UCRL-16517

University of California

Ernest O. Lawrence Radiation Laboratory

ON FINITE GYRO-RADIUS CORRECTIONS TO THE HYDROMAGNETIC EQUATIONS FOR A VLASOV PLASMA

Special thesis

UCRL-16517

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

AEC Contract No. W-7405-eng-48

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(Ph. D. Thesis)

October 15, 1965

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ABSTRACT

A systematic procedure for expanding velocity moments of the Vlasov equation in the mass-to-charge ratio is presented and its relation to the strong magnetic field expansion of the Vlasov equation and to the guiding center description of the particle motion is discussed. It is shown that the expansion of the Vlasov equation implies a condition on the form of the distribution function at the initial time. Scaling of these and Maxwell's equations is studied, and both the ordering scheme leading to ordinary hydromagnetics and the so-called "finite gyro-radius" (FGR) ordering are considered.

A hydromagnetic description of a two-component plasma is easily found from the moment equations for the separate components. For the first of the above orderings FGR effects produce small corrections to ordinary hydromagnetics which are found through first order for an arbitrary plasma configuration. For FGR ordering, "finite gyro-radius" terms, which include second-order corrections to the pressure tensor, appear in even the lowest-order approximation to the moment equations.

The corresponding modified hydromagnetic equations are derived and, for the special case of low plasma pressure and uniform magnetic field, previous results are recovered.

Because of transport along the field lines, closed moment equations are not obtained in general for either ordering. In addition, for FGR ordering and non-uniform magnetic field, an analogous but independent closure problem arises because of transport across the field lines. For this reason applicability of the hydromagnetic equations for FGR ordering to finite-pressure systems is limited. Closed hydromagnetic equations are obtained, however, for low-frequency, finite-pressure interchange modes, and these equations are used to find finite-pressure modifications of the well known FGR stabilization of these modes. Closed equations are also obtained for the (finite-pressure) "firehose" instability and FGR corrections to the condition of marginal stability are found. A hydromagnetic description of finite-pressure "long-term" equilibria, and "minimum-B" configurations in particular, is also obtained.

CHAPTER I

Introduction

In spite of interest in "finite gyro-radius" corrections to the hydromagnetic equations, no complete derivation of these effects has been given for a Vlasov plasma. In particular, the corrections to the pressure equations of Chew, Goldberger, and Low (CGL),[1] and second-order corrections to the pressure tensor have not been discussed in general. Previous work has been either incomplete[2, 3] or complicated by the inclusion of collisions and restricted to a local Maxwellian velocity distribution in lowest order (e.g. references 4 and 5). Rosenbluth and Simon,[6] in work partly concurrent with ours, discuss the complete set of moment equations describing weak instabilities of a low-pressure plasma in a uniform magnetic field. The present work includes an alternate derivation of their basic equations and, for certain interchange motions, an extension of these equations to include the effects of finite plasma pressure.

We outline here a systematic expansion of the moment equations for a Vlasov plasma in a strong magnetic field. This expansion is the same as the usual B^{-1} expansion of the Vlasov equation, and is equivalent to treating the particle motion as adiabatic. The form of the expansion depends on how the time scale, electric field, plasma pressure, and other quantities are assumed to behave in the limit of small gyro-radius and short gyro-period. Ordering of the time scale of interest with respect to the gyro-period is of particular importance because the accuracy with which the plasma acceleration is required, hence the importance of the so called "finite gyro-radius" effects, depends on how long the motion is followed. A careful examination of ordering is therefore essential if the expansion procedure is to be unambiguous.

Our object is to present a systematic and self-contained discussion of the expansion of the moment equations for a Vlasov plasma in a strong magnetic field, and in this way to find all "finite gyro-radius" corrections to the ordinary hydromagnetic equations. No a priori assumptions are made concerning the plasma pressure or form of the distribution function, and careful attention is paid to the ordering of the various important quantities. Our discussion is more general than those that have previously appeared,[6,7] since effects of finite plasma pressure and magnetic field curvature are considered. We apply the expansion directly to the exact velocity moments of the Vlasov equation rather than to the Vlasov equation itself. Although the latter procedure is more usual,[1,3,6] the former[2,8] is simpler in many respects, if moment equations are desired, because unneeded details of the velocity distribution are eliminated at the outset.

Details of the distribution function, and effects such as wave-particle resonances which depend on these details, are lost if the distribution function is described by a few of its velocity moments. Thus, although resonance effects may be important and may be associated with finite gyro-radius effects, they are beyond the scope of a purely hydromagnetic approach and are not included in the present discussion. Our interest is in extending the hydromagnetic equations to include finite gyro-radius effects; we give examples (but attempt no survey) of applications of the equations obtained.

One area of recent and current interest in which finite gyro-radius effects are important is that of "universal" instabilities.[9] These instabilities are characterized by long but finite wavelengths along the magnetic field and are most unstable when their transverse wavelengths are comparable to the gyro-radii. Because, in the absence of collisions, closed moment equations are not obtained in general for plasma motion along the field lines, because our discussion assumes small gyro-radius, and also because the moment equations do not describe wave-particle resonances, these instabilities will not be discussed here, although an approximate hydromagnetic description of those that do not depend on resonances is possible.[10]

A macroscopic or fluid description of a gas or plasma is possible only if there is a strong constraint on the motion of the particles that inhibits their random motion. From a formal point of view, the problem of obtaining a fluid description of a gas or plasma is that of closing the infinite set of coupled equations obtained by taking velocity moments of the relevant kinetic equation (e. g., the Boltzmann or Vlasov equations).

Strong collisions provide such a constraint, maintaining the distribution close to a local Maxwellian; closed moment equations are then obtained, for example, from the well-known Chapman - Enskog expansion[11, 12] of the Boltzmann equation, the expansion parameter being the ratio of collision time or mean free path to macroscopic times or distances. When this parameter is small a given particle makes many collisions while traversing a macroscopic distance.

For a Vlasov plasma in a strong magnetic field, gyration about the field lines replaces collisions as the dominating feature of the particle motion;[1, 8] under certain circumstances, closed moment equations may be obtained from an expansion in the ratio of gyro-period or gyro-radius to the macroscopic time or distance (the adiabatic parameter ϵ). This is the well known expansion of the Vlasov equation in inverse powers of magnetic field strength, equivalent to treating the particle motion as adiabatic. The expansion may equally well be carried out on the exact velocity moments of the Vlasov equation; this latter procedure will be followed in this paper.

The adiabatic expansion or guiding center picture provides an asymptotic description of the particle motion valid in the limit $\epsilon \to 0$. When this expansion is applied to the moment equations, it is the zero-order terms that are of primary interest, although higher-order corrections may also be considered. The zero-order equations require first- or higher-order terms in the expansion of the particle motion; they are not simply equivalent to the zero-order guiding center motion. Their particular form depends on how the time scale, electric field, plasma pressure and other quantities are assumed to behave in the limit $\epsilon \to 0$.

Single-fluid hydromagnetic equations are easily obtained for a fully ionized two-component plasma from the separate moment equations for each component.[13] If the time scale of interest is not too long (of order ϵ^{-1} times the gyro-period τ_g), the single-fluid equations obtained from the lowest-order approximation to the moment equations are just the hydromagnetic equations of Chew, Goldberger and Low.[1]

Deviations from CGL theory arising from higher-order corrections to the moment equations (corresponding to higher-order terms of the adiabatic expansion of the particle motion) are usually referred to as "finite gyro-radius" (FGR) effects. These effects are the subject of this paper.

The FGR terms of the moment equations are small because the gyro-radius is assumed to be small in order to apply the adiabatic expansion. However, depending on how the time scale is related to the gyro-period in the limit $\epsilon \to 0$, these terms will be either small corrections to CGL theory or essential modifications of that theory. If the time scale is short ($\sim\tau_g/\epsilon$) FGR corrections to CGL theory are small. For long time scales (τ_g/ϵ^2), on the other hand, the FGR terms must be included even in the lowest-order moment equations; CGL theory is then no longer applicable.[14] This is the most interesting case, and the one usually considered, because the "finite gyro-radius" effects are important even if the gyro-radius is very small. These "FGR" effects are thus associated with the length of the time scale rather than with the size of the gyro-radius; we suggest that they are more appropriately described by the term "finite drift time" rather than the conventional term "finite gyro-radius." In keeping with this convention, however, we retain the designation "FGR" for these effects.

If closed moment equations are obtained for a Vlasov plasma in a strong magnetic field, it is because the velocity transverse to the magnetic field of every guiding center is close to the "E × B" velocity. Guiding centers initially on neighboring field lines then tend to remain on neighboring field lines. Motion of the guiding centers along the field lines is not strongly constrained by the magnetic field. Because of special symmetries, however, this parallel guiding center motion may be unimportant, and closed moment equations may then be obtained. Thus, in addition to those required for adiabatic particle motion, two independent conditions must be satisfied if closed moment equations describing a Vlasov plasma in a strong magnetic field are to be obtained: motion of the guiding centers along the field lines must be unimportant, and guiding centers initially on neighboring field lines must remain on neighboring field lines. The former of these conditions is well known, being satisfied, for example, if the magnetic field direction is constant in space and time, if there are no gradients in this direction, and if there is no plasma motion along the field lines. The latter condition depends on how the time scale of interest is ordered with respect to the gyro-period. It is therefore related to the importance of FGR effects and is important for the subject of this paper. Even if the moment equations are not closed, however, they may form a useful guide to the processes that may be expected.

Scaling of the single-particle equation of motion, the velocity moments of the Vlasov equation, and Maxwell's equations is discussed in the first chapter following this introduction. Two orderings of the dimensionless parameters which characterize the solutions of these equations are of primary interest. The first, which will be referred to as "CGL ordering," characterizes motion occuring on a time scale of order τ_g/ϵ. If this ordering is assumed, CGL hydromagnetic theory is obtained from the lowest-order approximation to the moment equations; FGR effects are then small. The second of these orderings characterizes motions occuring on a time scale of order τ_g/ϵ^2 and slow enough to appear as static equilibria on the shorter time scale of CGL ordering: "weak" instabilities[14] are motions of this type. Because of the long time scale associated with this ordering, "finite gyro-radius" effects are important, and the CGL equations must be modified, even in the limit $\epsilon \to 0$. The conventional designation "FGR" will be used for this ordering.

We will apply the small ϵ expansion directly to the exact velocity moments of the Vlasov equation. The expansion of the Vlasov equation itself is more familiar, however, and will be useful for the discussion of the moment equations. It is briefly reviewed in Chapter III. This review follows Thompson,[3] and indicates how his method must be modified if FGR rather than CGL ordering is assumed. This modified expansion has also been discussed by Kennel[15] and, for the special case of a low-pressure plasma in a uniform magnetic field, by Rosenbluth and Simon.[6] We also discuss in Chapter III a condition on the form of the initial distribution function which is necessary for the small ϵ expansion to apply. This condition is illustrated in Appendix B.

In Chapter IV an expansion procedure for the exact velocity moments of the Vlasov equation is discussed, these equations are put in a form convenient for application of this expansion, and their relation to single-fluid hydromagnetic equations is indicated. The expansion is carried out and the results are discussed in the remaining chapters.

First-order FGR corrections to CGL theory, obtained when CGL ordering is assumed, are discussed in Chapter V. These FGR effects are small. The equations obtained resemble the hydrodynamics of a non-ideal fluid. The CGL pressure equations are modified by the appearance of transverse heat flows and other effects. The transverse heat flows involve fourth moments of the zero-order velocity distribution. In the absence of collisions this distribution is not necessarily Maxwellian, and at the initial time the fourth moments must be specified independently of the density and kinetic temperatures. Their time dependence is given by relations similar to the CGL pressure equations.

In Chapters VI through X the lowest-order approximation to the moment equations for FGR ordering is discussed. The results of the appropriate expansions for the general case are presented in Chapter VI. These results are specialized to the case of a magnetic field of constant direction and their relation to the guiding center description of the particle motion is discussed.

As mentioned above, for FGR ordering the moment equations are closed only under rather special circumstances. The most important case for which they are closed is that of a low-pressure plasma in a uniform magnetic field. This case has been discussed by Rosenbluth and Simon,[6] who find a very simple formulation of the moment equations from an expansion of the Vlasov equation. Their basic equations, and the equivalent single-fluid hydromagnetic equations, are obtained in

Chapter VII as a special case of the results of Chapter VI. We thus provide an alternate derivation of the Rosenbluth-Simon equations which emphasizes their relation to equations describing more general configurations.

In Chapter VIII the Rosenbluth-Simon equations (and the equivalent single-fluid hydromagnetic equations) are extended to include the effects of finite plasma pressure for certain interchange motions. It is shown that under certain circumstances the variational methods of Rosenbluth and Simon also may be extended to apply to these finite pressure modes.[16] Closed moment equations are not obtained for more general finite pressure modes with constant magnetic field direction.

Another finite pressure mode is discussed in Chapter IX: the "firehose" instability[17] of an Alfven wave traveling along a uniform magnetic field in a uniform plasma with sufficiently large pressure in the direction of the magnetic field. Near the condition of marginal stability, FGR ordering applies; simple closed hydromagnetic equations which include the FGR effects are obtained for this mode, and FGR corrections to the stability condition are found. Our results contradict those of Yajima and Taniuti,[18] who failed to include all the FGR terms and found the FGR effects to be strongly stabilizing. Sato,[19] in work concurrent with ours, has obtained FGR corrections to this instability from a direct solution of the Vlasov equation which assumes Gaussian equilibrium distributions. His results are a special case of ours, but his conclusion that the FGR effects can not be described by modified hydromagnetic equations is incorrect. We also find the particle orbits for this mode, and discuss the relation between FGR modifications of these orbits and the FGR terms of the hydromagnetic equations.

In Chapter X the equilibrium moment equations are discussed and, in particular, the equations describing configurations which are in equilibrium over the long times associated with FGR ordering. This discussion is motivated by Northrop and Whiteman's[20] discussion of finite-pressure "minimum B" equilibria for which the pressure is a function of the magnetic field strength B only. They discuss these equilibria from both the CGL equations and from the general theory of adiabatic particle motion, and find the same class of solutions by both methods. The CGL equations give only "short term" equilibrium ($\tau \sim \tau_g/\epsilon$); they do not differentiate between "long term" equilibrium ($\tau \sim \tau_g/\epsilon^2$) and slow, low-frequency motion. To find these equilibria from adiabatic theory, however, Northrop and Whiteman used some (but not all) of the conditions for "long term" equilibrium and assumed the electric field to vanish through first order. We extend their discussion by applying the hydromagnetic equations for "long term" equilibrium to this class of configurations, and in this way include the effect of first-order electric fields.

Albegraic details of the calculations presented in Chapters IV through VI are outlined in Appendix A. In Appendix B we discuss a contradiction, noted by Northrop,[21] between expressions for the off-diagonal components of the pressure tensor found from the small ϵ expansion and results which are obviously correct for a simple special case. This contradiction arises because the special case considered does not satisfy the initial condition found in Chapter III. It is shown by calculation of the particle orbits that for this special case the pressure fluctuates rapidly, whereas, if the initial condition is satisfied, it does not, and has the value given by the small ϵ expansion. This calculation also completes Kaufman's[2] instructive discussion of the relation between the collisionless viscosity and the particle orbits by including the effects of magnetic induction.

Some of the material of the present paper has been reported previously by the author.[21a] This material, which includes part of Chapter II, most of the results presented in Chapters IV and V, and some of the results of Chapters VI and VII, is covered in greater detail here.

CHAPTER II

Ordering of the Dimensionless Parameters

2.1 Scaling of the Particle Motion

In the following sections of this paper we discuss the velocity moments of the Vlasov equation in the limit of large magnetic field or small m/e, corresponding to adiabatic particle motion. To make the ordering of small quantities unambiguous, and to clafiry the situations for which "finite gyro-radius" effects are important, we discuss in this section the dimensionless forms of the equations.

Since the particles of a Vlasov plasma move independently in the Vlasov fields, we first review the scaling of the equation of motion of a single charged particle in given fields.[22] We consider, therefore, the equation of motion of a particle of mass m and charge e, moving in a magnetic field $\underset{\sim}{B}$ and electric field $\underset{\sim}{E}$ with projections $\underset{\sim}{E}_{\perp}$ and $E_{\parallel}$ perpendicular and parallel to $\underset{\sim}{B}$, respectively, and gravitational field $\underset{\sim}{g}$. To put this equation in dimensionless form we introduce the dimensionless quantities

$$\underset{\sim}{B}' = \underset{\sim}{B}/B_0, \quad \underset{\sim}{E}'_{\perp} = \underset{\sim}{E}_{\perp}/E_{\perp 0}, \quad E'_{\parallel} = E_{\parallel}/E_{\parallel 0}, \quad \underset{\sim}{g}' = \underset{\sim}{g}/g_0, \text{ and } \underset{\sim}{v}' = \underset{\sim}{v}/v_0,$$

where B_0, $E_{\perp 0}$, $E_{\parallel 0}$, g_0, and v_0 are characteristic magnitudes of $\underline{B}$, $\underline{E}_\perp$, $\underline{E}_\parallel$, $\underline{g}$, and $\underline{v}$, respectively. In addition we introduce the time τ and distance L which characterize the variation of $\underline{E}$ and $\underline{B}$ in time and space, and scale time with respect to the variations of $\underline{E}$ and $\underline{B}$ seen by the moving particle. Our dimensionless time variable is therefore

$$t' = t(\tau^{-1} + v_0/L).$$

When written in terms of these dimensionless variables the equation of motion may be put in the form

$$\left(\frac{1}{\Omega_0\tau} + \frac{R_g}{L}\right)\frac{d\underline{v}'}{dt'} = \underline{v}' \times \underline{B}' + \frac{cE_{\perp 0}}{v_0 B_0}\underline{E}'_\perp + \frac{R_g}{L}\left(\frac{g_0 L}{v_0^2}\right)\underline{g}'$$
$$+ \frac{cE_{\parallel 0}}{v_0 B_0} E'_\parallel \underline{b}, \tag{2. 1}$$

where $\Omega_0 = eB_0/mc$ and $R_g = v_0/\Omega_0$ are the characteristic gyro-frequency and gyro-radius of the particles and $\underline{b} = \underline{B}/B$.

Our basic assumption is that both $(\Omega_0\tau)^{-1}$ and R_g/L are small so that the adiabatic expansion[23] and guiding center description[24, 25] of the particle motion are applicable. Among other things we will be interested in situations such as low-frequency waves for which $(\Omega_0\tau)^{-1} \ll R_g/L$. Our basic expansion parameter is $\epsilon = R_g/L$. The parameter $L/v_0\tau$ then relates $(\Omega_0\tau)^{-1}$ to ϵ by the identity $(\Omega_0\tau)^{-1} =$

$\epsilon (L/v_0\tau)$.

Equation (2. 1) shows that the acceleration of the particle is of order ϵ^{-1} . The transverse acceleration is of this order because of the rapid gyration of the particles, but, since the velocity must remain of order v_0 for the expansion to apply and the gyration only slightly affects the parallel motion, the parallel acceleration must be of non-negative order in ϵ . The parameters entering Eq. (2. 1) are therefore restricted as follows:

$$L/v_0\tau \sim \epsilon^{n_1} , \; cE_{\perp 0}/v_0B_0 \sim \epsilon^{n_2} , \; g_0L/v_0^2 \sim \epsilon^{n_3} , \; cE_{\parallel 0}/v_0B_0 \sim \epsilon^{n_4+1} , \; n_i \geq 0 . \qquad (2.\ 1a)$$

The guiding center description of the particle motion represents an asymptotic expansion of the particle motion valid in the limit $\epsilon \to 0$. The value of $\epsilon = mcv_0/eB_0L$ appropriate to a given physical system may be decreased in a number of equivalent ways: by increasing B while keeping the geometry (i. e. , L), v_0 , and m/e fixed; by increasing L while keeping B , v_0 and m/e unchanged; by decreasing v_0 or m/e without changing the other quantities; or by some combination of these operations. The other dimensionless parameters may be related to ϵ in any way consistent with the conditions (2. 1a), Maxwell's equations, and the plasma dynamics.

In addition to these parameters, which appear explicitly in Eq. (2.1), others are required to characterize the plasma motion; for example Maxwell's equations introduce the pressure parameter β and the relativistic parameter v_0/c, consideration of a two-component plasma introduces the ion-electron mass ratio. Other parameters, such as the ratio $L_{\parallel}/L_{\perp}$ of gradients parallel and perpendicular to B may also be important.

Of particular importance is the total time scale of interest, since the accuracy with which the velocity or acceleration must be determined, hence the importance of "finite gyro-radius" effects, depends on how long the motion is followed. The total time lapse τ_0, which need not be of the same order as τ, will be specified by the parameter $L/v_0\tau_0$.

Our basic expansion parameter is ϵ. For convenience we refer to

$$L/v_0\tau_0, \quad L/v_0\tau, \quad cE_{0\perp,\parallel}/v_0B_0, \quad g_0L/v_0^2, \quad \beta, \; v_0/c$$

and any other parameters of interest, as "auxiliary parameters." The first three of these are the most important. The total time scale is specified by $L/v_0\tau_0$, the scale of time derivatives by $L/v_0\tau$, while $cE_{\perp 0}/v_0B_0$ is directly related to both the transverse guiding center and transverse plasma velocities. Since the only zero-order contribution to these velocities is the "$\underset{\sim}{E} \times \underset{\sim}{B}$" drift, they are of zero order only if $cE_{\perp 0}/v_0B_0 \sim 1$.

The Vlasov equation and its velocity moments describe the same physical process as the single-particle equation of motion; hence all these equations scale with the same parameters and in a similar way. This scaling may be found by writing the equations in dimensionless form, or, more simply, from a comparison of Eq. (2. 1) with its dimensional form. From this comparison it is clear that the ordering of terms in any of the equations may be determined by regarding $\underset{\sim}{v}$ and ∇ as independent of ϵ and the other parameters, m/e and Ω^{-1} as proportional to ϵ, and $\partial/\partial t$, $E_{\perp,\parallel}$ and g as proportional to $L/v_0\tau$, $cE_{0\perp,\parallel}/v_0B_0$, and g_0L/v_0^2, respectively. Thus, for example, the dimensionless form of the transverse equation of motion for one plasma component, Eq. (4. 3), is

$$\underset{\sim}{u}'_{\perp} - \frac{cE_{\perp 0}}{v_0B_0}\underset{\sim}{u}'_E = \epsilon\frac{\underset{\sim}{b}}{B'} \times \left[\frac{L}{v_0\tau}\frac{\partial \underset{\sim}{u}'}{\partial t''} + \underset{\sim}{u}' \cdot \nabla'\underset{\sim}{u}' + \frac{\nabla \cdot \underset{\sim}{P}'}{\rho}\right], \tag{2. 3}$$

where $\underset{\sim}{u}' = \underset{\sim}{u}/v_0$, $\underset{\sim}{P}' = \underset{\sim}{P}/n_0mv_0^2$, and $t'' = t/\tau$.

It is evident that this equation may be obtained either by transforming to the dimensionless variables, or directly from Eq. (4. 3) by use of the above rules. The dimensionless forms of the other moment equations are similar to Eq. (2. 3).

