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Authors
Garren, A.
Henrieh, L.
Northrop, T.
et al.

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NONADIABATIC EFFECTS IN SINGLE-PARTICLE ORBITS

by

A. Garren, L. Henrich, T. Northrop, R. Riddell, and L. Smith

University of California Radiation Laboratory
Berkeley and Livermore, California

The containment of a charged particle in a mirror field has been studied analytically and numerically. Formulas are presented which give the change in the magnetic moment as it passes from one end of the machine to the other. By following particles for many mirror reflections it is found that although the magnetic moment does change, this change is for most particles oscillatory, rather than random or cumulative. Numerically, these particles appear to be bound indefinitely.

As is well known, a single charged particle could be contained in a static-field mirror machine indefinitely if the adiabatic invariant, $J = |\vec{v} \times \vec{B}|^2/B^3$, were rigorously constant.[1] The containment of a particle for long periods is a crucial requirement for the mirror machines, particularly for those involving high-energy injection, in which case the charge density builds up slowly from a low initial value.

Although certain results have been obtained analytically by Kruskal[2] and Gardner,[3] for the variation of J these are essentially negative, in that a proof has been constructed that J is constant to all orders in an expansion in powers of the radius of gyration. However, J is not expected to be a rigorous invariant, and since its variation is important in determining the time of containment for particles in mirror machines, we have carried out a more detailed investigation, including numerical computations.

In these calculations the vector potential for the (vacuum) magnetic field was chosen to be axially symmetric, time-independent, and of a form in reasonable agreement with that found in the interior of the experimental device "Felix". Specifically, we have

$$A_\theta(r, z) = \frac{LB_0}{2\pi} \{\frac{\rho}{2} - a \cos \zeta \, I_1(\rho)\} ,$$

where L = distance between mirrors, $\rho = 2\pi r/L$, $\zeta = 2\pi z/L$, B_0 is the magnetic field at $\rho = 0$, $\zeta = \pi/2$, and I_1 is the Bessel function of imaginary argument and order one. Further, for the two known rigorous constants of the motion we use the dimensionless parameters

$$V = \frac{2\pi v}{\omega_0 L} \qquad \text{and} \qquad P_\theta = \frac{4\pi^2 p_\theta}{m\omega_0 L^2} ,$$

where v is the velocity, p_θ the canonical angular momentum in conventional units, and $\omega_0 = eB_0/mc$.

Earlier numerical calculations of orbits showed that J generally undergoes a large transient change as the particle traverses the central region, but that as it approaches the opposite "mirror", J approximately returns to its value at the preceding end.[4] ΔJ, the residual change in J between subsequent reflections, is a function of the value of δ and λ as the particle passes through the median plane (see Fig. 1), as well as P_θ and V; we have chosen to systematize the calculations by considering $\Delta J(\lambda)$, keeping δ, P_θ, and V fixed. A plot of $\Delta J/J$ vs. λ is shown in Fig. 2. This curve is rigorously antisymmetric about $\lambda = \pm \pi/2$, owing to the symmetry in A_θ about the median plane, and therefore has an average value of zero. If now the maximum of ln $|\Delta J/J|$ as a function of λ is plotted vs. 1/V, keeping δ and P_θ fixed, we find the curves of Fig. 3. Thus, $|\Delta J/J|_{max}$ is very well approximated by a function of the form[a] $ae^{-b/V}$. This function is consistent with Kruskal's theorem, being asymptotically zero to all powers of V. The dependence on V is very strong indeed; for example, a change in V from 0.5 to 0.2 decreases $(\Delta J/J)_{max}$ by a factor of 340!

