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Liaison of curves and bundles

by

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Abstract

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This thesis is devoted to the study of two central objects in algebraic geometry - algebraic curves and vector bundles - through the unifying lens of liaison theory. The liaison theory of curves originated in the work of M. Noether in the late XIX century, and has since become an instrumental tool in the classification of space curves and the study of Hilbert schemes in general. In Chapter I, we develop a biliaison theory of sheaves and prove several main results in analogy to those from the liaison theory of codimension two subvarieties. In Chapter II, we use the liaison of curves on surfaces with ordinary singularities to prove multiple results about homological invariants of general projections of curves into projective three-space. In Chapter III, we apply biliaison of sheaves to the study of vector bundles. A general program is outlined to describe the moduli of bundles on a projective variety of irregularity zero through a stratification approach. We take the first steps by classifying bundles in the biliaison class of the zero sheaf on projective spaces and describing the moduli. In particular, our results give a description of the moduli of bundles on the projective plane.

To Anca

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Introduction

Algebraic curves and vector bundles are two central objects in algebraic geometry. In this thesis, we generalize a classical tool in the study of space curves called *liaison* to the context of sheaves, and apply it to study vector bundles on projective varieties.

Linkage of curves in $\mathbf{P}^3_{\mathbb{C}}$

Every compact connected Riemann surface embeds as a projective curve in $\mathbf{P}^3_{\mathbb{C}}$. For this reason, the classification of space curves has been an important endeavor since the late XIX century. The technique of *linkage*, or *liaison*, originated from Noether's thesis [80].

Given a smooth curve $C \subset \mathbf{P}^3_{\mathbb{C}}$, Noether considered two hypersurfaces $V(F)$ of degree f and $V(G)$ of degree g containing C, such that $V(F)$ and $V(G)$ do not share any common components. The complete intersection $V(F) \cap V(G)$ contains C, as well as a (possibly empty) residual curve D. We say that the curves C and D are *linked*. For example, let C be the image of the 3-uple embedding $\mathbf{P}^1_{\mathbb{C}} \hookrightarrow \mathbf{P}^3_{\mathbb{C}}$ given by $[s : t] \mapsto [s^3 : s^2t : st^2 : t^3]$. As shown in Figure 1, the two conics $F = xz - y^2$ (red) and $G = xw - yz$ (orange) cut out the twisted cubic C (blue) and a line D (light yellow). This shows that the twisted cubic C and the line D are linked. Since $C = f \cdot H - D$ as (generalized) divisors on the surface $V(G)$, where H

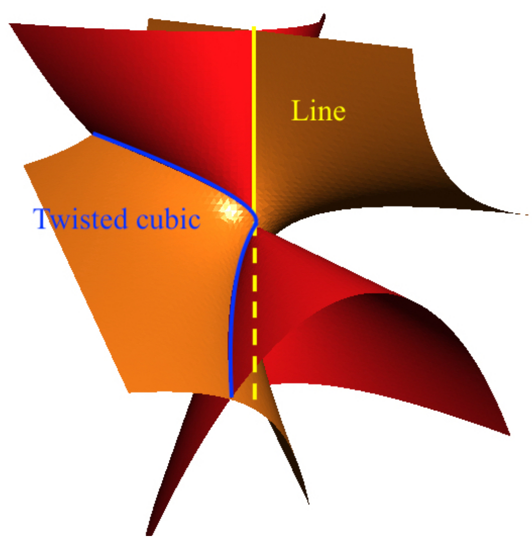

Figure 1: The twisted cubic is linked to a line

is the hyperplane class on $V(G)$, one can deduce the degree and genus of C from those of D using the adjunction formula (when $V(G)$ is singular we use [51, Prop 2.10] instead). It turns out that when the degree of C is small, and when the degrees f and g are taken to be as small as possible, the residual curve D often has strictly smaller degree than that of C. This observation enabled Noether to classify smooth curves in $\mathbf{P}^3_{\mathbb{C}}$ of degree at most 20 using an inductive strategy.

When the degree of C is large, it is possible that the residual curve D has strictly larger degree than that of C. We define the *linkage class* of a curve C to consist of all curves that can be obtained from C using finitely many links. Since the residual curve D of a smooth curve C in a complete intersection need not be smooth, we must enlarge our definition of a curve to include any pure codimension two (i.e. all associated points have codimension two) subschemes of $\mathbf{P}^3_{\mathbb{C}}$. Harris [43, p. 80] conjectured that a general smooth curve with large degree in $\mathbf{P}^3_{\mathbb{C}}$ (a general smooth curve in $\mathcal{M}_g$ with a general g^3_d) has minimal degree and genus in its linkage class. This conjecture was proven by Lazarsfeld and Rao [67]. The same paper contains an interesting structure theorem which we describe below.

There is a certain duality between two linked curves C and D (see Definition I.1.1), thus it makes sense to consider curves that are on the "same side" of this duality. We define an *even linkage class*, or a *biliaison class* of a curve $C \subset \mathbf{P}^3_{\mathbb{C}}$ to consist of all curves that can be obtained from C using an even number of links. If C_1 is linked to C_2 via the complete intersection $V(F,G)$, and C_2 and C_3 are linked via the complete intersection $V(F,GH)$, then we say C_3 is obtained from C_1 using the *basic double link* given by (F,H). Geometrically, we obtain C_3 by attaching to C_1 the complete intersection curve $V(F,H)$. Lazarsfeld and Rao proved that every curve in the even linkage class of a general curve C of large degree can be obtained from C using finitely many basic double links and a deformation preserving the first cohomology module. We refer to [67, Figure 1], reproduced here as Figure 2 for a visual description. The same structure theorem holds more generally for the even linkage

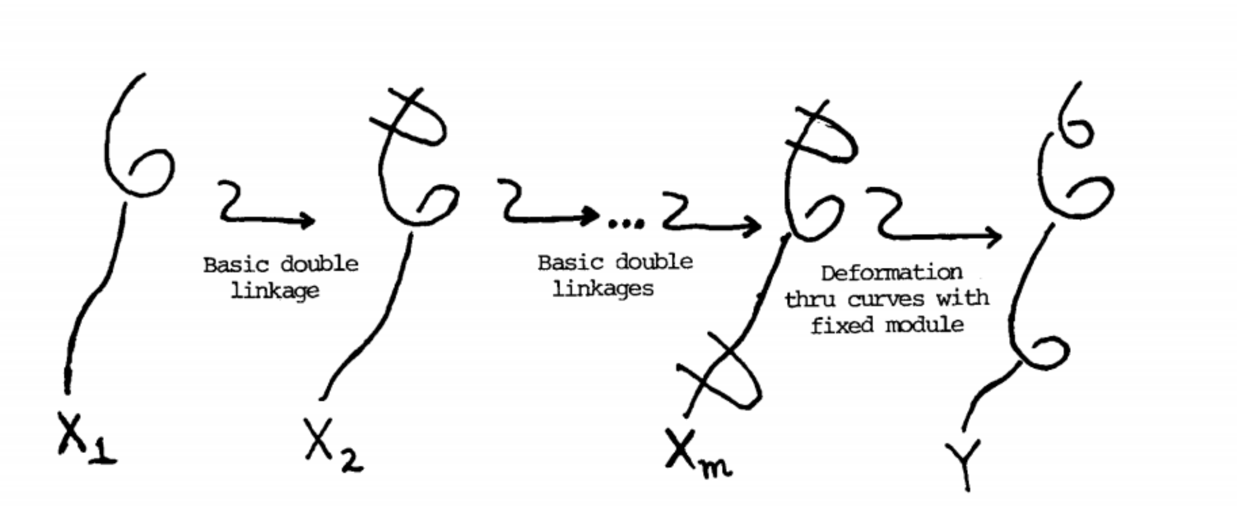

Figure 2: Structure of an even linkage class

class of any pure codimension two subschemes in $\mathbf{P}^n_k$, where k is any algebraically closed field. In Section I.1, we state the precise definitions, statements and references and provide a historical account of related results in liaison theory. We recommend the books [69] and [77] for introductions to this beautiful subject.

The key insight of our thesis is that a (bi)liaison theory can be developed for sheaves in analogy to subvarieties. Briefly, we define biliaison as a certain equivalence relation among coherent sheaves on a projective variety. We prove that there is a natural preorder on a biliaison class, and every biliaison class has minimal members. Moreover, the minimal sheaves in a biliaison class differ by a rational deformation preserving the intermediate cohomology modules, and every sheaf can be obtained from a minimal one using finitely many basic moves. We devote Chapter I to the treatment of this story. A list of our main results related to the biliaison of sheaves can be found in Chapter I summary.

The Hilbert scheme of smooth space curves

The aforementioned thesis of Noether won him the 1882 Steiner prize, shared with G. Halphen who also submitted a thesis [42]. The problem posed by the prize committee was to determine all possible values of degree and genus pairs (d, g) for smooth connected complete curves in $\mathbf{P}^3_{\mathbb{C}}$. The following results were known by the end of the XIX century.

1. There are smooth plane curves of genus $g = \frac{1}{2}(d-1)(d-2)$ for any degree $d \geq 1$.
2. (Castelnuovo [14]) If a smooth curve $C \subset \mathbf{P}^3_{\mathbb{C}}$ does not lie on any plane, then
$$g \leq \left\lfloor \frac{1}{4}d^2 - d + 1 \right\rfloor .$$
Any curve obtaining this bound lies on a quadric surface.
3. For each $a, b > 0$, there are smooth curves on the smooth quadric surface in $\mathbf{P}^3_{\mathbb{C}}$ with degree $d = a + b$ and genus $g = (a-1)(b-1)$.
4. On the singular quadric cone in $\mathbf{P}^3_{\mathbb{C}}$, if $d = 2a$ is even, then there are smooth complete intersections of the quadric cone with another surface of degree a. If $d = 2a + 1$ is odd, then any degree d curve on the quadric cone has genus $g = a^2 - a$.
5. (Halphen [42]) If a curve does not lie on any plane or quadric surface, then
$$g \leq \frac{1}{6}d(d-3) + 1.$$

A complete answer to the original prize problem was only established a century later by Gruson and Peskine [41]. These results are true more generally for smooth connected complete curves in $\mathbf{P}^3_k$, where k is any algebraically closed field. See Table 1 for a visualization of these statements.

6. (Gruson-Peskine [41, Cor 2.3]) For $d \geq 1$ and
$$\frac{1}{\sqrt{3}}d^{3/2} - d + 1 < g \leq \frac{1}{6}d(d-3) + 1,$$
there is a smooth curve of degree d and genus g on a smooth cubic surface in $\mathbf{P}^3_k$.
7. (Gruson-Peskine [41, Thm 1.1]) For $d \geq 1$ and
$$0 \leq g \leq \frac{1}{8}(d-1)^2,$$
there is a smooth curve of degree d and genus g on a smooth quartic surface with a double line in $\mathbf{P}^3_k$. See Figure 3 for an example of such a surface.

Figure 3: A quartic surface with a double line

With the recent advance of computer algebra, it is now possible to encode space curves explicitly as ideals in the polynomial ring over a finite field or the field of rational numbers. Our `SpaceCurves` package [100] in the computer algebra system Macaulay2 [32] can generate smooth curves of all possible (degree, genus) values in projective spaces. See [97] for a hands-on introduction to this package.

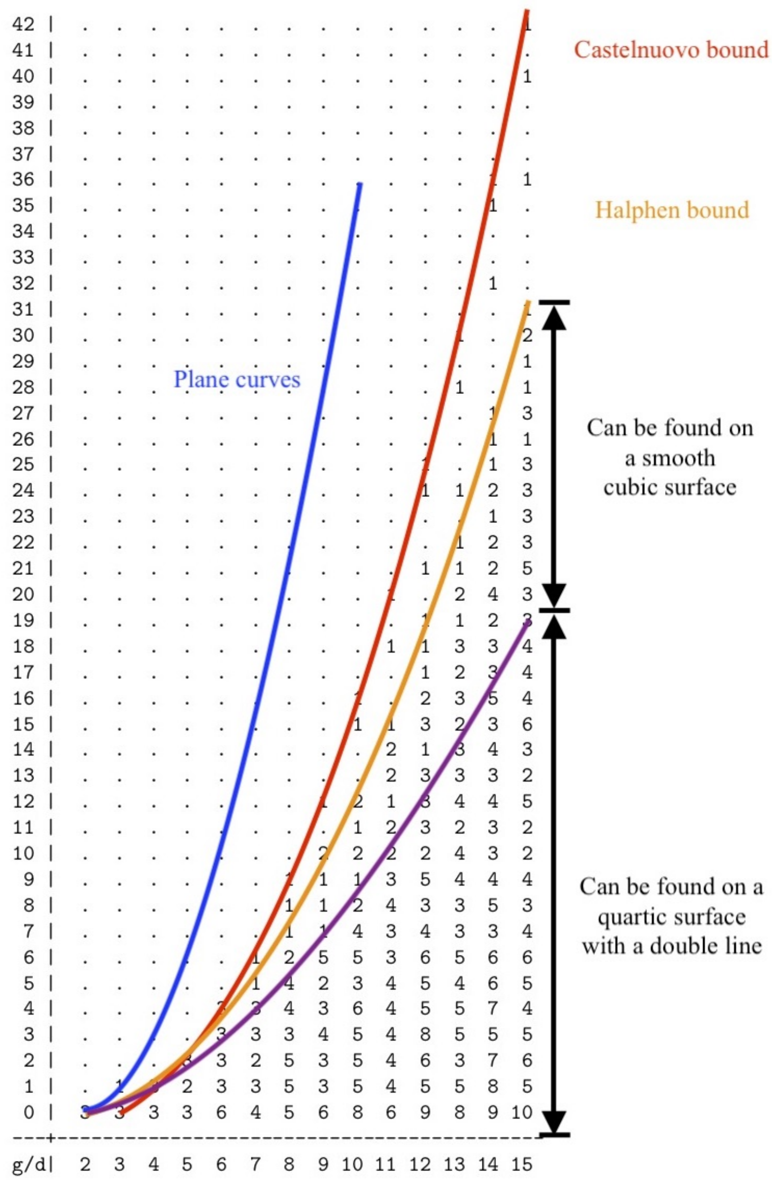

Table 1: Degree and genus of smooth space curves

Most plane curves are omitted from this plot. The numbers in the plot indicate how many distinct types of `Divisors` are implemented in the `SpaceCurves` package [100] that obtain the given (degree, genus) values.

One main method to produce curves in $\mathbf{P}^3_{\mathbb{C}}$ is to start with an abstract algebraic curve, embed it into a high dimensional projective space using the complete linear system of a very ample line bundle, and then project generally into $\mathbf{P}^3_{\mathbb{C}}$. Another important method is to generate curves as effective divisors in a linear system on a surface in $\mathbf{P}^3_{\mathbb{C}}$. In Chapter II, we combine both viewpoints and study curves lying on hypersurfaces with ordinary singularities in $\mathbf{P}^3_{\mathbb{C}}$. Ordinary surface singularities are exactly those that appear on the general projections of smooth surfaces from a higher dimensional projective space into $\mathbf{P}^3_{\mathbb{C}}$. The quartic surface with a double line depicted in Figure 3 is an example of a surface with ordinary singularities. We study the linkage of curves on such surfaces, and prove various results about general projections of smooth curves in $\mathbf{P}^3_{\mathbb{C}}$. See Chapter II summary for a list of our main results related to curves.

The milestone of determining the (degree, genus) values only marks the first obstacle in the classification of space curves. In the language of modern algebraic geometry, we would like to understand the geometry of the Hilbert scheme of curves in $\mathbf{P}^3_k$ for an algebraically closed field k. Consider the contravariant functor $\mathcal{C}^{\text{sm}}$ from the category of k-schemes to the category of sets, where $\mathcal{C}^{\text{sm}}(T)$ is defined to be the set

$$\left\{ C_T \subset \mathbf{P}^3_T \;\middle|\; \begin{array}{l} C_T \text{ is flat over T and } \forall t \in T, \\ \text{the fiber } C_{(t)} \text{ is a smooth connected curve of } \mathbf{P}^3_{k(t)} \end{array} \right\}$$

for any k-scheme T, and the map $\mathcal{C}^{\text{sm}}(T) \to \mathcal{C}^{\text{sm}}(S)$ corresponding to a morphism of k-schemes $S \to T$ sends a family C_T to its pullback $C_T \times_T S$. It follows from a well-known result of Grothendieck [39] that the functor $\mathcal{C}^{\text{sm}}$ is represented by a quasi-projective scheme $\mathcal{H}ilb^{\text{sm}}$, i.e. there is an isomorphism of functors $\mathcal{C}^{\text{sm}} \cong \text{Hom}(-, \mathcal{H}ilb^{\text{sm}})$. In particular, the k-points of $\mathcal{H}ilb^{\text{sm}}$ are in bijections with smooth connected curves in $\mathbf{P}^3_k$. Since every connected flat family of smooth curves have constant degree and genus, it follows that $\mathcal{H}ilb^{\text{sm}}$ can be broken up into connected components $\mathcal{H}ilb^{\text{sm}}_{d,g}$ labelled the degree d and genus g. The above results only tell us the values of (d, g) such that $\mathcal{H}ilb^{\text{sm}}_{d,g}$ is non-empty. In principle, we would like to know the number of irreducible components, as well as the dimensions of the components of each $\mathcal{H}ilb^{\text{sm}}_{d,g}$. However, even this seems to be out of reach. We know from deformation theory that $\dim \mathcal{H}ilb^{\text{sm}}_{d,g} \geq 4d$, which is a tight bound when d is small. We do not know of a general upper bound on the dimensions of components in $\mathcal{H}ilb^{\text{sm}}_{d,g}$. With regards to the number of components, Ein [20] showed that $\mathcal{H}ilb^{\text{sm}}_{d,g}$ is irreducible when $d \geq g+3$. It is an open conjecture that $\mathcal{H}ilb^{\text{sm}}_{d,g}$ is irreducible for $d \geq (g+9)/2$. In the general case, Ellia-Hirschowitz-Mezzetti [25] proved that the number of components of projectively normal curves in $\mathbf{P}^3_k$ with a fixed index of specialty is given by a Fibonacci number. This surprising result shows that the number of components of $\mathcal{H}ilb^{\text{sm}}_{d,g}$ cannot be bounded above by any polynomial in d and g in general! In addition to having too many components to keep track of, the Hilbert scheme $\mathcal{H}ilb^{\text{sm}}_{d,g}$ can have singularities. Mumford [78] described a component of $\mathcal{H}ilb^{\text{sm}}_{14,24}$ that parametrizes smooth curves on smooth cubic surfaces in $\mathbf{P}^3_k$ that is generically non-reduced. To make matters worse, Vakil [96] proved the striking result that every singularity type of finite type over $\mathbb{Z}$ appears on the Hilbert scheme of smooth curves in $\mathbf{P}^n_{\mathbb{Z}}$ for some values of n. It is unknown what singularity types can occur on $\mathcal{H}ilb^{\mathrm{sm}}_{d,g}$.

The above discoveries indicate that the global geometry of the Hilbert scheme of space curves is extremely complicated. However, not all hope is lost. The liaison theory of curves provides an alternative approach to understand the Hilbert scheme through stratification. First, let us extend $\mathcal{H}ilb^{\mathrm{sm}}$ to include all pure codimension two subschemes of $\mathbf{P}^3_k$, and let $\mathcal{H}ilb$ denote the corresponding Hilbert scheme. We partition $\mathcal{H}ilb$ by even linkage classes, which have been completely classified. The even linkage classes of curves C in $\mathbf{P}^3_k$ are in bijection with isomorphism classes of finite length graded modules M over the polynomial ring $S := k[x_0, \ldots, x_3]$ up to shift, where the bijection sends the class of the curve C to the class of the first cohomology module $H^1_*(\mathscr{I}_C)$. For each even linkage class represented by a finite length module M, we denote the corresponding piece of the Hilbert scheme by $\mathcal{H}ilb_M$. We can deterministically compute the Hilbert function H of minimal curves in the even linkage class of M. The structure theorem for even linkage classes allows us to obtain a systematic classification of the Hilbert functions H that occur in $\mathcal{H}ilb_M$ using the procedure of ascending elementary biliaisons (see Definition I.1.8) from minimal curves. Finally, it is known that the points in $\mathcal{H}ilb_M$ corresponding to curves with a given Hilbert function H form a unirational subspace [see 8], which we denote by $\mathcal{H}ilb_{M,H}$. In fact, the space $\mathcal{H}ilb_{M,H}$ supports a smooth scheme structure which represents the functor of flat families of curves in $\mathbf{P}^3_k$ with constant cohomologies. The dimension of $\mathcal{H}ilb_{M,H}$ is also computable from the Hilbert function H and the finite length module M. To summarize, we obtain a stratification

$$\mathcal{H}ilb = \bigsqcup_M \bigsqcup_H \mathcal{H}ilb_{M,H}$$

by smooth unirational schemes $\mathcal{H}ilb_{M,H}$, where the nonempty pieces $\mathcal{H}ilb_{M,H}$ that occur can be classified. This impressive program has been masterfully carried out in the book [69] by Martin-Deschamps and Perrin, and gives by far the most complete results to date on the classification of curves in $\mathbf{P}^3_k$.

Our successful extension of the liaison theory of curves to the context of sheaves suggests the possibility of a similar program for the moduli $\mathcal{M}$ of vector bundles (or torsion-free sheaves, reflexive sheaves etc.) on a projective variety of irregularity zero. In Chapter III, we take the initial steps by describing the piece $\mathcal{M}_0$ corresponding to bundles in the biliaison class of the zero sheaf on $\mathbf{P}^n_k$. Since every bundle on $\mathbf{P}^2_k$ is in the biliaison class of the zero sheaf, our results give a rather complete classification of bundles on $\mathbf{P}^2_k$. In the following, we provide some historical perspectives on vector bundles to highlight our results.

Vector bundles on projective spaces

Vector bundles are ubiquitous in topology and geometry. Algebraic geometers are particularly interested in algebraic vector bundles on algebraic varieties. If the ambient variety

X is smooth and proper over the field of complex numbers $\mathbb{C}$, then algebraic vector bundles on X are the same as holomorphic vector bundles on the corresponding compact complex manifold X^{an} by Serre's well-known GAGA principle [see 91].

The Picard group of a variety X is the group of isomorphism classes of line bundles on X, where the group operation is given by the tensor product. Grothendieck [37][38] proved that the Picard group, considered as a contravariant functor on the category of complete varieties, is representable by schemes. In other words, the Picard scheme is the moduli space of line bundles. Since line bundles govern maps to projective spaces, this theorem has far-reaching consequences in algebraic geometry, e.g. in the Brill-Noether theory of curves. We refer to Mumford [79] and Kleiman [63] for expositions on the Picard scheme. The effort to generalize the Picard scheme to a moduli of bundles of arbitrary rank was met with technical and essential difficulties. It is now well understood that the moduli space of all bundles in general does not exist as a scheme if we require a weak universal property [74, Theorem 1.7]. On the other hand, Mumford [79] proved that the coarse moduli space of semistable bundles of arbitrary rank on a smooth projective curve exists as a quasi-projective scheme. The key takeaway was that the notion of stability for bundles is closely related to the notion of stability in the context of geometric invariant theory. Around a decade later, Maruyama [71] proved that the coarse moduli space of rank two semistable bundles on a smooth projective surface exists as a quasi-projective scheme. A few years later, the existence of the coarse moduli space of semistable torsion-free sheaves of arbitrary rank on a smooth projective variety was finally established by Maruyama [72] and [74].

Let us for a moment restrict our attention to rank two bundles on $\mathbf{P}^2_k$. Deformation theory shows that the coarse moduli space $\mathcal{M}(2, c_1, c_2)^s$ of stable rank two bundles on $\mathbf{P}^2_k$ with given Chern classes c_1 and c_2 is smooth. Barth [5] showed that $\mathcal{M}(2, c_1, c_2)^s$ is connected and rational for c_1 even, and his student Hulek [58] showed the same for c_1 odd. Their arguments contained gaps which were pointed out and only partially fixed in [73] and [27]. Let $\mathscr{E}$ be a stable rank two bundle on $\mathbf{P}^2_k$ with $c_1(\mathscr{E}) = 0$. The key insight of Barth was that the multiplication map

$$\alpha : H^0(\mathscr{O}(1)) \otimes H^1(\mathscr{E}(-2)) \to H^1(\mathscr{E}(-1))$$

completely determines the bundle $\mathscr{E}$. Since $V := H^1(\mathscr{E}(-2))$ and $V^\vee := H^1(\mathscr{E}(-1))$ are Serre dual vector spaces of dimension $c_2(\mathscr{E})$, the map α can be thought of as a map

$$\psi : H^0(\mathscr{O}(1)) \to \operatorname{Sym}^2 V^\vee,$$

which is colloquially known as a net of quadrics on V. Barth classified the maps ψ that can arise from bundles $\mathscr{E}$, and called them *rank two nets of quadrics*. This way, the coarse moduli space $\mathcal{M}(2, c_1, c_2)^s$ can be constructed as the quotient of the space of rank two nets of quadrics on a vector space V of dimension c_2 by the action of $\mathrm{GL}(V)$. A similar story holds for stable rank two bundles $\mathscr{E}$ with $c_1(\mathscr{E}) = -1$. Since every rank two bundle can be normalized to have $c_1 = 0$ or -1, this beautiful unirational parametrization of $\mathcal{M}(2, c_1, c_2)^s$ allows us to produce a random stable rank two bundle on $\mathbf{P}^2_k$ in practice by taking a random rank two net of quadrics on a suitable vector space V.

One may ask the following questions.

1. What if we want to produce a random bundle $\mathscr{E}$ on $\mathbf{P}^2_k$ that is not semistable?
2. What if we want a random bundle $\mathscr{E}$ of rank $r > 2$?
3. What if we want a random bundle $\mathscr{E}$ where $h^0(\mathscr{E}(3)) \geq 3$?
4. What if we want a random bundle $\mathscr{E}$ that is generated in degree ≤ 6?
5. What if we want a random bundle $\mathscr{E}$ that satisfies all of the above?

Aside from the fact that Barth and Hulek's results work only for stable rank two bundles, which do not apply to bundles of type 1 and 2 above, there is another essential reason why it would not produce bundles of type 3 and 4 in practice. Since the moduli space $\mathcal{M}(2, c_1, c_2)^s$ is irreducible, a random bundle produced from a global parametrization will always exhibit the generic behavior. Looking for a special bundle that deviates from the general behavior is comparable to looking for a needle in a haystack. In Chapter III, we describe a stratification of the moduli space of bundles on $\mathbf{P}^2_k$ by Betti numbers, which are very fine homological invariants of bundles, where the pieces of the moduli corresponding to given Betti numbers are individually unirational.

Our results hold more generally for vector bundles on $\mathbf{P}^n_k$ in the biliaison class of the zero sheaf. In the spirit of the program outlined at the end of the previous section, let $\mathcal{M}_0$ denote the set of isomorphism classes of finite rank bundles on $\mathbf{P}^n_k$ in the biliaison class of the zero sheaf. We classify the possible Hilbert functions H that can occur in $\mathcal{M}_0$. For each Hilbert function H, we define a natural Zariski topology on $\mathcal{M}_{0,H}$, the subset of $\mathcal{M}_0$ consisting of classes of bundles with Hilbert function H. We then describe a stratification of $\mathcal{M}_{0,H}$ by quotients of rational varieties, and prove that the closed strata form a graded lattice given by the Betti numbers. The subspace $\mathcal{M}^{ss}_{0,H}$ corresponding to semistable bundles support a subscheme structure of the coarse moduli scheme $\mathcal{M}^{ss}_{\chi}$ of semistable bundles with Hilbert polynomial χ established by Maruyama. A similar stratification of $\mathcal{M}^{ss}_{0,H}$ holds in the coarse moduli scheme $\mathcal{M}^{ss}_{\chi}$. See Chapter III summary for a short list of our main results regarding bundles on projective spaces.

The above results are implemented in the Macaulay2 package `BundlesOnPn` [99], which generates all Betti numbers of bundles in $\mathcal{M}_0$ up to bounded regularity, as well as random bundles with given Betti numbers. In particular, the package can produce bundles with special features like those in item 1 - 5 listed above.

Disclaimer

Most figures of surfaces in this thesis are generated using the Surfer program by Oliver Labs. Chapter I is based on our papers [103] and [101]. Chapter II is based on our paper [98]. Chapter III is based on our paper [102].

Acknowledgments

I thank my advisor David Eisenbud for support and guidance over the years.
I thank Robin Hartshorne for many helpful discussions.
I also thank Bernd Sturmfels for suggestions that improved our exposition.

CHAPTER I

Liaison

Chapter I summary

We begin our thesis with a study of the biliaison theory of sheaves, which naturally generalizes the classical subject of even linkage of codimension two subvarieties.

In Section I.1, we survey the main results and history on even linkage classes of codimension two subvarieties. We also review the notion of Serre correspondence.

In Section I.2, we prove the existence of graded basic elements (Theorem I.2.10), extending results by Eisenbud-Evans [22] and Bruns [11] to the projective setting. We define the notion of m-reductions (Definition I.2.14) - the technical heart of the biliaison theory of sheaves, and prove a factorization theorem (Theorem I.2.15).

In Section I.3, we define biliaison of sheaves and prove the following main results.

(a) There are minimal sheaves in each biliaison class under a suitable preorder (Theorem I.3.26). Those that are minimal can be obtained from each other using a rational deformation preserving intermediate cohomology modules (Proposition I.3.21).

(b) Any sheaf can be obtained from a minimal one in its biliaison class using rigid deformations and certain basic moves (Theorem I.3.27).

(c) Biliaison classes of sheaves are in bijection with stable equivalence classes of primitive sheaves. This result is essentially due to Hartshorne [52], extending a well-known result of Rao [86].

The results (a) - (c) give satisfactory extensions of the main theorems on even linkage classes of codimension two subvarieties (cf. Theorem I.1.10). Furthermore, we also prove a sufficient criterion (Theorem I.3.28) for a sheaf to be minimal, generalizing the criterion of Lazarsfeld-Rao [67] for a curve to be minimal in $\mathbf{P}^3_k$.

I.1. Background

We review the main theorems and history of even linkage classes of codimension two subvarieties and recall basic facts about the Serre correspondence. Let X denote a Gorenstein projective variety over an infinite field k. All sheaves considered are coherent on X.

Even linkage of codimension two subvarieties

The notion of linkage originated in the work of M. Noether [80] in 1882. Using this technique, Noether gave a complete classification of curves in $\mathbf{P}^3_{\mathbb{C}}$ of degree at most 20. The modern definition of linkage is crystalized in the paper of Peskine-Szipro [83].

Definition I.1.1. Let Y and Z be two subschemes of X of codimension r. We say Y and Z are *linked* if there is a codimension r (global) complete intersection K in X containing Y and Z such that $\mathscr{I}_{Y/K} \cong \mathscr{H}om(\mathscr{O}_Z, \mathscr{O}_K)$ and $\mathscr{I}_{Z/K} \cong \mathscr{H}om(\mathscr{O}_Y, \mathscr{O}_K)$. A *linkage class* on X consists of all subschemes that can be obtained from each other using finitely many links. An *even linkage class* on X consists of all subschemes that can be obtained from each other using even numbers of links.

Theorem I.1.2 (Peskine-Szpiro)**.** *With notation as above, if Y and Z are linked, then Y is of pure codimension r (i.e. all associated points have codimension r) in X if and only if Z is. If Y is of pure codimension r in X and is contained in a complete intersection K of codimension r, then there is a unique subscheme $Z \subseteq K$ (possibly empty) such that Y is linked to Z through K. If Y and Z are linked, then Y is (locally) Cohen-Macaulay if and only if Z is.*

The first linkage class that was completely classified was the class of a codimension two complete intersection in $\mathbf{P}^n_k$. Peskine and Szpiro [83] showed that a codimension two subscheme Y of $\mathbf{P}^n_k$ is in the linkage class (equivalently even linkage class) of a complete intersection (equivalently the empty scheme) if and only if Y is arithmetically Cohen-Macaulay (ACM), i.e. the cone of Y in $\mathbf{A}^{n+1}_k$ is Cohen-Macaulay. This work generalizes results of Apery [1] and Gaeta [29]. In the following year, Ellingsrud [26] classified all codimension two ACM subvarieties of $\mathbf{P}^n_k$, and essentially described the relevant pieces of the Hilbert scheme. In a parallel spirit, we will classify in Chapter III the vector bundles in the biliaison class of the zero sheaf on $\mathbf{P}^n_k$ and describe their moduli.

When $X = \mathbf{P}^3_k$, we call a pure codimension two subscheme a *curve*. Let S denote the polynomial ring of X, and let $H^i_*(\mathscr{F})$ denote the graded S-module $\bigoplus_{n\in\mathbb{Z}} H^i(\mathscr{F}(n))$. A curve C is ACM if and only if its first cohomology module $H^1_*(\mathscr{I}_C)$ vanishes. The result of Peskine and Szpiro showed that whether the invariant $H^1_*(\mathscr{I}_C)$ is zero determines whether C belongs to the even linkage class of the empty scheme. Under the advice of Hartshorne, Rao set out to investigate the next simplest case of the disjoint union of two lines $Y = L_1 \sqcup L_2$, where $H^1_*(\mathscr{I}_Y) = k$. Rao [84] proved that a curve C is in the linkage class (equivalently even linkage class) of the disjoint union of two lines if and only if $H^1_*(\mathscr{I}_C)$ is isomorphic to k up to shift. When Rao submitted his paper, the referee pointed out that his method works more generally for any first cohomology module $H^1_*(\mathscr{I}_C)$. We have the following beautiful result connecting the geometric relation of linkage with the algebraic invariant given by the cohomology module.

Theorem I.1.3 (Rao [85])**.** *Two curves C and D in $\mathbf{P}^3_k$ are in the same linkage class if and only if $H^1_*(\mathscr{I}_C)$ and $H^1_*(\mathscr{I}_D)$ are isomorphic as graded S-modules up to shift and graded vector space dual. They are in the same even linkage class if and only if $H^1_*(\mathscr{I}_C)$ and $H^1_*(\mathscr{I}_D)$ are isomorphic as graded S-modules up to shift.*

Moreover, every finite length graded S-module M occurs as $H^1_*(\mathscr{I}_C)$ for some curve C up to shift. In particular, there is a bijection between the even linkage classes of curves in $\mathbf{P}^3_k$ and isomorphism classes of finite length graded S-modules up to shift.

A few year later, Rao established a fascinating connection between even linkage classes and stable equivalence of bundles on $\mathbf{P}^n_k$.

Theorem I.1.4 (Rao [86])**.** *The even linkage classes of Cohen-Macaulay pure codimension two subschemes of $\mathbf{P}^n_k$ are in bijection with the stable equivalence classes of bundles $\mathscr{E}$ up to shift, where $H^{n-1}_*(\mathscr{E}) = 0$.*

We say two bundles $\mathscr{E}_1$ and $\mathscr{E}_2$ on $\mathbf{P}^n_k$ are *stably equivalent* if $H^0_*(\mathscr{E}_1)$ and $H^0_*(\mathscr{E}_2)$ are stably equivalent as graded S-modules, i.e. they become isomorphic after taking direct sums with graded free S-modules. Note that the original formulation of Rao uses stable equivalence classes of bundles $\mathscr{E}$ where $H^1_*(\mathscr{E}) = 0$. We present the dual statement intentionally (see Definition I.3.13). When $X = \mathbf{P}^3_k$, a result of Horrocks [55] states that stable equivalence class of a bundle $\mathscr{E}$ is completely classified by its first cohomology module $H^1_*(\mathscr{E})$. This fact retroactively explains why even linkage classes of curves in $\mathbf{P}^3$ are determined by the first cohomology modules.

Armed with the philosophy that there is no way to get hands on a general curve, Harris [43, p. 80] conjectured that a general (abstract) curve (with a general g^3_d) for large degree d in $\mathbf{P}^3_k$ has minimal degree and genus in its linkage class. In particular, the inductive approach of Noether is doomed to fail for general curves of large degrees in $\mathbf{P}^3_k$. This conjecture was proven by Lazarsfeld and Rao.

Theorem I.1.5 (Lazarsfeld-Rao [67])**.** *If a curve C in $\mathbf{P}^3_k$ does not lie on a surface of degree $e(C)+3$, where $e(C) := \inf\{n \mid H^1(\mathscr{O}_C(n)) \neq 0\}$ is the index of specialty of C, then C has minimal degree and genus in its linkage class. A general (abstract) curve (with a general g^3_d) for $d \gg g$ satisfies this condition, and thus is minimal in its linkage class.*

Apart from settling the conjecture of Harris, the paper [67] included an interesting structure theorem of even linkage classes of a general curve of large degree. We describe what is called a *basic double link*. If C_1 is linked to C_2 via the complete intersection $V(f, g)$, and C_2

and C_3 are linked via the complete intersection $V(g, fh)$, then we say C_3 is obtained from C_1 using the basic double link given by (g, h). In a suitable sense, we obtain C_3 by attaching to C_1 the complete intersection curve $V(g, h)$. This procedure is a particular case of a liaison addition [see 90]. We refer to Figure 2 for a visual description.

Theorem I.1.6 (Lazarsfeld-Rao [67])**.** *If a curve C in $\mathbf{P}^3_k$ does not lie on a surface of degree $e(C)+3$, then every curve D in its even linkage class can be obtained from C using finitely many basic double links and a deformation preserving the first cohomology module.*

It is natural to ask whether every even linkage class of curves in $\mathbf{P}^3_k$ satisfies this structure theorem, i.e. can every curve be obtained from a curve of minimal degree in the even linkage class using finitely many basic double links and a deformation preserving the first cohomology module? The answer is affirmative. The structure theorem was proven for every even linkage class of curves in $\mathbf{P}^3_k$ by Martin-Deschamps and Perrin [69]. Independently and more generally, Ballico, Bolondi and Migliore [3] proved the structure theorem for every even linkage class of Cohen-Macaulay pure codimension two subvarieties in $\mathbf{P}^n_k$.

Theorem I.1.7 (Ballico-Bolondi-Migliore [3])**.** *Every Cohen-Macaulay pure codimension two subscheme Y of $\mathbf{P}^n_k$ is obtained by from one of minimal degree in its even linkage class using finitely many basic double links and a deformation preserving intermediate cohomology modules. In particular, all such subschemes of minimal degree in an even linkage class differ by a deformation preserving intermediate cohomology modules.*

For the case of curves in $\mathbf{P}^3_k$, Strano [95] showed that the deformation in the end of the structure theorem can be subsumed into linear equivalences on suitable surfaces.

Definition I.1.8. We say a subscheme Y of X is *minimal* if it has minimal degree in its even linkage class. If Y and Z are two pure codimension two subschemes of X, then an *elementary biliaison* of height h from Y to Z is given by a hypersurface K of X containing Y and Z such that $\mathscr{I}_{Y/K} \cong \mathscr{I}_{Z/K} \otimes \mathscr{O}(h)$ for some integer h. An elementary biliaison is *ascending* if the height is positive, and *descending* otherwise. It is easy to check that if there is an elementary biliaison from Y to Z, then Y and Z are in the same even linkage class.

If C_3 is obtained from C_1 using a basic double link (g, h) as above, then C_3 is clearly linearly equivalent to C_1 plus $V(g, h)$ considered as (generalized) divisors on the surface $V(g)$. We see that the notion of elementary biliaison has linear equivalence built-in, where we have the flexibility of taking any curve linearly equivalent to C_3 on $V(g)$.

Theorem I.1.9 (Strano[95])**.** *Any curve C in $\mathbf{P}^3_k$ is obtained from one of minimal degree in its even linkage class using finitely many ascending elementary biliaisons.*

In another direction, Nollet [81] removed the Cohen-Macaulay assumption for even linkage classes of pure codimension two in $\mathbf{P}^n_k$. Combining the progress made by Strano and Nollet, Hartshorne [52, 51] further generalized the structure theorem for even linkage classes of pure codimension two subschemes on an arithmetically Gorenstein scheme.

Theorem I.1.10 (Hartshorne [51, 52]). *Let X be a Gorenstein variety over an infinite field k and assume that $H^1_*(\mathscr{O}_X) = 0$.*

1. *Any two pure codimension two minimal subschemes in an even linkage class lie on an irreducible family.*
2. *Every pure codimension two subscheme Y can be obtained from a minimal one in its even linkage class using finitely many ascending elementary biliaisons.*
3. *Even linkage classes of pure codimension two subschemes of X are in bijection with stable equivalence classes of extraverti sheaves up to twist.*

We will use the more general notion of *primitive* sheaves instead of *extraverti* sheaves in this thesis (see Definition I.3.13). Although attributed to Hartshorne, the above theorem clearly crystalized decades of work by numerous mathematicians. We refer to the book [77] for a more complete story on this beautiful subject.

