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A LIEB-ROBINSON BOUND FOR QUANTUM SPIN CHAINS WITH STRONG ON-SITE IMPURITIES

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Abstract. We consider a quantum spin chain with nearest neighbor interactions and sparsely distributed on-site impurities. We prove commutator bounds for its Heisenberg dynamics which incorporate the coupling strengths of the impurities. The impurities are assumed to satisfy a minimum spacing, and each impurity has a non-degenerate spectrum. Our results are proven in a broadly applicable setting, both in finite volume and in thermodynamic limit. We apply our results to improve Lieb-Robinson bounds for the Heisenberg spin chain with a random, sparse transverse field drawn from a heavy-tailed distribution.

1. Introduction

Since the first demonstration of a finite group velocity for quantum spin systems in [LR72], Lieb-Robinson bounds have played an important role in proving fundamental results in condensed matter theory and quantum information theory [BHV06, NOS06, BnHVC09]. The question of whether they can be improved in systems with distinguishing features, such as disorder [BEO09, BO07, HSS12, GL16, EKS18], anomalous transport [DLLY14] or assumed rate of decay of interaction [TGS+19, EMNY20, WH20], has received considerable attention in recent years. There is also interest in comparing observed velocities in experiments with the estimates one can prove [The14]. An overview of proofs and applications of Lieb-Robinson bounds can be found in Section 3 of [NSY19].

In this paper we focus on spin chains. Propagation estimates of Lieb-Robinson type for classes of Hamiltonians are typically given in terms of a measure of the strength of the interactions, usually by using a norm. One naturally obtains such estimates that do not depend on the presence and magnitude of terms in the Hamiltonian supported on single sites since, by themselves, such terms do not generate propagation through the system. Here, we consider quantum spin chains with nearest neighbor interactions for which we are given such a propagation estimate that does not depend on single-site terms. We show that under certain conditions, taking into account single-site terms can lead to a sharper estimate. This improvement is manifested by a reduction of the pre-factor (amplitude of the propagation), and not in the Lieb-Robinson velocity. More specifically, we can exploit large on-site terms (such a magnetic fields) supported on a subset of sites for which we assume a minimum spacing between sites and a non-degeneracy condition on the eigenvalues of these single-site terms. As a consequence of our main result, Theorem 3.1, we show that with our set-up, for time t and observables A and B, there exists a constant $C(A, B, t)$ such that

$$\|[A(t), B]\| \leq \left(\frac{C(A, B, t)}{\lambda}\right)^N \tag{1.1}$$

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where N is the number of impurities which are well-separated from and between the supports of A and B, and $\lambda > 0$ is the minimum impurity strength. The precise statement is found in Corollary 3.2. The quantity $C(A, B, t)$ can be determined explicitly and is independent of the system size and, hence, the estimate in (1.1) also holds in the thermodynamic limit (see Section 4.4). Our result is non-trivial and gives a better bound for large λ than the standard Lieb-Robinson bound, which is independent of λ (see, e.g., [NSY19]). Our method does not allow us to derive an effect of the impurities on the Lieb-Robinson velocity itself. We note, however, that an improved standard Lieb-Robinson bound, a smaller velocity or faster spatial decay, would automatically propagate through our bound and similarly improve it while preserving the factor λ^{-N}.

In Section 4.5, we apply Theorem 3.1 to the Heisenberg model in the case when a sparse transverse field is coupled to the nearest neighbor interaction with i.i.d. couplings drawn from a heavy-tailed distribution. We show that with high probability, $\|[A(t), B]\|$ is much smaller than one would expect from the standard estimates, c.f. the commutator bound from Theorem 2.1.

2. Preliminaries and Notation

We consider the 1D lattice $\mathbb{Z}$ and associate a copy of $\mathbb{C}^D$, $D \geq 2$, to each lattice site. We equip $\mathbb{Z}$ with the natural distance $d(x, y) = |x - y|$ and define $d(X, Y) = \inf_{x\in X, y\in Y} |x - y|$ and $d(x, Y) = d(\{x\}, Y)$ for $x \in \mathbb{Z}$ and $X, Y \subset \mathbb{Z}$. For any finite $X \subset \mathbb{Z}$, we define

$$\mathcal{A}_X = \bigotimes_{x\in X} M_D(\mathbb{C}) \tag{2.1}$$

where $M_D(\mathbb{C})$ is the set of $D \times D$ matrices. When $X \subset Y$ and $|Y| < \infty$, we identify $\mathcal{A}_X$ as a subalgebra of $\mathcal{A}_Y$ by the map $A \mapsto A \otimes \mathbb{1}_{Y\setminus X}$, where $\mathbb{1}_{Y\setminus X}$ is the unit of $\mathcal{A}_{Y\setminus X}$. The algebra of local observables, the *local algebra* for short, is given by

$$\mathcal{A}_{\mathrm{loc}} = \bigcup_{\substack{X\subset\mathbb{Z}\\ |X|<\infty}} \mathcal{A}_X, \tag{2.2}$$

and we will refer to the operator norm completion of $\mathcal{A}_{\mathrm{loc}}$, denoted by $\mathcal{A}$ as the *quasi-local algebra*.

A mapping $\eta : \{X \subset \mathbb{Z} : |X| < \infty\} \to \mathcal{A}$ is an *interaction* if $\eta(X) = \eta(X)^* \in \mathcal{A}_X$ for all X. η is a *nearest neighbor* interaction when $\eta(X) \neq 0$ only if $X = \{x, x + 1\}$ for some $x \in \mathbb{Z}$. For a nearest neighbor interaction η we use the notation $\eta_{x,x+1} = \eta(\{x, x + 1\})$ and $\|\eta\| = \sup_{x\in\mathbb{Z}} \|\eta_{x,x+1}\|$.

Let $\Phi : \{X \subset \mathbb{Z} : |X| < \infty\} \to \mathcal{A}$ be a nearest neighbor interaction with $\|\Phi\| < \infty$. For $L \in \mathbb{N}$ we define

$$H_L(\Phi) \equiv H_L = \sum_{x=-L}^{L-1} \Phi_{x,x+1}. \tag{2.3}$$

We are interested in on-site perturbations of the Hamiltonian H_L, which we will refer to as impurities. To define these let $\emptyset \neq \mathcal{F} \subset \mathbb{Z}$ and define the *minimal spacing* of $\mathcal{F}$ as

$$\sigma_{\mathcal{F}} = \min\{|x - y| : x, y \in \mathcal{F}, x \neq y\}. \tag{2.4}$$

Later on, we have to assume $\sigma_{\mathcal{F}}$ is sufficiently large. Now for each $x \in \mathcal{F}$, let $V_x = V_x^* \in \mathcal{A}_{\{x\}}$ be an operator of the form

$$V_x = \sum_{j=1}^{D} \gamma_j^{(x)} P_j^{(x)} \tag{2.5}$$

where $P_j^{(x)}$ are the eigenprojectors of V_x, and assume the eigenvalues $\gamma_j^{(x)}$ are distinct at each site, i.e. for all $x \in \mathcal{F}$, $\gamma_i^{(x)} \neq \gamma_j^{(x)}$ for all $i \neq j$. We now consider perturbations of the original Hamiltonian H_L of the following form:

$$H_L(\vec{\lambda}) = H_L + \sum_{x \in \mathcal{F} \cap [-L,L]} \lambda_x V_x, \tag{2.6}$$

where $\lambda_x \in \mathbb{R} \setminus \{0\}$ for all $x \in \mathcal{F}$ and $\vec{\lambda} = (\lambda_x)_{x \in \mathcal{F} \cap [-L,L]}$ is the vector consisting of the coupling constants. We use the short-hand notation for the perturbation

$$V_L(\vec{\lambda}) = \sum_{x \in \mathcal{F} \cap [-L,L]} \lambda_x V_x. \tag{2.7}$$

For $L \in \mathbb{N}$, denote $\mathcal{A}_L = \mathcal{A}_{[-L,L]}$. We are interested in the Heisenberg evolution of an observable $A \in \mathcal{A}_L$, which for a Hamiltonian $H = H^* \in \mathcal{A}_L$ is defined by

$$\tau_t^H(A) = e^{itH} A e^{-itH}, \tag{2.8}$$

where $t \in \mathbb{R}$. Lastly, we fix some notation we frequently use in the following. For $X \subset [-L, L]$, we define an enlarged version of X by $X(n) = \{x \in [-L, L] : d(x, X) \leq n\}$. For an observable $A \in \mathcal{A}_L$ we denote by S_A the support of A, which we take to be the minimal length interval $[x, y]$ such that $A \in \mathcal{A}_{[x,y]}$.