In an attempt to understand the results, a variation-of-parameters approach to the problem was carried out. Keeping the lowest-order terms in the radius of curvature, we find

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[a] If the vector potential is modified by adding a harmonic term, $LB_0\beta/4\pi\,[\cos 2\zeta \cdot I_1(2\rho) - \rho]$, this simple form no longer fits the results (see Fig. 3). On the basis of the following analysis this can be qualitatively understood.

$$\frac{dJ}{dt} = -2av_{\parallel}{}^{2}(\vec{e}_B \cdot \nabla \vec{e}_B) \cdot \vec{e}_a \cos \Phi$$

$$- 2a^2 v_{\parallel} \omega \left[\frac{(\vec{e}_B)_r}{r} + \frac{\vec{e}_B \cdot \nabla \vec{B}}{2B} \right] \cos 2\Phi + 0(a^4) .$$

Here, a, the radius of the orbit, and $v_{\parallel}$, the velocity parallel to the field, are considered small and of the same order. $\vec{B}$ is the magnetic field, $\vec{B} = B\,\vec{e}_B$, and $\omega = eB/mc$; $\Phi = \int_0^t \omega dt + \phi(t = 0)$, where ϕ is the phase at $z = 0$. The field quantities are all to be measured at the guiding center of the orbit. This equation can be integrated by use of the unperturbed motion given by the adiabatic approximation. One then finds

$$\frac{\Delta J(z_0)}{J} = -\frac{4 \cos \phi}{\sin \delta} \int_0^{z_0} dz \sqrt{\left(1 - \frac{B}{B_0} \sin^2 \delta\right)} \frac{B_0 B}{B_z{}^2} \frac{\vec{e}_a \cdot \nabla B}{B} \times$$

$$\times \cos \int_0^{z} \frac{\omega dz'}{v} ,$$

where $\vec{e}_a$ = unit vector perpendicular to constant flux surface. We have omitted from the expression the integral of the 2Φ term because it is very small.

One sees that a sinusoidal dependence of $\Delta J/J$ on ϕ is obtained. By replotting Fig. 1 in terms of ϕ rather than λ, one finds a very accurately sinusoidal curve. Moreover, the integral is in reasonably good agreement with the exact results of Fig. 3, and also exhibits the characteristic transients mentioned in Reference 4.

In addition to the above information, however, one would like to know the cumulative effect of many traversals of the machine. For example, one might expect that in multiple passages J would change as in a random walk or in a regular fashion, perhaps merely oscillating so that the particle would be effectively bound. To investigate this question, orbits were computed for which the particle made many reflections.

To discuss the results it is helpful to point out some general properties of the trajectories, which are based on the constancy of V and P_θ and on the symmetries of the field. The motion in the ρ, ζ plane is given by a Hamiltonian from which θ has been eliminated:

$$H = \frac{1}{2}(P_\rho{}^2 + P_\zeta{}^2) + \frac{1}{2}\left(\frac{\psi - P_\theta}{\rho}\right)^2 = \frac{1}{2} V^2 = \text{const.} ,$$

where $P_\rho = \frac{1}{\omega_0}\frac{d\rho}{dt}$, $P_\zeta = \frac{1}{\omega_0}\frac{d\zeta}{dt}$, $\psi(\rho, \zeta) = \frac{1}{B_0}\rho A_\theta(\rho, \zeta)$.

Besides the constants of the motion V and P_θ, one needs three independent variables to describe a trajectory. These may be any three independent combinations of ρ, ζ, P_ρ, P_ζ. In particular we may use ζ and the two spherical polar angles of the velocity vector, δ and $\chi \equiv 3\pi/2 - \lambda$ (see Fig. 1).

For every orbit there are two associated orbits. One is the given orbit mirrored in the median plane. This follows from the symmetry of the field about that plane. The other is the same trajectory in the ρ, ζ plane but with the velocity reversed. This follows from the time and velocity independence of H. From the existence of the two associated orbits, one can show that there is a denumerable infinity of rigorously periodic orbits. The proof of the foregoing statements is too lengthy to be given here.