Our goal in Chapter I is to generalize the results in this section to biliaison of sheaves.

The Serre correspondence

The simplest Serre correspondence is between points and rank two bundles on a smooth surface. Let $\mathscr{E}$ be a rank two bundle and let $\mathscr{O} \xrightarrow{s} \mathscr{E}$ be a section. The image of the dual map $\mathscr{E}^* \xrightarrow{s^*} \mathscr{O}$ is an ideal sheaf $\mathscr{I}$. If $\mathscr{I} = \mathscr{O}$, then we have an extension

$$0 \to \wedge^2\mathscr{E}^* \to \mathscr{E}^* \to \mathscr{O} \to 0$$

given by the Koszul complex. If $\mathscr{I} \neq \mathscr{O}$, then the subscheme $V(\mathscr{I})$ defined by $\mathscr{I}$ has codimension at most two since it is locally defined by two equations. If $V(\mathscr{I})$ has codimension one, let Y be the largest codimension one component of $V(\mathscr{I})$. The map $\mathscr{E}^* \xrightarrow{s^*} \mathscr{O}$ factors as $\mathscr{E}^* \to \mathscr{I}(Y) \to \mathscr{O}$. Twisting by the line bundle $\mathscr{O}(Y)$, we obtain an exact sequence

$$0 \to \wedge^2\mathscr{E}^* \otimes \mathscr{O}(Y) \to \mathscr{E}^* \otimes \mathscr{O}(Y) \to \mathscr{J} \to 0$$

where $\mathscr{J}$ is the image ideal of the map $\mathscr{E}^* \otimes \mathscr{O}(Y) \to \mathscr{O}$. It follows that the subscheme defined by $\mathscr{J}$, if nonempty, has codimension exactly two. We say $V(\mathscr{J})$ is the vanishing scheme of a section of $\mathscr{E} \otimes \mathscr{I}(Y)$.

Definition I.1.11. Let X be a smooth surface, i.e. a smooth integral variety of dimension two over a field k. A zero dimensional subscheme Z of X satisfies the *Cayley-Bacharach property* with respect to the line bundle $\mathscr{L}$ if every effective divisor D in the complete linear system of $|\mathscr{L}|$ that contains a subscheme $Z' \subset Z$ of co-length one must contain Z.

Theorem I.1.12 (Griffith-Harris [35]). *With assumptions as above, a zero dimension subscheme Z of X is the vanishing of a rank two bundle $\mathscr{E}$ where $\wedge^2\mathscr{E} = \mathscr{L}$, i.e. $\mathscr{I}_Z$ admits an extension of the form*

$$0 \to \mathscr{L}^* \to \mathscr{E} \to \mathscr{I}_Z \to 0$$

if and only if Z is a local complete intersection (l.c.i.) and satisfies the Cayley-Bacharach property with respect to the line bundle $\mathscr{L} \otimes \omega_X$.

The case when Z is non-reduced is proven by Catanese [15]. There are analogous results in higher dimensions due to Serre [93], Horrocks [56] and Hartshorne [44].

Theorem I.1.13 (Serre, Horrocks, Hartshorne)**.** *A codimension two subscheme Y of a smooth scheme X is the vanishing scheme of a section of a rank two bundle $\mathscr{E}$ where $\wedge^2\mathscr{E} = \mathscr{L}$, i.e. $\mathscr{I}_Y$ admits an extension of the form*

$$0 \to \mathscr{L}^* \to \mathscr{E}^* \to \mathscr{I}_Y \to 0$$

if and only if Y is a l.c.i. and $\omega_Y \cong \mathscr{L} \otimes \omega_X \otimes \mathscr{O}_Y$.

This result sets up a correspondence, called the *Serre correspondence*, between codimension two local complete intersections and rank two bundles on a smooth variety. It is Hartshorne's insight to relax the locally-free condition and consider reflexive sheaves instead. Recall that a coherent sheaf is *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{**}$ is an isomorphism. The following theorem of Hartshorne extended the Serre correspondence to a correspondence between rank two reflexive sheaves and generic local complete intersections curves in $\mathbf{P}^3_k$.

Theorem I.1.14 (Hartshorne [48])**.** *Fix an integer c_1. There is a bijection between*

1. *pairs $(\mathscr{E}, s)$, where $\mathscr{E}$ is a rank two reflexive sheaf on $\mathbf{P}^3_k$ with $c_1(\mathscr{E}) = c_1$, and*
2. *pairs (Y, ξ), where Y is a generic l.c.i. curve in $\mathbf{P}^3_k$ and $\xi \in H^0(\omega_Y(4 - c_1))$ is a section that generates the sheaf $\omega_Y(4 - c_1)$ everywhere except at finitely many points.*

We will extend the above definition in Definition I.3.10.

Notations and conventions

For the remainder of this chapter, we fix X to be a (locally) Cohen-Macaulay projective variety over an infinite field k, and let $\mathscr{O}(1)$ be a fixed very ample line bundle of X over k. All sheaves $\mathscr{F}$ in consideration are assumed to be coherent on X. We always use underlined letters such as $\underline{a}$ to denote a finite sequence of integers $(a_i)_{i=1}^u$, and write $\mathscr{O}(\underline{a})$ instead of $\bigoplus_{i=1}^u \mathscr{O}(-a_i)$ for brevity.

Sections of $\mathscr{F}(l) := \mathscr{F} \otimes \mathscr{O}(1)^{\otimes l}$ are called *sections of $\mathscr{F}$ in degree l*, or *twisted sections of $\mathscr{F}$*. If $s_1, \ldots, s_u$ are sections of $\mathscr{F}$ in degrees $a_1, \ldots, a_u$, then we denote by $(s_1, \ldots, s_u)$ the image subsheaf of the morphism $\varphi : \mathscr{O}(\underline{a}) \xrightarrow{s_1,\ldots,s_u} \mathscr{F}$. For a point $p \in X$, let $\mathscr{F}_p$ denote the stalk of $\mathscr{F}$ at p, and let $\mathscr{F}_{(p)}$ denote the fiber at p. We write $\mu(\mathscr{F}, p)$ for the dimension of the fiber $\mathscr{F}_{(p)}$ over the residue field $k(p)$. By Nakayama's lemma, the number of minimal generators of $\mathscr{F}_p$ over the local ring $\mathscr{O}_p$ is equal to $\mu(\mathscr{F}, p)$.

I.2. Graded Basic Elements

A basic element of a module is an element that is part of a minimal system of generators of the module at each prime. We also consider elements that are basic at primes up to a certain codimension or depth. In the affine case, results of Eisenbud-Evans [23] guarantee the existence of a basic element in a submodule if it contains enough minimal generators of the ambient module at each point. The theory surrounding basic elements has always been very technical, but its applications have shown themselves to be extremely useful to those willing to understand the material. To name a few consequences of the existence theorems, one can prove the Bass' cancellation theorem, the Forster-Swan theorem, Bass' stable range theorem and a theorem of Serre on free summands of a projective module [see 22].

We encounter new difficulties and new features in the projective setting. The heart of the proofs of the above theorems is [23, Lemma 3], which we call the "finite shrinking lemma". It states that after a unipotent change of coordinates, one can drop a generator of a submodule while maintaining basicness to a maximal extent over finitely many points. The original proof of this lemma does not go through in the projective case. We were able to find a different approach to prove the finite shrinking lemma in the projective setting, thus obtaining analogous results of Eisenbud-Evans [23] and Bruns [11] for projective schemes.

Furthermore, we prove a criterion on the factorization of what we call an m-reduction, yielding several geometric applications in codimension two. We define the Caylay-Bacharach index of a set of points in $\mathbf{P}^2_k$, and provide upper and lower bounds of this invariant in terms of the second Betti numbers of the points. We also prove that the Lazarsfeld-Rao procedure [67] of producing a curve in $\mathbf{P}^3_k$ from a bundle factors through a Serre correspondence if and only if the curve is a generic complete intersection. Finally, we show that every pure codimension two l.c.i. subscheme of $\mathbf{P}^n_k$ is the degeneracy locus of $(n-1)$ generically independent sections of a rank n bundle.

One key new feature of graded basic elements is the existence of degrees. In practice we often strive to find graded basic elements of smallest degrees possible. We will expand on this idea in Section I.3.

The finite shrinking lemma

For any $p \in X$, there exists a linear form $L \in H^0(\mathscr{O}(1))$ not vanishing at p since $\mathscr{O}(1)$ is very ample. If $s_1, \ldots, s_u$ are sections of a sheaf $\mathscr{F}$ in degrees $a_1, \ldots, a_u$ corresponding to the map $\varphi : \mathscr{O}(\underline{a}) \to \mathscr{F}$, then the image of φ_p is generated by $(s_1/L^{a_1})_p, \ldots, (s_u/L^{a_u})_p$ in $\mathscr{F}_p$. A different choice of the dehomogenization L would result in the scaling by units of $\mathscr{O}_p$. The image of $\varphi_{(p)}$ in $\mathscr{F}_{(p)}$ is generated by $(s_1/L^{a_1})_{(p)}, \ldots, (s_u/L^{a_u})_{(p)}$, which is a $k(p)$

vector-subspace of dimension $\mu(\mathscr{F},p)-\mu(\mathscr{F}/(s_1,\dots,s_u),p)$ by the right exact sequence

$$\bigoplus_{i=u}^{l}\mathscr{O}(-a_i)_{(p)}\xrightarrow{\varphi_{(p)}}\mathscr{F}_{(p)}\to(\mathscr{F}/(s_1,\dots,s_u))_{(p)}\to 0.$$

Definition I.2.1. A subsheaf $\mathscr{F}'$ of $\mathscr{F}$ is *w-basic* in $\mathscr{F}$ at $p\in X$ if

$$\mu(\mathscr{F}/\mathscr{F}',p)\leq\mu(\mathscr{F},p)-w.$$

If $s_1,\dots,s_u$ are twisted sections of $\mathscr{F}$, then we say they are *basic* in $\mathscr{F}$ at $p\in X$ if $(s_1,\dots,s_u)$ is u-basic in $\mathscr{F}$ at p.

By Nakayama's lemma, a subsheaf $\mathscr{F}'$ is w-basic in $\mathscr{F}$ at p if and only if $\mathscr{F}'_p$ contains w members of a minimal system of generators of $\mathscr{F}_p$ over $\mathscr{O}_p$.

The finite shrinking lemma says that we may, after a unipotent coordinate change, drop one section of the lowest degree while maintaining basicness to the maximal possible extent at finitely many points.

Lemma I.2.2 (Finite shrinking lemma)**.** *Let $s_1,\dots,s_u$ be sections of a coherent sheaf $\mathscr{F}$ on X in degrees $a_1\leq\cdots\leq a_u$. If $(s_1,\dots,s_u)$ is w_i-basic in $\mathscr{F}$ at $p_i\in X$ for $1\leq i\leq v$, then there are $r_j\in H^0(\mathscr{O}(a_j-a_1))$ for $2\leq j\leq u$ such that $(s_2+r_2\cdot s_1,\dots,s_u+r_u\cdot s_1)$ is $\min(u-1,w_i)$-basic in $\mathscr{F}$ at p_i for all i.*

Lemma I.2.2 is the projective version of Lemma 3 of [23], which is crucial in the proof of [11, Theorem 1]. The original proof breaks down in the projective case and requires a different strategy. The key difficulty is that the forms r_j must now be homogeneous of degrees a_j-a_1, as opposed to any forms in the affine case. The original proof involved finding a form r_j that vanishes on $p_1,\dots,p_{v-1}$ but not on p_v, assuming p_v is minimal. Such a form always exists in high enough degrees, but there is no guarantee for it to exist in degree a_j-a_1. We overcome this difficulty with the following lemma from linear algebra.

Lemma I.2.3. *Let $v_2,\dots,v_u$ be vectors in a vector space V over a field K. For any $2\leq j\leq u$, there is at most one $\lambda\in K$ where*

$$\dim\operatorname{span}\{v_2,\dots,v_j+\lambda\cdot v_1,\dots,v_u\}<\dim\operatorname{span}\{v_2,\dots,v_u\}.$$

Proof. If $v_j\in\operatorname{span}\{v_2,\dots,\hat{v}_j,\dots,v_u\}$, then clearly

$$\dim\operatorname{span}\{v_2,\dots,v_j+\lambda\cdot v_1,\dots,v_u\}\geq\dim\operatorname{span}\{v_2,\dots,\hat{v}_j,\dots v_u\}=\dim\operatorname{span}\{v_2,\dots,v_u\}.$$

Suppose $v_j\notin\operatorname{span}\{v_2,\dots,\hat{v}_j,\dots,v_u\}$. We may quotient V by $\operatorname{span}\{v_2,\dots,\hat{v}_j,\dots,v_u\}$ and denote the images of v_1 and v_j by $\overline{v_1}$ and $\overline{v_j}$. If the inequality in the statement holds, then we must have $\overline{v_j}+\lambda\cdot\overline{v_1}=0$. It follows that $\overline{v_1}$ is a nonzero multiple of $\overline{v_j}$ and λ is uniquely determined. □

For clarity of exposition, we use the following lemma to make a consistent choice of dehomogenization at finitely many points.

Lemma I.2.4. *Let $p_1, \dots, p_v \in X$ be finitely many points. There is a linear form $L \in H^0(\mathscr{O}(1))$ that does not vanish on any p_i.*

Proof. Sections that vanish on p_i form a proper subspace of $H^0(\mathscr{O}(1))$ for any i. The conclusion follows from the fact that a vector space over an infinite field is not the union of finitely many proper subspaces. □

Proof of Lemma I.2.2. Let $L \in H^0(\mathscr{O}(1))$ be a form that does not vanish at p_i for any i as in Lemma I.2.4. We prove the lemma by induction on v. Suppose $v = 1$, and $(s_1, \dots, s_u)$ is w_1-basic at p_1. If $u = w_1$, then any choice of r_j's would work. We may suppose $u > w_1$. If $(s_2, \dots, s_u)$ is w_1-basic, then we may choose $r_j = 0$ for all j. If not, then there exists some $2 \le l \le u$ such that $(s_l/L^{a_l})_{(p_1)}$ is in the span of $(s_{l+1}/L^{a_{l+1}})_{(p_1)}, \dots, (s_u/L^{a_u})_{(p_1)}$. In this case, we may take $r_j = 0$ for every $j \neq l$, and choose $r_l = L^{a_j - a_1}$. Since r_l has the image a nonzero unit in $k(p_1)$, we see $\mu(\mathscr{F}/(s_2, \dots, s_l + r_l s_1, \dots, s_u), p_1) = \mu(\mathscr{F}/(s_1, \dots, s_u), p_1)$. It follows that $(s_2, \dots, s_l + r_l s_1, \dots, s_u)$ is w_1-basic at p_1.

Now we prove the case $v > 1$. If $w_i = u$ for some i, then any choice of r_j's would satisfy the requirement at p_i. Thus we may consider the same problem at fewer points, and induction on the number of points v takes care of this case. Thus we may assume that $u > w_i$ for all i. By the induction hypothesis, there exist $r'_j \in H^0(\mathscr{O}(a_j - a_1))$ for $2 \le j \le u$ such that $s'_j := s_j + r'_j s_1$ generate a subsheaf that is w_i-basic in $\mathscr{F}$ at $p_1, \dots, p_{v-1}$. If $(s'_2, \dots, s'_u)$ is w_v-basic at p_v, then we are done. Suppose not, then $(s'_2, \dots, s'_u)$ is $(w_v - 1)$-basic in $\mathscr{F}$ at p_v. There exists $2 \le l \le u$ such that $(s'_l/L^{a_l})_{(p_v)}$ is in the span of $(s'_{l+1}/L^{a_l+1})_{(p_v)}, \dots, (s'_u/L^{a_u+1})_{(p_v)}$. We choose $r''_l := \lambda \cdot L^{a_l - a_1}$ for a nonzero $\lambda \in k$ yet to be determined. By the same reasoning in the paragraph above, the image of r''_l in $k(p_v)$ is a nonzero unit and thus $(s'_2, \dots, s'_l + r''_l s_1, \dots, s'_u)$ is w_v-basic at p_v for any nonzero $\lambda \in k$. At each of the $p_1, \dots, p_{v-1}$, there exists at most one $\lambda \in k$ where $(s'_2, \dots, s'_l + r''_l s_1, \dots, s'_u)$ fails to be w_i-basic at p_i by Lemma I.2.3. Since k is an infinite field, we may choose a nonzero $\lambda \in k$ such that $(s'_2, \dots, s'_l + r''_l s_1, \dots, s'_u)$ remains w_i-basic at p_i for all $1 \le i \le v - 1$. □

We do not know if Lemma I.2.2 remains true when k is finite.

Existence of graded basic elements

We now use the finite shrinking lemma to extend the results of [23] and [11] to the projective setting. We slightly generalize the existence theorem to "catch" multiple basic elements at once. Aside from that, our contribution here is mostly that of a translation. We include the complete proofs for the sake of rigor and being self-contained.

For any sheaf $\mathscr{F}$ on X, there is a presentation of the form $\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$, where $\mathscr{E}_1$ and $\mathscr{E}_0$ are locally free. The sheaves $\mathscr{E}_i$ can be chosen to be the direct sums of tensor powers of the very ample line bundle $\mathscr{O}(1)$.

Definition I.2.5. Let $\mathscr{E}_1 \xrightarrow{\varphi} \mathscr{E}_0$ be a map of locally free sheaves on X. The i-th minor ideal sheaf $\mathscr{I}_i(\varphi)$ is defined as the image ideal of the map $\wedge^i \mathscr{E}_1 \otimes \wedge^i \mathscr{E}_0^* \to \mathscr{O}$ corresponding to the i-th exterior map $\wedge^i \mathscr{E}_1 \to \wedge^i \mathscr{E}_0$.

Let $\mathscr{F}$ be a sheaf on X and let $\mathscr{E}_1 \xrightarrow{\varphi} \mathscr{E}_0 \to \mathscr{F} \to 0$ be a presentation of $\mathscr{F}$ by locally free sheaves $\mathscr{E}_1$ and $\mathscr{E}_0$. We define the i-th Fitting ideal $\mathscr{F}itt_i(\mathscr{F})$ of $\mathscr{F}$ to be $\mathscr{I}_{n-i}(\varphi)$, where $n = \operatorname{rank} \mathscr{E}_0$. Let $Z_i(\mathscr{F})$ be the subscheme corresponding to $\mathscr{F}itt_i(\mathscr{F})$.

Proposition I.2.6. *With notations as above, the ideal sheaf $\mathscr{F}itt_i(\mathscr{F})$ is well-defined and does not depend on the presentation chosen for any i. Furthermore, the subscheme $Z_i(\mathscr{F})$ contains exactly the points $p \in X$ where $\mu(\mathscr{F}, p) > i$.*

Proof. See [21, §20] for basic facts on Fitting ideals. □

Lemma I.2.7 ([cf 23, Lemma 4])**.** *Let $\mathfrak{C}$ be a set of points in X and let $\mathscr{F}'$ be a subsheaf of $\mathscr{F}$. Suppose $\mathscr{F}'$ is w-basic in $\mathscr{F}$ at all points that are the generalization of a point in $\mathfrak{C}$, then $\mathscr{F}'$ is w-basic at all but finitely many points in $\mathfrak{C}$.*

Proof. We claim that if $\mathscr{F}'$ is not w-basic at $p \in \mathfrak{C}$, then p is the generic point of a component of $Z_i(\mathscr{F})$ for some i. Since there are only finitely many ideals $\mathscr{F}itt_i(\mathscr{F}/\mathscr{F}')$, and each $Z_i(\mathscr{F})$ has only finitely many components by the noetherian property, the conclusion follows. Let $p \in \mathfrak{C}$ and $\mu(\mathscr{F}/\mathscr{F}', p) = s$. It follows that $p \in Z_{s-1}(\mathscr{F}/\mathscr{F}')$ and $p \notin Z_s(\mathscr{F}/F')$. Suppose p is not the generic point of a component, then there exists a point $q \in Z_{s-1}(\mathscr{F}/\mathscr{F}')$ that is a proper generalization of p. By assumption $\mu(\mathscr{F}/\mathscr{F}', q) \leq \mu(\mathscr{F}, q) - w$. Since q is a generalization of p, it follows that $q \notin Z_s(\mathscr{F}/\mathscr{F}')$. We conclude that $\mu(\mathscr{F}/\mathscr{F}', q) = \mu(\mathscr{F}/\mathscr{F}', p) = s$. On the other hand $\mu(\mathscr{F}, q) \leq \mu(\mathscr{F}, p)$ since q is a generalization of p. It follows that $\mathscr{F}'$ is w-basic at p. □

For an integer $m \geq 0$, let $\mathfrak{C}_m := \{p \in X \mid \dim \mathscr{O}_p \leq m\}$ to be the collection of all points of X of codimension at most m.

Lemma I.2.8. *Let $s_1, \ldots, s_u$ be sections of a sheaf $\mathscr{F}$ on X in degrees $a_1 \leq \cdots \leq a_u$. Let $t \geq 1$ be an integer, and suppose $(s_1, \ldots, s_u)$ is $\min(u, m+t-\dim \mathscr{O}_p)$-basic at all $p \in \mathfrak{C}_m$. The subsheaf $(s_1, \ldots, s_u)$ is $\min(u, m+t+1-\dim \mathscr{O}_p)$-basic at all but finitely many $p \in \mathfrak{C}_m$.*

Proof. Fix $0 \leq i \leq m$ and set $\mathfrak{C} := \mathfrak{C}_i - \mathfrak{C}_{i-1}$ be the set of points in X of codimension exactly i. If q is a generalization of a point in $\mathfrak{C}$, then $\mathscr{F}'$ is $\min(u, m+t-\dim \mathscr{O}_q)$-basic at q and therefore is $\min(u, m+t+1-i)$-basic at q. By Lemma I.2.7, there are only finitely many points p in $\mathfrak{C}$ where $\mathscr{F}'$ is not $\min(u, m+t+1-\dim \mathscr{O}_p)$-basic at p. □

Theorem I.2.9. *Let $s_1, \ldots, s_u$ be sections of a coherent sheaf $\mathscr{F}$ on X in degrees $a_1 \leq \cdots \leq a_u$. If $t \geq 1$ is an integer such that $(s_1, \ldots, s_u)$ is $\min(u, m+t-\dim \mathscr{O}_p)$-basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$, then there are $r_j \in H^0(\mathscr{O}(a_j - a_1))$ for $2 \leq j \leq u$ such that $(s_2 + r_2 s_1, \ldots, s_u + r_u s_1)$ is $\min(u-1, m+t-\dim \mathscr{O}_p)$-basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$.*

Proof. By Lemma I.2.8, there are only finitely many points $p_1, \dots, p_v$ in $\mathfrak{C}_m$ where $(s_1, \dots, s_u)$ is not $\min(u, m+t+1-\dim \mathscr{O}_p)$-basic. We apply Lemma I.2.2 and find forms $r_j \in H^0(\mathscr{O}(a_j - a_1))$ for $2 \leq j \leq u$ such that $(s_2 + r_2 s_1, \dots, s_u + r_u s_1)$ is $\min(u-1, m+t-\dim \mathscr{O}_p)$-basic at $p_1, \dots, p_v$. At all points p in $\mathfrak{C}_m - \{p_1, \dots, p_v\}$, the subsheaf $(s_2 + r_2 s_1, \dots, s_u + r_u s_1)$ is also $\min(u-1, m+t-\dim \mathscr{O}_p)$-basic in $\mathscr{F}$ since $(s_1, \dots, s_u)$ is $\min(u, m+t+1-\dim \mathscr{O}_p)$-basic. □

Theorem I.2.10 (Existence of graded basic elements)**.** *Let $s_1, \dots, s_u$ be sections of a sheaf $\mathscr{F}$ on X in degrees $a_1 \leq \dots \leq a_u$. Suppose $t \geq 1$ is an integer such that $(s_1, \dots, s_u)$ is $\min(u, m+t-\dim \mathscr{O}_p)$-basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$. One of the following holds.*

1. *If $u \leq t$, then $s_1, \dots, s_u$ are basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$.*
2. *If $u > t$, then there are t sections $s'_{u-t+1}, \dots, s'_u$ of $\mathscr{F}$ in degrees $a_{u-t+1}, \dots, a_u$ that are basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$. Moreover, for each $u-t+1 \leq i \leq u$, the section s'_i can be chosen to be of the form $s_i + r_{i-1}s_{i-1} + \dots + r_1 s_1$ for some $r_j \in H^0(\mathscr{O}(a_i - a_j))$.*

Proof. If $u \leq t$, then $u \leq m + t - \dim \mathscr{O}_p$ for all $p \in \mathfrak{C}_m$ and the statement is trivial. If $u > t$, then we may apply Theorem I.2.9 $(u-t)$-times to obtain the desired t sections. □

Factorization of reductions

Definition I.2.11. A sheaf $\mathscr{F}$ on X satisfies *Serre's condition* (S_m) if depth $\mathscr{F}_p$ is at least $\min(m, \dim \mathscr{O}_p)$ for all $p \in X$. We say $\mathscr{F}$ satisfies (S_m^+) if $\mathscr{F}_p$ is free over $\mathscr{O}_p$ for all $p \in \mathfrak{C}_m$ in addition to $\mathscr{F}$ satisfying (S_m).

Note that if $\mathscr{F}_p$ has finite projective dimension over $\mathscr{O}_p$ for all $p \in \mathfrak{C}_m$, then (S_m^+) is implied by (S_m) by the Auslander-Buchsbaum formula. For example, this is the case for all sheaves $\mathscr{F}$ if X is regular in codimension m.

Other than the case of bundles, the following (S_m^+) sheaves are of interest to us. If X is regular in codimension one, then $\mathscr{E}$ satisfies (S_1^+) if and only if it is torsion-free. In this case, a rank one sheaf $\mathscr{E}$ satisfies (S_1^+) if and only if it is isomorphic to $\mathscr{I} \otimes \mathscr{L}$, where $\mathscr{I}$ is a nonzero ideal sheaf and $\mathscr{L}$ is a line bundle. If X is regular in codimension two, then $\mathscr{E}$ satisfies (S_2^+) if and only if $\mathscr{E}$ is reflexive, i.e. the natural map $\mathscr{E} \to \mathscr{E}^{**}$ is an isomorphism.

The following proposition relates the property of being (S_m^+) with the notion of basicness.

Proposition I.2.12. *Let $s_1, \dots, s_u$ be twisted sections of an (S_m^+) sheaf $\mathscr{F}$ of rank r. The sections $s_1, \dots, s_u$ are basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$ if and only if the quotient $\mathscr{F}/(s_1, \dots, s_u)$ has rank $(r-u)$ and satisfies (S_m^+).*

Proof. If $\mathscr{F}/(s_1, \dots, s_u)$ has rank $r-u$ and satisfies (S_m^+), then $\mathscr{F}/(s_1, \dots, s_u)$ locally generated by $r-u$ elements at each $p \in \mathfrak{C}_m$. Since $(s_1, \dots, s_u)$ is at most u-basic in $\mathscr{F}$ at any $p \in X$, it follows that $s_1, \dots, s_u$ are basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$.

Conversely, suppose $s_1, \dots, s_u$ are basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$. Since $s_1/L^{a_1}, \dots, s_u/L^{a_u}$ form part of a basis of the free module $\mathscr{F}_p$ over $\mathscr{O}_p$ for all $p \in \mathfrak{C}_m$, the corresponding map $\varphi : \mathscr{O}(\underline{a}) \xrightarrow{s_1,\dots,s_u} \mathscr{F}$ is injective and $\mathscr{F}/(s_1, \dots, s_u)$ is locally-free at all points $p \in \mathfrak{C}_m$. Since X is Cohen-Macaulay, all associated points of X are minimal, and thus φ is injective globally. An application of the depth lemma to the exact sequence

$$0 \to \mathscr{O}(\underline{a}) \xrightarrow{\varphi} \mathscr{F} \to \mathscr{F}/(s_1, \dots, s_u) \to 0$$

yields the conclusion that $\mathscr{F}/(s_1, \dots, s_u)$ satisfies (S_m) and hence (S_m^+). □

Corollary I.2.13. *Let $m \geq 0$ be an integer and let $\mathscr{F}$ be an (S_m^+) sheaf on X of rank $r > m$. If $\varphi : \mathscr{O}(\underline{a}) \xrightarrow{\varphi} \mathscr{F}$ is a surjective map for some $\underline{a} = (a_i)_{i=1}^u$, then there is a rank $(r-m)$ summand $\mathscr{L}$ of $\mathscr{O}(\underline{a})$, involving the smallest $(r-m)$ integers a_i, such that $\mathscr{L} \to \mathscr{F}$ is injective and $\mathscr{F}/\mathscr{L}$ has rank m and satisfies (S_m^+).*

Proof. Let $(s_1, \dots, s_u)$ be the sections of $\mathscr{F}$ in degrees $a_1 \leq \cdots \leq a_u$ corresponding to φ. Since $(s_1, \dots, s_u)$ is $\min(u, r)$-basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$, it is $\min(u, m+(r-m)-\dim \mathscr{O}_p)$-basic at all $p \in \mathfrak{C}_m$. By Theorem I.2.10, there is a rank $(r-m)$ summand $\mathscr{L}$ of $\mathscr{O}(\underline{a})$ mapping to $\mathscr{F}$, corresponding to $(r-m)$ sections that are basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$. The conclusion follows from Proposition I.2.12. □

The affine version of the above corollary is due to Bruns [11].

Definition I.2.14. An *m-reduction* of $\mathscr{F}$ is an injective map of the form $\varphi : \mathscr{O}(\underline{a}) \to \mathscr{F}$ for some $\underline{a}$ such that $\operatorname{coker} \varphi$ satisfies (S_m^+). The *rank* of the reduction is the rank of the map φ (over the generic fiber). The *corank* of the reduction is defined to be $\operatorname{rank} \operatorname{coker} \varphi$.

We say an m-reduction $\varphi : \mathscr{O}(\underline{a}) \to \mathscr{F}$ *factors through* an n-reduction $\psi : \mathscr{O}(\underline{b}) \to \mathscr{F}$ if there is an injective map $\iota : \mathscr{O}(\underline{b}) \to \mathscr{O}(\underline{a})$ such that $\psi = \varphi \circ \iota$ and $\operatorname{coker} \iota \cong \mathscr{O}(\underline{c})$ for some $\underline{c}$. In this case, the induced map $\operatorname{coker} \iota \to \operatorname{coker} \psi$ is an m-reduction whose cokernel is isomorphic to $\operatorname{coker} \varphi$ by an application of the snake lemma to the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}(\underline{b}) & \xrightarrow{\psi} & \mathscr{F} & \longrightarrow & \operatorname{coker} \psi & \longrightarrow & 0 \\ & & \Big\downarrow \iota & & \Big\| & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathscr{O}(\underline{a}) & \xrightarrow{\varphi} & \mathscr{F} & \longrightarrow & \operatorname{coker} \varphi & \longrightarrow & 0. \end{array}$$

In order for a sheaf $\mathscr{E}$ to admit an m-reduction, it must satisfy (S_m^+) itself. Corollary I.2.13 states that if $\mathscr{E}$ has rank $r \geq m$ and satisfies (S_m^+), then it admits an m-reduction of corank m. The next theorem gives us a sufficient criterion when an m-reduction factors through another n-reduction.

Theorem I.2.15 (Factorization theorem). *Let $\varphi : \mathscr{O}(\underline{a}) \to \mathscr{F}$ be an m-reduction of corank u. There is an n-reduction of $\mathscr{F}$ of corank v through which φ factors if the following hold:*

1. *$u \leq v$ and $m \leq n$,*
2. *$\mathscr{F}$ satisfies (S_n^+),*
3. *$\mu(\operatorname{coker}\varphi, p) \leq v - n + \dim \mathscr{O}_p$ for all $p \in X$ such that $m \leq \dim \mathscr{O}_p \leq n$.*

We may take the factor map $\iota : \mathscr{O}(\underline{b}) \to \mathscr{O}(\underline{a})$ to be a summand of $\mathscr{O}(\underline{a})$ consisting of the smallest $(\operatorname{rank}\mathscr{F} - v)$ integers a_i.

Proof. Set $\operatorname{rank}\mathscr{F} = r$ and let $s_1, \ldots, s_{r-u}$ be twisted sections of $\mathscr{F}$ corresponding to φ. By Proposition I.2.12, the sections $s_1, \ldots, s_{r-u}$ are basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_m$. Since $\mu(\operatorname{coker}\varphi, p) \leq v - n + \dim \mathscr{O}_p$ at all $p \in \mathfrak{C}_n - \mathfrak{C}_m$ and $\mathscr{F}$ satisfies (S_n^+), we see that $(s_1, \ldots, s_{r-u})$ is $\min(r-u, n+(r-v)-\dim\mathscr{O}_p)$-basic in $\mathscr{F}$ at these points. It follows that $(s_1, \ldots, s_{r-u})$ is $\min(r-u, n+(r-v)-\dim\mathscr{O}_p)$-basic in $\mathscr{F}$ at all $p \in \mathfrak{C}_n$. By Theorem I.2.10, we can find a summand $\iota : \mathscr{O}(\underline{b}) \to \mathscr{O}(\underline{a})$ consisting of $(r-v)$ smallest a_i such that $\varphi \circ \iota : \mathscr{O}(\underline{b}) \to \mathscr{F}$ is an n-reduction of $\mathscr{F}$ of corank v. □

In the following, we explore some geometric applications of the factorization theorem.

Lemma I.2.16. *A coherent sheaf $\mathscr{F}$ on $\mathbf{P}_k^n$ is isomorphic to a nonzero ideal sheaf up to twisting by $\mathscr{O}(l)$ if and only if $\mathscr{F}$ has rank one and satisfies (S_1^+).*

Proof. If $\mathscr{I}$ is a nonzero ideal sheaf of a subscheme Z, then $\operatorname{rank}\mathscr{I} = 1$. If Z contains a one dimensional component, then we may twisted $\mathscr{I}$ down by the equation of the corresponding hypersurface and assume that Z is empty or has codimension at least two. If Z is empty then $\mathscr{I} \cong \mathscr{O}(l)$ which clearly satisfies (S_1^+). If Z has codimension at least two, then $\mathscr{I}$ is locally-free in codimension one and the exact sequence

$$0 \to \mathscr{I} \to \mathscr{O} \to \mathscr{O}_Z \to 0$$

shows that $\mathscr{I}$ satisfies (S_1^+) by the depth lemma.

Conversely, suppose $\mathscr{F}$ has rank one and satisfies (S_1^+). The natural map $\mathscr{F} \to \mathscr{F}^{**}$ is an injection, where $\mathscr{F}^{**}$ is a rank one reflexive sheaf. Since $\operatorname{Pic}(\mathbf{P}_k^n) = \mathbb{Z}$, it follows that $\mathscr{F}^{**} \cong \mathscr{O}(l)$ for some l. It follows that $\mathscr{F}(-l)$ is isomorphic to an ideal sheaf. □

If a zero dimensional subscheme Z of $\mathbf{P}_k^2$ satisfies Cayley-Bacharach property with respect to the line bundle $\mathscr{O}(l)$ (see Definition I.1.11), then we write Z satisfies (CB_l) for brevity.

Proposition I.2.17. *With notations as above, if Z satisfies (CB_l) then Z satisfies (CB_{l-1}).*

Proof. Let $Z' \subset Z$ be a subscheme of colength one. Suppose $f \in H^0(\mathscr{O}(l-1))$ gives a divisor $V(f)$ that contains Z'. Let $l \in H^0(\mathscr{O}(1))$ cut out a hyperplane avoiding Z. It follows that the hypersurface cut out by $V(f \cdot l)$ contains Z' and thus contains Z. By the choice of l, we conclude that Z is contained in $V(f)$. □

Definition I.2.18. Let Z be a zero dimensional subscheme of $\mathbf{P}_k^2$. The *Cayley-Bacharach index* of Z, denoted by $CB(Z)$, is the largest integer l such that Z satisfies (CB_l).

The next theorem says that the Cayley-Bacharach index of points in $\mathbf{P}^2_k$, a geometric invariant, can be bounded below and above in terms of the degrees of the second syzygies, which are algebraic invariants.

Theorem I.2.19 (Bounds on the Cayley-Bacharach index)**.** *Let Z be a zero dimensional l.c.i. subscheme of $\mathbf{P}^2_k$. Let S be the polynomial ring of $\mathbf{P}^2_k$ and let*

$$0 \to \bigoplus_{i=1}^{u} S(-a_i) \to \bigoplus_{i=1}^{u+1} S(-b_i) \to I_Z \to 0 \qquad (*)$$

be a minimal graded free S-resolution of the homogeneous ideal I_Z. Suppose $a_1 \leq \cdots \leq a_u$, then $a_1 - 3 \leq CB(Z) \leq a_u - 3$.

Proof. We sheafify the minimal free S-resolutions to obtain a 1-reduction of $\bigoplus_{i=1}^{u+1} \mathscr{O}(-b_i)$. Since $\mu(\mathscr{I}_Z, p) \leq 2$ for all $p \in X$ by the assumption that Z is a l.c.i., it follows from Theorem I.2.15 that there is an extension of the form

$$0 \to \mathscr{O}(-a_1) \to \mathscr{E} \to \mathscr{I}_Z \to 0$$

where $\mathscr{E}$ is (S_2^+) of rank 2. Since $\dim X = 2$, it follows that $\mathscr{E}$ is locally-free and Z satisfies (CB_{a_1-3}) by Theorem I.1.12.

Conversely, suppose Z satisfies (CB_{l-3}) and thus $0 \to \mathscr{O}(-l) \to \mathscr{E} \to \mathscr{I}_Z \to 0$ is an extension where $\mathscr{E}$ is locally free of rank 2. It follows that we have an exact sequence

$$0 \to \bigoplus_{i=1}^{u} \mathscr{O}(-a_i) \xrightarrow{\varphi} \mathscr{O}(-l) \oplus \bigoplus_{i=1}^{u+1} \mathscr{O}(-b_i) \to \mathscr{E} \to 0.$$

If $l > a_u$ then the map $\bigoplus_{i=1}^{u} \mathscr{O}(-a_i) \to \mathscr{O}(-l)$ is zero. This would mean that φ drops rank on Z, a contradiction to the fact that φ drops rank nowhere since $\mathscr{E}$ is locally-free. □

As a consequence, we see that the Cayley-Bacharach index of an (a, b)-complete intersection in $\mathbf{P}^2_k$ is exactly $a + b - 3$. Let Z be a zero dimensional l.c.i. subscheme of $\mathbf{P}^2_k$ of degree 8. Then Z lies on at least two linearly independent cubics C_1 and C_2. If Z does not lie on any conic, then C_1 and C_2 cut out a complete intersection K. It follows that Z is residual to one point p in K. Since K satisfies (CB_3), it follows that every cubic containing Z contains the residual point p as well. This is the classical Cayley-Bacharach theorem.

The next theorem says that the Lazarsfeld-Rao procedure [67] of producing a curve from a bundle factors through a Serre correspondence (see Theorem I.1.14) if and only if the curve is a generic complete intersection.

Theorem I.2.20. *Let C be a pure codimension two curve in $\mathbf{P}^3_k$, and let*

$$e(C) := \sup\{l \mid H^1(\mathscr{O}_C(l)) \neq 0\}$$

be the index of specialty of C. There is an extension of the form

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{I}_C \to 0,$$

for some $\underline{a} = (a_i)_{i=1}^u$ where $a_i \leq e(C) + 4$, and $\mathscr{E}$ is locally-free of rank $u+1$ such that $H^2_*(\mathscr{E}) = 0$. The 1-reduction $\mathscr{O}(\underline{a}) \to \mathscr{E}$ factors through a 2-reduction of $\mathscr{E}$ of corank two if and only if C is a generic complete intersection.