We emphasize that the ordering of terms in the asymptotic expansion of the particle motion (e. g., the drift velocity) and in the expansion of the moment equations, will depend on the particular ordering chosen for the auxiliary parameters. The appropriate ordering of terms in the expansions thus depends on specification of how τ_0 , τ , $E_{\perp 0}$, $E_{\parallel 0}$, g_0 , β , $L_{\parallel}/L_{\perp}$, and other parameters that may be important behave in the limit $\epsilon \rightarrow 0$.

2.2 CGL and FGR Ordering

The accuracy with which the velocity or acceleration is required to obtain the displacement of the plasma to a given order, hence the importance of the so-called "finite gyro-radius" effects, depends on the order of $L/v_0\tau_0$ with respect to ϵ . Since we make the approximation of adiabatic particle motion, which is an asymptotic description valid in the limit $\epsilon \rightarrow 0$, our primary interest is in the zero-order displacement of the plasma. If consideration is limited to "short" times of order $(\epsilon\,\Omega)^{-1}$, $L/v_0\tau_0 \sim 1$, only the zero-order plasma velocity will be important. In this case the lowest-order approximation to the moment equations leads to the familiar hydromagnetic theory of Chew, Goldberger, and Low.[1] Over "long" times of order $(\epsilon^2\Omega)^{-1}$, however, first-order velocities will also produce zero-order displacements and therefore must be included even in lowest order. These "long" times correspond to $L/v_0\tau_0 \sim \epsilon$.

The zero-order transverse plasma and guiding center velocities are just the "E × B" drift produced by the zero-order electric field, hence zero-order displacement of the plasma occurs during times of order $(\epsilon\Omega)^{-1}$ only if $cE_{0\perp}/v_0 B_0 \sim 1$. Similarly, time derivatives are important on this time scale only if $L/v_0\tau \sim 1$ and gravitational effects will be negligible unless $g_0 L/v_0^2 \sim 1$. Motion occuring during times of order $(\epsilon\Omega)^{-1}$, and in particular motion described by CGL hydromagnetics, is therefore characterized by the "CGL ordering"

$$L/v_0\tau_0 \sim L/v_0\tau \sim cE_{0\perp}/v_0B_0 \sim g_0L/v_0^2 \sim 1, \quad cE_{0\parallel}/v_0B_0 \sim \epsilon . \qquad (2.\ 3)$$

For these motions the important plasma accelerations are of order v_0^2/L: the pressure tensor and other stresses are therefore required only in zero order.

If the time scale is extended to be of order $(\epsilon^2\Omega)^{-1}$, however, first-order velocities and accelerations as small as $\epsilon^2 v_0^2/L$ also contribute to the zero-order displacement. The CGL equations are then no longer a consistent approximation to the moment equations, but must be modified even in lowest order.[14] These modifications, which are usually referred to as "finite gyro-radius" effects, take the form of first- and second-order corrections to the pressure tensor, corresponding corrections to the CGL pressure equations, and suitable modifications of the generalized Ohm's law.[13]

The ordering of τ_0 with respect to ϵ may be considered from the point of view of a mirror confinement system. Guiding center motion in such a system may be characterized by the period of oscillation along the field lines $\tau_\ell \sim (\epsilon\Omega)^{-1}$, and the "drift" time $\tau_D \sim (\epsilon^2\Omega)^{-1}$ required for a particle to drift a zero-order distance accross the field lines in the absence of a strong electric field.

The CGL hydromagnetic theory describes motion occuring during the "longitudinal" time τ_ℓ, a strong $\underset{\sim}{E}_\perp$ being necessary for non-negligible transverse motion to occur during this time. This corresponds, in lowest order, to regarding the gyro-period as infinitely short $(\sim \epsilon\tau_\ell)$ and the "drift" time and as infinitely long $(\sim \tau_\ell/\epsilon)$ compared with the time of interest.

Extension of the theory to describe motion occuring during the drift time τ_D, on the other hand, corresponds to regarding both the gyro-period $(\sim \epsilon^2\tau_D)$ and "longitudinal" time τ_ℓ $(\sim \epsilon\tau_D)$ as infinitely small and τ_D as finite, the time of observation being increased with τ_D as $\epsilon \to 0$ so that τ_D remains finite with respect to the time scale τ_0 rather than being regarded as infinitely long. It might be more appropriate, therefore, to refer to the modifications of the lowest-order theory (e.g., CGL theory) that must be made when the time scale is extended to be of order τ_D as "finite drift-time" effects, and reserve the term "finite gyro-radius" for non-adiabatic effects, or at least for effects which are unimportant in the limit $\epsilon \to 0$.

A further indication of the inappropriateness of the term "finite gyro-radius" in this connection is that the current which enters CGL theory includes the effects of the first-order transverse drifts. These drifts would seem to be as much effects of "finite gyro-radius" as the first-order plasma velocity and small accelerations which become important on the longer time scale.

It is conventional, however, to describe these effects by the term "finite gyro-radius."[6, 7, 14] In keeping with convention we use the designation "FGR" for these effects, but prefer to associate the words "finite drift-time" with this designation, rather than "finite gyro-radius."

Processes occuring during the drift time are of particular interest if no motion occurs on the longitudinal time scale, corresponding to a static CGL equilibrium. In this paper discussion of motion occuring during the drift time will be limited to this case, which includes low frequency waves, "weak" instabilities, and "long term" equilibrium. These motions are characterized by a time scale of order $(\epsilon^2 \Omega)^{-1}$ and velocities of order ϵv_0, and therefore by the "FGR ordering"

$$L/v_0\tau_0 \sim L/v_0\tau \sim cE_{0\perp}/v_0B_0 \sim u_\parallel/v_0 \sim \epsilon \, . \qquad (2.4)$$

It should be emphasized that the appropriate ordering, as well as the value of ϵ, depends on the process under consideration, rather than only on the properties of the plasma. The interchange mode of a plasma in a weak gravitational field, for example, may be characterized by FGR ordering while the magnetoaccoustic wave propagating across the magnetic field will be described by CGL ordering. The appropriate "low β"

condition also depends on the motion considered.

2.3 Scaling of Maxwell's Equations

The dimensionless forms of Maxwell's equations may be written

$$\epsilon\,(cE_0/v_0B_0)\,(\Omega_+\Omega_-/\omega_p^2)\,\nabla'\cdot\underset{\sim}{E}' = \Delta n' , \tag{2.5}$$

$$\nabla'\times\underset{\sim}{E}' = -\,(v_0B_0/cE_0)\,(L/v_0\tau)\,\partial\underset{\sim}{B}'/\partial t'' , \tag{2.6}$$

$$\nabla'\times\underset{\sim}{B}' = \frac{\beta}{\epsilon}\underset{\sim}{j}' + \left(\frac{cE_0}{v_0B_0}\right)\left(\frac{L}{v_0\tau}\right)\left(\frac{v_0}{c}\right)^2\frac{\partial\underset{\sim}{E}'}{\partial t''} . \tag{2.7}$$

In these equations E_0 characterizes the contribution of $\underset{\sim}{E}$ to $\nabla\cdot\underset{\sim}{E}$ or $\nabla\times\underset{\sim}{E}$; $\Delta n' = \rho_c/n_0e$ and $\underset{\sim}{j}' = c\underset{\sim}{j}/n_0ev_0$ with ρ_c and $\underset{\sim}{j}$ the charge and current densities; and ω_p is the plasma frequency $(ne^2/m_-)^{\frac{1}{2}}$.

These equations involve two new independent dimensionless parameters; the three paramers $\Omega_+\Omega_-/\omega_p^2$, β, and v_0/c being related by the identity

$$\beta(\Omega_+\Omega_-/\omega_p^2) = (v_0/c)^2 . \tag{2.8}$$

The assumption of non-relativistic motion already implies an expansion in v_0/c; ordering of this parameter with respect to ϵ is discussed below.

Equation (2.5) shows that the charge separation $\Delta n'$ is small for high plasma frequency; because we assume low frequencies and small gradients, $\Delta n'$ will be small except for very low densities as shown by Eq. (2.5) and the relation (2.8). Thus

$$\begin{aligned} \Delta n' &\sim \epsilon (cE_0 / v_0 B_0) (B^2 / n_0 mc^2) \\ &\sim (\epsilon / \beta) (cE_0 / v_0 B_0) (v_0 / c)^2 . \end{aligned} \tag{2.9}$$

The importance of the charge separation is indicated by the magnitude of the electrostatic stress. This stress is of order

$$\begin{aligned} E^2 &\sim \rho v_0^2 (cE_0 / v_0 B_0)^2 \beta^{-1} (v_0 / c)^2 \\ &\sim \rho u_0^2 (cE_0 / v_0 B_0)^2 (B_0^2 / n_0 m_+ c^2) . \end{aligned} \tag{2.10}$$

For CGL ordering only stresses of order $N_0 m v_0^2$ are required in lowest order and the electrostatic stress is negligible unless

$$B_0^2 / n_0 m_+ c^2 \sim 1 \quad \text{or} \quad \beta \sim (v_0 / c)^2 . \tag{2.11}$$

This condition holds also for FGR ordering, even though stresses of order $\epsilon^2 n_0 m v_0^2$ become important in lowest order, because then $cE_0 / v_0 B_0 \sim \epsilon$.

Thus the electrostatic stresses are negligible in lowest order, and the approximation of quasi charge-neutrality applicable, if $(v_0/c)^2 \sim \epsilon\beta$ or smaller. Note that $(v_0/c)^2 \sim \epsilon$ is the condition that relativistic corrections to the plasma motion be negligible in the limit $\epsilon \to 0$.

The strength of the magnetic field produced by a plasma current of given order is indicated by Eq. (2.7). If $\beta \sim 1$ the zero-order field is produced by the first-order current; for this reason the zero-order plasma dynamics, e.g., CGL theory, requires the first-order current. Furthermore, for $\beta \sim 1$ there can be no zero-order current if the particle motion is to be adiabatic. The restriction this places on the applicability of our equations is discussed in Section 4.8.

Equation (2.7) may be written in the form

$$\nabla' \times \underset{\sim}{B}' = \frac{\beta}{\epsilon}\left[\underset{\sim}{j}' + \frac{\epsilon}{\beta}\left(\frac{cE_0}{v_0 B_0}\right)\left(\frac{v_0}{c}\right)^2\left(\frac{L}{v_0 \tau}\right)\frac{\partial \underset{\sim}{E}'}{\partial t''}\right] . \tag{2.12}$$

Thus, since $L/v_0\tau$ cannot be large, the displacement current is small whenever $\Delta n'$ is.

CHAPTER III

Expansion of the Vlasov Equation

3.1 Expansion Procedure.

Our interest in this paper is in the macroscopic equations obtained by taking velocity moments of the Vlasov equation, and, in particular, the small ϵ expansion of these equations. This expansion may be carried out either before or after taking the velocity moments. We will follow the latter procedure[2, 8] although the former procedure is more usual.[1, 3, 26] However, because the two procedures are equivalent, it is useful to have the form of the expansion of the Vlasov equation in mind when the expansion of the moment equations is considered. Furthermore, the Vlasov equation and its expansion clarify the relation between the expansion of the moment equations and the adiabatic expansion of the particle motion.

In this section, therefore, we review briefly the small ϵ expansion of the Vlasov equation following the discussion given by Thompson,[3] and briefly discuss the initial conditions necessary for its applicability.

The small ϵ expansion of the Vlasov equation,

$$\partial f(\underset{\sim}{r}, \underset{\sim}{v}, t)/\partial t + \underset{\sim}{u} \cdot \nabla f + (e/m)[\underset{\sim}{E} + (\underset{\sim}{v}/c) \times \underset{\sim}{B}] \cdot \nabla_{\underset{\sim}{v}} f = 0, \qquad (3.1)$$

is carried out by transforming from the velocity variable $\underset{\sim}{v}$ to the variable $\underset{\sim}{C} = \underset{\sim}{v} - \underset{\sim}{u}(\underset{\sim}{r}, t)$, where the velocity $\underset{\sim}{u}(\underset{\sim}{r}, t)$ is as yet unspecified, and by representing $\underset{\sim}{C}$ by its cylindrical components $C_{\parallel}(\underset{\sim}{r}, \underset{\sim}{v}, t)$, $C_{\perp}(\underset{\sim}{r}, \underset{\sim}{v}, t)$, and $\phi(\underset{\sim}{r}, \underset{\sim}{v}, t)$:

$$\underset{\sim}{C} = C_{\parallel}\underset{\sim}{e}_1 + C_{\perp}(\underset{\sim}{e}_2 \cos\phi + \underset{\sim}{e}_3 \sin\phi),$$

where the $\underset{\sim}{e}_i$ are the basis vectors of a local Cartesian coordinate system with $\underset{\sim}{e}_1$ in the direction of the magnetic field. The vector $\underset{\sim}{e}_2$ may be taken in the direction $\underset{\sim}{e}_1 \cdot \nabla \underset{\sim}{e}_1$, but this is not necessary.

When transformed to the variables $\underset{\sim}{r}, C_{\parallel}, C_{\perp}, \phi$, and t, Eq. (3.1) takes the form

$$\partial f(\underset{\sim}{r}, C_{\parallel}, C_{\perp}, \phi, t)/\partial\phi = \Omega^{-1}\mathfrak{D} f, \qquad (3.2)$$

where Ω is the gyro-frequency eB/mc and

$$\mathfrak{D} = D^* + \left[\frac{e}{m}E_{\parallel} - (D^*\underset{\sim}{u})_{\parallel} + \frac{C_{\perp}^2}{2}\nabla\cdot\underset{\sim}{b}\right]\frac{\partial}{\partial C_{\parallel}} - \frac{C_{\perp}}{2}(\nabla_{\perp}\cdot\underset{\sim}{u}^*)\frac{\partial}{\partial C_{\perp}}$$

$$+ \cos\phi\left\{C_{\perp}\frac{\partial}{\partial x_2} + C_{\perp}\left[(D^*\underset{\sim}{b})_2 - \left(\frac{\partial \underset{\sim}{u}}{\partial x_2}\right)_{\parallel}\right]\frac{\partial}{\partial C_{\parallel}} - \left[(D^*\underset{\sim}{u}^*)_2\right.\right.$$

$$\left.\left. - \Omega(\underset{\sim}{u} - \underset{\sim}{u}_E)_3\right]\frac{\partial}{\partial C_{\perp}}\right\} + \sin\phi\left\{C_{\perp}\left[\frac{\partial}{\partial x_3} + C_{\perp}(D^*\underset{\sim}{b})_3 - \left(\frac{\partial \underset{\sim}{u}}{\partial x_3}\right)_{\parallel}\right]\frac{\partial}{\partial C_{\parallel}}\right.$$

$$\left. - \left[(D^*\underset{\sim}{u}^*)_3 + \Omega(\underset{\sim}{u} - \underset{\sim}{u}_E)_2\right]\frac{\partial}{\partial C_{\perp}}\right\}$$

$$+ \frac{\cos 2\phi}{2} \underset{\sim}{I}_\delta : \left\{ (C_\perp^2 \nabla \underset{\sim}{b}) \frac{\partial f}{\partial C_\parallel} + C_\perp (\nabla \underset{\sim}{u}^*) \frac{\partial}{\partial C_\perp} \right\}$$

$$+ \frac{\sin 2\phi}{2} I_\gamma : \left\{ (C_\perp^2 \nabla \underset{\sim}{b}) \frac{\partial f}{\partial C_\parallel} - C_\perp (\nabla \underset{\sim}{u}^*) \frac{\partial}{\partial C_\perp} \right\}$$

$$+ \left\{ \frac{1}{2} \underset{\sim}{b} \cdot \nabla \times \underset{\sim}{u}^* - \frac{\cos \phi}{C_\perp} \left[\Omega(\underset{\sim}{u} - \underset{\sim}{u}_E)_2 - (D^* \underset{\sim}{u}^*)_3 \right] \right.$$

$$- \frac{\sin \phi}{C_\perp} \left[\Omega(\underset{\sim}{u} - \underset{\sim}{u}_E)_3 + (D^* \underset{\sim}{u}^*)_2 \right] + \left[\frac{\cos 2\phi}{2} I_\gamma - \frac{\sin 2\phi}{2} \underset{\sim}{I}_\delta \right] :$$

$$\left. \nabla \underset{\sim}{u}^* \right\} \frac{\partial}{\partial \phi}, \qquad (3.3)$$

with $D^* = \partial/\partial t + \underset{\sim}{u} \cdot \nabla + C_\parallel \underset{\sim}{b} \cdot \nabla$, $\underset{\sim}{u}^* = \underset{\sim}{u} + C_\parallel \underset{\sim}{b}$, $I_\gamma = \frac{1}{2}(\underset{\sim}{e}_2 \underset{\sim}{e}_3 + \underset{\sim}{e}_3 \underset{\sim}{e}_2)$, and $\underset{\sim}{I}_\delta = \frac{1}{2}(\underset{\sim}{e}_2 \underset{\sim}{e}_2 - \underset{\sim}{e}_3 \underset{\sim}{e}_3)$.

The small ϵ expansion of the Vlasov equation is based on the form (3.2) and the assumption that $\Omega^{-1} \mathcal{D} f \sim \epsilon f$.

This condition will be satisfied if all the functions entering this equation are slowly varying in space and time and if $\underset{\sim}{u} = \underset{\sim}{u}_E + 0(\epsilon v_0)$. These conditions reflect the equivalence of this expansion to the adiabatic expansion of the particle motion. The condition that f be slowly varying in time is discussed in Section 3.3.

The operator $\mathcal{D}$ takes its simplest form if $\underset{\sim}{u}_\perp = \underset{\sim}{u}_E$, and it is convenient to choose $u_\parallel = \langle v_\parallel f^{(0)} \rangle / \langle f^{(0)} \rangle$. These choices are usually made. Our discussion of the moment equations, however, corresponds to taking $\underset{\sim}{u}$ equal to the actual flow velocity of the plasma component

$\langle \underset{\sim}{v} f \rangle / \langle f \rangle$. For this reason we leave $\underset{\sim}{u}$ unspecified in Eq. (3. 3) .

If $\Omega^{-1} \mathfrak{D} f \sim \epsilon f$, the expansion of Eq. (3. 2) is straightforward. The distribution function f is expanded in the form $f^{(0)} + f^{(1)} + \ldots$. In lowest order Eq. (3. 2) reduces to the familiar statement

$$\partial f^{(0)} / \partial \phi = 0 , \tag{3. 4}$$

and the first-order terms of Eq. (3. 2) are

$$\partial f^{(1)} / \partial t = \Omega^{-1} f^{(0)} . \tag{3. 5}$$

Since $f^{(1)}$ must be single valued in ϕ , integration of Eq. (3. 5) over all ϕ gives

$$0 = \oint d\phi \, \partial f^{(1)} / \partial \phi = \Omega^{-1} \oint d\phi \, \mathfrak{D} \; f^{(0)},$$

or

$$\overline{\mathfrak{D}} f^{(0)} = 0 . \tag{3. 6}$$

Equation (3. 6) is the zero-order Vlasov equation. It describes the zero-order distribution of guiding centers and is easily derived, in fact, from the conservation of guiding centers, the zero-order guiding center motion, and the conservation of the magnetic moment $\mu = mC_{\perp}^2/2B$. That the conservation of magnetic moment is contained in Eq. (3. 6) is made evident by transforming from the variable $C_{\perp}$ to μ , the coefficient of $\partial f^{(0)} / \partial \mu$ in the transformed equation vanishing by use of Maxwell's induction equation.

Integration of Eq. (3.5) with respect to ϕ gives $f^{(1)}$ in the form

$$f^{(1)} = \Omega^{-1}\left(\int^{\phi} d\phi' \mathcal{D}\right) f^{(0)} + h^{(1)}(C_\perp, C_\parallel, \underset{\sim}{r}, t). \qquad (3.7)$$

Thus only the ϕ-dependent part of $f^{(1)}$ is determined by Eq. (3.5).

This suggests that $f = f^{(0)} + f^{(1)} + \ldots$ be divided into ϕ-dependent and ϕ-independent parts:

$$f(C_\perp, C_\parallel, \phi, \underset{\sim}{r}, t) = f_\phi(C_\perp, C_\parallel, \phi, \underset{\sim}{r}, t) + h(C_\perp, C_\parallel, \underset{\sim}{r}, t),$$

with

$$\oint f_\phi \, d\phi = 0, \qquad (3.8)$$

and

$$f^{(i)} = f_\phi^{(i)} + h^{(i)}, \qquad \oint f_\phi^{(i)} d\phi = 0.$$

Integration with respect to ϕ of the expansion of Eq. (3.2) then gives

$$f_\phi^i = \Omega^{-1} \int_{\phi_i}^{\phi} \mathcal{D} f^{i-1} d\phi', \qquad (3.9)$$

where the ϕ_i are determined by the condition that the ϕ-averages of $f_\phi^{(i)}$ vanish, and

$$\bar{\mathcal{D}} h^{(i)} = -\frac{1}{2\pi} \oint \mathcal{D} f_\phi^i d\phi = -\frac{1}{2\pi} \oint d\phi \mathcal{D} \frac{1}{\Omega} \int_{\phi_i}^{\phi} \mathcal{D} \; f^{i-1} d\phi'. \qquad (3.10)$$

Equation (3.9) is a recursion relation expressing $f_\phi^{(i)}$ in terms of $f^{(i-1)}$ while Eq. (3.10) is the i^{th} order Vlasov equation. Solution of the Vlasov equation for h through i^{th} order thus immediately gives f_ϕ through order $i + 1$.

The zero-order Vlasov equation (3.6) and the expression (3.9) for $f_\phi^{(1)}$ are quite simple. Thompson[3] discusses these equations, and from them derives the CGL hydromagnetic equations and the contributions of $f_\phi^{(1)}$ to the pressure tensor and heat flow vectors.

Except for very simple configurations, however, evaluation of even the first-order Vlasov equation and the expression for $f_\phi^{(2)}$ is tedious. Carrying out the expansion directly on the exact moment equations is algebraically simpler in many ways because details of the distribution function and the complications of the velocity transformation are eliminated at the outset, and gives results in a form convenient for iteration to higher order.

The characteristic feature of the expansion of the Vlasov equation is the separation of f into ϕ-dependent and ϕ-independent parts. The manner in which this feature carries over into the small ϵ expansion of the moment equations will be discussed in Section 4.1.

3.2 FGR Ordering and "FGR" Effects.

The above outline of the expansion procedure is oversimplified in that the operator $\mathfrak{D}$ is itself given by an expansion in ϵ because of the expansions of $\underset{\sim}{E}$, $\underset{\sim}{B}$, and $\underset{\sim}{u}$. The ordering of the various terms of $\mathfrak{D}$ depends on the ordering of the auxiliary parameters with respect to ϵ.

Thompson[3] implicitly assumes CGL ordering, and therefore obtains the CGL equations in lowest order. For FGR ordering, however, the expansion takes on a quite different form. This form is discussed in greater detail by Kennel[15] and, for the special case of a low-β plasma in a uniform magnetic field, by Rosenbluth and Simon.[6] We illustrate the form of the expansion for FGR ordering by discussing the lowest-order Vlasov equation for the case of straight field lines, transverse motion, but finite β.

