K,\kappa\}_G}=f_{(K_1,\kappa_1)}+\cdots+f_{(K_n,\kappa_n)}$ is an orthogonal idempotent decomposition and since $f_{\{K,\kappa\}_G}$ is central in E_G^c, we also have a decomposition

$$E_G^c f_{\{K,\kappa\}_G} = f_{\{K,\kappa\}_G}E_G^c f_{\{K,\kappa\}_G} = \bigoplus_{i,j=1}^n f_{(K_i,\kappa_i)}E_G^c f_{(K_j,\kappa_j)}$$

into k-submodules. Moreover, by Part (a) and its left-sided version we have, for $b_{ij} \in k[_{(G,K_i,\kappa_i)}\Gamma_{(G,K_j,\kappa_j)}]$,

$$\omega(b_{ij}) = f_{\{K,\kappa\}_G} b_{ij} f_{\{K,\kappa\}_G} = f_{(K_i,\kappa_i)} b_{ij} f_{(K_j,\kappa_j)} \in f_{(K_i,\kappa_i)} E_G^c f_{(K_j,\kappa_j)} \,.$$

Since ω is an isomorphism of k-modules, also each ω_{ij} is an isomorphism of k-modules.

(e) If $j = l$ then we have $b_{ij}b_{jm} \in k[_{(G,K_i,\kappa_i)}\Gamma_{(G,K_m,\kappa_m)}]$ by Proposition 2.6(d). Moreover, in this case, by the same argument as in the proof of Part (d), and since $f_{\{K,\kappa\}_G}$ is central in E_G^c, we have

$$\begin{aligned} \omega_{ij}(b_{ij})\omega_{jm}(b_{jm}) &= f_{(K_i,\kappa_i)} b_{ij} f_{(K_j,\kappa_j)} b_{jm} f_{(K_m,\kappa_m)} = f_{(K_i,\kappa_i)} b_{ij} f_{\{K,\kappa\}_G} b_{jm} f_{(K_m,\kappa_m)} \\ &= f_{(K_i,\kappa_i)} f_{\{K,\kappa\}_G} b_{ij} b_{jm} f_{(K_m,\kappa_m)} = f_{(K_i,\kappa_i)} b_{ij} b_{jm} f_{(K_m,\kappa_m)} = \omega_{im}(b_{ij}b_{jm}) \,. \end{aligned}$$

On the other hand, if $j \neq l$ then $\omega_{ij}(b_{ij})\omega_{lm}(b_{lm}) = f_{(K_i,\kappa_i)} b_{ij} f_{(K_j,\kappa_j)} f_{(K_l,\kappa_l)} b_{lm} f_{(K_m,\kappa_m)} = 0$, since $f_{(K_j,\kappa_j)} f_{(K_l,\kappa_l)} = 0$.

(f) This is an immediate consequence of Part (e) by choosing $i \in \{1, \dots, n\}$ such that $(K,\kappa) = (K_i,\kappa_i)$ and considering ω_{ii}. □

Now we are ready to prove Theorem 6.2.

6.7 Proof *of Theorem* 6.2. Let $(K,\kappa) \in \mathcal{M}_G^G$. By Corollary 6.4, it suffices to show that there exists a k-algebra isomorphism between $E_G^c f_{\{K,\kappa\}_G}$ and $\mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)})$ which maps $f_{(K_i,\kappa_i)}$ to the diagonal idempotent matrix e_i, where $(K_1,\kappa_1), (K_2,\kappa_2), \dots, (K_n,\kappa_n)$ enumerate the elements of $\{K,\kappa\}_G$.

For each $i = 1, \dots, n$, we choose an element $(U_i,\phi_i) \in \mathcal{M}_{G\times G}^c$ with $l_0(U_i,\phi_i) = (K,\kappa)$ and $r_0(U_i,\phi_i) = (K_i,\kappa_i)$, and set $x_i := \left[\frac{G\times G}{U_i,\phi_i}\right] \in E_G^{c,\{K,\kappa\}}$, for $i = 1 \dots, n$. For any two elements $i,j \in \{1, \dots, n\}$ we claim that the k-module homomorphism

$$\sigma_{ij} \colon k\Gamma_{(G,K,\kappa)} \to k[_{(G,K_i,\kappa_i)}\Gamma_{(G,K_j,\kappa_j)}] \,, \quad a \mapsto x_i^{\mathrm{op}} a x_j \,,$$

is an isomorphism. In fact, first of all, $x_i^{\mathrm{op}} a x_j \in k[_{(G,K_i,\kappa_i)}\Gamma_{(G,K_j,\kappa_j)}]$ by Proposition 2.6(d). Secondly, the map $b \mapsto x_i b x_j^{\mathrm{op}}$ is an inverse to σ_{ij}, since $x_i \cdot x_i^{\mathrm{op}} = e_{(K,\kappa)}$, $x_i^{\mathrm{op}} \cdot x_i = e_{(K_i,\kappa_i)}$, and $e_{(K,\kappa)} a = a e_{(K,\kappa)} = a$ and $e_{(K_i,\kappa_i)} b = b e_{(K_j,\kappa_j)} = b$ by Proposition 4.2(c) and (d). Moreover, we have, for $a, a' \in k\Gamma_{(G,K,\kappa)}$ and $i,j,l \in \{1, \dots, n\}$,

$$\sigma_{ij}(a)\sigma_{jl}(a') = \sigma_{il}(aa') \,, \tag{18}$$

since $x_i^{\mathrm{op}} a x_j x_j^{\mathrm{op}} a' x_l = x_i^{\mathrm{op}} a e_{(K_j,\kappa_j)} a' x_l = x_i^{\mathrm{op}} a a' x_l$, again by Proposition 4.2(c) and (d). Taking the direct sum of the maps σ_{ij} and using the direct sum decomposition in (17) we obtain a k-module isomorphism

$$\sigma \colon \mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)}) \to E_G^{c,\{K,\kappa\}} \,.$$

Thus, we have a k-module isomorphism

$$\rho := \omega \circ \sigma \colon \mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)}) \xrightarrow{\sim} E_G^{c,\{K,\kappa\}} \xrightarrow{\sim} E_G^c f_{\{K,\kappa\}_G}\,, \quad (a_{ij}) \mapsto \sum_{i,j=1}^{n} \omega_{ij}(\sigma_{ij}(a_{ij}))\,.$$

For $(a_{ij}), (a'_{lm}) \in \mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)})$, Equations (16) and (18) now imply that ρ is a k-algebra isomorphism:

$$\begin{aligned}
\rho((a_{ij}))\rho((a'_{lm})) &= \Big(\sum_{i,j=1}^{n} \omega_{ij}(\sigma_{ij}(a_{ij}))\Big)\Big(\sum_{l,m=1}^{n} \omega_{lm}(\sigma_{lm}(a'_{lm}))\Big)\\
&= \sum_{i,j,l,m=1}^{n} \omega_{ij}(\sigma_{ij}(a_{ij}))\cdot\omega_{lm}(\sigma_{lm}(a'_{lm})) = \sum_{i,j,m=1}^{n} \omega_{ij}(\sigma_{ij}(a_{ij}))\cdot\omega_{jm}(\sigma_{jm}(a'_{jm}))\\
&= \sum_{i,j,m=1}^{n} \omega_{im}\Big(\sigma_{ij}(a_{ij})\cdot\sigma_{jm}(a'_{jm})\Big) = \sum_{i,j,m=1}^{n} \omega_{im}\Big(\sigma_{im}(a_{ij}a'_{jm})\Big)\\
&= \sum_{i,m=1}^{n} \omega_{im}\Big(\sigma_{im}(\sum_{j=1}^{n} a_{ij}a'_{jm})\Big) = \omega(\sigma((a_{ij})\cdot(a'_{lm}))) = \rho((a_{ij})\cdot(a'_{lm}))\,.
\end{aligned}$$

Finally, one has $\rho(e_i) = \omega_{ii}(\sigma_{ii}(1)) = f_{(K_i,\kappa_i)}e_{(K_i,\kappa_i)}f_{(K_i,\kappa_i)} = f_{(K_i,\kappa_i)}$ as desired. □

7 The structure of $\Gamma_{(G,K,\kappa)}$

In this section we show that, for $(K,\kappa) \in \mathcal{M}_G^G$, the group $\Gamma_{(G,K,\kappa)}$ is an extension of a certain subgroup of the outer automorphism group $\mathrm{Out}(G)$ with the group $(G/K)^*$, under the assumption that κ is faithful. By passing from G to $G/\ker(\kappa)$ one could avoid this assumption at the cost of more complicated notation. In subsequent sections we will only need to consider the case where κ is faithful.

7.1 Let $(K,\kappa) \in \mathcal{M}_G^G$ and assume that κ is faithful. Then $K \leqslant Z(G)$. For $\eta \in \mathrm{Aut}(G/K)$ we set

$$U_\eta := \{(g_1,g_2) \in G\times G \mid \eta(g_2K) = g_1K\}\,.$$

Clearly, U_η satisfies

$$p_1(U_\eta) = p_2(U_\eta) = G\,, \quad \text{and} \quad k_1(U_\eta) = k_2(U_\eta) = K\,. \tag{19}$$

Conversely, every subgroup $U \leqslant G\times G$ satisfying (19) is of the form U_η for some $\eta \in \mathrm{Aut}(G/K)$. Moreover, if $(h_1,h_2) \in G\times G$ then

$${}^{(h_1,h_2)}U_\eta = U_{\eta'}\,,$$

where $\eta' := c_{h_1K\cdot\eta(h_2^{-1}K)} \circ \eta \in \mathrm{Aut}(G/K)$. This implies that the map $\eta \mapsto U_\eta$ induces a bijection between the outer automorphism group $\mathrm{Out}(G/K)$ and the $G\times G$-conjugacy classes of subgroups $U \leqslant G \times G$ satisfying (19). It is easy to see that, for $\eta_1, \eta_2 \in \mathrm{Aut}(G/K)$, one has

$$U_{\eta_1 \circ \eta_2} = U_{\eta_1} * U_{\eta_2}\,.$$

Thus, we obtain a well-defined group homomorphism

$$\pi\colon \Gamma_{(G,K,\kappa)} \to \mathrm{Out}(G/K)\,, \quad \left[\frac{G \times G}{U_\eta, \phi}\right] \mapsto \bar{\eta}\,,$$

where $\bar{\eta} := \eta\mathrm{Inn}(G/K)$.

Let $\alpha \in Z^2(G/K, K)$ be a cocycle that describes the central extension $1 \to K \to G \to G/K \to 1$. We denote by $\mathrm{Aut}^\circ(G/K)$ the subgroup of $\mathrm{Aut}(G/K)$ consisting of those elements η satisfying

$$[\kappa \circ \alpha \circ (\eta^{-1} \times \eta^{-1})] = [\kappa \circ \alpha] \in H^2(G/K, \mathrm{tor}A)\,.$$

In other words, $\mathrm{Aut}^\circ(G/K)$ is the stabilizer of $[\kappa \circ \alpha]$ in $\mathrm{Aut}(G/K)$ with respect to the natural action of $\mathrm{Aut}(G/K)$ on $H^2(G/K, \mathrm{tor}A)$. Note that the inner automorphism group $\mathrm{Inn}(G/K)$ of G/K fixes $[\alpha] \in H^2(G/K, K)$ and therefore also $[\kappa \circ \alpha] \in H^2(G/K, \mathrm{tor}A)$. In fact, if $\sigma\colon G/K \to G$ is a section of the canonical epimorphism $\pi\colon G \to G/K$ such that (12) holds then the cocycle $\alpha \circ (c_z^{-1} \times c_z^{-1})$ is defined by the section ${}^z\sigma\colon x \mapsto g\sigma(z^{-1}xz)g^{-1}$, for every $z := gK \in G/K$. We set

$$\mathrm{Out}^\circ(G/K) := \mathrm{Aut}^\circ(G/K)/\mathrm{Inn}(G/K) \leqslant \mathrm{Out}(G/K)\,.$$

It follows from Proposition 5.3 that, for given $\eta \in \mathrm{Aut}(G/K)$, there exists $\varphi \in (U_\eta)^*$ such that $l_0(U_\eta, \phi) = r_0(U_\eta, \phi) = (K, \kappa)$ if and only if $\eta \in \mathrm{Aut}^\circ(G/K)$. In other words, the image of π is equal to $\mathrm{Out}^\circ(G/K)$.

Recall that $U_{\mathrm{id}} = \Delta_K(G) = \Delta(G)(K \times 1)$. For every $\theta \in (G/K)^*$ we can define a homomorphism $\tilde{\theta} \in (\Delta_K(G))^*$ by

$$\tilde{\theta}(gk, g) := \kappa(k)\theta(gK)\,,$$

for $k \in K$ and $g \in G$. Note that $\tilde{1} = \phi_\kappa$ from Definition 4.1. It is straightforward to verify that the map

$$\iota\colon (G/K)^* \to \Gamma_{(G,K,\kappa)}\,, \quad \theta \mapsto \left[\frac{G \times G}{\Delta_K(G), \tilde{\theta}}\right]\,,$$

is an injective group homomorphism and that $\ker(\pi) = \mathrm{im}(\iota)$.

The following proposition is now immediate from the preceding discussion.

7.2 Proposition *Let $(K, \kappa) \in \mathcal{M}_G^G$ and assume that κ is faithful. With the notation from 7.1 one has a short exact sequence*

$$1 \longrightarrow (G/K)^* \stackrel{\iota}{\longrightarrow} \Gamma_{(G,K,\kappa)} \stackrel{\pi}{\longrightarrow} \mathrm{Out}^\circ(G/K) \longrightarrow 1\,.$$

7.3 Remark (a) If $\left[\frac{G\times G}{U,\phi}\right] \in \Gamma_{(G,K,\kappa)}$ with $U = U_\eta$ for some $\eta \in \mathrm{Aut}^\circ(G/K)$ and if $\theta \in (G/K)^*$ then, with the explicit formula from Corollary 2.5, it is easy to verify that

$$\left[\frac{G\times G}{U,\phi}\right]\left[\frac{G\times G}{\Delta_K(G),\tilde{\theta}}\right]\left[\frac{G\times G}{U,\phi}\right]^{\mathrm{op}} = \left[\frac{G\times G}{\Delta_K(G),\widetilde{\theta\circ\eta^{-1}}}\right],$$

which shows that the induced action of $\mathrm{Out}^\circ(G/K)$ on $(G/K)^*$ is the usual canonical action.

(b) We do not know if the sequence in Proposition 7.2 splits.

8 Reduced pairs and the simple $\bar{E}_G$-modules

We keep the notation from the previous sections. Thus, k denotes a commutative ring. We also fix a finite group G for this section and we will classify the simple $\bar{E}_G$-modules. This is the last step before the classification of the simple fibered biset functors.