Proof. The existence of the extension $0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \xrightarrow{p} \mathscr{I}_C \to 0$ is a result of Lazarsfeld-Rao [67, Lemma 1.1]. The map $\mathscr{O}(\underline{a}) \to \mathscr{E}$ a 1-reduction of $\mathscr{E}$. The pure codimension two subscheme C is a generic complete intersection if and only if $\mu(\mathscr{I}_C, p) \leq 2$ for all $p \in \mathfrak{C}_2$. If a Serre correspondence exits, then $\mathscr{I}_C$ is the quotient of a rank two reflexive sheaf and thus $\mu(\mathscr{I}_C, p) \leq 2$ for all $p \in \mathfrak{C}_2$. The converse follows from Theorem I.2.15. □

Theorem I.2.21. *Let V be a pure codimension two l.c.i. subscheme of $\mathbf{P}^n_k$. There exists a rank n bundle $\mathscr{E}$ and an exact sequence $0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{I}_V \to 0$ for some $\underline{a}$.*

Proof. Let Z be linked to V by an (s,t)-complete intersection K as in Definition I.1.1. Since V is of pure codimension two and Cohen-Macaulay, so is Z. Let $\mathscr{O}(\underline{b}) \xrightarrow{\varphi} \mathscr{I}_Z$ be a surjection. Since Z is Cohen-Macaulay of codimension two, it follows that $(\ker\varphi)_p$ is free for all $p \in \mathbf{P}^n_k$, i.e. $\ker\varphi$ is locally-free. Let $\mathscr{K}_\bullet$ be the Koszul complex of $\mathscr{O}_K$, and let $\mathscr{F}_\bullet : 0 \to \ker\varphi \to \mathscr{O}(\underline{b}) \to \mathscr{O}$ be a locally free resolution of $\mathscr{O}_Z$. The mapping cone of the natural map $\alpha : \mathscr{K}_\bullet \to \mathscr{F}_\bullet$ dualizes to a locally free resolution of $\mathscr{O}_V$ after shifting by $\mathscr{O}(-s-t)$ [see 83, Proposition 2.5]. We obtain an exact sequence of sheaves

$$0 \to \mathscr{O}(\underline{a})^*(-s-t) \to \mathscr{E}^*(-s-t) \oplus \mathscr{O}(-s) \oplus \mathscr{O}(-t) \to \mathscr{I}_V \to 0.$$

If $\operatorname{rank}\mathscr{E} + 2 < n$, then we may add trivial complexes of the form $0 \to \mathscr{O} \xrightarrow{\sim} \mathscr{O} \to 0 \to 0$. If $\operatorname{rank}\mathscr{E} + 2 > n$, then the conclusion follows from Theorem I.2.15 since $\mathscr{I}_V$ is locally generated by at most 2 elements. □

In particular, any smooth curve in $\mathbf{P}^3_k$ is the degeneracy locus of two sections of a rank 3 bundle on $\mathbf{P}^3_k$. In fact, we only need to require that the curve is pure codimension two Cohen-Macaulay, a generic complete intersection and locally an almost complete intersection.

I.3. Biliaison of Sheaves

In this section, we define the notion of biliaison of sheaves and extend the classical results reviewed in Section I.1 to the context of sheaves.

For readers who are interested in vector bundles on projective varieties, and who wish to get a gist of the ideas in this article without too much commutative algebra, it is advisable to replace all occurrences of "(S_m^+) sheaves" with "bundles".

The biliaison theory for the special case of rank two reflexive sheaves on $X = \mathbf{P}_k^3$ was established by Buraggina in [13] using the Serre correspondence and results from the linkage theory of curves. Our method provides a substantially simplified treatment as well as stronger theorems even in this special case.

Lattice structure

Definition I.3.1. Recall from Definition I.2.14 that an m-reduction of a sheaf $\mathscr{E}$ is an injective map of the form $\varphi : \mathscr{O}(\underline{a}) \to \mathscr{E}$ where $\operatorname{coker}\varphi$ satisfies (S_m^+). We define the *shape* of the reduction φ to be the sequence $\underline{a}$ sorted in **ascending** order.

If we consider the shapes of all m-reductions $\phi : \mathscr{O}(\underline{a}) \to \mathscr{E}$ of the same sheaf $\mathscr{E}$, we see that they are partially ordered in a natural way. We define this partial order more generally on the set of finite non-decreasing sequences of integers.

Definition I.3.2. Let $\underline{a}$ and $\underline{b}$ be two finite non-decreasing sequences of integers.

1. For an integer l, we define $\Sigma(\underline{a}, l)$ to be the number of entries of $\underline{a}$ that is $\leq l$.
   Note that the non-decreasing function $\Sigma(\underline{a}, -) : \mathbb{Z} \to \mathbb{N}$ determines the non-decreasing sequence $\underline{a}$.
2. We write $\underline{a} \leq \underline{b}$ if $\Sigma(\underline{a}, l) \leq \Sigma(\underline{b}, l)$ for all $l \in \mathbb{Z}$.
3. Let $\underline{a} \vee \underline{b}$ be the non-decreasing sequence $\underline{c}$ determined by the property that

$$\Sigma(\underline{c}, l) = \min(\Sigma(\underline{a}, l), \Sigma(\underline{b}, l)), \quad \forall l \in \mathbb{Z}.$$

4. Let $\underline{a} \wedge \underline{b}$ be the non-decreasing sequence $\underline{c}$ determined by the property that

$$\Sigma(\underline{c}, l) = \max(\Sigma(\underline{a}, l), \Sigma(\underline{b}, l)), \quad \forall l \in \mathbb{Z}.$$

Let $\mathfrak{S}$ denote the set of finite non-decreasing sequences of integers.
It is easy to see that the poset $(\mathfrak{S}, \leq)$ is a lattice with meet $\vee$ and join $\wedge$.

Example I.3.3. If $\underline{a} = (1, 3, 4)$ and $\underline{b} = (2, 2)$, then $\underline{a} \wedge \underline{b} = (1, 2, 4)$ and $\underline{a} \vee \underline{b} = (2, 3)$.

Here is an equivalent way to determine $\underline{a} \vee \underline{b}$ and $\underline{a} \wedge \underline{b}$ without writing down $\Sigma(\underline{a}, -)$ and $\Sigma(\underline{b}, -)$. We illustrate on the previous example. First append ∞ to the shorter sequence till the lengths match up: $\underline{a} = (1, 3, 4)$ and $\underline{b} = (2, 2, \infty)$. Then $\underline{a} \vee \underline{b}$ and $\underline{a} \wedge \underline{b}$ are given by the position-wise maximum and minimum, with ∞ interpreted as a non-entry.

Theorem I.3.4 (Semilattice theorem)**.** *For a fixed $m \geq 1$, the shapes of m-reductions of a given sheaf $\mathscr{E}$ is a subsemilattice of $\mathfrak{S}$. When $m = 1$, the shapes of 1-reductions of a given sheaf $\mathscr{E}$ is a sublattice of $\mathfrak{S}$.*

A *semilattice* is a partially ordered set (poset) with meet. A *subsemilattice* is a subposet inheriting the same meet from the ambient semilattice, analogously for a sublattice. The conclusions of Theorem I.3.4 follow immediately from Theorem I.3.6 and Theorem I.3.7, whose proofs will occupy the remainder of the subsection.

The next lemma is a generalization of [81, Lemma 3.6], where the quotients are required to have rank one and satisfy (S_1^+). Essentially we prove that m-reductions are open among the affine variety of morphisms.

Lemma I.3.5. *Let ϕ, ψ be m-reduction of $\mathscr{E}$ with shapes $(a_i)_{i=1}^u$ and $(b_i)_{i=1}^v$. Denote by $J \subseteq \{1, \ldots, \min(u, v)\}$ the subset of indices where $a_j = b_j$ for all $j \in J$. There are m-reductions ϕ', ψ' of $\mathscr{E}$ with shapes $(a_i)_{i=1}^u$ and $(b_i)_{i=1}^v$, such that if $s'_1, \ldots, s'_u$ and $t'_1, \ldots, t'_v$ are twisted sections of $\mathscr{E}$ corresponding to ϕ' and ψ' respectively, then $s'_j = t'_j$ for all $j \in J$.*

Proof. Let $s_1, \ldots, s_u$ be sections of $\mathscr{E}$ in degrees $a_1, \ldots, a_u$ corresponding to ϕ. Let ϵ be an arbitrary index. We claim that for a general choice of sections s'_ϵ of $\mathscr{E}$ in degree a_ϵ, the map $\phi' : \mathscr{O}(\underline{a}) \to \mathscr{E}$ given by $s_1, \ldots, s'_\epsilon, \ldots, s_u$ is an m-reduction of $\mathscr{E}$. The conclusion of the lemma then follows by replacing the sections s_j and t_j by a common general section of $\mathscr{E}$ of degree $a_j = b_j$ for every $j \in J$.

Let V be the finite dimensional k-vector space $H^0(\mathscr{E}(a_\epsilon))$, and let $\mathbf{A} = \operatorname{Spec} \operatorname{Sym} V^*$ be the affine space parametrizing these sections. Consider the scheme $X' = X \times_k \mathbf{A}$ and the pullback sheaves $\mathscr{E}'$ of $\mathscr{E}$ as well as $\mathscr{O}(\underline{a})'$ of $\mathscr{O}(\underline{a})$ from X. There is a map $\Phi : \mathscr{O}(\underline{a})' \to \mathscr{E}'$ defined by the following property. Suppose p is a k-point of $\mathbf{A}$ corresponding to the sections s'_ϵ in V, then the fiber $\Phi_p : \mathscr{O}(\underline{a}) \to \mathscr{E}$ at p is given by the sections $s_1, \ldots, s'_\epsilon, \ldots, s_u$. Let Z denote the subscheme of X' cut out by the Fitting ideal $\mathscr{F}itt_{r-u}(\operatorname{coker} \Phi)$ (see Definition I.2.5). The subscheme Z contains points in X' where $\operatorname{coker} \Phi$ cannot be generated by $\leq r - u$ elements locally. Since X' is proper over $\mathbf{A}$, semicontinuity of fiber dimensions over the target implies that the points $p \in A$ such that Z_p has dimension $< \dim X - m$ are open in A. Since X is a projective variety over a field, dimension and codimension are complementary by Noether normalization. It follows that the set of points $p \in \mathbf{A}$ such that $\operatorname{codim}(Z_p, X \otimes_k k(p)) > m$ form an open set U. If $p \in \mathbf{A}$ is such a point, then $\operatorname{coker} \Phi_p$ has rank r on $X \otimes_k k(p)$ and therefore we have an exact sequence

$$0 \to \bigoplus_{i=1}^{u} \mathscr{O}(-a_i) \otimes_k k(p) \to \mathscr{E} \otimes_k k(p) \to \operatorname{coker} \Phi_p \to 0.$$

Since $\operatorname{coker}\Phi_p$ is locally-free in codimension m, it satisfies (S_m^+) by an application of the depth lemma to the above sequence. Finally, the existence of ϕ gives us a k-point of U. Since k is infinite, it follows that the k-points of U are dense in V. □

Theorem I.3.6. *For a fixed $m \geq 1$, if there are m-reductions ϕ, ψ of $\mathscr{E}$ with shapes $(a_i)_{i=1}^u$ and $(b_i)_{i=1}^v$, then there is an m-reduction ξ of $\mathscr{E}$ with shape $(a_i)_{i=1}^u \wedge (b_i)_{i=1}^v$.*

Proof. By induction, we may reduce to proving the following statement. Set ϵ to be the largest integer in the interval $[0, \min(u, v)]$ such that $a_i = b_i$ for all $1 \leq i \leq \epsilon$. If $\epsilon = \min(u, v)$, then $(a_i)_{i=1}^u$ is a subsequence of $(b_i)_{i=1}^v$ or vice versa, and their join is just the longer sequence among the two. The statement of the theorem is true in this case. Assume without loss of generality that $\epsilon < \min(u, v)$ and $b_{\epsilon+1} > a_{\epsilon+1}$. We claim that there exists an m-reduction of $\mathscr{E}$ with shape $a_1, \ldots, a_{\epsilon+1}, b_{\epsilon+2}, \ldots, b_v$.

Step 1: Using Lemma I.3.5, we may assume ϕ and ψ are given by twisted sections $s_1, \ldots, s_u$ and $t_1, \ldots, t_v$ respectively, such that $s_j = t_j$ for all $1 \leq j \leq \epsilon$.

Step 2: Let Y be any subvariety of X of codimension $< m$, we claim that $(\operatorname{coker}\psi)_Y$ is torsion-free on Y. Since $\operatorname{coker}\psi$ satisfies (S_m^+), we see that $(\operatorname{coker}\psi)_Y$ is locally-free in codimension one on Y. By Krull's principal ideal theorem, if a module M has a zerodivisor r, then any minimal prime P above (r) has height one. In particular, the image of r in the localization would remain a zerodivisor on M_P. Since $(\operatorname{coker}\psi)_Y$ is locally-free in codimension one on Y, we conclude that $(\operatorname{coker}\psi)_Y$ must be torsion-free on Y.

Step 3: We claim that $t_1, \ldots, t_v, s_{\epsilon+1}$ are basic in $\mathscr{E}$ at all points of codimension $\leq m-1$. Suppose not, let $y \in X$ be a point of codimension $\leq m-1$ and let $Y = \overline{\{y\}}$ be the corresponding subvariety. If $t_1, \ldots, t_v, s_{\epsilon+1}$ are not basic in $\mathscr{E}$ at y, then the image of the corresponding map $\psi'_Y : \bigoplus_{i=1}^v \mathscr{O}_Y(-b_i) \oplus \mathscr{O}_Y(-a_{\epsilon+1}) \to \mathscr{E}_Y$ has rank v on Y, the same rank as $\operatorname{im}\psi_Y$. We obtain the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^v \mathscr{O}_Y(-b_i) & \xrightarrow{\psi_Y} & \mathscr{E}_Y & \longrightarrow & (\operatorname{coker}\psi)_Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha} & & \| & & \downarrow{\scriptstyle\beta} & & \\
0 & \longrightarrow & \operatorname{im}\psi'_Y & \longrightarrow & \mathscr{E}_Y & \longrightarrow & \operatorname{coker}\psi'_Y & \longrightarrow & 0.
\end{array}$$

The upper complex is exact because it is exact at the generic point y of Y and $\bigoplus_{i=1}^v \mathscr{O}_Y(-b_i)$ is torsion-free. The snake lemma implies that $\operatorname{coker}\alpha \cong \ker\beta$, which vanishes at the generic point y of Y since $\operatorname{rank}\operatorname{im}\psi'_Y = v$, therefore $\ker\beta$ is torsion on Y. Since $(\operatorname{coker}\psi)_Y$ is torsion-free by the above step, we conclude that $\ker\beta = 0$. This means that ψ'_Y factors through ψ_Y. However, any map from $\mathscr{O}_Y(-a_{\epsilon+1})$ to $\mathscr{O}_Y(-b_i)$ is zero for $i > \epsilon$ since X is integral and $\mathscr{O}(1)$ is very ample. It follows that $\mathscr{O}_Y(-a_{\epsilon+1}) \to \mathscr{E}_Y$ factors through $\bigoplus_{i=1}^{\epsilon} \mathscr{O}_Y(-b_i) \to \mathscr{E}_Y$. This means that $t_1, \ldots, t_\epsilon, s_{\epsilon+1}$ are not basic at y, which is a contradiction since $t_j = s_j$ for all $1 \leq j \leq \epsilon$ and $s_1, \ldots, s_{\epsilon+1}$ are basic at y.

Step 4: Let Z be the subscheme defined by the Fitting ideal $\mathscr{F}itt_{r-1}(\mathscr{E}/(t_1, \ldots, t_v, s_{\epsilon+1}))$, where $r = \operatorname{rank}\operatorname{coker}\psi$. The subscheme Z contains all points in X where $\mathscr{E}/(t_1, \ldots, t_v, s_{\epsilon+1})$ cannot be generated by $\leq r-1$ elements. Since $\mathscr{E}$ is locally-free in codimension m, we see that Z contains no point in X of codimension $\leq m-1$ by the previous step. Therefore Z contains at most finitely many points in X of codimension m. Let B denote this finite set of points of codimension m in X where $t_1, \ldots, t_v, s_{\epsilon+1}$ are not basic in $\mathscr{E}$.

The idea is to fix the basicness of $s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+2}, \ldots, t_v$ at one point in B at a time by modifying a section t_i to $t_i + r_i t_{\epsilon+1}$ for some suitable $r_i \in H^0(\mathscr{O}(b_i - b_{\epsilon+1}))$, without worsening the basicness at the remaining points in B.

At each point $x \in B$, if $s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+2}, \ldots, t_v$ are basic in $\mathscr{E}$ then we do nothing. If not, we can find t_i for $\epsilon + 2 \leq i \leq v$ such that $s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+2}, \ldots, t_i$ have the same basicness in $\mathscr{E}$ at x as $s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+2}, \ldots, t_{i-1}$. Since $\mathscr{O}(1)$ is very ample, there exists a form $r_i \in H^0(\mathscr{O}(b_i - b_{\epsilon+1}))$ that does not vanish at x. Let $\lambda \in k$ be an undetermined nonzero scalar, then $s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+2}, \ldots, t'_i, \ldots, t_v$ are basic in $\mathscr{E}$ at x, where $t'_i = t_i + \lambda r_i t_{\epsilon+1}$. By Lemma I.2.3, for all but finitely many choices of λ the sections $s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+2}, \ldots, t'_i, \ldots, t_v$ maintain the same amount of basicness as $s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+2}, \ldots, t_v$ at the remaining points in B. We choose such a nonzero λ and go to the next point in B with the modified sections $s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+2}, \ldots, t'_i, \ldots, t_v$ as input, and carry out the same procedure. Eventually, we arrive at sections $s_1, \ldots, s_{\epsilon+1}, t'_{\epsilon+2}, \ldots, t'_v$ that are basic in $\mathscr{E}$ at all points in B, where $t'_i = t_i + r_i t_{\epsilon+1}$ for some $r_i \in H^0(\mathscr{O}(b_i - b_{\epsilon+1}))$. The sections $s_1, \ldots, s_{\epsilon+1}, t'_{\epsilon+2}, \ldots, t'_v$ are basic in $\mathscr{E}$ at all points of codimension $\leq m$ outside of B since $t_1, \ldots, t_v, s_{\epsilon+1}$ are basic in $\mathscr{E}$ at these points. It follows that the map $\xi : \bigoplus_{i=1}^{\epsilon+1} \mathscr{O}(-a_i) \oplus \bigoplus_{i=\epsilon+2}^{v} \mathscr{O}(-b_i) \to \mathscr{E}$ corresponding to $s_1, \ldots, s_{\epsilon+1}, t'_{\epsilon+2}, \ldots, t'_v$ is an m-reduction of $\mathscr{E}$. □

Theorem I.3.6 is a generalization of [3, Lemma 2.1]. The above proof for (S_m^+) sheaves is more subtle. At its core, Theorem I.3.6 is about the codimension of the ideal of certain minors of matrix extensions. Note that this procedure gives us a way to construct new bundles from old ones.

The next theorem is similar in spirit with Theorem I.3.6, but the proof requires a slightly different approach. We include the proof here for the sake of completeness.

Theorem I.3.7. *If there are 1-reductions ϕ, ψ of $\mathscr{E}$ with shapes $(a_i)_{i=1}^u$ and $(b_i)_{i=1}^v$, then there is a 1-reduction ξ of $\mathscr{E}$ with shape $(a_i)_{i=1}^u \vee (b_i)_{i=1}^v$.*

Proof. Without loss of generality, assume that $u \leq v$. By the remark below Example I.3.3, we see that $(a_i)_{1=1}^u \vee (b_i)_{i=1}^v = (a_i)_{1=1}^u \vee (b_i)_{i=1}^u$. Certainly $\psi' : \bigoplus_{i=1}^v \mathscr{O}(-a_i) \to \mathscr{E}$ is an m-reduction of $\mathscr{E}$ if ψ is. We thus reduce to the case where $u = v$.

Let $D(\phi, \psi)$ denote the number of indicies where the shapes of ϕ and ψ differ. We prove the assertion by induction on $D(\phi, \psi)$. When $D(\phi, \psi) = 0$ there is nothing to prove. Suppose $D(\phi, \psi) > 0$. Let $s_1, \ldots, s_u$ and $t_1, \ldots, t_u$ be sections of $\mathscr{E}$ corresponding to ϕ and ψ respectively. Let $J \subseteq \{1, \ldots, u\}$ be the subset of indicies j where $a_j = b_j$. By Lemma I.3.5, we may assume that $s_j = t_j$ for all $j \in J$. We claim that there is an index $\epsilon \in \{1, \ldots, u\} - J$ where $s_1, \ldots, s_u, t_\epsilon$ are basic in $\mathscr{E}$ at the generic point η of X. Suppose not, then every map $\mathscr{O}(\underline{a}) \oplus \mathscr{O}(-b_i) \to \mathscr{E}$ factors through $\phi : \mathscr{O}(\underline{a}) \to \mathscr{E}$ by the same argument in step 3 of Theorem I.3.7. This would mean that $\psi : \mathscr{O}(\underline{b}) \to \mathscr{E}$ factors through

$\phi : \mathscr{O}(\underline{a}) \to \mathscr{E}$. But the factor map $\mathscr{O}(\underline{b}) \to \mathscr{O}(\underline{a})$ must drop rank along the determinant hypersurface since $\underline{a} \neq \underline{b}$, and thus so must ψ. This is a contradiction to the fact that ψ does not drop rank in codimension one. Thus we find ϵ such that $s_1, \ldots, s_u, t_\epsilon$ are basic in $\mathscr{E}$ at the generic point. We may assume without loss of generality that $b_\epsilon > a_\epsilon$, otherwise we reverse the role of ϕ and ψ. The same argument in step 4 of Theorem I.3.6 shows that $s_1, \ldots, s_u, t_\epsilon$ are basic in $\mathscr{E}$ at all but finitely many codimension one points in X, and thus so are $s_1, \ldots, \hat{s}_\epsilon, \ldots, s_u, t_\epsilon$. Carrying out the same procedure in step 4 of Theorem I.3.6, we can find a suitable $r \in H^0(\mathscr{O}(b_\epsilon - a_\epsilon))$ such that $s_1, \ldots, \hat{s}_\epsilon, \ldots, s_u, t'_\epsilon$ are basic at all points in X of codimension ≤ 1, where $t'_\epsilon = t_\epsilon + r \cdot s_\epsilon$. The corresponding map $\xi : \mathscr{O}(-b_\epsilon) \oplus \bigoplus_{i \neq \epsilon} \mathscr{O}(-a_i) \to \mathscr{E}$ is thus a 1-reduction. Since $D(\phi, \xi) < D(\phi, \psi)$, by induction we find a 1-reduction η of $\mathscr{E}$ with shape $\underline{a} \vee \underline{c} = \underline{a} \vee \underline{b}$, where $\underline{c}$ is the shape of ξ. □

Example I.3.8. Continuing Example I.3.3. Suppose there are extensions

$$0 \to \mathscr{O}(-1) \oplus \mathscr{O}(-3) \oplus \mathscr{O}(-4) \to \mathscr{E} \to \mathscr{F} \to 0$$

$$0 \to \mathscr{O}(-2) \oplus \mathscr{O}(-2) \to \mathscr{E} \to \mathscr{F}' \to 0$$

where $\mathscr{F}, \mathscr{F}'$ are locally-free. By Theorem I.3.6, there is an extension of the form

$$0 \to \mathscr{O}(-1) \oplus \mathscr{O}(-2) \oplus \mathscr{O}(-4) \to \mathscr{E} \to \mathscr{F}'' \to 0$$

where $\mathscr{F}''$ is locally-free. By Theorem I.3.7, there is an extension of the form

$$0 \to \mathscr{O}(-2) \oplus \mathscr{O}(-3) \to \mathscr{E} \to \mathscr{F}''' \to 0$$

where $\mathscr{F}'''$ is (S_1^+). We do not know if we can always make $\mathscr{F}'''$ locally-free.

The weak structure theorem

In this subsection, we define the biliaison equivalence of sheaves. We prove a weak version of the structure theorem for a biliaison class, which says that (S_m^+) sheaves in a biliaison class can be obtained from one another using rigid deformations and other basic moves.

Definition I.3.9. If there is an extension of the form

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{F} \to 0,$$

then we say $\mathscr{F}$ is a *descendant* of $\mathscr{E}$ and $\mathscr{E}$ is an *ancestor* of $\mathscr{F}$. Let $\mathfrak{D}(\mathscr{E})$ denote the collection of all descendants of $\mathscr{E}$, and let $\mathfrak{D}_m(\mathscr{E})$ denote the collection of all descendants of $\mathscr{E}$ that satisfy (S_m^+). Two sheaves $\mathscr{F}$ and $\mathscr{G}$ are *related*, written $\mathscr{F} \sim \mathscr{G}$, if they share a common ancestor. The equivalence relation among sheaves on X generated by $\sim$ is called *biliaison equivalence*.

Note that if $\mathscr{F} \in \mathfrak{D}_m(\mathscr{E})$, then $\mathscr{E} \oplus \mathscr{O}(\underline{a})$ satisfies (S_m^+) for some $\underline{a}$. By the characterization of depth using vanishing of local cohomologies, we have $\operatorname{depth} \mathscr{E}_x = \operatorname{depth}(\mathscr{E} \oplus \mathscr{O}(\underline{a}))_x$ for any $x \in X$. We see that $\mathscr{E}$ satisfies (S_m^+) if and only if $\mathscr{E} \oplus \mathscr{O}(\underline{a})$ satisfies (S_m^+) for some $\underline{a}$. Therefore a sufficient and necessary condition for $\mathfrak{D}_m(\mathscr{E})$ to be nonempty is that $\mathscr{E}$ satisfies (S_m^+).

Definition I.3.9 is motivated by the following well-known theorem of Rao [86], strengthened in [81] and [52].

Theorem (Rao-Nollet-Hartshorne)**.** *Suppose $H^1_*(\mathscr{O}_X) = 0$. Two pure codimension two subschemes Y, Z of X are evenly linked if and only if $\mathscr{I}_Y, \mathscr{I}_Z(\delta) \in \mathfrak{D}_1(\mathscr{F})$ for a sheaf $\mathscr{F}$ and an integer δ. If X is Gorenstein in codimension two, then $\mathscr{F}$ can be chosen to be reflexive. If X is regular, then Y is Cohen-Macaulay if and only if Z is Cohen-Macaulay if and only if $\mathscr{F}$ can be chosen to be locally-free.*

In order to discuss the structure theorem for a biliaison class of sheaves, we need to generalize the notions of Serre correspondence (cf. Theorem I.1.14) and elementary biliaison (cf. Definition I.1.8).

Definition I.3.10. An (S_m^+)-*Serre correspondence* is an m-reduction of a sheaf $\mathscr{E}$ of the form $\varphi : \mathscr{O}(-a) \to \mathscr{E}$ for some integer a.

An *elementary (S_m^+)-biliaison from $\mathscr{F}$ to $\mathscr{G}$* is a pair of (S_m^+)-Serre correspondences $\phi : \mathscr{O}(-a) \to \mathscr{F}$ and $\psi : \mathscr{O}(-b) \to \mathscr{G}$ where $\operatorname{coker} \phi \cong \operatorname{coker} \psi$. The *height* of the elementary biliaison (ϕ, ψ) is the integer $a - b$. The elementary (S_m^+)-biliaison is *increasing* if the height is positive, and *decreasing* otherwise.

Let $\mathscr{E}$ be a sheaf on X, let T be a rational variety and let $p : X \times_k T \to X$ be the natural projection. If for some $\underline{a}$ there is a map $\Phi_T : p^*\mathscr{O}(\underline{a}) \to p^*\mathscr{E}$ that is fiber-wise injective over T, then we call $\operatorname{coker} \Phi_T$ a *rigid family* of sheaves on X.

If there are extensions $0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{F} \to 0$ and $0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{G} \to 0$ for the same $\underline{a}$ and sheaf $\mathscr{E}$, then $\mathscr{F}$ and $\mathscr{G}$ belong in a rigid family by the proof of Lemma I.3.5. The converse is true trivially. In particular, if there is an elementary biliaison of height zero from $\mathscr{F}$ to $\mathscr{G}$, then $\mathscr{F}$ and $\mathscr{G}$ belong in a rigid family.

Example I.3.11. There is an elementary (S_1^+)-biliaison between two rank two reflexive sheaves $\mathscr{E}$ and $\mathscr{E}'$ in $\mathbf{P}^3_k$ if and only if they correspond to the same curve C in $\mathbf{P}^3_k$ as in Theorem I.1.14.

We briefly explain how elementary (S_m^+)-biliaisons of sheaves generalize elementary biliaisons of codimension two subvarieties (see Definition I.1.8). Let $\mathscr{I}_Y$ and $\mathscr{I}_Z$ be ideal sheaves of pure codimension two subschemes Y and Z of X. If there is an elementary (S_0^+)-biliaison between $\mathscr{I}_Y$ and $\mathscr{I}_Z(\delta)$, then we have exact sequences

$$0 \to \mathscr{O}(-a) \to \mathscr{I}_Y \to \mathscr{I}_{Y/K} \to 0$$

$$0 \to \mathscr{O}(-a+\delta) \to \mathscr{I}_Z(\delta) \to \mathscr{I}_{Z/K}(\delta) \to 0,$$

where K is a hypersurface in the linear system $|\mathscr{O}(a)|$ containing Y, Z such that $\mathscr{I}_{Y/K} \cong \mathscr{I}_{Z/K}(\delta)$ [52, Prop 3.5]. Conversely, if there is a hypersurface K in $|\mathscr{O}(a)|$ containing Y, Z such that $\mathscr{I}_{Y/K} \cong \mathscr{I}_{Z/K}(\delta)$, the maps $\mathscr{O}(-a) \to \mathscr{I}_Y$ and $\mathscr{O}(-a+\delta) \to \mathscr{I}_Z(\delta)$ form an elementary (S_0^+)-biliaison of height δ from $\mathscr{I}_Y$ to $\mathscr{I}_Z(\delta)$.

The main result of this section is that (S_m^+) sheaves in a biliaison class can be obtained from one another using finitely many elementary biliaisons, rigid deformations and at most one m-reduction.

Theorem I.3.12 (Weak structure theorem). *For $m \geq 1$, if $\mathscr{F}$ and $\mathscr{G}$ are (S_m^+) sheaves in the same biliaison class, then there are (S_m^+) sheaves $\mathscr{F} = \mathscr{F}_0, \ldots, \mathscr{F}_l = \mathscr{G}$, such that $\mathscr{F}_i$ and $\mathscr{F}_{i+1}$ are related in one of the following ways:*

(a) there is an elementary (S_{m-1}^+)-biliaison from $\mathscr{F}_i$ to $\mathscr{F}_{i+1}$,

(b) $\mathscr{F}_i$ and $\mathscr{F}_{i+1}$ belong in a rigid family,

(c) there is an m-reduction $\phi : \mathscr{O}(\underline{a}) \to \mathscr{F}_{i+1}$ with $\operatorname{coker}\phi \cong \mathscr{F}_i$ or vice versa.

We need (c) at most once. If $\operatorname{rank}\mathscr{F} = \operatorname{rank}\mathscr{G}$, then we do not need (c).

Proof. Since biliaison equivalence is generated by the relation $\sim$, it is enough to prove the assertion when $\mathscr{F}$ and $\mathscr{G}$ have the same ancestor. By definition, there are extensions

$$0 \to \mathscr{O}(\underline{a}) \xrightarrow{\phi} \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{F} \to 0$$

$$0 \to \mathscr{O}(\underline{a}') \xrightarrow{\psi} \mathscr{E} \oplus \mathscr{O}(\underline{b}') \to \mathscr{G} \to 0.$$

We may consider two m-reductions of the same sheaf $\mathscr{E}' := \mathscr{E} \oplus \mathscr{O}(\underline{b}) \oplus \mathscr{O}(\underline{b}')$ given by

$$\phi \oplus \mathrm{Id} : \mathscr{O}(\underline{a}) \oplus \mathscr{O}(\underline{b}') \to \mathscr{E}'$$

$$\psi \oplus \mathrm{Id} : \mathscr{O}(\underline{a}') \oplus \mathscr{O}(\underline{b}) \to \mathscr{E}'.$$

In doing so, we reduce to the following: the cokernels of two m-reductions $\phi : \mathscr{O}(\underline{a}) \to \mathscr{E}$ and $\psi : \mathscr{O}(\underline{b}) \to \mathscr{E}$ of the same sheaf $\mathscr{E}$ are related by finitely many steps in the manner $(a)-(c)$.

Let $\underline{a} = (a_i)_{i=1}^u$ and $\underline{b} = (b_i)_{i=1}^v$, and let $s_1, \ldots, s_u$ and $t_1, \ldots, t_v$ be twisted sections of $\mathscr{E}$ corresponding to ϕ and ψ respectively. We proceed by induction on $D(\phi, \psi)$, the number of indices where $(a_i)_{i=1}^u$ and $(b_i)_{i=1}^v$ differ, including those i where only one of a_i, b_i is defined. If $D(\phi, \psi) = 0$, then $\operatorname{coker}\phi$ and $\operatorname{coker}\psi$ are related in manner (b) by the proof of Lemma I.3.5.

Suppose $D(\phi, \psi) > 0$. Let ϵ be the largest integer in $[0, \min(u, v)]$ where $a_i = b_i$ for all $1 \leq i \leq \epsilon$. By Lemma I.3.5 again, we may replace ϕ and ψ in manner (b) and assume that $s_i = t_i$ for $1 \leq i \leq \epsilon$. If $\epsilon = \min(u, v)$, then $(a_i)_{i=1}^u$ is a subsequence of $(b_i)_{i=1}^v$ (or vice versa), and we see that $\operatorname{coker}\phi$ and $\operatorname{coker}\psi$ are related in manner (c).

We now discuss the case where $\epsilon < \min(u, v)$. Interchanging ϕ and ψ if necessary, we may assume $a_{\epsilon+1} < b_{\epsilon+1}$. By the proof of Theorem I.3.7, there is an m-reduction $\xi : \mathscr{O}(\underline{c}) \to \mathscr{E}$ corresponding to twisted sections $s_1, \ldots, s_{\epsilon+1}, t'_{\epsilon+2}, \ldots, t'_v$, where $t'_i = t_i + r_i \cdot t_{\epsilon+1}$ for some $r_i \in H^0(\mathscr{O}(-b_i - b_{\epsilon+1}))$. Since $D(\phi, \xi)$ is smaller than $D(\phi, \psi)$, by the induction hypothesis the sheaves $\operatorname{coker}\phi$ and $\operatorname{coker}\xi$ are related by finitely many steps of manner $(a) - (c)$.

To finish the proof, we show that $\operatorname{coker}\psi$ and $\operatorname{coker}\xi$ are related in manner (a). In step (3) of the proof of Theorem I.3.7, we showed that $t_1, \ldots, t_v, s_{\epsilon+1}$ are basic in $\mathscr{E}$ at all points of codimension $\le m-1$. There are (S^+_{m-1})-Serre correspondences

$$u : \mathscr{O}(-a_{\epsilon+1}) \xrightarrow{s_{\epsilon+1}} \operatorname{coker}\psi, \quad v : \mathscr{O}(-b_{\epsilon+1}) \xrightarrow{t_{\epsilon+1}} \operatorname{coker}\xi$$

such that

$$\operatorname{coker} u = \frac{\mathscr{E}}{(t_1, \ldots, t_v, s_{\epsilon+1})} \cong \frac{\mathscr{E}}{(s_1, \ldots, s_{\epsilon+1}, t_{\epsilon+1}, t'_{\epsilon+2}, \ldots, t'_v)} = \operatorname{coker} v. \qquad \square$$

The converse of Theorem I.3.12 is obviously true, i.e. if two (S^+_m) sheaves are related by finitely many steps of (a) - (c), then they belong to the same biliaison class.

There is a dual notion of elementary biliaisons, see for example [13, Definition 4.7]. A word of caution that this dual notion **does not generalize** elementary biliaisons of subvarieties. We say there is a *dual elementary* (S^+_m)*-biliaison* from $\mathscr{F}$ to $\mathscr{G}$ if there is a pair of (S^+_m)-Serre correspondences $\phi : \mathscr{O}(-a) \to \mathscr{E}$ and $\psi : \mathscr{O}(-b) \to \mathscr{E}$ for some sheaf $\mathscr{E}$ and integers a, b such that $\operatorname{coker}\phi \cong \mathscr{F}$ and $\operatorname{coker}\psi \cong \mathscr{G}$. We remark that Theorem I.3.12 holds trivially with dual elementary (S^+_m)-biliaisons instead, due to the fact that an m-reduction remains an m-reduction when we restrict to a summand. Our results in Theorem I.3.27 give stronger statements than those in [13] even for the special case of rank two reflexive sheaves in $X = \mathbf{P}^3_k$.

If we restrict to the speical case of rank one (S^+_1) sheaves on $\mathbf{P}^n_k$, then Theorem I.3.12 recovers a weak version of the structure theorem for even linkage classes of codimension two subvarieties (cf. Theorem I.1.7). In the next subsection, we will prove a stronger structure theorem under an additional assumption on X which we now define.

Definition I.3.13. The following are defined relative to the very ample line bundle $\mathscr{O}(1)$.

1. We say a sheaf $\mathscr{F}$ is *primitive* if $\operatorname{Ext}^1(\mathscr{F}, \mathscr{O}(l)) = 0$ for all $l \in \mathbb{Z}$.
2. We say X is *primitive* if $\mathscr{O}_X$ is primitive.
3. Two sheaves $\mathscr{F}$ and $\mathscr{F}'$ are *stably equivalent* if $\mathscr{F} \oplus \mathscr{O}(\underline{a}) \cong \mathscr{F}' \oplus \mathscr{O}(\underline{b})$ for some $\underline{a}, \underline{b}$.

If $H^1(\mathscr{O}_X) = 0$, then X is primitive relative to any large enough multiple of an ample line bundle by Serre vanishing and Serre duality. If X is subcanonical, i.e. $\omega_X \cong \mathscr{O}(l)$ for some integer l, then a sheaf $\mathscr{F}$ is primitive if and only if $H^{n-1}_*(\mathscr{F}) = 0$ by Serre duality, where $n = \dim X$.

Note that X is primitive if and only if $H^1_*(\mathscr{O}_X) = 0$. Under this assumption, biliaison equivalence is also called *psi-equivalence* in [52], and is closely related to stable equivalence. We recall some useful facts from the same paper.

Proposition I.3.14. *Suppose X is primitive.*

1. *If $\mathscr{F}$ is a descendant of $\mathscr{E}$ and $\mathscr{G}$ is a descendant of $\mathscr{F}$, then $\mathscr{G}$ is a descendant of $\mathscr{E}$.*
2. *If $\mathscr{F}$ and $\mathscr{G}$ have a common descendant, then $\mathscr{F}$ and $\mathscr{G}$ have a common ancestor.*
3. *The relation $\sim$ is an equivalence relation, thus coincides with biliaison equivalence.*
4. *If $\mathscr{E}$ is primitive and shares a common descendant with $\mathscr{F}$, then $\mathscr{F}$ is a descendant of $\mathscr{E}$. Thus all sheaves in the biliaison class of a primitive sheaf $\mathscr{E}$ are descendants of $\mathscr{E}$.*
5. *Two primitive sheaves in the same biliaison class are stably equivalent.*
6. *If $\mathscr{E}$ is primitive and $\mathscr{F}$ is a sheaf in the biliaison class of $\mathscr{E}$ that satisfy (S_m^+), then $\mathscr{E}$ also satisfies (S_m^+).*
7. *If X is Gorenstein in codimension one (G_1) and $\mathscr{F}$ satisfies (S_1^+), then $\mathscr{F}$ is the descendant of a primitive sheaf. In particular, biliaison classes that contain an (S_1^+) sheaf are in bijection with the stable equivalence classes of primitive (S_1^+) sheaves.*

Proof. 1. [52, Lemma 2.4].