We consider, therefore, the operator $\mathfrak{D}$ under the following conditions: $\underset{\sim}{b}$ constant in space and time, $\underset{\sim}{b} \cdot \nabla = 0$, $\underset{\sim}{u}_\perp = \underset{\sim}{u}_E$, $f(C_\parallel) = f(-C_\parallel)$, and FGR ordering. Under these conditions $\bar{\mathfrak{D}}$ is of first order. The zero-order Vlasov equation (3.6), which should read

$$\bar{\mathfrak{D}}^{(0)} f^{(0)} = 0, \tag{3.11}$$

is therefore automatically satisfied. The first order Vlasov equation becomes, since $\mathfrak{D}^{(1)} h^{(1)} \sim \epsilon^2$,

$$\bar{\mathfrak{D}}^{(1)} f^{(0)} = -\frac{1}{2\pi} \oint d\phi\, \mathfrak{D}^{(0)} \frac{1}{\Omega} \int_{\phi_1}^{\phi} \mathfrak{D}^{(0)} f^{(0)} d\phi' . \tag{3.12}$$

For CGL ordering the equation for $f^{(0)}$ has the form (3.11). When FGR ordering is considered this equation is modified in form by the "FGR" term on the right-hand side of Eq. (3.12). This term arises from the first-order drifts and appears in Eq. (3.12) because of the long time scale associated with FGR ordering.

For the special case under consideration here,

$$\mathfrak{D}^{(0)} = C_{\perp}\left[\cos\phi\frac{\partial}{\partial x_2} + \sin\phi\frac{\partial}{\partial x_3}\right], \qquad (3.13)$$

and the lowest-order Vlasov Equation (3.12) becomes

$$\frac{\partial f^{(0)}}{\partial t} + \underset{\sim}{u}_E^{(1)} \cdot \nabla f^{(0)} - \frac{C_{\perp}}{2}(\nabla \cdot \underset{\sim}{u}_E^{(1)}) \frac{\partial f^{(1)}}{\partial C_{\perp}} = -\frac{C_{\perp}^2}{2\Omega} \underset{\sim}{b} \cdot (\nabla B \times \nabla f^{(0)}). \qquad (3.14)$$

The FGR term on the right-hand side of Eq. (3.14) may be understood by recalling that under the present assumptions the lowest-(i.e., first) order drift velocity is just $\underset{\sim}{u}_E + \underset{\sim}{v}_B$, where $\underset{\sim}{v}_B = (C_{\perp}^2/2\Omega)\underset{\sim}{b} \times \nabla B/B$ is the well-known "gradient B" drift.[22] The FGR term of Eq. (3.14) is thus $-\underset{\sim}{v}_B \cdot \nabla f$, and the FGR modification of the zero-order Vlasov equation, in this case, is obtained simply by substituting the first-order drift velocity $\underset{\sim}{u}_E + \underset{\sim}{v}_B$ for the zero-order drift velocity $\underset{\sim}{u}_E^{(0)}$ appearing in Eq. (3.11).

When Eq. (3.14) is written in terms of μ instead of $C_{\perp}$, the coefficient of $\partial f/\partial\mu$ again vanishes. For the special case considered here the magnetic moment is therefore conserved over times of order $(\epsilon^2\Omega)^{-1}$, even in the presence of strong transverse magnetic field gradients $\nabla_\perp B \sim \epsilon B/R_g$. Equation (3.14) thus describes particles of constant μ which move along the field lines with the constant velocity $v_\parallel$ and in the transverse direction with the first-order guiding center velocity $\underset{\sim}{u}_E + \underset{\sim}{v}_B$. Zero-order curvature of the field lines produces additional first-order drifts and first-order variations of μ, hence additional FGR modifications of Eq. (3.10).

3.3 Initial Conditions.

The expansion of the Vlasov equation outlined above is based on the assumption that $\Omega^{-1}\mathfrak{D} f \sim \epsilon f$. This condition requires that both f and the fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$ be slowly varying in space and time, and thus that the particle motion be adiabatic. Adiabatic particle motion is not sufficient, however, to insure that f varies slowly in time; its form must be such that it does not fluctuate rapidly because of the gyration of the particles. We show in this section that this condition constrains the form of f at the initial time. This constraint is illustrated in Appendix B.

The constraint on f(0) follows from Eqs. (3.4). In particular, for $i = 1$ and $t = 0$, and by use of Eq. (3.6),

$$f_\phi^{(1)}(0) = \Omega^{-1}\int_{\phi_1}^{\phi}(\mathfrak{D} - \bar{\mathfrak{D}})h^{(0)}(0) . \qquad (3.15)$$

We note from Eq. (3.3) that Eq. (3.15) does not involve the inital time derivatives of h(t). Equation (3.15) therefore gives the first-order ϕ-dependence of f(0) in terms of $h^{(0)}$, $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\partial\underset{\sim}{E}/\partial t$, and $\partial\underset{\sim}{B}/\partial t$ at the initial time.

This argument is easily carried to any order. For this purpose it is convenient to introduce the operator .

$$\mathcal{D}^* = \Omega^{-1}\left[\int_0^\phi d\phi'\mathcal{D} - \frac{1}{2\pi}\oint d\phi \int_0^\phi d\phi'\mathcal{D}\right]. \qquad (3.16)$$

By use of this operator, Eqs. (3.9) may be written

$$f_\phi^{(i)} = \mathcal{D}^* h^{(i-1)} + \mathcal{D}^*\mathcal{D}^* h^{(i-2)} + \ldots + (\mathcal{D}^*)^i h^{(0)}, \qquad (3.17)$$

and summing over all i gives the formal result

$$f_\phi = \sum_i f_\phi^{(i)} = [\mathcal{D}^* + \mathcal{D}^*\mathcal{D}^* + \ldots]\,h. \qquad (3.18)$$

The time derivatives of h are easily eliminated from Eqs. (3.17) and (3.18) by use of Eqs. (3.10), but this introduces arbitrarily high time derivatives of $\underset{\sim}{E}$ and $\underset{\sim}{B}$. Equation (3.18) then gives a formal expression for $f_\phi(0)$ in terms of $h(\underset{\sim}{r}, 0)$, $\underset{\sim}{E}(\underset{\sim}{r}, t)$, and $\underset{\sim}{B}(\underset{\sim}{r}, t)$.

The time dependence of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ is determined by the initial values of $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\partial\underset{\sim}{E}/\partial t$, $\partial\underset{\sim}{B}/\partial t$, and the distribution function for each species of particle. Equation (3.18) therefore yields, in principle, a constraint on $f_\phi(0)$. This constraint is much more restrictive than the requirement that the initial current be consistent with the given field $\underset{\sim}{B}(0)$.

In the following chapters of this paper we consider the small-ϵ expansion of the exact velocity moments of the Vlasov equation, equivalent to moments of the expansion outlined in this section. The constraint on $f_\phi(0)$ carries over to that expansion as a constraint on the initial values of the moments of f_ϕ. These moments, however, will be expressed (by use of equations equivalent to moments of Eq. (3.18)) in terms of the moments of h and the flow velocity. The constraint on the initial values of these moments is therefore automatically satisfied. The constraint on $f_\phi(0)$ must not be overlooked, however, when an actual distribution function corresponding to a given solution of the moment equations is desired; this is illustrated in Appendix B.

CHAPTER IV

Expansion Procedure for the Moment Equations

4.1 ϕ-Dependent and ϕ-Independent Moments.

In this chapter we consider application of the small ϵ expansion directly to the exact velocity moments of the Vlasov equation describing one plasma component, and put these equations into a convenient form. The relation of these equations to the single-fluid description of a two-component plasma is also discussed. The actual expansions are carried out in the following chapters.

Expansion of the exact moment equations is equivalent, of course, to the expansion of the Vlasov equation reviewed in the previous chapter. The characteristic feature of that expansion is the separation of the distribution function f into ϕ-dependent and ϕ-independent parts f_ϕ and h. A straightforward recursion equation is obtained relating f_ϕ in any order to $\underline{E}$, $\underline{B}$, and the lower order f. On the other hand, h is given by a differential equation in the variables $\underline{r}$, $\underline{v}$, and t which involves ϵ only through the appearance of f_ϕ.

Velocity moments of the expanded Vlasov equation will therefore yield recursion equations expressing the moments of f_ϕ in any order in terms of $\underset{\sim}{E}$, $\underset{\sim}{B}$, and moments of lower order, and differential equations in $\underset{\sim}{r}$ and t for the moments of h which involve ϵ only through the moments of f_ϕ.

Since f_ϕ is defined so that its average over ϕ vanishes, non-vanishing velocity moments of f_ϕ are obtained only when it is multiplied by a ϕ-dependent velocity function. Non-vanishing moments of h, on the other hand, are obtained only when the velocity function has a non-zero ϕ-average.

Thus, corresponding to the separation of f into the parts f_ϕ and h, the velocity moments may be divided into those of the form $\langle \alpha(\phi) f \rangle$, where $\alpha(\phi)$ is a velocity function with a vanishing ϕ-average, and $\langle \beta f \rangle$, where β is a velocity function independent of ϕ. (The brackets $\langle \; \rangle$ indicate integration over all velocities.) We refer to moments of the first type as "ϕ-dependent" since they depend on f_ϕ only and correspond to ϕ-dependant velocity functions. Moments of the second type will be called "ϕ-independent". The terms "ϕ-dependent" and "ϕ-independent" describe the velocity function used to form the particular moment. The moments themselves, of course, are not functions of ϕ.

When the exact velocity moments of the Vlasov equation are expanded in ϵ, therefore, they will separate into recursion equations for the ϕ-dependent moments and differential equations for the ϕ-independent moments in which ϵ does not appear explicitly.

4.2 Velocity Moments of the Vlasov Equation.

The velocity moments of the Vlasov equation describing one plasma component may be written

$$\partial\rho / \partial t + \nabla \cdot \rho \underset{\sim}{u} = 0, \tag{4.1}$$

$$(d\underset{\sim}{u}/dt)_{\parallel} - g_{\parallel} + \rho^{-1}(\nabla \cdot \underset{\sim}{P})_{\parallel} - (e/m)E_{\parallel} = 0, \tag{4.2}$$

$$\underset{\sim}{u}_{\perp} - \underset{\sim}{u}_{E} = \Omega^{-1}\underset{\sim}{b} \times [d\underset{\sim}{u}/dt - \underset{\sim}{g} + \rho^{-1}\nabla \cdot \underset{\sim}{P}], \tag{4.3}$$

$$\underset{\sim}{G} \equiv d\underset{\sim}{P}/dt + \underset{\sim}{P}\nabla \cdot \underset{\sim}{u} + \nabla \cdot \underset{\sim}{Q} + [\underset{\sim}{P} \cdot \nabla\underset{\sim}{u}]^{s} = \Omega[\underset{\sim}{P} \times \underset{\sim}{b}]^{s}, \tag{4.4}$$

$$\underset{\sim}{H} \equiv d\underset{\sim}{Q}/dt + \underset{\sim}{Q}\nabla \cdot \underset{\sim}{u} + \nabla \cdot \underset{\sim}{R} + [\underset{\sim}{Q} \cdot \nabla\underset{\sim}{u}]^{s} - [\rho^{-1}\underset{\sim}{P}\nabla \cdot \underset{\sim}{P}]^{s} = \Omega[\underset{\sim}{Q} \times \underset{\sim}{b}]^{s}, \tag{4.5}$$

$$d\underset{\sim}{R}/dt + \underset{\sim}{R}\nabla \cdot \underset{\sim}{u} + \nabla \cdot \underset{\sim}{S} + [\underset{\sim}{R} \cdot \nabla\underset{\sim}{u}]^{s} - [\rho^{-1}\underset{\sim}{Q}\nabla \cdot \underset{\sim}{P}]^{s} = \Omega[\underset{\sim}{R} \times \underset{\sim}{b}]^{s}, \tag{4.6}$$

and

$$d\underset{\sim}{M}_{N}/dt + \underset{\sim}{M}_{N}\nabla \cdot \underset{\sim}{u} + \nabla \cdot \underset{\sim}{S} + [\underset{\sim}{M}_{N} \cdot \nabla\underset{\sim}{u}]^{s} + [\rho^{-1}\underset{\sim}{M}_{N-1}\nabla \cdot \underset{\sim}{P}]^{s} = \Omega[\underset{\sim}{M}_{N} \times \underset{\sim}{b}]^{s}, \tag{4.7}$$

where $d/dt = \partial/\partial t + \underset{\sim}{u} \cdot \nabla$, and the various moments of the distribution function f are defined as follows:

the mass and number densities $\rho = mn = m\langle f\rangle$;

the flow velocity $\underset{\sim}{u} = \rho^{-1} m\langle \underset{\sim}{v} f\rangle$;

the pressure tensor $\underset{\sim}{P} = m\langle \underset{\sim}{\nu}\underset{\sim}{\nu} f\rangle$, with $\underset{\sim}{\nu} = \underset{\sim}{v} - \underset{\sim}{u}$;

the heat flux tensor $\underset{\sim}{Q} = m\langle \underset{\sim}{\nu}\underset{\sim}{\nu}\underset{\sim}{\nu} f\rangle$;

and the higher moments $\underset{\sim}{R} = m\langle \underset{\sim}{\nu}\underset{\sim}{\nu}\underset{\sim}{\nu}\underset{\sim}{\nu} f\rangle$, $\underset{\sim}{S} = m\langle \underset{\sim}{\nu}\underset{\sim}{\nu}\underset{\sim}{\nu}\underset{\sim}{\nu}\underset{\sim}{\nu} f\rangle$,

and $\underset{\sim}{M}_{N} = m\langle \underset{\sim}{\nu} \ldots \underset{\sim}{\nu} f\rangle$ with N factors of $\underset{\sim}{\nu}$, $N \geq 2$.

Following the notation of the previous chapters, e and m are the charge and mass of the particles, $\underset{\sim}{E}$ and $\underset{\sim}{B}$ the electric and magnetic fields, $\underset{\sim}{g}$ the gravitational acceleration, $\underset{\sim}{b}$ is the unit vector in the direction $\underset{\sim}{B}$, $\Omega = eB/mc$, $\underset{\sim}{u}_E = c\underset{\sim}{E} \times \underset{\sim}{B}/B^2$, and the subscripts $\perp$ and $\parallel$ indicate projections perpendicular and parallel to $\underset{\sim}{B}$.

The superscript s indicates that the quantity is to be symmetrized by adding to it all cyclic permutations of its vector factors (or tensor indices). Thus

$$[\underset{\sim}{P} \times \underset{\sim}{b}]^s = \underset{\sim}{P} \times \underset{\sim}{b} - \underset{\sim}{b} \times \underset{\sim}{P},$$

$$\{[\underset{\sim}{Q} \cdot \nabla \underset{\sim}{u}]^s\}_{ijk} = Q_{ijn}\partial u_k/\partial x_n + Q_{kin}\partial u_j/\partial x_n + Q_{jkn}\partial u_i/\partial x_n .$$

We note that this operation produces a symmetric tensor only when applied to a tensor which is already symmetric with respect to all but one vector factor or tensor index.

The derivations of the continuity equation (4.1) and of the equation of motion, which we have written in the form (4.2) and (4.3), need not be reviewed here. The remaining equations are not so familiar; their derivation is outlined in Appendix A.

It is easily verified that these equations do in fact separate in the manner described in Section 4.1. Recursion relations are obtained for, and only for, those moments which appear in the dominant terms of the moment equations. These terms, on the left hand side of Eq. (4.3) and the right hand side of Eqs. (4.4) through (4.7), are those resulting from the gyration term of the Vlasov equation, hence involving only ϕ-dependent moments.

From each of the tensor equations (4.4) through (4.7) we thus obtain both recursion relations for the ϕ-dependent moments and differential equations for the ϕ-independent moments which appear in the tensors $\underset{\sim}{P}, \underset{\sim}{Q}, \underset{\sim}{R} \cdots$. This separation is carried out below.

The transverse velocity $\underset{\sim}{u}_\perp$ is not simply related to ϕ-dependent and ϕ-independent moments, since $\underset{\sim}{u}$ enters into the definition of ϕ, but $\underset{\sim}{u}_\perp$ resembles the ϕ-dependent moments in being given by the recursion equation (4.3).

In zero order ($\Omega \to \infty$) we see from the moment equations that $\underset{\sim}{u}_\perp = \underset{\sim}{u}_E$ and that all the ϕ-dependent moments vanish. This corresponds to the zero-order particle motion, and to the zero-order solution of the Vlasov equation.

4.3 The Pressure Tensor.

In order to express the tensors $\underset{\sim}{P}, \underset{\sim}{Q}, \underset{\sim}{R}, \cdots$ in terms of ϕ-dependent and ϕ-independent moments, it is convenient to introduce the local, orthonormal, right-handed basis vectors $\underset{\sim}{e}_1 = \underset{\sim}{b}$, $\underset{\sim}{e}_2$, and $\underset{\sim}{e}_3$. We may choose $\underset{\sim}{e}_2$ to be along the principal radius of curvature of $\underset{\sim}{B}$, but this is not necessary. We will, in fact, obtain representations of the tensors in which $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_3$ do not appear.

Components of the velocity moment tensors in the $\underset{\sim}{e}_i$ coordinate system are easily expressed in terms of ϕ-dependent and ϕ-independent moments. The six components of $\underset{\sim}{P}$ may be written:

$$p_{\parallel} \equiv P_{11} = m\langle \nu_{\parallel}^2 f\rangle ,$$

$$p_{\perp} \equiv (1/2)(P_{22} + P_{33}) = (m/2)\langle \nu_{\perp}^2 f\rangle ,$$

$$P_{12} = m\langle \nu_{\perp} \nu_{\parallel} \cos\phi\, f\rangle , \qquad (4.8)$$

$$P_{13} = m\langle \nu_{\perp} \nu_{\parallel} \sin\phi\, f\rangle ,$$

$$P_{23} = (m/2)\langle \nu_{\perp}^2 \sin 2\phi\, f\rangle ,$$

$$\text{and} \quad P_{\delta} \equiv (1/2)(P_{22} - P_{33}) = (m/2)\langle \nu_{\perp}^2 \cos 2\phi\, f\rangle .$$

Thus $p_{\perp}$ and $p_{\parallel}$ are ϕ-independent while P_{12}, P_{13}, P_{23}, and P_{δ} are ϕ-dependent. In terms of these components the tensor $\underset{\sim}{P}$ may be written

$$\underset{\sim}{P} = p_{\perp} \underset{\sim}{I}_{\perp} + p_{\parallel} \underset{\sim}{b}\underset{\sim}{b} + \underset{\sim}{\Pi} , \qquad (4.9)$$

where $\underset{\sim}{I}_{\perp}$ is the transverse projection operator $\underset{\sim}{e}_2\underset{\sim}{e}_2 + \underset{\sim}{e}_3\underset{\sim}{e}_3 = \underset{\sim}{I} - \underset{\sim}{b}\underset{\sim}{b}$ and

$$\underset{\sim}{\Pi} \doteq P_{12}(\underset{\sim}{e}_1\underset{\sim}{e}_2)^s + P_{13}(\underset{\sim}{e}_1\underset{\sim}{e}_3)^s + P_{23}(\underset{\sim}{e}_2\underset{\sim}{e}_3)^s + P_{\delta}(\underset{\sim}{e}_2\underset{\sim}{e}_2 - \underset{\sim}{e}_3\underset{\sim}{e}_3) \qquad (4.10)$$

represents the contribution of the ϕ-dependent moments to $\underset{\sim}{P}$.

These relations may be expressed more compactly by use of the quantities

$$\underset{\sim}{I}_\delta \equiv (1/2)(\underset{\sim}{e}_2\underset{\sim}{e}_2 - \underset{\sim}{e}_3\underset{\sim}{e}_3)$$

and $$\underset{\sim}{I}_\gamma \equiv (1/2)(\underset{\sim}{e}_2\underset{\sim}{e}_3 + \underset{\sim}{e}_3\underset{\sim}{e}_2) = (1/2)[\underset{\sim}{e}_2\underset{\sim}{e}_3]^s . \qquad (4.11)$$

Thus Eqs. (4.8) may be written

$$p_{\parallel} = \underset{\sim}{P}:\underset{\sim}{b}\underset{\sim}{b}, \qquad P_{23} = (1/2)\underset{\sim}{P}:\underset{\sim}{I}_\gamma ,$$

$$p_{\perp} = (1/2)\underset{\sim}{P}:\underset{\sim}{I}_{\perp}, \qquad P_\delta = (1/2)\underset{\sim}{P}:\underset{\sim}{I}_\delta , \qquad (4.12)$$

and since

$$P_{12}\underset{\sim}{e}_2 + P_{13}\underset{\sim}{e}_3 = (\underset{\sim}{P}\cdot\underset{\sim}{b})_{\perp} ,$$

$\underset{\sim}{\Pi}$ takes the form

$$\underset{\sim}{\Pi} = P_\gamma\underset{\sim}{I}_\gamma + P_\delta\underset{\sim}{I}_\delta + [(\underset{\sim}{P}\cdot\underset{\sim}{b})_{\perp}\underset{\sim}{b}]^s . \qquad (4.13)$$

The recursion relations for the ϕ-dependent components of $\underset{\sim}{P}$ are obtained by using the representation (4.13) to evaluate $(\underset{\sim}{P}\times\underset{\sim}{b})^s$ in Eq. (4.4). The results are

$$P_\delta = -(4\Omega)^{-1}\underset{\sim}{G}:\underset{\sim}{I}_\gamma , \qquad P_{23} = (4\Omega)^{-1}\underset{\sim}{G}:\underset{\sim}{I}_\delta ,$$

and (4.14)

$$(\underset{\sim}{b}\cdot\underset{\sim}{P})_{\perp} = \Omega^{-1}\underset{\sim}{b}\times(\underset{\sim}{b}\cdot\underset{\sim}{G}) .$$

Substitution of these results into Eq. (4.13) gives a representation for $\underset{\sim}{\Pi}$ in terms of $\underset{\sim}{G}$

$$\underset{\sim}{\Pi} = (1/4\Omega)(\underset{\sim}{I}_\gamma \underset{\sim}{I}_\delta - \underset{\sim}{I}_\delta \underset{\sim}{I}_\gamma) : \underset{\sim}{G} + \Omega^{-1}\{[\underset{\sim}{b} \times (\underset{\sim}{b} \cdot \underset{\sim}{G})]\underset{\sim}{b}\}^s$$

$$= (1/4\Omega)\underset{\sim}{b} \times (2\underset{\sim}{G} - \underset{\sim}{I}_\perp \underset{\sim}{I}_\perp : \underset{\sim}{G}) \cdot \underset{\sim}{I}_\perp + \Omega^{-1}\{[\underset{\sim}{b} \times (\underset{\sim}{b} \cdot \underset{\sim}{G})]\}^s. \quad (4.15)$$

The first of these expressions follows immediately from Eq. (4.13). The second, which is convenient because it does not involve $\underset{\sim}{e}_2$ or $\underset{\sim}{e}_3$, is easily verified by evaluation of its components. The derivations of these results are outlined in Appendix A.