Now suppose there exists a continuous surface in our three-dimensional phase space which terminates inside the mirror and which has the property that a particle starting on it remains on it. On the basis of our theorems such surfaces, if they exist, have definite properties. If we consider their intersections with the $\zeta = 0$ plane, these should be closed curves such that for every point (δ, χ) on one of them, $(\pi - \delta, \chi)$, $(\delta, -\chi)$ and $(\pi - \delta, -\chi)$ lie on it as well. If we plot such curves on a polar plot in which δ is the radius and χ the azimuth, then it can be shown that if there is a closed curve of an intersecting surface encircling the pole, then all orbits that once intersect the median plane outside this curve are permanently contained.

To determine the possible existence of such invariant surfaces within the machine, we have followed a variety of orbits with digital computers. Most of these were for $V = 0.4$, $P_\theta = 0.1$, which corresponds to orbits whose diameters are about 1/4 that of a typical machine, and which come fairly close to the axis. Each orbit was followed for a number of traversals of the median plane, in one case as many as forty. At each traversal of the median plane we plotted δ and χ, as well as the associated points listed above. This plot is shown in Fig. 4, in which δ or $\pi - \delta$ (whichever is smaller) is plotted radially, and χ azimuthally.

The orbits fall into two main classes, which may tentatively be called stable and unstable. The "stable" orbits intersect the median plane in points that may easily be joined by smooth curves. Each curve is designated by a capital letter, and the points for successive traversals of each orbit are numbered. These smooth curves exhibit the symmetry properties characteristic of the curves in which invariant surfaces intersect the median plane, as discussed in the preceding paragraphs. There are two types of stable orbits, viz., B, C, M; and

L, J, N, P. The data for these orbits seem to be consistent with the hypothesis that they lie on the invariant surfaces defined above, at least as far as we have carried the investigation.

The "unstable" orbits are not shown in Fig. 4--they all lie in the cross-hatched area (note that Fig. 4 does not show any $\delta < 50^{o}$). Successive points on the orbits skip about in a rather erratic way, and in two cases were followed long enough to see the particles escape through the ends. All the "unstable" orbits were inside L, the innermost circumpolar curve. If L really is an invariant surface, then it is rigorously true that all orbits outside it (i.e., with larger δ) are permanently contained. Hence we may tentatively identify curve L as the effective loss cone for V = 0.4, $P_\theta = +0.1$. It has an average δ of about 65^{o}, which may be compared to the value predicted if one assumes the constancy of μ, which is 54^{o}.

Inside L there appear to be stable islands such as B, C. These are the regions surrounding the intersection with the median plane of the simplest kind of periodic orbits, namely those which always traverse the median plane at $\chi = \pi$ and two values of δ: δ_0 and $\pi - \delta_0$. We call these intersections fixed points. For example, the one at the center of the B, C system belongs to an orbit that makes exactly four turns between transits through the median plane. A similar system represented by orbit M surrounds the fixed point belonging to the orbit that makes three turns between transits. An orbit belonging to a fixed point is shown in Fig. 5. In this case there are no fixed points outside curve M, for the number of turns per transit as the orbits approach those lying within the median plane entirely is about 2-1/2 (this is close to the value predicted by assuming adiabatic invariance). The other class of fixed points with $\chi = 0$ seem to be surrounded by unstable orbits, if they are inside curve L.

We have also made similar plots for the intersections of the trajectories at planes for other values of ζ, and again one sees the characteristics of invariant surfaces.

We have studied to some extent the variation with V and P_θ. As might be expected from the strong dependence of $\Delta J/J$ on V discussed previously, the loss cone defined by the curves corresponding to L also varies:

V	P_θ	δ_{min}	$\delta_{adiabatic}$	ζ_{max}
0.2	+ 0.1	$\sim 55^{o}$	54^{o}	$\sim \pi$
0.4	+ 0.1	$\sim 65^{o}$	54^{o}	$\sim \pi/2$
0.6	+ 0.1	$\sim 80^{o}$	54^{o}	$\sim \pi/4$
0.4	+ 0.4	$\sim 65^{o}$	54^{o}	$\sim \pi/2$

If a term $LB_0\beta/4\pi\,(I_1(2\rho)\cos(2\zeta) - \rho)$ is added to A_θ, tending to flatten the field near the median plane, the closed curves like P, J, L, etc. become more circular, but δ_{min} is not appreciably altered. However, ζ_{max} is increased appreciably.