2. [52, Lemma 2.5].
3. The relation $\sim$ is evidently reflexive and symmetric. (1) and (2) show that $\sim$ is transitive.
4. Given $0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{G} \to 0$ and $0 \to \mathscr{O}(\underline{a}') \to \mathscr{F} \oplus \mathscr{O}(\underline{b}') \to \mathscr{G} \to 0$, the map $\mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{G}$ lifts to a map $\mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{F} \oplus \mathscr{O}(\underline{b}')$ since $\operatorname{Ext}^1(\mathscr{E} \oplus \mathscr{O}(\underline{b}), \mathscr{O}(\underline{a}')) = 0$. By the horseshoe lemma we have an extension
$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \oplus \mathscr{O}(\underline{a}') \to \mathscr{F} \oplus \mathscr{O}(\underline{b}') \to 0.$$
It follows that $\mathscr{F} \oplus \mathscr{O}(\underline{b}')$ is a descendant of $\mathscr{E}$. Since $\mathscr{F}$ is a descendant of $\mathscr{F} \oplus \mathscr{O}(\underline{b}')$, we conclude from (1) that $\mathscr{F}$ is a descendant of $\mathscr{E}$.
5. By (4) we have an exact sequence $0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{E}' \to 0$. This sequence is split since $\operatorname{Ext}^1(\mathscr{E}', \mathscr{O}(\underline{a})) = 0$. We conclude that $\mathscr{E} \oplus \mathscr{O}(\underline{b}) \cong \mathscr{E}' \oplus \mathscr{O}(\underline{a})$.
6. By (4), the sheaf $\mathscr{F}$ is a descendant of $\mathscr{E}$. Thus $\mathscr{E}$ satisfies (S_m^+) by the remark below Definition I.3.9.
7. Consider a surjection $\mathscr{O}(\underline{a}) \to \mathscr{F}$ with kernel $\mathscr{K}$. There is a surjection $\operatorname{Hom}_*(\mathscr{K}, \mathscr{O}_X) \to \operatorname{Ext}^1_*(\mathscr{F}, \mathscr{O}_X)$. Since X satisfies (G_1) and (S_2), and $\mathscr{K}$ satisfies (S_2), we see that $\mathscr{K}$ is reflexive. We conclude that $\operatorname{Hom}_*(\mathscr{K}, \mathscr{O}_X)$ and $\operatorname{Ext}^1_*(\mathscr{F}, \mathscr{O}_X)$ are finitely generated modules over $H^0_*(\mathscr{O}_X)$. We may then find an extension
$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{F} \to 0,$$

where the map α in the long exact sequence

$$\cdots \to \mathrm{Hom}_*(\mathscr{O}(\underline{a}), \mathscr{O}_X) \xrightarrow{\alpha} \mathrm{Ext}^1_*(\mathscr{F}, \mathscr{O}_X) \to \mathrm{Ext}^1_*(\mathscr{E}, \mathscr{O}_X) \to 0$$

corresponds to generators of the module $\mathrm{Ext}^1_*(\mathscr{F}, \mathscr{O}_X)$ [see 52, Prop 2.1]. □

From the lower terms of the spectral sequence of Ext

$$0 \to H^1_*(\mathscr{E}^*) \to \mathrm{Ext}^1_*(\mathscr{E}, \mathscr{O}_X) \to H^0_*(\mathscr{E}xt^1(\mathscr{E}, \mathscr{O}_X)) \to H^2_*(\mathscr{E}^*) \to \dots$$

we see that *extraverti* sheaves in the sense of [52] are primitive. The converse need not be true as extraverti sheaves are exactly primitive sheaves whose classes contain the ideal sheaf of a pure codimension two subvariety up to twist. In our article, we are not concerned with the properties of the varieties defined by the ideal sheaves in a biliaison class, thus we resort to the more general definition of primitive sheaves.

Minimal sheaves

In this subsection, we assume that X is primitive and Gorenstein in codimension one (G_1). We define a natural preorder among sheaves in a biliaison class and prove that there is always a minimal (S_m^+) member, generalizing the fact that there is always a minimal subvariety in an even linkage class. We then prove a stronger structure theorem for (S_m^+) sheaves in a biliaison class, which is an analogue of Theorem I.1.7. Finally, we deduce a sufficient criterion for an (S_m^+) sheaf to be minimal.

Definition I.3.15. We say a sheaf $\mathscr{F}$ is *very primitive* if $\mathscr{F}$ is primitive and does not admit a non-trivial direct summand of the form $\mathscr{O}(\underline{a})$.

Recall that coherent sheaves on X form a Krull-Schimdt category [2]. Thus any primitive sheaf $\mathscr{F}$ is of the form $\mathscr{F}' \oplus \mathscr{O}(\underline{a})$ for some very primitive sheaf $\mathscr{F}'$ and finite integer sequence $\underline{a}$, and this decomposition is unique up to isomorphism. It follows from Proposition I.3.14 that a very primitive sheaf is unique up to isomorphism in its biliaison class.

We define the following invariant for sheaves in the biliaison class of a primitive sheaf.

Definition I.3.16. Let $\mathscr{E}$ be a very primitive sheaf, and let $\mathscr{F}$ be in the biliaison class of $\mathscr{E}$. It follows from Proposition I.3.14 that $\mathscr{F} \in \mathfrak{D}(\mathscr{E})$, i.e. there is an extension of the form

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{F} \to 0.$$

We define the Σ *function* of $\mathscr{F}$ to be $\Sigma(\mathscr{F}, l) := \Sigma(\underline{b}, l) - \Sigma(\underline{a}, l)$.

Proposition I.3.17. *The function $\Sigma(\mathscr{F}, -)$ is well-defined for any sheaf $\mathscr{F}$ in the biliaison class of a primitive sheaf. In particular, the Σ function is well defined for any sheaf $\mathscr{F}$ in the biliaison class of an (S_1^+) sheaf.*

Proof. We need to show that the function $\Sigma(\mathscr{F},-)$ does not depend on the extension

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{F} \to 0.$$

Suppose $0 \to \mathscr{O}(\underline{a}') \to \mathscr{E}' \oplus \mathscr{O}(\underline{b}') \to \mathscr{F} \to 0$ is another extension where $\mathscr{E}'$ is very primitive. The surjection $\mathscr{E}' \oplus \mathscr{O}(\underline{b}') \to \mathscr{F}$ lifts to a map $\mathscr{E}' \oplus \mathscr{O}(\underline{b}') \to \mathscr{E} \oplus \mathscr{O}(\underline{b})$ since $\mathscr{E}$ is primitive and X is primitive. We have a surjection $\mathscr{E}' \oplus \mathscr{O}(\underline{b}') \oplus \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b})$ with kernel $\mathscr{O}(\underline{a}')$ by the horseshoe lemma. Since $\operatorname{Ext}^1(\mathscr{E} \oplus \mathscr{O}(\underline{b}), \mathscr{O}(\underline{a}')) = 0$, the above surjection splits and we obtain an isomorphism

$$\mathscr{E}' \oplus \mathscr{O}(\underline{b}') \oplus \mathscr{O}(\underline{a}) \cong \mathscr{E} \oplus \mathscr{O}(\underline{b}) \oplus \mathscr{O}(\underline{a}').$$

Since $\mathscr{E}$ and $\mathscr{E}'$ are both very primitive, we have that $\mathscr{O}(\underline{b}') \oplus \mathscr{O}(\underline{a}) \cong \mathscr{O}(\underline{b}) \oplus \mathscr{O}(\underline{a}')$ by the uniqueness of the Krull-Schimdt decomposition. It follows that $\Sigma(\mathscr{F},l) = \Sigma(\underline{b},l) - \Sigma(\underline{a},l) = \Sigma(\underline{b}',l) - \Sigma(\underline{a}',l)$. The last statement follows from Proposition I.3.14 (7). □

Given the Hilbert function of a very primitive sheaf $\mathscr{E}$, the data of the Σ function of a sheaf $\mathscr{F}$ in the biliaison class of $\mathscr{E}$ is equivalent to the data of the Hilbert function of $\mathscr{F}$. The next proposition says we can compute the Σ function of $\mathscr{F}$ from the Σ function of any ancestor of $\mathscr{F}$.

Proposition I.3.18. *Suppose $\mathscr{F}, \mathscr{G} \in \mathfrak{D}(\mathscr{E})$ for some very primitive sheaf $\mathscr{E}$. If there is an extension $0 \to \mathscr{O}(\underline{a}) \to \mathscr{G} \oplus \mathscr{O}(\underline{b}) \to \mathscr{F} \to 0$, then*

$$\Sigma(\mathscr{F} \oplus \mathscr{O}(\underline{a}), l) = \Sigma(\mathscr{G} \oplus \mathscr{O}(\underline{b}), l), \quad \forall l \in \mathbb{Z}.$$

Proof. Given an extension $0 \to \mathscr{O}(\underline{a}') \to \mathscr{E} \oplus \mathscr{O}(\underline{b}') \to \mathscr{F} \to 0$, the map $\mathscr{E} \oplus \mathscr{O}(\underline{b}') \to \mathscr{F}$ lifts to a map $\mathscr{E} \oplus \mathscr{O}(\underline{b}') \to \mathscr{G} \oplus \mathscr{O}(\underline{b})$ and we obtain an exact sequence

$$0 \to \mathscr{O}(\underline{a}') \to \mathscr{E} \oplus \mathscr{O}(\underline{b}') \oplus \mathscr{O}(\underline{a}) \to \mathscr{G} \oplus \mathscr{O}(\underline{b}) \to 0.$$

We conclude that $\Sigma(\mathscr{G} \oplus \mathscr{O}(\underline{b}), l) = \Sigma(\underline{b}', l) + \Sigma(\underline{a}, l) - \Sigma(\underline{a}', l) = \Sigma(\mathscr{F} \oplus \mathscr{O}(\underline{a}), l)$. □

We now prove some simple but important observations on Σ functions of (S_1^+) sheaves. For every sheaf $\mathscr{F}$ there is a natural surjection $\bigoplus_{l\in\mathbb{Z}} \mathscr{O}(-l)^{f(l)} \to \mathscr{F}$ given by sections of $\mathscr{F}$ in all degrees, where $f(l) = h^0(\mathscr{F}(l))$. For any integer $a \in \mathbb{Z}$, we define $\mathscr{F}_{\leq a}$ to be the image subsheaf of the restriction $\bigoplus_{l\leq a} \mathscr{O}(-l)^{f(l)} \to \mathscr{F}$. We say $\mathscr{F}_{\leq a}$ is the subsheaf of $\mathscr{F}$ generated by sections of degree $\leq a$. Our notations were chosen to be consistent in the sense that $\Sigma(\mathscr{O}(\underline{a}), l) = \Sigma(\underline{a}, l) = \operatorname{rank} \mathscr{O}(\underline{a})_{\leq l}$.

Proposition I.3.19. *Let $m \geq 1$ and let $\mathscr{F} \in \mathfrak{D}_m(\mathscr{E})$ for a very primitive sheaf $\mathscr{E}$. Let r be the minimal rank of all sheaves in $\mathfrak{D}_m(\mathscr{E})$, and define $e := \inf\{l \mid H^0(\mathscr{E}(l)) \neq 0\}$.*

1. *If $\mathscr{E} \neq 0$, then e is an integer.*

2. *$\Sigma(\mathscr{F}, l) = 0$ for $l \ll 0$ and $\Sigma(\mathscr{F}, l) \geq 0$ for all $l < e$.*
3. *$\Sigma(\mathscr{F}, l) \geq r - \operatorname{rank} \mathscr{E}$ for all $l \geq e$.*
4. *$\Sigma(\mathscr{F}, l) = \operatorname{rank} \mathscr{F} - \operatorname{rank} \mathscr{E}$ for all $l \gg 0$.*

Proof. 1. Since $\mathscr{E}$ satisfies (S_1^+), the map $\mathscr{E} \to \mathscr{E}^{**}$ is injective. We have $H^0(\mathscr{E}^{**}(l)) = 0$ for $l \ll 0$ by Serre duality and Serre vanishing. It follows that $H^0(\mathscr{E}(l)) = 0$ for $l \ll 0$ and e is an integer.

2. Let $0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{F} \to 0$ be an extension. It is clear that the restricted map $\phi : \mathscr{O}(\underline{a})_{\leq l} \to \mathscr{E} \oplus \mathscr{O}(\underline{b})$ is an m-reduction. In fact, $\mathscr{O}(\underline{a})_{\leq l}$ maps into $\mathscr{O}(\underline{b})_{\leq l}$ since $\mathscr{E} \oplus \mathscr{O}(\underline{b})_{>l}$ has no sections of degree $\leq l$. It follows that $\psi : \mathscr{O}(\underline{a})_{\leq l} \to \mathscr{O}(\underline{b})_{\leq l}$ is an m-reduction, as its cokernel fits in an exact sequence

$$0 \to \operatorname{coker} \psi \to \operatorname{coker} \phi \to \mathscr{O}(\underline{b})_{>l} \to 0.$$

We conclude that $\Sigma(\mathscr{F}, l) = \operatorname{rank} \mathscr{O}(\underline{b})_{\leq l} - \operatorname{rank} \mathscr{O}(\underline{a})_{\leq l} \geq 0$.

3. Suppose $l \geq e$. Similar to the above, there is an m-reduction $\psi : \mathscr{O}(\underline{a})_{\leq l} \to \mathscr{E} \oplus \mathscr{O}(\underline{b})_{\leq l}$. Since $\Sigma(\mathscr{F}, l) + \operatorname{rank} \mathscr{E} = \operatorname{rank} \operatorname{coker} \psi \geq r$, we see that $\Sigma(\mathscr{F}, l) \geq r - \operatorname{rank} \mathscr{E}$.

4. This is true for any l greater than the maximum of all entries of $\underline{a}$ and $\underline{b}$. $\square$

The invariant Σ allows us to define a preorder on the biliaison class of a primitive sheaf.

Definition I.3.20. If $\mathscr{E}$ is a very primitive sheaf and $\mathscr{F}, \mathscr{G} \in \mathfrak{D}(\mathscr{E})$, we write $\mathscr{F} \preceq \mathscr{G}$ if $\Sigma(\mathscr{F}, l) \leq \Sigma(\mathscr{G}, l)$ for all $l \in \mathbb{Z}$. This defines a preorder on the biliaison class of $\mathscr{E}$.

A preorder is a relation that is reflexive and transitive. Every preorder has an associated partial order, obtained by modding out equivalences where $\mathscr{F} \preceq \mathscr{G}$ and $\mathscr{G} \preceq \mathscr{F}$. The associated poset of a biliaison class with respect to the preorder $\preceq$ is exactly the poset of Σ functions under the partial order of point-wise comparison. The next proposition characterizes when two sheaves in a biliaison class have the same Σ function.

Proposition I.3.21. *The following are equivalent for two sheaves $\mathscr{F}$ and $\mathscr{G}$ whose biliaison classes admit primitive sheaves.*

1. *$\mathscr{F}$ and $\mathscr{G}$ are in a rigid family,*
2. *$\mathscr{F}$ and $\mathscr{G}$ are in the same biliaison class, and $\mathscr{F} \preceq \mathscr{G}$ as well as $\mathscr{G} \preceq \mathscr{F}$.*

Proof. If $\mathscr{F}$ and $\mathscr{G}$ are in the same biliaison class and have the same Σ functions, then there are extensions

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{F} \to 0$$

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \oplus \mathscr{O}(\underline{b}) \to \mathscr{G} \to 0$$

for a very primitive sheaf $\mathscr{E}$ by the proof of Proposition I.3.17. The proof of Lemma I.3.5 shows that $\mathscr{F}$ and $\mathscr{G}$ lie in a rigid family, parametrized by an open subscheme of the affine scheme $\mathrm{Hom}(\mathscr{O}(\underline{a}), \mathscr{E} \oplus \mathscr{O}(\underline{b}))$.

Conversely, if $\mathscr{F}$ and $\mathscr{G}$ lie in a rigid family, then there is a not necessarily primitive sheaf $\mathscr{E}$ and extensions

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{F} \to 0$$

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{G} \to 0.$$

It follows from Proposition I.3.18 that $\Sigma(\mathscr{F}, l) = \Sigma(\mathscr{G}, l) = \Sigma(\mathscr{E}, l) - \Sigma(\underline{a}, l)$ for all $l \in \mathbb{Z}$. □

As a direct corollary to Theorem I.3.4, we see that the associated poset of (S_m^+) sheaves in a biliaison class is a meet-semilattice.

Theorem I.3.22. *For $m \geq 1$, the Σ functions of (S_m^+) sheaves in a biliaison class form a meet-semilattice, i.e. a poset with meet. For $m = 1$, the Σ functions of (S_1^+) sheaves in a biliaison class form a lattice.*

Proof. Let $\mathscr{F}$ and $\mathscr{G}$ be two (S_m^+) sheaves in the same biliaison class. We find a sheaf $\mathscr{E}$ and m-reductions $\phi : \mathscr{O}(\underline{a}) \to \mathscr{E}$ and $\psi : \mathscr{O}(\underline{b}) \to \mathscr{E}$ where $\operatorname{coker} \phi \cong \mathscr{F}$ and $\operatorname{coker} \psi \cong \mathscr{G}$ by Proposition I.3.14. By Theorem I.3.6, we find an m-reduction $\xi : \mathscr{O}(\underline{c}) \to \mathscr{E}$ where $\underline{c} = \underline{a} \wedge \underline{b}$. By Proposition I.3.18, we see that $\Sigma(\operatorname{coker} \xi, l) = \min(\Sigma(\mathscr{F}, l), \Sigma(\mathscr{G}, l))$. The first conclusion follows. The second conclusion follows analogously from Theorem I.3.7. □

In the following, we show that the meet-semilattice of Σ functions of (S_m^+) sheaves in a biliaison class is always bounded below.

Definition I.3.23. For $m \geq 1$, a *minimal (S_m^+) sheaf* is a sheaf that is minimal among all (S_m^+) sheaves in its biliaison class with respect to the preorder $\preceq$.

We make several remarks regarding this definition.

First, any two minimal (S_m^+) sheaves in a biliaison class lie in a rigid family by Proposition I.3.21. Since we assume that X is primitive in this section, all minimal (S_m^+) sheaves in a biliaison class have the same intermediate cohomology modules and Hilbert functions.

Second, note that the Chern classes of minimal sheaves have smallest degrees (with respect to pairing with complementary powers of the hyperplane class H) among all (S_m^+) sheaves in their biliaison classes. This can be seen by an elementary computation from short exact sequences.

Third, since $\operatorname{rank} \mathscr{F} = \operatorname{rank} \mathscr{E} + \Sigma(\mathscr{F}, l)$ for $l \gg 0$, where $\mathscr{E}$ is a very primitive sheaf in the biliaison class of $\mathscr{F}$, we conclude that $\mathscr{F} \preceq \mathscr{G}$ implies $\operatorname{rank} \mathscr{F} \leq \operatorname{rank} \mathscr{G}$. In particular, minimal (S_m^+) sheaves must have minimal rank among all (S_m^+) sheaves in its biliaison class.

Fourth, one might ask if there is a minimal member among (S_m^+) sheaves **of a given rank** in a biliaison class. However, such a sheaf might not exist unless the given rank is minimal. Consider the biliaison class of the zero sheaf, one immediately sees that there is no minimal rank one bundle as we have the bundle $\mathscr{O}(l)$ for any $l \gg 0$.

Last but not least, in the linkage theory of pure codimension two subvarieties of $\mathbf{P}^n_k$, a variety is minimal in its even linkage class if and only if its ideal sheaf is minimal among all rank one (S_1^+) sheaves in its biliaison class with respect to the preorder $\preceq$. We will see from the next proposition that these ideal sheaves are in fact minimal among (S_1^+) sheaves of **all** ranks in its biliaison class, i.e. they are minimal (S_1^+) sheaves.

Proposition I.3.24. *If $m \geq 1$ and $\mathscr{F} \preceq \mathscr{G}$ for all (S_m^+) sheaves $\mathscr{G}$ of minimal rank in the biliaison class of $\mathscr{F}$, then $\mathscr{F}$ is a minimal (S_m^+) sheaf.*

Proof. Clearly the condition implies that $\mathscr{F}$ has minimal rank among (S_m^+) sheaves in its biliaison class. Suppose $\mathscr{E}$ is any (S_m^+) sheaf in the biliaison class of $\mathscr{F}$, then can find a sheaf $\mathscr{G}$ where $\mathscr{G} \preceq \mathscr{E}$ and $\mathscr{G} \preceq \mathscr{F}$ by Theorem I.3.22. Since $\operatorname{rank} \mathscr{F} = \operatorname{rank} \mathscr{G}$ is minimal, by assumption we see that $\mathscr{F} \preceq \mathscr{G}$, and thus $\mathscr{F} \preceq \mathscr{E}$. □

Migliore [76] proved that every even linkage class of curves in $\mathbf{P}^3_k$ has a minimal member. This result was extended in [10] to every even linkage class of pure codimension two Cohen-Macaulay subvarieties of $\mathbf{P}^n_k$ has a minimal member. Nollet [81] generalized this further to pure codimension r subvarieties and removed the Cohen-Macaulay assumption, and described an algorithm to construct the minimal ideal sheaves given a primitive sheaf as input. This algorithm was based on calculations in [69] for the case of space curves. Combined with Proposition I.3.24, we obtain many examples of minimal sheaves.

Corollary I.3.25. *There is a minimal (S_1^+) sheaf in every biliaison class that admits an (S_1^+) sheaf on $\mathbf{P}^n_k$.*

We prove a generalization of the above result for (S_m^+) sheaves on any projective variety X satisfying our assumptions.

Theorem I.3.26 (Existence of minimal sheaves)**.** *For $m \geq 1$, there is a minimal (S_m^+) sheaf in every biliaison class that admits an (S_m^+) sheaf.*

Proof. Let $\mathscr{E}$ be a very primitive sheaf satisfying (S_m^+). If $\mathscr{E} = 0$, then the zero sheaf is the minimal (S_m^+) sheaf. If $\mathscr{E} \neq 0$, then the zero sheaf is not in $\mathfrak{D}_m(\mathscr{E})$. Since $m \geq 1$, any sheaf in $\mathfrak{D}_m(\mathscr{E})$ is torsion-free and thus has positive rank. Let r be the minimal rank of sheaves in $\mathfrak{D}_m(\mathscr{E})$. Let $\mathscr{F}_1 \in \mathfrak{D}_m(\mathscr{E})$ be a sheaf of rank r. If $\mathscr{F}_1$ is not minimal, then there exists a sheaf $\mathscr{G} \in \mathfrak{D}_m(\mathscr{E})$ where $\mathscr{F}_1 \not\preceq \mathscr{G}$. By Theorem I.3.22, there exists a sheaf $\mathscr{F}_2 \in \mathfrak{D}_m(\mathscr{E})$ such that $\mathscr{F}_2 \preceq \mathscr{F}_1$ and $\mathscr{F}_2 \preceq \mathscr{G}$. Since $\mathscr{F} \not\preceq \mathscr{G}$, we must have $\mathscr{F}_2 \prec \mathscr{F}_1$. Since $\operatorname{rank} \mathscr{F}_2 \leq \operatorname{rank} \mathscr{F}_1$, we see that $\operatorname{rank} \mathscr{F}_2 = r$ as well. Suppose to the contrary that $\mathfrak{D}_m(\mathscr{E})$ has no minimal member, arguing analogously, we obtain an infinite descending chain of rank r sheaves $\mathscr{F}_1 \succ \mathscr{F}_2 \succ \cdots$. They give an infinite descending chain of Σ functions $\Sigma(\mathscr{F}_1, -) > \Sigma(\mathscr{F}_2, -) > \cdots$. Set $e := \inf\{l \mid H^0(\mathscr{E}(l)) \neq 0\}$. By Proposition I.3.19, e is an integer and $\Sigma(\mathscr{F}_i, l) = 0$ for $l \ll 0$, $\Sigma(\mathscr{F}_i, l) \geq 0$ for $l < e$, $\Sigma(\mathscr{F}_i, l) \geq r - \operatorname{rank} \mathscr{E}$ for $l \geq e$ and

$\Sigma(\mathscr{F}_i, l) = r - \operatorname{rank}\mathscr{E}$ for $l \gg 0$. We see that it is impossible to have an infinite descending chain of such functions $\Sigma(\mathscr{F}_1, -) > \Sigma(\mathscr{F}_2, -) > \cdots$ satisfying the above properties. The assertion of the theorem follows. □

The existence of minimal sheaves allows us to strengthen the structure theorem.

Theorem I.3.27 (Strong structure theorem)**.** *Suppose X is primitive and $\mathscr{F}$ is an (S_m^+) sheaf for $m \geq 1$. There are (S_m^+) sheaves $\mathscr{F} = \mathscr{F}_0, \ldots, \mathscr{F}_l$ in the biliaison class of $\mathscr{F}$, such that $\mathscr{F}_l$ is a minimal (S_m^+) sheaf, and $\mathscr{F}_i, \mathscr{F}_{i+1}$ are related in one of the following manner:*

- *(a) there is a descending elementary (S_{m-1}^+)-biliaison from $\mathscr{F}_i$ to $\mathscr{F}_{i+1}$,*
- *(b) $\mathscr{F}_i$ and $\mathscr{F}_{i+1}$ belong in a rigid family,*
- *(c) there is an m-reduction $\phi : \mathscr{O}(\underline{a}) \to \mathscr{F}_i$ with $\operatorname{coker}\phi \cong \mathscr{F}_{i+1}$ for some $\underline{a}$.*

If $\mathscr{F}$ is of minimal rank among (S_m^+) sheaves in its biliaison class, then we do not need (c). Note that a rigid family preserves Hilbert functions and intermediate cohomology modules.

Proof. Let $\mathscr{G}$ be a minimal (S_m^+) sheaf in the biliaison class of $\mathscr{F}$, whose existence follows from Theorem I.3.26. If we follow the proof of Theorem I.3.12, we see that the elementary (S_{m-1}^+)-biliaison involved at every step is decreasing. □

In fact, when $m = 1$, we do not need deformations by rigid families in manner (b) at all. A proof of this can be given based on [52, Proposition 3.6].

Although Theorem I.3.26 gives us a theoretical guarantee that minimal (S_m^+) sheaves exist, it does not tell us how to produce or identify them in practice. The next theorem solves this problem by giving a sufficient condition for a sheaf to be a minimal (S_m^+) sheaf. This generalizes the sufficient condition for a curve in $\mathbf{P}_k^3$ to be minimal proven in [67].

Theorem I.3.28 (Sufficient condition for minimal sheaves)**.** *Let $m \geq 1$, and let $\mathscr{F}$ be an (S_m^+) of minimal rank in its biliaison class. If $\mathscr{F}$ admits an extension*

$$0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{F} \to 0$$

where $\mathscr{E}$ is primitive and $H^0(\mathscr{F}(l)) = 0$ for all $l < \max(\underline{a})$, then $\mathscr{F}$ is a minimal (S_m^+) sheaf.

Proof. If $\mathscr{G}$ is another (S_m^+) sheaf in the biliaison class of $\mathscr{F}$, then it admits an extension of the form

$$0 \to \mathscr{O}(\underline{c}) \to \mathscr{E} \oplus \mathscr{O}(\underline{d}) \to \mathscr{G} \to 0$$

for some $\underline{c}$ and $\underline{d}$ since $\mathscr{E}$ is primitive. By Proposition I.3.18, we need to show that

$$\operatorname{rank}(\mathscr{O}(\underline{a}) \oplus \mathscr{O}(\underline{d}))_{\leq l} \geq \operatorname{rank}\mathscr{O}(\underline{c})_{\leq l}, \quad \forall l \in \mathbb{Z}.$$

We separate into two cases, where $l \geq \max(\underline{a})$ and $l < \max(\underline{a})$.

Case $l \geq \max(\underline{a})$: We have an exact sequence $0 \to \mathscr{O}(\underline{c})_{\leq l} \to \mathscr{E} \oplus \mathscr{O}(\underline{d}) \to \mathscr{G}' \to 0$, where $\mathscr{G}'$ is an extension of $\mathscr{G}$ with $\mathscr{O}(\underline{c})_{>l} := \mathscr{O}(\underline{c})/\mathscr{O}(\underline{c})_{\leq l}$. In particular, the sheaf $\mathscr{G}'$ satisfies (S_m^+). Since any map $\mathscr{O}(\underline{c})_{\leq l} \to \mathscr{O}(\underline{d})_{>l} := \mathscr{O}(\underline{d})/\mathscr{O}(\underline{d})_{\leq l}$ is zero, the injection $\mathscr{O}(\underline{c})_{\leq l} \to \mathscr{E} \oplus \mathscr{O}(\underline{d})$ lands inside $\mathscr{E} \oplus \mathscr{O}(\underline{d})_{\leq l}$. We obtain an exact sequence $0 \to \mathscr{O}(\underline{c})_{\leq l} \to \mathscr{E} \oplus \mathscr{O}(\underline{d})_{\leq l} \to \mathscr{G}'' \to 0$, where $\mathscr{G}''$ sits in an exact sequence $0 \to \mathscr{G}'' \to \mathscr{G}' \to \mathscr{O}(\underline{d})_{>l} \to 0$. By the depth lemma, the sheaf $\mathscr{G}''$ also satisfies (S_m^+). Now $\operatorname{rank} \mathscr{O}(\underline{a})_{\leq l} = \operatorname{rank} \mathscr{O}(\underline{a}) = \operatorname{rank} \mathscr{E} - \operatorname{rank} \mathscr{F}$ by the assumption on a. Since $\mathscr{F}$ has minimal rank among sheaves in $\mathfrak{D}_m(\mathscr{E})$, we conclude that

$$\operatorname{rank} \mathscr{E} + \operatorname{rank} \mathscr{O}(\underline{d})_{\leq l} - \operatorname{rank} \mathscr{O}(\underline{c})_{\leq l} = \operatorname{rank} \mathscr{G}'' \geq \operatorname{rank} \mathscr{F} = \operatorname{rank} \mathscr{E} - \operatorname{rank} \mathscr{O}(\underline{a})_{\leq l}.$$

It follows that $\operatorname{rank}(\mathscr{O}(\underline{a}) \oplus \mathscr{O}(\underline{d}))_{\leq l} \geq \operatorname{rank} \mathscr{O}(\underline{c})_{\leq l}$.

Case $l < \max(\underline{a})$: The cokernel $\mathscr{F}'$ of $\mathscr{O}(\underline{a})_{\leq l} \to \mathscr{E}$ is an extension of $\mathscr{F}$ by $\mathscr{O}(\underline{a})_{>l}$. Since $H^0(\mathscr{F}(n)) = 0$ for all $n \leq l$, the same is true for $\mathscr{F}'$. We have the exact sequence

$$0 \to \mathscr{O}(\underline{a})_{\leq l} \oplus \mathscr{O}(\underline{d})_{\leq l} \to \mathscr{E} \oplus \mathscr{O}(\underline{d}) \to \mathscr{F}' \oplus \mathscr{O}(\underline{d})_{>l} \to 0.$$

The composition $\mathscr{O}(\underline{c})_{\leq l} \to \mathscr{E} \oplus \mathscr{O}(\underline{d}) \to \mathscr{F}' \oplus \mathscr{O}(\underline{d})_{>l}$ is zero since $\mathscr{F}'$ has no sections in degree $\leq l$. It follows that the injection $\mathscr{O}(\underline{c})_{\leq l} \to \mathscr{E}$ factors through $\mathscr{O}(\underline{a})_{\leq l} \oplus \mathscr{O}(\underline{d})_{\leq l}$, and we conclude that $\operatorname{rank}(\mathscr{O}(\underline{a}) \oplus \mathscr{O}(\underline{d}))_{\leq l} \geq \operatorname{rank} \mathscr{O}(\underline{c})_{\leq l}$. □

The next theorem is a necessary condition for an (S_m^+) sheaf of minimal rank in its biliaison class to be a minimal (S_m^+) sheaf.

Theorem I.3.29 (Necessary condition for minimal sheaves). *Suppose $\mathscr{F}$ is a minimal (S_m^+) sheaf for some $m \geq 1$. Let $0 \to \mathscr{O}(\underline{a}) \to \mathscr{E} \to \mathscr{F} \to 0$ be an extension, where $\mathscr{E}$ is primitive of rank $\geq m$. If $\mathscr{O}(\underline{c}) \to \mathscr{E}$ is any surjection, then $\mathscr{O}(\underline{c}') \preceq \mathscr{O}(\underline{a})$, where $\underline{c}'$ consists of the largest* $(\operatorname{rank} \mathscr{E} - m)$ *entries of $\underline{c}$.*

Proof. By Corollary I.2.13, there is always an m-reduction $\phi : \mathscr{O}(\underline{c}') \to \mathscr{E}$ of corank m. It follows that $\mathscr{O}(\underline{c}') \preceq \mathscr{O}(\underline{a})$ since $\mathscr{F}$ is a minimal (S_m^+) sheaf. □

We remark that the necessary condition in Theorem I.3.29 is not tight in general. The following is an example on how one could use this theorem in practice.

Example I.3.30. Suppose $\mathscr{F}$ is a sheaf of minimal rank r in $\mathfrak{D}_m(\mathscr{E})$, where $\mathscr{E}$ is primitive of rank $\geq m$. Let there be an extension of the form

$$0 \to \mathscr{O}(-2)^v \to \mathscr{E} \to \mathscr{F} \to 0.$$

If $\mathscr{E}$ is generated in degree 1, then there is a surjection $\mathscr{O}(-1)^N \to \mathscr{E}$ for some large N. Since there is an m-reduction of the form $\phi : \mathscr{O}(-1)^u \to \mathscr{E}$, where $u = \operatorname{rank} \mathscr{E} - m \leq \operatorname{rank} \mathscr{E} - r = v$, Theorem I.3.29 says that $\mathscr{F}$ cannot be a minimal (S_m^+) sheaf since there must be an m-reduction of $\mathscr{E}$ with shape $(\underbrace{1, \ldots, 1}_{u}) \wedge (\underbrace{2, \ldots, 2}_{v}) = (\underbrace{1, \ldots, 1}_{u}, \underbrace{2, \ldots, 2}_{v-u})$.

CHAPTER II

Curves

Chapter II summary

In this chapter, we study general projections of curves from higher dimensional projective spaces into $\mathbf{P}^3$ via the linkage theory of curves on surfaces with ordinary singularities.

In Section II.1, we review necessary background for this chapter. We state the multiple-point formulas proven by Kleiman [60] and Kleiman-Lipman-Ulrich [59], and revisit the classical theorem on the singularities of general projections of smooth projective surfaces into $\mathbf{P}^3$. We also briefly introduce the theory of generalized divisors developed by Hartshorne [51] in order for us to discuss divisors on singular surfaces.

In Section II.2, we describe of the geometry of linked curves on a surface X with ordinary singularities (Theorem II.2.4). We show that smooth curves that are evenly linked on X are in fact linearly equivalent if X is singular (Corollary II.2.10), a drastic contrast to the case when X is smooth. We compute certain homological invariants of a curve C on X in terms of cohomologies of divisors on S, provided that C is preserved by the normalization of the surface or is linked to such a curve. Examples of homological invariants of C include the Hilbert function $h^0(\mathscr{I}_C(n))$ (Proposition II.2.15), the Rao function $h^1(\mathscr{I}_C(n))$ (Proposition II.2.16), the specialty function $h^1(\mathscr{O}_C(n))$ (Proposition II.2.14) and the dimension of the tangent space $h^0(\mathscr{N}_C)$ in the Hilbert scheme (Proposition II.2.17).

In Section II.3, we use the results in Section II.2 to study general projections of curves lying on a rational normal scroll $S(a,b) \subseteq \mathbf{P}^{a+b+1}$. We compute the dimension of the family of curves in $\mathbf{P}^3$ arising from various projections of curves varying in a given linear system on $S(a,b)$, as well as the dimension of the tangent spaces in the Hilbert scheme. We show that the difference between these two dimensions is a linear function in a and b, which does not depend on the linear system chosen (Theorem II.3.4). Last but not least, we determine all maximal rank curves on a ruled cubic surface (Theorem II.3.12). Consequently, we find that the linear projections of all but finitely many linear equivalence classes of arithmetically Cohen-Macaulay (ACM) curves on the cubic scroll $S(1,2) \subseteq \mathbf{P}^4$ fail to have maximal rank in $\mathbf{P}^3$. These examples are interesting in view of the recent progress made on the Maximal Rank Conjecture by Larson [66].

Throughout this chapter, we work over the field of complex numbers $\mathbb{C}$, which we suppress from our notations. By a curve we mean a one-dimensional projective scheme without point components, isolated or embedded. By a point on a finite type $\mathbb{C}$-scheme we always mean a closed point. We use $g(C)$ to denote the arithmetic genus of a curve C.

II.1. Background

This section contains the necessary background for this chapter. First, we recall the multiple-point formulas proven by Kleiman [60] and Kleiman-Lipman-Ulrich [59]. Next, we revisit the classical theorem on the singularities of general projections of smooth projective surfaces into $\mathbf{P}^3$. Finally, we review the theory of generalized divisors developed by Hartshorne [51] in order for us to discuss divisors on singular surfaces.

Multiple point formulas

Let $f : X \to Y$ of a finite morphism between finite type $\mathbb{C}$-schemes.

Definition II.1.1 (Multiple-point and ramification loci)**.** Let N_r denote the subset of points $y \in Y$ such that $f^{-1}(y) := X \times_Y k(y)$ contains a zero dimensional subscheme of length r. By fiber continuity, the subset N_r is closed in Y. Let $M_r := f^{-1}(N_r)$ denote the closed subset in X. We call M_r and N_r the *source and target r-fold points* of f respectively.

The central theme of the theory of multiple-point formulas is to find appropriate scheme structures on M_r and N_r, and to determine the classes $[M_r]$ and $[N_r]$ in the Chow ring (or other cohomology rings). For example, such a formula can give the degree and genus of the curve of trisecant lines of a given space curve C, as well as the number of quadrisecant lines. This subject started in 1850 and still inspires current research, see [60, §V] for a survey on the subject of multiple-point formulas, see [61] for an approach using iterations and see [62] for an approach using Hilbert schemes.

In codimension one, Kleiman-Lipman-Ulrich [59] considered the subscheme structures on N_r given by the Fitting ideals $\mathscr{F}itt_{r-1}^{\mathscr{O}_Y}(f_*\mathscr{O}_X)$ (see Definition I.2.5) and the corresponding preimage subscheme structures on M_r. Under mild assumptions, the subschemes M_r and N_r are Cohen-Macaulay, and their classes $[M_r]$ and $[N_r]$ are compatible with those coming from iteration [61]. We summarize here some of the results of [61] and [59].

Definition II.1.2. With notations as above, let R_i be the subscheme of X defined by the fitting ideal $\mathscr{F}itt_{i-1}^{\mathscr{O}_X}(\Omega_{X/Y})$. From the conormal sequence $f^*\Omega^1_{Y/\mathbb{C}} \to \Omega^1_{X/\mathbb{C}} \to \Omega_{X/Y} \to 0$, we see that the scheme R_i is supported at points in X where the differential ∂f drops rank by at least i. We call $R := R_1$ the *ramification locus.*

Theorem II.1.3 (Kleiman-Lipman-Ulrich)**.**

(i) Suppose f is locally of flat dimension 1 and birational onto its image, then N_1 is equal to the scheme-theoretical image $f(X)$ and $f_*[M_1] = [N_1]$.

(ii) Suppose furthermore that Y satisfies Serre's condition (S_2). The ideal sheaf $\mathscr{I}_{N_2/Y}$ is equal to $\operatorname{ann}_{\mathscr{O}_Y}(f_*\mathscr{O}_X/\mathscr{O}_Y)$ and the ideal sheaf $\mathscr{I}_{N_2/N_1}$ is equal to $\operatorname{ann}_{\mathscr{O}_{N_1}}(f_*\mathscr{O}_X/\mathscr{O}_{N_1})$. Each component of M_2 has codimension 1 and maps onto a component of N_2. Each component of N_2 has codimension 2. The fundamental cycles of these two schemes are

related by the equation $f_*[M_2] = 2[N_2]$. *The* $\mathcal{O}_Y$*-modules* $\mathcal{O}_{N_2}$ *and* $f_*\mathcal{O}_{M_2}$ *are perfect of grade 2.*

(iii) For $r > 0$, *suppose furthermore that* Y *satisfies Serre's condition* (S_r) *and* $R_2 = \varnothing$. *If each component of* N_r *has codimension* r, *then* $\mathcal{O}_{N_r}$ *and* $f_*\mathcal{O}_{M_r}$ *are perfect* $\mathcal{O}_Y$*-modules of grade* r. *Each component of* M_r *has codimension* $r-1$ *and maps onto a component of* N_r, *and the fundamental cycles of these two schemes are related by the equation* $f_*[M_r] = r[N_r]$.

Suppose f *satisfies all the assumptions above for all* $r > 0$. *If* f *is a local complete intersection and* X *has no embedded components, then*

$$[M_{r+1}] = f^*f_*[M_r] - rc_1(\nu)[M_r].$$

Here $\nu = f^*\mathcal{T}_Y - \mathcal{T}_X$ *is the virtual normal bundle in the Grothendieck group* $K(X)$.

General linear projections of smooth surfaces into $\mathbf{P}^3$

Every smooth projective surface $S \subseteq \mathbf{P}^N$ for $N > 5$ can be projected isomorphically into $\mathbf{P}^5$, but not further down in general. The following is a summary of the classical results on the singularities of general linear projections of S into $\mathbf{P}^3$.

Theorem II.1.4 (Classical projection theorem). *Let* $S \subseteq \mathbf{P}^5$ *be a non-degenerate smooth projective surface that is not the Veronese surface, then the following are true for a general linear projection* $f : S \to \mathbf{P}^3$.

1. *The map* f *is birational onto its scheme-theoretical image* X, *which is an integral hypersurface in* $\mathbf{P}^3$.
2. *The second ramification locus* R_2 *and the quadruple-point loci* M_4 *and* N_4 *are empty.*
3. *The target double-point locus* N_2 *is an integral curve and is exactly the singular locus of* X. *The target triple-point locus* N_3 *is a reduced set of points and is exactly the singular locus of* N_2. *Each point of* N_3 *is an ordinary triple point of* N_2 *and of* X.
4. *The source double-point locus* M_2 *is an integral curve mapping generically 2-1 to* N_2. *The source triple-point locus* M_3 *is a reduced set of points and is exactly the singular locus of* M_2. *Each point of* M_3 *is a simple node of* M_2. *Three distinct points of* M_3 *map to each point of* N_3.
5. *The ramification locus* R *is a set of reduced points and consisting exactly of the ramification points of the double cover* $M_2 \to N_2$. *The images of points in* R *are exactly the pinch points of* X.
6. *Every point of* $N_2 - N_3 - f(R_1)$ *is an ordinary double point of* X.