Since Φ is a nearest neighbor interaction, the dynamics generated by Φ satisfy a Lieb-Robinson bound. We parametrize the bound by a parameter $\mu > 0$ which is the rate of spatial decay, and the strength $\|\Phi\|$ of the nearest neighbor interaction. The relevant statement of this commutator bound, Theorem 2.1 below, is implied by Corollary 2.2 of [NRSS09].

Theorem 2.1. *Suppose Φ is a nearest neighbor interaction with $\|\Phi\| < \infty$. For all $\mu > 0$, there exist $C_0, v > 0$ depending on μ and $\|\Phi\|$ such that for any operator of the form*

$$\Psi_L = \sum_{x=-L}^{L} \Psi_x, \ \ \Psi_x \in \mathcal{A}_{\{x\}}, \tag{2.9}$$

if $A, B \in \mathcal{A}_L$, then for all $t \in \mathbb{R}$:

$$\|[\tau_t^{H_L(\Phi)+\Psi_L}(A), B]\| \leq C_0 \|A\| \|B\| (e^{v|t|} - 1) e^{-\mu d(S_A, S_B)}. \tag{2.10}$$

We note that since our definition of support uses a single interval, the bound in (2.10) does not depend on the support sizes of A and B.

Assuming $C_0 \geq 1$ will simplify the form of the constants in Theorem 3.1 without loss of generality. And the proof of Corollary 2.2 in [NRSS09] shows that the constants C_0 and v in Theorem 2.1 can be taken as

$$C_0 = \frac{10 c_\mu}{K_\mu} \text{ and } v = 8 e^\mu K_\mu \|\Phi\| \tag{2.11}$$

where

$$c_\mu = \sum_{x\in\mathbb{Z}} e^{-\mu x}\frac{1}{(1+|x|)^2} \text{ and } K_\mu = \sup_{x,y\in\mathbb{Z}}\sum_{z\in\mathbb{Z}}\frac{e^{-\mu(|x-z|+|y-z|-|x-y|)}(1+|x-y|)^2}{(1+|x-z|)^2(1+|z-y|)^2}. \quad (2.12)$$

Assumption 2.2. *In the following, we will always assume Φ is a nearest neighbor interaction with $\|\Phi\| < \infty$. For any $\mu > 0$, we will take $C_0 \geq 1$ and $v > 0$ as defined in (2.11).*

3. Main results

The main result of this work is the following theorem.

Theorem 3.1. *For any $\mu > 0$, if $\sigma_{\mathcal{F}} > \max\{1/\mu, 2\}$, then there exists a constant $C > 0$ such that for all $L \in \mathbb{N}$ and $A, B \in \mathcal{A}_L$ with $\max S_A + 3 < \min S_B - 3$,*

$$\|[\tau_t^{H_L(\vec{\lambda})}(A), B]\| \leq \frac{C^N}{\prod_{x\in Z}|\lambda_x|\Gamma_x}\|A\|\|B\|\, G_N(t)\, F_N\big(d(S_A, S_B)\big) \quad (3.1)$$

for all $t \in \mathbb{R}$ and $\vec{\lambda} = (\lambda_x)_{x\in Z}$, where $N = |Z|$ for $Z = [\max S_A + 3, \min S_B - 3] \cap \mathcal{F}$,

$$\Gamma_x = \min_{i\neq j}|\gamma_i^{(x)} - \gamma_j^{(x)}| \quad (3.2)$$

and

$$G_N(t) = v|t|(1+v|t|)^{N-1}e^{v|t|} \qquad F_N(d) = (\mu d)^N e^{-\mu d}. \quad (3.3)$$

The constant C can be taken as

$$C = \frac{444 C_0^2 e^{5\mu}}{\mu(1-e^{-\mu})}\|\Phi\|\binom{D}{2}^2 \quad (3.4)$$

We prove Theorem 3.1 by modifying the estimate in (2.10) by an inductive argument. In principle, we could assume that, under the assumptions of Theorem 2.1, (2.10) holds for a monotone rapidly decreasing function f instead of $e^{-\mu d}$, e.g. $f(d) = e^{-\mu d\log(d)}$ as in [WH20], and derive a similar result without significant changes to the proof.

The velocity term of a Lieb-Robinson bound generally diverges with the strength of the interaction, so it is significant that large on-site terms can lead to a stronger estimate in (2.10). However, our result does not show that a sparse field decreases the Lieb-Robinson velocity. This is because our method introduces a prefactor polynomial in t and the d to the commutator bound which, for large times, diverges with the number of field sites. Proposition 4.5 shows that in the case $N = 1$, we can choose F_1 to have a marginally better dependence on $d(S_A, S_B)$. We leave open the question of whether the bound in (3.1) would hold if $G_N(t) = e^{v|t|}$ and $F_N(d) = e^{-\mu d}$.

Corollary 3.2, below, follows immediately from Theorem 3.1 and makes the statement in (1.1) precise. Let $\mathcal{T}_x : \mathcal{A} \to \mathcal{A}$ denote the translation operator which maps $\mathcal{A}_{\{0\}}$ to $\mathcal{A}_{\{x\}}$.

Corollary 3.2. *Let $\mu > 0$, $\sigma_{\mathcal{F}} > \max\{1/\mu, 2\}$. Suppose $V_x = \mathcal{T}_x(V_0)$ and $\lambda_x = \lambda$ for all $x \in \mathcal{F}$. Then for all $A, B \in \mathcal{A}_L$ with $\max S_A + 3 < \min S_B - 3$ and $Z = [\max S_A + 3, \min S_B - 3] \cap \mathcal{F}$, we obtain*

$$\|[\tau_t^{H_L(\vec{\lambda})}(A), B]\| \leq \left(K\frac{\mu d(S_A, S_B)(1+v|t|)}{\lambda}\right)^N \|A\|\|B\|\, e^{v|t|}\, e^{-\mu d(S_A, S_B)}. \quad (3.5)$$

where we have set $N = |Z|$. The constant $K > 0$ depends on μ, D, Φ and V_x.

4. Proofs

When the context is unambiguous, we omit the L dependence from the notation, since all estimates will be independent of L, and write $H_L = H$.

4.1. **Auxiliary results.** For $x \in \mathcal{F} \cap [-L+2, L-2]$ we define

$$\hat{H}_x = \sum_{y=-L}^{x-2} \Phi_{y,y+1} + \sum_{j=1}^{D} P_j^{(x)} (\Phi_{x-1,x} + \Phi_{x,x+1}) P_j^{(x)} + \sum_{y=x+1}^{L-1} \Phi_{y,y+1} \tag{4.1}$$

and set for $\vec{\lambda} = \big(\lambda_x\big)_{x\in Z}$

$$\hat{H}_x(\vec{\lambda}) = \hat{H}_x + V(\vec{\lambda}). \tag{4.2}$$

We will use the short-hand notations

$$\overline{V}_x(\vec{\lambda}) = \sum_{y\neq x} \lambda_y V_y, \quad \tau_t = \tau_t^{H(\vec{\lambda})}, \quad \text{and} \quad \hat{\tau}_t^x = \tau_t^{\hat{H}_x(\vec{\lambda})}. \tag{4.3}$$

Lemma 4.1. *Let $A, B \in \mathcal{A}_L$ and $x \in \mathcal{F} \cap [-L+2, L-2]$ such that $\max S_A < x < \min S_B$. Then*

$$[\hat{\tau}_t^x(A), B] = 0. \tag{4.4}$$

Proof. From the definition of $\hat{H}_x$ it follows that V_x commutes with $\hat{H}_x$ and with V_y as well when $y \neq x$. And so, since $\max S_A < x$

$$\hat{\tau}_t^x(A) = \tau_t^{\hat{H}_x + \overline{V}_x(\vec{\lambda})} \tau_t^{\lambda_x V_x}(A) = \tau_t^{\hat{H}_x + \overline{V}_x(\vec{\lambda})}(A). \tag{4.5}$$