8.1 Notation We define the subset $\mathcal{R}_G = \mathcal{R}_k^A(G)$ of $\mathcal{M}_G^G$ by

$$\mathcal{R}_G = \mathcal{R}_k^A(G) := \{(K,\kappa) \in \mathcal{M}_G^G \mid e_{(K,\kappa)} \notin I_G\}\,.$$

We call $(K,\kappa) \in \mathcal{M}_G$ a *reduced* pair if $(K,\kappa) \in \mathcal{R}_G$. Thus, every pair in $\mathcal{M}_G \smallsetminus \mathcal{M}_G^G$ is by definition not reduced. Note that if $(K,\kappa),(K',\kappa') \in \mathcal{M}_G^G$ are G-linked (cf. Definition 5.1(b)) by a pair $(U,\phi) \in \mathcal{M}_{G\times G}^c$ with $(K,\kappa) = l_0(U,\phi)$ and $(K',\kappa') = r_0(U,\phi)$ then (K,κ) is reduced if and only if (K',κ') is reduced. In fact,

$$e_{(K',\kappa')} = \left[\frac{G\times G}{U,\phi}\right]^{\mathrm{op}} \mathop{\cdot}_{G} \left[\frac{G\times G}{U,\phi}\right] = \left[\frac{G\times G}{U,\phi}\right]^{\mathrm{op}} \mathop{\cdot}_{G} e_{(K,\kappa)} \mathop{\cdot}_{G} \left[\frac{G\times G}{U,\phi}\right],$$

by Proposition 4.2(c) and (d). We write $\mathcal{R}_G/\sim$ for the set of linkage classes in $\mathcal{R}_G$.

Note also that for $(K,\kappa) \in \mathcal{M}_G^G$, one has

$$(K,\kappa) \leqslant (K',\kappa') \in \mathcal{R}_G \implies (K,\kappa) \in \mathcal{R}_G\,, \tag{20}$$

by Proposition 4.2(c).

In the sequel we will denote the image of an element $b \in E_G$ in $\bar{E}_G$ by $\bar{b}$.

8.2 Lemma *The ideal I_G of E_G is generated as a k-module by the standard basis elements $\left[\frac{G\times G}{U,\phi}\right]$ with $(U,\phi) \in \mathcal{M}_{G\times G}$ satisfying*

(i) *$p_1(U) \neq G$ or*

(ii) *$p_1(U) = G$ and $l_0(U,\phi) \notin \mathcal{R}_G$.*

Proof First we show that every element in I_G can be written as a k-linear combination of elements $\left[\frac{G\times G}{U,\phi}\right]$ with $(U,\phi) \in \mathcal{M}_{G\times G}$ satisfying (i) or (ii). To that end, let H be a finite group with $|H| < |G|$ and let $(V,\psi) \in \mathcal{M}_{G\times H}$ and $(W,\mu) \in \mathcal{M}_{H\times G}$. It suffices to show that $\left[\frac{G\times H}{V,\psi}\right]\underset{H}{\cdot}\left[\frac{H\times G}{W,\mu}\right]$ can be written as such a linear combination. But every summand occurring in this product is of the form $\left[\frac{G\times G}{V*W',\psi*\mu'}\right]$ for some $(W',\mu')\in\mathcal{M}_{H\times G}$. We may assume that $p_1(V*W')=G$. Otherwise (i) holds and we are done. So we have $l(V*W',\psi*\mu')=(G,K,\kappa)$ for some $(K,\kappa)\in\mathcal{M}_G^G$. We need to show that $(K,\kappa)\notin\mathcal{R}_G$. By Proposition 2.6(a) and (b), we have $l(V,\psi)=(G,K',\kappa')$ with $(K',\kappa')\leqslant(K,\kappa)$. Thus, if $(K,\kappa)\in\mathcal{R}_G$ then (20) implies that $(K',\kappa')\in\mathcal{R}_G$ and we obtain the contradiction

$$e_{(K',\kappa')}=\left[\frac{G\times H}{V,\psi}\right]\underset{H}{\cdot}\left[\frac{G\times H}{V,\psi}\right]^{\mathrm{op}}\in I_G\,.$$

Conversely, assume that $(U,\psi)\in\mathcal{M}_{G\times G}$ with $p_1(U)<G$. Then the decomposition in Proposition 2.8 implies that $\left[\frac{G\times G}{U,\phi}\right]\in I_G$. Also, if $p_1(U)=G$ and $(K,\kappa):=l_0(U,\phi)\notin\mathcal{R}_G$ then $e_{(K,\kappa)}\in I_G$ and $\left[\frac{G\times G}{U,\phi}\right]=e_{(K,\kappa)}\underset{G}{\cdot}\left[\frac{G\times G}{U,\phi}\right]\in I_G$. □

8.3 Remark Since I_G is stable under the map $-^{\mathrm{op}}\colon E_G\to E_G$, we may replace the conditions (i) and (ii) in the previous lemma also by

(i') $p_2(U)\neq G$ or

(ii') $p_2(U)=G$ and $r_0(U,\phi)\notin\mathcal{R}$.

8.4 Proposition (a) *One has*

$$E_G^{\mathrm{c}}+I_G=E_G$$

and

$$E_G^{\mathrm{c}}\cap I_G=\bigoplus_{\substack{\{K,\kappa\}_G\in\mathcal{M}_G^G/\sim\\ (K,\kappa)\notin\mathcal{R}_G}} f_{\{K,\kappa\}_G}E_G^{\mathrm{c}}\,. \tag{21}$$

(b) *The canonical epimorphism $E_G\to\bar{E}_G$ maps the subalgebra*

$$\bigoplus_{\{K,\kappa\}_G\in\mathcal{R}_G/\sim} f_{\{K,\kappa\}_G}E_G^{\mathrm{c}}$$

of E_G^{c} isomorphically onto $\bar{E}_G$.

(c) *For each $(K,\kappa)\in\mathcal{R}_G$, the map*

$$k\Gamma_{(G,K,\kappa)}\to\bar{f}_{(K,\kappa)}\bar{E}_G\bar{f}_{(K,\kappa)}\,,\quad a\mapsto\bar{f}_{(K,\kappa)}\bar{a}\bar{f}_{(K,\kappa)}\,,$$

is a k-algebra isomorphism.

Proof (a) Lemma 8.2 and Remark 8.3 imply immediately that a standard basis element $\left[\frac{G\times G}{U,\phi}\right]$ of E_G belongs to E_G^{c} if $(U,\phi)\in\mathcal{M}_{G\times G}^{\mathrm{c}}$, and it belongs to I_G if $(U,\phi)\notin\mathcal{M}_{G\times G}^{\mathrm{c}}$. This implies the first equation.

Lemma 8.2 implies that $E_G^c \cap I_G$ is generated as k-module by the canonical basis elements $\left[\frac{G\times G}{U,\phi}\right]$ with $(U,\phi) \in \mathcal{M}^c_{G\times G}$ with $l_0(U,\phi) \notin \mathcal{R}_G$. Thus,

$$E_G^c \cap I_G = \bigoplus_{\substack{\{K,\kappa\}_G \in \mathcal{M}_G^G/\sim \\ (K,\kappa)\notin \mathcal{R}_G}} E_G^{c,\{K,\kappa\}}\,.$$

But, by the property in (20) and by Lemma 6.6, the latter direct sum is equal to the direct sum in Equation (21).

Part (b) follows immediately from Part (a) and Corollary 6.4, and Part (c) follows immediately from Part (b) and Lemma 6.6(f). □

Now we are ready to classify the simple $\bar{E}_G$-modules. Set

$$\mathcal{S}_G = \mathcal{S}_k^A(G) := \{((K,\kappa),[V]) \mid (K,\kappa) \in \mathcal{R}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,\kappa)})\}\,.$$

We call two elements $(K,\kappa,[V]),(K',\kappa',[V']) \in \mathcal{S}_G$ *equivalent*, if (K,κ) and (K',κ') are G-linked and $[V]$ corresponds to $[V']$ via the canonical bijection $\mathrm{Irr}(k\Gamma_{(G,K',\kappa')}) \xrightarrow{\sim} \mathrm{Irr}(k\Gamma_{(G,K,\kappa)})$ from 6.1(c). If $\tilde{\mathcal{R}}_G$ denotes a set of representatives of the linkage classes of $\mathcal{R}_G$ then

$$\tilde{\mathcal{S}}_G := \{((K,\kappa),[V]) \mid (K,\kappa) \in \tilde{\mathcal{R}}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,\kappa)})\}$$

is a set of representatives of the equivalence classes of $\mathcal{S}_G$. By the canonical isomorphism from Proposition 8.4(c), we can view each simple $k\Gamma_{(G,K,\kappa)}$-module V as a simple $\bar{f}_{(K,\kappa)}\bar{E}_G\bar{f}_{(K,\kappa)}$-module, and we can view $\bar{E}_G\bar{f}_{(K,\kappa)}$ as $(\bar{E}_G, k\Gamma_{(G,K,\kappa)})$-bimodule.

8.5 Corollary *With the above notation, the map*

$$((K,\kappa),[V]) \mapsto \widetilde{V} := \bar{E}_G\bar{f}_{(K,\kappa)} \otimes_{k\Gamma_{(G,K,\kappa)}} V\,. \tag{22}$$

induces a bijection between the set of equivalence classes of $\mathcal{S}_G$ *and* $\mathrm{Irr}(\bar{E}_G)$.

Proof By the isomorphism in Proposition 8.4(b), one obtains a bijection between $\mathrm{Irr}(\bar{E}_G)$ and the set of pairs $((K,\kappa),[W])$ with $(K,\kappa) \in \tilde{\mathcal{R}}_G$ and $[W] \in \mathrm{Irr}(f_{\{K,\kappa\}_G}E_G^c)$. By the isomorphism $\rho\colon \mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)}) \to f_{\{K,\kappa\}_G}E_G^c$ used in the proof of Theorem 6.2, one obtains a bijection between $\mathrm{Irr}(f_{\{K,\kappa\}_G}E_G^c)$ and $\mathrm{Irr}(\mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)}))$. Finally, since $\mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)})$ and $k\Gamma_{(G,K,\kappa)}$ are Morita equivalent, $\mathrm{Irr}(\mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)}))$ is in bijection with $\mathrm{Irr}(k\Gamma_{(G,K,\kappa)})$. If $(K,\kappa) \in \tilde{\mathcal{R}}_G$, $[V] \in \mathrm{Irr}(k\Gamma_{(G,K,\kappa)})$, and $(K,\kappa) = (K_i,\kappa_i)$ in the enumeration of $\{K,\kappa\}_G$ in Theorem 6.2, then we use the Morita equivalence given by tensoring with the bimodule $\mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)})e_i$, where $e_i := \mathrm{diag}(0,\ldots,0,1,0,\ldots,0)$, and $k\Gamma_{(G,K,\kappa)}$ is identified with $e_i\mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)})e_i$ via $a \mapsto e_iae_i$. Thus, $[V] \in \mathrm{Irr}(k\Gamma_{(G,K,\kappa)})$ corresponds to the class of $\mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)})e_i \otimes_{k\Gamma_{(G,K,\kappa)}} V$ in $\mathrm{Irr}(\mathrm{Mat}_n(k\Gamma_{(G,K,\kappa)}))$. Choosing $x_i := e_{(K,\kappa)}$ in the definition of the isomorphism ρ (in the proof of Theorem 6.2), the isomorphism ρ transports this latter irreducible module to the irreducible module $f_{\{K,\kappa\}_G}E_G^cf_{(K,\kappa)} \otimes_{k\Gamma_{(G,K,\kappa)}} V = E_G^cf_{(K,\kappa)} \otimes_{k\Gamma_{(G,K,\kappa)}} V$, since $f_{\{K,\kappa\}_G}E_G^cf_{(K,\kappa)} = E_G^cf_{\{K,\kappa\}_G}f_{(K,\kappa)} = E_G^cf_{(K,\kappa)}$. And finally, via the isomorphism in Proposition 8.4(b), the latter simple module corresponds to the module $\widetilde{V}$ in the statement of the corollary. ◻

The following proposition gives a necessary condition and a sufficient condition for a pair $(K,\kappa) \in \mathcal{M}_G^G$ to be reduced. For special cases of A, we will see in Section 10 that the sufficient condition in Part (b) is also necessary.

8.6 Proposition *Let $(K,\kappa) \in \mathcal{M}_G^G$.*

(a) *If $(K,\kappa) \in \mathcal{R}_G$ then κ is faithful and $K \leqslant Z(G)$.*

(b) *If κ is faithful and $K \leqslant G'$ then $(K,\kappa) \in \mathcal{R}_G$.*

Proof (a) By the decomposition in Proposition 2.8 we obtain that $\ker(\kappa) = 1$. Then, $K \leqslant Z(G)$ by Proposition 1.3(b).

(b) Assume that $e_{(K,\kappa)} \in I_G$. Then, by the definition of I_G there exists a finite group H with $|H| < |G|$ and pairs $(V,\psi) \in \mathcal{M}_{G\times H}$ and $(W,\mu) \in \mathcal{M}_{H\times G}$ such that the standard basis element $e_{(K,\kappa)}$ occurs as a summand in $\left[\frac{G\times H}{V,\psi}\right] \mathop{\cdot}\limits_{H} \left[\frac{H\times G}{W,\mu}\right]$. By the Mackey formula in Corollary 2.5, it follows that there exists a pair $(W',\mu') \in \mathcal{M}_{H\times G}$ such that $(\Delta_K(G),\phi_\kappa) = (V * W', \psi * \mu')$. By Proposition 2.6, it follows that $l(V,\psi) = (G,K',\kappa')$ with $(K',\kappa') \leqslant (K,\kappa)$. Now, by Proposition 4.2(b), we have $e_{(K,\kappa)} \mathop{\cdot}\limits_{G} \left[\frac{G\times H}{V,\psi}\right] = \left[\frac{G\times H}{V',\psi'}\right]$ with $(V',\psi') \in \mathcal{M}_{G\times H}$ satisfying $l(V',\psi') = (G,K,\kappa)$. Proposition 1.3(e) applied to (V',ψ') implies that $|G| \leqslant |H|$, a contradiction. ◻

8.7 Example The converse of Part (b) in the previous Proposition does not hold in general: Let $A := \{\pm 1\}$, G the cyclic group of order 4, K its subgroup of order 2, and $\kappa\colon K \to A$ the unique injective homomorphism. Then an easy computation shows that $e_{(K,\kappa)} \notin I_G$. Thus, (K,κ) is reduced in G, but K is not contained in $G' = \{1\}$.

We conclude this section with some additional results around the notion of being reduced.

8.8 Proposition *Let G be a finite group and let $(K,\kappa) \in \mathcal{M}_G^G$. The following are equivalent:*

(i) $(K,\kappa) \notin \mathcal{R}_G$.