If S *is the Veronese surface, then a general linear projection of* S *into* $\mathbf{P}^3_{\mathbb{C}}$ *is called a Steiner surface (see Figure 4). In this case, the scheme* N_2 *is the union of three reduced lines* L_1, L_2, L_3 *meeting at a point* p. *The scheme* M_2 *is three reduced conics* C_1, C_2, C_3 *meeting* *each other at one point, where the three intersection points all map to p. The conic C_i maps 2-1 to L_i with two ramification points. The six branch points are the pinch points of X. In particular, all the results above are true except that M_2 and N_2 are reducible.*

Proof. See [35], [75] and [88] for modern expositions of these classical facts. □

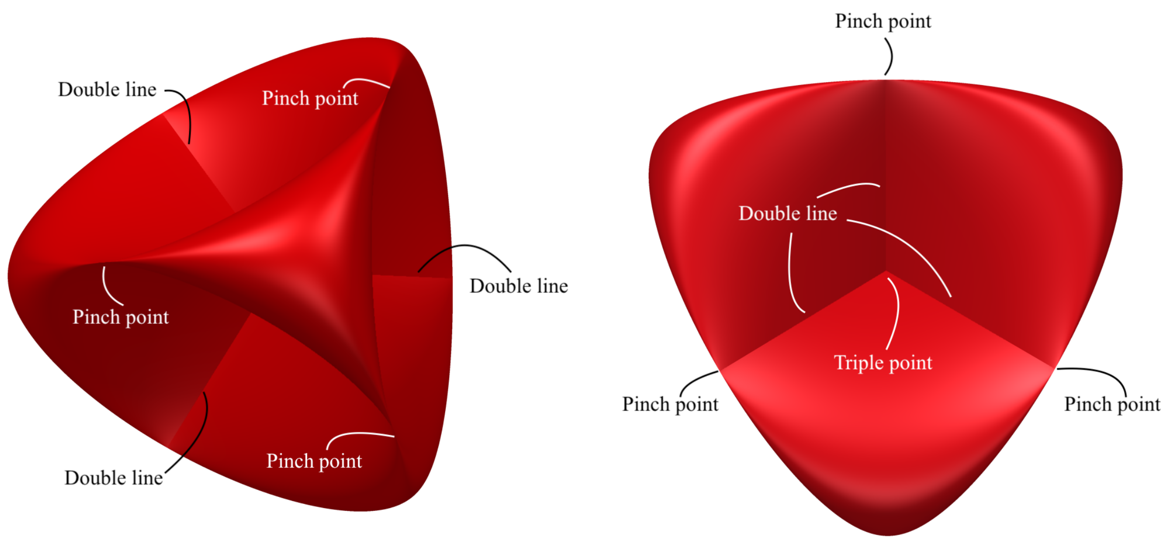

Figure 4: The Steiner surface

The following definition is made for surfaces whose singularities resemble those occurring on the general linear projections of smooth surfaces into $\mathbf{P}^3_{\mathbb{C}}$.

Definition II.1.5. An integral hypersurface $X \subseteq \mathbf{P}^3$ is said to have ordinary singularities if the singular locus of X is a curve C, such that if we put the reduced structure on C then the following hold.

1. Singular points of C are ordinary triple points (i.e. the origins of three linear branches of C with three distinct non-coplanar tangent directions).
2. A point on C is either a nodal point of X (i.e. the origins of two linear branches of X with two distinct tangent planes) or a pinch point of X (i.e analytically isomorphic to $x^2 - yz = 0$ at origin), and there are only finitely many pinch points on X.
3. Every triple point of C is an ordinary triple point of X (analytically isomorphic to $xyz = 0$ at origin).

An integral hypersurface $X \subseteq \mathbf{P}^3$ with ordinary singularities has a smooth normalization. The normalization may not be embedded in a projective space of which X is a linear projection, i.e. the pull back of the $\mathscr{O}_X(1)$ may not be very ample on the normalization. Such an example is given by the quartic surface with a double line considered by Gruson-Peskine [41] in order to determine all possible (degree, genus) pairs for smooth curves in $\mathbf{P}^3$.

Since S is smooth and $\mathbf{P}^3$ is Cohen-Macaulay, Theorem II.1.4 implies that a general linear projection $f : S \to \mathbf{P}^3$ satisfies all the assumptions in Theorem II.1.3. In particular, we have the following enumerative formulas.

Corollary II.1.6. *With notations be as above, let h be the class of $f^*\mathscr{O}_{\mathbf{P}^3}(1)$ in the Chow ring $A(S)$, and let c_i denote the Chern classes of tangent bundle $\mathcal{T}_S$. For two divisor classes $a, b \in A^1(S)$, let $a.b$ denote the intersection number. The following are true.*

(a) The class of M_2 in $A^1(S)$ is $(h.h - 4)h + c_1$.

(b) The degree of the curve N_2 is $\frac{1}{2}((h.h)^2 - 4h.h + c_1.h)$.

(c) There are short exact sequences

$$0 \to \mathscr{O}_X \to f_*\mathscr{O}_S \to \omega_{N_2}(4 - h.h) \to 0, \qquad (A)$$
$$0 \to \mathscr{O}_{N_2} \to f_*\mathscr{O}_{M_2} \to \omega_{N_2}(4 - h.h) \to 0. \qquad (B)$$

(d) The arithmetic genus of N_2 is equal to

$$\frac{1}{3}(h.h)^3 - 3(h.h)^2 + \frac{37}{6}(h.h) + \frac{1}{2}(h.h)(h.c_1) - 2(h.c_1) + \frac{1}{12}(c_1.c_1 + \deg c_2) + 1.$$

(e) The class of M_3 in $A_0(S)$ is equal to

$$((h.h)^2 - 12h.h + c_1.h + 44)h^2 + (2h.h - 24)c_1h + 4c_1^2 - 2c_2.$$

The number of triple points of X is one third the degree of $[M_3]$.

(f) The class of R_1 in $A_0(S)$ is equal to $6h^2 - 4hc_1 + c_1^2 - c_2$. The number of pinch points of X is the degree of R_1.

Proof. (a) $[M_2] = f^*f_*[M_1] - c_1(\nu) = (h.h - 4)h + c_1$.
(b) By push-pull formula, we have $M_2.h = 2\deg N_2 = (h.h - 4)h.h + c_1.h$.
(c) The $\mathscr{O}_X$-dual of the exact sequence $0 \to \mathscr{I}_{N_2/X} \to \mathscr{O}_X \to \mathscr{O}_{N_2} \to 0$ gives an exact sequence $0 \to \mathscr{O}_X \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}_{N_2/X}, \mathscr{O}_X) \to \mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{O}_{N_2}, \mathscr{O}_X) \to 0$. Note that $\mathscr{I}_{N_2/X} = \mathscr{H}om_{\mathscr{O}_X}(f_*\mathscr{O}_S, \mathscr{O}_X)$, and that $f_*\mathscr{O}_S$ is a reflexive $\mathscr{O}_X$-module since it satisfies (S2) and X is Gorenstein. It follows that we have isomorphisms

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{I}_{N_2/X}, \mathscr{O}_X) = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{H}om_{\mathscr{O}_X}(f_*\mathscr{O}_S, \mathscr{O}_X), \mathscr{O}_X) = f_*\mathscr{O}_S.$$

Since N_2 is Cohen-Macaulay, the third term in the exact sequence is isomorphic to

$$\mathscr{E}xt^1_{\mathscr{O}_X}(\mathscr{O}_{N_2}, \mathscr{O}_X) \cong \omega_{N_2} \otimes \omega_X^{-1} \cong \omega_{N_2}(4 - h.h).$$

We obtain the first exact sequence. Since $f_*\mathscr{I}_{M_2/S} = \mathscr{I}_{N_2/X}$, and $R^if_* = 0$ for $i > 0$ because f is affine, the snake lemma gives the second exact sequence. These two sequences are due to Roberts [89].
(d) This is a computation derived from sequence (A).

$$\begin{aligned} g(N_2) &= 1-\chi(\mathscr{O}_{N_2}) \\ &= 1+\chi(\omega_{N_2}) \\ &= 1+\chi((f_*\mathscr{O}_S)(h.h-4))-\chi(\mathscr{O}_{N_1}(h.h-4)) \\ &= 1+\chi(O_S((h.h-4)h))-\chi(\mathscr{O}_{\mathbf{P}^3}(h.h-4))+\chi(\mathscr{O}_{\mathbf{P}^3}(-4)) \\ &= 1+\frac{1}{2}(h.h-4)h.((h.h-4)h+c_1)+\frac{1}{12}(c_1.c_1+\deg c_2)-\frac{1}{6}(h.h-1)(h.h-2)(h.h-3)-1 \\ &= \frac{1}{3}(h.h)^3-3(h.h)^2+\frac{37}{6}(h.h)+\frac{1}{2}(h.h)(h.c_1)-2(h.c_1)+\frac{1}{12}(c_1.c_1+\deg c_2)+1 \end{aligned}$$

In the second line, we used the fact that N_2 is Cohen-Macaulay and Serre duality holds. In the third line, we used the exact sequence (A). In the fourth line, we use the projection formula to conclude that

$$h^i((f_*\mathscr{O}_S)(h.h-4)) = h^i(f_*(\mathscr{O}_S((h.h-4)h))) = h^i(\mathscr{O}_S((h.h-4)h)$$

since π is affine. Since N_1 is a hypersurface of degree $h.h$, there is an exact sequence

$$0 \to \mathscr{O}_{\mathbf{P}^3}(-4) \to \mathscr{O}_{\mathbf{P}^3}(h.h-4) \to \mathscr{O}_{N_1}(h.h-4) \to 0.$$

On the next line, we applied Hirzebruch-Riemann-Roch on the smooth surface S to the line bundle $(h.h-4)h$ and used the fact that $\chi(\mathscr{O}_{\mathbf{P}^3}(d)) = \frac{1}{6}(d+3)(d+2)(d+1)$. It is somewhat curious that this expression always ends up being an integer.
(e) The triple-point formula yields

$$\begin{aligned} [M_3] &= f^*f_*[M_2]-2c_1(\nu)[M_2]+2c_2(\nu) \\ &= ((h.h)^2-12h.h+c_1.h+44)h^2+(2h.h-24)c_1h+4c_1^2-2c_2. \end{aligned}$$

The number of triple points is the degree of $[N_3]$, which is one third the degree of $[M_3]$ since $f_*[M_3] = 3[N_3]$.
(f) Applying Porteous formula to the transpose of the map $\partial f : f^*\Omega_{\mathbf{P}^3/\mathbb{C}} \to \Omega_{S/\mathbb{C}}$ yields the result. Since ramification points map bijectively to pinch points, the number of pinch points is given by the degree of $[R_1]$. □

Generalized divisors

Let $\pi : S \to X$ be a finite birational morphism from a smooth projective surface to an integral singular hypersurface in $\mathbf{P}^3$. Let h denote the class of $\pi^*\mathscr{O}_X(1)$ in $A^1(S)$. We briefly review the definitions and basic operations of generalized divisors established in [51].

Definition II.1.7. Generalized divisors on a Gorenstein scheme X are reflexive fractional ideals of rank one. In particular, curves on Gorenstein surfaces are exactly the effective generalized divisors. An almost Cartier divisor is a generalized divisor that is locally principal away from a closed subset of codimension at least two. Let Cart, ACart and GDiv denote the group of Cartier divisors, the group of almost Cartier divisors and the set of generalized divisors. Let Pic, APic and GPic be the corresponding isomorphism classes.

For example, any curve not supported on any components of the singular locus of a Gorenstein surface in dimension zero is an almost Cartier divisor, but may not be Cartier.

There is a pullback map $\pi^* : \mathrm{ACart}(X) \to \mathrm{ACart}(S) = \mathrm{Cart}(S)$ defined by sending a reflexive fractional ideal $\mathscr{I}$ of rank one to $((\mathscr{O}_S \cdot \mathscr{I})^{-1})^{-1}$. Here $\mathscr{J}^{-1}$ is defined to be $(\mathscr{O}_S :_{\mathscr{K}_S} \mathscr{J})$ for a fractional ideal $\mathscr{J}$, and is isomorphic to its dual $\mathscr{J}^*$. One can verify that π^* is a group homomorphism that descends to $\pi^* : \mathrm{APic}(X) \to \mathrm{Pic}(S)$.

Theorem II.1.8 (Hartshorne-Polini [53, Thm 4.1])**.**

1. *There is a morphism of groups* $\varphi : \mathrm{APic}(X) \to \mathrm{Cart}\, M_2/\pi^* \,\mathrm{Cart}\, N_2$.
2. *There is an exact sequence of groups*

$$0 \to \mathrm{APic}(X) \to \mathrm{Pic}(S) \oplus \mathrm{Cart}\, M_2/\pi^* \,\mathrm{Cart}\, N_2 \to \mathrm{Pic}\, M_2/\pi^* \,\mathrm{Pic}\, N_2 \to 0.$$

The first map of the short exact sequence is given by $\pi^* \oplus \varphi$ *and the second map is given by the difference of the two maps*

$$\mathrm{Pic}(S) \to \mathrm{Pic}\, M_2/\pi^* \,\mathrm{Pic}\, N_2 \text{ and } \mathrm{Cart}\, M_2/\pi^* \,\mathrm{Cart}\, N_2 \to \mathrm{Pic}\, M_2/\pi^* \,\mathrm{Pic}\, N_2.$$

The map $\varphi : \mathrm{APic}(X) \to \mathrm{Cart}\, M_2/\pi^* \,\mathrm{Cart}\, N_2$ is rather important, so we review how it is defined. First we need to observe the following.

Proposition II.1.9. *The map* $\pi : S \to X$ *is the blowup of* X *with center* N_2.

Proof. Recall that $\mathscr{I}_{N_2/X}$ is the conductor of the normalization and thus $\pi_* \mathscr{I}_{M_2/S} = \mathscr{I}_{N_2/X}$. There is an isomorphism of schemes:

$$\mathrm{Bl}_{N_2} X = \mathrm{Proj} \bigoplus_{i=0}^{\infty} \mathscr{I}^i_{N_2/X} \cong \mathrm{Proj} \bigoplus_{i=0}^{\infty} \mathscr{I}^i_{M_2/S} = \mathrm{Bl}_{M_2} S.$$

But $\mathrm{Bl}_{M_2} S = S$ since M_2 is a divisor. □

If C is a curve on X not supported on any components of N_2, then we reserve the notation $\widetilde{C}$ for the proper transform of C on S.

Let D be an almost Cartier divisor, then $[D] = [C_1] - [C_2]$ for two almost Cartier divisors C_1 and C_2 not supported on any components of N_2 by [51, Prop 2.11]. Then $\widetilde{C_1}$ and $\widetilde{C_2}$ are two Cartier divisors on S intersecting M_2 properly, and thus restricts to Cartier divisors $\alpha_1 = \widetilde{C_1} \cap M_2|_{M_2}$ and $\alpha_2 = \widetilde{C_2} \cap M_2|_{M_2}$ on M_2. We define $\varphi[D]$ to be the image of $\alpha_1 - \alpha_2$ in $\mathrm{Cart}\, M_2/N_2$. The map $\varphi : \mathrm{APic}(X) \to \mathrm{Cart}\, M_2/\pi^* \,\mathrm{Cart}\, N_2$ is well-defined by [53, Prop 2.3].

II.2. Curves on Surfaces with Ordinary Singularities

In this section, let X be an integral hypersurface with ordinary singularities in $\mathbf{P}^3$ and let $\pi : S \to X$ be its normalization. We use the M_r and N_r to denote the source and target r-fold points of the composition $f : S \to X \hookrightarrow \mathbf{P}^3$. We determine the homological invariants of a curve C on X in terms of those of its preimage on S.

Linkage on surfaces with ordinary singularities

Recall from Section I.1 that two curves C and D are said to be *linked* on X if D and $\mathscr{O}_X(m) - C$ are linearly equivalent as generalized divisors on X for some positive integer m. As before, we use $\widetilde{C} \subseteq S$ to denote the proper transform of a curve $C \subseteq X$ only when C is not supported on any components of N_2.

Proposition II.2.1. *If C is a curve on X not supported on any components of N_2, then π^*C and $\widetilde{C}$ coincide as Cartier divisors on S.*

Proof. Recall that any reflexive sheaf over S extends uniquely from an open set with a codimension 2 complement [51, Prop 1.11]. Since $\pi^{-1}(C \cap N_2)$ is codimension two in S and $\widetilde{C}$ is equal to π^*C on the open set $S - \pi^{-1}(C \cap N_2)$, it follows that $\widetilde{C}$ is equal to π^*C on the whole of S. □

Corollary II.2.2. *If D is an effective almost Cartier divisor on X, then $\pi_*\pi^*[D] = [D]$ as Chow classes on X. In particular $(\pi^*[D]).h = \deg D$.*

Proof. Since D is almost Cartier, by [51, Prop 2.11] D is linearly equivalent to $C_1 - C_2$, where C_1, C_2 are effective almost Cartier divisors not supported on any components of N_2. Then $[D] = [C_1] - [C_2]$ as Chow classes. Now $\pi_*\pi^*[C_i] = \pi_*[\widetilde{C_i}] = [C_i]$ since $\widetilde{C_i} \to C_i$ is degree one on every component. It follows that $\pi_*\pi^*[D] = [D]$, and the push-pull formula yields the last claim. □

If N_2 were almost Cartier on X, then $\pi_*\pi^*[N_2] = [N_2]$ by the above proposition. However, since $\mathscr{O}_S \cdot \mathscr{I}_{N_2/X} = \mathscr{I}_{M_2/S}$, we must have $\pi^*[N_2] = [M_2]$. This contradicts the fact that $f_*[M_2] = 2[N_2]$. Therefore the generalized divisor N_2 is not almost Cartier on X. Consequently, neither is $N_2 + C$ for any almost Cartier divisor C.

Lemma II.2.3. *If D is a Cartier divisor on X not supported on any components of N_2, then $\pi^*(D \cap N_2) = (\pi^*D) \cap M_2 = \widetilde{D} \cap M_2$ as Cartier divisors of M_2.*

Proof. Suppose D is defined locally by (U_i, f_i). Since D is not supported on N_2, the local sections f_i restrict to non-zerodivisors in $H^0(U_i \cap N_2, \mathscr{K}_{N_2})$. The corresponding local sections $(\pi^{-1}(U_i), \pi^\# f_i)$ of $\mathscr{K}_S$ define a Cartier divisor π^*D not supported on M_2, and thus restrict to non-zerodivisors in $H^0(\pi^{-1}(U_i) \cap M_2, \mathscr{K}_{M_2})$. The three Cartier divisors on M_2 are equal because they are all defined by the data $(\pi^{-1}(U_i) \cap M_2, \pi^\# f_i)$. □

Theorem II.2.4. *Let C and D be two curves on X not supported on any components of N_2, then C and D are linked by $\mathscr{O}_X(m)$ if and only if*

1. $[\widetilde{C}] + [\widetilde{D}] = mh$, *where h is the class of $\pi^*\mathscr{O}_X(1)$;*
2. $\widetilde{C} \cap M_2 + \widetilde{D} \cap M_2$ *is a Cartier divisor of M_2 in $\pi^* \operatorname{Cart} N_2$.*

Proof. Suppose C and D are linked by $\mathscr{O}_X(m)$, where neither are supported on any component of N_2. Since both are almost Cartier divisors and the pullback of almost Cartier divisors is a group homomorphism, we must have (1). Lemma II.2.3 implies we must have (2). Conversely, C and D satisfy (1) and (2). For any codimension 2 point $x \in X$, the map $\varphi_x : \operatorname{APic}(\mathscr{O}_{X,x}) \to \operatorname{Cart} M_{2,x}/\pi^* \operatorname{Cart} N_{2,x}$ is an isomorphism by [53, Thm 3.1]. Since (2) is satisfied, $C + D$ has image 0 in $\operatorname{APic}(\mathscr{O}_{X,x})$ for every codimension 2 point $x \in X$ and thus $C + D$ is Cartier. Since both $C + D$ and $\mathscr{O}_X(m)$ pull back to mh, we conclude that they are isomorphic since the map $\pi^* : \operatorname{Pic}(X) \to \operatorname{Pic}(S)$ is injective over the complex numbers by [53, Thm 4.5]. We refer to Figure 5 for an illustration. □

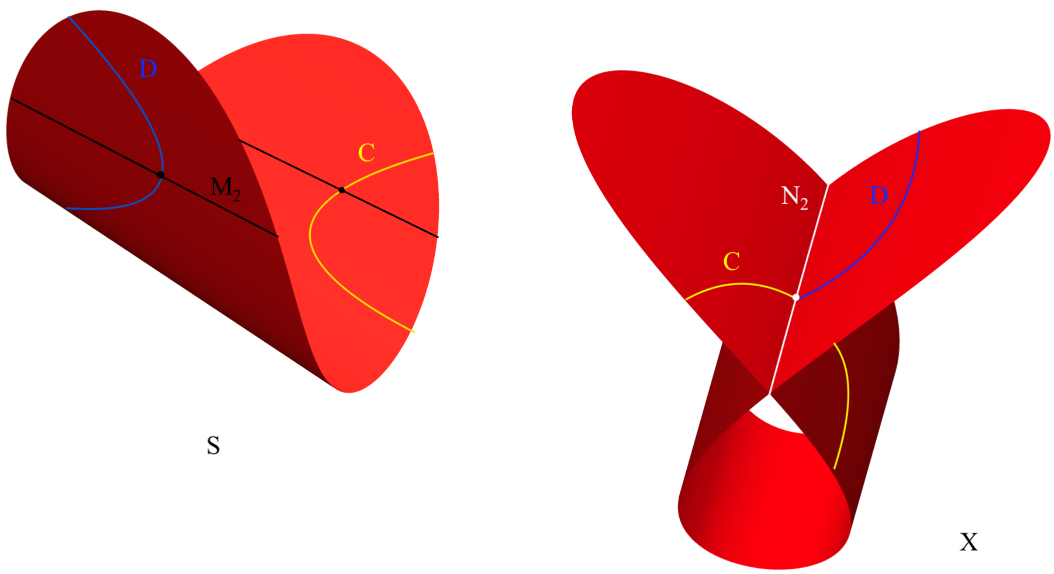

Figure 5: Linked curves must meet at involution points

Theorem II.2.5. *Suppose $\pi : S \to X$ is induced by a general linear projection $f : S \to \mathbf{P}^3$ of a nondegenerate smooth surface S in $\mathbf{P}^5$. Suppose C is an integral curve on S meeting M_2 transversely avoiding R_1. If $m = 2C.M_2/h.M_2$ is a positive integer such that $mH - C$ is effective and $h^0(\mathscr{O}_S(mH - C)) > C.M_2$, then there is a curve D on S such that $\pi(C)$ and $\pi(D)$ are linked on X.*

Proof. Since C is a Cartier divisor on S, it meets M_2 at a Cartier divisor. Since C meets M_2 transversely, the intersection $C \cap M_2$ is a reduced set of points $p_1, \ldots, p_n$ outside R_1 and M_3 where $n = C.M_2$. Thus each point p_i has precisely one corresponding reduced point q_i in $f^{-1}f(p_i)$. It follows that $p_i + q_i = \pi^{-1}\pi(p_i)$ is a Cartier divisor on M_2 in the image of $\pi^* \operatorname{Cart} N_2$. By the proposition above, we need to find an effective divisor D in the class of $mH - C$ which meets M_2 at $q_1, \ldots, q_n$. The choice of m guarantees that $D.M_2 = C.M_2 = n$. This is always possible if $h^0(\mathscr{O}_S(mH - C)) > n$ considering the exact sequence

$$0 \to H^0(\mathscr{I}_Q(mH - C)) \to H^0(\mathscr{O}_S(mH - C)) \to H^0(\mathscr{O}_Q),$$

where Q is the subscheme of S consisting of the reduced points $q_1, \ldots, q_n$. □

Preserved curves

Definition II.2.6. We say a curve C on X is *preserved* if a curve C' on S maps isomorphically to C.

Proposition II.2.7. *Let C be a preserved curve on X. If M_2 is irreducible, then so is N_2, and C is not supported on N_2.*

Proof. The map $M_2 \to N_2$ is not injective on the level of topological spaces. If it were, then $\pi : S \to X$ would be a homeomorphism of topological spaces. But both S and X are reduced, which would imply that π is an isomorphism, contrary to our assumption. If C contains N_2 set-theoretically, then any curve C' on S mapping onto C must contain M_2 set-theoretically, and thus cannot be mapped isomorphically to C. □

Preserved curves may be supported on N_2 when M_2 is not irreducible. If X is the union of two planes meeting at a line $L = N_2$ in $\mathbf{P}^3$, and S is its normalization given by the disjoint union of two planes, then L is preserved since it is the isomorphic image of any one of the two lines on S.

Proposition II.2.8. *Let C be a curve on X not supported on any component of N_2, then C is preserved if and only if $\widetilde{C} \to C$ is an isomorphism if and only if $C \cap N_2$ is a Cartier divisor on C. In particular, smooth curves on X not supported on any components of N_2 are preserved.*

Proof. Since C is not supported on any components of N_2, the proper transform $\widetilde{C}$ on S is isomorphic to the blowup of C at $C \cap N_2$ by the universal property of the blowup. If C is a preserved curve, then the curve on S that maps isomorphically to C must be its proper transform $\widetilde{C}$. It follows that $\widetilde{C}$ is isomorphic to C if and only if $C \cap N_2$ is a Cartier divisor on C. □

Theorem II.2.9. *For an almost Cartier curve C on X, we define the rational number*

$$m := \frac{2(\pi^*C).M_2 - 2(g(C) - g(\pi^*C))}{h.M_2}.$$

Here the genus of a divisor is defined by the adjunction formula, which agrees with the arithmetic genus when the divisor is effective.

1. *If C is linked to a preserved curve D by $\mathscr{O}_X(n)$ for some $n > 0$, then $n = m$.*
2. *Conversely, if m is a positive integer, then any nonzero section of $\mathscr{I}_{C/X}(m)$ defines a curve D that is either preserved or is supported on a component of N_2.*

Proof. Suppose C is linked to a preserved curve D by $\mathscr{O}_X(n)$ for some n. Then the arithmetic genus of C and D are related by

$$g(D)-g(C) = \frac{1}{2}(h^2+n-4)(\deg D-\deg C) = \frac{1}{2}(h^2+n-4)(h^2n-2\deg C) \tag{II.1}$$

using liaison theory, see for example [69, III Prop.1.2]. Since $\pi^* : \mathrm{APic}(X) \to \mathrm{Pic}(S)$ is a group homomorphism, we have $\widetilde{D} = nh - \pi^*C$ in $\mathrm{Pic}(S)$. The adjunction formula on S yields

$$2g(\widetilde{D}) - 2 = \widetilde{D}.(\widetilde{D} + c_1) = (nh - \pi^*C).(nh - \pi^*C + c_1). \tag{II.2}$$

Since $\widetilde{D} \to D$ is an isomorphism, we have $g(\widetilde{D}) = g(D)$. Combining (1) and (2), we have an equality

$$(h^2+n-4)(nh^2-2\deg C)+2g(C)-2 = (nh-\pi^*C).(nh-\pi^*C+c_1).$$

Now $\pi^*C.h = \deg C$ since C is almost Cartier by Corollary II.2.2 and $M_2 = (h^2 - 4)h - c_1$ by Theorem II.1.3. After substitution, we arrive at the linear equation in n

$$nh.M_2 = 2M_2.\pi^*C - 2(g(C) - g(\pi^*C)).$$

Since $h.M_2 = 2\deg N_2 \neq 0$, we see that n must be equal to

$$n = \frac{M_2.\pi^*C - (g(C) - g(\pi^*C))}{h.M_2}.$$

This proves the necessary direction of the theorem.

Conversely, suppose m is a positive integer and let D be defined by a nonzero section of $\mathscr{I}_{C/X}(m)$. If D is not supported on any components of N_2, then we have an exact sequence of sheaves on D

$$0 \to \mathscr{O}_D \to \pi_*\mathscr{O}_{\widetilde{D}} \to \mathscr{K} \to 0,$$

where K is supported on the zero dimension scheme $D \cap N_2$. By the choice of m, we see that $g(D) = g(\widetilde{D})$ by the same computation as above. Therefore $\chi(\mathscr{O}_D) = \chi(\mathscr{O}_{\widetilde{D}})$ and $\chi(\mathscr{K}) = 0$. Since $\mathscr{K}$ is supported on a dimension zero subscheme it follows that $\mathscr{K} = 0$. We conclude that $\widetilde{D} \to D$ is an isomorphism. ∎

Note that we do not write $\widetilde{C}$ since C could be supported on a component of N_2 although it is almost Cartier, in which case the strict transform $\widetilde{C}$ is undefined.

Corollary II.2.10. *For an almost Cartier divisor D on X, there is at most one integer m where $[D + \mathscr{O}_X(m)]$ contains a preserved curve.*

Proof. Suppose $[D_1] = [D + \mathscr{O}_X(n_1)]$ and $[D_2] = [D + \mathscr{O}_X(n_2)]$ are two classes that contain preserved curves D_1 and D_2. Then for $l \gg 0$, the class $[\mathscr{O}_X(l) - D_1]$ contains an almost Cartier curve C. Since C is linked to both D_1 and D_2, we must have $n_1 = n_2$ by Theorem II.2.9 (1). □

Corollary II.2.11. *Let $X \subset \mathbf{P}^3$ be an integral hypersurface with ordinary surface singularities. If the singular locus of X is irreducible, then any two smooth curves in the same biliaison class on X are linearly equivalent.*

Proof. This is true for two smooth curves not supported on the singular locus N_2 by the previous corollary. If two smooth curves are supported on N_2, since N_2 is irreducible, the two curves must be the same. □

The situation is very different on a smooth projective surface S with an ample divisor h. If C is any divisor on S, then for any $m \gg 0$, the linear system $|C + mh|$ is basepoint-free and contains a smooth curve by Bertini's theorem.

Homological invariants

In this subsection, we study homological invariants of curves on the singular surface X.

Lemma II.2.12. *Let C be a curve with an effective line bundle $\mathscr{L}$. If C is reduced, then $h^0(\mathscr{L}^{-1}) = 0$.*

Proof. Let $p : \widetilde{C} \to C$ be the normalization, then $\widetilde{C}$ is a disjoint union of nonsingular curves. Since $p^*\mathscr{L}$ is effective on each component of $\widetilde{C}$, it follows that $h^0(\widetilde{C}, (p^*\mathscr{L})^{-1}) = 0$. There is an injection of $\mathscr{O}_C \hookrightarrow p_*\mathscr{O}_{\widetilde{C}}$ and therefore

$$H^0(C, \mathscr{L}^{-1}) \hookrightarrow H^0(C, (p_*\mathscr{O}_{\widetilde{C}}) \otimes \mathscr{L}^{-1}) \cong H^0(\widetilde{C}, (p^*\mathscr{L})^{-1}) = 0.$$ □

The assumption that C is of pure dimension 1 and reduced cannot be dropped, as the following two counter-examples demonstrate. (1) A line with an embedded point has sections in infinitely many negative degrees. (2) Let E be the exceptional curve of the blowup of $\mathbf{P}^2$ at origin, then any curve D in $|3E|$ is non-reduced. If H is the very ample line bundle of conics through the origin, then $h^0(\mathscr{O}_D(-H)) \neq 0$ by a simple computation.

Corollary II.2.13. *Let S be a smooth surface with a very ample line bundle $\mathscr{L}$ whose class in $A^1(S)$ is h. If $h^0(\mathscr{L}) = 4$ or $h^0(\mathscr{L}) \geq 6$, then $h^1(\mathscr{L}^n) = h^2(\mathscr{L}^n) = 0$ for $n > h.h - 4$.*

Proof. If $h^0(\mathscr{L}) = 4$, then S can be embedded as a hypersurface in $\mathbf{P}^3$ and $h^1(\mathscr{L}^n) = 0$ for all n. If $h^0(\mathscr{L}) \geq 6$, then a general choice of four sections gives map $f : S \to \mathbf{P}^3$ that satisfies the Corollary II.1.6. Let X be the image hypersurface and consider the short exact sequence

$$0 \to \mathscr{O}_X \to f_*\mathscr{O}_S \to \omega_{N_2}(4 - h.h) \to 0.$$

The long exact sequence of cohomologies yields

$$0 \to H^1((f_*\mathscr{O}_S) \otimes \mathscr{O}_X(n)) \to H^1(\omega_{N_2}(4 - h.h + n)).$$

Since $f_*(\mathscr{O}_S \otimes f^*\mathscr{O}_X(n)) = (f_*\mathscr{O}_S) \otimes \mathscr{O}_X(n)$ and f is affine, it follows that the left term is just $H^1(\mathscr{L}^n)$. Since N_2 is Cohen-Macaulay, the right term is dual to $H^0(\mathscr{O}_{N_2}(-n+h.h-4))$, which vanishes if N_2 is reduced and $n > h.h - 4$ by the previous lemma. Since X is a hypersurface of degree $h.h$, we have $h^2(\mathscr{O}_X(n)) = h^0(\mathscr{O}_X(-n + h.h - 4))$, which vanishes if $n > h.h - 4$. It follows from the long exact sequence of cohomologies that $h^2(\mathscr{L}^n) =$ for $n > h.h - 4$ since ω_{N_2} has one-dimesional support. □

Proposition II.2.14. *If D is a preserved curve on X and $h^1(\mathscr{O}_S(nh)) = 0$ for some n, then $h^1(\mathscr{O}_D(n)) = h^2(\mathscr{O}_S(nh - \widetilde{D})) - h^2(\mathscr{O}_S(nh))$.*

Proof. We have an exact sequence

$$H^1(\mathscr{O}_S(nh)) \to H^1(\mathscr{O}_{\widetilde{D}}(nh)) \to H^2(\mathscr{I}_{\widetilde{D}/S}(nh)) \to H^2(\mathscr{O}_S(nh)) \to 0.$$

Note that $H^1(\mathscr{O}_{\widetilde{D}}(nh)) = H^1(\mathscr{O}_D(n))$ since D is preserved. □

Proposition II.2.15. *If C is a curve linked to a preserved curve D on X by $\mathscr{O}_X(m)$ and $h^1(\mathscr{O}_S((h.h{+}m{-}n{-}4)h) = 0$, then*

$$h^0(\mathscr{I}_{C/\mathbf{P}^3}(n)) = h^0(\mathscr{I}_T(n)) + h^0(\mathscr{O}_S(nh - \pi^*C - M_2)) - h^0(\mathscr{O}_S((n-m)h - M_2)).$$

Here T is the $(h.h, m)$-complete intersection in $\mathbf{P}^3$ linking C and D, and

$$h^0(\mathscr{I}_T(n)) = \binom{n-m+3}{3} + \binom{n-h.h+3}{3} - \binom{n-m-h.h+3}{3}.$$

Proof. By the linkage theory of curves [see 69, Prop 1.2], we have

$$h^0(\mathscr{I}_{C/\mathbf{P}^3}(n)) = h^0(\mathscr{I}_T(n)) + h^1(\mathscr{O}_D(h^2 + m - n - 4)).$$

From the Koszul complex

$$0 \to \mathscr{O}_{\mathbf{P}^3}(-m-h^2) \to \mathscr{O}_{\mathbf{P}^3}(-m) \oplus \mathscr{O}_{\mathbf{P}^3}(-h^2) \to \mathscr{I}_T \to 0$$

it follows that

$$h^0(\mathscr{I}_T(n)) = \binom{n-m+3}{3} + \binom{n-h^2+3}{3} - \binom{n-m-h^2+3}{3}.$$

By Proposition II.2.14, if $h^1(\mathscr{O}_S((h^2{+}m{-}n{-}4)h)) = 0$, then

$$h^1(\mathscr{O}_D(h^2{+}m{-}n{-}4)) = h^2(\mathscr{O}_S((h^2{+}m{-}n{-}4)h{-}\widetilde{D}){-}h^2(\mathscr{O}_S((h^2{+}m{-}n{-}4)h)$$

$$= h^0(\mathscr{O}_S(\widetilde{D}{-}(h^2{+}m{-}n{-}4)h{-}c_1)){-}h^0(\mathscr{O}_S(-(h^2{+}m{-}n{-}4)h{-}c_1))$$
$$= h^0(\mathscr{O}_S(nh{-}\pi^*C{-}M_2)){-}h^0(\mathscr{O}_S((n{-}m)h{-}M_2)).$$

□

Proposition II.2.16. *Let C be a preserved curve on X that is linked to a preserved curve D by $\mathscr{O}_X(m)$. If $h^1(\mathscr{O}_S(lh)) = 0$ for all l, then*

$$\begin{aligned} h^1(\mathscr{I}_{C/\mathbf{P}^3}(n)) =& h^0(\mathscr{O}_S(nh)){-}h^0(\mathscr{O}_S(nh{-}\widetilde{C})){+}h^1(\mathscr{O}_S(nh{-}\widetilde{C})){-}h^0(\mathscr{O}_T(n)) \\ &+h^0(\mathscr{O}_S(nh{-}\widetilde{C}{-}M_2)){-}h^0(\mathscr{O}_S((n{-}m)h{-}M_2)) \end{aligned}$$

Here T is a $(m, h.h)$-complete intersection curve in $\mathbf{P}^3$ as before.

Proof. Since $h^1(\mathscr{O}_S(lh)) = 0$ for all l by assumption, the formulas for $h^0(\mathscr{I}_{C/\mathbf{P}^3}(n))$ and $h^2(\mathscr{I}_{C/\mathbf{P}^3}(n)) = h^1(\mathscr{O}_C(n))$ are given by the previous two propositions. Thus we have

$$\begin{aligned} h^1(\mathscr{I}_{C/\mathbf{P}^3}(n)) &= h^0(\mathscr{O}_C(n)){-}h^0(\mathscr{O}_{\mathbf{P}^3}(n)){+}h^0(\mathscr{I}_{C/\mathbf{P}^3}(n)) \\ &= h^0(\mathscr{O}_C(n)){-}h^0(\mathscr{O}_{\mathbf{P}^3}(n)){+}h^0(\mathscr{I}_T(n)){+}h^1(\mathscr{O}_D(h.h{+}m{-}n{-}4)) \\ &= h^0(\mathscr{O}_C(n)){-}h^0(\mathscr{O}_T(n)){+}h^1(\mathscr{O}_D(h.h{+}m{-}n{-}4)). \end{aligned}$$

Since $h^1(\mathscr{O}_S(l)) = 0$ for all l, we have

$$h^0(\mathscr{O}_C(n)) = h^0(\mathscr{O}_C(nh)) = h^0(\mathscr{O}_S(nh)){-}h^0(\mathscr{I}_{\widetilde{C}/S}(nh)){+}h^1(\mathscr{I}_{\widetilde{C}/S}(nh)).$$

We also rewrite $h^1(\mathscr{O}_D(h.h{+}m{-}n{-}4))$ as in the proof of the previous proposition. □

The following useful proposition by Gruson-Peskine allows us to compute the dimension of the sections of the normal bundle of a smooth curve on the singular surface X. By deformation theory, this is equal to the dimension of the tangent space of the Hilbert scheme at the closed point corresponding to the curve.