4.4 The Pressure Equations.

The differential equations for $p_\perp$ and $p_\parallel$, the ϕ-independent components of the pressure tensor $\underset{\sim}{P}$, obtained by forming the contractions of Eq. (4.4) with $\frac{1}{2}\underset{\sim}{I}_\perp$ and $\frac{1}{2}\underset{\sim}{b}\underset{\sim}{b}$, are

$$dp_\perp/dt + p_\perp \nabla \cdot \underset{\sim}{u} + \nabla \cdot \underset{\sim}{q}^\perp + (\underset{\sim}{P} \cdot \nabla \underset{\sim}{u}) : \underset{\sim}{I}_\perp + \underset{\sim}{P} : \underset{\sim}{b} d\underset{\sim}{b}/dt + \underset{\sim}{b} \cdot \underset{\sim}{Q} : \nabla \underset{\sim}{b} = 0 \quad (4.16)$$

and

$$\tfrac{1}{2}dp_\parallel/dt + \tfrac{1}{2}p_\parallel \nabla \cdot \underset{\sim}{u} + \nabla \cdot \underset{\sim}{q}^\parallel + (\underset{\sim}{P} \cdot \nabla \underset{\sim}{u}) : \underset{\sim}{b}\underset{\sim}{b} - \underset{\sim}{P} : \underset{\sim}{b} d\underset{\sim}{b}/dt - \underset{\sim}{b} \cdot \underset{\sim}{Q} : \nabla \underset{\sim}{b} = 0, \quad (4.17)$$

where $\underset{\sim}{q}^\perp$ and $\underset{\sim}{q}^\parallel$ are the fluxes of transverse and parallel thermal energy $\frac{1}{2}\underset{\sim}{Q} : \underset{\sim}{I}_\perp$ and $\frac{1}{2}\underset{\sim}{Q} : \underset{\sim}{b}\underset{\sim}{b}$.

The usual energy equation of kinetic theory is obtained by adding Eqs. (4.16) and (4.17), which gives one-half the trace of Eq. (4.4):

$$dK/dt + K\nabla \cdot \underset{\sim}{u} + \nabla \cdot \underset{\sim}{q} + \underset{\sim}{P} : \nabla \underset{\sim}{u} = 0, \quad (4.18)$$

where $K = p_\perp + \frac{1}{2}p_\parallel = (m/2)\langle v^2 f\rangle$ is the total thermal energy density and $\underset{\sim}{q} = \underset{\sim}{q}^\perp + \underset{\sim}{q}^\parallel$ is the total heat flow vector. The first four terms in Eq. (4.16) and in Eq. (4.17) are similar in form to this energy equation. These terms represent the rate of increase of the thermal energy density of a given fluid element, the rate of increase of the volume of the fluid element, the transport of thermal energy out of the fluid element, and the work done on the fluid element by the fluid motion. The remaining terms occur with opposite signs in Eqs. (4.16) and (4.17), and represent an exchange of energy between the transverse and parallel thermal motion. The first of these terms arises from the rotation of the magnetic field direction seen by a moving fluid element; the origin of the second is more difficult to visualize.

4.5 The Heat Flux Tensor.

The heat flux tensor $\underset{\approx}{Q}$ has ten independent components. It may be expressed in terms of the quantities

$$\underset{\sim}{q}^\parallel \equiv (m/2)\langle \underset{\sim}{v} v_\parallel^2 f\rangle ,$$

$$\underset{\sim}{q}^\perp \equiv (m/2)\langle \underset{\sim}{v} v_\perp^2 f\rangle ,$$

$$\underset{\sim}{q}^\delta \equiv (m/2)\langle \underset{\sim}{v} v_\perp^2 \cos 2\phi\rangle , \qquad (4.19)$$

and

$$\underset{\sim}{q}^\gamma \equiv (m/2)\langle \underset{\sim}{v} v_\perp^2 \sin 2\phi\rangle .$$

It is easily verified from these definitions that $\underset{\sim}{q}_\perp^\perp$, $\underset{\sim}{q}_\perp^\delta$, and $\underset{\sim}{q}_\perp^\gamma$, are related by the identity

$$\mathbf{q}_{\perp} = \mathbf{I}_{\gamma} \cdot \mathbf{q}^{\gamma} + \mathbf{I}_{\delta} \cdot \mathbf{q}^{\delta} , \qquad (4.20)$$

hence only two of these three vectors are required in representation of $\mathbf{Q}$.

The flows of parallel and transverse thermal energies are $\mathbf{q}^{\parallel}$ and $\mathbf{q}^{\perp}$, respectively. The other vectors represent more complex fluxes.

It is evident that the only ϕ-independent components of $\mathbf{Q}$ are the parallel heat flows $q^{\parallel}_{\parallel}$ and $q^{\perp}_{\parallel}$. The tensor $\mathbf{Q}$ therefore has the form

$$\mathbf{Q} = 2q^{\parallel}_{\parallel}(\mathbf{b}\,\mathbf{b}\,\mathbf{b}) + q^{\perp}_{\parallel}(\mathbf{b}\mathbf{I}_{\perp})^{s} + \mathbf{Q}_{\phi} , \qquad (4.21)$$

where $\mathbf{Q}_{\phi}$ represents the contribution of the ϕ-dependent moments. It may be written in the form, easily verified by evaluation of the components,

$$\begin{aligned} \mathbf{Q}_{\phi} = {} & 2(\mathbf{b}\,\mathbf{b}\,\mathbf{q}^{\parallel}_{\perp})^{s} + (\mathbf{I}_{\perp}\mathbf{q}^{\perp}_{\perp} + \mathbf{I}_{\gamma}\mathbf{q}^{\gamma}_{\perp} + \mathbf{I}_{\delta}\mathbf{q}^{\delta}_{\perp}) \\ & + q^{\delta}_{\parallel}(\mathbf{b}\mathbf{I}_{\delta})^{s} + q^{\gamma}_{\parallel}(\mathbf{b}\mathbf{I}_{\gamma})^{s} . \end{aligned} \qquad (4.22)$$

The second term of this expression is actually symmetric, although this is not self-evident, because of the identity Eq. (4.20).

By use of this representation for $\mathbf{Q}_{\phi}$ Eq. (4.5) may be inverted to give the recursion relations for the ϕ-dependent components of $\mathbf{Q}$:

$$\underset{\sim}{q}_{\perp}^{\parallel} = (1/2\Omega)\,\underset{\sim}{b} \times (\underset{\sim}{H} : \underset{\sim}{b}\underset{\sim}{b}), \tag{4.23}$$

$$\underset{\sim}{q}_{\perp}^{\perp} = (1/2\Omega)\,\underset{\sim}{b} \times (\underset{\sim}{H} : \underset{\sim}{I}_{\perp}), \tag{4.24}$$

$$\underset{\sim}{q}_{\perp}^{\delta} = (1/6\Omega)\,\underset{\sim}{b} \times (\underset{\sim}{H} : \underset{\sim}{I}_{\gamma}) + (2/3)\underset{\sim}{I}_{\delta} \cdot \underset{\sim}{q}^{\perp}, \tag{4.25}$$

$$\underset{\sim}{q}_{\perp}^{\gamma} = (1/6\Omega)\,\underset{\sim}{b} \times (\underset{\sim}{H} : \underset{\sim}{I}_{\delta}) + (2/3)\underset{\sim}{I}_{\gamma} \cdot \underset{\sim}{q}^{\perp}, \tag{4.26}$$

$$q_{\parallel}^{\delta} = -(1/4\Omega)\,\underset{\sim}{b} \cdot \underset{\sim}{H} : \underset{\sim}{I}_{\gamma}, \tag{4.27}$$

$$q_{\parallel}^{\gamma} = (1/4\Omega)\,\underset{\sim}{b} \cdot \underset{\sim}{H} : \underset{\sim}{I}_{\delta}. \tag{4.28}$$

These recursion relations may be substituted back into Eq. (4. 22) to give $\underset{\sim}{Q}_{\phi}$ in forms analogous to the representations (4. 15) for $\underset{\sim}{\Pi}$. The resulting expressions are too cumbersome to be very useful, however, and will not be given here; a convenient representation of the first-order terms of $\underset{\sim}{Q}_{\phi}$ is given in Section 5. 3.

The "equations of motion" for the ϕ-independent components of $\underset{\sim}{Q}$, $q_{\parallel}^{\parallel}$ and $q_{\parallel}^{\perp}$, are just the contractions of Eq. (4. 5) with $\frac{1}{2}\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}$ and $\frac{1}{2}\underset{\sim}{I}_{\perp}\underset{\sim}{b}$:

$$dq_{\parallel}^{\parallel}/dt + q_{\parallel}^{\parallel}\nabla \cdot \underset{\sim}{u} - 3\underset{\sim}{q}_{\perp}^{\parallel} \cdot (d\underset{\sim}{b}/dt) + \tfrac{1}{2}\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b} \vdots \left[\nabla \cdot \underset{\sim}{R} + (\underset{\sim}{Q} \cdot \nabla \underset{\sim}{u})^{s} - \frac{(\underset{\sim}{P}\nabla \cdot \underset{\sim}{P})^{s}}{\rho} \right] = 0, \tag{4.29}$$

$$dq_{\parallel}^{\perp}/dt + q_{\parallel}^{\perp}\nabla \cdot \underset{\sim}{u} + (2\underset{\sim}{q}_{\perp}^{\parallel} - \underset{\sim}{q}_{\perp}^{\perp}) \cdot d\underset{\sim}{b}/dt + \tfrac{1}{2}\underset{\sim}{b}\underset{\sim}{I}_{\perp} \vdots \left[\nabla \cdot \underset{\sim}{R} + (\underset{\sim}{Q} \cdot \nabla \underset{\sim}{u})^{s} - \frac{(\underset{\sim}{P}\nabla \cdot \underset{\sim}{P})^{s}}{\rho} \right] = 0. \tag{4.30}$$

These equations are not often useful because, in the absence of collisions, closed moment equations which require the parallel heat flows are obtained only in special cases. Equilibrium configurations, however, may be described in terms of $p_\perp$ and $p_\parallel$; Eqs. (4. 29) and (4. 30) then take the form of constraints on the form of the equilibrium distribution function.

4. 6 The Fourth Moments.

Evaluation of $\underset{\sim}{Q}$ requires the fourth velocity moment tensor $\underset{\sim}{R}$. Of the fifteen independent moments entering this tensor only three are ϕ-independent:

$$R_1 \equiv m\langle \nu_\parallel^4 f\rangle , \qquad R_2 \equiv (m/2)\langle \nu_\parallel^2 \nu_\perp^2 f\rangle , \text{ and } R_3 \equiv (m/2)\langle \nu_\perp^4 f\rangle .$$

The tensor $\underset{\sim}{R}$ may be written

$$\underset{\sim}{R} = R_1 \underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b} + (R_2/2)[\underset{\sim}{b}(\underset{\sim}{b}\underset{\sim}{I}_\perp)^s]^s + (R_3/16) \sum_{i=2,3} [\underset{\sim}{e}_i(\underset{\sim}{e}_i\underset{\sim}{I}_\perp)^s]^s + \underset{\sim}{R}_\phi , \tag{4. 31}$$

where $\underset{\sim}{R}_\phi$ represents the contribution of the twelve ϕ-dependent moments. Representation and evaluation of $\underset{\sim}{R}_\phi$ is straightforward but will not be required here.

The ϕ-independent components R_i are required, however; they are determined by the differential equations obtained from Eq. (4. 6):

$$dR_1/dt + R_1 \nabla\cdot\underset{\sim}{u} - 4\underset{\sim}{b}\underset{\sim}{b}:\underset{\sim}{R}:\underset{\sim}{b}d\underset{\sim}{b}/dt + \underset{\sim}{b}\underset{\sim}{b}:\underset{\sim}{S}^*:\underset{\sim}{b}\underset{\sim}{b} = 0,$$

$$dR_2/dt + R_2 \nabla\cdot\underset{\sim}{u} + (\underset{\sim}{I}_\perp - \underset{\sim}{b}\underset{\sim}{b}):\underset{\sim}{R}:\underset{\sim}{b}d\underset{\sim}{b}/dt + \tfrac{1}{2}\underset{\sim}{I}_\perp:\underset{\sim}{S}^*:\underset{\sim}{b}\underset{\sim}{b} = 0,$$

and
$$dR_3/dt + R_3 \nabla\cdot\underset{\sim}{u} - 2\underset{\sim}{I}_\perp:\underset{\sim}{R}:d\underset{\sim}{b}d/dt + \tfrac{1}{2}\underset{\sim}{I}_\perp:\underset{\sim}{S}^*:\underset{\sim}{I}_\perp = 0, \tag{4. 32}$$

with
$$\underset{\sim}{S}^* \equiv \nabla\cdot\underset{\sim}{S} + [\underset{\sim}{R}\cdot\nabla\underset{\sim}{u}]^s - [\rho^{-1}\underset{\sim}{Q}\nabla\cdot\underset{\sim}{P}]^s .$$

The relationship between given values of the R_i and the form of distribution function is most easily seen by comparing R_1 and R_3 to their values for distributions of Gaussian form, and R_2 to its value for a distribution which separates in $\nu_\perp$ and $\nu_\parallel$. We therefore introduce the quantities

$$\begin{aligned} R_1' &= \rho R_1/3p_\parallel^2, \\ R_2' &= \rho R_2/p_\parallel p_\perp, \\ R_3' &= \rho R_3/4p_\perp^2. \end{aligned} \tag{4.33}$$

The R_i' have been defined so that for a Gaussian distribution of $\nu_\parallel$ or $\nu_\perp$ $R_1' = 1$ or $R_3' = 1$, respectively, while $R_2' = 1$ if $f = f_1(\nu_\parallel)f_2(\nu_\perp)$.

By a simple modification of the proof of the Schwarz inequality,[27] it is straightforward to show that

$$R_1' \geq 1/3 \qquad \text{and} \qquad R_3' \geq 1/4, \tag{4.34}$$

the equalities corresponding to a δ-function distribution of the velocity. R_1' and R_3' are thus measures of the spread of the distributions of $\nu_\parallel$ and $\nu_\perp$ about their mean square values. Large values of R_1' and R_3' occur if only a small fraction of the particles make a significant contribution to the pressure, R_1' and R_3' being proportional to α for a distribution of the form $\delta(\nu^2) + \alpha f_1(\nu)$, $\alpha \ll 1$.

The quantity R_2' is related to the correlation between the transverse and parallel thermal motions. It is evident from its definition that $R_2' \geq 0$, the equality corresponding to a situation in which the transverse and parallel thermal motions are carried out by different groups of particles [i. e., to a distribution of the form $\delta(\nu_\perp)f_1(\nu_\parallel) + \delta(\nu_\parallel)f_2(\nu_\perp)$]. If there is no correlation between the transverse and parallel thermal motions, f is separable in $\nu_\perp$ and $\nu_\parallel$, and $R_2' = 1$. As is the case for R_1' and R_3', large values of R_2' correspond to situations in which only a small fraction of the particles contribute to the pressure.

4.7 The Single-Fluid Equations.

Before discussing the expansion of the moment equations we will briefly review the relation between the equation of motion for each component of a two-component plasma and the equivalent single-fluid equations.[13]

The two-component plasma is described by the variables ρ_+, $\underset{\sim}{u}_+$, $\underset{\sim}{P}_+$, ρ_-, $\underset{\sim}{u}_-$, and $\underset{\sim}{P}_-$, where the subscripts + and - refer to ions and electrons of charge and mass $\pm e$ and $m_\pm$. The single-fluid equations are obtained by transforming from these variables to the total density $\rho_t = \rho_+ + \rho_-$, the charge density $\rho_c = e(n_+ - n_-)$, the mass velocity $\underset{\sim}{V} = (1/\rho_t)(\rho_+\underset{\sim}{u}_+ + \rho_-\underset{\sim}{u}_-)$, and the current $\underset{\sim}{j} = (e/c)(n_+\underset{\sim}{u}_+ - n_-\underset{\sim}{u}_-)$.

When this is done the two continuity equations yield conservation equations for the total mass and charge,

$$\partial\rho_t/\partial t + \nabla\cdot\rho_t\underset{\sim}{V} = 0 \quad \text{and} \quad \partial\rho_c/\partial t + \nabla\cdot\underset{\sim}{j} = 0. \qquad (4.35)$$

The sum of the ion and electron equations of motion expresses conservation of momentum for the fluid as a whole:

$$\rho_t D\underset{\sim}{V}/Dt = -\nabla\cdot\underset{\sim}{P}_t + \underset{\sim}{j}\times\underset{\sim}{B} + \rho_c\underset{\sim}{E} + \rho_t\underset{\sim}{g}, \tag{4.36}$$

where $D/Dt = \partial/\partial t + \underset{\sim}{V}\cdot\nabla$, and

$$\underset{\sim}{P}_t = \underset{\sim}{P}_+ + \rho_+(\underset{\sim}{u}_+ - \underset{\sim}{V})(\underset{\sim}{u}_+ - \underset{\sim}{V}) + \underset{\sim}{P}_- + \rho_-(\underset{\sim}{u}_- - \underset{\sim}{V})(\underset{\sim}{u}_- - \underset{\sim}{V}) \tag{4.37}$$

is the total pressure referred to the velocity $\underset{\sim}{V}$.

In addition to the momentum equation (4.36), the generalized Ohm's law,[13] obtained by subtracting m_+/m_- times the electron equation of motion from the ion equation of motion, is required. The resulting equation is rather complicated in general, but it becomes very much simplified if either the charge separation or electron inertia is negligible.

For small charge separation an expansion may be made in $\Delta n' = \rho_c/n_0 e$. To lowest order in this expansion we set $n_+ = n_-$ in the moment equations, drop Poisson's equation for $\underset{\sim}{E}$, neglect the displacement current, and find $\nabla\cdot\underset{\sim}{j} = 0$. In this approximation the generalized Ohm's law takes the form

$$\underset{\sim}{E} + \underset{\sim}{V}\times\underset{\sim}{B} = \frac{m_+ - m_-}{m_+ + m_-}\frac{\underset{\sim}{j}}{ne}\times\underset{\sim}{B} + \frac{m^*}{e}\left\{\frac{1}{n}\nabla\cdot\left(\frac{\underset{\sim}{P}_+}{m_+} - \frac{\underset{\sim}{P}_-}{m_-}\right)\right.$$
$$\left. + \frac{D}{Dt}\left(\frac{\underset{\sim}{j}}{ne}\right) + \frac{\underset{\sim}{j}}{ne}\cdot\nabla\left[\underset{\sim}{V} - \frac{m_+ - m_-}{m_+ + m_-}\frac{\underset{\sim}{j}}{ne}\right]\right\}, \tag{4.38}$$

where $m^* = m_+ m_-/(m_+ + m_-)$ is the reduced mass.

Equations (4.36) and (4.38) are the basis of the "single-fluid" description of a quasi-neutral two-component plasma. To obtain a closed theory the total pressure $\underset{\sim}{P}_t$ must be determined. This is the object of the expansion carried out in the following chapters of this paper.

If n_- and $\underset{\sim}{u}_-$ remain finite in the limit $m_- \to 0$, electron inertia is unimportant and the single-fluid equations take on a simple form independently of the approximation of quasi charge-neutrality. In this limit the electron pressure may remain finite,

$$\rho_t = \rho_+ , \qquad \underset{\sim}{V} = \underset{\sim}{u}_+ ,$$

and

$$\underset{\sim}{P}_t = \underset{\sim}{P}_+ + \underset{\sim}{P}_- . \tag{4.39}$$

The ion forms of Eqs. (4.2) and (4.3) may then be written

$$\underset{\sim}{V}_\perp = \underset{\sim}{u}_E + \frac{\underset{\sim}{b}}{\Omega_+} \times \left[\frac{D\underset{\sim}{V}}{Dt} + \frac{\nabla \cdot \underset{\sim}{P}^+}{\rho t} - \underset{\sim}{g} \right] \tag{4.40}$$

and

$$0 = (D\underset{\sim}{V}/Dt)_{\parallel} + \rho_t^{-1} (\nabla \cdot \underset{\sim}{P}^+)_{\parallel} - g_{\parallel} - (e/m_+) E_{\parallel} . \tag{4.41}$$

These are the transverse and parallel components of Ohm's law appropriate to this case; they are easily transformed into the more usual forms

$$\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B} = (m_+/e)[D\underset{\sim}{V}/Dt - \underset{\sim}{g} + \rho_t^{-1} \nabla \cdot \underset{\sim}{P}_+] \tag{4.42}$$

and

$$\underset{\sim}{E} + \underset{\sim}{V} \times \underset{\sim}{B} = (1/n_+ e)[\underset{\sim}{j} \times \underset{\sim}{B} - \nabla \cdot \underset{\sim}{P}_-] . \qquad (4.43)$$

Equation (4.43) is also obtained from Eq. (4.38) in the limit $m_- \rightarrow 0$, but it is not dependent on the approximation of small charge separation as is Eq. (4.38).

One advantage of the single-fluid equations over the corresponding two-fluid equations is that $\underset{\sim}{j}$, $\underset{\sim}{E}$, and ρ_c may be eliminated from the single-fluid equations by use of Maxwell's curl B equation, Poisson's equation, and the generalized Ohm's law. This is very easily accomplished if charge separation, displacement current, and electron inertia are negligible. Then

$$\underset{\sim}{j} \times \underset{\sim}{B} = -\nabla_\perp (B^2/2) + B^2 \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}, \qquad (4.44)$$

$$DB/Dt + B\nabla_\perp \cdot \underset{\sim}{V} = -\underset{\sim}{b} \cdot \{\nabla \times [(ne)^{-1}(B^2 \underset{\sim}{b} \cdot \nabla \underset{\sim}{b} - \nabla_\perp B^2 - \nabla \cdot \underset{\sim}{P}_-)]\}, \qquad (4.45)$$

and

$$D\underset{\sim}{b}/Dt = (\underset{\sim}{b} \cdot \nabla \underset{\sim}{V})_\perp + \{\nabla \times [(ne)^{-1}(B^2 \underset{\sim}{b} \cdot \nabla \underset{\sim}{b} - \nabla_\perp B^2 - \nabla \cdot \underset{\sim}{P}_-)]\}_\perp . \qquad (4.46)$$

Because the transverse motion of every particle approaches $\underset{\sim}{u}_E$ as $m \rightarrow 0$, neglecting electron inertia is a good approximation if the plasma motion is transverse to the magnetic field. Charge separation and displacement current are unimportant unless the plasma density is very low.

If there is plasma motion along the field lines, however, electron inertia may be important, but closed moment equations are obtained only in special cases. For the present discussion, therefore, these motions are of limited interest.

4.8 Expansion Procedure.

The small ϵ expansion of the moment equations is carried out by using the recursion relations to eliminate the ϕ-dependent moments from the differential equations for the ϕ-independent moments. Thus, for example, $\underset{\sim}{P}$ is found from Eqs. (4.15) through (4.17) by using Eqs. (4.25) through (4.28) to eliminate $\underset{\sim}{Q}_\phi$. This procedure corresponds to the determination of f_ϕ by the recursion relation (3.9) when the small ϵ expansion is applied directly to the Vlasov equation.