Finally, orbits with P_θ sufficiently negative (these encircle the axis) are rigorously contained. In that case the effective potential for the ρ, ζ motion, $1/2[(\psi - P_\theta)/\rho]^2$, is such that equipotential lines up to the total particle energy are closed and the potential decreases inwards. The present research was inspired in large part by the fact that when the particles do not encircle the axis the equipotential surface is an open ended trough.

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FIGURES

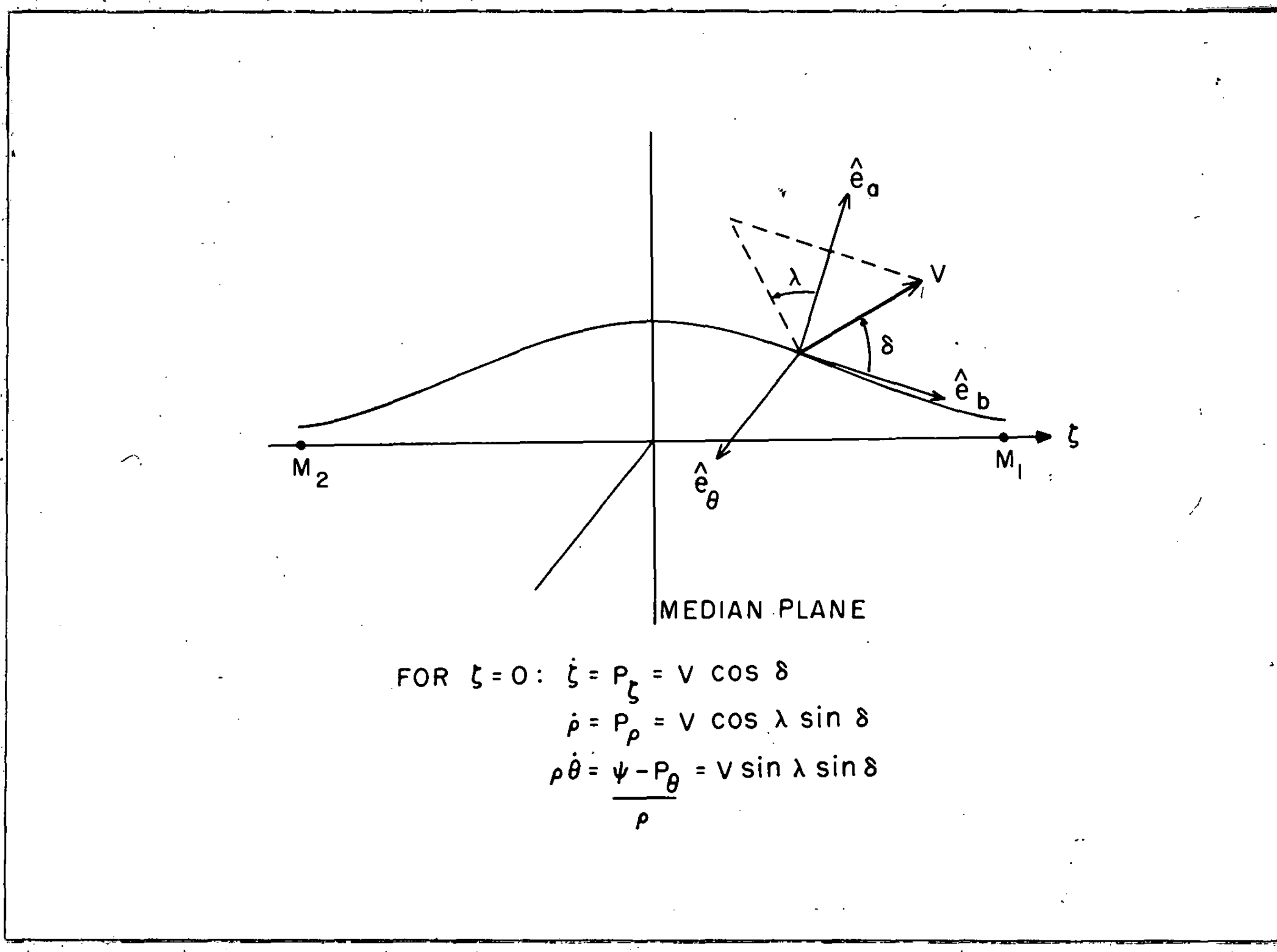
Fig. 1. Definition of angles δ and λ.