Proposition II.2.17 (Gruson-Peskine)**.** *Let C be a smooth connected curve on X not supported on N_2, whose proper transform $\widetilde{C}$ avoids R_1. Then there is an exact sequence of bundles*

$$0 \to \mathscr{N}_{\widetilde{C}/S} \to \mathscr{N}_{C/\mathbf{P}^3} \to \mathscr{O}_S(4h - c_1) \otimes \mathscr{O}_{\widetilde{C}} \to 0.$$

Proof. Technically we should pullback all sheaves to $\widetilde{C}$ or pushforward all sheaves to C, but we omit this from the notations since $\pi : \widetilde{C} \to C$ is an isomorphism. The map of sheaves

$$\mathcal{T}_S \otimes \mathscr{O}_{\widetilde{C}} \to \mathcal{T}_X \otimes \mathscr{O}_C$$

is injective and locally split since the map $S \to X$ is an immersion away from R_1. Thus the dual morphism

$$\Omega_X \otimes \mathscr{O}_C \to \Omega_S \otimes \mathscr{O}_{\widetilde{C}}$$

is surjective. On the other hand, we have a surjection of sheaves

$$\Omega_{\mathbf{P}^3} \otimes \mathscr{O}_C \to \Omega_X \otimes \mathscr{O}_C$$

and thus the composition

$$\Omega_{\mathbf{P}^3} \otimes \mathscr{O}_C \to \Omega_S \otimes \mathscr{O}_{\widetilde{C}}$$

is surjective. Applying the snake lemma to the diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{I}_{C/\mathbf{P}^3} \otimes \mathscr{O}_C & \longrightarrow & \Omega_{\mathbf{P}^3} \otimes \mathscr{O}_C & \longrightarrow & \Omega_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathscr{I}_{\widetilde{C}/S} \otimes \mathscr{O}_{\widetilde{C}} & \longrightarrow & \Omega_S \otimes \mathscr{O}_{\widetilde{C}} & \longrightarrow & \Omega_{\widetilde{C}} & \longrightarrow & 0, \end{array}$$

we conclude that $\mathscr{I}_{C/\mathbf{P}^3} \otimes \mathscr{O}_C \to \mathscr{I}_{\widetilde{C}/S} \otimes \mathscr{O}_{\widetilde{C}}$ is a surjection of bundles. Dualizing, we obtain a locally split exact sequence of bundles

$$0 \to \mathscr{N}_{\widetilde{C}/S} \to \mathscr{N}_{C/\mathbf{P}^3} \to \mathscr{N}_1 \to 0.$$

Taking top wedge power, we see that

$$\wedge^2 \mathscr{N}_{C/\mathbf{P}^3} = \mathscr{N}_{C/S} \otimes \mathscr{N}_1.$$

Taking third wedge power of the conormal sequence

$$0 \to \mathscr{N}^*_{C/\mathbf{P}^3} \to \Omega_{\mathbf{P}^3} \otimes \mathscr{O}_C \to \Omega_C \to 0,$$

it follows that

$$\mathscr{O}_C(-4) = \omega_{\mathbf{P}^3} \otimes \mathscr{O}_C = \wedge^3(\Omega_{\mathbf{P}^3} \otimes \mathscr{O}_C) = \wedge^2 \mathscr{N}^*_{C/\mathbf{P}^3} \otimes \omega_C.$$

Combining, we arrive at

$$\mathscr{N}_1 = (\wedge^2 \mathscr{N}_C) \otimes \mathscr{N}^*_{C/S} = \mathscr{I}_{C/S} \otimes \omega_C(4) = \mathscr{I}_{C/S} \otimes \mathscr{O}_S(C) \otimes \omega_S \otimes \mathscr{O}_C(4) = \mathscr{O}_S(4h - c_1) \otimes \mathscr{O}_C. \quad \square$$

II.3. Projections of Curves on Rational Normal Scrolls

In this section, we apply results of Section II.2 to study rational normal scrolls $S(a,b) \subseteq \mathbf{P}^{a+b+1}$ and their general linear projections $f : S(a,b) \to \mathbf{P}^3$. We refer the readers to [24] and [45, §V.2] for basic facts of rational normal scrolls and ruled surfaces.

Let $a \leq b$ be two positive integers and let $\mathscr{E}$ be the rank two bundle $\mathscr{O}_{\mathbf{P}^1}(a) \oplus \mathscr{O}_{\mathbf{P}^1}(b)$ on $\mathbf{P}^1$. Then $p : \mathrm{Proj}(\mathrm{Sym}\,\mathscr{E}) \to \mathbf{P}^1$ is a ruled surface over $\mathbf{P}^1$. The surface $S := \mathrm{Proj}(\mathrm{Sym}\,\mathscr{E})$ has a tautological bundle $\mathscr{O}_S(1)$ which is very ample and embeds S into $\mathbf{P}^{a+b+1}$ with image $S(a,b)$. The surface S is isomorphic to the Hirzebruch surface H_e where $e := b-a$. Let η denote the unique $(-e)$-curve on S and let $\mathfrak{f}$ denote the class of a fiber of $p : S \to \mathbf{P}^1$, then the Chow ring of S is given by

$$A(S) = A(\mathbf{P}^1)[\eta]/(\eta^2 - c_1(\mathscr{E})\eta + c_2(\mathscr{E})) = \mathbb{Z}[\mathfrak{f},\eta]/(\eta^2 + (b-a)\mathfrak{f}\eta, \mathfrak{f}^2).$$

In the following, we express the divisors in the coordinates given by the basis $\{\eta, \mathfrak{f}\}$ and use the shorthand $\mathscr{O}_S(c,d)$ for $\mathscr{O}_S(c\eta + d\mathfrak{f})$. In particular, the class of $O_S(1)$ is $h = \eta + b\mathfrak{f}$. From the exact sequences

$$0 \to \mathscr{T}_{S/\mathbf{P}^1} \to \mathscr{T}_S \to \mathscr{T}_{\mathbf{P}^1} \to 0$$
$$0 \to \mathscr{O}_S \to (p^*\mathscr{E}^*)(1) \to \mathscr{T}_{S/\mathbf{P}^1} \to 0,$$

we conclude that

$$\begin{aligned} c(\mathscr{T}_S) &= p^*c(\mathscr{T}_{\mathbf{P}^1})c((p^*\mathscr{E}^*)(1)) \\ &= (1+2\mathfrak{f})(1+(b-a)\mathfrak{f}+2\eta+(b-a)\mathfrak{f}\eta+\eta^2) \\ &= (1+2\mathfrak{f})(1+(b-a)\mathfrak{f}+2\eta) \\ &= 1+(b-a+2)\mathfrak{f}+2\eta+4\eta\mathfrak{f}. \end{aligned}$$

Therefore $c_1 = 2\eta+(b-a+2)\mathfrak{f}$ and $c_2 = 4\eta\mathfrak{f}$.

A straightforward substitution of the above calculations into the formulas of Corollary II.1.6 yields the following.

Proposition II.3.1. *Suppose $(a,b) \neq (1,1)$. Let $f : S \to \mathbf{P}^3$ be a general linear projection of $S(a,b) \subseteq \mathbf{P}^{a+b+1}$ with image X. Then*

$$\begin{aligned} \deg N_2 &= \frac{1}{2}(b+a-2)(b+a-1), \\ g(N_2) &= \frac{1}{6}(b+a-3)(b+a-4)(2b+2a-1), \\ \deg N_3 &= \frac{1}{3}(b+a-2)(b+a-3)(b+a-4), \\ \deg R_1 &= 2b+2a-4. \end{aligned}$$

The curve N_2 is integral of degree $\deg N_2$ and genus $g(N_2)$, and the number of singular points of N_2 is equal to $\deg N_3$. The curve M_2 is reduced and connected and maps 2-1 to N_2, ramified over $2a+2b-4$ pinch points of X.

The next proposition is a well-known fact about rational normal scrolls.

Proposition II.3.2. *Let* $S = S(a,b) \subseteq \mathbf{P}^{a+b+1}$, *then*

$$H^1(\mathscr{I}_S(l)) = H^2(\mathscr{I}_S(l)) = H^1(\mathscr{O}_S(l)) = 0, \quad \forall \in \mathbb{Z}.$$

Proof. The ideal I of $S(a,b)$ in $\mathbf{P}^{a+b+1}$ is defined by the maximal minors of the matrix

$$\begin{pmatrix} x_0 & \dots & x_{a-1} & y_0 & \dots & y_{b-1} \\ x_1 & \dots & x_a & y_1 & \dots & y_b \end{pmatrix}$$

in the ring $R := \mathbb{C}[x_0, \dots, x_a, y_1, \dots, y_b]$. Since the matrix is 1-generic, the R-module R/I admits a minimal free R-resolution given by the Eagon-Northcott complex which is of length $a+b-1$. In particular, it follows from local duality that

$$H^i_m(R/I) = \operatorname{Ext}_R^{a+b+2-i}(R/I, R(-a-b-2))^\vee = 0, \quad i = 1, 2.$$

Here $H^i_m(-)$ denotes the i-th local cohomology supported on the irrelevant ideal m of R. The local-to-sheaf exact sequence gives $H^1_m(R/I) \cong \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{I}_S(n))$ and $H^2_m(R/I) \cong \bigoplus_{l\in\mathbb{Z}} H^1(\mathscr{O}_S(l))$. We conclude that $H^1(\mathscr{I}_S(l)) = H^1(\mathscr{O}_S(l)) = 0$ for all integers l. It also follows that $H^2(\mathscr{I}_S(l)) = 0$ for all l by the short exact sequence

$$0 \to \mathscr{I}_S \to \mathscr{O}_{\mathbf{P}^{a+b+1}} \to \mathscr{O}_S \to 0.$$

□

Lemma II.3.3. *A nontrivial divisor* $D = c\eta + d\mathfrak{f}$ *on* $S(a,b)$ *is effective if and only if* $c \geq 0, d \geq 0$. *An effective divisor* D *has natural cohomology, i.e.* $h^1(\mathscr{O}_S(D)) = h^2(\mathscr{O}_S(D)) = 0$, *if and only if* $d \geq c(b-a) - 1$.

Proof. Suppose D is effective and $c < 0$, then $p(D)$ is a finite set of points on $\mathbf{P}^1$ since $D.\mathfrak{f} < 0$. It follows that D is concentrated on finitely many fibers, and is thus linearly equivalent to $d\mathfrak{f}$ for some $d > 0$. This is a contradiction to $c < 0$. It follows that effective divisors must have $c \geq 0$.

Suppose $D = c\eta + d\mathfrak{f}$ is divisor where $c \geq 0$. Since $D.\mathfrak{f} \geq 0$, Grauert's theorem implies that $R^i p_*(\mathscr{L}(D)) = 0$ for all $i > 0$. Therefore the Lerray spectral sequence degenerates and $H^i(\mathscr{L}(D)) \cong H^i(\pi_*\mathscr{L}(D))$. By projection formula, we have

$$\begin{aligned} h^i(\pi_*\mathscr{L}(D)) &= h^i(\pi_*(\mathscr{O}_S(c) \otimes \pi^*\mathscr{O}_{\mathbf{P}^1}(d))) \\ &= h^i((\operatorname{Sym}^c \mathscr{E}) \otimes \mathscr{O}_{\mathbf{P}^1}(d)) \\ &= \sum_{i=0}^{c} h^i(\mathscr{O}_{\mathbf{P}^1}(-i(b-a)+d)). \end{aligned}$$

It follows that D is effective if and only if $d \geq 0$. It also follows that an effective divisor D has natural cohomology if and only if $d - c(b-a) > -2$. □

Theorem II.3.4. *Let C be a smooth connected curve on X not supported on N_2 and whose proper transform $\widetilde{C}$ avoids R_1. Suppose $\widetilde{C}$ is of class $c\eta + d\mathfrak{f}$, then $h^0(\mathscr{N}_{C/\mathbf{P}^3})$ and $h^1(\mathscr{N}_{C/\mathbf{P}^3})$ can be computed explicitly in terms of a, b, c, d. In particular, if $c \leq 3$ or $d < 4b$, then $h^1(\mathscr{N}_{C/\mathbf{P}^3}) = 0$ and C corresponds to a smooth point in the Hilbert scheme of curves in $\mathbf{P}^3$. If $c \geq 4$ and $d \geq c(b-a) - 1 + 4a$, then*

$$\begin{aligned} h^0(\mathscr{N}_{C/\mathbf{P}^3}) &= \frac{1}{2}(ac^2 - bc^2 + ac - bc) + cd + 6a + 6b + c + d - 3 \\ &= \dim|C| + 6a + 6b - 3 - 4\deg C. \\ h^1(\mathscr{N}_{C/\mathbf{P}^3}) &= \frac{1}{2}(ac^2 - bc^2 - 7ac - bc) + cd + 6a + 6b + c - 3d - 3 \\ &= \dim|C| + 6a + 6b - 3. \end{aligned}$$

Proof. Since $\widetilde{C}$ is a smooth curve, we have $d \geq c(b-a)$ by [45, Cor V.2.18]. Lemma II.3.3 implies that $\mathscr{O}_S(\widetilde{C})$ has natural cohomology. The short exact sequence

$$0 \to \mathscr{O}_S \to \mathscr{O}_S(\widetilde{C}) \to \mathscr{O}_{\widetilde{C}} \otimes \mathscr{O}_S(\widetilde{C}) \to 0$$

implies that $H^1(\mathscr{N}_{\widetilde{C}/S}) = H^1(\mathscr{O}_{\widetilde{C}} \otimes \mathscr{O}_S(\widetilde{C})) = 0$ since $H^1(\mathscr{O}_S) = H^2(\mathscr{O}_S) = 0$. The short exact sequence from Proposition II.2.17

$$0 \to \mathscr{N}_{\widetilde{C}/S} \to \mathscr{N}_{C/\mathbf{P}^3} \to \mathscr{O}_S(4h - c_1) \otimes \mathscr{O}_{\widetilde{C}} \to 0$$

thus implies that $H^1(\mathscr{N}_{C/\mathbf{P}^3}) = H^1(\mathscr{O}_S(4h - c_1) \otimes \mathscr{O}_{\widetilde{C}})$ since the sheaves are supported on a curve. Finally, consider the short exact sequence

$$0 \to \mathscr{I}_{\widetilde{C}/S}(4h - c_1) \to \mathscr{O}_S(4h - c_1) \to \mathscr{O}_S(4h - c_1) \otimes \mathscr{O}_{\widetilde{C}} \to 0.$$

The divisor $4h - c_1$ has coordinates $(2, 3b + a - 2)$ in the basis $\{\eta, f\}$, which is effective with natural cohomology by Lemma II.3.3. We conclude that

$$H^1(\mathscr{N}_{C/\mathbf{P}^3}) \cong H^1(\mathscr{O}_S(4h - c_1) \otimes \mathscr{O}_{\widetilde{C}}) \cong H^2(\mathscr{I}_{\widetilde{C}/S}(4h - c_1)) \cong H^0(\mathscr{O}_S(C - 4h))^\vee.$$

If either $c < 4$ or $d < 4b$, then $h^0(\mathscr{O}_S(C - 4h)) = 0$. If $c \geq 4$, then

$$h^0(\mathscr{O}_S(C - 4h)) = h^0(\mathscr{O}_S(c - 4, d - 4b)) = \sum_{i=0}^{c-4} h^0(\mathscr{O}_{\mathbf{P}^1}(-i(b-a) + d - 4b)).$$

Since $\chi(\mathscr{N}_{C/\mathbf{P}^3}) = 4\deg C$, it follows that

$$h^0(\mathscr{N}_{C/\mathbf{P}^3}) = 4(d + ac) + \sum_{i=0}^{c-4} h^0(\mathscr{O}_{\mathbf{P}^1}(-i(b-a) + d - 4b)).$$

If $d \geq c(b-a) - 1 + 4a$, then $\mathscr{O}_S(c-4, d-4b)$ is effective with natural cohomology by Lemma II.3.3. We apply Riemann-Roch formula on S to obtain

$$h^1(\mathscr{N}_{C/\mathbf{P}^3}) = \chi(\mathscr{O}_S(C-4h)) = \frac{1}{2}(ac^2 - bc^2 - 7ac - bc) + cd + 6a + 6b + c - 3d - 3.$$

Recall that the intersection formula states that

$$(-C).4h = \chi(\mathscr{O}_S) - \chi(\mathscr{O}_S(C)) - \chi(-4h) + \chi(\mathscr{O}_S(C-4h)).$$

It follows that $h^1(\mathscr{N}_{C/\mathbf{P}^3}) = \dim|C| + 6a + 6b - 3 - 4\deg C$ and $h^0(\mathscr{N}_{C/\mathbf{P}^3}) = \dim|C| + 6a + 6b - 3$. □

The dimension of the family of surfaces X obtained from linear projections of $S(a,b) \subseteq \mathbf{P}^{a+b+1}$ into $\mathbf{P}^3$ is given by

$$\dim \mathbb{G}\mathrm{r}(a+b-3, a+b+1) + \dim \mathrm{PGL}(3,\mathbb{C}) - \dim \mathrm{Aut}(S(a,b)).$$

Recall that there is an exact sequence of groups for the Hirzebruch surface H_e

$$0 \to H^0(\mathscr{O}_{\mathbf{P}^1}(e)) \rtimes \mathbb{C}^* \to \mathrm{Aut}(H_e) \to \mathrm{PGL}(2,\mathbb{C}) \to 0.$$

In particular, we deduce that $\dim \mathrm{Aut}(S(a,b)) = b - a + 5$. It follows that there is a $(5a+3b+2)$-dimension family of integral surfaces of degree $a+b$ in $\mathbf{P}^3$ arising as general linear projections of the scroll $S(a,b) \subseteq \mathbf{P}^{a+b+1}$. If we vary the curve $\widetilde{C}$ in the linear system as well, we end up with a family of curves in $\mathbf{P}^3$ of dimension $\dim|C| + 5a + 3b + 2$. Suppose linear system $\widetilde{C} = c\eta + df$ satisfies $c \geq 4$ and $d \geq c(b-a) - 1 + 4a$, then the difference between $h^0(\mathscr{N}_{C/\mathbf{P}^3})$ and the dimension of the family of the curves is $a + 3b - 5$, which does not depend on the class of $\widetilde{C}$. There are two possibilities in this situation. Either the family of curves are not dense in the component of the Hilbert scheme they belong to, or there is a highly nonreduced component of the Hilbert scheme whose general member is given by such a curve. We are not able to determine which is the case.

Maximal rank curves on the ruled cubic surface

In this subsection, we continue the study of curves on a ruled cubic surface initiated by Hartshorne [51]. Consider the general projection of $S(1,2) \subseteq \mathbf{P}^4$ into $\mathbf{P}^3$, its image X is a ruled cubic surface with singularity a double line N_2 that has 2 pinch points on it by Proposition II.3.1. The map $\pi : S \to X$ is an isomorphism away from the conic M_2 and the line N_2, and maps M_2 generically 2-1 to N_2 branched over the two pinch points of X (see Figure 6).

Since all integral ruled cubic surfaces that are not cones all differ by a coordinate change in $\mathbf{P}^3$ [see 41], we can afford to work explicitly. In doing so, we verify the formulas and

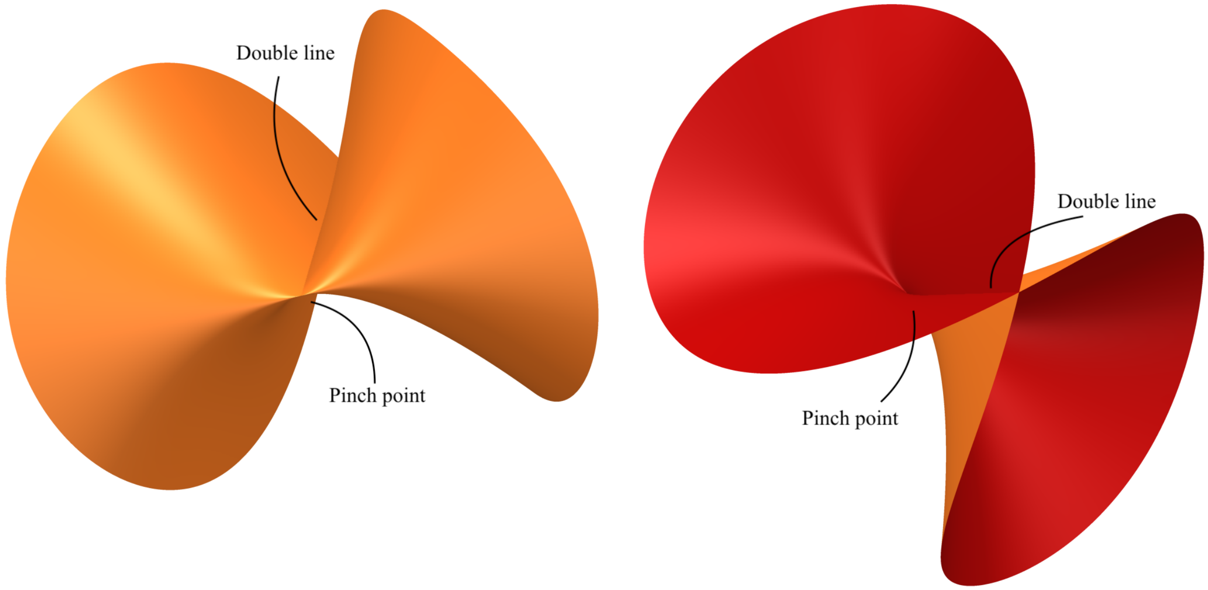

Figure 6: The ruled cubic surface

see the geometry more clearly. Let S be the blowup of $\mathbf{P}^2 = \operatorname{Proj}\mathbb{C}[s,t,u]$ at the point $O = V(s,t)$, embedded by the complete linear system h of proper transforms of conics passing through O. Then S is parametrized as $[tu : t^2 : st : su : s^2]$, and its defining equations in $\mathbf{P}^4 = \operatorname{Proj}\mathbb{C}[x,y,z,w,v]$ are given by the 2×2-minors of the matrix

$$M = \begin{bmatrix} x & y & z \\ w & z & v \end{bmatrix}.$$

Let $\pi : S \to \operatorname{Proj}\mathbb{C}[x,y,w,v]$ be the projection of S away from the point $V(x,y,w,v)$. The image X is parametrized by $[tu : t^2 : su : s^2]$ and has the defining equation $x^2v - y^2w$. The double locus M_2 on S is defined by $V(x,w) \cap S = [0 : t^2 : st : 0 : t^2]$ which maps 2-1 to the line $N_2 = V(x,w) = [0 : t^2 : 0 : s^2]$. Since X is the image of the ruled surface S, X itself is spanned by lines and thus the name ruled cubic surface. Let $\sigma : M_2 \to N_2$ be the involution defined by

$$[0 : t^2 : st : 0 : t^2] \mapsto [0 : t^2 : -st : 0 : t^2].$$

The locus R_1 consists of the two ramification points of $M_2 \to N_2$, which are $V(x,z,w,v)$ and $V(x,y,z,w)$.

By [53, Thm 4.1, 4.5], there is an exact sequence of groups

$$0 \to \operatorname{APic}(X) \to \operatorname{Pic}(S) \oplus \operatorname{Cart} M_2/\pi^*\operatorname{Cart} N_2 \to \underbrace{\operatorname{Pic} M_2/\pi^*\operatorname{Pic} N_2}_{\mathbb{Z}/2\mathbb{Z}} \to 0.$$

In this way, we describe an almost Cartier divisor on X by a triple (c, d, α), where (c, d) is the class $c\eta + d\mathfrak{f}$ in $\mathrm{Pic}(S)$ as before and α is a Cartier divisor of M_2, subject to the constraint that $d \equiv \deg \alpha \bmod 2$.

Definition II.3.5. For any divisor $\alpha \in \mathrm{Cart}\, M_2$, we set $\overline{\alpha} \in \mathrm{Cart}\, M_2$ to be the effective divisor of least degree such that $\alpha \equiv \overline{\alpha} \bmod \pi^* \mathrm{Cart}\, N_2$. More explicitly, if $\alpha = \sum_i P_i - \sum_j Q_j$ where $P_i \neq Q_j$, then we remove all the pairs $P + \sigma(P)$ from $\sum_i P_i + \sum_j \sigma(Q_j)$ to obtain $\overline{\alpha}$.

Proposition II.3.6 (Hartshorne [51, Prop 6.5])**.** *A class $(c, d, \alpha) \in \mathrm{APic}(X)$ is effective if and only if one of the following is true:*

1. $c > 0, d > 0$, *or*
2. $d = \alpha = 0, c > 0$, *or*
3. $c = 0$, $d > 0$ *and* $\deg \overline{\alpha} \leq d$.

A word of caution that the basis $\{\mathfrak{f} + \eta, -\eta\}$ instead of $\{\eta, \mathfrak{f}\}$ was used for $\mathrm{Pic}(S)$ in [51].

With the effectiveness criterion in hand, it is relatively easy to classify classes that contain a preserved curve on X.

Proposition II.3.7. *A class (c, d, α) contains a preserved curve if and only if one of the following is true:*

1. $c > 0, d > 0$ *and* $d = \deg \overline{\alpha}$, *or*
2. $d = \alpha = 0, c > 0$, *or*
3. $c = 0, d > 0$ *and* $d = \deg \overline{\alpha}$.

Proof. For (3), take c disjoint points on M_2 that contain no pairs of involution points, then the sum of the c-fibers through them will be preserved. For (2) take a multiple structure on the (-1)-curve, which does not intersect M_2 and is therefore preserved. For (1) take the union of curves in (2) and (3).

Now we argue that (1) - (3) is necessary. Suppose D is a preserved curve, then the linear system corresponding to the projection separates points and tangent vectors on $\widetilde{D}$. Thus $\widetilde{D}$ cannot meet pairs of involution points $P_i \neq \sigma(P_i)$. If $\widetilde{D}$ meets a point Q in the branch locus R_i, then it must meet it with multiplicity one. Otherwise the tangent space of $\widetilde{D}$ will contain the tangent line of Γ at Q, which is the line $\overline{OQ}$ where O is the point of projection. In this case the linear system H would fail to separate tangent vectors of $\widetilde{D}$ at Q. Taking $\alpha = \widetilde{D} \cap M_2$, it is clear that $\deg \overline{\alpha} = \deg \alpha = d$. □

Theorem II.3.8. *Every smooth connected curve C on X is linked to a preserved curve D except for the line N_2 and the (-1)-curve η.*

Proof. Every link of N_2 is not almost Cartier, and thus must be supported on N_2. In particular, no such curve can be preserved. Let C be a smooth irreducible curve of the class (c, d, α). Then either (1) $c > 0, d > 0, \deg \overline{\alpha} = d$, or (2) C is the exceptional curve where $d = \alpha = 0$ and $c = 1$, or (3) C is a ruling where $c = 0, d > 0$ and $\deg \overline{\alpha} = d$. The linked divisor $D = dH - C = (d - c, d, \sigma(\alpha))$ contains a preserved curve for case (1) and (3) by the previous proposition. □

More explicitly, let $C \neq N_2$ be a smooth connected curve in the class (c, d, α). If C is not E then $d > 0$ and its proper transform $\widetilde{C}$ meets M_2 at d points α on M_2 containing no pairs of involution points. The preserved curve D can be constructed as the image of the sum of d fibers passing through $\sigma(\alpha)$ and $(d - c)$ multiples of the exceptional curve η that does not meet the double curve M_2.

Definition II.3.9. A closed subscheme V of $\mathbf{P}^n$ is said to have maximal rank if for every $d \geq 0$ the map

$$H^0(\mathscr{O}_{\mathbf{P}^n}(d)) \to H^0(\mathscr{O}_V(d))$$

has maximal rank, i.e. is either injective or surjective.

Note that having maximal rank is the same as having either $H^0(\mathscr{I}_V(d)) = 0$ or $H^1(\mathscr{I}_V(d)) = 0$ for every $d \geq 0$. Examples of maximal rank varieties include ACM curves C in $\mathbf{P}^3_k$, which are characterized by the vanishing of $H^1(\mathscr{I}_C(d))$ for all d.

Proposition II.3.10. *Let C be a curve of class (c, d) on $S(1, 2) \subseteq \mathbf{P}^4$, then*

$$h^1(\mathscr{I}_{C/\mathbf{P}^4}(c-1)) = 0,$$

$$h^1(\mathscr{I}_{C/\mathbf{P}^4}(n)) = \sum_{i=0}^{n-c} h^0(\mathscr{O}_{\mathbf{P}^1}(d+i-2n-2)), \quad \forall n \geq c, \tag{A}$$

$$h^1(\mathscr{I}_{C/\mathbf{P}^4}(n)) = \sum_{i=0}^{c-n-2} h^0(\mathscr{O}_{\mathbf{P}^1}(2n+i+1-d)), \quad \forall n \leq c-2. \tag{B}$$

Proof. First we note that there is a short exact sequence

$$0 \to \mathscr{I}_{S/\mathbf{P}^4} \to \mathscr{I}_{C/\mathbf{P}^4} \to j_* \mathscr{I}_{C/S} \to 0$$

where $j : S \hookrightarrow \mathbf{P}^4$ is the inclusion. Since $H^1(\mathscr{I}_{S/\mathbf{P}^4}(n)) = H^2(\mathscr{I}_{S/\mathbf{P}^4}(n)) = 0$ for all n by Proposition II.3.2, we conclude that $H^1(\mathscr{I}_{C/\mathbf{P}^4}(n) \cong H^1(\mathscr{I}_{C/S}(n))$ for all n. Neither $(c-1)h - C$ nor $C - (c-1)h - c_1$ are effective since their first coordinates are -1. It follows that

$$\begin{aligned} h^1(\mathscr{I}_{C/S}(c-1)) &= h^1(\mathscr{O}_S((c-1)h-C) \\ &= -\chi(\mathscr{O}_S((c-1)h-C)) \\ &= \frac{1}{2}((c-1)h-C).((c-1)h-C+c_1)+1 = 0. \end{aligned}$$

If $n \geq c$, then the first coordinate of $nh - C$ is nonnegative and thus $h^1(\mathscr{I}_C(n))$ is given by

$$h^1(\mathscr{O}_S(n-c, 2n-d)) = \sum_{i=0}^{n-c} h^1(\mathscr{O}_{\mathbf{P}^1}(2n-d-i)) = \sum_{i=0}^{n-c} h^0(\mathscr{O}_{\mathbf{P}^1}(d+i-2n-2))$$

using the same computations in Lemma II.3.3. If $n \leq c-2$, then the first coordinate of $C - nh - c_1$ is nonnegative and thus

$$\begin{aligned} h^1(\mathscr{I}_{C/S}(n)) &= h^1(\mathscr{O}_S(nh - C)) \\ &= h^1(\mathscr{O}_S(C - nh - c_1)) \\ &= h^1(\mathscr{O}_S(c-n-2, d-2n-3)) \\ &= \sum_{i=0}^{c-n-2} h^1(\mathscr{O}_{\mathbf{P}^1}(d-2n-3-i)) \\ &= \sum_{i=0}^{c-n-2} h^0(\mathscr{O}_{\mathbf{P}^1}(2n+i+1-d)). \end{aligned}$$

□

Corollary II.3.11. *Let C be a curve of class (c,d) on $S(1,2)$.*

1. *If $d \geq 2c+2$, then $h^1(\mathscr{I}_{C/\mathbf{P}^4}(n))$ is nonzero and strictly decreasing on the interval $[c, d-c-2]$ with value (A) and vanishes elsewhere.*
2. *If $b < 2c+2$, then $h^1(\mathscr{I}_{C/\mathbf{P}^4}(n))$ is nonzero and strictly increasing on the interval $[d-c+1, c-2]$ with value (B) and vanishes elsewhere.*

In particular, C is ACM if and only if $2c-2 \leq d \leq 2c+1$.

Theorem II.3.12 (Classification of maximal curves)**.** *Apart from N_2, which is obviously of maximal rank, a smooth curve C of the class (c,d,α) on X has maximal rank if and only if (c,d) is one of the following:*

$$\{(1,0),(1,1),(1,2),(2,2),(2,3),(2,4),(3,3),(3,4)\}.$$

Among them only $(1,0)$, $(1,1)$ and $(1,2)$ are ACM.

Proof. Let C be a smooth curve on X that is not N_2. Suppose $c > 3$. Since $\widetilde{C}$ is smooth, it follows that $d \geq c$ by [45, Cor V.2.18]. The expressions of Proposition II.2.16 simplifies to

$$\begin{aligned} h^1(\mathscr{I}_{C/\mathbf{P}^3}(3)) =& h^0(\mathscr{O}_S(3h)) - \cancelto{0}{h^0(\mathscr{O}_S(3h-\widetilde{C}))} + h^1(\mathscr{O}_S(3h-\widetilde{C})) - h^0(\mathscr{O}_T(3)) \\ &+ \cancelto{0}{h^0(\mathscr{O}_S(3h-\widetilde{C}-M_2))} - \cancelto{0}{h^0(\mathscr{O}_S((3-d)h-M_2))} \\ =& 22 + h^1(\mathscr{O}_S(3-c, 6-d)) - 19 \geq 3. \end{aligned}$$

Since C lies on a cubic surface, it follows that if C is maximal rank then we must have $c \leq 3$. On the other hand,

$$\begin{aligned} h^1(\mathscr{I}_{C/\mathbf{P}^3}(3)) &= h^0(\mathscr{O}_C(3)) - h^0(\mathscr{O}_{\mathbf{P}^3}(3)) + h^0(\mathscr{I}_{C/\mathbf{P}^3}(3)) \\ &\geq \chi(\mathscr{O}_C(3)) - 20 + h^0(\mathscr{I}_{C/\mathbf{P}^3}(3)). \end{aligned}$$

It remains to estimate $H^0(\mathscr{O}_C(3))$. By Riemann-Roch, we have

$$h^0(\mathscr{O}_C(3)) \geq \chi(\mathscr{O}_C(3)) = 3\widetilde{C}.h + 1 - g(C) = \frac{1}{2}c^2 + \frac{7}{2}c + (4-c)d.$$

This is an increasing function in d, thus if $d \geq 5$ then

$$h^0(\mathscr{O}_C(3)) \geq \frac{1}{2}c^2 - \frac{3}{2}c + 20.$$

Since $c \leq 3$, it follows that $h^0(\mathscr{O}_C(3)) \geq 20$ unless $c = 1, d = 5$. But for $(c,d) = (1,5)$, we see that $3h - C$ is effective and thus C lies on another cubic surface, i.e. $h^0(\mathscr{I}_{C/\mathbf{P}^3}(3)) \geq 2$. This shows that if $c \leq 3$ and $d \geq 5$ then $h^1(\mathscr{I}_C(3)) > 0$, therefore a maximal rank curve C must satisfy $c \leq 3$ and $d \leq 4$. We compute $h^0(\mathscr{I}_{C/\mathbf{P}^3}(n))$ and $h^1(\mathscr{I}_{C/\mathbf{P}^3}(n))$ by hand using Proposition II.2.15 and Proposition II.2.16 for these finitely many cases and verify the claim. We omit these computations. □

It follows that for $c > 3$ and $2c-2 \leq d \leq 2c+1$, the general projections C of smooth ACM curves in the linear system $|c\eta + d\mathfrak{f}|$ on $S(1,2)$ do not have maximal rank in $\mathbf{P}^3$. Since having maximal rank is an open condition, it follows that no projection of these smooth ACM curves into $\mathbf{P}^3$ have maximal rank. These curves have degree $c+d$ and genus $-\frac{1}{2}c^2 + cd - \frac{1}{2}c - d + 1$. Since $d \sim 2c$, we see that $g(C) \sim \frac{1}{6}(\deg C)^2$ for $c \gg 0$.

The computations in this chapter are rather involved, and a priori leave lots of room for error. Fortunately, we were able to verify the formulas and computations using the Macaulay2 package `SpaceCurves` developed by the author. We invite the readers who are interested in computer algebra to explore this package with the article [97].

CHAPTER III

Bundles

Chapter III summary

In the final chapter of this thesis, we use the biliaison theory of sheaves developed in Chapter I to study vector bundles on projective spaces.

In Section III.1, we provide a conceptual proof of a result of Buraggina [13] that indecomposable rank two bundles on $\mathbf{P}^3_k$ are minimal reflexive sheaves (Theorem III.1.1). We also prove that the Horrock-Mumford bundle on $\mathbf{P}^4_{\mathbb{C}}$ is a minimal reflexive sheaf (Theorem III.1.2).

In Section III.2, we study bundles in the biliaison class of the zero sheaf on $\mathbf{P}^n_k$. These are exactly the bundles $\mathscr{E}$ such that

$$H^i(\mathscr{E}(t)) = 0, \quad \forall t \in \mathbb{Z}, \quad \forall 1 \leq i \leq n-2. \tag{†}$$

For example, all bundles on $\mathbf{P}^2_k$ satisfy this condition trivially. We classify the Betti numbers of bundles on $\mathbf{P}^n_k$ satisfying (†) for any rank r (Theorem III.2.4), generalizing results from Bohnhorst and Spindler [7] where $r = n$. Accordingly, we classify the Hilbert functions of such bundles (Theorem III.2.11), and introduce a compact way to represent (Definition III.2.12) and to generate them up to a bounded regularity (Proposition III.2.15). We then give an example to show that the semistability of such a bundle is not determined by its Betti numbers in general (Example III.2.18), in contrast to the case when $r = n$ discussed in [7].

In Section III.3, we describe the moduli of bundles in the biliaison class of the zero sheaf. We define a natural topology on $\mathcal{M}_0(H)$, the set of isomorphism classes of bundles in the biliaison class of the zero sheaf on $\mathbf{P}^n_k$ with Hilbert function H. We show that $\mathcal{M}_0(H)$ is irreducible and unirational (Proposition III.3.16). We then describe a stratification of $\mathcal{M}_0(H)$ by quotients of rational varieties and show that the closed strata form a graded lattice given by the Betti numbers (Theorem III.3.17). If we restrict to semistable bundles, then we obtain a corresponding stratification of the coarse moduli space.

For the remainder of this chapter, we work on the projective space $\mathbf{P}^n_k$, where k is an algebraically closed field. All sheaves in consideration are coherent on $\mathbf{P}^n_k$.

III.1. Minimal Bundles

We say a bundle is *minimal* if it is a minimal (S_n^+) sheaf where $n = \dim \mathbf{P}_k^n$. The next theorem is due to Buraggina [13], based on computations in [70]. We provide a conceptual proof of the same result.

Theorem III.1.1 (Buraggina [13]). *A rank two bundle $\mathscr{E}$ on $\mathbf{P}_k^3$ is a minimal (S_2^+) (i.e. reflexive) sheaf if and only if it is indecomposable.*

Proof. If $\mathscr{E}$ is decomposable, then it is the direct sum of two line bundles. In this case, the minimal reflexive sheaf in the class of $\mathscr{E}$ is the zero sheaf. Suppose $\mathscr{E}$ is indecomposable, then the zero sheaf is not in the biliaison class of $\mathscr{E}$ since $H_*^1(\mathscr{E}) \neq 0$. First we show that $\mathscr{E}$ is a minimal bundle. By Proposition I.3.24, we only need to show that $\mathscr{E} \preceq \mathscr{E}'$ for any other rank two bundle $\mathscr{E}'$ in its biliaison class.

Let $M := H_*^1(\mathscr{E})$ and $c_1 := c_1(\mathscr{E})$. The Horrocks' technique of eliminating homology shows that there are universal extensions killing $H_*^1(\mathscr{E})$ and $H_*^2(\mathscr{E})$

$$0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{O}(\underline{a}) \to 0$$

$$0 \to \mathscr{O}(-\underline{a} + c_1) \to \mathscr{G} \to \mathscr{E} \to 0,$$

which fit into the display

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathscr{O}(-\underline{a}+c_1) & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{E} & \longrightarrow & 0 \\
 & & \| & & \downarrow d & & \downarrow \\
0 & \longrightarrow & \mathscr{O}(-\underline{a}+c_1) & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{F} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \mathscr{O}(\underline{a}) & = & \mathscr{O}(\underline{a}) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}$$

of a monad $\mathscr{O}(-\underline{a} + c_1) \to \mathscr{H} \to \mathscr{O}(\underline{a})$ [6].

By Horrocks' criterion of splitting, we see that $\mathscr{H} \cong \mathscr{O}(\underline{b})$ for some $\underline{b}$ as it has no H_*^1 nor H_*^2. If we chose $H_*^0(\mathscr{O}(\underline{a})) \to M$ to be a minimal system of generators, then a result in [54] shows that the map $d : \mathscr{G} \to \mathscr{H}$ would not split off summands. It follows that $0 \to H_*^0(\mathscr{G}) \to H_*^0(\mathscr{H}) \to H_*^0(\mathscr{O}(\underline{a}) \to M$ are the first steps of a minimal free resolution of M, and $\mathscr{G} \cong \widetilde{\Omega^2 M}$, where $\Omega^2 M$ denotes the second minimal syzygy of M. The same statements apply to $\mathscr{E}'$, and $H_*^1(\mathscr{E}') \cong H_*^1(\mathscr{E}) = M$. Since $\mathscr{G}$ is primitive, we conclude from Proposition I.3.18 that $\Sigma(\mathscr{E}, l) = \Sigma(\mathscr{E}', l)$.

If $\mathscr{F}$ is a reflexive sheaf in the biliaison class of $\mathscr{E}$ such that $\mathscr{F} \preceq \mathscr{E}$, then $\mathscr{F}$ has rank at most two. Since $\mathscr{F}$ is neither zero or $\mathscr{O}(l)$, it must have rank exactly two. Since $\mathscr{E}$ is obtained using finitely many ascending elementary (S_2^+)-biliaison and rigid deformations by Theorem I.3.27, we see that $c_3(\mathscr{F}) \leq c_3(\mathscr{E})$. It follows from [48, Prop. 2.6] that $\mathscr{F}$ is in fact a bundle, and thus $\mathscr{E} \preceq \mathscr{F}$ by the above. □

Let us summarize the three distinct cases of biliaison classes of bundles on $\mathbf{P}^3_k$. The finite length module $M = H^1_*(\mathscr{E})$ uniquely determines the stable equivalence class of a primitive bundle $\mathscr{E}$ [55], and therefore uniquely determines the biliaison class of a bundle $\mathscr{E}$ on $\mathbf{P}^3_k$. There are three possibilities.