Since this procedure is carried out order by order, we expand in ϵ all quantities appearing in the moment equations, writing them in the form $\rho = \rho^{(1)} + \rho^{(2)} + \ldots$, and regarding m/e and Ω^{-1} as proportional to ϵ. As discussed in Chapter II, the particular form these expansions will take depends on the ordering of the auxiliary parameters with respect to ϵ.

To simplify notation, the expansions of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ (hence also $\underset{\sim}{b} = \underset{\sim}{e}_1$, $\underset{\sim}{e}_2$, and $\underset{\sim}{e}_3$) will not be carried out explicitly. The order to which these quantities will be required in any expression will be evident by inspection.

It was shown in Chapter II that the relation between the expansions of $\underset{\sim}{E}$ and $\underset{\sim}{B}$ and the expansions of the moments depends primarily on the ordering of the pressure parameter β and the relativistic parameter v_o/c with respect to ϵ. Three cases are of interest:

$$\beta \sim 1, \qquad \beta \sim \epsilon \gg v_0^2/c^2, \qquad \text{and} \qquad \beta \sim \epsilon \text{ with } \frac{1}{\beta}\frac{v_0^2}{c^2} \sim 1 .$$

The approximation of quasi charge-neutrality is applicable in the first two cases. It is not in the third, however, and that case will be considered briefly in Section 7. 3.

Our primary interest is in the finite pressure case $\beta \sim 1$. A current of given order then generates a magnetic field of one lower order [see Eq. (2. 7)]. The zero-order current must therefore vanish, and $\underset{\sim}{B}^{(0)}$ is determined by $\underset{\sim}{j}^{(1)}$. For finite β zero-order currents produce field gradients of order B/R_g, in violation of the basic requirements for adiabatic particle motion.

Because charge separation is negligible for this case, and the zero-order transverse velocity of every particle is the same, $\underset{\sim}{j}_{\perp}^{(0)}$ vanishes automatically. The requirement that $j_{\parallel}^{(0)}$ also vanish, however, restricts our equations to situations in which the parallel motion of the electrons closely follows that of the ions. For these motions electron inertia may be neglected, at least in lowest order, and the simple single-fluid equations of the previous section may be applied. Effects of finite electron mass are considered, for example, in references 10 and 28.

Because the magnetic field does not strongly constrain the motion of the particles along the field lines, the small ϵ expansion is not sufficient to produce closed moment equations. Closed equations are sometimes obtained by ignoring the parallel heat flows on the grounds that very weak collisions will inhibit them but have no other effects of importance. To obtain closed equations in the absence of collisions, however, an additional expansion is required. Usually gradients of $\underset{\sim}{b}$ and gradients along $\underset{\sim}{b}$ are considered small compared with gradients transverse to $\underset{\sim}{b}$. Closed equations are then obtained if $u_{\parallel}$, $q_{\parallel}^{\perp}$, and other parallel transport terms are assumed proportional to $L_{\parallel}/L_{\perp}$, and $L_{\parallel}/L_{\perp}$ is of order ϵ. To lowest order in this additional expansion $L_{\parallel} = u_{\parallel} = q_{\parallel}^{\perp} = q_{\parallel}^{\parallel} = 0$. This case is discussed for CGL ordering in Section 5.6, and for FGR ordering in Section 6.6, and Chapters VII and VIII. For FGR ordering additional closure problems arise because of the transverse collisionless heat flows; these are independent of, but analogous to, those associated with parallel motion.

For the special case of an Alfven wave travelling along a uniform magnetic field in a uniform plasma, linearization of the equations is sufficient for closure; this property is independent of the expansion in ϵ. Low-frequency stability of this Alfven wave is discussed in Chapter IX.

The assumption of equilibrium, in addition to the small ϵ expansion, is also sufficient to produce closed moment equations. Equilibria may be described in terms of $p_\perp$ and $p_\parallel$, the differential equations for the ϕ-independent moments (e.g., the pressure equations) then become constraints on the higher moments of the ϕ-independent part of the distribution function. The equilibrium moment equations and, in particular, "long-term" equilibria, are discussed in Chapter X.

CHAPTER V

Finite Gyro-Radius Corrections to the CGL Equations

5.1 The CGL Equations.

In this chapter we discuss the zero- and first-order corrections to the moment equations assuming CGL ordering (i.e., we consider the time scale of interest to be of order $(\epsilon\Omega)^{-1}$, hence short compared to the drift time $\tau_D \sim (\epsilon^2\Omega)^{-1}$). The zero-order approximation to the moment equations, which we discuss in this section, is then the familiar CGL hydromagnetic theory.

First-order corrections to these zero-order equations are presented in the following sections of the chapter. Since these effects vanish with ϵ, they may more appropriately be regarded as effects of "finite" gyro-radius than the modifications of the zero-order moment equations necessary when the time scale is extended to be of order τ_D. These latter effects are discussed in the following chapters of this paper.

The effects to be discussed in the following sections of this chapter are equivalent to the first-order terms in the expansion of the Vlasov equation, and analogous to the first-order terms of the Chapman-Enskog[11] expansion. The equations obtained in this chapter have been incorrectly applied to low-frequency waves characterized by FGR ordering.[7, 18]

The difference in form between the first-order equations of this chapter and the zero-order equations obtained for FGR ordering are therefore of special interest. The latter are described in Chapters VI through X.

If CGL ordering is assumed, the zero-order terms of the single-fluid equation of motion (4. 36) are

$$\rho_t D\underset{\sim}{V}/Dt = -\nabla\cdot\underset{\sim}{P}_t + \underset{\sim}{j}\times\underset{\sim}{B}, \tag{5. 1}$$

where charge separation has been assumed to be small ($\beta \gg \epsilon v_0^2/c^2$), ρ_t, $\underset{\sim}{V}$, $\underset{\sim}{P}$, and $\underset{\sim}{B}$ are taken to zero order, $\underset{\sim}{j}$ is first order, and

$$\underset{\sim}{P}_t^{(0)} = P_\perp^{(0)}\underset{\sim}{I}_\perp + P_\parallel^{(0)}\underset{\sim}{b}\underset{\sim}{b} \tag{5. 2}$$

with $P_\perp = p_\perp^+ + p_\perp^-$ and $P_\parallel = p_\parallel^+ + p_\parallel^-$.

The zero-order terms of the generalized Ohm's law simply state that $\underset{\sim}{V}^{(0)} = \underset{\sim}{u}_E$ or

$$\underset{\sim}{E} + \underset{\sim}{V}\times\underset{\sim}{B} = 0 . \tag{5. 3}$$

The pressure components $p_\perp^{+(0)}$, $p_\perp^{-(0)}$, $p_\parallel^{+(0)}$, and $p_\parallel^{-(0)}$ are determined by the zero-order terms of the pressure equations (4. 16) and (4. 17). By use of the zero-order forms of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ and the equality of $\underset{\sim}{u}^{+(0)}$ and $\underset{\sim}{u}^{-(0)}$, the zero-order terms of the ion and electron pressure equations may be added to give the CGL pressure equations[1, 8] in the form

$$DP_{\perp}/Dt + P_{\perp}(\nabla\cdot\mathbf{V} + \nabla_{\perp}\cdot\mathbf{V}) = -B\nabla\cdot(q_{\parallel}^{\perp}\mathbf{b}/B), \tag{5.4}$$

and

$$DP_{\parallel}/Dt + P_{\parallel}(3\nabla\cdot\mathbf{V} - 2\nabla_{\perp}\cdot\mathbf{V}) = -2\nabla\cdot(q_{\parallel}^{\parallel}\mathbf{b}) + 2q_{\parallel}^{\perp}\nabla\cdot\mathbf{b}, \tag{5.5}$$

where, in these equations only, $q_{\parallel}^{\perp}$ and $q_{\parallel}^{\parallel}$ are added over ions and electrons.

Equations (5.1) through (5.5), together with the continuity equation and Maxwell's curl B and induction equations, comprise the familiar CGL hydromagnetic theory. These equations are not closed in general, because of the parallel heat flows.

If the terms onthe right hand sides of Eqs(5.4) and (5.5) vanish or are neglected, these equations are easily integrated by use of the continuity equation and the zero-order induction equation in the form (4.45) to give the well known CGL double adiabatic conditions

$$D(P_{\perp}/\rho B)/Dt = 0$$

and

$$D(B^2 P_{\parallel}/\rho^3) \times Dt = 0. \tag{5.6}$$

5.2 The Pressure Tensor.

The CGL equations may be extended to give the plasma velocity through first order and the current through second order by including the first-order corrections to $\mathbf{P}$ in the single-fluid equation of motion and the first-order terms in the generalized Ohm's law.

The ϕ-dependent components of $\mathbf{P}^{(1)}$ (i.e., $\mathbf{\Pi}^{(1)}$) are found from the recursion formulae (4.14). By use of the zero-order forms for $\mathbf{P}$

and $\underset{\sim}{Q}$ these equations give

$$P_{\delta}^{(1)} = -(2\Omega)^{-1}\underset{\sim}{I}_{\gamma}:[p\,\nabla\underset{\sim}{u} + q_{\parallel}^{\perp}\nabla\underset{\sim}{b}]^{(0)},$$

$$P_{23}^{(1)} = +(2\Omega)^{-1}\underset{\sim}{I}_{\delta}:[p\,\nabla\underset{\sim}{u} + q_{\parallel}^{\perp}\nabla\underset{\sim}{b}]^{(0)}, \tag{5.7}$$

and

$$(\underset{\sim}{b}\cdot\underset{\sim}{P})_{\perp} = \Omega^{-1}\underset{\sim}{b}\times[(p_{\parallel} - p_{\perp})d\underset{\sim}{b}/dt + p_{\parallel}\underset{\sim}{b}\cdot\nabla\underset{\sim}{u} + p_{\perp}(\nabla\underset{\sim}{u})\cdot\underset{\sim}{b}$$
$$+ 2(q_{\parallel}^{\parallel} - q_{\parallel}^{\perp})\underset{\sim}{b}\cdot\nabla\underset{\sim}{b} + \nabla q_{\parallel}^{\perp}]^{(0)}. \tag{5.8}$$

The time derivative $d\underset{\sim}{b}/dt$ may be eliminated from $(\underset{\sim}{b}\cdot\underset{\sim}{P})_{\perp}$ by use of Maxwell's induction equation in the form of Eq. (4.46):

$$(\underset{\sim}{b}\cdot\underset{\sim}{P})_{\perp} = \Omega^{-1}\underset{\sim}{b}\times[(2p_{\parallel} - p_{\perp})\underset{\sim}{b}\cdot\nabla\underset{\sim}{u} + p_{\perp}(\nabla\underset{\sim}{u})\cdot\underset{\sim}{b} + 2(q_{\parallel}^{\parallel} - q_{\parallel}^{\perp})\underset{\sim}{b}\cdot\nabla\underset{\sim}{b} + \nabla q_{\parallel}^{\perp}]^{(0)}. \tag{5.9}$$

The terms of Eqs. (5.7) and (5.9) proportional to $\nabla\underset{\sim}{u}$ are the collisionless viscosity terms. They are well known in the isotropic case (e.g., references 11 and 12) and have been given for an anisotropic $\underset{\sim}{P}^{(0)}$ by Kaufman[2] and by Thompson.[3]

Thompson's calculation is outlined in Chapter II. His results for P_{12} and P_{13} are incorrect, however, because of an algebraic error; the correct expressions are easily obtained from his distribution function.

Kaufman has shown how the components of $\underset{\sim}{\Pi}^{(1)}$ are related to the first-order corrections to the particle orbits in the adiabatic approximation by studying these orbits for special cases. His calculation of P_{12}

and P_{13} is incomplete, however, because he assumes a constant magnetic field. This is inconsistent with a nonzero $(\underset{\sim}{b} \cdot \nabla \underset{\sim}{u})_{\perp}$ and Maxwell's induction equation. Calculation of the additional contribution to $\underset{\sim}{\Pi}^{(1)}$ from the particle orbits is outlined in Appendix B.

The remaining terms of Eqs. (5.7) and (5.9) represent stresses produced by the zero-order heat flows. Because of these stresses $\underset{\sim}{\Pi}^{(1)}$ need not vanish even if $\underset{\sim}{u}^{(0)} = 0$. In Chapter IX we show that because of these terms an equilibrium parallel heat flow tends to destabilize a low frequency Alfven wave.

Explicit representations for $\underset{\sim}{\Pi}^{(1)}$ may be obtained by substituting these results into Eqs. (4.15). Thus

$$\underset{\sim}{\Pi}^{(1)} = (2\Omega)^{-1}(\underset{\sim}{I}_{\gamma}\underset{\sim}{I}_{\delta} - \underset{\sim}{I}_{\delta}\underset{\sim}{I}_{\gamma}):(p_{\perp}\nabla\underset{\sim}{u} + q_{\parallel}^{\perp}\nabla\underset{\sim}{b})^{(0)} + [\underset{\sim}{b}(\underset{\sim}{b}\cdot\underset{\sim}{P})_{\perp}^{(1)}]^{s}, \qquad (5.10)$$

or

$$\underset{\sim}{\Pi}^{(1)} = (2\Omega)^{-1}\underset{\sim}{b} \times \{p_{\perp}[(\nabla\underset{\sim}{u})^{s}\cdot\underset{\sim}{I}_{\perp} - \underset{\sim}{I}_{\perp}\nabla_{\perp}\cdot\underset{\sim}{u}]^{(0)} + q_{\parallel}^{\perp}[(\nabla\underset{\sim}{b})^{s} - \underset{\sim}{I}_{\perp}\nabla\cdot\underset{\sim}{b}]\}$$
$$+ [\underset{\sim}{b}(\underset{\sim}{b}\cdot\underset{\sim}{P})_{\perp}^{(1)}]^{s}. \qquad (5.11)$$

Equation (5.11) is often useful because $\underset{\sim}{I}_{\gamma}$ and $\underset{\sim}{I}_{\delta}$ have been eliminated.

5.3 The Heat Flux Tensor and Fourth Moments.

The evaluation of $\underset{\sim}{Q}_{\phi}^{(1)}$ from the recursion formulae (4.23) through (4.28) is outlined in Appendix A. The results are

$$q_{\perp}^{\parallel} = (2\Omega)^{-1} b \times [\nabla R_2 - \rho^{-1} p_{\parallel} \nabla \cdot P + (R_1 - 3R_2) b \cdot \nabla b$$

$$+ 2(2q_{\parallel}^{\parallel} - q_{\parallel}^{\perp}) b \cdot \nabla u + 2q_{\parallel}^{\perp} (\nabla u) \cdot b]^{(0)} , \qquad (5.12)$$

$$q_{\perp}^{\perp(1)} = (2\Omega)^{-1} b \times [\nabla R_3 - 4\rho^{-1} p_{\perp} \nabla \cdot P + 4b \cdot T]^{(0)} , \qquad (5.13)$$

$$q_{\perp}^{\gamma(1)} = \tfrac{1}{2} I_{\gamma} \cdot q_{\perp}^{\perp(1)} , \qquad q_{\perp}^{\delta} = \tfrac{1}{2} I_{\delta} \cdot q_{\perp}^{\perp(1)} , \qquad (5.14)$$

$$q_{\parallel}^{\gamma(1)} = (2\Omega)^{-1} I_{\delta} : T^{(0)} , \qquad q_{\parallel}^{\delta(1)} = -(2\Omega)^{-1} I_{\gamma} : T^{(0)} , \qquad (5.15)$$

where

$$T = (R_2 - R_3/4) \nabla b + q_{\parallel}^{\perp} \nabla u .$$

If the terms $q_{\parallel}^{\perp}$ and $q_{\parallel}^{\parallel}$ are neglected, the transverse heat flows given above agree with those given by Thompson,[3] who assumed that $q_{\parallel}^{\perp(0)}$ and $q_{\parallel}^{\parallel(0)}$ vanish. Thompson's fluxes appear to differ from (5.12) and (5.13) because they are referred to the zero-order velocity $u_E + u_{\parallel}^{(0)} b$, whereas ours are referred to u. The corresponding difference in P is of order $[u - u^{(0)}]^2$ hence does not appear in Eqs. (5.7) and (5.8).

A representation of the tensor $Q^{(1)}$ is obtained by substituting Eqs. (5.12) through (5.15) into Eqs. (4.21) and (4.22). It is easily verified that the resulting tensor may be expressed in the form

$$Q = q_{\parallel}^{\parallel} bbb + q_{\parallel}^{\perp} (b I_{\perp})^s + (q_{\perp}^{\parallel} bb)^s + (1/2)(q_{\perp}^{\perp} I_{\perp})^s$$

$$+ (2\Omega)^{-1} [b (I_{\gamma} I_{\delta} - I_{\delta} I_{\gamma}) : T]^s + O(\epsilon^2) . \qquad (5.16)$$

In the presence of strong collisions, the zero-order distribution function is constrained to a local Maxwellian form. The zero-order pressure is then isotropic, and the fourth moments R_i appearing in the expressions for $\underset{\sim}{Q}_\phi$ are determined by the pressure and density:

$$R_1 = 3p^2/\rho, \qquad R_2 = p^2/\rho, \qquad \text{and} \qquad R_3 = 4p^2/\rho.$$

For this case of a local Maxwellian $f^{(0)}$ the sum of $\underset{\sim}{q}_\perp^\perp$ and $\underset{\sim}{q}_\perp^\parallel$ gives the well-known transverse collisionless heat flow[4]

$$\underset{\sim}{q}_\perp = \underset{\sim}{q}_\perp^\perp + \underset{\sim}{q}_\perp^\parallel = (5/2\rho m\Omega)\, p\underset{\sim}{b} \times \nabla(p/n). \qquad (5.17)$$

In the absence of collisions, however, $f^{(0)}$ need not be Maxwellian and independent equations are required to determine the $R_i(0)$. These equations are the zero-order terms of Eqs. (4.32):

$$dR_1/dt + R_1(5\nabla\cdot\underset{\sim}{u} - 4\nabla_\perp\cdot\underset{\sim}{u}) = 8q_\parallel^\parallel\rho^{-1}(\nabla\cdot\underset{\sim}{P})_\parallel + 4S_2\nabla\cdot\underset{\sim}{b} - \nabla\cdot(S_1\underset{\sim}{b}),$$

$$dR_2/dt + R_2(3\nabla\cdot\underset{\sim}{u} - \nabla_\perp\cdot\underset{\sim}{u}) = 2q_\parallel^\parallel\rho^{-1}(\nabla\cdot\underset{\sim}{P})_\parallel + (S_2 - S_3)\nabla\cdot\underset{\sim}{b} - \nabla\cdot(S_2\underset{\sim}{b}), \qquad (5.18)$$

and

$$dR_3/dt + R_3(\nabla\cdot\underset{\sim}{u} + 2\nabla_\perp\cdot\underset{\sim}{u}) = -2S_3\nabla\cdot\underset{\sim}{b} - 2\nabla\cdot(S_3\underset{\sim}{b}),$$

where all quantities are taken to zero order and the S_i are the ϕ-independent components of the fifth velocity moment tensor $\underset{\sim}{S}$:

$$S_1 = m\langle\nu_\parallel^5\rangle, \qquad S_2 = (m/2)\langle\nu_\parallel^3\nu_\perp^2\rangle, \qquad S_3 = (m/2)\langle\nu_\parallel\nu_\perp^4\rangle.$$

In the absence of collisions the S_i are not determined by a closed set of the moment equations. If $f^{(0)}$ is assumed to be of anisotropic Gaussian form, Eqs. (5.18) yield no new information, but are consistent with the CGL pressure equations.

These equations for the $R_i^{(0)}$ are similar in form to the CGL pressure equations and, if the terms on their right hand sides vanish, they may be integrated in the same way to obtain "adiabatic" conditions for the $R_i^{(0)}$. These adiabatic conditions take the form

$$dR_i'^{(0)}/dt = 0, \tag{5.19}$$

where the R_i', defined by Eqs. (4.33), are related to the form of the distribution function. In this special case, therefore, the form of the distribution is preserved by the plasma motion. In particular, $f^{(0)}$ will remain Gaussian, but not necessarily isotropic, if it has this form initially.

5.4 The Pressure Equations.

We consider now the first-order terms of the pressure equations (4.16) and (4.17), and of the energy equation (4.18). Of particular interest is the term $\underset{\approx}{P} : \nabla \underset{\approx}{u}$ of Eq. (4.18) which represents the energy transfer between the fluid and thermal motions. The corresponding terms of Eqs. (4.16) and (4.17) give separately the energy transfers between the fluid motion and the transverse and parallel components of the thermal motion.

By use of Eqs. (4. 9), (5. 9), and (5. 10) we obtain

$$\underset{\sim}{P} : \nabla \underset{\sim}{u} = p_\perp \nabla_\perp \cdot \underset{\sim}{u} + p_\parallel (\underset{\sim}{b} \cdot \nabla \underset{\sim}{u}) \cdot \underset{\sim}{b} + 2(P_\parallel - P_\perp) a_1 - \Gamma + O(\epsilon^2), \qquad (5.20)$$

where

$$a_1 = 2\Omega^{-1} \underset{\sim}{b} \cdot [(\underset{\sim}{b} \cdot \nabla \underset{\sim}{u}) \times (\nabla \underset{\sim}{u}) \cdot \underset{\sim}{b}],$$

and

$$\Gamma = 2\Omega^{-1} q_\parallel^\perp (\nabla \underset{\sim}{u}) : (\underset{\sim}{I}_\gamma \underset{\sim}{I}_\delta - \underset{\sim}{I}_\delta \underset{\sim}{I}_\gamma) : \nabla \underset{\sim}{b}$$

$$+ \Omega^{-1} \underset{\sim}{b} \cdot \{[\underset{\sim}{b} \cdot \nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u}) \cdot \underset{\sim}{b}] \times [(2q_\parallel^\parallel - q_\parallel^\perp) \underset{\sim}{b} \cdot \nabla \underset{\sim}{b} + \nabla q_\parallel^\perp]\}.$$

In zero order only the first two terms of this expression are present; they represent the anisotropic compressional heating. The third term arises from the contribution of the collisionless viscosity to P_{12} and P_{13}. (There is no contribution from the parts of P_{23} and P_δ proportional to $\nabla \underset{\sim}{u}$.) This term is somewhat analogous to ordinary viscous dissipation in that it arises from the contribution of the collisionless viscosity to $\underset{\sim}{P} : \nabla \underset{\sim}{u}$ and is quadratic in $\underset{\sim}{u}$. The net transfer of energy between the fluid and thermal motions thus does not vanish in general, as might be expected for a collisionless system. This collisionless energy transfer is not dissipational however, as it may have either sign. The collisionless viscosity modifies the propagation of waves, but does not cause them to be damped; it may even be a mechanism for wave propagation. A special example of such a "viscosity wave" appears in Section 9. 2.

The term Γ of Eq. (5.20) represents the energy transfer between the fluid and thermal motions arising from the first-order stresses produced by the zero-order heat flows. Because zero-order heat flow cannot occur in the presence of strong collisions, these terms have no analogy in ordinary hydrodynamics.