(ii) *There exist a finite group H with $|H| < |G|$ and $(L,\lambda) \in \mathcal{M}_H^H$ such that $(G,K,\kappa) \sim (H,L,\lambda)$.*

Proof (i) $\Rightarrow$ (ii): If $(K,\kappa) \notin \mathcal{R}_G$ then $e_{(K,\kappa)} \in I_G$ and there exists a group H with $|H| < |G|$ and pairs $(U,\phi) \in \mathcal{M}_{G\times H}$, $(V,\psi) \in \mathcal{M}_{H\times G}$ such that the standard basis element $e_{(K,\kappa)}$ of $B_k^A(G,G)$ occurs with non-zero coefficient in the tensor product $\left[\frac{G\times H}{U,\phi}\right] \mathop{\cdot}\limits_{H} \left[\frac{H\times G}{V,\psi}\right]$. The Mackey formula and Proposition 2.6(a),(b) imply that $p_1(U) = G$ and that $(K,\kappa) \geqslant l_0(U,\phi)$. We set $\tilde{H} := p_2(U)$ and write $e_{(K,\kappa)} \mathop{\cdot}\limits_{G} \left[\frac{G\times \tilde{H}}{U,\phi}\right] = \left[\frac{G\times \tilde{H}}{\tilde{U},\tilde{\phi}}\right]$ with $(\tilde{U},\tilde{\phi}) \in \mathcal{M}_{G\times\tilde{H}}$. Then

$$|\tilde{H}| \leqslant |H| < |G|, \quad p_2(\tilde{U}) = \tilde{H}, \quad \text{and} \quad l(\tilde{U},\tilde{\phi}) = (G,K,\kappa)$$

by Proposition 4.2(b). Thus, $(G,K,\kappa) \underset{(\tilde U,\tilde\phi)}{\sim} (\tilde H,\tilde L,\tilde\lambda)$ with $(L,\tilde\lambda) := r(\tilde U,\tilde\phi) \in \mathcal{M}_{\tilde H}^{\tilde H}$.

(ii)⇒(i): Let $(U,\phi) \in \mathcal{M}_{G\times H}$ with $l(U,\phi) = (G,K,\kappa)$ and $r(U,\phi) = (H,L,\lambda)$. Then Proposition 4.2(d), implies $e_{(K,\kappa)} = \left[\frac{G\times H}{U,\phi}\right] \cdot_H \left[\frac{G\times H}{U,\phi}\right]^{\mathrm{op}} \in I_G$, and therefore $(K,\kappa) \notin \mathcal{R}_G$. □

Since the condition in Proposition 8.8(ii) is independent of k, we have the following immediate corollary.

8.9 Corollary *Let G and H be finite groups. The subset $\mathcal{R}_k^A(G)$ of $\mathcal{M}_G$ is independent of k.*

8.10 Definition Let G and H be finite groups. A pair $(U,\phi) \in \mathcal{M}_{G\times M}^c$ is called *reduced* if $l_0(U,\phi) \in \mathcal{R}_G$ and $r_0(U,\phi) \in \mathcal{R}_H$. By Corollary 8.9, this notion does not depend on k.

The following proposition justifies this terminology.

8.11 Proposition *Let G and H be finite groups, let $(U,\phi) \in \mathcal{M}_{G\times H}^c$, and set $(K,\kappa) := l_0(U,\phi)$ and $(L,\lambda) := r_0(U,\phi)$. The following are equivalent:*

(i) *(U,ϕ) is reduced.*

(ii) *$|G| = |H|$ and $\left[\frac{G\times H}{U,\phi}\right]$ does not factor through a group of order smaller than $|G| = |H|$, i.e., there exists no group I of order smaller than $|G| = |H|$ such that $\left[\frac{G\times H}{U,\phi}\right] \in B_k^A(G,I) \cdot_I B_k^A(I,H)$.*

Proof (i)⇒(ii): Since $\left[\frac{G\times H}{U,\phi}\right] \cdot_H \left[\frac{G\times H}{U,\phi}\right]^{\mathrm{op}} = e_{(K,\kappa)}$ by Proposition 4.2(d), and since $(K,\psi) \in \mathcal{R}_G$, we obtain $|H| \geqslant |G|$. Similarly one proves that $|G| \geqslant |H|$ to obtain $|G| = |H|$. If $\left[\frac{G\times H}{U,\phi}\right]$ factors through a group of order smaller than $|G|$, so does $e_{(K,\kappa)} = \left[\frac{G\times H}{U,\phi}\right] \cdot_H \left[\frac{G\times H}{U,\phi}\right]^{\mathrm{op}}$ which contradicts $(K,\psi) \in \mathcal{R}_G$.

(ii)⇒(i): By Proposition 4.2(c) we have $\left[\frac{G\times H}{U,\phi}\right] = e_{(K,\psi)} \cdot_G \left[\frac{G\times H}{U,\phi}\right] \cdot_H e_{(L,\lambda)}$. Now, (ii) immediately implies (i). □

9 Simple A-Fibered Biset Functors

We keep the notation from the previous sections: A is an abelian group, k is a commutative ring, $\mathcal{C} = \mathcal{C}_k^A$ is the category introduced in Section 3, and $\mathcal{F} = \mathcal{F}_k^A$ is the category of k-linear functors from $\mathcal{C}_k^A$ to ${}_k\mathsf{Mod}$. For a finite group G, we continue to write $E_G := B_k^A(G,G) = \mathrm{End}_{\mathcal{C}}(G,G)$ and $\bar E_G = E_G/I_G$, where I_G denotes the ideal of E_G of all morphisms factoring through groups of order strictly smaller than $|G|$. The goal of this section is the classification of the simple objects in $\mathcal{F}$.

9.1 Recall from Proposition 3.7 that for every finite group G and every simple E_G-module V one obtains a simple functor $S_{G,V} \in \mathcal{F}$ and that every simple functor arises that way. Also, recall from Corollary 8.5 that if $(K,\kappa) \in \mathcal{R}_G$ and if V is an irreducible $k\Gamma_{(G,K,\kappa)}$-module then

$$\widetilde{V} := \bar{E}_G \bar{f}_{(K,\kappa)} \otimes_{k\Gamma_{(G,K,\kappa)}} V \tag{23}$$

is an irreducible E_G-module. Therefore, we obtain a simple functor

$$S_{(G,K,\kappa,V)} := S_{G,\widetilde{V}} \in \mathcal{F}\,.$$

It is clear that if $V' \cong V$ as $k\Gamma_{(G,K,\kappa)}$-modules then $\widetilde{V} \cong \widetilde{V}'$ as E_G-modules, $L_{G,\widetilde{V}} \cong L_{G,\widetilde{V}'}$ as functors in $\mathcal{F}$, and therefore, $S_{G,\widetilde{V}} \cong S_{G,\widetilde{V}'}$ in $\mathcal{F}$ as the unique simple factors of $L_{G,\widetilde{V}}$ and $L_{G,\widetilde{V}'}$, respectively. We denote by $\mathrm{Irr}(\mathcal{F})$ the set of isomorphism classes $[S]$ of simple functors $S \in \mathcal{F}$. Moreover, we set

$$\mathcal{S} = \mathcal{S}_k^A := \{(G,K,\kappa,[V]) \mid G \in \mathrm{Ob}(\mathcal{C}), (K,\kappa) \in \mathcal{R}_G, [V] \in \mathrm{Irr}(k\Gamma_{(G,K,\kappa)})\}\,.$$

Two quadruples $(G,K,\kappa,[V])$ and $(H,L,\lambda,[W])$ in $\mathcal{S}$ will be called *linked* if $(G,K,\kappa) \sim (H,L,\lambda)$, cf. Definition 5.1(a), and

$$V \cong k[{}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}] \otimes_{k\Gamma_{(H,L,\lambda)}} W$$

as $k\Gamma_{(G,K,\kappa)}$-modules, cf. 6.1(c). In this case we write $(G,K,\kappa,[V]) \sim (H,L,\lambda,[W])$. Note that linkage is an equivalence relation on $\mathcal{S}$. We denote by $\bar{\mathcal{S}} = \bar{\mathcal{S}}_k^A$ the set of linkage classes $[G,K,\kappa,[V]]$ of $\mathcal{S}$. By all the above, we have now defined a function

$$\omega\colon \mathcal{S} \to \mathrm{Irr}(\mathcal{F})\,, \quad (G,K,\kappa,[V]) \mapsto [S_{(G,K,\kappa,V)}]\,. \tag{24}$$

The goal of this section is to prove the following theorem.

9.2 Theorem *The function ω from (24) induces a bijection*

$$\bar{\omega}\colon \bar{\mathcal{S}} \to \mathrm{Irr}(\mathcal{F})\,, \quad [G,K,\kappa,[V]] \mapsto [S_{(G,K,\kappa,V)}]\,.$$

Before we can prove the theorem we will need three Lemmas as preparation.

9.3 Lemma *Let G and H be finite groups and let $(K,\kappa) \in \mathcal{M}_G^G$ and $(L,\lambda) \in \mathcal{M}_H^H$.*

(a) *If $M \in \mathcal{F}$ and ${}_{(G,K,\kappa)}\Gamma_{H,L,\lambda} \neq \emptyset$ then, for every $x \in {}_{(G,K,\kappa)}\Gamma_{H,L,\lambda}$, the map*

$$f_{(L,\lambda)}M(H) \to f_{(K,\kappa)}M(G)\,, \quad m \mapsto xm\,,$$

is an isomorphism of k-modules with inverse $m \mapsto x^{\mathrm{op}}m$.

(b) *Assume that $(K,\kappa) \in \mathcal{R}_G$ and $(L,\lambda) \in \mathcal{R}_H$ and that $(G,K,\kappa) \sim (H,L,\lambda)$. Then $|G| = |H|$.*

Proof (a) This follows immediately from the relations

$$xx^{\mathrm{op}} = e_{(K,\kappa)}, \quad x^{\mathrm{op}}x = e_{(L,\lambda)}, \quad e_{(K,\kappa)}f_{(K,\kappa)} = f_{(K,\kappa)}, \quad e_{(L,\lambda)}f_{(L,\lambda)} = f_{(L,\lambda)}. \tag{25}$$

(b) Let $(U,\phi) \in \mathcal{M}_{G\times H}$ satisfy $l(U,\phi) = (G,K,\kappa)$ and $r(U,\phi) = (H,L,\lambda)$ and assume that $|G| > |H|$. Then $e_{(K,\kappa)} = \left[\frac{G\times H}{U,\phi}\right] \cdot_H \left[\frac{G\times H}{U,\phi}\right]^{\mathrm{op}} = e_{(K,\kappa)} \in I_G$, contradicting $(K,\kappa) \in \mathcal{R}_G$. Similarly, we obtain a contradiction if we assume $|G| < |H|$. ∎

9.4 Lemma *Let $(H,L,\lambda,[W])\in\mathcal{S}$ and let $\widetilde{W}$ be the irreducible $\bar{E}_H$-module defined as in (23).*

(a) *If G is a finite group with $|G|=|H|$ then $I_G\cdot S_{H,\widetilde{W}}(G)=\{0\}$ and $S_{H,\widetilde{W}}(G)$ can be viewed as $\bar{E}_G$-module.*

(b) *One has a $k\Gamma_{(H,L,\lambda)}$-module isomorphism $f_{(L,\lambda)}S_{H,\widetilde{W}}(H)\cong W$. Here, $f_{(L,\lambda)}S_{H,\widetilde{W}}(H)$ is viewed as $k\Gamma_{(H,L,\lambda)}$-module via the k-algebra isomorphism $k\Gamma_{(H,L,\lambda)}\to f_{(L,\lambda)}E_H^c f_{(L,\lambda)}$ from Lemma 6.6(f).*

Proof (a) We may assume that $S_{H,\widetilde{W}}(G)\neq\{0\}$. Then, with H, also G is minimal for $S_{H,\widetilde{W}}$, since $|G|=|H|$. The result now follows from Proposition 3.7(b).

(b) Recall from Lemma 3.5 that $S_{H,\widetilde{W}}(H)\cong\widetilde{W}=\bar{E}_H\bar{f}_{(L,\lambda)}\otimes_{k\Gamma_{(H,L,\lambda)}}W$ as E_H-modules. Thus, using Part (a) for $G=H$, we have $f_{(L,\lambda)}S_{H,\widetilde{W}}(H)=\bar{f}_{(L,\lambda)}S_{H,\widetilde{W}}(H)\cong\bar{f}_{(L,\lambda)}\bar{E}_H\bar{f}_{(L,\lambda)}\otimes_{k\Gamma_{(H,L,\lambda)}}W$ as $\bar{f}_{(L,\lambda)}\bar{E}_H\bar{f}_{(L,\lambda)}$-modules. Using the k-algebra isomorphism in Proposition 8.4(c), we obtain the desired isomorphism. □

Let G and H be finite groups. Generalizing the notation from 6.1(a), we define $B_k^{A,c}(G,H)$ as the k-span of the standard basis elements $\left[\frac{G\times H}{U,\phi}\right]$ of $B_k^A(G,H)$ with $(U,\phi)\in\mathcal{M}^c_{G\times H}$, i.e., such that $p_1(U)=G$ and $p_2(U)=H$.

9.5 Lemma *Let G and H be finite groups and let $(K,\kappa)\in\mathcal{M}_G^G$ and $(L,\lambda)\in\mathcal{M}_H^H$.*

(a) *If $|G|=|H|$ then the map*

$$\alpha\colon k\big[{}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}\big]\to f_{(K,\kappa)}B_k^{A,c}(G,H)f_{(L,\lambda)},\quad b\mapsto f_{(K,\kappa)}bf_{(L,\lambda)}\,,$$

is an isomorphism of $(k\Gamma_{(G,K,\kappa)},k\Gamma_{(H,L,\lambda)})$-bimodules, where $f_{(K,\kappa)}B_k^{A,c}(G,H)f_{(L,\lambda)}$ is viewed as $(k\Gamma_{(G,K,\kappa)},k\Gamma_{(H,L,\lambda)})$-bimodule via the isomorphism from Lemma 6.6(f).

(b) *Assume that $(H,L,\lambda,[W])\in\mathcal{S}$ and that $|G|=|H|$. Then there exists an epimorphism*

$$k\big[{}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}\big]\otimes_{k\Gamma_{(H,L,\lambda)}}W\to f_{(K,\kappa)}S_{H,\widetilde{W}}(G)$$

of $k\Gamma_{(G,K,\kappa)}$-modules, where $f_{(K,\kappa)}S_{H,\widetilde{W}}(G)$ is viewed as $k\Gamma_{(G,K,\kappa)}$-module via the isomorphism from Lemma 6.6(f).