We write $\hat{H}_x = \hat{H}_x^\ell + \hat{H}_x^r$ with $\hat{H}_x^\ell$ supported on $[-L, x]$, and $\hat{H}_x^r$ supported on $[x, L]$:

$$\begin{aligned} \hat{H}_x^\ell &= \sum_{y=-L}^{x-2} \Phi_{y,y+1} + \sum_{j=1}^{D} P_j^{(x)} \Phi_{x-1,x} P_j^{(x)} \\ \hat{H}_x^r &= \sum_{j=1}^{D} P_j^{(x)} \Phi_{x,x+1} P_j^{(x)} + \sum_{y=x+1}^{L-1} \Phi_{y,y+1} \end{aligned} . \tag{4.6}$$

Let $j \in \{1, ..., D\}$. By assumption, $P_j^{(x)} = |\psi_j^{(x)}\rangle\langle\psi_j^{(x)}|$ where $\psi_j^{(x)} \in \mathbb{C}^D$ is a unit norm eigenvector to the simple eigenvalue $\gamma_j^{(x)}$ of V_x. Expanding $\Phi_{x-1,x}$ into a sum of elementary tensors $\Phi_{x-1,x} = \sum_k \Phi_{x-1}^{(k)} \otimes \Phi_x^{(k)}$, $\Phi_{x-1}^{(k)} \in \mathcal{A}_{\{x-1\}}$ and $\Phi_x^{(k)} \in \mathcal{A}_{\{x\}}$, shows that

$$\begin{aligned} P_j^{(x)} \Phi_{x-1,x} P_j^{(x)} &= \sum_k \Phi_{x-1}^{(k)} \otimes \langle \psi_j^{(x)}, \Phi_x^{(k)} \psi_j^{(x)} \rangle P_j^{(x)} \\ &= \left(\sum_k \langle \psi_j^{(x)}, \Phi_x^{(k)} \psi_j^{(x)} \rangle \Phi_{x-1}^{(k)} \right) \otimes P_j^{(x)}. \end{aligned} \tag{4.7}$$

Similarly, there exists $\tilde{\Phi}_{x+1} \in \mathcal{A}_{\{x+1\}}$ such that $P_j^{(x)} \Phi_{x,x+1} P_j^{(x)} = P_j^{(x)} \otimes \tilde{\Phi}_{x+1}$, which proves that $[P_j^{(x)} \Phi_{x-1,x} P_j^{(x)}, P_j^{(x)} \Phi_{x,x+1} P_j^{(x)}] = 0$ and in turn $[\hat{H}_x^\ell, \hat{H}_x^r] = 0$. Hence

$$\left[\hat{H}_x^\ell + \sum_{y<x} \lambda_y V_y, \ \hat{H}_x^r + \sum_{y>x} \lambda_y V_y \right] = 0 \tag{4.8}$$

and since $\max S_A < x$,

$$\hat{\tau}_t^x(A) = \tau_t^{\hat{H}_x^\ell + \sum_{y<x} \lambda_y V_y}(A) \in \mathcal{A}_{[-L,x]}. \tag{4.9}$$

As $x < \min S_B$, this implies $[\hat{\tau}_t^x(A), B] = 0$. □

Proposition 4.2. *Let $A, B \in \mathcal{A}_L$, and suppose $x \in \mathcal{F} \cap [-L+2, L-2]$ such that $\max S_A + 1 < x < \min S_B$. Then for all $t \in \mathbb{R}$*

$$\|[\tau_t(A), B]\| \leq \frac{1}{|\lambda_x|\Gamma_x} \sum_{\substack{j,k \in \{1,\dots,D\} \\ j \neq k}} \left(\|f_x^{jk}(0,t)\| + \int_0^{|t|} ds \, \|\frac{d}{ds} f_x^{jk}(s,t)\| \right) \tag{4.10}$$

where we have set for $s \in \mathbb{R}$

$$\begin{aligned} f_x^{jk}(s,t) &= [[\tau_s^{\hat{H}_x + \overline{V}_x(\vec{\lambda})}(R_{jk}^{(x)}), \hat{\tau}_s^x \tau_{t-s}(A)], B] \\ R_{jk}^{(x)} &= P_j^{(x)}(\Phi_{x-1,x} + \Phi_{x,x+1}) P_k^{(x)} \end{aligned}. \tag{4.11}$$

Proof. Without loss of generality, we assume $t > 0$. Lemma 4.1 implies that $[\hat{\tau}_t^x(A), B] = 0$. Hence Duhamel's formula gives

$$\begin{aligned} [\tau_t(A), B] &= [\tau_t(A), B] - [\hat{\tau}_t^x(A), B] \\ &= i \int_0^t ds \, [\hat{\tau}_s^x([H(\vec{\lambda}) - \hat{H}_x(\vec{\lambda}), \tau_{t-s}(A)]), B]. \end{aligned} \tag{4.12}$$

We write

$$H(\vec{\lambda}) - \hat{H}_x(\vec{\lambda}) = \sum_{\substack{j,k \in \{1,\dots,D\}: \\ j \neq k}} R_{jk}^{(x)} \tag{4.13}$$

with $R_{jk}^{(x)} = P_j^{(x)}(\Phi_{x-1,x} + \Phi_{x,x+1}) P_k^{(x)}$ for $j, k \in \{1, \dots, D\}$, $j \neq k$. Since $[\hat{H}_x + \overline{V}_x(\vec{\lambda}), V_x] = 0$, we obtain for $s \in \mathbb{R}$

$$\hat{\tau}_s^x(R_{jk}^{(x)}) = \tau_s^{\hat{H}_x + \overline{V}_x(\vec{\lambda})} \tau_s^{\lambda_x V_x}(R_{jk}^{(x)}) = \sum_{\substack{j,k \in \{1,\dots,D\} \\ j \neq k}} e^{is\lambda_x(\gamma_j^{(x)} - \gamma_k^{(x)})} \tau_s^{\hat{H}_x + \overline{V}_x(\vec{\lambda})}(R_{jk}^{(x)}). \tag{4.14}$$

This implies

$$\begin{aligned} \int_0^t ds \, [\hat{\tau}_s^x([H(\vec{\lambda}) - \hat{H}_x(\vec{\lambda}), \tau_{t-s}(A)]), B] &= \int_0^t ds \, [[\hat{\tau}_s^x(H(\vec{\lambda}) - \hat{H}_x(\vec{\lambda})), \hat{\tau}_s^x \tau_{t-s}(A)]), B] \\ &= \sum_{\substack{j,k \in \{1,\dots,D\} \\ j \neq k}} \int_0^t ds \, e^{is\lambda_x(\gamma_j^{(x)} - \gamma_k^{(x)})} f_x^{jk}(s,t) \end{aligned} \tag{4.15}$$

where we have set $f_x^{jk}(s,t) = [[\tau_s^{\hat{H}_x + \overline{V}_x(\vec{\lambda})}(R_{jk}^{(x)}), \hat{\tau}_s^x \tau_{t-s}(A)], B]$. For $j, k \in \{1, \dots, D\}$ with $j \neq k$ integration by parts yields

$$\begin{aligned} &\Big\| \int_0^t ds \, e^{is\lambda_x(\gamma_j^{(x)} - \gamma_k^{(x)})} f_x^{jk}(s,t) \Big\| \\ \leq &\frac{1}{|\lambda_x|} \frac{1}{|\gamma_j^{(x)} - \gamma_k^{(x)}|} \left(\|f_x^{jk}(t,t)\| + \|f_x^{jk}(0,t)\| + \int_0^t ds \, \|\frac{d}{ds} f_x^{jk}(s,t)\| \right). \end{aligned} \tag{4.16}$$

Next we show $f_x^{jk}(t,t) = 0$. Since V_x commutes with $\hat{H}_x + \overline{V}_x(\vec{\lambda})$, we can rewrite

$$\|f_x^{jk}(t,t)\| = \big\|[[\tau_t^{\hat{H}_x + \overline{V}_x(\vec{\lambda})}(R_{jk}^{(x)}), \hat{\tau}_t^x(A)], B]\big\| = \big\|[\hat{\tau}_t^x\big([\tau_{-t}^{\lambda_x V_x}(R_{jk}^{(x)}), A]\big), B]\big\|. \tag{4.17}$$

Now $\tau_{-t}^{\lambda_x V_x}(R_{jk}^{(x)}) = e^{it\lambda_x(\gamma_k^{(x)}-\gamma_j^{(x)})}R_{jk}^{(x)}$, and so

$$\big\|[\hat\tau_t^x\big([\tau_{-t}^{\lambda_x V_x}(R_{jk}^{(x)}),A]\big),B]\big\| = \big\|[\hat\tau_t^x\big([R_{jk}^{(x)},A]\big),B]\big\| = 0 \tag{4.18}$$

where the last equality follows from the assumption that $S_A \cap [x-1,x+1] = \emptyset$. □

We will need the explicit form of the derivative $\frac{d}{ds}f_x^{jk}(s,t)$ in the proof of Theorem 3.1.