The complete equation for the total thermal energy K through first order is

$$dK/dt + K\nabla\cdot\mathbf{u} + p_\perp \nabla_\perp\cdot\mathbf{u} + p_\parallel(\mathbf{b}\cdot\nabla\mathbf{u})\cdot\mathbf{b} + \nabla\cdot\mathbf{q} + (p_\parallel - p_\perp)^{(0)}\alpha_1 = \Gamma. \qquad (5.21)$$

By use of the representation (5.16) for $\mathbf{Q}^{(1)}$, the pressure equations (4.16) and (4.17) through first order become

$$dp_\perp/dt + p_\perp(\nabla\cdot\mathbf{u} + \nabla_\perp\cdot\mathbf{u}) + \nabla\cdot\mathbf{q}_\perp^{\perp(1)} - p_\perp^{(0)}\alpha_1 + \alpha_2 = - B\nabla\cdot(q_\parallel^\perp\mathbf{b}/B) + \Gamma_\perp + O(\epsilon^2), \qquad (5.22)$$

and

$$(1/2)\,dp_\parallel/dt + p_\parallel[(3/2)\nabla\cdot\mathbf{u} - \nabla_\perp\cdot\mathbf{u}] + \nabla\cdot\mathbf{q}_\perp^{\parallel(1)} + p_\parallel^{(0)}\alpha_1 - \alpha_2 = - \nabla\cdot(q_\parallel^\parallel\mathbf{b}) - q_\parallel^\perp\mathbf{b}\cdot\nabla B/B + \Gamma - \Gamma_\perp + O(\epsilon^2), \qquad (5.23)$$

where Γ is defined following Eq. (5.20),

$$\Gamma_{\perp} = 2\Omega^{-1}\underset{\sim}{b}\cdot\{q_{\parallel}^{\perp}(\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}) \times [(\nabla\underset{\sim}{u})\cdot\underset{\sim}{b} - 2\underset{\sim}{b}\cdot\nabla\underset{\sim}{u}] + \underset{\sim}{b}\cdot(\nabla\underset{\sim}{u}) \times \nabla q_{\parallel}^{\perp}\},$$

and

$$\alpha_2 = \Omega^{-1}\underset{\sim}{b}\cdot[(\nabla R_2 - \rho^{-1}p_{\parallel}\nabla p_{\perp}) \times (\underset{\sim}{b}\cdot\nabla\underset{\sim}{b})]^{(0)}.$$

The energy transfer produced by the collisionless viscosity is again given by the α_1 terms, which now include also the contribution of the collisionless viscosity to $[\underset{\sim}{P}:(\underset{\sim}{b}d\underset{\sim}{b}/dt)]^{(1)} = \underset{\sim}{\Pi}^{(1)}:[\underset{\sim}{b}(\underset{\sim}{b}\cdot\nabla\underset{\sim}{u})]^{(0)}$. We note that even if the contribution of these energy transfers to dK/dt vanishes because $p_{\parallel} = p_{\perp}$, their contributions to $dp_{\parallel}/dt$ and $dp_{\perp}/dt$ separately may not vanish, and have opposite signs. If α_1 is nonzero, therefore, the collisionless viscosity tends to produce anisotropy in the pressure.

The α_2 terms of Eqs. (5.22) and (5.23) represent a transfer of energy between the parallel and transverse thermal motions which arises from $\underset{\sim}{b}\cdot\underset{\sim}{Q}:\nabla\underset{\sim}{b}$ in Eqs. (4.16) and (4.17). First-order effects of the zero-order parallel heat flows are represented by Γ and $\Gamma_{\perp}$.

5.5 The Single-Fluid Equations.

Except for $q_{\parallel}^{\perp}$ and $q_{\parallel}^{\parallel}$ through first order and the $S_i^{(0)}$, Eqs. (4.1) through (4.3), (5.7), (5.9), (5.12), (5.13), (5.18), (5.22), and (5.23), for each component, together with Maxwell's equations, determine the transverse motion of each component through second order and the other quantities through first order.

The corresponding single-fluid equations for a two-component plasma, which determine the plasma velocity through first order and $\underset{\sim}{j}$ through order ϵ^2, are obtained by taking $\underset{\sim}{P}$, $\underset{\sim}{E}$, and $\underset{\sim}{B}$ through first order in Eq. (4. 36) and including the first-order terms of the generalized Ohm's law (4. 43). The electrostatic force is negligible unless $\beta \sim \epsilon (v_0/c)^2$ and will not be considered here. These equations apply in the limit of small electron mass but non-negligible electron pressure. The electron thermal velocity is then large, of order $(m_+/m_-)^{\frac{1}{2}}$ times the ion thermal velocity, and the electron gyro-radius of order $(m_-/m_+)^{\frac{1}{2}}$ compared to the ion gyro-radius.

The transverse flow velocities $\underset{\sim}{u}_\perp^+$ and $\underset{\sim}{u}_\perp^-$, on the other hand, are both of order $\underset{\sim}{u}_E$ and we assume that $u_\parallel^+$ and $u_\parallel^-$ are also of this same order. The electron collisionless viscosity terms are therefore negligible, but the current produced by the electron pressure gradient and the electron heat flow is of the same order as the corresponding ion effects.

The pressure tensor appearing in the equation of motion (4. 36) is therefore

$$\underset{\sim}{P}_t^{(1)} = P_\perp \underset{\sim}{I}_\perp + P_\parallel \underset{\sim}{b}\underset{\sim}{b} + \underset{\sim}{\Pi}_+^{(1)} + O(\epsilon^2), \qquad (5.24)$$

where $\underset{\sim}{\Pi}_+^{(1)}$ is given by Eq. (5.10) or (5.11).

Equations for $P_\perp$ and $P_\parallel$ are obtained by adding together the ion and electron forms of the pressure equations (5. 22) and (5. 23), taking account of the first-order difference between $\underset{\sim}{u}_+$ and $\underset{\sim}{u}_-$, and the terms of $\underset{\sim}{Q}_-^{(1)}$ not proportional to $\underset{\sim}{u}_-$.

The fourth moments enter these equations for $P_\perp$ and $P_\parallel$ in the form $R_i^* = R_i^+ - (m_-/m_+)R_i^{-(0)}$. The equations for the R_i^* have exactly the same form as Eqs. (5.18), since the ion and electron forms of these zero-order equations are identical.

Because the zero-order ion and electron pressures are required separately in the first-order pressure equations, the generalized Ohm's law, and $\underset{\sim}{\Pi}^{(1)}$, the zero-order equations for either $p_\perp^+$ and $p_\parallel^+$ or $p_\perp^-$ and $p_\parallel^-$ are required in addition to the first-order equations for $P_\perp$ and $P_\parallel$.

The extension of CGL theory to determine the plasma velocity through first order is thus comprised by the equation of motion with the pressure tensor of Eq. (5.24), the generalized Ohm's law (4.43) including first-order terms, the two first-order pressure equations for $P_\perp$ and $P_\parallel$, two zero-order (i.e., CGL) pressure equations, and the three zero-order equations for the R_i^*.

These equations simplify considerably if $\underset{\sim}{b}$ is constant in space and time and if the properties of the plasma are independent of position along the field lines; they form a closed set if $u_\parallel$, $q_\parallel^\perp$, and $q_\parallel^\parallel$ vanish initially. These quantities then remain zero, and $p_\parallel^+$, $p_\parallel^-$, R_1^*, R_2^*, $\underset{\sim}{q}_\perp^\parallel$, and the S_i are no longer required.

The transverse components of the equation of motion become

$$\rho\, D\underset{\sim}{V}/Dt = -\nabla_\perp (P_\perp + \tfrac{1}{2}B^2) - \underset{\sim}{\lambda} + \rho\, \underset{\sim}{g}, \tag{5.25}$$

where

$$\underset{\sim}{\lambda} = + \nabla \cdot \underset{\sim}{\Pi}^{(1)} = -\underset{\sim}{I}_{\delta} \cdot \nabla [(p_{\perp}^{+}/\Omega) \underset{\sim}{I}_{\gamma} : \nabla \underset{\sim}{V}]^{(0)} + \underset{\sim}{I}_{\gamma} \cdot \nabla [(p_{\perp}^{+}/\Omega) \underset{\sim}{I}_{\delta} : \nabla \underset{\sim}{V}]^{(0)} , \qquad (5.26)$$

or

$$\lambda = -(\nabla \cdot \underset{\sim}{V}) \underset{\sim}{b} \times \nabla (p_{\perp}^{+}/2\Omega) + (p_{\perp}^{+}/2\Omega) \underset{\sim}{b} \times \nabla^2 \underset{\sim}{V} + \underset{\sim}{b} \times (\nabla \underset{\sim}{V})^{s} \cdot \nabla (p_{\perp}^{+}/2\Omega) . \qquad (5.27)$$

The first-order pressure equation may be written

$$DP_{\perp}/Dt + 2P_{\perp} \nabla_{\perp} \cdot \underset{\sim}{V} + \nabla_{\perp} \cdot \underset{\sim}{q}_{\perp}^{\perp t} + (ne)^{-1} [2\underset{\sim}{j} \cdot \nabla n/n - \underset{\sim}{j} \cdot \nabla p_{\perp}^{-}] = 0, \qquad (5.28)$$

where the last term arises from the difference between $\underset{\sim}{u}_{-}$ and $\underset{\sim}{V}$, and

$$\underset{\sim}{q}_{\perp}^{\perp t} = (2\Omega_{+})^{-1} \underset{\sim}{b} \times \{ \nabla B_3^{*} - 2\rho_{+}^{-1} \nabla [P_{\perp} (P_{\perp} - 2p_{\perp}^{-})] \}^{(0)} \qquad (5.29)$$

is the total transverse flow of transverse thermal energy.

The CGL equation for $p_{\perp}^{-(0)}$ becomes

$$Dp_{\perp}^{-(0)}/Dt + 2p_{\perp}^{-(0)} \nabla \cdot \underset{\sim}{V} = 0, \qquad (5.30)$$

and the fourth moment R_3^{*} is determined by the conservation of $R_{3-}^{'}$ and $R_{3+}^{'}$,

$$D(\rho R_3^{*}/p_{\perp}^{2})/Dt = 0. \qquad (5.31)$$

The transverse motion of the fluid is described through first order by Eqs. (5.25) and (5.28) through (5.31).

The first-order energy equation (5. 28) differs from the CGL pressure equation because of the last two terms. The heat flow of Eq. (5. 29) vanishes if $p_\perp^+ = p_\perp^-$ and $R'_{3+} = R'_{3-}$ in zero order. These zero-order equalities are preserved by the plasma motion. In this case the first-order pressure equation differs from the CGL form only by the last term. If the electron pressure is small, on the other hand, this latter term is negligible but the unbalanced ion heat flow causes Eq. (5. 28) to differ from the CGL form.

In certain cases, such as perturbations of a uniform equilibria or one-dimensional configurations, the first-order pressure equation reduces to the CGL form for all distribution functions, and the CGL adiabatic condition holds through first order. In general, however, the adiabatic condition is modified in first order because of the first-order corrections to the pressure equation and to the generalized Ohm's law.

This closed set of equations differs from those obtained on the assumption that $f^{(0)}$ is an isotropic Maxwellian[29] in that the scalar pressure is replaced by $P_\perp$, and that the heat flow $\underset{\sim}{q}_\perp$ of Eq. (5. 17) is replaced by $\underset{\sim}{q}_\perp^\perp$ of Eq. (5. 29) with R'_{3+} and R'_{3-} conserved. Conservation of R'_i implies that $f^{(0)}$ will remain Gaussian, but not necessarily isotropic, if it has this form initially. The two energy equations then differ in form only by a factor 5/4 in the expressions for the heat flows.

The above equations may be used to find FGR corrections to the frequency of a magneto-acoustic wave propagating across a uniform magnetic field in a uniform plasma. It is easily verified that the first-order corrections vanish. For this simple case (linear perturbations of a uniform plasma) the second-order terms of the moment equations are easily found. These terms give a second-order correction to the frequency which does not vanish, and differs from that found by Talwar[30] (who included only first-order FGR terms in his equations) by a numerical factor which depends on the fourth moments of the equilibrium distribution.

CHAPTER VI

Lowest-Order Hydromagnetic Equations for FGR Ordering

6.1 FGR Ordering.

In the remaining chapters of this paper we discuss the lowest-order approximation to the moment equations obtained if the time scale is extended to be of order $\tau_D \sim (\epsilon^2 \Omega)^{-1}$. This discussion will be restricted to situations in which no zero-order motion occurs on the time scale of CGL ordering, i.e. to motion characterized by the FGR ordering of Eq. (2.4).

Because such motions are slow compared to the thermal and Alfven velocities, they will occur only under special circumstances. We may consider, for example, the gravitational interchange instability. Ordinary hydromagnetic (CGL) theory gives for the frequency of this mode

$$\omega = (\rho' g)^{\frac{1}{2}} \sim \epsilon (gL/v_0^2)^{\frac{1}{2}} \Omega \qquad (6.1)$$

where ρ' is the equilibrium density gradient characterized by the scale distance L. The stability condition is

$$\rho' g \geq 0 . \qquad (6.2)$$

It is evident that ω is of order $\epsilon\Omega$ and CGL ordering applies only for strong gravitational accelerations, $(gL/v_0^2)^{\frac{1}{2}} \sim 1$. If it is assumed that $(gL/v_0^2)^{\frac{1}{2}} \sim \epsilon$, corresponding to a very weak gravitational acceleration, the interchange mode is characterized by FGR ordering and the results (6.1) and (6.2) no longer apply.[14] The FGR corrections to the weak gravitational interchange instability are well known for low β;[6,7,14] they are discussed for finite β in Section 8.3.

The relation $gL/v_0^2 \sim \epsilon^2$ may therefore be associated with FGR ordering. This is not a necessary condition for FGR ordering to apply, however, as a mode only slightly influenced by gravity might be of low frequency even in the presence of a strong gravitational field.

More generally, a necessary (but not sufficient) condition for FGR ordering is the CGL equilibrium condition

$$\underset{\sim}{j}^{(1)} \times \underset{\sim}{B}^{(0)} - \nabla \cdot \underset{\sim}{P}_t^{(0)} + (\rho \underset{\sim}{g})^{(0)} = 0,$$

or

$$-\nabla_\perp (P + \tfrac{1}{2}B^2) + (B^2 + P_\perp - P_\parallel)\underset{\sim}{b}\cdot\nabla\underset{\sim}{b} - [\underset{\sim}{b}\cdot\nabla P_\parallel - (P_\parallel - P_\perp)\underset{\sim}{b}\cdot\nabla B]\underset{\sim}{b}$$
$$+ \rho\underset{\sim}{g} = 0, \qquad (6.3)$$

where charge separation has been neglected. Unless $\underset{\sim}{b}\cdot\nabla B$ and $\underset{\sim}{b}\cdot\nabla P_\parallel$ are small, this condition is a strong constraint on $P_\perp$, $P_\parallel$, and $\underset{\sim}{B}$. Except under rather special circumstances, therefore, FGR ordering will be associated with weak parallel gradients.

The FGR ordering (2.4) thus does not specify all important aspects or properties of slow, low-frequency motion. For the interchange mode it must be supplemented by the condition of weak gravity; for many waves by the condition of long parallel wavelengths; for the Alfvèn mode discussed in Chapter IX by the condition that the frequency obtained from ordinary hydromagnetic theory be small; and for long term equilibrium by the condition (6.3) itself.

We will be interested in the low-frequency interchange mode, but not in effects of strong gravitational accelerations. We therefore extend our definition of FGR ordering to include the condition of weak gravity. We do not, however, include the condition of weak parallel gradients. Furthermore, in keeping with our requirements that the motion be slow, we assume that $q_{\parallel}^{\perp}$, $q_{\parallel}^{\parallel}$, S_i, and higher parallel transport terms are of first order. Our extended definition of FGR ordering is therefore

$$\begin{aligned} L/v_0\tau_0 \sim L/v_0\tau \sim cE_{0\perp}/v_0B_0 \sim (gL)^{\frac{1}{2}}/v_0 \\ \sim \langle \nu_\perp^{2m}\nu_\parallel^{2n+1}f\rangle /n_0v_0^{2m+2n+1} \sim \epsilon, \end{aligned} \tag{6.4}$$

where $m, n \geq 0$.

Because the transverse guiding center velocity is directly determined by the electric and magnetic fields, the condition $cE_{0\perp}/v_0 B_0 \sim \epsilon \ll 1$ is sufficient to insure slow transverse plasma motion. In the direction of the magnetic field, however, only the acceleration of the guiding centers is determined by the fields. The parallel motion is therefore sensitive to variations in the fields, particularly $E_{\parallel}$ and $\underset{\sim}{b} \cdot \nabla \underset{\sim}{B}$. This is especially true if both $L_{\parallel}$ and the time scale are large, for then the acceleration acts over a long time. Thus, if $\tau \sim (\epsilon^2 \Omega)^{-1}$ and $L_{\parallel} \sim R_g/\epsilon^2$, even a parallel electric field of order $\epsilon^3 v_0 B_0/c$ may drive a first-order parallel current.

A variety of low-frequency waves exist which propagate nearly transverse to the magnetic field. Waves of this type provide mechanisms for numerous instabilities which recently have been extensively studied;[9,10,15] These waves, however, are beyond the scope of a purely hydromagnetic treatment unless collisions are present, and they will not be discussed here.

6.2 Finite Drift-Time Effects.

As discussed in Chapter II, extension of the time scale from $\tau_{\ell} = (\epsilon\Omega)^{-1}$ to $\tau_D = \tau/\epsilon$ changes the ordering of terms in the small ϵ expansion of the particle orbits and the moment equations, with the result that the zero-order approximation to the moment equations differs from the CGL equations. These modifications are important, even in the limit $\epsilon \to 0$, because during the drift time τ_D very small accelerations ($\sim\epsilon^2 v_0^2/L$) and velocities ($\sim\epsilon v_0$) contribute to the zero-order displacement of the plasma. It was suggested in Section 2. 2 that these effects be described by the term "finite drift time" rather than the conventional term "finite gyro-radius."

The change in ordering may be illustrated by the displacement produced by the first-order guiding center drifts. During the "longitudinal" time τ these first-order velocities produce only first-order displacements of the guiding centers. If the time scale is extended to be of order τ_D, however, these displacements become of zero order. Guiding centers initially on the same field line then end up on quite separate field lines; this represents a serious breakdown of the fluid properties of the plasma. As a result, for a time scale of order τ_D and, in particular, for FGR ordering, a hydromagnetic description of even the transverse plasma motion is possible only in special cases.

For FGR ordering the plasma acceleration is order $\epsilon^2 v_0^2/L$ (this may be seen explicitly from Eq. (2. 3)). The lowest-order approximations to the equations of motion (4. 2) and (4. 3) for each plasma component are therefore obtained by evaluating the terms of Eq. (4. 2) through second order and those of (4. 3) through third order. These equations then become

$$[d\underset{\sim}{u}^{(1)}/dt]_{\parallel} + \rho^{-1}(\nabla\cdot\underset{\sim}{P})_{\parallel} - (e/m)E_{\parallel} = 0(\epsilon^3 v_0^2/L)\,, \qquad (6.5)$$

and

$$u_{\perp}^{(1)} - u_{E}^{(1)} = \Omega^{-1} b \times [\, du^{(1)}/dt + \rho^{-1}\nabla\cdot P - g] = 0(\epsilon^4 v_0) . \qquad (6.6)$$

In these equations $u_{\parallel}$ appears only in first order, E, B, P, and ρ are required through second order, and $u_{\perp}$ (hence $j_{\perp}$) is obtained through third order. They differ in form from the zero-order equations obtained for CGL ordering only by the appearance of Π (which is required through second order).

It is well known that the guiding center equations equivalent to CGL theory are:

$$v_D = u_E + \Omega^{-1} b \times [\, (v_{\perp}^2/2B)\nabla B + D^* v_D - g]^{(0)} + 0(\epsilon^2 v_0) , \qquad (6.7)$$

where v_D is the guiding center drift velocity and

$$D^* = \partial/\partial t + (u_E + v_{D\parallel}^{(0)} b)\cdot\nabla$$

is the total time derivative following the guiding center motion; the equation for the parallel guiding center motion

$$D^*(v_{D\parallel}) = (e/m)E_{\parallel} + g_{\parallel} - (v_{\perp}^2/2B) b\cdot\nabla B + u_E\cdot(D^* b) + 0(\epsilon v_0^2/L) ; \quad (6.8)$$

the relation between the transverse flow velocity $u(r)$ and the average transverse drift velocity of guiding centers at r

$$u_{\perp}(r) = \langle v_D(r) \rangle_{\perp} + (ne)^{-1}(\nabla \times \mu_t)_{\perp} + 0(\epsilon^2 v_0), \qquad (6.9)$$

where $\mu_t(r)$ is the total magnetic moment of particles with guiding centers at r; and the conservation of the magnetic moment $\mu = mv_{\perp}^2/2B$. The relation for the parallel velocity components corresponding to (6.9) is[25]

$$u_{\parallel}(r) = \langle v_D(r) \rangle_{\parallel} + (ne)^{-1}(\nabla \times \mu_t) + 0(\epsilon^2 v_0), \qquad (6.10)$$

but this relation is not required for CGL theory because only $u_{\parallel}^{(0)}$ appears in that theory.

The first- and second-order terms of Π appearing in Eqs. (6.5) and (6.6) therefore correspond to the second- and third-order transverse guiding center drifts, the first- and second-order corrections to the parallel guiding center equation (6.8), the second- and third-order corrections to the relation (6.7), and the first-order terms of Eq. (6.10).

The zero-order transverse drift is just u_E while first-order drifts are produced by ∇B and the zero-order guiding center acceleration. We would expect the first modifications of the electric field drift to be of order $(v_0/\Omega)^2 \nabla\nabla E$, corresponding to effects caused by variation of E over the particle orbit. For FGR ordering this term is of third order, hence the only modification of the electric field drift required. Corrections to the ∇B drift are also to be expected.

The acceleration term of Eq. (6.7) is just the drift corresponding to the electric field equivalent to the zero-order guiding center acceleration.

We would expect this form to remain unchanged in higher order. The modification of this term would then consist in substituting the guiding center acceleration through second order for the zero-order acceleration, and in including the higher-order electric field drift. Since $\underset{\sim}{u}_E$ and $\partial/\partial t$ are small for FGR ordering, only the curvature drift ($\sim v_\parallel^2 \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}$) would contribute to this latter term; the transverse acceleration $(\partial/\partial t + \underset{\sim}{u}_E \cdot \nabla)\underset{\sim}{u}_E$ is already of second order, and the corresponding drift is therefore unmodified.

In the equivalent moment equations (6.5) and (6.6), all these complications appear in the ϕ-dependent components $\underset{\sim}{\Pi}$ of the pressure tensor. It is therefore not surprising that $\underset{\sim}{\Pi}^{(2)}$ is quite complicated, but its calculation is straightforward and not prohibitively tedious.

Fundamental difficulties arise in the evaluation of $p_\perp$ and $p_\parallel$, however, because of the breakdown of the fluid properties of the plasma referred to above. These difficulties are analogous to, but independent of, those which arise in CGL theory because of transport along the field lines.