Proof (a) First we treat the case that ${}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$ is non-empty and we pick an element x from it. Consider the diagram of left $k\Gamma_{(G,K,\kappa)}$-module homomorphisms,

$$\begin{array}{ccc}
k\big[{}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}\big] & \xrightarrow{\;-\,\underset{H}{\cdot}\,x^{\mathrm{op}}\;} & k\Gamma_{(G,K,\kappa)} \\
\Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\wr} \\
f_{(K,\kappa)}B_k^{A,c}(G,H)f_{(L,\lambda)} & \xrightarrow{\;-\,\underset{H}{\cdot}\,x^{\mathrm{op}}f_{(K,\kappa)}\;} & f_{(K,\kappa)}E_G^c f_{(K,\kappa)}
\end{array}$$

where the right vertical map is the isomorphism $a \mapsto f_{(K,\kappa)} a f_{(K,\kappa)}$ from Lemma 6.6(f). By Lemma 4.5 we have

$$f_{(K,\kappa)} x f_{(L,\lambda)} = x f_{(L,\lambda)} \quad \text{and} \quad f_{(L,\lambda)} x^{\mathrm{op}} f_{(K,\kappa)} = x^{\mathrm{op}} f_{(K,\kappa)} \,. \tag{26}$$

This implies that the diagram is commutative. And together with the relations in (25) it implies that the top horizontal map is an isomorphism with inverse $- \underset{G}{\cdot} x$ and that the bottom horizontal map is an isomorphism with inverse $- \underset{G}{\cdot} x f_{(L,\lambda)}$. Thus, also α is an isomorphism.

Next assume that ${}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$ is empty. Let $(U,\phi) \in \mathcal{M}^c_{G\times H}$ and set $b := \left[\frac{G\times H}{U,\phi}\right] \in B_k^{A,c}(G,H)$, a general standard basis element. It suffices to show that $f_{(K,\kappa)} b f_{(L,\lambda)} = 0$. Set $(K',\kappa') := l_0(U,\phi) \in \mathcal{M}_G^G$ and $(L',\lambda') := r_0(U,\phi) \in \mathcal{M}_H^H$. Then $b = e_{(K',\kappa')} b e_{(L',\lambda')}$. By Proposition 4.4, we may assume that $(K',\kappa') \leqslant (K,\kappa)$ and $(L',\lambda') \leqslant (L,\lambda)$. We will show that a pair (U,ϕ) with these conditions cannot exist. Assume it does. Then

$$e_{(K,\kappa)} \left[\frac{G\times H}{U,\phi}\right] e_{(L,\lambda)} = \left[\frac{G\times H}{V,\psi}\right]$$

for some $(V,\psi) \in \mathcal{M}_{G\times H}$ satisfying $l(V,\psi) = (G,K,\kappa)$ and $l(V,\psi) = (H,L,\lambda)$ by Proposition 4.2(b). Thus $(V,\psi) \in {}_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}$, a contradiction.

(b) Recall from 3.4 that, for every finite group I, the evaluation $L_{H,\widetilde{W}}(I)$ is given by

$$L_{H,\widetilde{W}}(I) = B_k^A(I,H) \otimes_{E_H} \widetilde{W} = B_k^A(I,H) \otimes_{E_H} \bar{E}_H \bar{f}_{(L,\lambda)} \otimes_{k\Gamma_{(H,L,\lambda)}} W \,.$$

Similar to the functor $L_{H,\widetilde{W}}$ we define a functor $M_{H,W} \in \mathcal{F}$ via

$$M_{H,W}(I) = B_k^A(I,H) \otimes_{E_H} E_H f_{(L,\lambda)} \otimes_{k\Gamma_{(H,L,\lambda)}} W = B_k^A(I,H) f_{(L,\lambda)} \otimes_{k\Gamma_{(H,L,\lambda)}} W \,.$$

The functor is defined on morphisms in the same way as $L_{H,\widetilde{W}}$, namely via composition in the category $\mathcal{C}$ from the left. Since the $(E_H, k\Gamma_{(H,L,\lambda)})$-bimodule $\bar{E}_H \bar{f}_{(L,\lambda)}$ is a factor module of the $(E_H, k\Gamma_{(H,L,\lambda)})$-bimodule $E_H f_{(L,\lambda)}$, we obtain an epimorphism of functors $M_{H,W} \to L_{H,\widetilde{W}}$. Composing it with the natural epimorphism $L_{H,\widetilde{W}} \to S_{H,\widetilde{W}}$ we obtain an epimorphism $\pi\colon M_{H,W} \to S_{H,\widetilde{W}}$ of A-fibered biset functors. In particular we have an epimorphism

$$\pi_G\colon B_k^A(G,H) f_{(L,\lambda)} \otimes_{k\Gamma_{(H,L,\lambda)}} W \to S_{H,\widetilde{W}}(G)$$

of E_G-modules. If $(U,\phi) \in \mathcal{M}_{G\times H} \smallsetminus \mathcal{M}^c_{G\times H}$ then $\pi_G(\left[\frac{G\times H}{U,\phi}\right] \otimes w) = 0$ for all $w \in W$. In fact, this follows from the decomposition of $\left[\frac{G\times H}{U,\phi}\right]$ as in Proposition 2.8 and from $|G| = |H|$, since $S_{H,\widetilde{W}}(I) = \{0\}$ for all finite groups I with $|I| < |G| = |H|$. Thus, we also obtain an epimorphism

$$B_k^{A,c}(G,H) f_{(L,\lambda)} \otimes_{k\Gamma_{(H,L,\lambda)}} W \to S_{H,\widetilde{W}}(G)$$

of E_G^c-modules, and after multiplying with the idempotent $f_{(K,\kappa)}$ we obtain an epimorphism

$$f_{(K,\kappa)}B_k^{A,c}(G,H)f_{(L,\lambda)}\otimes_{k\Gamma_{(H,L,\lambda)}}W\to f_{(K,\kappa)}S_{H,\widetilde{W}}(G)$$

of $f_{(K,\kappa)}E_G^c f_{(K,\kappa)}$-modules. Using the isomorphism from Part (a) and the k-algebra isomorphism $k\Gamma_{(G,K,\kappa)}\to f_{(K,\kappa)}E_G^c f_{(K,\kappa)}$ from Proposition 8.4(c), we obtain the desired epimorphism of $k\Gamma_{(G,K,\kappa)}$-modules. □

9.6 Proof of Theorem 9.2. (a) First we show that the map ω in (24) is surjective. So let $S\in\mathcal{F}$ be simple. Choose a finite group G of minimal order such that $S(G)\neq\{0\}$. Then, by Proposition 3.7(a), $U:=S(G)$ is an irreducible E_G-module and $S\cong S_{G,U}$. Since $S(H)=0$ for all finite groups H with $|H|<|G|$, the module U is annihilated by I_G and comes via inflation from an irreducible $\bar{E}_G$-module which we again denote by U. By Corollary 8.5, we obtain $U\cong\bar{E}_G\bar{f}_{(K,\kappa)}\otimes_{k\Gamma_{(G,K,\kappa)}}V$ for some $(K,\kappa)\in\mathcal{R}_G$ and some irreducible $k\Gamma_{(G,K,\kappa)}$-module V. Thus, $U\cong\widetilde{V}$ as defined in (23) and $S\cong S_{G,U}\cong S_{G,\widetilde{V}}=S_{(G,K,\kappa,V)}$.

(b) Next we show that if $(G,K,\kappa,[V]),(H,L,\lambda,[W])\in\mathcal{S}$ are linked then $S_{(G,K,\kappa,V)}\cong S_{(H,L,\lambda,W)}$. First, $(G,K,\kappa,[V])\sim(H,L,\lambda,[W])$ implies $|G|=|H|$, by Lemma 9.3(b). Further, by Lemma 9.4(a), $S_{H,\widetilde{W}}(G)$ is an $\bar{E}_G$-module. Since $f_{(L,\lambda)}S_{H,\widetilde{W}}(H)\cong W$, by Lemma 9.4(b), and since $(G,K,\kappa)\sim(H,L,\lambda)$, we obtain $\bar{f}_{(K,\kappa)}S_{H,\widetilde{W}}(G)\neq 0$, by Lemma 9.3(a). Moreover, since $(G,K,\kappa,[V])\sim(H,L,\lambda,[W])$, we have $V\cong k[_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}]\otimes_{k\Gamma_{(H,L,\lambda)}}W$ and Lemma 9.5(b) implies $\mathrm{Hom}_{k\Gamma_{(G,K,\kappa)}}(V,\bar{f}_{(K,\kappa)}S_{H,\widetilde{W}}(G))\neq\{0\}$. But

$$\begin{aligned}\mathrm{Hom}_{k\Gamma_{(G,K,\kappa)}}\Big(V,\bar{f}_{(K,\kappa)}S_{H,\widetilde{W}}(G)\Big)&\cong\mathrm{Hom}_{k\Gamma_{(G,K,\kappa)}}\Big(V,\mathrm{Hom}_{\bar{E}_G}(\bar{E}_G\bar{f}_{(K,\kappa)},S_{H,\widetilde{W}}(G))\Big)\\ &\cong\ \mathrm{Hom}_{\bar{E}_G}\Big(\bar{E}_G\bar{f}_{(K,\kappa)}\otimes_{k\Gamma_{(G,K,\kappa)}}V,S_{H,\widetilde{W}}(G)\Big)\cong\mathrm{Hom}_{\bar{E}_G}(\widetilde{V},S_{H,\widetilde{W}}(G))\,.\end{aligned}$$

Thus, $\mathrm{Hom}_{\bar{E}_G}(\widetilde{V},S_{H,\widetilde{W}}(G))\neq\{0\}$. Since both $\widetilde{V}$ and $S_{H,\widetilde{W}}(G)$ are simple E_G-modules, we obtain $\widetilde{V}\cong S_{H,\widetilde{W}}(G)$. Now Proposition 3.7(d) implies that $S_{G,\widetilde{V}}=S_{H,\widetilde{W}}$.

(c) Finally, we show that if $(G,K,\kappa,[V]),(H,L,\lambda,[W])\in\mathcal{S}$ satisfy $S_{(G,K,\kappa,[V])}\cong S_{(H,L,\lambda,[W])}$ then $(G,K,\kappa,V)\sim(H,L,\lambda,W)$. Note that, since G is a minimal subgroup for $S_{G,\widetilde{V}}$ and H is a minimal subgroup for $S_{H,\widetilde{W}}$ (by Proposition 3.7(d)), we have $|G|=|H|$. Thus, I_G annihilates $S_{H,\widetilde{W}}(G)$, and $S_{H,\widetilde{W}}(G)$ is an $\bar{E}_G$-module, by Lemma 9.4(a). Since $S_{G,\widetilde{V}}\cong S_{H,\widetilde{W}}$, we have isomorphisms $\bar{f}_{(K,\kappa)}S_{H,\widetilde{W}}(G)\cong\bar{f}_{(K,\kappa)}S_{G,\widetilde{V}}(G)\cong V$ as $k\Gamma_{(G,K,\kappa)}$-modules, by Proposition 9.4(b). By Lemma 9.5(b), there exists an epimorphism $k[_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}]\otimes_{k\Gamma_{(H,L,\lambda)}}W\to\bar{f}_{(K,\kappa)}S_{H,\widetilde{W}}(G)\cong V$ of $k\Gamma_{(G,K,\kappa)}$-modules. In particular, $_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}\neq\emptyset$. Since $k[_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}]\otimes_{k\Gamma_{(H,L,\lambda)}}W$ is a simple $k\Gamma_{(G,K,\kappa)}$-module (see 6.1(c)), we obtain $k[_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}]\otimes_{k\Gamma_{(H,L,\lambda)}}W\cong V$ as $k\Gamma_{(G,K,\kappa)}$-modules. Thus, $(G,K,\kappa,[V])\sim(H,L,\lambda,[W])$, and the proof of Theorem 9.2 is complete. □

The next proposition shows that the evaluation $S(H)$ of a simple functor S parametrized by the quadruple $(G,K,\kappa,[V])$ vanishes, unless H has a section that is related to the triple (G,K,κ) in very strong sense.

9.7 Proposition *Let $(G, K, \kappa, [V]) \in \mathcal{S}$ and let $\widetilde{V}$ be the associated irreducible E_G-module from (23). Assume that H is a finite group such that $S_{G,\widetilde{V}}(H) \neq \{0\}$. Then there exist subgroups $H_1 \trianglelefteq H_2 \leqslant H$ such that $I := H_2/H_1$ has the following property: There exists a pair $(L, \lambda) \in \mathcal{M}_I^I$ with faithful λ, such that $(G, K, \kappa) \sim (I, L, \lambda)$, $G/K \cong I/L$, $K \cap G' \cong L \cap I'$, and $|I| \geqslant |G|$.*

Proof Recall that $S_{G,\widetilde{V}} = L_{G,\widetilde{V}}/J_{G,\widetilde{V}}$. Since $S_{G,\widetilde{V}}(H) \neq \{0\}$, we have a proper inclusion

$$J_{G,\widetilde{V}}(H) \subset L_{G,\widetilde{V}}(H) = B_k^A(H, G) \otimes_{E_G} \widetilde{V} = B_k^A(H, G) \otimes_{E_G} \bar{E}_G \bar{f}_{(K,\kappa)} \otimes_{k\Gamma_{(G,K,\kappa)}} V\,.$$