Lemma 4.3. *Let $A,B \in \mathcal{A}_L$, $\sigma_{\mathcal{F}} \geq 2$ and $x \in \mathcal{F} \cap [-L+2, L-2]$. Then for all $j,k \in \{1,...,D\}$ with $j \neq k$ and $s,t \in \mathbb{R}$,*

$$\begin{aligned}\frac{d}{ds}f_x^{jk}(s,t) = i\Big[\Big[\tau_s^{\hat H_x+\overline V_x(\vec\lambda)}([\hat H_x, R_{jk}^{(x)}]), \hat\tau_s^x\tau_{t-s}(A)\Big],B\Big] \\ - i\sum_{\substack{l,r\in\{1,...,D\}\\ l\neq r}}\Big[\Big[\tau_s^{\hat H_x+\overline V_x(\vec\lambda)}(R_{jk}^{(x)}), [\hat\tau_s^x(R_{lr}^{(x)}), \hat\tau_s^x\tau_{t-s}(A)]\Big],B\Big].\end{aligned} \tag{4.19}$$

Proof. We recall that $f_x^{jk}(s,t) = [[\tau_s^{\hat H_x+\overline V_x(\vec\lambda)}(R_{jk}^{(x)}), \hat\tau_s^x\tau_{t-s}(A)], B]$. First, we compute

$$\begin{aligned}\frac{d}{ds}\tau_s^{\hat H_x+\overline V_x(\vec\lambda)}(R_{jk}^{(x)}) &= i\big[\hat H_x + \overline V_x(\vec\lambda), \tau_s^{\hat H_x+\overline V_x(\vec\lambda)}(R_{jk}^{(x)})\big] \\ &= i\tau_s^{\hat H_x+\overline V_x(\vec\lambda)}\big([\hat H_x, R_{jk}^{(x)}]\big)\end{aligned} \tag{4.20}$$

where the last equality in (4.20) follows from $[\overline V_x(\vec\lambda), R_{jk}^{(x)}] = 0$ since $d(x, \mathcal{F}\setminus\{x\}) \geq 2$ and the support satisfies $S_{R_{jk}^{(x)}} \subset [x-1,x+1]$. Secondly, recalling the definition of τ, $\hat\tau^x$ in (4.3) and (4.13), we obtain

$$\begin{aligned}\frac{d}{ds}\hat\tau_s^x\tau_{t-s}(A) &= -i\,\hat\tau_s^x\big([H(\vec\lambda) - \hat H_x(\vec\lambda), \tau_{t-s}(A)]\big) \\ &= -i\sum_{\substack{l,r\in\{1,...,D\}\\ l\neq r}}[\hat\tau_s^x(R_{lr}^{(x)}), \hat\tau_s^x\tau_{t-s}(A)].\end{aligned} \tag{4.21}$$

Then the lemma follows from (4.20) and (4.21) and the product rule applied to the derivative of the s-dependent part of $f_x^{jk}(s,t)$. □

4.2. **Case of a single impurity.** In the following, we prove Theorem 3.1 in the case when there is only one impurity in between the supports of A and B. To do so, we need to estimate the terms on the right hand side of (4.10).

Lemma 4.4. *Let $A,B \in \mathcal{A}_L$ and suppose $\sigma_{\mathcal{F}} \geq 2$. Let $x \in \mathcal{F}$ such that $\max S_A + 3 < x < \min S_B - 3$. Then for all $j,k \in \{1,...,D\}$ with $j \neq k$ and $s,t \in \mathbb{R}$*

$$\|\frac{d}{ds}f_x^{jk}(s,t)\| \leq C_\mu\binom{D}{2}\|A\|\|B\|\|\Phi\|\mu d(x-3, S_B)ve^{v|t|}e^{-\mu d(S_A,S_B)} \tag{4.22}$$

$$C_\mu = 218\frac{C_0^2e^{5\mu}}{\mu(1-e^{-\mu})}. \tag{4.23}$$

Proof. As before, assume $\|A\| = \|B\| = 1$ and $t > 0$. We use the expression for the derivative obtained in Lemma 4.3. We consider the norm of the first term in the

latter and further rewrite this term. The definitions of $R_{jk}^{(x)}$ and $\hat{H}_x$ imply $R_{jk}^{(x)} = e^{-is(\gamma_j^{(x)}-\gamma_k^{(x)})}\tau_s^{\lambda_x V_x}(R_{jk}^{(x)})$ and $[\hat{H}_x, V_x] = 0$. Therefore,

$$[\hat{H}_x, R_{jk}^{(x)}] = e^{-is(\gamma_j^{(x)}-\gamma_k^{(x)})}\tau_s^{\lambda_x V_x}([\hat{H}_x, R_{jk}^{(x)}]) \tag{4.24}$$

from which we can rewrite, using $[\hat{H}_x + \overline{V}_x(\vec{\lambda}), V_x] = 0$,

$$\|[[\tau_s^{\hat{H}_x+\overline{V}_x(\vec{\lambda})}([\hat{H}_x, R_{jk}^{(x)}]), \hat{\tau}_s^x\tau_{t-s}(A)], B]\| = \big\|\big[\big[[\hat{H}_x, R_{jk}^{(x)}], \tau_{t-s}(A)\big], \hat{\tau}_{-s}^x(B)\big]\big\|. \tag{4.25}$$

The assumption $\max S_A + 3 < x$ implies $\max S_A < \min S_{[\hat{H}_x, R_{jk}^{(x)}]} - 1 < \max S_{[\hat{H}_x, R_{jk}^{(x)}]} < \min S_B$, and so we are in position to apply Corollary 5.3 with $W = [\hat{H}, R_{jk}^{(x)}]$. This implies

$$(4.25) \leq C'\|[\hat{H}_x, R_{jk}^{(x)}]\|e^{v|t|}d(x-3, S_B)e^{-\mu d(S_A,S_B)}. \tag{4.26}$$

where $C' = 72C_0^2\frac{e^{6\mu}}{1-e^{-\mu}}$ is chosen as in Corollary 5.3 since $\operatorname{diam} S_W = 4$ in this case, and since $|t-s| + |s| = t$ for $0 < s < t$. Moreover $\|[\hat{H}_x, R_{jk}^{(x)}]\| \leq 6\|\Phi\|^2$ since $\|R_{jk}^{(x)}\| \leq 2\|\Phi\|$. Taking this together with (4.25) and (4.26), and using from (2.11) and (2.12) that $v = 8e^{\mu}K_{\mu}\|\Phi\|$ and $K_{\mu} \geq 1$, we obtain

$$\begin{aligned}\|[[\tau_s^{\hat{H}_x+\overline{V}_x(\vec{\lambda})}([\hat{H}, R_{jk}^{(x)}]), \hat{\tau}_s^x\tau_{t-s}(A)], B]\| &\leq 6C'\|\Phi\|^2e^{vt}d(x-3, S_B)e^{-\mu d(S_A,S_B)} \\ &\leq 72C_0^2\frac{e^{5\mu}}{\mu(1-e^{-\mu})}\|\Phi\|(8e^{\mu}K_{\mu}\|\Phi\|)e^{vt}\mu d(x-3, S_B)e^{-\mu d(S_A,S_B)} \\ &= 72C_0^2\frac{e^{5\mu}}{\mu(1-e^{-\mu})}\|\Phi\|ve^{vt}\mu d(x-3, S_B)e^{-\mu d(S_A,S_B)}.\end{aligned} \tag{4.27}$$