Fig. 2. Change in the adiabatic invariant as a function of the phase with which a particle passes through the median plane.

Fig. 3. Change in the adiabatic invariant between reflections as a function of the particle velocity.

Fig. 4. Values of δ and χ with which various orbits intersect the median plane.

Fig. 5. A self-reversing orbit in the r, z plane belonging to a "fixed point."



$\hat{e}_a$
V
λ
δ
$\hat{e}_b$
ζ
M_2
M_1
$\hat{e}_\theta$
MEDIAN PLANE
FOR $\zeta = 0$: $\dot{\zeta} = P_\zeta = V \cos \delta$
$\dot{\rho} = P_\rho = V \cos \lambda \sin \delta$
$\rho\dot{\theta} = \dfrac{\psi - P_\theta}{\rho} = V \sin \lambda \sin \delta$

Fig. 1

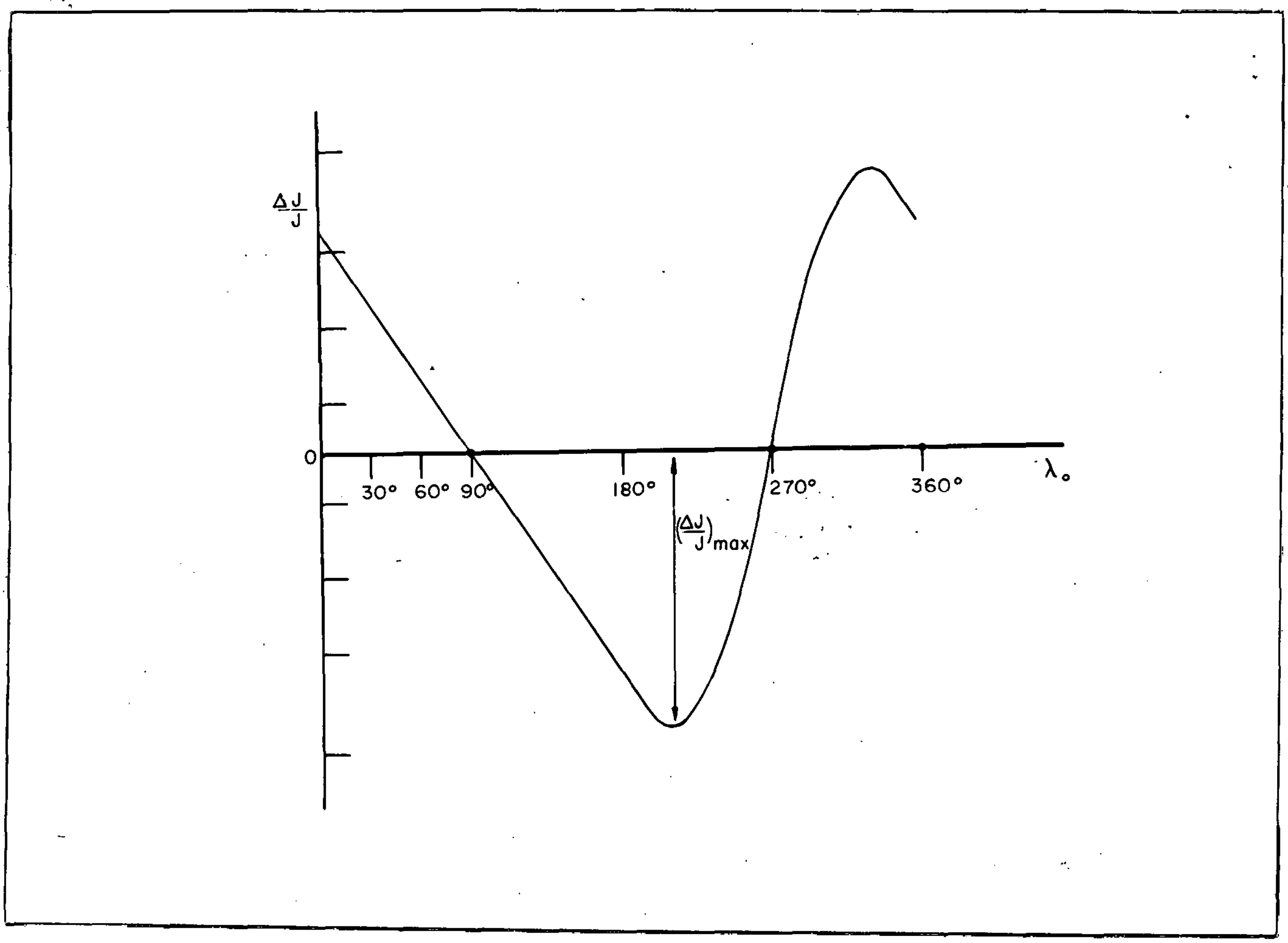

$\frac{\Delta J}{J}$
0
30°
60°
90°
180°
270°
360°
λ_o
$\left(\frac{\Delta J}{J}\right)_{max}$