1. The minimal bundles of a biliaison class have rank two if and only if M satisfies the condition in [16].
2. The minimal bundle of a biliaison class is the zero sheaf if and only if $M = 0$.
3. The minimal bundles of all other biliaison classes have rank three.

Perhaps not surprisingly, we show that the Horrocks-Mumford bundle is minimal.

Theorem III.1.2. *The Horrocks-Mumford bundle $\mathscr{F}$ on $\mathbf{P}^4_{\mathbb{C}}$ is minimal, both as a bundle and as an (S_2^+) (i.e. reflexive) sheaf.*

Proof. Since rank one reflexive sheaves on $\mathbf{P}^n_k$ are just line bundles $\mathscr{O}(l)$, and $\mathscr{F}$ is not in the biliaison class of the zero sheaf, it has minimal rank among reflexive sheaves in its biliaison class. Let $H \subseteq \mathrm{SL}(5,\mathbb{C})$ be the Heisenberg group. Let $V = \mathrm{Map}(\mathbb{Z}/5,\mathbb{C})$ and let $V_1,\ldots,V_4$ be the four irreducible representations of H arising from V as in [57]. Let $W = \mathrm{Hom}_H(V_1, \wedge^2 V)$. The Horrocks-Mumford bundle $\mathscr{F}$ is the homology of the monad

$$\mathscr{O}(2)\otimes V_1 \xrightarrow{p} \wedge^2\mathcal{T}\otimes W \xrightarrow{q} \mathscr{O}(3)\otimes V_3.$$

We show that $\ker q$ is primitive. By the short exact sequence

$$0 \to \ker q \to \wedge^2\mathcal{T}\otimes W \to \mathscr{O}(3)\otimes V_3 \to 0,$$

it suffices to show that $\wedge^2\mathcal{T}$ is primitive. Consider the Koszul complex

$$0 \to \mathscr{O} \to \mathscr{O}(1)\otimes V \to \mathscr{O}(2)\otimes\wedge^2 V \to \mathscr{O}(3)\otimes\wedge^3 V \xrightarrow{d} \mathscr{O}(4)\otimes\wedge^4 V \to \mathscr{O}(5)\otimes\wedge^5 V \to 0,$$

where $\ker d \cong \wedge^2\mathcal{T}$. We see that $\mathrm{Ext}^2(\mathrm{im}\, d, \mathscr{O}(l)) = 0$ for all l by the short exact sequence

$$0 \to \mathrm{im}\, d \to \mathscr{O}(4)\otimes\wedge^4 V \to \mathscr{O}(5)\otimes\wedge^5 V \to 0,$$

and thus $\mathrm{Ext}^1(\wedge^2\mathcal{T}, \mathscr{O}(l)) = 0$ for all l by the short exact sequence

$$0 \to \wedge^2\mathcal{T} \to \mathscr{O}(3)\otimes\wedge^3 V \to \mathrm{im}\, d \to 0.$$

Finally, we have $H^0(\mathscr{F}(l)) = 0$ for $l < 0$ [57, §4]. Since the maximum degree of $\mathscr{O}(2)\otimes V_1$ is -2, the conclusion follows from Theorem I.3.28 applied to the extension

$$0 \to \mathscr{O}(2)\otimes V_1 \to \ker q \to \mathscr{F} \to 0.$$ □

Note that $H^0(\ker q(-2)) \cong V_1$, thus all minimal bundles in the biliaison class of $\mathscr{F}$ are equivalent under the action of $\mathrm{PGL}(5,\mathbb{C})$. In particular, all reflexive sheaves in the biliaison class of the Horrocks-Mumford bundle are constructed from it using finitely many ascending elementary (S_1^+)-biliaisons, rigid deformations and extensions by line bundles.

III.2. Bundles on $\mathbf{P}_k^n$ with Vanishing Lower Cohomologies

Our work in Chapter I paves the way for a program to describe the moduli $\mathcal{M}$ (in a broad sense) of (e.g. semistable, stable) torsion-free sheaves (or bundles) on a smooth projective variety X with irregularity zero.

Program. First, we partition $\mathcal{M}$ by biliaison equivalence into pieces $\mathcal{M}_{\mathscr{E}}$ that are labeled by stable equivalence classes of primitive sheaves. The major difficulty here is that we need to wisely choose a very ample line bundle and classify the stable equivalence classes of primitive sheaves on X. Next, we partition each $\mathcal{M}_{\mathscr{E}}$ into pieces $\mathcal{M}_{\mathscr{E},\Sigma}$ by the discrete invariant of Σ functions, which form a bounded below semilattice. Another major difficulty here is the classification of the Σ functions of sheaves in each biliaison class, especially those of minimal sheaves (or bundles). If we know the Σ function of a minimal sheaf (or bundle) in $\mathcal{M}_{\mathscr{E}}$, then we can systematically determine the numerical invariants of sheaves (or bundles) in $\mathcal{M}_{\mathscr{E}}$, as well as the dimensions of the pieces $\mathcal{M}_{\mathscr{E},\Sigma}$, and the dimensions of tangent spaces etc.

If we restrict to rank one torsion free sheaves on $X = \mathbf{P}_k^3$, which are exactly the ideal sheaves of curves up to twist, then the appropriate moduli $\mathcal{M}$ is the Hilbert scheme of curves and this program has been successfully carried out in [69]. The goal of the remainder of this chapter is to describe $\mathcal{M}_0$, the moduli of bundles in the biliaison class of the zero sheaf on $X = \mathbf{P}_k^n$. Since we restrict to the biliaison class of the zero sheaf, we may consider Hilbert functions H instead of the Σ functions as they encode the same information.

From here on we work on $\mathbf{P}_k^n$, and use $R := k[x_0, \ldots, x_n]$ to denote the polynomial ring of $\mathbf{P}_k^n$. As before, we write $H_*^i(\mathscr{F})$ for the R-module $\bigoplus_{t\in\mathbb{Z}} H^i(\mathscr{F}(t))$. For a finite integer sequence $\underline{a} = (a_i)_{i=1}^u$, we write $R(\underline{a})$ instead of $\bigoplus_{i=1}^u R(-a_i)$ and write $\mathscr{O}(\underline{a})$ instead of $\bigoplus_{i=1}^u \mathscr{O}(-a_i)$. We denote by $\mathcal{M}_0$ the set of isomorphism classes of finite rank bundles in the biliaison class of the zero sheaf on $\mathbf{P}_k^n$. We shall slightly abuse the terminology by saying $\mathscr{E} \in \mathcal{M}_0$ to mean that a bundle $\mathscr{E}$ is in the biliaison class of the zero sheaf.

We start with a standard observation on the relation between the vanishing of lower cohomologies of a coherent sheaf and the projective dimension of its section module.

Proposition III.2.1. *Let M be a finitely generated graded R-module. Then* $\operatorname{proj.dim}_R M \leq 1$ *if and only if* $M \cong H_*^0(\widetilde{M})$ *and* $H_*^i(\widetilde{M}) = 0$ *for all* $1 \leq i \leq n-2$.

Proof. Let $H_m^i(-)$ denote the i-th local cohomology module supported at the homogeneous maximal ideal m of R. There is a four-term exact sequence

$$0 \to H_m^0(M) \to M \to H_*^0(\widetilde{M}) \to H_m^1(M) \to 0$$

along with isomorphisms $H_m^{i+1}(M) \cong H_*^i(\widetilde{M})$ for $1 \leq i \leq n$. By the vanishing criterion of local cohomology, we have $\operatorname{depth} M = \inf\{i \mid H_m^i(M) \neq 0\}$. Finally, the Auslander-Buchsbaum formula states that $\operatorname{proj.dim} M = n + 1 - \operatorname{depth} M$. The statement follows. □

Corollary III.2.2. *A bundle $\mathscr{E}$ on $\mathbf{P}_k^n$ is in $\mathcal{M}_0$ if and only if $\mathscr{E}$ satisfies (†).*

Definition III.2.3. Let $\mathscr{E}$ be a rank r bundle on $\mathbf{P}_k^n$ satisfying (†). By Proposition III.2.1, the R-module $H^0_*(\mathscr{E})$ admits a unique (up to isomorphism) minimal graded free R-resolution

$$0 \to R(\underline{a}) \xrightarrow{\phi} R(\underline{b}) \to H^0_*(\mathscr{E}) \to 0 \tag{$*$}$$

for some $\underline{a} = (a_i)_{i=1}^{l}$ and $\underline{b} = (b_i)_{i=1}^{l+r}$. We make the convention to always sort $\underline{a}$ and $\underline{b}$ **in ascending** order, and define $(\underline{a}, \underline{b})$ to be *the Betti numbers* of $\mathscr{E}$.

Note that $\mathscr{E}$ is isomorphic to a direct sum of line bundles if and only if $H^0_*(\mathscr{E})$ is a free R-module if and only if $l = 0$ and the sequence $\underline{a}$ is empty.

The resolution $(*)$ of graded R-modules sheafifies to a resolution

$$0 \to \mathscr{O}(\underline{a}) \xrightarrow{\varphi} \mathscr{O}(\underline{b}) \to \mathscr{E} \to 0 \tag{$\star$}$$

of $\mathscr{E}$ by direct sums of line bundles. Conversely, a resolution $(\star)$ of $\mathscr{E}$ by direct sums of line bundles gives rise to a free resolution $(*)$ of the R-module $H^0_*(\mathscr{E})$ under the functor $H^0_*(-)$. With this understanding, we shall speak of these two resolutions of modules and sheaves interchangeably. In particular, the morphism φ is called *minimal* if and only if the corresponding map of R-modules ϕ is minimal, i.e. $\phi \otimes_R k = 0$.

Classification of Betti numbers

In this subsection we classify the Betti numbers of bundles in $\mathcal{M}_0$. For a pair $(\underline{a}, \underline{b})$, we write $\mathcal{M}_0(\underline{a}, \underline{b})$ for the subset of isomorphism classes of bundles with Betti numbers $(\underline{a}, \underline{b})$.

Theorem III.2.4 (Classification of Betti numbers)**.** *Let $\underline{a} = (a_1, \dots, a_l)$ and $\underline{b} = (b_1, \dots, b_{l+r})$ be two sequences of integers in ascending order for some $l \geq 0$ and $r > 0$. The set $\mathcal{M}_0(\underline{a}, \underline{b})$ is nonempty if and only if $\underline{a}$ is empty or*

$$r \geq n \text{ and } a_i > b_{n+i} \text{ for } i = 1, \dots, l. \tag{$\mathbf{A}$}$$

In this case, we have $\operatorname{coker} \varphi \in \mathcal{M}_0(\underline{a}, \underline{b})$ *for a general minimal map* $\varphi \in \operatorname{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$.

This generalizes the results of Bohnhorst and Spindler [7] for $r = n$. Likewise, we say a pair of ascending sequences of integers $(\underline{a}, \underline{b})$ is *admissible* if it satisfies the equivalent conditions of Theorem III.2.4. The fact that a bundle $\mathscr{E}$ satisfying (†) that is not a direct sum of line bundles must have rank $r \geq n$ also follows from the Evans-Griffith splitting criterion [28, Theorem 2.4].

In order to prove Theorem III.2.4, we need two lemmas regarding depth of minors of matrices.

Let S denote a noetherian ring and let $\phi : S^p \to S^q$ be a map between two free S-modules. For any integer r, the ideal $I_r(\phi)$ of $(r \times r)$-minors of ϕ is defined as the image of the map $\wedge^r S^p \otimes_S (\wedge^r S^q)^* \to S$, which is induced by the map $\wedge^r \phi : \wedge^r S^p \to \wedge^r S^q$.

Similarly, let $\varphi : \bigoplus_{i=1}^p \mathscr{O}_{\mathbf{P}_A^n}(-a_i) \to \bigoplus_{i=1}^q \mathscr{O}_{\mathbf{P}_A^n}(-b_i)$ be a morphism of sheaves on $\mathbf{P}_A^n$ over a noetherian ring A. Set $S := A[x_0, \ldots, x_n]$ and let $\phi : \bigoplus_{i=1}^p S(-a_i) \to \bigoplus_{i=1}^q S(-b_i)$ denote the corresponding morphism of graded free S-modules given by $H^0_*(\varphi)$. For any integer r, we define $I_r(\varphi) = I_r(\phi)$ as an ideal in S.

The depth of a proper ideal I in a noetherian ring S is defined to be the length of a maximal regular sequence in I. The depth of the unit ideal is by convention $+\infty$. Recall that if S is Cohen-Macaulay, then $\operatorname{depth} I = \operatorname{codim} I$ for every proper ideal I.

Lemma III.2.5. *Let A be a finitely generated integral domain over k, and let S be a finitely generated A-algebra. Suppose $\phi : S^q \to S^p$ is a morphism of free S-modules with $p \geq q$. For a prime P of A, let ϕ_P denote the morphism $\phi \otimes_A k(P)$ over the fiber ring $S \otimes_A k(P)$. For any integer d, the set of primes P in A such that* $\operatorname{depth} I_q(\phi_P) \geq d$ *is open in A.*

Proof. Note that $I_q(\phi) = I_q(\phi^*)$. Let $\mathscr{K}_\bullet(\phi^*)$ be the Eagon-Northcott complex associated to ϕ^* as in [19]. Note that the formation of the Eagon-Northcott complex is compatible with taking fibers, i.e. $\mathscr{K}_\bullet(\phi^*) \otimes_A k(P) = \mathscr{K}_\bullet(\phi^* \otimes_A k(P))$. For each prime ideal P of A, we have $\operatorname{depth} I_q(\phi_P^*) \geq d$ if and only if $\mathscr{K}_\bullet(\phi^*) \otimes_A k(P)$ is exact after position $p - q + 1 - d$ by the main theorem in [19]. The statement of the lemma follows from the general fact that the exactness locus of a family of complexes is open, see E.G.A. IV 9.4.2 [36]. □

Lemma III.2.6. *Let S be a standard graded finitely generated k-algebra, i.e. S is generated in degree* 1 *over $S_0 = k$. Let $\phi : \bigoplus_{i=1}^q S(-a_i) \to \bigoplus_{i=1}^p S(-b_i)$ be a morphism of graded free S-modules with $p \geq q$, and assume that ϕ is minimal, i.e. $\phi \otimes_S k = 0$. If relative to some bases, the matrix of ϕ has a block of zeros of size $u \times v$, then*

$$\operatorname{codim} I_q(\phi) \leq p - q + 1 - \inf(u + v, p + 1) + \inf(u + v, q).$$

Proof. For the case of generic matrices over an algebraically closed field, this is a result of Giusti-Merle [31]. We fix once for all bases of the domain and target of ϕ, and let $Z \subset \{1, \ldots, p\} \times \{1, \ldots, q\}$ be the $u \times v$ rectangle where the matrix of ϕ has zero entries. Consider the polynomial ring $A := k\left[\{x_{ij}\}_{1 \leq j \leq q}^{1 \leq i \leq p}\right] / (x_{ij} \mid (i,j) \in Z)$, which is the coordinate ring of the affine space of $(p \times q)$-matrices with a zero block of size $u \times v$ in position Z. Let $\psi : A(-1)^q \to A^p$ be the morphism given by the generic matrix (x_{ij}), then

$$\operatorname{codim} I_q(\psi) = p - q + 1 - \inf(u + v, p + 1) + \inf(u + v, q)$$

by [31, Theorem 1.3].

The general case follows from a theorem of Serre. The map ϕ corresponds to a morphism of k-algebras $A \to S$, where x_{ij} is sent to the entry of the matrix of ϕ relative to the fixed bases. In particular, note that $I_q(\phi) = SI_q(\psi)$. Let m and m' denote the homogeneous maximal ideals of A and S respectively. Since all entries of ϕ are in m' by assumption, we have an induced morphism on the localizations $A_m \to S_{m'}$ where $S_{m'}m \subseteq m'$. Let P be a prime above $A_m I_q(\psi)$ of the least codimension. Since $S_{m'}P \subseteq m'$, Serre's result on superheight on prime ideals in a regular local ring [94] implies that $\operatorname{codim} S_{m'}P \leq \operatorname{codim} P$. Now $I_q(\psi)$ and $SI_q(\psi)$ are homogeneous and $S_{m'}I_q(\psi) \subseteq S_{m'}P$, therefore we conclude that

$$\begin{aligned} \operatorname{codim} SI_q(\psi) &= \operatorname{codim} S_{m'}I_q(\psi) \\ &\leq \operatorname{codim} S_{m'}P \\ &\leq \operatorname{codim} P \\ &= \operatorname{codim} A_m I_q(\psi) \\ &= \operatorname{codim} I_q(\psi) \\ &= p - q + 1 - \inf(u+v, p+1) + \inf(u+v, q). \end{aligned}$$ □

The following is a simple fact that allows us to translate between bundles and homogeneous matrices whose ideals of maximal minors have maximal depth.

Proposition III.2.7. *Let $\underline{a} = (a_1, \ldots, a_l)$ and $\underline{b} = (b_1, \ldots, b_{l+r})$ for some integers $l > 0$ and $r \geq 0$. For a map $\varphi \in \operatorname{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$, the cokernel of φ is a rank r bundle on $\mathbf{P}^n_k$ if and only if $\operatorname{depth} I_l(\varphi) \geq n+1$. In this case, we have a resolution of $\mathscr{E} := \operatorname{coker} \varphi$ by direct sums of line bundles*

$$0 \to \mathscr{O}(\underline{a}) \xrightarrow{\varphi} \mathscr{O}(\underline{b}) \to \mathscr{E} \to 0.$$

Proof. The rank of $\operatorname{coker} \varphi$ is r if and only if $I_r(\varphi)$ is nonzero if and only if φ is injective at the generic point of $\mathbf{P}^n_k$ if and only if φ is injective. The ideal $I_r(\varphi)$ cuts out points on $\mathbf{P}^n_k$ where $\operatorname{coker} \varphi$ is not locally free of rank r. Thus $\operatorname{coker} \varphi$ is a rank r bundle if and only if $I_r(\varphi)$ is the unit ideal or is m-primary, where m is the homogeneous maximal ideal of R. In either case $\operatorname{depth} I_r(\varphi) \geq n+1$. □

Proof of Theorem III.2.4. If $\underline{a}$ is empty, then $\mathscr{E} \cong \mathscr{O}(\underline{b})$ has Betti numbers $(\underline{a}, \underline{b})$. Suppose $\underline{a}$ is nonempty and $(\underline{a}, \underline{b})$ satisfies condition (**A**). Consider the minimal map $\varphi : \mathscr{O}(\underline{a}) \to \mathscr{O}(\underline{b})$ given by the following matrix

$$\begin{array}{c} \\ b_1 \\ \vdots \\ \vdots \\ b_{n+1} \\ \vdots \\ \vdots \\ \vdots \\ b_{l+r} \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & \cdots & a_l \end{array} \\ \left(\begin{array}{ccc} x_0^{a_1-b_1} & 0 & 0 \\ \vdots & \ddots & 0 \\ \vdots & & x_0^{a_l-b_l} \\ x_n^{a_1-b_{n+1}} & & \vdots \\ 0 & \ddots & \vdots \\ 0 & 0 & x_n^{a_l-b_{l+n}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array} \begin{array}{c} \\ b_1 \\ \vdots \\ b_l \\ \vdots \\ \vdots \\ b_{l+n} \\ \vdots \\ b_{l+r}. \end{array}$$

Since φ drops rank nowhere on $\mathbf{P}^n_k$, we conclude that $\mathscr{E} := \operatorname{coker}\varphi$ is a rank r bundle with a resolution by direct sums of line bundles

$$0 \to \mathscr{O}(\underline{a}) \xrightarrow{\varphi} \mathscr{O}(\underline{b}) \to \mathscr{E} \to 0$$

by Proposition III.2.7. Since φ is minimal, it follows from Proposition III.2.1 that $\mathscr{E}$ represents an isomorphism class in $\mathcal{M}_0(\underline{a}, \underline{b})$.

Conversely, suppose $\mathcal{M}_0(\underline{a}, \underline{b})$ is nonempty and $\underline{a}$ is nonempty. Then there is a minimal map $\varphi \in \operatorname{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$ where $\operatorname{coker}\varphi$ is a rank r bundle $\mathscr{E}$. Since φ is minimal, it follows that $I_l(\varphi) \subseteq I_1(\varphi) \subseteq m$ is a proper ideal. By Proposition III.2.7, we have depth $I_l(\varphi) = n+1$. By the main theorem in [19], we have depth $I_l(\varphi) \leq l + r - l + 1 = r + 1$. It follows that we must have $r \geq n$. Now suppose on the contrary that there is an index $1 \leq i \leq l$ where $a_i \leq b_{n+i}$. Since φ is minimal, we see that the $(n+i, i)$-th entry in the matrix of φ must be zero. In fact, since $\underline{a}$ and $\underline{b}$ are in ascending order, we must have a block of zeros of size $(l + r - n - i + 1) \times i$ as the following

$$\begin{array}{c} \\ b_1 \\ \vdots \\ \vdots \\ b_{n+i} \\ \vdots \\ \vdots \\ \vdots \\ b_{l+r} \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & \cdots & a_i & \cdots & a_l \end{array} \\ \left(\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ 0 & \cdots & 0 & & \\ \vdots & & \vdots & & \\ \vdots & & \vdots & & \\ \vdots & & \vdots & & \\ 0 & \cdots & 0 & & \end{array} \right) \end{array}.$$

By Lemma III.2.6, we conclude that

$$\begin{aligned} \operatorname{depth} I_l(\varphi) &\leq l + r - l + 1 - \inf(l + r - n + 1, l + r + 1) + \inf(l + r - n + 1, l) \\ &= r + 1 - (l + r - n + 1) + l \\ &= n. \end{aligned}$$

This is a contradiction to the fact that depth $I_l(\varphi) = n + 1$.

Now we prove the last statement. It is obvious when $\underline{a}$ is empty, so we assume $\underline{a}$ is nonempty. The set $\operatorname{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$ has the structure of the closed points of an affine space $\mathbf{A}^N$. The subset of minimal maps is an affine subspace $\mathbf{A}^M$. There is a tautological morphism

$$\Phi : \bigoplus_{i=1}^{l} \mathscr{O}_{\mathbf{P}^n_{A^M}}(-a_i) \to \bigoplus_{i=1}^{l+r} \mathscr{O}_{\mathbf{P}^n_{\mathbf{A}^M}}(-b_i),$$

where the fiber Φ_P for a closed point P of $\mathbf{A}^M$ is given by the minimal map that P corresponds to. By Lemma III.2.5, the set U of points in $\mathbf{A}^M$ where depth $I_l(\Phi_P) \geq n + 1$ is open. Since there is a morphism $\varphi \in \operatorname{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$ whose cokernel is a bundle $\mathscr{E} \in \mathcal{M}_0(\underline{a}, \underline{b})$, by Proposition III.2.7 the map φ corresponds to a closed point in U. It follows that U is open and dense in $\mathbf{A}^M$. □

Recall that the category of bundles on $\mathbf{P}^n_k$ is a Krull-Schmidt category [2], i.e. every bundle $\mathscr{E}$ admits a decomposition $\mathscr{E} \cong \mathscr{E}_0 \oplus \mathscr{O}(\underline{c})$, unique up to isomorphism, where $\mathscr{E}_0$ has no line bundle summands.

Corollary III.2.8. *Let $\mathscr{E} \in \mathcal{M}_0(\underline{a}, \underline{b})$ for some $\underline{a}$ nonempty. If $\mathscr{E} \cong \mathscr{E}_0 \oplus \mathscr{O}(\underline{c})$ is the Krull-Schmidt decomposition of $\mathscr{E}$, then $n \leq \operatorname{rank} \mathscr{E}_0 \leq \max\{j \mid a_l > b_{l+j}\}$.*

Proof. Set $s := \max\{j \mid a_l > b_{l+j}\}$ and define $\underline{b}' := b_1, \dots, b_s$. Let $\pi : \mathscr{O}(\underline{b}) \to \mathscr{O}(\underline{b}')$ be the coordinate projection. If $\varphi \in \operatorname{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}'))$ is a minimal map whose cokernel is a bundle $\mathscr{E}$, then we claim that $\varphi' := \pi \circ \varphi$ is a minimal map in $\operatorname{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$ whose cokernel is a bundle $\mathscr{E}'$. To see this, observe that since $a_l \leq b_{l+i}$ for $s < i \leq r$ and φ is minimal, the last $(r-s)$ rows of the matrix representing φ relative to any bases are zero. In particular, we have $I_l(\varphi) = I_l(\pi \circ \varphi)$. By Proposition III.2.7, the cokernel of φ' is a bundle. It follows from the snake lemma that $\mathscr{E} \cong \mathscr{E}' \oplus \mathscr{L}$, where $\mathscr{L}$ is the kernel of the projection π. This shows that $\operatorname{rank} \mathscr{E}_0 \leq s$. Observe that $\mathscr{E}_0$ also satisfies (†) and thus $\operatorname{rank} \mathscr{E}_0 \geq n$ by Theorem III.2.4. □

Finiteness

In this subsection, we show that there are only finitely many possible Betti numbers for bundles in $\mathcal{M}_0$ with given rank, first Chern class and bounded regularity.

Recall that a coherent sheaf $\mathscr{F}$ on $\mathbf{P}^n_k$ is said to be d-regular if $H^i(\mathscr{F}(d-i)) = 0$ for all $i > 0$ [see 79, Lecture 14]. The *Castelnuovo-Mumford regularity* (shorthand regularity) of $\mathscr{F}$ is the least integer d such that $\mathscr{F}$ is d-regular. By the semicontinuity of cohomologies, being d-regular is an open condition for a family of coherent sheaves on $\mathbf{P}^n_k$. The notion of regularity also exists for graded R-modules [see 21].

If $\mathscr{E} \in \mathcal{M}_0(\underline{a}, \underline{b})$, then $\operatorname{reg} \mathscr{E} = \max(\underline{b}, \underline{a} - 1)$. Since the regularity depends only on the Betti numbers, we define $\operatorname{reg}(\underline{a}, \underline{b}) := \max(\underline{b}, \underline{a} - 1)$ for any admissible pair $(\underline{a}, \underline{b})$.

Proposition III.2.9. *There are only finitely many possible Betti numbers $(\underline{a}, \underline{b})$ of rank r bundles on $\mathbf{P}^n_k$ satisfying (†) with fixed first Chern class c_1 and regularity $\leq d$.*

Proof. Set $\underline{a} = (a_i)_{i=1}^l$ and $\underline{b} = (b_i)_{i=1}^{l+r}$ for some $l \geq 0$. Since $c_1 = \sum_{i=1}^l a_i - \sum_{i=1}^{l+r} b_i$, the statement is evidently true for direct sums of line bundles. Thus we may consider the case

$l > 0$. Since a_i and b_i are bounded above by $d+1$, we only need to show that l is bounded above and b_1 is bounded below. Consider the following inequalities

$$\begin{aligned} l &\leq \sum_{i=1}^{l}(a_i - b_{i+n}) \\ &= c_1 + \sum_{i=1}^{n} b_i + \sum_{i=l+n+1}^{l+r} b_i \\ &\leq c_1 + r \cdot d. \end{aligned}$$

And similarly,

$$\begin{aligned} b_1 &= -c_1 - \sum_{i=2}^{n} b_i - \sum_{i=l+n+1}^{l+r} b_i + \sum_{i=1}^{l}(a_i - b_i) \\ &\geq -c_1 - (r-1)\cdot d + l. \end{aligned}$$

□

This generalizes the observation of Dionisi-Maggesi [17] for the case $n = r = 2$.

Hilbert functions of bundles

In this subsection, we classify the Hilbert functions of bundles in $\mathcal{M}_0$. We introduce an efficient way to represent and generate them up to a bounded regularity.

Recall that the *Hilbert function* of a bundle $\mathscr{E}$ on $\mathbf{P}^n_k$ is the function $H_{\mathscr{E}}(t) : \mathbb{Z} \to \mathbb{Z}$ given by $H_{\mathscr{E}}(t) := \dim_k H^0(\mathscr{E}(t))$. For any function $H : \mathbb{Z} \to \mathbb{Z}$, we define $\mathcal{M}_0(H)$ to be the subset of $\mathcal{M}_0$ consisting of isomorphism classes of bundles with Hilbert function H.

Definition III.2.10. The *numerical difference* of a function $H : \mathbb{Z} \to \mathbb{Z}$ is a function $\partial H : \mathbb{Z} \to \mathbb{Z}$ given by $\partial H(t) := H(t) - H(t-1)$. We inductively define $\partial^{i+1} H := \partial\partial^i H$.

Note that if $H : \mathbb{Z} \to \mathbb{Z}$ is a function such that $H(t) = 0$ for $t \ll 0$, then H can be recovered by its i-th difference $\partial^i H$ for any $i \geq 0$.

Theorem III.2.11. *A function $H : \mathbb{Z} \to \mathbb{Z}$ is the Hilbert function of a rank r bundle $\mathscr{E} \in \mathcal{M}_0$ if and only if*

1. *$\partial^n H(t) = 0$ for $t \ll 0$ and $\partial^n H(t) = r$ for $t \gg 0$,*
2. *$\partial^n H(t+1) < \partial^n H(t)$ implies that $\partial^n H(t+1) \geq n$.*

Proof. Let $\mu(\underline{d}, t)$ denote the number of times an integer t occurs in the sequence $\underline{d}$.

($\Longrightarrow$): Suppose $\mathscr{E}$ is a rank r bundle in $\mathcal{M}_0(H)$. The Grothendieck-Riemann-Roch formula states that

$$\chi(\mathscr{E}(t)) = \int_{\mathbf{P}^n} \mathrm{ch}(\mathscr{E}(t)) \cdot \mathrm{td}(T_{\mathbf{P}^n}).$$

A routine computation shows that the leading coefficient of the Hilbert polynomial $\chi(\mathscr{E}(t))$ is $r \cdot t^n/n!$. Since the Hilbert function H eventually agrees with the Hilbert polynomial, we see that $\partial^n H(t) = 0$ for $t \ll 0$ and $\partial^n H(t) = r$ for $t \gg 0$.

Let $(\underline{a}, \underline{b})$ be the Betti numbers of $\mathscr{E}$. If $\underline{a}$ is empty, then $\mathscr{E}$ is a direct sum of line bundles and $\partial^n H$ is monotone nondecreasing and thus satisfies both conditions. We prove the case where $\underline{a}$ is non-empty. Consider the minimal free resolution

$$0 \to R(\underline{a}) \to R(\underline{b}) \to H^0_*(\mathscr{E}) \to 0.$$

A simple calculation shows that $\partial^{n+1} H(R(-a), t)$ is the delta function at a. It follows from the minimal resolution that $\partial^{n+1} H(t) = \mu(\underline{b}, t) - \mu(\underline{a}, t)$. Suppose $\partial^n H(t+1) < \partial^n H(t)$ for some t, then $\partial^{n+1} H(t+1) < 0$ and thus $\mu(\underline{a}, t+1) > 0$. Let j be the largest index where $a_j = t+1$. By Theorem III.2.4, we have $a_j > b_{j+n}$ and therefore

$$\partial^n H(t+1) = \sum_{i \le t+1} \partial^{n+1} H(i) = \sum_{i \le t+1} (\mu(\underline{b}, i) - \mu(\underline{a}, i)) \ge j + n - j = n.$$

($\Longleftarrow$): Conversely, suppose H satisfies the conditions of the theorem. We define the ascending sequences of integers $\underline{\alpha}$ and $\underline{\beta}$ by the property that for all $t \in \mathbb{Z}$,

$$\mu(\underline{\alpha}, t) = \max\{0, \partial^n H(t-1) - \partial^n H(t)\}, \quad \mu(\underline{\beta}, t) = \max\{0, \partial^n H(t-1) - \partial^n H(t)\}.$$

By the first condition on H, the sequences $\underline{\alpha}$ and $\underline{\beta}$ are finite. Furthermore, if $\underline{\alpha}$ has length l then $\underline{\beta}$ has length $l + r$. The second condition on H implies that $a_i \ge b_{i+n}$ for all $1 \le i \le l$. Since $\underline{\alpha}$ and $\underline{\beta}$ share no common entries by construction, it follows that $a_i > b_{i+n}$ for all $1 \le i \le l$. By Theorem III.2.4, there is a rank r bundle $\mathscr{E}$ in $\mathcal{M}_0$ with Betti numbers $(\underline{\alpha}, \underline{\beta})$. The Hilbert function of $\mathscr{E}$ is H by the reasoning of the previous direction. □

The above theorem suggests that we use the finitely many intermediate values of $\partial^n H$ to encode the infinitely many values of the Hilbert function H.

Definition III.2.12. A finite sequence of integers $\underline{B} = B_1, \ldots, B_m$ for some $m \ge 1$ is called a *bundle sequence of rank* r on $\mathbf{P}^n_k$ if it satisfies the following:

1. $B_i > 0$ for $1 \le i \le m$,
2. $B_m = r$ and $B_{m-1} \ne r$,
3. $B_{i+1} < B_i$ implies $B_{i+1} \ge n$.

If $\mathscr{E}$ is a rank r bundle in $\mathcal{M}_0(H)$ for some Hilbert function H, then we set

$$s_0 := \inf\{t \mid \partial^n H(t) \ne 0\}, \quad s_1 := \sup\{t \mid \partial^n H(t) \ne r\}.$$

The sequence $\partial^n H(s_0), \partial^n H(s_0+1), \ldots, \partial^n H(s_1+1)$ is a bundle sequence of rank r by Theorem III.2.11, which we call the *bundle sequence of H and of $\mathscr{E}$*.

By Theorem III.2.11, there is a one-to-one correspondence between the set of Hilbert functions of rank r bundles in $\mathcal{M}_0$ **up to shift** and the set of bundles sequences of rank r. The ambiguity of shift disappears if we deal with normalized bundles.

Definition III.2.13. We say a rank r bundle on $\mathbf{P}^n_k$ is *normalized* if $-r < c_1(\mathscr{E}) \leq 0$. Since $c_1(\mathscr{E}(t)) = c_1(\mathscr{E}) + r \cdot t$, it follows that every bundle can be normalized after twisting by the line bundle $\mathscr{O}(-\lceil c_1(\mathscr{E})/r \rceil)$.

We define the *degree* of a bundle sequence $\underline{B} = B_1, \ldots, B_m$, denoted by $\deg \underline{B}$, to be the sum $B_1 + \cdots + B_m$.

Proposition III.2.14. *If a normalized rank r bundle $\mathscr{E} \in \mathcal{M}_0$ has bundle sequence $\underline{B}$, then* $\operatorname{reg} \mathscr{E} \geq \lceil \deg \underline{B}/r \rceil - 2$.

Proof. Suppose $\mathscr{E}$ has Betti numbers $(\underline{a}, \underline{b})$ and Hilbert function H. We set $c := \max(a_l, b_{l+r})$ and $s_1 := \sup\{t \mid \partial^n H(t) \neq r\}$. It follows from the short exact sequence

$$0 \to R(\underline{a}) \to R(\underline{b}) \to H^0_*(\mathscr{E}) \to 0$$

that $s_1 < c$. We have

$$\begin{aligned} c_1(\mathscr{E}) &= \sum_{i=1}^{l} a_i - \sum_{i=1}^{l+r} b_i = -\sum_t t \cdot \partial^{n+1} H(t) = -\sum_t t \cdot (\partial^n H(t) - \partial^n H(t-1)) \\ &= \sum_{t \leq s_1+1} t \cdot \partial^n H(t-1) - \sum_{t \leq s_1+1} t \cdot \partial^n H(t) \\ &= \sum_{t \leq s_1} \partial^n H(t) - (s_1+1) \cdot r = \deg \underline{B} - (s_1+2) \cdot r \geq \deg \underline{B} - (c+1) \cdot r. \end{aligned}$$

Since $\mathscr{E}$ is normalized, we must have $c \geq \lceil \deg \underline{B}/r \rceil - 1$. Finally, regularity $\mathscr{E}$ is c or $c-1$ depending on whether $b_{l+r} \geq a_l - 1$ or not. □

Proposition III.2.15. *If $\underline{B} = B_1, \ldots, B_m$ is a bundle sequence of rank r and degree d, then $\underline{B}' = B_2, \ldots, B_m$ is a bundle sequence of rank r and degree $d - B_1$.*

It follows from Proposition III.2.14 and Proposition III.2.15 that we can inductively generate, in the form of bundle sequences, all Hilbert functions of normalized bundles in $\mathcal{M}_0$ up to any bounded regularity. The generation of these bundle sequences can be reduced to a partition problem with constraints.

Example III.2.16. The following are all bundles sequences of rank 4 and degree 9 on $\mathbf{P}^3_k$

$$\{(1^5, 4), (1^3, 2, 4), (1^2, 3, 4), (1, 2^2, 4), (2, 3, 4), (5, 4)\}.$$

Here we use t^j to denote the sequence of j copies of t.

Table 2: All rank 2 bundle sequences on $\mathbf{P}_k^2$ of regularity ≤ 2

$c_1 = 0$

c_2	Bundle sequence	reg	semistable
-4	1,1,1,1	2	no
-1	1,1	1	no
0	$\varnothing$	0	yes
0	1,1,1,3	2	no
1	1,3	1	yes
2	1,2,3	2	yes
2	4	1	yes
3	1,1,4	2	yes
3	3,3	2	yes
4	2,4	2	yes
5	1,5	2	yes
6	6	2	yes

$c_1 = -1$

c_2	Bundle sequence	reg	semistable
-6	1,1,1,1,1	2	no
-2	1,1,1	1	no
0	1	0	no
1	1,1,3	2	no
1	3	1	yes
2	2,3	2	yes
3	1,4	2	yes
5	5	2	yes

Example III.2.17. Table 2 lists all the rank 2 bundles sequences of regularity ≤ 2 on $\mathbf{P}_k^2$. These completely classify the Hilbert functions of normalized rank two bundles of regularity ≤ 2 on $\mathbf{P}_k^2$.

Semistability

In this subsection, we address the following question. Do the Betti numbers determine the semistability of a bundle in $\mathcal{M}_0$? If so, is there a criterion?

Here we use μ-semistability, where $\mu(\mathscr{F}) := c_1(\mathscr{F})/\operatorname{rank}(\mathscr{F})$ for any torsion-free coherent sheaf $\mathscr{F}$ on $\mathbf{P}_k^n$. The results are similar for Hilbert polynomial semistability as in [72].

For $r < n$, all rank r bundles $\mathscr{E} \in \mathcal{M}_0$ are direct sums of line bundles by Theorem III.2.4, which are not semistable except for $\mathscr{O}(d)^r$. The main result in [7] states that if $\mathscr{E} \in \mathcal{M}_0$ has rank $r = n$, then $\mathscr{E}$ is semistable if and only if its Betti numbers $(\underline{a}, \underline{b})$ satisfy $b_1 \geq \mu(\mathscr{E}) = (\sum_{i=1}^{l} a_i - \sum_{i=1}^{l+n} b_i)/n$. The latter condition is obviously necessary.

The following example demonstrates that for $r > n$, the semistability of a bundle in $\mathcal{M}_0$ is not determined by its Betti numbers in general.

Example III.2.18. For any $r > n$, consider $(\underline{a}, \underline{b})$ where

$$a_1 = 2, \quad b_i = \begin{cases} 0 & 1 \leq i < r \\ 1 & r \leq i \leq r+1. \end{cases}$$

Let φ and ψ be two maps in $\mathrm{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$ defined by the matrices

$$(0, \ldots, x_0^2, \ldots, x_{n-1}^2, x_n, 0)^T, \quad (0, \ldots, x_0^2, \ldots, x_{n-2}^2, x_{n-1}, x_n)^T$$

respectively. Then $\mathscr{E}_1 := \operatorname{coker}\varphi$ and $\mathscr{E}_2 := \operatorname{coker}\psi$ are rank r bundles satisfying (†) with Betti numbers $(\underline{a}, \underline{b})$ by Proposition III.2.7. Furthermore, it is easy to see that $\mathscr{E}_1 \cong \mathscr{E}_1' \oplus \mathscr{O}(-1) \oplus \mathscr{O}^{r-n-1}$ and $\mathscr{E}_2 \cong \mathscr{E}_2' \oplus \mathscr{O}^{r-n}$ for some rank n bundles $\mathscr{E}_1'$ and $\mathscr{E}_2'$ respectively. Since $\mu(\mathscr{E}_1) = \mu(\mathscr{E}_2) = 0$, it is clear that $\mathscr{E}_1$ is not semistable. On the other hand, the bundle $\mathscr{E}_2'$ is semistable by the criterion for the case $r = n$ stated above. Since both $\mathscr{E}_2'$ and $\mathscr{O}^{r-n}$ are semistable bundles with $\mu = 0$, it follows that so is $\mathscr{E}_2$.