For the norm of the second term in (4.19), we first fix the indices $l \neq r$ of the term $R_{lr}^{(x)}$. We use $\tau_s^{\lambda_x V_x}(R_{jk}^{(x)}) = e^{is\lambda_x(\gamma_j^{(x)}-\gamma_k^{(x)})}R_{jk}^{(x)}$ to rewrite

$$\|[[\tau_s^{\hat{H}_x+\overline{V}_x(\vec{\lambda})}(R_{jk}^{(x)}), [\hat{\tau}_s^x(R_{lr}^{(x)}), \hat{\tau}_s^x\tau_{t-s}(A)]], B]\| = \big\|\big[\big[R_{jk}^{(x)}, [R_{lr}^{(x)}, \tau_{t-s}(A)]\big], \hat{\tau}_{-s}^x(B)\big]\big\|. \tag{4.28}$$

Next we apply Jacobi's identity for commutators,

$$[[X, Y], Z] = -[[Y, Z], X] - [[Z, X], Y], \quad X, Y, Z \in \mathcal{A}_L, \tag{4.29}$$

and obtain

$$\begin{aligned}&\big\|\big[\big[R_{jk}^{(x)}, [R_{lr}^{(x)}, \tau_{t-s}(A)]\big], \hat{\tau}_{-s}^x(B)\big]\big\| \\ \leq&2\|R_{jk}^{(x)}\|\big\|\big[\big[R_{lr}^{(x)}, \tau_{t-s}(A)\big], \hat{\tau}_{-s}^x(B)\big]\big\| + 2\big\|[R_{jk}^{(x)}, \hat{\tau}_{-s}^x(B)]\big\|\big\|[R_{lr}^{(x)}, \tau_{t-s}(A)]\big\|.\end{aligned} \tag{4.30}$$

For the first norm in the above we use again Corollary 5.3 with $W = R_{lr}^{(x)}$ to estimate

$$2\|R_{jk}^{(x)}\|\big\|\big[\big[R_{lr}^{(x)}, \tau_{t-s}(A)\big], \hat{\tau}_{-s}^x(B)\big]\big\| \leq 8C'\|\Phi\|^2e^{vt}d(x-2, S_B)e^{-\mu d(S_A,S_B)} \tag{4.31}$$

$$\leq 72C_0^2\frac{e^{5\mu}}{\mu(1-e^{-\mu})}\|\Phi\|ve^{vt}\mu d(x-2, S_B)e^{-\mu d(S_A,S_B)}. \tag{4.32}$$

We note that we used (2.11) and (2.12) as we did in the inequalities in (4.27) to obtain the last inequality in (4.31). For the second term on the r.h.s of (4.30) we use the a

priori Lieb-Robinson from Theorem 2.1 with $\chi = \Phi$, $\Psi_L = \overline{V}_x(\vec{\lambda})$ and $\mu > 0$ chosen as before. This results in the bound

$$2\big\|[R^{(x)}_{jk}, \hat{\tau}^x_{-s}(B)]\big\|\big\|[R^{(x)}_{lr}, \tau_{t-s}(A)]\big\| \leq 8C_0^2 e^{3\mu}\|\Phi\|^2 e^{vt} e^{-\mu d(S_A,S_B)} \tag{4.33}$$

$$\leq \frac{C_0^2 e^{2\mu}}{\mu}\|\Phi\| v e^{vt}\mu d(x-2, S_B) e^{-\mu d(S_A,S_B)} \tag{4.34}$$

Taking together the computations in (4.28) – (4.34), we obtain for all $l \neq r$,

$$\|[[\tau_s^{\hat{H}_x + \overline{V}_x(\vec{\lambda})}(R^{(x)}_{jk}), [\hat{\tau}^x_s(R^{(x)}_{lr}), \hat{\tau}^x_s \tau_{t-s}(A)]], B]\|$$

$$\leq \frac{73 C_0^2 e^{5\mu}}{\mu(1-e^{-\mu})}\|\Phi\| v e^{vt}\mu d(x-2, S_B) e^{-\mu d(S_A,S_B)}. \tag{4.35}$$

Summing over $l \neq r$ and using (4.27) in (4.19), we obtain

$$\|\frac{d}{ds} f^{jk}_x(s,t)\| \leq \frac{218 C_0^2 e^{5\mu}}{\mu(1-e^{-\mu})}\binom{D}{2}\|\Phi\| v e^{vt}\mu d(x-3, S_B) e^{-\mu d(S_A,S_B)}. \tag{4.36}$$

independently of s, j,k and λ_x.

□

Proposition 4.5. *Let $A, B \in \mathcal{A}_L$ and $\sigma_{\mathcal{F}} \geq 2$. Suppose $x \in \mathcal{F}$ and $\max S_A + 3 < x < \min S_B - 3$. Then, for all $t \in \mathbb{R}$,*

$$\|[\tau_t^{H(\vec{\lambda})}(A), B]\| \leq \frac{C}{|\lambda_x|\Gamma_x}\|A\|\|B\| G_1(t)\mu \min\{d(x-3, S_B), d(x+3, S_A)\} e^{-\mu d(S_A,S_B)} \tag{4.37}$$

where $C = \frac{444 C_0^2 e^{5\mu}}{\mu(1-e^{-\mu})}\|\Phi\|\binom{D}{2}^2$ and $G_1(t) = v|t|e^{v|t|}$.

Proof. Since $\|[\tau_t^{H(\vec{\lambda})}(A), B]\| = \|[\tau_{-t}^{H(\vec{\lambda})}(B), A]\|$, the roles of A and B in the proof are symmetric and we may assume $\min\{d(x+3, S_A), d(x-3, S_B)\} = d(x-3, S_B)$. Suppose $\|A\| = \|B\| = 1$. Jacobi's identity (4.29) implies

$$\|f^{jk}_x(0,t)\| \leq 2\|R^{(x)}_{jk}\|\|[\tau_t(A), B]\| \leq 4\|\Phi\| C_0 \|A\|\|B\|(e^{v|t|}-1) e^{-\mu d(S_A,S_B)} \tag{4.38}$$

where we used Theorem 2.1 and $\|R^{(x)}_{jk}\| \leq 2\|\Phi\|$ in the last inequality. Applying (4.38) and Lemma 4.4 to the right-hand side of the inequality (4.10) yields

$$\|[\tau_t^{H(\vec{\lambda})}(A), B]\| \leq \frac{2\binom{D}{2}}{|\lambda_x|\Gamma_x}\Big(4\|\Phi\| C_0 (e^{v|t|}-1) e^{-\mu d(S_A,S_B)}$$

$$+ C_\mu \binom{D}{2}\|\Phi\| v|t| e^{v|t|}\mu d(x-3, S_B) e^{-\mu d(S_A,S_B)}\Big)$$

$$\leq \frac{444 C_0^2 e^{5\mu}}{\mu(1-e^{-\mu})}\|\Phi\|\binom{D}{2}^2 v|t|e^{v|t|}\mu d(x-3, S_B) e^{-\mu d(S_A,S_B)} \tag{4.39}$$

which is the bound in (4.37). □

4.3. **Multiple impurities.** We recall the following real-valued functions from (3.3):

$$F_n(d) = (\mu d)^n e^{-\mu d}, \qquad G_n(t) = v|t|(1+v|t|)^{n-1} e^{v|t|}. \tag{4.40}$$

It is easy to see that F_n is decreasing on $[n/\mu, \infty)$ and that $G_n \leq G_{n+1}$ on $[0, \infty)$.

Proof of Theorem 3.1. Let $\sigma_{\mathcal{F}} \geq \min\{2, 1/\mu\}$ and $Z = \{x_1, \ldots, x_N\} \subset \mathcal{F}$, ordered according to $x_1 < \cdots < x_N$. We prove by induction on $n \in \{1, \ldots, N\}$ the statement:

> For all $A, B \in \mathcal{A}_L$ with $\max S_A + 3 < x_1 < x_n < \min S_B - 3$, and for all times $t \in \mathbb{R}$
>
> $$\|[\tau_t(A), B]\| \leq \frac{C^n}{\prod_{j=1}^n |\lambda_{x_j}| \Gamma_{x_j}} \|A\| \|B\| G_n(|t|) F_n(d(S_A, S_B)) \tag{4.41}$$
>
> with F_n, G_n defined above in (4.40).