By the explicit description of $J_{G,\widetilde{V}}(H)$ in Lemma 3.5, and since $\bar{E}_G \bar{f}_{(K,\kappa)} = \bar{f}_{\{K,\kappa\}} \bar{E}_G \bar{f}_{(K,\kappa)}$, there exist elements $(U, \phi) \in \mathcal{M}_{G\times H}$ and $(W, \psi) \in \mathcal{M}_{H\times G}$ such that the standard basis elements

$$x := \left[\frac{G \times H}{U, \phi}\right] \in B_k^A(G, H) \quad \text{and} \quad y := \left[\frac{H \times G}{W, \psi}\right] \in B_k^A(H, G)$$

satisfy $x \mathbin{\cdot_H} y \mathbin{\cdot_G} f_{\{K,\kappa\}} \notin I_G$. This implies that there exists $(\tilde{K}, \tilde{\kappa}) \in \{K, \kappa\}_G$ such that $x \mathbin{\cdot_H} y \mathbin{\cdot_G} f_{(\tilde{K},\tilde{\kappa})} \notin I_G$. By the decomposition in Proposition 2.8 we obtain immediately that $r(W, \psi) = (G, \tilde{\tilde{K}}, \tilde{\tilde{\kappa}})$ with faithful $\tilde{\tilde{\kappa}}$. Moreover, since $x \mathbin{\cdot_H} y \mathbin{\cdot_G} f_{(\tilde{K},\tilde{\kappa})} \notin I_G$ and $y = y \mathbin{\cdot_G} e_{(\tilde{\tilde{K}},\tilde{\tilde{\kappa}})}$, Proposition 4.4 implies that $(\tilde{\tilde{K}}, \tilde{\tilde{\kappa}}) \leqslant (\tilde{K}, \tilde{\kappa})$. Again by Proposition 4.4, we have $y \mathbin{\cdot_G} f_{(\tilde{K},\tilde{\kappa})} = y \mathbin{\cdot_G} e_{(\tilde{K},\tilde{\kappa})} f_{(\tilde{K},\tilde{\kappa})}$. Thus, replacing y with the standard basis element $y \mathbin{\cdot_G} e_{(\tilde{K},\tilde{\kappa})}$ and using the dual version of Proposition 4.2(b), we may assume that $r(W, \psi) = (G, \tilde{K}, \tilde{\kappa})$ with $(\tilde{K}, \tilde{\kappa}) \in \{K, \kappa\}_G$. Since $p_2(W) = G$ and κ is faithful, one can decompose y according to Proposition 2.8 as

$$y = \mathrm{Ind}_{H_2}^H \mathbin{\cdot_{H_2}} \mathrm{Inf}_I^{H_2} \mathbin{\cdot_I} \left[\frac{I \times G}{\widetilde{W}, \tilde{\psi}}\right]$$

with $H_2 := p_1(W)$ and $H_1 := \ker(\psi_1)$, where $\psi_1 \in k_1(W)^*$ is defined as in 1.2, and $I := H_2/H_1$. Note that $l(\widetilde{W}, \tilde{\psi}) = (I, L, \lambda)$ with faithful $\lambda \in L^*$ and that $r(\widetilde{W}, \tilde{\psi}) = r(W, \psi) = (G, \tilde{K}, \tilde{\kappa})$. Thus we have $(I, L, \lambda) \sim (G, \tilde{K}, \tilde{\kappa}) \sim (G, K, \kappa)$. Parts (c) and (d) of Proposition 1.3 now imply that $H/L \cong G/K$ and that $L \cap I' \cong K \cap G'$. Finally, since $x \mathbin{\cdot_H} y \mathbin{\cdot_G} f_{\{K,\kappa\}} \notin I_G$, and y factors through I we also have $|G| \leqslant |I|$. □

9.8 Remark It will frequently happen that G and H are non-isomorphic finite groups and that $(K, \kappa) \in \mathcal{R}_G$ and $(L, \lambda) \in \mathcal{R}_H$ are reduced pairs such that $(G, K, \kappa) \sim (H, L, \lambda)$. All one needs is a pair $(U, \phi) \in \mathcal{M}_G$ with $l(U, \phi) = (G, K, \kappa)$ and $r(U, \phi) = (H, L, \lambda)$. If W is an irreducible $k\Gamma_{(H,L,\lambda)}$-module and $V := k[_{(G,K,\kappa)}\Gamma_{(H,L,\lambda)}] \otimes_{k\Gamma_{(H,L,\lambda)}} W$ is the corresponding irreducible $k\Gamma_{(G,K,\kappa)}$-module, then $S := S_{G,\widetilde{V}} \cong S_{H,\widetilde{W}}$ and the minimal groups G and H for S are non-isomorphic, answering a question of Serge Bouc (cf. [R12, Conjecture 2.16]) to the negative. See for instance [R12, Example 12], giving an example of $(U, \phi) \in \mathcal{M}_{G\times H}$ where G is the quaternion group of order 8 and H the dihedral group of order 8. For this example to work, A needs to be an abelian group which contains an element of order 4. A pair (U, ϕ) with ϕ of order 4 is explicitly constructed. In this example, $(K, \kappa) = (Z(G), \kappa)$ and $(L, \lambda) = (Z(H), \lambda)$, where κ and λ are injective homomorphisms to A. Note that (K, κ) and (L, λ) are reduced by Proposition 8.6(b).

10 Reduced pairs revisited

In the previous section, we parametrized the simple fibered biset functors by equivalence classes of quadruples $(G, K, \kappa, [V])$, where (K, κ) is reduced in $\mathcal{M}_G$. The indirect definition of being reduced makes it very difficult to determine which pairs (K, κ) are reduced in $\mathcal{M}_G$. The goal of this section is to establish a more explicit characterization (see Corollary 10.13) of being reduced under additional assumptions on A. These assumptions are satisfied in all the cases of interest to us; for instance when A is the multiplicative group of an algebraically closed field F.

10.1 Hypothesis The group A has the following property: There exists a (unique) set π of primes such that for every $n \in \mathbb{N}$, the n-torsion part $\{a \in A \mid a^n = 1\}$ of A is cyclic of order n_π. Here, n_π denotes the π-part of n.

10.2 Remark (a) By π' we will denote the subset of primes which is complementary to π. For an abelian group B we denote the π-torsion subgroup of B by B_π.

(b) Note that Hypothesis 10.1 implies that $\mathrm{tor}A$ is divisible, i.e., for every $a \in \mathrm{tor}A$ and every $n \in \mathbb{N}$ there exists $b \in \mathrm{tor}A$ such that $b^n = a$.

(c) The important part of A is its torsion subgroup $\mathrm{tor}A$. Nothing in what follows changes if we replace A with $\mathrm{tor}A$, see Proposition 10.3(c). But we want to keep the freedom to choose A as the multiplicative group of a field or of an integral domain. Then Hypothesis 10.1 means that, for any given prime p, if the field has a primitive p-th root of unity then it must have a primitive p^n-th root of unity for every n. It is not difficult to see that Hypothesis 10.1 is equivalent to having an isomorphism $\mathrm{tor}A \cong (\mathbb{C}^\times)_\pi \cong (\mathbb{Q}/\mathbb{Z})_\pi$, but we want to keep the freedom to choose the ring or field more naturally depending on the situation.

The following propositions list some consequences of Hypothesis 10.1 on A.

10.3 Proposition *Assume that* $\mathrm{tor}A$ *is divisible and that, for every* $n \in \mathbb{N}$*, the* n*-torsion group* A_n *is finite. Furthermore, let* G *be a finite group acting trivially on* A*.*

(a) *The group* $B^2(G, A)$ *of 2-coboundaries has a complement in the group* $Z^2(G, A)$ *of 2-cocycles.*

(b) *The group* $H^2(G, A)$ *is finite and has exponent dividing* $|G|$*.*

(c) *The canonical map* $H^2(G, \mathrm{tor}A) \to H^2(G, A)$ *is an isomorphism and every 2-cohomology class in* $H^2(G, A)$ *can be represented by a 2-cocycle with values in* $A_{|G|}$*.*

Proof Parts (a) and (b) are proved in the same way as Theorem 11.15 in [I76], using Lemma [I76, Lemma 11.14]. The proof of Theorem 11.15 in [I76] also shows that $C^2(G,A)_{|G|} \cdot B^2(G,A) = Z^2(G,A)$. This proves the surjectivity of the canonical map $H^2(G, \mathrm{tor}A) \to H^2(G,A)$ and the existence part of the statement in Part (c). The injectivity of the above map follows immediately from the long cohomology sequence, see also Remark 5.4(b). □

Recall the notation $B^* := \mathrm{Hom}(B,A)$ for any abelian group B.

10.4 Proposition *Assume Hypothesis 10.1, let S be a finite group and let B be a finite abelian group.*

(a) *One has $|B^*| = |B|_\pi$.*

(b) *For every subgroup C of B, the sequence of natural maps*

$$1 \to (B/C)^* \to B^* \to C^* \to 1$$

is exact.

(c) *One has $\bigcap_{\mu \in B^*} \ker(\mu) = B_{\pi'}$.*

(d) *The canonical map*

$$\epsilon\colon B_\pi \to (B^*)^*, \quad b \mapsto (\mu \mapsto \mu(b)),$$

is an isomorphism.

(e) *Let $T \leqslant S$ be an abelian subgroup with $|T| = |T|_\pi$. The restriction map $S^* \to T^*$ is surjective if and only if $S' \cap T = \{1\}$.*

(f) *The cohomology group $H^2(S, \mathrm{tor}A)$ is finite and the group $B^2(S, \mathrm{tor}A)$ of coboundaries has a complement in the group $Z^2(S, \mathrm{tor}A)$ of cocycles. Here, we assume that S acts trivially on $\mathrm{tor}A$.*

Proof (a) This follows immediately from Hypothesis 10.1 and the structure theorem for finite abelian groups.

(b) Since $\mathrm{Hom}(-,A)$ is left exact, the sequence is exact everywhere, except possibly at C^*. By Part (a), this implies that the restriction map $B^* \to C^*$ has image of order

$$|B^*|/|(B/C)^*| = |B|_\pi/|B/C|_\pi = |C|_\pi = |C^*|.$$

Thus, the map $B^* \to C^*$ is surjective.

(c) Since A has trivial π'-torsion, it is clear that $B_{\pi'} \subseteq \ker(\mu)$ for every homomorphism $\mu \in B^*$. Conversely, assume that $b \in B$ has order n with $n_\pi \neq 1$. Let $a \in A$ be an element of order n_π (which exists by Hypothesis 10.1). Then $\nu\colon \langle b\rangle \to A$, $b \mapsto a$, defines a group homomorphism. By Part (b), ν can be extended to a homomorphism $\mu \in B^*$. Thus, we have $\mu(b) = a \neq 1$ for some $\mu \in B^*$.

(d) By Part (c), the map ϵ is injective. By Part (a), we have $|B_\pi| = |B|_\pi = |B^*| = |B^*|_\pi = |(B^*)^*|$. Thus, ϵ is also surjective.

(e) Assume first that $S' \cap T$ contains an element $t \neq 1$. Since T is a π-group, there exists a homomorphism $\nu_1 \colon \langle t \rangle \to A$ with $\nu_1(t) \neq 1$. By Part (b), this homomorphism can be extended to a homomorphism $\nu_2 \in T^*$. Then $\nu_2(t) \neq 1$ and, since $t \in S'$, the map ν_2 is not in the image of $S^* \to T^*$.

Now assume that $S' \cap T = \{1\}$. Then $S'T/S'$ is canonically isomorphic to T and every homomorphism $S'T/S' \to A$ can be extended to a homomorphism $S/S' \to A$, by Part (b). Thus, $S^* \to T^*$ is surjective.

(f) This follows immediately from Proposition 10.3. □

10.5 Notation Let S be a group and let B be an abelian group. We set

$$M(S) := H^2(S, \mathrm{tor}A)\,,$$

regarding $\mathrm{tor}A$ endowed with the trivial S-action. If $A = \mathbb{C}^\times$, the unit group of the complex numbers, then $M(S)$ is the well-known Schur multiplier of S. There exists a natural group homomorphism

$$\begin{aligned} \Psi \colon H^2(S,B) &\to \mathrm{Hom}(B^*, M(S))\,, \\ [\alpha] &\mapsto (\mu \mapsto [\mu \circ \alpha])\,. \end{aligned}$$

Here, we denote by $[\alpha]$ the cohomology class of a cocycle α. As usual, for an abelian group C we denote by

$$\mathrm{Ext}(C, B)$$

the subgroup of $H^2(C,B)$ consisting of cohomology classes of cocycles $\alpha \colon C \times C \to B$ satisfying $\alpha(c_1, c_2) = \alpha(c_2, c_1)$ for all $c_1, c_2 \in C$. These cohomology classes correspond to abelian extensions of C by B. Here we assume again that C acts trivially on B. Note that one has a natural group homomorphism

$$\begin{aligned} \iota \colon \mathrm{Ext}(S/S', B) &\to H^2(S,B)\,, \\ [\alpha] &\mapsto [\alpha \circ (\nu \times \nu)]\,, \end{aligned}$$

where $\nu \colon S \to S/S'$ denotes the natural epimorphism. It is shown in [K87, Lemma 2.1.17] that ι is injective and that the image of ι is equal to the subgroup

$$H^2_0(S,B) \leqslant H^2(S,B)$$

which is defined as the set of cohomology classes whose corresponding group extensions $1 \to B \to X \to S \to 1$ satisfy $B \leqslant Z(X)$ and $B \cap X' = \{1\}$.

10.6 Remark Assume that G is a group and that K is a normal abelian subgroup of G. Moreover set $\tilde{K} := K \cap G'$. Then the maps defined in 10.5 give rise to a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & & & \mathrm{Hom}((K/\tilde{K})^*, M(G/K)) & & \\
 & & & & & & \uparrow{\scriptstyle \epsilon_4} & & \\
1 \to & \mathrm{Ext}(G/KG', K) & \xrightarrow{\iota_1} & H^2(G/K, K) & \xrightarrow{\Psi_1} & & \mathrm{Hom}(K^*, M(G/K)) & \to 1 \\
 & \uparrow{\scriptstyle \epsilon_1} & & \uparrow{\scriptstyle \epsilon_2} & & & \uparrow{\scriptstyle \epsilon_3} & \\
1 \to & \mathrm{Ext}(G/KG', \tilde{K}) & \xrightarrow{\iota_2} & H^2(G/K, \tilde{K}) & \xrightarrow{\Psi_2} & & \mathrm{Hom}(\tilde{K}^*, M(G/K)) & \to 1 \\
 & & & & & & \uparrow & \\
 & & & & & & 1 &
\end{array}
$$

Here, ϵ_1, ϵ_2 and ϵ_3 are induced by the inclusion $\tilde{K} \to K$ and ϵ_4 is induced by the natural epimorphism $K \mapsto K/\tilde{K}$. Also, in the domains of the maps ι_1 and ι_2, we identify $(G/K)/(G/K)'$ with G/KG' in the obvious way.

10.7 Proposition *Assume that A satisfies Hypothesis 10.1. Let G be a finite group, let K be a subgroup of $Z(G)$ with $|K|_\pi = |K|$, and set $\tilde{K} := K \cap G'$.*

(a) *The two rows and the right column in the diagram in Remark 10.6 are exact.*

(b) *Let $\alpha \in Z^2(G/K, K)$ be a cocycle describing the central extension $1 \to K \to G \to G/K \to 1$. Then, in the diagram in Remark 10.6, one has $(\epsilon_4 \circ \Psi_1)([\alpha]) = 1$.*

Proof (a) The column is exact since, $\mathrm{Hom}(-, M(G/K))$ is left exact and since $1 \to (K/\tilde{K})^* \to K^* \to \tilde{K}^* \to 1$ is exact by Proposition 10.4(b).

We only show exactness of the first row. The exactness proof for the second one is analogous. For the exactness of the first row we refer to the proof of [K87, Theorem 2.1.19] in the case $A = \mathbb{C}^\times$. The same proof works for A satisfying Hypothesis 10.1. The injectivity of ι_1 and that $\mathrm{im}(\iota_1) = H_0^2(G/K, K)$ is proved in [K87, Lemma 2.1.17]. The exactness at $H^2(G/K, K)$ can be proved with the same arguments as in [K87, Theorem 2.1.19], using Proposition 10.4(e) and that K is a π-group. The surjectivity of Ψ_1 is proved with the same arguments as in the proof of [K87, Theorem 2.1.19], using the results in Proposition 10.4(d) and (f).