The main reason to discuss semistability is that we might hope for a coarse moduli structure on the set $\mathcal{M}_0(\underline{a}, \underline{b})$. However, the above example illustrates the difficulty. In Section III.3 we will define a topology on $\mathcal{M}_0(\underline{a}, \underline{b})$, where the semistable bundles form an open subspace $\mathcal{M}_0(\underline{a}, \underline{b})^{ss}$. The space $\mathcal{M}_0(\underline{a}, \underline{b})^{ss}$ supports the structure of a subscheme of $\mathcal{M}(\chi)$, the coarse moduli space of semistable torsion-free sheaves with Hilbert polynomial χ, whose existence is established by Maruyama [see 72].

III.3. A Description of the Moduli

The set $\mathcal{M}_0$ is the disjoint union of $\mathcal{M}_0(H)$ for all possible Hilbert functions H which are classified by Theorem III.2.11. In this section we define a natural topology on $\mathcal{M}_0(H)$ and study how $\mathcal{M}_0(H)$ is stratified by bundles with different Betti numbers. In the following, we fix a Hilbert function H satisfying the conditions of Theorem III.2.11.

The graded lattice of Betti numbers

In this subsection we show that all possible Betti numbers of bundles in $\mathcal{M}_0(H)$ form a graded lattice, such that those with bounded regularity form a finite sublattice.

Definition III.3.1. We define $\mathcal{B}etti(H)$ to be the set of Betti numbers $(\underline{a}, \underline{b})$ of bundles in $\mathcal{M}_0(H)$. There is a grading $\mathcal{B}etti(H) = \bigsqcup_q \mathcal{B}etti^q(H)$, where

$$\mathcal{B}etti^q(H) := \{(\underline{a}, \underline{b}) \in \mathcal{B}etti(H) \mid \underline{a} \text{ and } \underline{b} \text{ have exactly } q \text{ entries in common}\}.$$

We remark that $\mathcal{B}etti(H)$ is infinite in general without restrictions on regularity. This is due to the fact that the Hilbert function H only bounds regularity from below (see Proposition III.2.14) but not above, as the following example demonstrates.

Example III.3.2. Let $(\underline{a}, \underline{b}) \in \mathcal{B}etti(H)$. For some arbitrarily large integer c, regarded as a singleton sequence, the pair $(\underline{a}, \underline{b}) + c$, defined in according to Definition III.3.4, is admissible by Theorem III.2.4. Note that any bundle with these Betti numbers has a line bundle summand by Corollary III.2.8.

Proposition III.3.3. *There is a unique element in $\mathcal{B}etti^0(H)$, which we denote by $(\underline{\alpha}, \underline{\beta})$.*

Proof. The construction of an element in $\mathcal{B}etti^0(H)$ is given in the proof of Theorem III.2.11. Recall from the proof of Theorem III.2.11 that $\partial^{n+1}H(t) = \mu(\underline{\alpha}, t) - \mu(\underline{\beta}, t)$. The uniqueness of $(\underline{\alpha}, \underline{\beta})$ follows from the fact that either $\mu(\underline{\alpha}, t) = 0$ or $\mu(\underline{\beta}, t) = 0$ by assumption. □

We now define a partial order on all pairs of increasing sequences of integers.

Definition III.3.4. Let $\underline{a}, \underline{b}, \underline{c}$ be three finite sequences of integers in ascending order. The sum $\underline{a} + \underline{c}$ is defined be the sequence obtained by appending $\underline{c}$ to $\underline{a}$ and sorting in ascending order. It is clear that this operation is associative.

We define $(\underline{a}, \underline{b}) + \underline{c}$ to be the pair $(\underline{a} + \underline{c}, \underline{b} + \underline{c})$. If $(\underline{a}', \underline{b}') = (\underline{a}, \underline{b}) + \underline{c}$ for some $\underline{c}$, then we say $(\underline{a}, \underline{b})$ is a *generalization* of $(\underline{a}', \underline{b}')$ and write $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$.

It is a consequence of Theorem III.2.4 that admissibility is stable under generalization.

Lemma III.3.5. *If $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$ and $(\underline{a}', \underline{b}')$ is admissible, then so is $(\underline{a}, \underline{b})$.*

Proof. By induction, it suffices to prove the case where $\underline{a}'$ and $\underline{b}'$ have a common entry c at index p and q respectively, and that $(\underline{a}, \underline{b})$ is obtained from $(\underline{a}', \underline{b}')$ by removing a'_p and b'_q. We may assume that p and q are the largest indices where $a'_p = c$ and $b'_q = c$ respectively. For $i < p$, we have $a_i = a'_i$. But $i + n < q$ and $b_{i+n} = b'_{i+n}$ for $i < p$ since $q > n + p$ by Theorem III.2.4. Therefore $a_i > b_{i+n}$ for $i < p$. In this case, $b_{l+n} = b'_{l+n}$ and $a_l > b_{l+n}$. For $i > p$, we have $a_{i-1} = a'_i > c$. In this case, either $i+n \leq q$, in which case $b_{i+n-1} \leq c < a_{i-1}$; or $i+n > q$, and $b_{i+n-1} = b'_{i+n}$ thus $b_{i+n-1} < a_{i-1}$. We conclude that $(\underline{a}, \underline{b})$ is also admissible. □

Corollary III.3.6. *Every $(\underline{a}, \underline{b})$ in $\mathcal{B}etti(H)$ is of the form $(\underline{\alpha}, \underline{\beta}) + \underline{c}$ for some $\underline{c}$.*

The main theorem of this subsection is the following.

Theorem III.3.7. *The set $\mathcal{B}etti(H)$ has the structure of a graded lattice given by the partial order $\rightsquigarrow$ and the grading $\mathcal{B}etti(H) = \bigsqcup_q \mathcal{B}etti^q(H)$.*

For the clarity of the proof, we first establish two lemmas.

Lemma III.3.8. *If c and d are two distinct integers (considered as singleton sequences) such that both $(\underline{a}, \underline{b}) + c$ and $(\underline{a}, \underline{b}) + d$ are admissible, then so is $(\underline{a}, \underline{b}) + c + d$.*

Proof. The lemma is simple, but the notations may make it appear more complicated than it is. Nonetheless, we include a proof here for the sake of completeness.

For an ascending sequence $\underline{d}$ and an integer t, let $p(\underline{d}, t)$ denote the largest index i where $d_i = t$. We may assume $c < d$, and write $(\underline{a}', \underline{b}') := (\underline{a}, \underline{b}) + c$, $(\underline{a}'', \underline{b}'') := (\underline{a}, \underline{b}) + d$ and $(\underline{a}''', \underline{b}''') := (\underline{a}, \underline{b}) + c + d$.

Since $(\underline{a}', \underline{b}')$ is admissible, we have $p(\underline{a}', c) < p(\underline{b}', c) - n$. Since $c < d$, it follows that $p(\underline{a}''', c) = p(\underline{a}', c)$ and $p(\underline{b}''', c) = p(\underline{b}', c)$. We conclude that $p(\underline{a}''', c) < p(\underline{b}''', c) - n$. Since $(\underline{a}'', \underline{b}'')$ is admissible, we have $p(\underline{a}'', d) < p(\underline{b}'', d) - n$. Since $c < d$, it follows that $p(\underline{a}''', d) = p(\underline{a}'', d) + 1$ and $p(\underline{b}''', d) = p(\underline{b}'', d) + 1$. We conclude that $p(\underline{a}''', d) < p(\underline{b}''', d) - n$. Finally, we show that $(\underline{a}''', \underline{b}''')$ is admissible. For $i < p(\underline{a}''', d)$, we have $i + n < p(\underline{b}''', d)$ and thus $a'''_i = a'_i > b'_{i+n} = b'''_{i+n}$. For $p(\underline{b}''', c) - n < i$, we have $p(\underline{a}''', c) < i$ and thus $a'''_i = a''_{i-1} > b''_{i+n-1} = b'''_{i+n}$. For $p(\underline{a}''', d) \leq i \leq p(\underline{b}''', c) - n$, we have $a_i \geq d > c \geq b_{i+n}$. □

Lemma III.3.9. *If $\underline{c}$ is an integer sequence and d is an integer (considered as a singleton sequence) not appearing in $\underline{c}$, such that both $(\underline{a}, \underline{b}) + \underline{c}$ and $(\underline{a}, \underline{b}) + d$ are admissible, then so is $(\underline{a}, \underline{b}) + \underline{c} + d$.*

Proof. By Lemma III.3.5 and Lemma III.3.8, the pair $(\underline{a}, \underline{b}) + c_1 + d$ is admissible. Applying Lemma III.3.8 again with $(\underline{a}, \underline{b}) + c_1$ in place of $(\underline{a}, \underline{b})$, we see that $(\underline{a}, \underline{b}) + c_1 + c_2 + d$ is admissible. By induction it follows that $(\underline{a}, \underline{b}) + \underline{c} + d$ is admissible. □

Proof of Theorem III.3.7. If $(\underline{a}', \underline{b}') \in \mathcal{B}etti^i(H)$ and $(\underline{a}, \underline{b}) \in \mathcal{B}etti^j(H)$ such that $(\underline{a}', \underline{b}') = (\underline{a}, \underline{b}) + \underline{c}$ for some $\underline{c}$, then obviously $i \geq j$. The cover relations in $\mathcal{B}etti(H)$ are given exactly by adding singleton sequences. It follows that $(\mathcal{B}etti(H), \rightsquigarrow)$ is a graded poset.

Suppose $(\underline{a}, \underline{b})$ and $(\underline{a}', \underline{b}')$ are in $\mathcal{B}etti(H)$. By Corollary III.3.6, there are sequences $\underline{c}$ and $\underline{c}'$ such that $(\underline{a}, \underline{b}) = (\underline{\alpha}, \underline{\beta}) + \underline{c}$ and $(\underline{a}', \underline{b}') = (\underline{\alpha}, \underline{\beta}) + \underline{c}'$. We define $\min(\underline{c}, \underline{c}')$ to be the descending integer sequence where an integer t occurs $\min(\mu(\underline{c}, t), \mu(\underline{c}', t))$ times, and similarly for $\max(\underline{c}, \underline{c}')$.

Clearly $(\underline{a}, \underline{b}) + \min(\underline{c}, \underline{c}') \rightsquigarrow (\underline{a}, \underline{b}) + \underline{c}$ and thus is admissible by Lemma III.3.5. It follows that $(\underline{a}, \underline{b}) + \min(\underline{c}, \underline{c}')$ is the meet of $(\underline{a}, \underline{b})$ and $(\underline{a}', \underline{b}')$ in $\mathcal{B}etti(H)$.

We claim that $(\underline{a}, \underline{b}) + \max(\underline{c}, \underline{c}')$ is admissible, and thus it is the join of $(\underline{a}, \underline{b})$ and $(\underline{a}', \underline{b}')$ in $\mathcal{B}etti(H)$. To see this, we may replace $(\underline{a}, \underline{b})$ by $(\underline{a}, \underline{b}) + \min(\underline{c}, \underline{c}')$ and assume that $\underline{c}$ and $\underline{c}'$ have no common entries. By Lemma III.3.9, we see that $(\underline{a}, \underline{b}) + \underline{c} + c'_1$ is admissible. Applying Lemma III.3.9 again with $(\underline{a}, \underline{b}) + c'_1$ in place of $(\underline{a}, \underline{b})$, we conclude that $(\underline{a}, \underline{b}) + c'_1 + \underline{c} + c'_2$ is admissible. By induction, it follows that $(\underline{a}, \underline{b}) + \underline{c}' + \underline{c}$ is admissible. □

For any integer d, let $\mathcal{B}etti(H)_{\le d}$ denote the subset of Betti numbers of bundles that are d-regular. The set $\mathcal{B}etti(H)_{\le d}$ inherits a grading $\bigoplus_{q \ge 0} \mathcal{B}etti^q(H)_{\le d}$, where $\mathcal{B}etti^q(H)_{\le d} := \mathcal{B}etti^q(H) \cap \mathcal{B}etti(H)_{\le d}$.

Corollary III.3.10. *For any integer d, the set $\mathcal{B}etti(H)_{\le d}$ is a finite graded lattice isomorphic to the lattice of subsequences of some sequence $\underline{c}$.*

Proof. If $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$, then $\operatorname{reg}(\underline{a}, \underline{b}) \le (\underline{a}', \underline{b}')$. If $(\underline{a}'', \underline{b}'')$ is the join of $(\underline{a}, \underline{b})$ and $(\underline{a}', \underline{b}')$ in $\mathcal{B}etti(H)$, then the regularity of $(\underline{a}'', \underline{b}'')$ is the maximum of those of $(\underline{a}, \underline{b})$ and $(\underline{a}', \underline{b}')$ by the construction in the proof of Theorem III.3.7. It follows that $\mathcal{B}etti(H)_{\le d}$ is a graded lattice. The finiteness of $\mathcal{B}etti(H)_{\le d}$ follows from Proposition III.2.9. Thus there is a maximum element of the form $(\underline{\alpha}, \underline{\beta}) + \underline{c}$ for some sequence $\underline{c}$. By Lemma III.3.5, we see that

$$\mathcal{B}etti^q(H)_{\le d} = \{(\underline{\alpha}, \underline{\beta}) + \underline{c}' \mid \underline{c}' \text{ is a subsequence of } \underline{c} \text{ of length } q\}.$$ □

Example III.3.11. Let H be the Hilbert function of a normalized bundle on $\mathbf{P}^3_k$ with bundle sequence $(5, 4)$. With the same notation as in Example III.2.16, the minimal element of $\mathcal{B}etti(H)$ is given by $\underline{\alpha} = (0)$ and $\underline{\beta} = (-1^5)$. The maximum element of $\mathcal{B}etti(H)_{\le 2}$ is $(\underline{\alpha}, \underline{\beta}) + \underline{c}$, where $\underline{c} = (0, 1, 2)$. In particular,

$$\mathcal{B}etti^q(H)_{\le 2} = \{(\underline{\alpha}, \underline{\beta}) + \underline{c}' \mid \underline{c}' \text{ is a subsequence of } (0, 1, 2) \text{ of length } q\}$$

and $\mathcal{B}etti(H)_{\le 2}$ is isomorphic to the lattice of subsequences of $(0, 1, 2)$.

The stratification

In this subsection, we define a natural topology on $\mathcal{M}_0(H)$. We then describe the stratification of $\mathcal{M}_0(H)$ by locally closed subspaces $\mathcal{M}_0(\underline{a}, \underline{b})$.

Definition III.3.12. Let $(\underline{a}, \underline{b}) \in \mathcal{B}etti(H)$. Let $\mathbf{A}(\underline{a}, \underline{b})$ denote the structure of the affine space on the vector space $\mathrm{Hom}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$. The minimal maps form an affine subspace $\mathbf{A}^0(\underline{a}, \underline{b})$ in $\mathbf{A}(\underline{a}, \underline{b})$. We define the subset of matrices whose maximal minors have maximal depth

$$\mathbf{Mat}(\underline{a}, \underline{b}) := \{\varphi \in \mathbf{A}(\underline{a}, \underline{b}) \mid \operatorname{depth} I_l(\varphi) \geq n+1\},$$
$$\mathbf{Mat}^0(\underline{a}, \underline{b}) := \{\varphi \in \mathbf{A}^0(\underline{a}, \underline{b}) \mid \operatorname{depth} I_l(\varphi) \geq n+1\}.$$

As in the proof of Theorem III.2.4, the subset $\mathbf{Mat}(\underline{a}, \underline{b})$ and $\mathbf{Mat}^0(\underline{a}, \underline{b})$ are open subvarieties of $\mathbf{A}(\underline{a}, \underline{b})$ and $\mathbf{A}^0(\underline{a}, \underline{b})$ respectively. For $\mathbf{A} = \mathbf{A}(\underline{a}, \underline{b})$ and $\mathbf{A}^0(\underline{a}, \underline{b})$, the tautological morphism

$$\Phi : \bigoplus_{i=1}^{l} \mathscr{O}_{\mathbf{P}^n_{\mathbf{A}}}(-a_i) \to \bigoplus_{i=1}^{l+r} \mathscr{O}_{\mathbf{P}^n_{\mathbf{A}}}(-b_i)$$

gives a tautological family of sheaves $\mathscr{E} := \operatorname{coker} \Phi$ over $\mathbf{A}$, which pulls back to a family of bundles $\mathscr{E}(\underline{a}, \underline{b})$ and $\mathscr{E}^0(\underline{a}, \underline{b})$ satisfying (†) over $\mathbf{Mat}(\underline{a}, \underline{b})$ and $\mathbf{Mat}^0(\underline{a}, \underline{b})$ respectively by Proposition III.2.7.

Let $G(\underline{a}, \underline{b})$ denote the algebraic group $\mathrm{Aut}(\mathscr{O}(\underline{a})) \times \mathrm{Aut}(\mathscr{O}(\underline{b}))$. The natural action $\rho : G(\underline{a}, \underline{b}) \times \mathbf{A}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b})) \to \mathbf{A}(\mathscr{O}(\underline{a}), \mathscr{O}(\underline{b}))$ given by $(f, g) \times \varphi \mapsto f \circ \varphi \circ g$ is a morphism of algebraic varieties. The action ρ leaves the subspace of minimal maps invariant. Since the change of coordinates does not change the ideal of maximal minors, it follows that the open subvarieties $\mathbf{Mat}(\underline{a}, \underline{b})$ and $\mathbf{Mat}^0(\underline{a}, \underline{b})$ are stable under the $G(\underline{a}, \underline{b})$-action.

Lemma III.3.13. *Two maps $\varphi, \psi \in \mathbf{Mat}(\underline{a}, \underline{b})$ are in the same $G(\underline{a}, \underline{b})$-orbit if and only if* $\operatorname{coker} \varphi \cong \operatorname{coker} \psi$.

Proof. Clearly if φ, ψ are in the same $G(\underline{a}, \underline{b})$-orbit then $\operatorname{coker} \varphi \cong \operatorname{coker} \psi$. Conversely, let $\mathscr{E} := \operatorname{coker} \varphi$ and $\mathscr{E}' := \operatorname{coker} \psi$. Then the isomorphism of the R-modules $H^0_*(\mathscr{E}) \cong H^0_*(\mathscr{E}')$ lifts to an isomorphism of free resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(\underline{a}) & \xrightarrow{\varphi} & R(\underline{b}) & \longrightarrow & H^0_*(\mathscr{E}) & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow{\scriptstyle \cong} & & {\scriptstyle g}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & R(\underline{a}) & \xrightarrow{\varphi'} & R(\underline{b}) & \longrightarrow & H^0_*(\mathscr{E}') & \longrightarrow & 0. \end{array}$$

It follows that φ, φ' are in the same $G(\underline{a}, \underline{b})$-orbit. □

Proposition III.2.7 and Lemma III.3.13 imply that the set $\mathcal{M}_0(\underline{a}, \underline{b})$ supports the structure of the quotient topological space $\mathbf{Mat}^0(\underline{a}, \underline{b})/G(\underline{a}, \underline{b})$. Similarly, we let $\mathcal{M}_0(\underline{a}, \underline{b})^{\rightsquigarrow}$ denote the subset of $\mathcal{M}_0$ consisting of isomorphism classes of bundles $\mathscr{E}$ that admit a (not necessarily minimal) free resolution of the form

$$0 \to R(\underline{a}) \to R(\underline{b}) \to H^0_*(\mathscr{E}) \to 0.$$

Then Lemma III.3.13 also implies that the set $\mathcal{M}_0(\underline{a},\underline{b})^{\rightsquigarrow}$ supports the structure of the quotient topological space $\mathbf{Mat}(\underline{a},\underline{b})/G(\underline{a},\underline{b})$. Clearly the inclusion of sets $\mathcal{M}_0(\underline{a},\underline{b}) \subseteq \mathcal{M}_0(\underline{a},\underline{b})^{\rightsquigarrow}$ is an inclusion of topological spaces.

Lemma III.3.14. *If $(\underline{a},\underline{b}) \rightsquigarrow (\underline{a}',\underline{b}')$ in $\mathcal{B}etti(H)$, then $\mathcal{M}_0(\underline{a},\underline{b})^{\rightsquigarrow}$ is a subspace of $\mathcal{M}_0(\underline{a}',\underline{b}')^{\rightsquigarrow}$. In particular, $\mathcal{M}_0(\underline{a},\underline{b})$ is a subspace of $\mathcal{M}_0(\underline{a}',\underline{b}')^{\rightsquigarrow}$.*

Proof. Let $(\underline{a}',\underline{b}') = (\underline{a},\underline{b}) + \underline{c}$ for some $\underline{c}$. Consider an injective morphism $\iota : \mathbf{Mat}(\underline{a},\underline{b}) \to \mathbf{Mat}(\underline{a}',\underline{b}')$ given by $\varphi \mapsto \varphi \oplus \mathrm{Id}_{\mathscr{L}(\underline{c})}$. It is not hard to see that the ideal of maximal minors does not change under this map, and thus ι is well-defined. Suppose φ, ψ are two morphisms in $\mathbf{Mat}(\underline{a},\underline{b})$ such that $\varphi \oplus \mathrm{Id}_{\mathscr{L}(\underline{c})}$ and $\psi \oplus \mathrm{Id}_{\mathscr{L}(\underline{c})}$ are in the same $G(\underline{a}',\underline{b}')$-orbit. It follows that $\operatorname{coker}\varphi \oplus \mathrm{Id}_{\mathscr{L}(\underline{c})} \cong \operatorname{coker}\psi \oplus \mathrm{Id}_{\mathscr{L}(\underline{c})}$. Since $\operatorname{coker}\varphi \cong \operatorname{coker}\varphi \oplus \mathrm{Id}_{\mathscr{L}(\underline{c})}$ and $\operatorname{coker}\psi \cong \operatorname{coker}\psi \oplus \mathrm{Id}_{\mathscr{L}(\underline{c})}$, we conclude that $\operatorname{coker}\varphi \cong \operatorname{coker}\psi$. It follows from Lemma III.3.13 that φ and ψ are in the same $G(\underline{a},\underline{b})$-orbit. This shows that the composition

$$\mathbf{Mat}(\underline{a},\underline{b}) \to \mathbf{Mat}(\underline{a}',\underline{b}') \to \mathcal{M}_0(\underline{a}',\underline{b}')^{\rightsquigarrow}$$

induces an injection of topological spaces on the quotient $\mathcal{M}_0(\underline{a},\underline{b})^{\rightsquigarrow} \hookrightarrow \mathcal{M}_0(\underline{a}',\underline{b}')^{\rightsquigarrow}$. $\square$

For each integer d, the set $\mathcal{B}etti(H)_{\leq d}$ is a finite lattice by Corollary III.3.10 and thus has a maximum element $(\underline{a}',\underline{b}')$. It follows from Lemma III.3.14 that every d-regular bundle $\mathscr{E}$ in $\mathcal{M}_0(H)$ admits a (not necessarily minimal) free resolution of the form

$$0 \to L(\underline{a}') \to L(\underline{b}') \to H^0_*(\mathscr{E}) \to 0.$$

Let $\mathcal{M}_0(H)_{\leq d}$ be the subspace of $\mathcal{M}_0(H)$ consisting of isomorphism classes of d-regular bundles. Then by Lemma III.3.13, the set $\mathcal{M}_0(H)_{\leq d}$ supports the structure of the quotient topological space $\mathbf{Mat}(\underline{a}',\underline{b}')/G(\underline{a}',\underline{b}')$.

It follows from Lemma III.3.14 and the construction above that if $d < d'$, then $\mathcal{M}_0(H)_{\leq d}$ is a subspace of $\mathcal{M}_0(H)_{\leq d'}$. Finally, we can now define a topology on $\mathcal{M}_0(H)$ by

$$\mathcal{M}_0(H) = \varinjlim_{d} \mathcal{M}_0(H)_{\leq d}.$$

Proposition III.3.15. *For each integer d, the subspace $\mathcal{M}_0(H)_{\leq d}$ is open in $\mathcal{M}_0(H)$.*

Proof. We need to show that $\mathcal{M}_0(H)_{\leq d}$ is open in $\mathcal{M}_0(H)_{\leq d'}$ for any d'. Let $(\underline{a}',\underline{b}')$ be the maximum element in $\mathcal{B}etti(H)_{\leq d'}$, and consider the quotient map $\pi : \mathcal{M}(\underline{a}',\underline{b}') \to \mathcal{M}_0(H)_{\leq d'}$. By the semicontinuity of cohomologies, the fibers of the tautological family $\mathscr{E}(\underline{a}',\underline{b}')$ are d-regular over an open subset of $\mathcal{M}(\underline{a}',\underline{b}')$. It follows that $\mathcal{M}_0(H)_{\leq d}$ is the image of this open subset under π, and thus is an open subspace of $\mathcal{M}_0(H)_{\leq d'}$. $\square$

Proposition III.3.16. *The topological space $\mathcal{M}_0(H)$ is irreducible and unirational.*

Proof. For $d \gg 0$, the subspace $\mathcal{M}_0(H)_{\leq d}$ is dense in $\mathcal{M}_0(H)$. Since $\mathcal{M}_0(H)_{\leq d}$ is the quotient of $\mathbf{Mat}(\underline{a}', \underline{b}')$, where $(\underline{a}', \underline{b}')$ is the maximum element of $\mathcal{B}etti(H)_{\leq d}$, it follows that $\mathcal{M}_0(H)_{\leq d}$ is irreducible and unirational, and thus so is $\mathcal{M}_0(H)$. □

The main result of this subsection is the following.

Theorem III.3.17. *The closed strata $\overline{\mathcal{M}_0(\underline{a}, \underline{b})}$ in $\mathcal{M}_0(H)$ form a graded lattice dual to $\mathcal{B}etti(H)$ under the partial order of inclusion. Furthermore, the intersection of two closed strata $\overline{\mathcal{M}_0(\underline{a}, \underline{b})}$ and $\overline{\mathcal{M}_0(\underline{a}', \underline{b}')}$ is again a closed stratum $\overline{\mathcal{M}_0(\underline{a}'', \underline{b}'')}$, where $(\underline{a}'', \underline{b}'')$ is the join of $(\underline{a}, \underline{b})$ and $(\underline{a}', \underline{b}')$ in the lattice $\mathcal{B}etti(H)$.*

The theorem needs several standard lemmas on the behavior of resolutions in families with constant Hilbert functions. We include proofs here for the lack of appropriate references.

Lemma III.3.18. *Let $\mathscr{E}' \in \mathcal{M}_0(\underline{a}', \underline{b}')$ and suppose $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$. Then there is a family of bundles $\mathscr{E}$ on $\mathbf{P}^n_k$ over a dense open set $U \subset \mathbf{A}^1$ containing the origin $0 \in \mathbf{A}^1$, such that $\mathscr{E}_0 \cong \mathscr{E}'$ and $\mathscr{E}_t \in \mathcal{M}_0(\underline{a}, \underline{b})$ for any closed point $0 \neq t \in U$.*

Proof. Suppose $(\underline{a}', \underline{b}') = (\underline{a}, \underline{b}) + \underline{c}$. By Lemma III.3.5, the pair $(\underline{a}, \underline{b})$ is admissible. Let $\psi \in \mathbf{Mat}^0(\underline{a}', \underline{b}')$ be a minimal presentation of $\mathscr{E}'$, and let $\varphi \in \mathbf{Mat}^0(\underline{a}, \underline{b})$ be a minimal presentation of a bundle $\mathscr{E}$. Set $\varphi' = \varphi \oplus \mathrm{Id}_{\mathscr{O}(\underline{c})}$ and consider the morphism $\Phi : \mathscr{O}(\underline{b}') \times \mathbf{A}^1 \to \mathscr{O}(\underline{a}') \times \mathbf{A}^1$ whose fiber over a closed point $t \in \mathbf{A}^1$ is given by $\Phi_t := \psi + t \cdot \varphi'$. By Lemma III.2.5, the morphism $\Phi_t \in \mathbf{Mat}(\underline{a}', \underline{b}')$ for all closed points t in an open dense set $U \subset \mathbf{A}^1$ containing 0. This shows that $\operatorname{coker} \Phi_t \in \mathcal{M}_0(\underline{a}, \underline{b})^{\rightsquigarrow}$ for $t \in U$. We show that in fact $\operatorname{coker} \Phi_t \in \mathcal{M}_0(\underline{a}, \underline{b})$ for all $0 \neq t \in U$. Let $t \neq 0$ be any closed point of U. Since ψ is minimal and φ' induces an isomorphism on the common summand $\mathscr{O}(\underline{c}) \xrightarrow{\sim} \mathscr{O}(\underline{c})$, it follows that Φ_t also splits off the common summand $\mathscr{O}(\underline{c}) \xrightarrow{t} \mathscr{O}(\underline{c})$. Since φ does not split off any common summands other than those of $\mathscr{O}(\underline{c})$, neither does Φ_t by Nakayama's lemma. It follows that the free resolution

$$0 \to R(\underline{a}') \xrightarrow{\Phi_t} R(\underline{b}') \to H^0(\mathscr{E}_t) \to 0$$

contains a minimal one of the form

$$0 \to R(\underline{a}) \to R(\underline{b}) \to H^0_*(\mathscr{E}_t) \to 0.$$ □

Lemma III.3.19. *Let $\mathscr{E}$ be a family of bundles on $\mathbf{P}^n_k$ satisfying (†) parametrized by a variety T. If all fibers of $\mathscr{E}$ have the same Hilbert function H, then general fibers have the same Betti numbers $(\underline{a}, \underline{b})$, and $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$ for the Betti numbers $(\underline{a}', \underline{b}')$ of any fiber $\mathscr{E}_t$.*

Proof. Let $t \in T$ be a closed point. We may base change to $\operatorname{Spec} \mathscr{O}_{T,t}$ and reduce to the case where T is an affine local domain. Let m be the maximal ideal of T with residue field k and set $R_T := T[x, y, z]$ and $R := k[x, y, z]$. The module $E := \bigoplus_{l \in \mathbb{Z}} H^0(\mathscr{E}(l))$ is finitely generated over R_T since $\mathscr{E}$ is a bundle. Since the fibers over T have the same Hilbert functions, it follows that E is flat over T. If $\bigoplus_{i=1}^{l+r} R(-b_i') \xrightarrow{d} E \otimes_T k$ is a minimal system of generators, then by Nakayama's lemma over generalized local rings, it lifts to a system of generators $\bigoplus_{i=1}^{l+r} R_T(-b_i') \xrightarrow{d_T} E$. Since E is flat over T, so is $\ker d_T$ and thus $(\ker d_T) \otimes_T k \cong \ker d$. Applying this procedure again, we find a free resolution of E

$$F_\bullet : \quad 0 \to \bigoplus_{i=1}^{l} R_T(-a_i') \to \bigoplus_{i=1}^{l+r} R_T(-b_i') \to E \to 0$$

that specializes to a minimal free resolution of $E \otimes_T k$. It follows that $F_\bullet \otimes_T k(T)$ is a free resolution of the generic fiber which contains a minimal free resolution of the form

$$0 \to \bigoplus_{i=1}^{j} R_T(-a_i) \otimes_T k(T) \to \bigoplus_{i=1}^{j+r} R_T(-b_i) \otimes_T k(T) \to E \otimes_T k(T) \to 0.$$

We conclude that the general fibers $\mathscr{E}_t$ have the Betti numbers $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$. □

Lemma III.3.20. *For $(\underline{a}, \underline{b}), (\underline{a}', \underline{b}') \in \mathcal{B}etti(H)$ the following are equivalent.*

1. $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$,
2. $\overline{\mathcal{M}_0(\underline{a}, \underline{b})} \supseteq \mathcal{M}_0(\underline{a}', \underline{b}')$,
3. $\mathcal{M}_0(\underline{a}', \underline{b}') \cap \overline{\mathcal{M}_0(\underline{a}, \underline{b})} \neq \varnothing$,
4. $\mathcal{M}_0(\underline{a}, \underline{b}) \subseteq \mathcal{M}_0(\underline{a}', \underline{b}')^{\rightsquigarrow}$.

Here all closures are taken within $\mathcal{M}_0(H)$.

Proof. (1) $\Longrightarrow$ (2): Suppose $(\underline{a}', \underline{b}') = (\underline{a}, \underline{b}) + \underline{c}$. Let $\varphi \in \mathbf{Mat}^0(\underline{a}, \underline{b})$ and $\psi \in \mathbf{Mat}^0(\underline{a}', \underline{b}')$. Consider the line $\Phi : \mathbf{A}^1 \hookrightarrow \mathbf{A}(\underline{a}', \underline{b}')$ defined $\Phi(t) := \psi + t \cdot \varphi'$, where $\varphi' = \varphi \oplus \mathrm{Id}_{\mathscr{L}(\underline{c})}$. For an open set $U \subset \mathbf{A}^1$ containing 0, the image $\Phi(t)$ is contained in $\mathbf{Mat}(\underline{a}', \underline{b}')$. By Lemma III.3.18, the image of $\Phi(t)$ in the quotient $\mathcal{M}_0(\underline{a}', \underline{b}')^{\rightsquigarrow}$ lies in $\mathcal{M}_0(\underline{a}, \underline{b})$ for $t \neq 0$. It follows that the image of ψ in $\mathcal{M}_0(\underline{a}', \underline{b}')$ is contained in the closure of $\mathcal{M}_0(\underline{a}, \underline{b})$ inside the space $\mathcal{M}_0(\underline{a}', \underline{b}')^{\rightsquigarrow}$. Since ψ represents an arbitrary point of $\mathcal{M}_0(\underline{a}', \underline{b}')$, we conclude that $\mathcal{M}_0(\underline{a}', \underline{b}')$ is contained in the closure of $\mathcal{M}_0(\underline{a}, \underline{b})$ in $\mathcal{M}_0(\underline{a}', \underline{b}')^{\rightsquigarrow}$, and therefore the same is true inside $\mathcal{M}_0(H)$.

(2) $\Longrightarrow$ (3) is trivial.

(1) $\Longrightarrow$ (4) is proven in Lemma III.3.14.

(3) $\Longrightarrow$ (1): Let $d := \max(\mathrm{reg}(\underline{a}, \underline{b}), \mathrm{reg}(\underline{a}', \underline{b}'))$. Let $(\underline{a}'', \underline{b}'')$ denote the maximum element of $\mathcal{B}etti(H)_{\leq d}$. Let $\pi : \mathbf{Mat}(\underline{a}'', \underline{b}'') \to \mathcal{M}_0(H)_{\leq d}$ be the quotient map and set V to be the preimage of $\overline{\mathcal{M}_0(\underline{a}, \underline{b})}$ under π, endowed with the structure of a (reduced) subvariety of $\mathbf{Mat}(\underline{a}'', \underline{b}'')$. Let $\mathscr{E}$ be the pullback of the tautological family of bundles $\mathscr{E}(\underline{a}'', \underline{b}'')$ on $\mathbf{Mat}(\underline{a}'', \underline{b}'')$ to V. Since $\mathcal{M}_0(\underline{a}, \underline{b})$ is dense in $\overline{\mathcal{M}_0(\underline{a}, \underline{b})}$, it follows that the fiber $\mathscr{E}_v$ over a general point $v \in V$ has Betti numbers $(\underline{a}, \underline{b})$. If p is a point in $\mathcal{M}_0(\underline{a}', \underline{b}')$ that is in the closure of $\mathcal{M}_0(\underline{a}, \underline{b})$ and q is a point in $\pi^{-1}(p)$, then $q \in V$ and $\mathscr{E}_q$ has Betti numbers $(\underline{a}', \underline{b}')$. Finally, an application of Lemma III.3.19 to the family $\mathscr{E}$ gives $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$.

(4) $\Longrightarrow$ (1): If $\mathscr{E}$ is a bundle with a free resolution of the form

$$0 \to L(\underline{b}') \to L(\underline{a}') \to H^0_*(E) \to 0,$$

then it contains as a summand the minimal free resolution of $\mathscr{E}$

$$0 \to L(\underline{b}) \to L(\underline{a}) \to H^0_*(E) \to 0$$

with a direct complement of the form

$$0 \to L(\underline{c}) \xrightarrow{\sim} L(\underline{c}) \to 0$$

for some $(\underline{a}', \underline{b}') = (\underline{a}, \underline{b}) + \underline{c}$. It follows that $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{a}', \underline{b}')$. $\square$

Proof of Theorem III.3.17. The first statement follows directly from Lemma III.3.20. For the same reason, it is clear that $\overline{\mathcal{M}_0(\underline{a}'', \underline{b}'')}$ is in the intersection of $\overline{\mathcal{M}_0(\underline{a}, \underline{b})}$ and $\overline{\mathcal{M}_0(\underline{a}', \underline{b}')}$. Let p be a closed point in the intersection of $\overline{\mathcal{M}_0(\underline{a}, \underline{b})}$ and $\overline{\mathcal{M}_0(\underline{a}', \underline{b}')}$. We assume $p \in \mathcal{M}_0(\underline{c}, \underline{d})$ for some $(\underline{c}, \underline{d}) \in \mathcal{B}etti(H)$ since $\mathcal{M}_0(H)$ is the disjoint union of these subspaces. By Lemma III.3.20, it follows that $(\underline{a}, \underline{b}) \rightsquigarrow (\underline{c}, \underline{d})$ and $(\underline{a}', \underline{b}') \rightsquigarrow (\underline{c}, \underline{d})$. Since $(\underline{a}'', \underline{b}'') \rightsquigarrow (\underline{c}, \underline{d})$ by the definition of join, another application of Lemma III.3.20 shows that $p \in \overline{\mathcal{M}_0(\underline{a}'', \underline{b}'')}$. $\square$

Last but not least, we discuss the semistable case where the description of the stratification holds within the coarse moduli space.

By [73, Theorem 4.2], semistability is open for a family of torsion-free sheaves. Furthermore, the set of semistable torsion sheaves with a given Hilbert polynomial χ is bounded in the sense of Maruyama, and thus have bounded regularity by [73, Theorem 3.11]. Let $\mathcal{M}_0(H)^{ss}$ and $\mathcal{M}_0(\underline{a}, \underline{b})^{ss}$ denote the subset of isomorphism classes of semistable bundles in $\mathcal{M}_0(H)$ and $\mathcal{M}_0(\underline{a}, \underline{b})$ respectively. It follows that $\mathcal{M}_0(H)^{ss}$ and all $\mathcal{M}_0(\underline{a}, \underline{b})^{ss}$ are contained in $\mathcal{M}_0(H)_{\leq d}$ for some large enough integer d. Since $\mathcal{M}_0(\underline{a}, \underline{b})^{ss}$ is open in $\mathcal{M}_0(\underline{a}, \underline{b})$ and $\mathcal{M}_0(H)^{ss}$ is open in $\mathcal{M}_0(H)$ by the similar reasoning as in Theorem III.3.15, it follows that the stratification of $\mathcal{M}_0(H)^{ss}$ by $\mathcal{M}_0(\underline{a}, \underline{b})^{ss}$ has the same description as given in Theorem III.3.17.

Let $\mathcal{M}(\chi)$ denote the coarse moduli space of semistable sheaves on $\mathbf{P}^n_k$ with Hilbert polynomial χ. We show that the spaces $\mathcal{M}_0(H)^{ss}$ and $\mathcal{M}_0(\underline{a}, \underline{b})^{ss}$ are subschemes of $\mathcal{M}(\chi)$. Let $\mathbf{Mat}^0(\underline{a}, \underline{b})^{ss}$ denote the open subscheme of $\mathbf{Mat}^0(\underline{a}, \underline{b})$ over which the fibers of the tautological family of bundles $\mathscr{E}^0(\underline{a}, \underline{b})$ are semistable. By the property of the coarse moduli space, there is a map $p_0 : \mathbf{Mat}^0(\underline{a}, \underline{b})^{ss} \to \mathcal{M}(\chi)$ inducing the family of semistable bundles. By Lemma III.3.13, the isomorphism classes of the fibers are exactly given by the $G(\underline{a}, \underline{b})$-orbits. Therefore $\mathcal{M}_0(\underline{a}, \underline{b})^{ss}$ is a subscheme of $\mathcal{M}(\chi)$ with the image subscheme structure of p_0. Similarly, the space $\mathcal{M}_0(H)_{\leq d}$ is also a subscheme of $\mathcal{M}(\chi)$. Since $\mathcal{M}_0(H)^{ss}$ is an open subspace of $\mathcal{M}_0(H)_{\leq d}$ for some $d \gg 0$, the same is true for $\mathcal{M}_0(H)^{ss}$.

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