Then Theorem 3.1 is proven as the case when $n = N$. For $n = 1$, (4.41) follows directly from Proposition 4.5.

Now suppose that (4.41) is correct for $n-1 < N$ and all $D, E \in \mathcal{A}_L$ with $\max S_D + 3 < x_1$ and $x_{n-1} < \min S_E - 3$, and for all times $t \in \mathbb{R}$. Let A, B in $\mathcal{A}_L$ such that $\max S_A + 3 < x_1$ and $x_n < \min S_B - 3$. Without loss of generality, we assume $\|A\| = \|B\| = 1$ and $t > 0$. We apply Proposition 4.2 in the case when $x = x_n$ to get

$$\|[\tau_t(A), B]\| \leq \frac{1}{|\lambda_{x_n}| \Gamma_{x_n}} \sum_{\substack{j,k \in \{1,\ldots,D\} \\ j \neq k}} \left(\|f_{x_n}^{jk}(0,t)\| + \int_0^t ds \, \|\frac{d}{ds} f_{x_n}^{jk}(s,t)\| \right). \tag{4.42}$$

As in bound (4.38) in the proof of Proposition 4.5, we estimate

$$\begin{aligned} \|f_{x_n}^{jk}(0,t)\| &\leq 2\|R_{jk}^{(x)}\| \|[\tau_t(A), B]\| \\ &\leq \frac{4\|\Phi\|}{\mu} \frac{C^{n-1}}{\prod_{j=1}^{n-1} |\lambda_{x_j}| \Gamma_{x_j}} G_{n-1}(t) F_n(d(S_A, S_B)) \end{aligned} \tag{4.43}$$

where the last inequality follows from the induction hypothesis and $\|R_{jk}^{(x)}\| \leq 2\|\Phi\|$. We proceed as in inequalities (4.25), (4.30) in the proof of Lemma 4.4 to bound

$$\begin{aligned} \|\frac{d}{ds} f_{x_n}^{jk}(s,t)\| \leq \|[[[\hat{H}_{x_n}, R_{jk}^{(x_n)}], \tau_{t-s}(A)], \hat{\tau}_{-s}^{x_n}(B)]\| \\ + 2 \sum_{l \neq r} \|R_{jk}^{(x_n)}\| \|[[R_{lr}^{(x_n)}, \tau_{t-s}(A)], \hat{\tau}_{-s}^{x_n}(B)]\| \\ + 2 \sum_{l \neq r} \|[R_{jk}^{(x_n)}, \hat{\tau}_{-s}^{x_n}(B)]\| \|[R_{lr}^{(x_n)}, \tau_{t-s}(A)]\|. \end{aligned} \tag{4.44}$$

Next we estimate the three terms on the right hand side of the above inequality individually. First we deal with $\|[[[\hat{H}_{x_n}, R_{jk}^{(x_n)}], \tau_{t-s}(A)], \hat{\tau}_{-s}^{x_n}(B)]\|$. We set $W = [\hat{H}_{x_n}, R_{jk}^{(x_n)}]$. Since $\sigma_{\mathcal{F}} \geq 1/\mu$, $\max S_A + 3 < x - 1$ and $S_W \subset [x_n - 2, x_n + 2]$, we obtain $\min S_W - 1 > \max S_A + (n-1)/\mu$. And F_{n-1} restricted to $[(n-1)/\mu, \infty)$ is a monotone decreasing function. So we apply Lemma 5.2 to W using $k = \max\{3, (n-1)/\mu\}$, $g = G_{n-1}$, $f = F_{n-1}$ and the commutator bound in (4.41) as the assumed commutator bound in (5.4) to get

$$\|[[W, \tau_{t-s}(A)], \hat{\tau}_{-s}^{x_n}(B)]\| \leq \left(24 C_0 \frac{e^{\mu}}{1 - e^{-\mu}} \right) C_* \|W\| G_{n-1}(t) e^{v|s|} h_\mu(S_A, S_W, S_B) \tag{4.45}$$

where h_μ is defined as in (5.7) and $C_* = \frac{C^{n-1}}{\prod_{j=1}^{n-1} |\lambda_{x_j}| \Gamma_{x_j}}$. Furthermore, with these choices and the facts that $\mu d(S_A, S_W) \geq 1$ and $d(S_A, S_W) + \operatorname{diam}(S_W) + d(S_W, S_B) = d(S_A, S_B)$,

$$h_\mu(S_A, S_W, S_B) \leq \frac{3e^{5\mu}}{\mu} (\mu d(S_A, S_B))^n e^{-\mu d(S_A, S_B)} = \frac{3e^{5\mu}}{\mu} F_n(d(S_A, S_B)). \tag{4.46}$$

And so, using the fact that $\|[\hat{H}_{x_n}, R_{jk}^{(x_n)}]\| \leq 6\|\Phi\|^2$, we insert this in (4.45) to yield

$$\begin{aligned}&\|[[[\hat{H}_{x_n}, R_{jk}^{(x_n)}], \tau_{t-s}(A)], \hat{\tau}_{-s}^{x_n}(B)]\| \\ &\leq 432\, C_0 \frac{e^{6\mu}}{\mu(1-e^{-\mu})} \frac{C^{n-1}}{\prod_{j=1}^{n-1} |\lambda_{x_j}|\Gamma_{x_j}} \|\Phi\|^2 G_{n-1}(t-s) e^{v|s|} F_n(d(S_A, S_B))\end{aligned} \tag{4.47}$$

independently of $j, k \in \{1, .., D\}$ with $j \neq k$.

Secondly, we bound $\|[[R_{lr}^{(x_n)}, \tau_{t-s}(A)], \hat{\tau}_{-s}^{x_n}(B)]\|$. Choosing $W = R_{lr}^{(x_n)}$ and recalling $S_{R_{lr}^{(x_n)}} = [x_n - 1, x_n + 1]$ and $\|R_{lr}^{(x_n)}\| \leq 2\|\Phi\|$, we obtain along the very same lines as above

$$\begin{aligned}&\|[[R_{lr}^{(x_n)}, \tau_{t-s}(A)], \hat{\tau}_{-s}^{x_n}(B)]\| \\ &\leq 144\, C_0 \frac{e^{6\mu}}{\mu(1-e^{-\mu})} \frac{C^{n-1}}{\prod_{j=1}^{n-1} |\lambda_{x_j}|\Gamma_{x_j}} \|\Phi\| G_{n-1}(t-s) e^{v|s|} F_n(d(S_A, S_B))\end{aligned} \tag{4.48}$$

independently of $j, k, l, r \in \{1, .., D\}$.

Thirdly, we estimate $\|[R_{jk}^{(x_n)}, \hat{\tau}_{-s}^{x_n}(B)]\| \|[R_{lr}^{(x_n)}, \tau_{t-s}(A)]\|$. To do so, we use $S_{R_{jk}^{(x_n)}}, S_{R_{lr}^{(x_n)}} = [x_n - 1, x_n + 1]$, $\|R_{lr}^{(x_n)}\|, \|R_{jk}^{(x_n)}\| \leq 2\|\Phi\|$ and apply the induction hypothesis to $\|[R_{lr}^{(x_n)}, \tau_{t-s}(A)]\|$ and Theorem 2.1 to $\|[R_{jk}^{(x_n)}, \hat{\tau}_{-s}^{x_n}(B)]\|$. This results in

$$\begin{aligned}&\|[R_{jk}^{(x_n)}, \hat{\tau}_{-s}^{x_n}(B)]\| \|[R_{lr}^{(x_n)}, \tau_{t-s}(A)]\| \\ &\leq 4\|\Phi\|^2 C_0 \frac{C^{n-1}}{\prod_{j=1}^{n-1} |\lambda_{x_j}|\Gamma_{x_j}} G_{n-1}(t-s) F_{n-1}(d(S_A, x_n) - 1)) (e^{v|s|} - 1) e^{-\mu d(x_n+1, S_B)} \\ &\leq 4\|\Phi\|^2 C_0 \frac{e^{2\mu}}{\mu} \frac{C^{n-1}}{\prod_{j=1}^{n-1} |\lambda_{x_j}|\Gamma_{x_j}} G_{n-1}(t-s) e^{v|s|} F_n(d(S_A, S_B))\end{aligned} \tag{4.49}$$

independently of $j, k, l, r \in \{1, ..., D\}$.