(b) There exists a function $\rho\colon G/K \to G$ such that

$$\rho(x)\rho(y) = \alpha(x, y)\rho(xy)\,,$$

for all $x, y \in G/K$. Every element in $(K/\tilde{K})^*$ is of the form $\bar{\mu}$ for a homomorphism $\mu\colon K \to A$ with $\mu|_{\tilde{K}} = 1$. By the definition of Ψ_1 and ϵ_4, it suffices to show that $[\mu \circ \alpha] = 1$

in $M(G/K)$, for each such μ. From Proposition 10.4(e) we know that $\bar{\mu}$ can be extended to a homomorphism $G/\tilde{K} \to A$. Then, also μ can be extended to a homomorphism $\tilde{\mu}\colon G \to A$. Applying $\tilde{\mu}$ to the above equation and setting $\nu(x) := \tilde{\mu}(\rho(x))$, for $x \in G/K$, we obtain

$$\nu(x)\nu(y) = \mu(\alpha(x, y))\nu(xy)$$

for all $x, y \in G/K$, showing that $[\mu \circ \alpha] = 1$. □

The motivation for the following definition is given in Proposition 10.9. It will be heavily used in Lemma 10.10.

10.8 Definition Let S be a group and let B be an abelian group (endowed with the trivial S-action). For $n \in \mathbb{N}$, let $F(S^{2n}, B)$ denote the abelian group of functions from S^{2n} to B. For each $\alpha \in Z^2(S, B)$, we define functions $\alpha_n \in F(S^{2n}, B)$, $n \in \mathbb{N}$, recursively by setting

$$\alpha_1(s_1, s_2) := \alpha(s_1, s_2)\alpha(s_2, s_1)^{-1}\alpha(s_2s_1, s_1^{-1}s_2^{-1})^{-1}\alpha(s_1s_2, s_1^{-1}s_2^{-1})\alpha(1, 1)^{-1},$$

for $s_1, s_2 \in S$, and by

$$\begin{aligned}\alpha_n(s_1, \ldots, s_{2n}) :=& \alpha_{n-1}(s_1, \ldots, s_{2n-2})\alpha_1(s_{2n-1}, s_{2n})\cdot \\ &\cdot \alpha([s_1, s_2] \cdots [s_{2n-3}, s_{2n-2}], [s_{2n-1}, s_{2n}]),\end{aligned}$$

for $n \geqslant 2$.

10.9 Proposition *Let S be a group, let B be an abelian group (with trivial S-action), let $\alpha \in Z^2(S, B)$ and let $n \in \mathbb{N}$. The function $\alpha_n\colon S^{2n} \to B$ has the following properties:*

(a) *If*

$$1 \longrightarrow B \stackrel{\iota}{\longrightarrow} T \stackrel{\pi}{\longrightarrow} S \longrightarrow 1 \tag{27}$$

is a short exact sequence of groups with $\iota(B) \leqslant Z(T)$ and if $\sigma\colon S \to T$ is a section of π (i.e., $\pi \circ \sigma = \mathrm{id}_S$) such that

$$\sigma(s_1)\sigma(s_2) = \iota(\alpha(s_1, s_2))\sigma(s_1s_2) \tag{28}$$

for all $s_1, s_2 \in S$, then one has

$$\begin{aligned}&[\sigma(s_1), \sigma(s_2)] \cdots [\sigma(s_{2n-1}), \sigma(s_{2n})] \\ &\qquad = \iota(\alpha_n(s_1, s_2, \ldots, s_{2n}))\sigma([s_1, s_2] \cdots [s_{2n-1}, s_{2n}]),\end{aligned} \tag{29}$$

for all $s_1, \ldots, s_{2n} \in S$.

(b) *One has $(\alpha\beta)_n = \alpha_n\beta_n$ for all $\alpha, \beta \in Z^2(S, B)$.*

(c) *If $f\colon \tilde{S} \to S$ is a group homomorphism then $(\alpha \circ (f \times f))_n = \alpha_n \circ (f \times \cdots \times f)$.*

(d) *If also C is an abelian group with trivial S-action and $f\colon B \to C$ is a group homomorphism then $(f \circ \alpha)_n = f \circ \alpha_n$.*

(e) *If S is abelian and α is symmetric (i.e., $\alpha(s_1, s_2) = \alpha(s_2, s_1)$ for all $s_1, s_2 \in S$) then α_n is the constant function with value $\alpha(1,1)^{-1}$.*

(f) *If there exists a function $\mu\colon S \to B$ such that $\alpha(s,t) = \mu(s)\mu(t)\mu(st)^{-1}$ for all $s, t \in S$ then*

$$\alpha_n(s_1, \ldots, s_{2n}) = \mu([s_1, s_2] \cdots [s_{2n-1}, s_{2n}])^{-1},$$

for all $s_1, \ldots, s_{2n} \in S$.

Proof (a) Assume that we have a short exact sequence as in (27) with $\iota(B) \leqslant Z(T)$ and a section σ of π such that Equation (28) holds for all $s_1, s_2 \in S$. Then Equation (28) implies

$$\sigma(1) = \iota(\alpha(1,1)) \tag{30}$$

and

$$\sigma(s)^{-1} = \iota(\alpha(s, s^{-1})^{-1}\alpha(1,1)^{-1})\sigma(s^{-1}) \tag{31}$$

for all $s \in S$. We prove Equation (29) by induction on n. For $n = 1$, Equation (28) yields

$$\begin{aligned}[\sigma(s_1), \sigma(s_2)] &= \sigma(s_1)\sigma(s_2)\Big(\sigma(s_2)\sigma(s_1)\Big)^{-1} \\ &= \sigma(s_1 s_2)\iota\alpha(s_1, s_2)\sigma(s_2 s_1)^{-1}\iota\alpha(s_2, s_1)^{-1}.\end{aligned}$$

Now applying Equation (31), for $s = s_2 s_1$, to the third factor in the last expression, we obtain the desired formula. For the induction step from $n-1$ to n we see by first applying the induction hypothesis and the case $n = 1$, then Equation (28), that

$$\begin{aligned}&[\sigma(s_1), \sigma(s_2)] \cdots [\sigma(s_{2n-1}), \sigma(s_{2n})] \\ &= \iota\alpha_{n-1}(s_1, \ldots, s_{2n-2})\sigma\Big([s_1, s_2] \cdots [s_{2n-3}, s_{2n-2}]\Big)\iota\alpha_1(s_{2n-1}, s_{2n})\sigma([s_{2n-1}, s_{2n}]) \\ &= \sigma\Big([s_1, s_2] \cdots [s_{2n-1}, s_{2n}]\Big)\iota\alpha\Big([s_1, s_2] \cdots [s_{2n-3}, s_{2n-2}], [s_{2n-1}, s_{2n}]\Big) \cdot \\ &\qquad \cdot \iota\alpha_{n-1}(s_1, \ldots, s_{2n-2})\iota\alpha_1(s_{2n-1}, s_{2n}).\end{aligned}$$

Parts (b), (c), (d), and (f) follow immediately from the definition of α_n using induction on n. And Part (e) is an easy consequence of Equations (29) and (30). □

10.10 Lemma *Assume Hypothesis 10.1 on A. Let G be a finite group, let $K \leqslant Z(G)$ be a π-group, let $\kappa \in K^*$ be faithful and set $\tilde{K} := K \cap G'$. Then there exists a short exact sequence of groups*

$$1 \longrightarrow \tilde{K} \xrightarrow{\ \tilde{\iota}\ } \tilde{G} \xrightarrow{\ \tilde{\pi}\ } G/K \longrightarrow 1 \tag{32}$$

with $\tilde{\iota}(\tilde{K}) \leqslant Z(\tilde{G}) \cap \tilde{G}'$ and an element $(M, \mu) \in \mathcal{M}_{G \times \tilde{G}}$ such that, after identifying $\tilde{K}$ and $\tilde{\iota}(\tilde{K})$ via $\tilde{\iota}$, one has

$$p_1(M) = G\,,\ k_1(M) = K\,,\ \mu_1 = \kappa\,, \quad \text{and} \quad p_2(M) = \tilde{G}\,,\ k_2(M) = \tilde{K}\,,\ \mu_2 = \kappa|_{\tilde{K}}\,.$$

In particular, with $\tilde{\kappa} := \kappa|_{\tilde{K}}$, one has $(\tilde{K}, \tilde{\kappa}) \in \mathcal{R}_{\tilde{G}}$, by Proposition 8.6(b).

Proof Consider the short exact sequence

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} G/K \to 1$$

with ι the inclusion and π the canonical surjection. Let $\sigma\colon G/K \to G$ be a section of π and let $\alpha \in Z^2(G/K, K)$ denote the cocycle satisfying Equation (28) for all $s_1, s_2 \in G/K$. We consider the commutative diagram from Remark 10.6. By Proposition 10.7(b) we have $\epsilon_4(\Psi_1(\bar{\alpha})) = 1$. As the right hand column and bottom row of the diagram are exact, there exists $\beta \in Z^2(G/K, \tilde{K})$ such that $\epsilon_3(\Psi_2([\beta])) = \Psi_1([\alpha])$. It follows that $\epsilon_2([\beta])^{-1} \cdot [\alpha] \in \ker(\Psi_1)$ and, by the exactness of the top row, there exists a symmetric $\gamma \in Z^2(G/KG', K)$ such that $\iota_1([\gamma]) \cdot \epsilon_2([\beta]) = [\alpha]$. After changing β and γ by a coboundary, we may assume that $\beta(1,1) = 1$ and that $\gamma(1,1) = 1$. After changing the section σ, we may also assume that

$$\alpha(s_1, s_2) = \gamma(\bar{s}_1, \bar{s}_2)\beta(s_1, s_2)$$

for all $s_1, s_2 \in G/K$, where $\bar{s}$ denotes the image of an element $s \in G/K$ under the natural epimorphism $G/K \to G/KG'$. It follows that $\alpha(1,1) = 1$ and that $\sigma(1) = 1$. By Proposition 10.9(e), we have

$$\gamma_n(\bar{s}_1, \ldots, \bar{s}_{2n}) = 1 \tag{33}$$

and by parts (b) and (c) of the same proposition, we obtain

$$\alpha_n(s_1, \ldots, s_{2n}) = \beta_n(s_1, \ldots, s_{2n}) \tag{34}$$

for all $s_1, \ldots, s_{2n} \in G/K$ and all $n \in \mathbb{N}$.

Next, we define the group $\tilde{G}$, using the cocycle β, as the set $\tilde{K} \times G/K$ with multiplication

$$(\tilde{k}_1, s_1)(\tilde{k}_2, s_2) := (\beta(s_1, s_2)\tilde{k}_1\tilde{k}_2, s_1 s_2)$$

for $\tilde{k}_1, \tilde{k}_2 \in \tilde{K}$ and $s_1, s_2 \in G/K$. We obtain a short exact sequence as in (32) with $\tilde{\iota}(\tilde{k}) := (\tilde{k}, 1)$ and $\tilde{\pi}(\tilde{k}, s) := s$, for $\tilde{k} \in \tilde{K}$ and $s \in G/K$. With the section

$$\tilde{\sigma}\colon G/K \to \tilde{G}, \quad s \mapsto (1, s),$$

of $\tilde{\pi}$ we obtain

$$\tilde{\sigma}(s_1)\tilde{\sigma}(s_2) = \tilde{\iota}(\beta(s_1, s_2))\tilde{\sigma}(s_1 s_2) \tag{35}$$

for all $s_1, s_2 \in G/K$. Note that since $\beta(1,1) = 1$, also $\beta(s,1) = \beta(1,s) = 1$ for all $s \in S$. This implies that $(1,1)$ is the identity of $\tilde{G}$ and that $\tilde{\iota}(\tilde{K}) \leqslant Z(\tilde{G})$.

Next, we define $M \leqslant G \times \tilde{G}$ by

$$M := \{(g, (\tilde{k}, s)) \in G \times \tilde{G} \mid \pi(g) = s\}.$$

We will identify $\tilde{K}$ with $\tilde{\iota}(\tilde{K})$ and view $\tilde{K}$ as a subgroup of $\tilde{G}$. It is now clear from the definition of M that

$$p_1(M) = G, \quad p_2(M) = \tilde{G}, \quad k_1(M) = K, \quad k_2(M) = \tilde{K}.$$

Next, we want to define $\mu \in M^*$ such that μ extends $\kappa \times 1 \in (K \times 1)^*$. By Proposition 10.4(e), it suffices to show that $(K \times 1) \cap M' = 1$. So let $k \in K$ and assume that $(k, (1,1)) \in M'$. Then we can write $(k,(1,1))$ as a product of n commutators $[(g_i, (\tilde{k}_i, s_i)), (g_i', (\tilde{k}_i', s_i'))]$ of elements in M, $i = 1, \ldots, n$. Note that, since the above elements belong to M, we have $\pi(g_i) = s_i$ and $\pi(g_i') = s_i'$ for $i = 1, \ldots, n$. Note also that, since K is central in G and $\tilde{K}$ is central in $\tilde{G}$, we may replace g_i by $\sigma(\pi(g_i)) = \sigma(s_i)$, g_i' by $\sigma(s_i)$ and we may replace $\tilde{k}_i$ and $\tilde{k}_i'$ by 1. This way, we see that we can write

$$(k,(1,1)) = [(\sigma(s_1), \tilde{\sigma}(s_1)), (\sigma(s_1'), \tilde{\sigma}(s_1'))] \cdots [(\sigma(s_n), \tilde{\sigma}(s_n)), (\sigma(s_n'), \tilde{\sigma}(s_n'))]$$

for certain element $s_1, s_1', \ldots, s_n, s_n' \in G/K$. It follows from Proposition 10.9 and Equation (34) that

$$\begin{aligned} &[(\sigma(s_1), \tilde{\sigma}(s_1)), (\sigma(s_1'), \tilde{\sigma}(s_1'))] \cdots [(\sigma(s_n), \tilde{\sigma}(s_n)), (\sigma(s_n'), \tilde{\sigma}(s_n'))] \qquad (36)\\ =&\Big(\alpha_n(s_1, \ldots, s_n')\sigma([s_1, s_1'] \cdots [s_n, s_n']), \big(\beta_n(s_1, \ldots, s_n'), \tilde{\sigma}([s_1, s_1'] \ldots [s_n, s_n'])\big)\Big)\\ =&\Big(\beta_n(s_1, \ldots, s_n')\sigma([s_1, s_1'] \cdots [s_n, s_n']), \big(\beta_n(s_1, \ldots, s_n'), \tilde{\sigma}([s_1, s_1'] \ldots [s_n, s_n'])\big)\Big). \end{aligned}$$

This implies that $\tilde{\sigma}([s_1, s_1'] \ldots [s_n, s_n']) = 1$ and that $\beta_n(s_1, \ldots, s_n') = 1$. Moreover, the first of the last two equation implies that $[s_1, s_1'] \cdots [s_n, s_n'] = 1$ and that $\sigma([s_1, s_s'] \cdots [s_n, s_n']) = \sigma(1) = 1$. Altogether, we obtain that $k = 1$.