These difficulties, and the situations for which they may be avoided, are easily understood in terms of the first-order guiding-center drift velocity. For FGR ordering this velocity. For FGR ordering this velocity is

$$\underset{\sim}{v}_{D\perp}^{(1)} = \underset{\sim}{u}_E^{(1)} + \Omega^{-1}\underset{\sim}{b} \times [(\nu_\perp^2/2B)\nabla B + (v_{D\parallel})^2 \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}] . \qquad (6.11)$$

Since $\underset{\sim}{v}_{D\perp}^{(1)}$ depends on the thermal velocities $v_\perp$ and $v_{D\parallel}$, guiding centers initially on the same field line may develop large transverse separations over a time of order τ_D. Unless these separations are actually small, or unless all gradients in their direction are small, closed hydromagnetic equations are not obtained even in the absence of parallel transport.

The most important case for which the first of these conditions is fulfilled is a low β plasma in a uniform magnetic field. Then $\underset{\sim}{v}_{D\perp}^{(1)} = \underset{\sim}{u}_E^{(1)}$ and in many ways the plasma again behaves as an ideal fluid. This case has been discussed by Rosenbluth and Simon,[6] and is the subject of Chapter VII.

If small perturbations of a uniform magnetic field are considered, the guiding center separation produced by $\underset{\sim}{v}_{D\perp}^{(1)}$ is of the same order as the amplitude of the perturbations; the linearized moment equations are then closed with respect to transverse transport, even for finite β.

The moment equations are also closed with respect to transverse transport if all zero-order transverse gradients are in the direction $\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}$. The second term of Eq. (6.11) is then orthogonal to all transverse gradients and therefore unimportant. This type of configuration is illustrated by the finite β interchange modes discussed in Chapter VIII, and by the special "long term" equilibria discussed in Section 10.3.

6.3 Pressure Equations for FGR Ordering.

The ϕ-independent moments $p_\perp$ and $p_\parallel$ for each plasma component are determined by the pressure equations (4.16) and (4.17). The zero-order terms of these equations have the forms (5.4) and (5.5). These zero-order terms vanish for FGR ordering because $\partial/\partial t$, $\underset{\sim}{u}$, $q_\parallel^\perp$, and $q_\parallel^\parallel$ are of first order; equations for $p_\perp^{(0)}$ and $p_\parallel^{(0)}$ are then obtained from the first-order terms of Eqs. (4.16) and (4.17). The first-order terms have the forms (5.22) and (5.23) obtained for CGL ordering, except that for FGR ordering $\underset{\sim}{u}^{(0)}$, $q_\parallel^{\perp(0)}$ and $q_\parallel^{\parallel(0)}$ all vanish, and $p_\perp$ and $p_\parallel$ are taken to zero order wherever they appear. The lowest-order pressure equations for FGR ordering are thus

$$dp_\perp^{(0)}/dt + p_\perp^{(0)}(\nabla\cdot\underset{\sim}{u} + \nabla_\perp\cdot\underset{\sim}{u})^{(1)} + \nabla\cdot\underset{\sim}{q}_\perp^{\perp(1)} + \alpha_2 = -B\nabla\cdot(q_\parallel^{\perp(1)}\underset{\sim}{b}/B) \tag{6.12}$$

and

$$dp_\parallel^{(0)}/dt + p_\parallel^{(0)}(3\nabla\cdot\underset{\sim}{u} - 2\nabla_\perp\cdot\underset{\sim}{u})^{(1)} + \nabla\cdot\underset{\sim}{q}_\perp^{\parallel(1)} - \alpha_2 = -\nabla\cdot(q_\parallel^{\parallel(1)}\underset{\sim}{b}) + q_\parallel^{\perp(1)}\nabla\cdot\underset{\sim}{b}, \tag{6.13}$$

where α_2 is defined following Eq. (5.23) and, for FGR ordering,

$$\underset{\sim}{q}_\perp^{\perp(1)} = (2\Omega)^{-1}\underset{\sim}{b}\times[\nabla R_3 - 4\rho^{-1}p_\perp\nabla\cdot\underset{\sim}{P} + (4R_2 - R_3)\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}]^{(0)}, \tag{6.14}$$

and

$$\underset{\sim}{q}_\perp^{\parallel(1)} = (2\Omega)^{-1}\underset{\sim}{b}\times[\nabla R_2 - \rho^{-1}p_\parallel\nabla\cdot\underset{\sim}{P} + (R_1 - 3R_2)\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}]^{(0)}. \tag{6.15}$$

The important terms of Eqs. (6.12) and (6.13) are the divergences of the first-order transverse heat flows; these now contribute to $p_{\perp}^{(0)}$ and $p_{\parallel}^{(0)}$ because of the long time scale associated with FGR ordering. Note that these terms introduce the $R_i^{(0)}$ into the equations for $p_{\perp}^{(0)}$ and $p_{\parallel}^{(0)}$.

For FGR ordering similar modifications must be made to Eqs. (5.18) for the $R_i^{(0)}$, these quantities now being determined by the first-order terms of Eqs. (4.32), and the first-order fifth moments $\underset{\sim}{S}^{(1)}$ are required to find $R_i^{(0)}$. The ϕ-dependent components of $\underset{\sim}{S}^{(1)}$ may be found from Eq. (4.7) in the same way that $\underset{\sim}{Q}_{\phi}^{(1)}$ is obtained from Eq. (4.5). In this way the ϕ-independent sixth moments enter the equations for the $R_i^{(0)}$. Equations for the sixth moments bring in yet higher moments. Closed moment equations are obtained only if this sequence terminates.

This infinite sequence of equations represents, of course, the breakdown of the fluid properties of the plasma discussed in the previous section; it is readily verified that termination occurs for the special cases discussed in that section: For $\underset{\sim}{B}$ = constant $\nabla \cdot \underset{\sim}{q}_{\perp}^{\perp}$ and $\nabla \cdot \underset{\sim}{q}_{\perp}^{\parallel}$ are independent of the R_i; linearization about a constant magnetic field produces equations which depend only on the unperturbed values of the R_i; while if all transverse gradients are assumed to be in the direction $\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}$, both $\nabla \cdot \underset{\sim}{q}_{\perp}^{\perp}$ and $\nabla \cdot \underset{\sim}{q}_{\perp}^{\parallel}$ vanish if $j_{\parallel}^{(1)}$ does.

The equations of motion (6.5) and (6.6) require the pressure tensor, hence $p_\perp$ and $p_\parallel$, through second order. The second- and third-order terms of the pressure equations are therefore required in addition to the first-order terms just discussed. Evaluation of these terms in general is straightforward, but tedious, and the resulting pressure equations are quite complicated. It is easy to see, however, that the higher-order terms introduce no new closure problems; if the sequence of equations for $p_\perp$ and $p_\parallel$ closes in lowest order, it closes in every order, but in higher order involves higher velocity moments.

Fortunately, use of these complicated higher-order corrections to the pressure equations may be avoided, because, in those cases for which the moment equations are closed with respect to parallel transport, the equivalent single-fluid equations may be formulated without use of the higher-order corrections to $p_\perp$ and $p_\parallel$. This formulation is discussed in Sections (6.5) and (6.6).

6.4 The Second-Order Pressure Tensor.

In contrast to $p_\perp$ and $p_\parallel$, evaluation of $\underset{\sim}{\Pi}$ through second order causes no difficulty; it is only necessary to evaluate $\underset{\sim}{G}$ in the recursion equations (4.14) or (4.15) through first order.

The tensor $\underset{\sim}{G}$ may be written in the form

$$\underset{\sim}{G} = d(\underset{\sim}{P}_d)/dt + \underset{\sim}{P}_d \nabla\cdot\underset{\sim}{u} + (\underset{\sim}{P}_d\cdot\nabla\underset{\sim}{u})^s + \nabla\cdot[q_\parallel^\parallel \underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b} + q_\parallel^\perp(\underset{\sim}{b}\underset{\sim}{I}_\perp)^s]$$
$$+ d\underset{\sim}{\Pi}/dt + \underset{\sim}{\Pi}\nabla\cdot\underset{\sim}{u} + [\underset{\sim}{\Pi}\cdot\nabla\underset{\sim}{u}]^s + \nabla\cdot\underset{\sim}{Q}_\phi, \qquad (6.16)$$

where $\underset{\sim}{P}_d = p_\perp \underset{\sim}{I}_\perp + p_\parallel \underset{\sim}{b}\underset{\sim}{b}$. In zero order only the first four terms of this expression are non-vanishing. For CGL ordering they yield the $\underset{\sim}{\Pi}^{(1)}$ of Eqs. (5.8) and (5.7), (5.10), or (5.11).

For FGR ordering these terms vanish in zero order and $\underset{\sim}{\Pi}^{(1)}$ vanishes. Then

$$\underset{\sim}{G}^{(1)} = d(\underset{\sim}{P}_d)/dt + \underset{\sim}{P}_d \nabla\cdot\underset{\sim}{u} + (\underset{\sim}{P}_d\cdot\nabla\underset{\sim}{u})^s + \nabla\cdot[q_\parallel^\parallel \underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b} + q_\parallel^\perp(\underset{\sim}{b}\underset{\sim}{I}_\perp)^s]$$
$$+ \nabla\cdot\underset{\sim}{Q}_\phi^{(1)}, \tag{6.17}$$

where now $\underset{\sim}{P}_d = \underset{\sim}{P}_d^{(0)} = p_\perp^{(0)}\underset{\sim}{I}_\perp + p_\parallel^{(0)}\underset{\sim}{b}\underset{\sim}{b}$, $\underset{\sim}{u} = \underset{\sim}{u}^{(1)}$, and $\underset{\sim}{Q}_\phi^{(1)}$ is given by Eq. (5.16) with $\underset{\sim}{u}^{(0)} = 0$. All terms of $\underset{\sim}{G}^{(1)}$ except $\nabla\cdot\underset{\sim}{Q}_\phi^{(1)}$ are thus identical in form to the zero-order terms of Eq. (6.16). Except for the contribution from $\nabla\cdot\underset{\sim}{Q}_\phi^{(1)}$, therefore, $\underset{\sim}{\Pi}^{(2)}$ is given by Eq. (5.8) and (5.7), (5.10) or (5.11) with $\underset{\sim}{u}^{(0)}$, $q_\parallel^{\perp(0)}$, and $q_\parallel^{\parallel(0)}$ replaced by their first-order values. (The form (5.9) for $\underset{\sim}{b}\cdot\underset{\sim}{\Pi}$ no longer applies, however, because first-order corrections to the generalized Ohm's law are now required.)

We denote this contribution to $\underset{\sim}{\Pi}^{(2)}$, of the same form as the $\underset{\sim}{\Pi}^{(1)}$ of Eqs. (5.7) and (5.8), by $\underset{\sim}{\Pi}_1^{(2)}$ and the additional terms arising from $\nabla\cdot\underset{\sim}{Q}_\phi^{(1)}$ by $\underset{\sim}{\Pi}_2^{(2)}$. This additional term occurs only in second order.

From Eq. (5.16) we find, as outlined in Appendix A,

$$\underset{\sim}{b} \cdot \underset{\sim}{\Pi}_2^{(2)} = -(\alpha/\Omega)\{[(\nabla \underset{\sim}{b})^s - \underset{\sim}{I}_\perp \nabla \cdot \underset{\sim}{b}] \cdot [(\nabla \alpha)/\alpha - \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}]$$

$$- \nabla \cdot [(\nabla \underset{\sim}{b})^s - \underset{\sim}{I}_\perp \nabla \cdot \underset{\sim}{b}]\} + \Omega^{-1} \underset{\sim}{b} \times \{2\underset{\sim}{b} \cdot \nabla q_\parallel^\perp \qquad (6.19)$$

$$+ (2q_\parallel^\perp - \tfrac{1}{2}q_\perp^\perp) \cdot (\nabla \underset{\sim}{b} + \underset{\sim}{I}_\perp \nabla \cdot \underset{\sim}{b})\},$$

and

$$(\underset{\sim}{\Pi}_2^{(2)})_{\perp\perp} = (4\Omega)^{-1}\{(\underset{\sim}{T}_1)^s - \underset{\sim}{I}_\perp \underset{\sim}{I}_\perp : \underset{\sim}{T}_1 + \underset{\sim}{b} \times [(\underset{\sim}{T}_2)^s - \underset{\sim}{I}_\perp \underset{\sim}{I}_\perp : \underset{\sim}{T}_2]\}, \quad (6.20)$$

where

$$\alpha \equiv (1/2\Omega)(R_2 - R_3/4),$$

$$\underset{\sim}{T}_1 \equiv \alpha\{2(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b})(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}) - 2\nabla_\perp(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}) - (\nabla \times \underset{\sim}{b}) \cdot \underset{\sim}{b}\underset{\sim}{b} \times \nabla \underset{\sim}{b}$$

$$- [\nabla \cdot \underset{\sim}{b} + 2(\underset{\sim}{b} \cdot \nabla \alpha)/\alpha] \nabla \underset{\sim}{b}\},$$

and

$$\underset{\sim}{T}_2 \equiv (4\underset{\sim}{q}_\perp^\parallel - \underset{\sim}{q}_\perp^\perp)\underset{\sim}{b} \cdot \nabla \underset{\sim}{b} + \nabla_\perp \underset{\sim}{q}_\perp^\perp .$$

Some attempts (e.g., references 7 and 18) to extend the hydromagnetic equations to include the "finite gyro radius" effects important for FGR ordering have omitted $\underset{\sim}{\Pi}_2^{(2)}$ because it was not recognized that the collisionless viscosity terms were actually of second order in this case. Although it is now well understood that the second-order terms are required,[6] we are aware of no general derivation of $\underset{\sim}{\Pi}^{(2)}$. Rosenbluth and Simon[6] find those terms of $\underset{\sim}{\Pi}^{(2)}$ necessary in the "flute approximation"; their equations are reviewed in Chapter VII.

Because $\underset{\sim}{\Pi}^{(2)}$ is rather complicated in general, we restrict further discussion to cases for which closed moment equations are obtained. Before discussing these cases, in Chapters VII through X, we consider in the remainder of this chapter the single-fluid formulation of the moment equations for FGR ordering.

6.5 Single-Fluid Equations for FGR Ordering.

For FGR ordering the lowest-order approximation to the single-fluid momentum equation is obtained by use of the pressure tensor through second order in Eq. (4.36). The electrostatic force (and displacement current) will be negligible unless $\beta \sim \epsilon^{-1}(v_0/c)^2$. That case will be considered briefly in Section 7.2; except in that section these effects will be ignored.

The momentum equation (4.36) is then

$$\rho_t^{(0)} D\underset{\sim}{V}^{(1)}/Dt = -\nabla\cdot\underset{\sim}{P}_t + \underset{\sim}{j}\times\underset{\sim}{B} + \rho_t^{(0)}\underset{\sim}{g}^{(2)} + 0(\epsilon^3\rho v_0^2/L), \quad (6.21)$$

and, if electron inertia is neglected, the total pressure is

$$\underset{\sim}{P}_t = P_\perp \underset{\sim}{I}_\perp + P_\parallel \underset{\sim}{b}\underset{\sim}{b} + \underset{\sim}{\Pi}_+^{(2)} . \quad (6.22)$$

Only the first-order fluid velocity $\underset{\sim}{V}^{(1)}$ appears in Eq. (6.21), whereas $\underset{\sim}{j}_\perp$ appears through third order and $p_\perp$ and $\underset{\sim}{B}$ through second order.

If electron inertia is neglected, the generalized Ohm's law is given by Eq. (4. 42). In the present case terms of order $\epsilon^3 v_0 B_0/c$ must be included in this equation, corresponding to the evaluation of $\underset{\sim}{B}$ through second order by Eqs. (4. 45) and (4. 46). The fluid velocity $\underset{\sim}{V}$ therefore appears in the Ohm's law through third order, rather than only in first order as in Eq. (6. 21). For FGR ordering, therefore, a straightforward application of the single-fluid equations is not possible in general.

The lowest-order terms of the Ohm's law (4. 42) may be used to evaluate $\underset{\sim}{B}^{(0)}$; these terms are

$$\underset{\sim}{E}^{(1)} + \underset{\sim}{V}^{(1)} \times \underset{\sim}{B}^{(0)} = (ne)^{-1} \nabla p_\perp^+ \cdot \qquad (6.23)$$

We will show in the next section that for the special case of constant magnetic field direction the condition (6. 3), necessary for FGR ordering to apply, may be used to obtain equations in which $B^{(1)}$ and $B^{(2)}$, as well as $P_\perp^{(1)}$ and $P_\perp^{(2)}$, do not appear; in this way a set of simple single-fluid equations will be obtained. Simple single-fluid equations are also obtained for the Alfven mode discussed in Chapter IX.

6. 5 Magnetic Field of Constant Direction.

All the above equations are very much simplified if $\underset{\sim}{b}$ is constant in space and time. These simplifications correspond to the vanishing of the "curvature" drift in Eq. (6. 11) and terms in Eq. (6. 9) arising from variation of the planes of the particle orbits (which in general are not transverse to $\underset{\sim}{b}$).

Furthermore, the transverse and parallel plasma motions separate if there are no gradients in the direction of the magnetic field; only this transverse motion will be considered here. For this case Π_{12} and Π_{13} are not required and the transverse components of $\underset{\sim}{\Pi}$ reduce to

$$\underset{\sim}{\Pi}^{(2)} = (p_{\perp}^{(0)}/2\Omega)(\underset{\sim}{I}_{\gamma}\underset{\sim}{I}_{\delta} - \underset{\sim}{I}_{\delta}\underset{\sim}{I}_{\gamma}):\nabla\underset{\sim}{V}^{(1)} + (2\Omega)^{-1}(\underset{\sim}{I}_{\gamma}\underset{\sim}{I}_{\gamma} + \underset{\sim}{I}_{\delta}\underset{\sim}{I}_{\delta}):\nabla\underset{\sim}{q}^{*} \qquad (6.24)$$

or

$$\underset{\sim}{\Pi}^{(2)} = (p_{\perp}^{(0)}/2\Omega)\underset{\sim}{b}\times[(\nabla\underset{\sim}{V})^{s} - \underset{\sim}{I}_{\perp}\nabla_{\perp}\cdot\underset{\sim}{V}]^{(1)} + (2\Omega)^{-1}[(\nabla\underset{\sim}{q}^{*})^{s} - \underset{\sim}{I}_{\perp}\nabla_{\perp}\cdot\underset{\sim}{q}^{*}], \qquad (6.25)$$

where the contributions of $\underset{\sim}{\Pi}_{2}^{(2)}$ are represented by

$$\underset{\sim}{q}^{*} \equiv \Omega^{-1}[(p_{\perp}\nabla p_{\perp})/\rho - (\nabla R_{3})/4]^{(0)}.$$

Note that

$$\underset{\sim}{q}^{*} = \Omega^{-1}\{\nabla[(p_{\perp}^{2}/\rho) - R_{3}/4] - p_{\perp}\nabla(p_{\perp}/\rho)\}. \qquad (6.26)$$

The factor $[(p_{\perp}^{2}/\rho) - R_{3}/4]$ vanishes if the distribution of $\nu_{\perp}$ is Gaussian; thus $\underset{\sim}{q}^{*}$ vanishes if the distribution is Gaussian and the temperature constant.

We may put $\underset{\sim}{\Pi}^{(2)}$ in a more interesting form by noting that, for the present case,

$$(\underset{\sim}{u} - \underset{\sim}{u}_{E})^{(1)} \equiv \underset{\sim}{u}_{p} = (\rho\Omega)^{-1}\underset{\sim}{b}\times\nabla p_{\perp}^{(0)}. \qquad (6.27)$$

The contribution of the $\nabla p_\perp$ terms of $\underset{\sim}{q}^*$ to $\underset{\sim}{\Pi}^{(2)}$ is therefore

$$(2\Omega)^{-1}(\underset{\sim}{I}_\gamma\underset{\sim}{I}_\gamma + \underset{\sim}{I}_\delta\underset{\sim}{I}_\delta):[p_\perp(\nabla u_p) \times \underset{\sim}{b} + (\underset{\sim}{u}_p \times \underset{\sim}{b})\nabla p]$$
$$= (2\Omega)^{-1}[\underset{\sim}{I}_\gamma\underset{\sim}{I}_\delta - \underset{\sim}{I}_\delta\underset{\sim}{I}_\gamma]:\nabla\underset{\sim}{u}_p + \rho[\underset{\sim}{u}_p\underset{\sim}{u}_p - \tfrac{1}{2}\underset{\sim}{I}_\perp(\underset{\sim}{u}_p)^2] . \tag{6.28}$$

The pressure tensor thus takes the form

$$\underset{\sim}{P} = \underset{\sim}{P}_E + \rho\underset{\sim}{u}_p\underset{\sim}{u}_p \tag{6.29}$$

where, from the form of Eq. (6.29),

$$\underset{\sim}{P}_E = [p_\perp - \tfrac{1}{2}\rho(\underset{\sim}{u}_p)^2]\underset{\sim}{I}_\perp + (p_\perp^{(0)}/2\Omega)(\underset{\sim}{I}_\gamma\underset{\sim}{I}_\delta - \underset{\sim}{I}_\delta\underset{\sim}{I}_\gamma):\nabla\underset{\sim}{u}_E^{(1)}$$
$$+ (8\Omega)^{-1}(\underset{\sim}{I}_\gamma\underset{\sim}{I}_\gamma + \underset{\sim}{I}_\delta\underset{\sim}{I}_\delta):\nabla(\Omega^{-1}\nabla R_3)^{(0)} \tag{6.30}$$

may be recognized as the pressure referred to the velocity $\underset{\sim}{u}_E$ rather than $\underset{\sim}{V} = \underset{\sim}{u}_E + \underset{\sim}{u}_p$.

In the $\underset{\sim}{u}_E$ reference frame, therefore, the pressure tensor is given by the usual scalar term $(m/2)\langle\langle(\underset{\sim}{v}_\perp - \underset{\sim}{u}_E)^2 f\rangle\underset{\sim}{I}_\perp$, the ordinary collisionless viscosity terms which, as shown by Kaufman,[2] are related to electric field gradients, and an additional second-order term depending on gradients of the magnetic field strength and the fourth moment R_3.

Returning to the $\underset{\sim}{\Pi}^{(2)}$ of Eqs. (6.24) and (6.25), we see that $\underset{\sim}{\Pi}_1^{(2)}$ and the contribution of $\nabla p_\perp^{(0)}$ to $\underset{\sim}{q}^*$ together represent the effects of the electric field on the particle orbits and the difference between the flow velocity and electric field drift, while the remaining term of $\underset{\sim}{q}^*$ arises from modifications of the particle orbits produced by magnetic field gradients, and from stresses associated with the second derivatives of the distribution function.