Inserting the bounds (4.47), (4.48) and (4.49) into (4.44), yields

$$\|\frac{d}{ds} f_{x_n}^{jk}(s,t)\| \leq \frac{218 C_0 e^{5\mu}}{\mu(1-e^{-\mu})} \binom{D}{2} \|\Phi\| \frac{C^{n-1}}{\prod_{j=1}^{n-1} |\lambda_{x_j}|\Gamma_{x_j}} F_n(d(S_A, S_B)) v G_{n-1}(t-s) e^{v|s|}. \tag{4.50}$$

where we have used the fact that $v = 8e^{\mu} K_{\mu} \|\Phi\|$. Further plugging this and (4.43) in (4.42), we end up with

$$\begin{aligned}\|[\tau_t(A), B]\| \leq \frac{444 C_0 e^{5\mu}}{\mu(1-e^{-\mu})} \|\Phi\| \frac{C^{n-1}}{\prod_{j=1}^{n} |\lambda_{x_j}|\Gamma_{x_j}} \binom{D}{2}^2 F_n(d(S_A, S_B)) \\ \times \left(G_{n-1}(t) + v \int_0^t \mathrm{d}s\, G_{n-1}(t-s) e^{v|s|}. \right)\end{aligned} \tag{4.51}$$

Using the definition of $G_{n-1}(t)$ and $0 \leq t - s \leq t$, we see

$$G_{n-1}(t) + v \int_0^t ds\; G_{n-1}(t-s) e^{vs} \leq vt e^{vt} \sum_{j=0}^{n-1} \binom{n-1}{j} (vt)^j = G_n(t). \tag{4.52}$$

Inserting this in (4.51) proves (4.41). Finally, the statement (4.41) with $n = N$ gives Theorem 3.1. □

4.4. **Thermodynamic limit.** We observe that the constants which have appeared so far do not depend on L. This implies that the statements in Proposition 4.5 and Theorem 3.1 hold with $\tau^{H_L(\vec{\lambda})}$ replaced with the thermodynamic limit $\tau : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ defined by pointwise limit

$$\tau_t(A) = \lim_{L\to\infty} \tau_t^{H_L(\vec{\lambda})}(A). \tag{4.53}$$

4.5. **Disordered spin chain.** We now apply our results to a specific example to show that in 1D, the presence of a sparse disordered field can imply that for fixed t, with high probability, the Lieb-Robinson bound from Theorem 2.1 is not sharp. Let $\mu > 0$ be fixed, and take $\mathcal{F} = \sigma\mathbb{Z}$, where $\sigma = \lceil \max\{1/\mu, 2\} \rceil$. Consider the Heisenberg spin $S = 1/2$ chain with sparse transverse field and open boundary conditions on $[-L, L]$ for $L \in \mathbb{N}$

$$H_L(\vec{\lambda}) = -J \sum_{n=-L}^{L-1} \sum_{j=1}^{3} \sigma_n^j \sigma_{n+1}^j + \sum_{x\in\mathcal{F}\cap[-L,L]} \lambda_x \sigma_x^3 \tag{4.54}$$

where σ^j, $j = 1, ..., 3$ are the standard Pauli matrices and $J > 0$. Let $\mu > 0$ be fixed. Then Theorem 2.1 gives constants C_0 and v such that

$$\|[\tau_t^{H_L(\Phi)+\Psi_L}(A), B]\| \leq C_0 \|A\| \|B\| e^{v|t|} e^{-\mu d(S_A, S_B)} \tag{4.55}$$

for all $A, B \in \mathcal{A}_L$ and times $t \in \mathbb{R}$.

We want to improve this bound by making the couplings λ_x randomly chosen from a heavy-tailed distribution. At each x, let $\lambda_x \in [1, \infty)$ be drawn from the long-range distribution given by the density $m_a(r) = \frac{a}{r^{1+a}}$, $r \in [1, \infty)$ for some $0 < a < 1/2$. Since $\mathcal{F}$ is countably infinite and uniformly spaced, we can prove in this situation the following large deviation bound: For any $b \in (a, 1)$ and $\varepsilon > 0$, there exists $L_0 \in \mathbb{N}$ and $c > 0$ such that for all $L \geq L_0$

$$\mathbb{P}\Big(|\{x \in \mathcal{F} \cap [-L-3, L+3] : \lambda_x \geq \varepsilon(2L+1)\}| \geq (2L+1)^{1-b} \Big) \geq 1 - e^{-c\varepsilon^{-a}(2L+1)^{1-a}}, \tag{4.56}$$

see e.g. [Hoe63]. Equation (4.56) does not depend on the precise form of the density m_a but only on its tail.

Suppose $A \in \mathcal{A}_{\{-L\}}$, $B \in \mathcal{A}_{\{L\}}$. Setting $\varepsilon = C(1 + v|t|)(2L + 1)$, Theorem 3.1 implies the following bound.

Proposition 4.6. *Let $H_L(\vec{\lambda})$ be the Heisenberg spin chain with random transverse field defined in (4.54). Then, for all $A \in \mathcal{A}_{\{-L\}}$, $B \in \mathcal{A}_{\{L\}}$,*

$$\|[\tau_t^{H_L(\vec{\lambda})}(A), B]\| \leq \|A\| \|B\| e^{v|t|} e^{-2\mu L} e^{-(2L+1)^{1-b} \ln(2L+1)} \tag{4.57}$$

with probability $1 - e^{-c\varepsilon^{-a}(2L+1)^{1-a}}$ as derived in (4.56).

When L is sufficiently large so that $\exp(-(2L+1)^{1-b} \ln(2L+1)) < C_0$, the commutator bound in (4.57) is strictly sharper than the bound from Theorem 2.1.

5. Appendix: Double commutator bound

In this appendix we prove the double commutator bound which we use in the proof of Theorem 3.1. Our proof is a straightforward argument which we include for completeness. Let Φ_1 and Φ_2 denote two interactions such that $\Phi_i(X) \neq 0$ only for $X \subset \mathbb{Z}$ with $\mathrm{diam}(X) = \max\{|x-y| : x, y \in X\} \leq 1$ and

$$\|\Phi_i\| = \sup_x \|\Phi_i(\{x, x+1\})\| < \infty \tag{5.1}$$

for $i = 1, 2$. We assume that $\|\Phi_1\| = \|\Phi_2\| = \|\Phi\|$, and we note that Φ_1 and Φ_2 are nearest neighbor interactions where arbitrary large on-site terms are added. Let $\tau_t^{(1)}$ and $\tau_t^{(2)}$ be the Heisenberg dynamics on $[-L, L]$ generated by Φ_1 and Φ_2, respectively.

Let $\rho_x : \mathcal{A}_{\{x\}} \to \mathbb{C}$ denote the normalized trace $\rho_x(A) = \frac{1}{D}\mathrm{tr}(A)$, and for any finite subset Z, let id_Z denote the identity map on $\mathcal{A}_Z$. Then for any finite set $X \subset [-L, L]$ we define $\mathbb{E}_X = \mathrm{id}_X \otimes \bigotimes_{y \in [-L,L]\setminus X} \rho_y$, which has the following approximation property.

Lemma 5.1. [BHV06, NSW12, Cor. 3.1] *Suppose $A \in \mathcal{A}_L$, and suppose there exist $X \subset [-L, L]$ and $\varepsilon > 0$ such that for all $B \in \mathcal{A}_{[-L,L]\setminus X}$,*

$$\|[A, B]\| \leq \varepsilon \|A\| \|B\|. \tag{5.2}$$

Then $\|(\mathrm{id}_{[-L,L]} - \mathbb{E}_X)(A)\| \leq \varepsilon \|A\|$.

Lastly, we denote

$$\mathbb{E}_r = \mathbb{E}_{[-L,r)}. \tag{5.3}$$

for any $r \in (-L, L]$.