Fig. 2

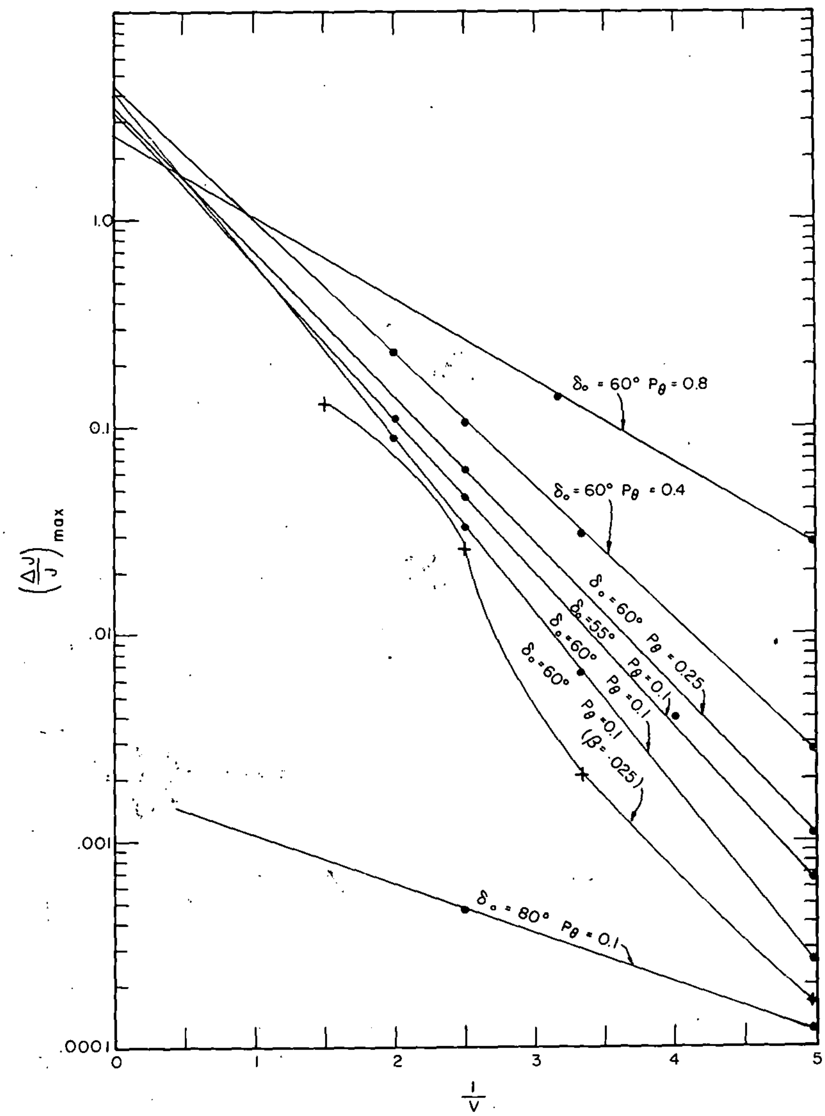

δo = 60° Pθ = 0.8
δo = 60° Pθ = 0.4
δo = 60° Pθ = 0.25
δo = 55° Pθ = 0.1
δo = 60° Pθ = 0.1
δo = 60° Pθ = 0.1 (β = .025)
δo = 80° Pθ = 0.1
(ΔJ/J)max
1/v
1.0
0.1
.01
.001
.0001
0
1
2
3
4
5