Now we know that there exists $\mu \in M^*$ with $\mu|_{K \times 1} = \kappa \times 1$ and we choose any such μ. Then clearly $\mu_1 = \kappa$.

Next, we show that

$$\Delta(\tilde{K}) = \{(\tilde{k}, (\tilde{k}, 1)) \mid \tilde{k} \in \tilde{K}\} \leqslant M'.$$

Let $\tilde{k} \in \tilde{K}$. Since $\tilde{K} \in G'$, there exist elements $g_1, g_1', \ldots, g_n, g_n' \in G$ such that $\tilde{k} = [g_1, g_1'] \cdots [g_n, g_n']$. Since K is central in G, we may assume that $g_i = \sigma(s_i)$ and $g_i' = \sigma(s_i')$ for $i = 1, \ldots, n$ and elements $s_1, \ldots, s_n' \in S$. Thus, we have

$$\begin{aligned} \tilde{k} &= [\sigma(s_1), \sigma(s_1')] \cdots [\sigma(s_n), \sigma(s_n')]\\ &= \beta_n(s_1, \ldots, s_n')\sigma([s_1, s_1'] \cdots [s_n, s_n']), \end{aligned}$$

by Proposition 10.9 and Equation (34). This implies that $\sigma([s_1, s_1'] \cdots [s_n, s_n']) \in \tilde{K}$ and, consequently, that $[s_1, s_1'] \cdots [s_n, s_n'] = 1$. Thus $\sigma([s_1, s_1'] \cdots [s_n, s_n']) = 1$ and $\tilde{k} = \beta_n(s_1, \ldots, s_n')$. With Equation (36), we see now that

$$(\tilde{k}, (\tilde{k}, 1)) = [(\sigma(s_1), \tilde{\sigma}(s_1)), (\sigma(s_1'), \tilde{\sigma}(s_1'))] \cdots [(\sigma(s_n), \tilde{\sigma}(s_n)), (\sigma(s_n'), \tilde{\sigma}(s_n'))],$$

since $\tilde{\sigma}(1) = 1$. Therefore $\Delta(\tilde{K}) \leqslant M'$.

Since $\Delta(\tilde{K}) \leqslant M'$, we have

$$1 = \mu(\tilde{k}, (\tilde{k}, 1)) = \mu_1(\tilde{k})\mu_2^{-1}(\tilde{k})$$

for all $\tilde{k} \in \tilde{K}$. This implies that $\mu_2 = \mu_1|_{\tilde{K}} = \kappa|_{\tilde{K}}$. Finally, since $\Delta(\tilde{K}) \leqslant M'$, we have $(\tilde{k}, (\tilde{k}, 1)) \in (G \times \tilde{G})' = G' \times \tilde{G}'$ and therefore $(\tilde{k}, 1) \in \tilde{G}'$ for all $\tilde{k} \in \tilde{K}$. Thus $\tilde{K} \leqslant \tilde{G}'$. □

10.11 Definition Let G, K, κ, $\tilde{G}$, and (M,μ) be as in Lemma 10.10. By abuse of notation, we define the A-fibered $(G,\tilde{G})$-biset

$$\mathrm{Ins}_{\tilde{G}}^{G} := \left(\frac{G\times\tilde{G}}{M,\mu}\right) \,,$$

the *insertion* from $\tilde{G}$ to G (which inserts the section $K/\tilde{K}$ into $\tilde{G}$), and we define the A-fibered $(\tilde{G},G)$-biset

$$\mathrm{Del}_{\tilde{G}}^{G} := (\mathrm{Ins}_{\tilde{G}}^{G})^{\mathrm{op}} \,,$$

the *deletion* from G to $\tilde{G}$ (which deletes the section $K/\tilde{K}$ from G).

Note that (G,K,κ) does not determine $\tilde{G}$ and (M,μ) uniquely, so that $\mathrm{Ins}_{\tilde{G}}^{G}$ and $\mathrm{Del}_{\tilde{G}}^{G}$ are not well-defined elements. But all that matters for our purposes is the existence of $\tilde{G}$ and (M,μ) with the properties from Lemma 10.10, for given (G,K,κ).

Next we decompose the biset X from Proposition 2.8 further. Recall the properties of X mentioned in the paragraph following Proposition 2.8.

Recall from Definition 8.10 that, for finite groups G and H, we call a pair $(U,\phi)\in\mathcal{M}_{G\times H}^{c}$ *reduced* if $l_0(U,\phi)\in\mathcal{R}_G$ and $r_0(U,\phi)\in\mathcal{R}_H$.

10.12 Proposition *Let G and H be finite groups, let $(U,\phi)\in\mathcal{M}_{G\times H}$, and set $X := \left(\frac{G\times H}{U,\phi}\right)$. Assume that $p_1(U)=G$, $p_2(U)=H$, set $K:=k_1(U)$, $L:=k_2(U)$, and assume that $\kappa:=\phi_1\in K^*$ and $\lambda:=\phi_2\in L^*$ are faithful. Then there exists a decomposition*

$$X \cong \mathrm{Ins}_{\tilde{G}}^{G}\otimes_{A\tilde{G}} Y\otimes_{A\tilde{H}}\mathrm{Del}_{\tilde{H}}^{H}$$

with the following properties:

(a) *The group $\tilde{G}$ is a central extension of G/K by $\tilde{K}:=K\cap G'$ and the group $\tilde{H}$ is a central extension of H/L by $\tilde{L}:=L\cap H'$.*

(b) *One has an isomorphism $Y\cong\left(\frac{\tilde{G}\times\tilde{H}}{\tilde{U},\tilde{\phi}}\right)$ for a reduced pair $(\tilde{U},\tilde{\phi})\in\mathcal{M}_{\tilde{G}\times\tilde{H}}$ satisfying*

$$k_1(\tilde{U})=\tilde{K}\,,\ k_2(\tilde{U})=\tilde{L}\,,\ \tilde{\phi}_1=\kappa|_{\tilde{K}}\,,\ \tilde{\phi}_2=\lambda|_{\tilde{L}}\,.$$

Proof We define $Y:=\mathrm{Del}_{\tilde{G}}^{G}\otimes_{AG}X\otimes_{AH}\mathrm{Ins}_{\tilde{H}}^{G}$. All the statements of the proposition follow from Lemma 10.10, Proposition 4.2(b) and (c) and from the associativity of the tensor product. □

Assuming Hypothesis 10.1, we have now a simple criterion for a pair to be reduced, thus simplifying the classification of simple fibered biset functors.

10.13 Corollary *Assume that A satisfies Hypothesis 10.1, let G be a finite group, and let $(K,\kappa)\in\mathcal{M}_G^G$. Then (K,κ) is reduced in G if and only if K is a cyclic π-subgroup contained in $Z(G)\cap G'$ and κ is faithful.*

Proof It was already proved in Proposition 8.6(b) that the condition is sufficient for (K,κ) to be reduced in G. Conversely, assume that (K,κ) is reduced in G. By Proposition 8.6(a), we know that κ is faithful and $K \leqslant Z(G)$. This implies that K is a cyclic π-group, by the hypothesis on A. Assume that K is not contained in G'. Applying Proposition 10.12 to the the pair $(\Delta_K(G),\phi_\kappa)$ shows that $e_{(K,\kappa)} \in I_G$, since $|\tilde{G}| < |G|$. □

Combining Propositions 2.8 and 10.12, we obtain the following refined decomposition of an arbitrary transitive fibered biset under Hypothesis 10.1.

10.14 Theorem *Assume that A satisfies Hypothesis 10.1. Let G and H be finite groups, let $(U,\phi) \in \mathcal{M}_{G\times H}$, and let*

$$\hat{K} \leqslant \tilde{K} \leqslant K \leqslant P \leqslant G\,,\ \hat{L} \leqslant L \tilde{\leqslant} L \leqslant Q \leqslant H\,,\ \kappa := \phi_1 \in K^*\,,\ \text{and } \lambda := \phi_2 \in L^*$$

be defined as in Proposition 1.3.

There exists a decomposition

$$\left(\frac{G\times H}{U,\phi}\right) \cong \mathrm{Ind}_P^G \otimes \mathrm{Inf}_{P/\hat{K}}^P \otimes \mathrm{Ins}_{\tilde{G}}^{P/\hat{K}} \otimes X \otimes \mathrm{Del}_{\tilde{H}}^{Q/\hat{L}} \otimes \mathrm{Def}_{Q/\hat{L}}^Q \otimes \mathrm{Res}_Q^H\,,$$

where $X \cong \left(\frac{\tilde{G}\times\tilde{H}}{\tilde{U},\tilde{\phi}}\right)$ is a transitive A-fibered $(\tilde{G},\tilde{H})$-biset such that $(\tilde{U},\tilde{\phi}) \in \mathcal{M}^c_{\tilde{G}\times\tilde{H}}$ is reduced, where $\tilde{G}$ is a central extension of P/K by $\tilde{K}/\hat{K}$ and $\tilde{H}$ is a central extension of Q/L by $\tilde{L}/\hat{L}$, and $\tilde{\phi}_1 = \bar{\kappa}|_{\tilde{K}/\hat{K}}$ and $\tilde{\phi}_2 = \bar{\lambda}|_{\tilde{L}/\hat{L}}$. Here, $\bar{\kappa} \in (K/\hat{K})^*$ and $\bar{\lambda} \in (L/\hat{L})^*$ are defined as the unique homomorphisms inflating to κ and λ, respectively.

11 Examples

In this section, we look at several examples of fibered biset functors. We give a criterion for a fibered biset functor to be simple and we apply it to some functors. As before, A is a multiplicatively written abelian group and k denotes a commutative ring.

More precisely, we will identify the simple functors $S_{(\{1\},\{1\},1,k)}$ for some choices of A and k as well-known objects in the representation theory of finite groups. Note that if G is the trivial group then $(K,\kappa) = (\{1\},1)$ is the only reduced pair and $\Gamma_{(\{1\},\{1\},1)}$ is again the trivial group, so that k is naturally a $k\Gamma_{(\{1,\},\{1\},1)}$-module.

For any functor $F \in \mathcal{F}_k^A$ and any class $\mathcal{H} \subseteq \mathrm{Ob}(\mathcal{C})$ of finite groups we define subfunctors $\mathcal{I}_{F,\mathcal{H}}$ and $\mathcal{K}_{F,\mathcal{H}}$ of F as follows. For a finite group G we set

$$\mathcal{I}_{F,\mathcal{H}}(G) := \sum_{\substack{H\in\mathcal{H}\\ x\in B_k^A(G,H)}} \mathrm{im}\Big(F(x)\colon F(H) \to F(G)\Big) \subseteq F(G)\,,$$

$$\mathcal{K}_{F,\mathcal{H}}(G) := \bigcap_{\substack{H\in\mathcal{H}\\ y\in B_k^A(H,G)}} \ker\Big(F(y)\colon F(G) \to F(H)\Big) \subseteq F(G)\,.$$

It is clear that $\mathcal{I}_{F,\mathcal{H}}$ and $\mathcal{K}_{F,\mathcal{H}}$ are subfunctors of F.

The following proposition gives a criterion for F to be simple. It holds for more general functor categories with the same proof. See also [TW, Theorem 3.1] for a similar result for Mackey functors.

11.1 Proposition *Let $F \in \mathcal{F}_k^A$ be an A-fibred biset functor over k and let H be a finite group such that $F(H) \neq \{0\}$. Then, F is a simple functor if and only if the following three conditions are satisfied:*

(i) *$F(H)$ is a simple E_H-module.*

(ii) $\mathcal{I}_{F,\{H\}} = F$.

(iii) $\mathcal{K}_{F,\{H\}} = 0$.

Proof We first assume that F is a simple functor. Then $F(H)$ is a simple E_H-module by Proposition 3.7(a). Further note that $\mathcal{I}_{F,\{H\}}(H) = F(H)$ and $\mathcal{K}_{F,\{H\}}(H) = \{0\}$ (use $x = y = \mathrm{id}_H$ in the definition of $\mathcal{I}_{F,\{H\}}(H)$ and $\mathcal{K}_{F,\{H\}}(H)$). Thus $\mathcal{I}_{F,\{H\}}$ is a non-zero subfunctor of F and $\mathcal{K}_{F,\{H\}}$ is a proper subfunctor of F. Since F is simple, the properties (ii) and (iii) follow.

Next assume that the conditions (i)–(iii) are satisfied and let $L \subseteq F$ be a subfunctor of F. Then, $L(H)$ is an E_H-submodule of $F(H)$. Since the latter is simple, by (i), we have $L(H) = \{0\}$ or $L(H) = F(H)$. First assume that $L(H) = \{0\}$. We'll show that this implies that $L = 0$. In fact, for every finite group G we then have

$$\begin{aligned}
\{0\} = \mathcal{K}_{F,\{H\}}(G) &= \bigcap_{y \in B_k^A(H,G)} \ker\Big(F(y)\colon F(G) \to F(H)\Big) \\
&\supseteq L(G) \cap \bigcap_{y \in B_k^A(H,G)} \ker\Big(F(y)\colon F(G) \to F(H)\Big) \\
&= \bigcap_{\substack{H \in \mathcal{H} \\ y \in B_k^A(H,G)}} \ker\Big(F(y)\colon L(G) \to F(H)\Big) = L(G)\,,
\end{aligned}$$

since $F(y)(L(G)) \subseteq L(H) = \{0\}$ for all $y \in B_k^A(H,G)$. Finally, we assume that $L(H) = F(H)$ and will show that this implies $L = F$. In fact, for every finite group G we then have

$$\begin{aligned}
F(G) = \mathcal{I}_{F,\{H\}}(G) &= \sum_{x \in B_k^A(G,H)} \Big(F(x)\colon F(H) \to F(G)\Big)) \\
&= \sum_{x \in B_k^A(G,H)} \mathrm{im}\Big(F(x)\colon L(H) \to F(G)\Big) \subseteq L(G)\,,
\end{aligned}$$

since $F(x)(L(H)) \subseteq L(G)$, for all $x \in B_k^A(G,H)$. □

11A. THE FUNCTOR B_k^A

Canonically identifying a finite group G with $G \times \{1\}$, we have

$$B_k^A(G) = B_k^A(G \times \{1\}) = \mathrm{Hom}_{\mathcal{C}_k^A}(\{1\}, G)\,.$$

Thus, we can view B_k^A as the Yoneda functor $\mathrm{Hom}_{\mathcal{C}_k^A}(\{1\}, -)\colon \mathcal{C}_k^A \to {}_k\mathsf{Mod}$. For any A-fibered (G, H)-biset X, the map $B_k^A([X])\colon B_k^A(H) \to B_k^A(G)$ is induced by composition with $[X]$ on the left. The functor B_k^A is called the *A-fibered Burnside functor* over k. Note also that B_k^A is isomorphic in $\mathcal{F}_k^A$ to the functor $L_{\{1\},k}$, where $\{1\}$ is the trivial group and k is the regular $\bar{E}_{\{1\}}$-module, identifying $\bar{E}_{\{1\}} \cong E_{\{1\}} \cong k$ as k-algebras, cf. 3.4 and 3.6.