Lemma 5.2. *Let $A, B \in \mathcal{A}_L$ such that $\max S_A < \min S_B$. Suppose there exists $k \geq 0$ such that for all $T \in \mathcal{A}_L$ with $\min S_T > \max S_A + k$,*

$$\|[\tau_t^{(1)}(A), T]\| \leq C_* \|A\| \|T\| g(t) f(d(S_A, S_T)) \tag{5.4}$$

for some constant $C_* > 0$ and g, f real-valued functions, where f is monotone decreasing. Then, for all $W \in \mathcal{A}_L$ such that

$$\max S_A + k < \min S_W - 1 \leq \max S_W < \min S_B \tag{5.5}$$

we have

$$\|[[W, \tau_t^{(1)}(A)], \tau_s^{(2)}(B)]\| \leq \left(24 C_0 \frac{e^\mu}{1 - e^{-\mu}}\right) C_* \|A\| \|B\| \|W\| g(t) e^{v|s|} h_\mu(S_A, S_W, S_B) \tag{5.6}$$

where

$$\begin{aligned} h_\mu(S_A, S_W, S_B) = f(d(S_A, S_B)) + f(d(S_A, S_W) - 1) e^{-\mu d(S_W, S_B)} \\ + \sum_{m=1}^{D_W} f(d(S_A, S_W) + m - 2) e^{-\mu(D_W - m)} \end{aligned} \tag{5.7}$$

and $D_W = d(S_W, S_B) + \mathrm{diam}\, S_W + 1$.

Proof. Denote $\mathrm{id} = \mathrm{id}_{[-L,L]}$. Without loss of generality, we suppose $\|A\| = \|B\| = 1$ and we denote $b = \min S_B$. For any $X \subset [-L, L]$ and $P, R \in \mathcal{A}_X$ we obtain $\mathbb{E}_X(PQR) =$

$P\mathbb{E}_X(Q)R$ for $Q \in \mathcal{A}_L$. Since by assumption $W \in \mathrm{ran}(\mathbb{E}_b)$, we use the latter with $X = [-L, b)$, $Q = \tau_t^{(1)}(A)$ and P, R to be $\mathbb{1}$ and W alternatingly to get

$$\mathbb{E}_b([W, \tau_t^{(1)}(A)]) = [W, \mathbb{E}_b(\tau_t^{(1)}(A))]. \tag{5.8}$$

Hence

$$\begin{aligned} \|[(\mathrm{id} - \mathbb{E}_b)([W, \tau_t^{(1)}(A)]), \tau_s^{(2)}(B)]\| &= \|[[W, (\mathrm{id} - \mathbb{E}_b)(\tau_t^{(1)}(A))], \tau_s^{(2)}(B)]\| \\ &\leq 4\|W\|\|(\mathrm{id} - \mathbb{E}_b)(\tau_t^{(1)}(A))\|. \end{aligned} \tag{5.9}$$

Next we bound $\|[\mathbb{E}_b([W, \tau_t^{(1)}(A)]), \tau_s^{(2)}(B)]\|$. We set $w = \min S_W - 1$ and note

$$[W, \tau_t^{(1)}(A)] = [W, (\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A))] \tag{5.10}$$

which implies

$$\begin{aligned} \|[\mathbb{E}_b([W, \tau_t^{(1)}(A)]), \tau_s^{(2)}(B)]\| &= \|[\mathbb{E}_b([W, (\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A))]), \tau_s^{(2)}(B)]\| \\ &= \|[[W, \mathbb{E}_b(\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A))], \tau_s^{(2)}(B)]\|. \end{aligned} \tag{5.11}$$

Using Jacobi's identity (4.29) we obtain

$$\begin{aligned} \|[[W, \mathbb{E}_b(\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A))], \tau_s^{(2)}(B)]\| \leq \|[\mathbb{E}_b(\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A)), [\tau_s^{(2)}(B), W]]\| \\ + \|[W, [\mathbb{E}_b(\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A)), \tau_s^{(2)}(B)]]\|. \end{aligned} \tag{5.12}$$

We first treat the term $T = \|[W, [\mathbb{E}_b(\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A)), \tau_s^{(2)}(B)]]\|$. By setting $X = \{x : d(x, S_B) \leq n\}$ in Lemma 5.1, we decompose $\tau_s^{(2)}(B) = \sum_{n=1}^{\infty} B(s, n)$ such that each $B(s, n) \in \mathcal{A}_{S_B(n)}$ and

$$\|B(s, n)\| \leq (2C_0 e^{\mu}) e^{v|s|} e^{-\mu n}. \tag{5.13}$$

Substituting this into T, yields

$$T \leq 2\|W\| \sum_{m=1}^{D_W} \sum_{n \geq D_W - m} \|[\Delta_m(\tau_t^{(1)}(A)), B(s, n)]\| \tag{5.14}$$

where $D_W = d(S_W, S_B) + \mathrm{diam}\, S_W + 1$ and

$$\Delta_m(\tau_t^{(1)}(A)) = \Big(\mathbb{E}_{w+m} - \mathbb{E}_{w+m-1}\Big)(\tau_t^{(1)}(A)). \tag{5.15}$$

Hence (5.9), (5.11), (5.12) and (5.14) give

$$\begin{aligned} \|[[W, \tau_t^{(1)}(A)], \tau_s^{(2)}(B)]\| \leq\ & 4\|W\|\|(\mathrm{id} - \mathbb{E}_b)(\tau_t^{(1)}(A))\| \\ & + 2\|\mathbb{E}_b(\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A))\|\|[\tau_s^{(2)}(B), W]\| \\ & + 2\|W\| \sum_{m=1}^{D_W} \sum_{n \geq D_W - m} \|[\Delta_m(\tau_t^{(1)}(A)), B(s, n)]\|. \end{aligned} \tag{5.16}$$

Now assumption (5.4) with Lemma 5.1 implies

$$\begin{aligned} & 4\|W\|\|(\mathrm{id} - \mathbb{E}_b)(\tau_t^{(1)}(A))\| + 2\|\mathbb{E}_b(\mathrm{id} - \mathbb{E}_w)(\tau_t^{(1)}(A))\|\|[\tau_s^{(2)}(B), W]\| \\ \leq & 8C_0 C_* \|W\| g(t) e^{v|s|} \Big(f(d(S_A, S_B)) + f(d(S_A, S_W) - 1) e^{-\mu d(S_W, S_B)} \Big) \end{aligned} \tag{5.17}$$

as well as

$$2\|W\|\sum_{m=1}^{D_W}\sum_{n\geq D_W-m}\|[\Delta_m(\tau_t^{(1)}(A)),B(s,n)]\|$$
$$\leq\left(\frac{16C_0e^{\mu}}{1-e^{-\mu}}\right)C_*g(t)e^{v|s|}\|W\|\sum_{m=1}^{D_W}f(d(S_A,S_W)+m-2)e^{-\mu(D_W-m)}. \tag{5.18}$$

Plugging (5.17) and (5.18) into (5.16), the assertion follows. □

Corollary 5.3. *Let $A,W,B\in\mathcal{A}_L$ such that*

$$\max S_A<\min S_W-1\leq\max S_W<\min S_B. \tag{5.19}$$

Then for all $s,t\in\mathbb{R}$

$$\|[[W,\tau_t^{(1)}(A)],\tau_s^{(2)}(B)]\|\leq$$
$$\left(72C_0^2\frac{e^{\mu(\operatorname{diam}S_W+2)}}{1-e^{-\mu}}\right)\|A\|\|B\|\|W\|e^{v(|t|+|s|)}d(\min S_W-1,S_B)e^{-\mu d(S_A,S_B)}. \tag{5.20}$$

Proof. This follows as a special case of Lemma 5.2 using the commutator bound from Theorem 2.1 and $k=0$. In this case,

$$h_\mu(S_A,S_W,S_B)=e^{-\mu d(S_A,S_B)}+e^{-\mu(d(S_A,S_W)+d(S_W,S_B)-1)}+\sum_{m=1}^{D_W}e^{-\mu(d(S_A,S_B)-1)}$$
$$\leq 3d(\min S_W-1,S_B)e^{\mu(\operatorname{diam}S_W+1)}e^{-\mu d(S_A,S_B)} \tag{5.21}$$

where we recall $D_W=d(\min S_W-1,S_B)$. □

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