Fig. 3

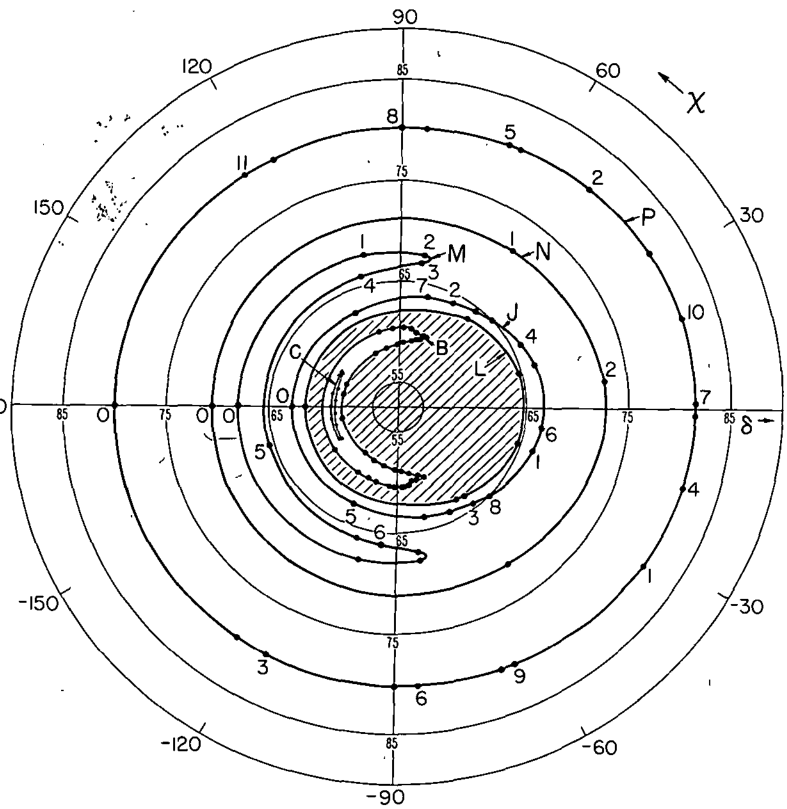


Fig. 4

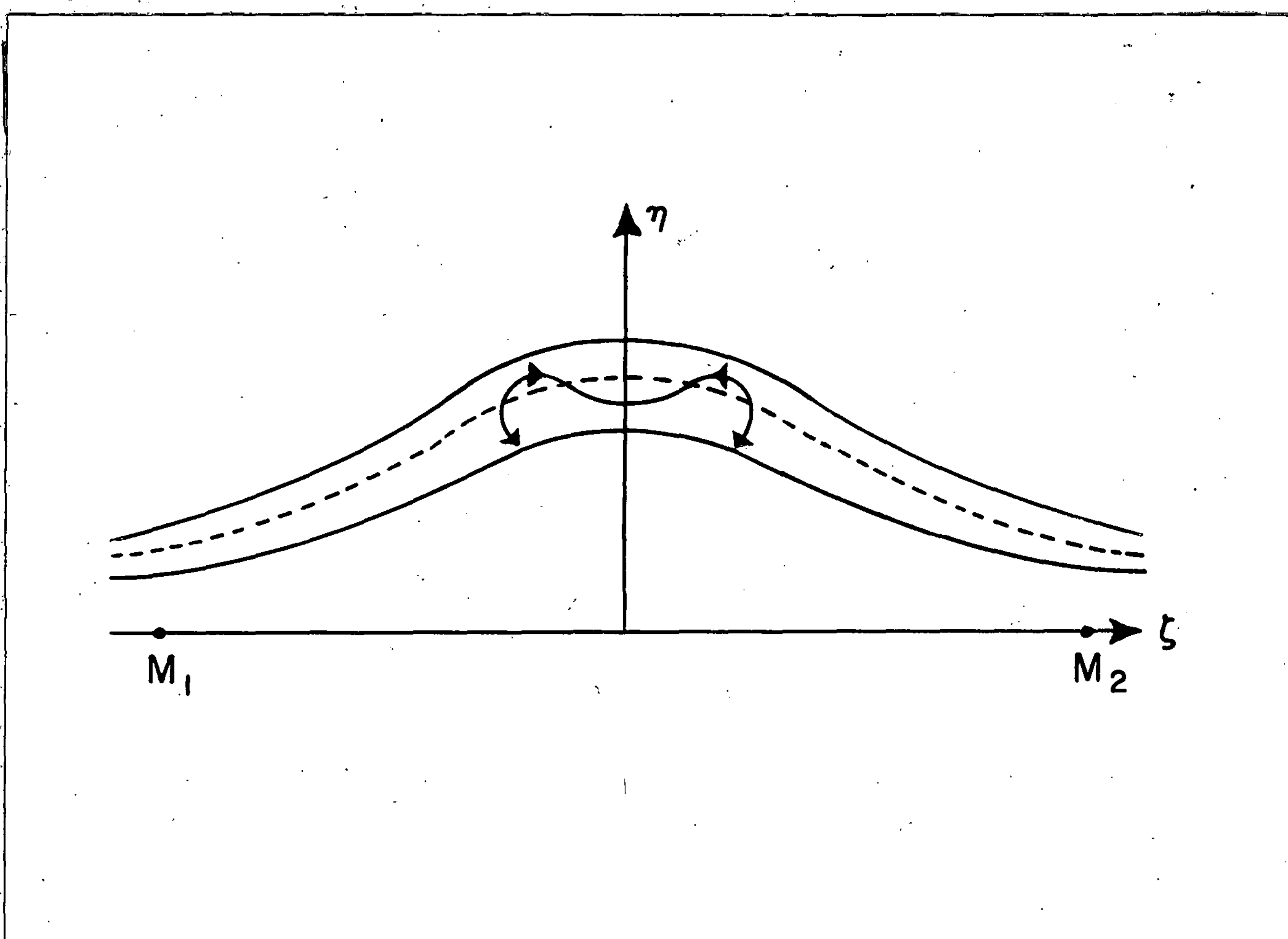


Fig. 5

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