Given a transitive A-fibered $(G, \{1\})$-biset $\left(\frac{G\times\{1\}}{U,\phi}\right)$, we simply write $[U, \phi]_G$ for its image in the $B_k^A(G)$. With this notation, the k-module $B_k^A(G)$ is freely generated by the elements $[U, \phi]_G$ as (U, ϕ) runs through a set of representatives of the G-orbits of $\mathcal{M}_G$.

11.2 Theorem *Assume that k is a field. The functor B_k^A in $\mathcal{F}_k^A$ is a projective and indecomposable object. It is a projective cover of the simple functor $S_{(\{1\},\{1\},1,k)}$.*

Proof The first statement is an easy consequence of the Yoneda-Lemma. The second statement follows from the first, the observation that the functors $L_{\{1\},k}$ and B_k^A are isomorphic, and the fact that $S_{\{1\},k} = S_{(\{1\},\{1\},1,k)}$ is the only simple factor of the functor $L_{\{1\},k}$. □

In the following subsections we identify the simple functor $S_{(\{1\},\{1\},1,k)}$ for various choices of A and k.

11B. THE FUNCTOR $kR_{\mathbb{C}}$

Let $A = \mathbb{C}^\times$. For a finite group G, we denote by $R_{\mathbb{C}}(G)$ the character ring of $\mathbb{C}[G]$-modules. Recall that there is a linearization map

$$\mathrm{lin}_G\colon B^{\mathbb{C}^\times}(G) \to R_{\mathbb{C}}(G), \quad [H, \phi]_G \mapsto \mathrm{ind}_H^G(\phi)\,,$$

and recall that it is surjective, by Brauer's induction theorem. It follows from [Bou10c, Theorem 1.1] and the explicit formula in Corollary 2.5 that

$$\mathrm{lin}_{G\times H}([X]) \otimes_{\mathbb{C}H} \mathrm{lin}_{H\times K}([Y]) = \mathrm{lin}_{G\times K}([X] \underset{H}{\cdot} [Y])$$

in $R_{\mathbb{C}}(G \times K)$. This implies first that $G \mapsto R_{\mathbb{C}}(G)$ gives rise to a $\mathbb{C}^\times$-fibered biset functor $R_{\mathbb{C}^\times}$ by mapping the standard basis element $\left[\frac{G\times H}{U,\phi}\right]$ of $B^{\mathbb{C}^\times}(G, H)$ to the map $R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G)$, $[M] \mapsto [\mathrm{Ind}_U^{G\times H}(\mathbb{C}_\phi) \otimes_{\mathbb{C}H} M]$; and secondly that the maps lin_G form a morphism of fibered biset functors. Clearly, one can extend scalars from $\mathbb{Z}$ to any commutative ring k and $\mathrm{lin}\colon kB^{\mathbb{C}^\times} \to kR_{\mathbb{C}}$ becomes a morphism in $\mathcal{F}_k^{\mathbb{C}^\times}$.

11.3 Theorem *Let k be an arbitrary field. The $\mathbb{C}^\times$-fibered biset functor $kR_\mathbb{C}$ over k is isomorphic to the simple functor $S_{(\{1\},\{1\},1,k)}$. Moreover,* $\mathrm{lin}\colon kB^{\mathbb{C}^\times} \to kR_\mathbb{C}$ *is a projective cover in the category $\mathcal{F}_k^{\mathbb{C}^\times}$.*

Proof It suffices to show that $kR_\mathbb{C} \in \mathcal{F}_k^{\mathbb{C}^\times}$ is a simple functor. Since $\mathrm{lin}\colon kB^{\mathbb{C}^\times} \to kR_\mathbb{C}$ is surjective, Theorem 11.2 then implies the rest. To see that $kR_\mathbb{C}$ is simple we apply the criterion from Proposition 11.1 with H chosen as the trivial group $\{1\}$. We will show that the three conditions (i)-(iii) hold.

Clearly, $kR_\mathbb{C}(\{1\}) \cong k$ is a simple module for $E_{\{1\}} \cong k$ and the first condition holds. To see that the second condition holds, we need to show that for every irreducible character χ of G there exists $x \in B_k^{\mathbb{C}^\times}(G,\{1\}) = B_k^{\mathbb{C}^\times}(G)$ such that $(kR_\mathbb{C}(x))(1) = \chi$, where $1 \in k = kR_\mathbb{C}(\{1\})$. But, with the above identification, $(kR_\mathbb{C}(x))(1) = \mathrm{lin}_G(x)$. And since lin_G is surjective, the second condition holds. Finally we show that the third condition holds, namely that $\mathcal{K}_{kR_\mathbb{C},\{1\}}(G) = 0$ for any finite group G. In other words, we need to show that for any non-zero k-linear combination f of simple $\mathbb{C}G$-modules, there exists $x \in B_k^{\mathbb{C}^\times}(\{1\},G) = B_k^{\mathbb{C}^\times}(G)$ such that $xf = (R_\mathbb{C}(x))(f) \neq 0 \in kR_\mathbb{C}(\{1\}) = k$. However, for $x \in B_k^{\mathbb{C}^\times}(G)$ and any $\mathbb{C}G$-module V, with the above identifications, we have $x[V] = \mathrm{lin}_G(x) \underset{G}{\cdot} [V] \in R_\mathbb{C}(\{1\})$, where $\underset{G}{\cdot}\colon R_\mathbb{C}(G) \times R_\mathbb{C}(G) \to k$ is induced by $(W,V) \mapsto \dim_\mathbb{C}(\widetilde{W} \otimes_{\mathbb{C}G} V)$, for left $\mathbb{C}G$-modules V and W, where $\widetilde{W}$ is the right $\mathbb{C}G$-module with W as underlying space and G-action $wg := g^{-1}w$ for $w \in W$ and $g \in G$. Thus, choosing $x \in B_k^{\mathbb{C}^\times}(G)$ such that $\mathrm{lin}_G(x)$ is the dual of an irreducible constituent occurring in f with non-zero coefficient $\alpha \in k$, we obtain $xf = \alpha \neq 0$. □

11C. The Functor $kR_\mathbb{Q}$

Let $A = \mathbb{Q}^\times$ and let $R_\mathbb{Q}(G)$ denote the character ring of finitely generated $\mathbb{Q}G$-modules, where G is a finite group. Since $\mathbb{Q}^\times \subset \mathbb{C}^\times$, the $\mathbb{C}^\times$-fibered biset functor $R_\mathbb{C}$ restricts to a $\mathbb{Q}^\times$-fibered biset functor. Considering $R_\mathbb{Q}(G)$ as subgroup of $R_\mathbb{C}(G)$, we see that $R_\mathbb{Q}$ becomes a subfunctor of $R_\mathbb{C}$ as $\mathbb{Q}^\times$-biset functor and thus is a $\mathbb{Q}^\times$-fibered biset functo in its on right.

On the other hand, any $\mathbb{Q}^\times$-fibered biset functor restricts to an ordinary biset functor. Further, recall that, by [Bou10b, Proposition 4.4.8], for a field k of characteristic zero, the functor $kR_\mathbb{Q} = k \otimes R_\mathbb{Q}$ is simple as a biset functor. Therefore, the $\mathbb{Q}^\times$-fibered biset functor $kR_\mathbb{Q}$ is also simple. Since $kR_\mathbb{Q}(\{1\}) \neq \{0\}$, the trivial group $\{1\}$ is a minimal group for this functor and the classification of simple fibered biset functors from Section 9 implies that it is parameterized by the quadruple $(\{1\},\{1\},1,k)$. The map $\mathrm{lin}_G\colon B_k^{\mathbb{C}^\times}(G) \to kR_{\mathbb{C}^\times}(G)$ from 11B restricts to the map

$$\mathrm{lin}_G\colon B_k^{\mathbb{Q}^\times}(G) \to R_\mathbb{Q}(G)\,, \quad [H,\phi]_G \mapsto \mathrm{ind}_H^G(\phi)\,,$$

and therefore defines a morphism of $\mathbb{Q}^\times$-fibered biset functors. It must be surjective, since $kR_{\mathbb{Q}^\times}$ is a simple functor in $\mathcal{F}_k^{\mathbb{Q}^\times}$. Thus, using Theorem 11.2, we now have the following result.

11.4 Theorem *Let k be a field of characteristic zero. Then there is an isomorphism*

$$S_{(\{1\},\{1\},1,k)} \cong kR_{\mathbb{Q}}$$

of $\mathbb{Q}^\times$-fibered biset functors. Moreover, $\mathrm{lin}\colon kB^{\mathbb{Q}^\times} \to kR_{\mathbb{Q}}$ *is a projective cover in the category $\mathcal{F}_k^{\mathbb{Q}^\times}$.*

11D. The Functor of Trivial Source Modules

Let F be an algebraically closed field of characteristic $p > 0$. For any finite group G, we denote by $T(G)$ the ring of trivial source $F[G]$-modules, also known as p-permutation FG-modules, see [Be95, Section 5.5], [Br85], or [T95, §27]. Again we have a linearization map

$$\mathrm{lin}_G\colon B^{F^\times}(G) \to T(G)\,, \quad [H,\phi]_G \mapsto [\mathrm{Ind}_H^G(F_\phi)]\,.$$

With the same arguments as at the beginning of Subsection 11B one obtains that T is an $F^\times$-fibered biset functor and that the maps lin_G form a morphism of $F^\times$-fibered biset functors. Note that lin_G is surjective (see [D75], [Bol98b], or [BK05]). Now let k be a field. Then the functor $kT := k \otimes T$ is a quotient of the functor $B_k^{F^\times}$. Since B_k^A has a unique simple quotient isomorphic to $S_{(\{1\},\{1\},1,k)}$ (see Theorem 11.2), also kT has a unique quotient functor, isomorphic to $S_{(\{1\},\{1\},1,k)}$.

We will now use Proposition 11.1 applied to $H := \{1\}$ in order to show that kT is not a simple functor. More precisely we show that $\mathcal{K}_{kT,\{1\}} \neq 0$. In fact, let $G = C_p \times C_p$ be the elementary abelian p-group of rank 2 and let $H_1, \ldots, H_{p+1}$ denote its subgroups of order p. Then, the map $\mathrm{lin}_G\colon B^{F^\times}(G) = B(G) \to T(G)$, $[G/P] \mapsto \mathrm{Ind}_P^G(F)$, is an isomorphism, since G is a p-group. Here $B(G)$ denotes the Burnside ring of G. Moreover, identifying $B^{F^\times}(\{1\}, G) = B(\{1\}, G)$ with $B(G)$, the map $kT([G/P])\colon kT(G) \to kT(\{1\}) = k$ is given by $[G/Q] \mapsto |P\backslash G/Q| \cdot 1_k$, for $Q \leqslant G$. Now it is easy to see that the element $p[G/G] - ([G/H_1] + \cdots + [G/H_{p+1}]) + [G/\{1\}] \in kB(G) \cong kT(G)$ is a non-zero element in $\mathcal{K}_{kT,\{1\}}(G)$. Thus, we have the following result.

11.5 Proposition *Let k be a field. Then the $F^\times$-fibered biset functor kT is not simple. It has a unique quotient functor and this quotient is isomorphic to $S_{(\{1\},\{1\},1,k)}$*

References

[Ba04] L. Barker: Fibred permutation sets and the idempotents and units of monomial Burnside rings. *J. Alg.* **281** (2004), 535–566.

[Be95] D. J. Benson: Representations and cohomology. I. Basic representation theory of finite groups and associative algebras. Second edition. Cambridge Studies in Advanced Mathematics, 30. Cambridge University Press, Cambridge, 1998.

[Bol89] R. Boltje: A canonical Brauer induction formula. *Astérisque* **181–182** (1990), 31–59.

[Bol98a] R. Boltje: A general theory of canonical induction formulae. *J. Alg.* **206** (1998) 293–343.

[Bol98b] R. Boltje: Linear source modules and trivial source modules. Group representations: cohomology, group actions and topology (Seattle, WA, 1996), 7–30, Proc. Sympos. Pure Math., 63, Amer. Math. Soc., Providence, RI, 1998.

[Bol01] R. Boltje: Monomial resolutions. *J. Algebra* **246** (2001), 811–848.

[BK05] R. Boltje, B. Külshammer: Explicit and canonical Dress induction. *Algebr. Represent. Theory* **8** (2005), 731–746.

[Bou96] S. Bouc: Foncteurs d'ensembles munis d'une double action. *J. Alg.* **183** (1996), 664–736.

[Bou10a] S. Bouc: The functor of units of Burnside rings for p-groups. *Comment. Math. Helv.* **82** (2007), 583–615.

[Bou10b] S. Bouc: Biset functors for finite groups. Lecture Notes in Mathematics, 1990. Springer-Verlag, Berlin, 2010.

[Bou10c] S. Bouc: Bisets as categories, and tensor product of induced bimodules. *Appl. Categ. Structures* **18** (2010), 517–521.

[BT00] S. Bouc, J. Thévenaz: The group of endo-permutation modules. *Invent. math.* **139** (2000), 275–349.

[BT15] S. Bouc:, J. Thévenaz: The representation theory of finite sets and correspondences. Preprint 2015. arXiv:1510.03034.

[Br85] M. Broué: On Scott modules and p-permutation modules: an approach through the Brauer morphism. *Proc. Amer. Math. Soc.* **93** (1985), 401–408.

[D71] A. Dress: The ring of monomial representations I. Structure theory. *J. Alg.* **18** (1971) 137–157.

[D75] A. Dress: Modules with trivial source, modular monomial representations and a modular version of Brauer's induction theorem. *Abh. Math. Sem. Univ. Hamburg* **44** (1975), 101–109 (1976).

[I76] M. Isaacs: Character theory of finite groups. Pure and Applied Mathematics, No. 69. Academic Press, New York-London, 1976.

[K87] G. Karpilovsky: The Schur multiplier. London Mathematical Society Monographs. New Series, 2. The Clarendon Press, Oxford University Press, New York, 1987.

[R12] N. Romero: Simple modules over Green biset functors. *J. Alg.* **367** (2012), 203–221.

[R13] N. Romero: On fibered biset functors with fibres of order prime and four. *J. Alg.* **387** (2013), 185–194.

[T95] J. Thévenaz: G -algebras and modular representation theory. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.

[T07] J. Thévenaz: Endo-permutation modules, a guided tour. Group representation theory, 115–147, EPFL Press, Lausanne, 2007.

[TW] J. Thévenaz, P. Webb: The structure of Mackey functors. *Trans. Amer. Math. Soc.* **347** (1995), 1865–1961.