For $\underset{\sim}{b}$ constant and $\underset{\sim}{b}\cdot\nabla = 0$, the curl of the momentum equation (6. 21) is in the direction $\underset{\sim}{b}$ and of magnitude

$$\underset{\sim}{b}\cdot\nabla\times(\rho D\underset{\sim}{V}^{(1)}/Dt) = -\underset{\sim}{b}\cdot\nabla\times(\nabla\cdot\underset{\sim}{\Pi}^{(2)}) + \underset{\sim}{b}\cdot\nabla\times(\rho\underset{\sim}{g}), \qquad (6.31)$$

where $\underset{\sim}{\Pi}^{(2)}$ is given by Eq. (6. 24) or (6. 25). Equation (6. 31) may be regarded as one equation to determine the two components of $\underset{\sim}{V}_\perp$ in terms of $B^{(0)}$, $p_\perp^{(0)}$, and $R_3^{(0)}$. Note that for FGR ordering all terms of this equation are of the same order.

The quantities $B^{(0)}$ and $p_\perp^{(0)}$ are determined by Eq. (6. 23), in the form of (4. 45), and Eq. (6. 12); these equations become

$$DB^{(0)}/Dt + B^{(0)}\nabla_\perp\cdot\underset{\sim}{V}_\perp^{(1)} = [(B/\rho^2\Omega)\underset{\sim}{b}\cdot(\nabla\rho\times\nabla p_\perp^+)]^{(0)} \qquad (6.32)$$

and

$$Dp_\perp^+/Dt + 2p_\perp^+\nabla_\perp\cdot\underset{\sim}{V}^{(1)} + \nabla\cdot\underset{\sim}{q}_\perp^{\perp +} = 0. \qquad (6.33)$$

If, for the moment, we assume that $R_3^{(0)}$ may be evaluated in terms of $B^{(0)}$, $p_\perp^{(0)}$, and $\underset{\sim}{V}_\perp$, Eqs. (6. 31) through (6. 33), plus one more expression relating $\underset{\sim}{V}_\perp$ to these quantities, will form a closed hydromagnetic description of the transverse plasma motion.

Another equation for $\underset{\sim}{V}_\perp$ is obtained by taking the divergence of the second-order terms of the momentum equation (6.21), but this equation involves the first- and second-order corrections to B and $P_\perp$. However, if it is assumed that temporal and spatial variations do not occur independently, a relation of the desired form may be obtained by use of the zero-order terms of Eq. (6.21), i.e., the condition for FGR ordering (6.3),

$$\nabla_\perp [p_\perp^+ + p_\perp^- + B^2/2]^{(0)} = 0$$

or

$$[p_\perp^+ + p_\perp^- + B^2/2]^{(0)} = \text{const.} \tag{6.34}$$

The desired equation for $\underset{\sim}{V}_\perp$ is obtained from Eq. (6.34), the lowest-order ion and electron pressure equations, and Maxwell's induction equation, and is most easily obtained by expressing these equations in terms of $\underset{\sim}{u}_E$. By use of the relations

$$\underset{\sim}{u}_p^{\pm} \cdot \nabla p_\perp^{\pm} = 0$$

and

$$\nabla \cdot \underset{\sim}{q}^{\pm} = (2\Omega)^{-1} \underset{\sim}{b} \cdot (\nabla B \times \nabla R_3^{\pm})^{(0)} - 2p_\perp^{\pm} \nabla \cdot \underset{\sim}{u}_\rho^{\pm},$$

the ion pressure equation (6.33) and the corresponding electron pressure equation give

$$(\partial/\partial t + \underset{\sim}{u}_E^{(1)}\cdot\nabla)(p_\perp^+ + p_\perp^-)^{(0)} + 2(p_\perp^+ + p_\perp^-)^{(0)}\nabla_\perp\cdot\underset{\sim}{u}_E^{(1)} = \Omega_+^{-1}\underset{\sim}{b}\cdot(\nabla B \times \nabla R_3^*)^{(0)}, \qquad (6.35)$$

where R_3^* is defined following Eq. (5.24); Maxwell's induction equation is simply

$$(\partial/\partial t + \underset{\sim}{u}_E^{(1)}\cdot\nabla)B^{(0)} + B^{(0)}\nabla_\perp\cdot\underset{\sim}{u}_E^{(1)} = 0. \qquad (6.36)$$

Combining (6.34), (6.35), and (6.36) gives the desired result

$$\nabla\cdot\underset{\sim}{V}_\perp^{(1)} = \nabla\cdot(\underset{\sim}{u}_E + \underset{\sim}{u}_p)^{(1)} = \frac{\underset{\sim}{b}\cdot[(\nabla B)\times\nabla R_3^*]}{\Omega_+ B(B^2 + 2P_\perp)} + \frac{\underset{\sim}{b}}{\rho\,\Omega_+}\cdot\left[\left(\frac{\nabla\rho}{\rho} + \frac{\nabla B}{B}\right)\times\nabla p_\perp^+\right]. \qquad (6.37)$$

Equations (6.31), (6.32), (6.33), (6.37), the lowest-order electron pressure equation, and the continuity equation for ρ describe the transverse motion of the plasma if $R_3^\pm$ can be evaluated, or if the terms involving $R_3^\pm$ vanish for some special reason.

Closed equations are obtained in the cases discussed in Section 6.2: for low β, $\nabla B^{(0)} = 0$ and $R_3^\pm$ no longer enters the equations; this case is discussed in the following chapter. Even for finite β only the equilibrium value of $R_3^{\pm}$ appears in the equations if they are linearized about an equilibrium for which B is constant. Closed equations are also obtained for motion of the interchange type, even for finite β; these interchange motions are discussed in Chapter VIII.

CHAPTER VII

The Low-Frequency Flute Approximation

7.1 The Flute Approximation.

In this chapter we discuss slow, low-frequency, transverse motion of a low β plasma in a uniform magnetic field. For this case the moment equations are closed; their single-fluid form is obtained by setting ∇B equal to zero in the equations of Section 6.6.

These transverse motions are of particular interest in the study of stability, and especially the stability of equilibrium configurations that depend on only one coordinate. The assumptions of low β, uniform magnetic field, transverse motion, and one-dimensional equilibrium together have been called the "flute approximation."[6] In this approximation all effects of line curvature are simulated by an artificial gravity.

For low β the plasma current is unimportant, and the plasma dynamics may be formulated in terms of the density of each component and Poisson's equation. Rosenbluth and Simon[6] have obtained a formulation of this type for the flute approximation and FGR ordering from moments of the expansion of the Vlasov equation outlined in Chapter II. Their equations are of remarkably simple form, especially when compared to the single-fluid equations used by Roberts and Taylor[7] to find the FGR corrections to the gravitational interchange instability. Rosenbluth and Simon show that their formulation is equivalent to single-fluid equations of the usual form, and obtain all components of the pressure tensor required in their approximation. Conversely, the Rosenbluth-Simon equations may be obtained from our expansion of the moment equations; this derivation is outlined in Section 7. 3.

The single-fluid equations for the flute approximation and FGR ordering are a special case of those of Section 6. 6. In the following section of this chapter we show that they also may be put in form which closely resembles, and in some respects is even simpler than, the two-fluid Rosenbluth-Simon equations. Effects of charge separation are easily included in these single-fluid equations.

The main content of reference 6 is a discussion of variational methods for determining stability from the eigenvalue equation obtained by linearizing the basic equations for the flute approximation and FGR ordering. In Section 7. 3 this eigenvalue equation is derived from the simple form of the single-fluid equations obtained in Section 7. 2. In Chapter VIII a very similar eigenvalue equation is obtained for certain finite-β interchange modes.

The moment equations are closed for low β, magnetic field of constant direction, and transverse plasma motion because then $\nabla B = 0$ and the first-order drift velocity of Eq. (6. 11) reduces to $\underset{\sim}{u}_E^{(1)}$; thus the plasma behaves in many ways as an ideal fluid. For FGR ordering, however, the lowest-order plasma velocity $u_\perp^{(1)}$ may differ from $u_E^{(1)}$ because of pressure gradients; the FGR effects result from this difference.

The condition for the ∇B drift to be negligible in first order, and that the terms which introduce R_3 into the single-fluid equations of Section 6.6 be unimportant, is just that ∇B be of first-order ($\nabla B^{(0)} = 0$). This implies that β be of order ϵ, as may be seen from the dimensionless form of Eq. (6.34), which is

$$\beta \nabla' P_\perp' + \tfrac{1}{2}\nabla (B')^2 \doteq 0(\epsilon^2 n_0 m v_0^2), \tag{7.1}$$

or from the dimensionless Maxwell's equation (2.7). The "low β" conditions appropriate for the flute approximation are thus

$$\beta \sim \epsilon, \qquad B^{(0)} = \text{const.}, \qquad \nabla \times E^{(1)} = \nabla \cdot u_E^{(1)} = 0. \tag{7.2}$$

The first-order terms of Eq. (7.1) relate $\nabla B^{(1)}$ to $\nabla P_\perp^{(0)}$, but this relation is not needed since $B^{(1)}$ is not required. By use of the conditions (7.2), the Ohm's law (6.23) yields

$$\nabla \cdot V^{(1)} = (\rho^2 \Omega)^{-1} \hat{b} \cdot (\nabla \rho \times \nabla p_\perp^+) . \tag{7.3}$$

This result is also obtained by setting $\nabla B^{(0)}$ equal to zero in Eq. (6.37).

7.2 The Single-Fluid Equations.

The single-fluid momentum equation appropriate for the flute approximation and FGR ordering is obtained by taking B constant in the evaluation of $\underset{\sim}{\Pi}^{(2)}$ and its divergence from Eq. (6.24) or (6.25). The divergence of $\underset{\sim}{\Pi}_1^{(2)}$ is then given, for example, by an expression of the form (5.26) with Ω constant. Another expression for $\nabla \cdot \underset{\sim}{\Pi}_1^{(2)}$, which considerably simplifies the momentum equation, will be discussed below.

Evaluation of $\nabla \cdot \underset{\sim}{\Pi}_2^{(2)}$ is straightforward for the present case. From the definition of $\underset{\sim}{q}^*$ following Eq. (6.25) we find

$$\underset{\sim}{b} \cdot \nabla \times \underset{\sim}{q}^* = - p_\perp^+ \nabla \cdot \underset{\sim}{u}_p^+ = - p_\perp^+ \nabla \cdot \underset{\sim}{V} .$$

By use of this result, Eq. (6.25), and the identity, valid for arbitrary α and $\underset{\sim}{q}$, that

$$\nabla \cdot [\alpha (\nabla \underset{\sim}{q})^S] = 2\nabla \cdot [\alpha (\nabla \underset{\sim}{q})^T] - \nabla \times (\alpha \nabla \times \underset{\sim}{q}), \qquad (7.4)$$

where the superscript T indicates the transpose, we obtain

$$\begin{aligned} \nabla \cdot \underset{\sim}{\Pi}_2^{(2)} &= (2\Omega)^{-1} [\nabla \nabla \cdot \underset{\sim}{q}^* - \nabla \times (\nabla \times \underset{\sim}{q}^*)] \\ &= (2\Omega)^{-1} \{\nabla \nabla \cdot \underset{\sim}{q}^* - [\nabla (\nabla \times \underset{\sim}{q}^*) \cdot \underset{\sim}{b}] \times \underset{\sim}{b}\} \qquad (7.5) \\ &= (2\Omega)^{-1} [\nabla \nabla \cdot \underset{\sim}{q}^* - \underset{\sim}{b} \times \nabla p_\perp^+ \nabla \cdot \underset{\sim}{V}] . \end{aligned}$$

For the flute approximation, then, the second-order terms of the single-fluid momentum equation (6.21) are

$$\rho \frac{D\underset{\sim}{V}}{Dt} + \underset{\sim}{\lambda} - \frac{\nabla \cdot \underset{\sim}{V}}{2\Omega} \underset{\sim}{b} \times \nabla p_{\perp}^{+} - \rho \underset{\sim}{g} = - \nabla \psi + 0\left(\frac{\epsilon^3 \rho v_0^2}{L}\right), \qquad (7.6)$$

where

$$\psi = p_{\perp} + \tfrac{1}{2}B^2 + (2\Omega)^{-1} \nabla \cdot \underset{\sim}{q}^{*}$$

and

$$\underset{\sim}{\lambda} = \nabla \cdot \{(p_{\perp}^{+}/2\Omega)\underset{\sim}{b} \times [(\nabla \underset{\sim}{V})^{s} - \underset{\sim}{I}_{\perp} \nabla \cdot \underset{\sim}{V}]\}.$$

The electrostatic force and displacement current have been neglected in Eq. (7.6). This is a good approximation unless $(1/\beta)(v_0/c)^2 \sim 1$. Modification of the single-fluid equations to include these effects will be discussed below.

Equation (7.6) is identical with the single-fluid equation (A.15) of reference 6 obtained by Rosenbluth and Simon, except that their unknown function ψ is now determined. Since ψ involves $p_{\perp}$ and B through second order, and also $R_3^{(0)}$, it is clear that only the curl of Eq. (7.6) is useful. The additional equation required for $\underset{\sim}{V}$ is just (7.3).

The set of single-fluid equations is completed by the continuity equation

$$D\rho^{(0)}/Dt + \rho^{(0)} \nabla \cdot \underset{\sim}{V}^{(1)} = 0 \qquad (7.7)$$

and the pressure equation

$$Dp_\perp^{+(0)}/Dt + 2p_\perp^{+(0)}\nabla\cdot V^{(1)} - 2p_\perp^{+}(\rho^2\Omega)^{-1}b\cdot[\nabla\rho \times \nabla p_\perp^{+}]^{(0)} = 0. \quad (7.8)$$

Equations (7.3), (7.7), (7.8), and the curl of Eq. (7.6) are the lowest-order hydromagnetic equations appropriate to the flute approximation, FGR ordering, and $(1/\beta)(v_0/c)^2 \ll 1$. They are identical with the single-fluid equations obtained by Rosenbluth and Simon for this case. The FGR effects appear in the terms arising from $\Pi^{(2)}$, the non-zero divergence of V, and the last term of the pressure equation.

These equations differ from those used by Roberts and Taylor[7] to discuss FGR modifications of the gravitational interchange instability at low β only by the term of Eq. (7.6) arising from $\Pi_2^{(2)}$. For interchange modes, however, $\nabla\rho^{(0)}$ and $\nabla p_\perp^{(0)}$ remain parallel, $\nabla\cdot V$ vanishes, and this additional term makes no contribution. Instead of using a pressure equation, Roberts and Taylor assumed the equilibrium temperature to be constant and neglected perturbations in the temperature. This is consistent with Eq. (7.8). Roberts and Taylor also included the acceleration and collisionless viscosity terms in the generalized Ohm's law. They found that these terms had a negligible effect, in agreement with Eq. (6.23).

The momentum equation (7.6) may be put in a simpler form by writing λ in the form, obtained from the expression following Eq. (7.6),

$$\begin{aligned}
\underset{\sim}{\lambda} &= + \nabla \cdot \{ (p^{+}/2\Omega) [(\nabla \underset{\sim}{V})^{s} - \underset{\sim}{I}_{\perp} \nabla \cdot \underset{\sim}{V}] \times \underset{\sim}{b} \} \\
&= + \underset{\sim}{b} \times \{ \nabla \cdot [(p_{\perp}^{+}/2\Omega) (\nabla \underset{\sim}{V})^{s}] - \nabla [p_{\perp}^{+}/2\Omega) \nabla \cdot \underset{\sim}{V}] \} \\
&= + \underset{\sim}{b} \times \{ (\nabla \underset{\sim}{V}) \cdot \nabla (p_{\perp}^{+}/\Omega) + (p_{\perp}^{+}/2\Omega) \nabla (\nabla \cdot \underset{\sim}{V}) \\
&\qquad - (\nabla \cdot \underset{\sim}{V}) \nabla (p_{\perp}^{+}/2\Omega) \} - \nabla [(p_{\perp}^{+}/2\Omega) (\nabla \times \underset{\sim}{V}) \cdot \underset{\sim}{b}] ,
\end{aligned} \tag{7.9}$$

where use has been made of the identity (7.4). From Eqs. (7.5) and (7.9) we obtain

$$\nabla \cdot \underset{\sim}{P}_{t} = \nabla \psi' + \Omega^{-1} \underset{\sim}{b} \times [(\nabla \underset{\sim}{V}) \cdot \nabla (p_{\perp}^{+}/\Omega) - (\nabla \cdot \underset{\sim}{V}) \nabla (p_{\perp}^{+}/\Omega)] , \tag{7.10}$$

and

$$\rho [D\underset{\sim}{V}/Dt - \underset{\sim}{g}] + \Omega^{-1} \underset{\sim}{b} \times [(\nabla \underset{\sim}{V}) \cdot \nabla (p_{\perp}^{+}/\Omega) - (\nabla \cdot \underset{\sim}{V}) \nabla (p_{\perp}^{+}/\Omega)] = - \nabla \psi' , \tag{7.11}$$

with

$$\psi' = \psi - (p_{\perp}^{+}/2\Omega) (\nabla \times \underset{\sim}{V}) \cdot \underset{\sim}{b} .$$

Equation (7.11), and especially its curl, is considerably simpler than Eq. (7.6) with the usual expression (5.26) for $\underset{\sim}{\lambda}$.

For the special assumptions of the flute approximation and FGR ordering, the transverse drift velocity of every particle is $\underset{\sim}{u}_{E}^{(1)}$. For this reason the continuity and pressure equations take their simplest forms when expressed in terms of $\underset{\sim}{u}_{E}$ rather than $\underset{\sim}{V}$. By use of Eq. (7.3), Eqs. (7.7) and (7.8) reduce to the simple forms

$$(\partial/\partial t + \underset{\sim}{u}_E^{(1)} \cdot \nabla)\,\rho^{(0)} = (\partial/\partial t + \underset{\sim}{u}_E^{(1)} \cdot \nabla)\, p_\perp^{+(0)} = 0. \tag{7.12}$$

Rosenbluth and Simon[6] point out that the single-fluid equation of motion (7.6) or (7.9) is also simplified by transforming from the variable $\underset{\sim}{V}$ to $\underset{\sim}{u}_E$. This may be accomplished by use of the relation

$$D(\underset{\sim}{u}_\perp - \underset{\sim}{u}_E)^{(1)}/Dt = D\underset{\sim}{u}_p/Dt = -(\rho\Omega)^{-1}\underset{\sim}{b} \times [(\nabla\underset{\sim}{V})\cdot\nabla p_\perp - (\nabla p_\perp)\nabla\cdot\underset{\sim}{V}], \tag{7.13}$$

obtained from Eqs. (6.27), (7.7), and (7.8), and the vector identity

$$\underset{\sim}{u}_p \cdot \nabla\underset{\sim}{u}_E = -(\rho\Omega)^{-1}\underset{\sim}{b} \times [(\nabla\underset{\sim}{u}_E)\cdot\nabla p_\perp^+ - (\nabla p_\perp^+)\nabla\cdot\underset{\sim}{u}_E]. \tag{7.14}$$

By use of these identities and the low β condition $\nabla\cdot\underset{\sim}{u}_E = 0$, Eq. (7.11) becomes

$$\rho(\partial/\partial t + \underset{\sim}{u}_E\cdot\nabla)\underset{\sim}{u}_E - \rho\underset{\sim}{g} - \Omega^{-1}\underset{\sim}{b} \times [(\nabla\underset{\sim}{u}_E)\cdot\nabla p_\perp^+] = -\nabla\psi'. \tag{7.15}$$

This equation is identical in form to the original momentum equation (7.11) except that the terms arising from $\nabla\cdot\underset{\sim}{\Pi}^{(2)}$ and not contributing to $\nabla\psi'$ are changed in sign. They are also simplified by the condition $\nabla\cdot\underset{\sim}{u}_E = 0$. These properties of the transformed equation were pointed out by Rosenbluth and Simon.

The curl of Eq. (7.15) may be written in the simple form, by use of (7.14),

$$\underset{\sim}{b}\cdot\nabla\times\rho[D\underset{\sim}{u}_E/Dt-\underset{\sim}{g}]=0, \tag{7.16}$$

where

$$D/Dt=\partial/\partial t+\underset{\sim}{u}_E\cdot\nabla+\underset{\sim}{u}_p^+\cdot\nabla.$$

In the derivation of Eq. (7.16) the electrostatic force and displacement current were neglected. If $\beta\sim(v_0/c)^2$, however, these effects become important; they are then easily included in Eq. (7.16).

The total electromagnetic force on the plasma is

$$\underset{\sim}{j}\times\underset{\sim}{B}+\rho_c\underset{\sim}{E}=(\nabla\times\underset{\sim}{B})\times\underset{\sim}{B}-(\partial\underset{\sim}{E}/\partial t)\times\underset{\sim}{B}/c+\underset{\sim}{E}\nabla\cdot\underset{\sim}{E}, \tag{7.17}$$

the last two terms having been neglected in the above discussion. This force is required through order $\epsilon^2 v_0^2/L$. Since

$$(1/c)\,\partial\underset{\sim}{E}/\partial t\sim(\nabla\cdot\underset{\sim}{E})\underset{\sim}{E}\sim(\epsilon^2/\beta)(v_0/c)^2,$$

and we assume $(1/\beta)(v_0/c)^2$ is of order unity or smaller, only $\underset{\sim}{B}^{(0)}$ and $\underset{\sim}{E}^{(1)}$ are required in these terms. In these terms, then, $\underset{\sim}{E}$ may be written $(B^{(0)}/c)\underset{\sim}{b}\times\underset{\sim}{u}_E^{(1)}$ with B and $\underset{\sim}{b}$ constant. We then obtain the identity

$$\underset{\sim}{E}^{(1)}\nabla\cdot\underset{\sim}{E}^{(1)}+(B/c)^2\underset{\sim}{u}_E^{(1)}\cdot\nabla\underset{\sim}{u}_E^{(1)}=\tfrac{1}{2}\nabla(\underset{\sim}{I}\gamma:\underset{\sim}{E}\,\underset{\sim}{E}). \tag{7.18}$$

The total electromagnetic stress thus takes the form

$$\underset{\sim}{j} \times B + \rho_c \underset{\sim}{E} = \tfrac{1}{2}\nabla[\underset{\sim}{I}_\gamma : \underset{\sim}{E}\underset{\sim}{E} - B^2] - (B/c)^2(\partial/\partial t + \underset{\sim}{u}_E^{(1)} \cdot \nabla)\underset{\sim}{u}_E^{(1)} + 0(\epsilon^3 v_0^2/L)\,, \tag{7.19}$$

and the momentum equation (7.16) becomes.

$$\underset{\sim}{b} \cdot \nabla \times \{(\rho + B^2/c^2)^{(0)}(\partial/\partial t + \underset{\sim}{u}_E^{(1)} \cdot \nabla)\underset{\sim}{u}_E^{(1)} + (\Omega^{-1}\underset{\sim}{b} \times \nabla p_\perp^+)^{(0)} \cdot \nabla \underset{\sim}{u}_E^{(1)} - \rho^{(0)}\underset{\sim}{g}^{(2)}\} = 0. \tag{7.20}$$

This equation, together with the continuity and pressure equations (7.12), and the condition $\nabla \cdot \underset{\sim}{u}_E^{(1)} = 0$, are the single-fluid hydromagnetic equations for the flute approximation and FGR ordering expressed in terms of the guiding center velocity $\underset{\sim}{u}_E^{(1)}$. These equations are considerably less complicated than those of Roberts and Taylor,[7] but they are also more general because the effects of temperature variation, charge separation, and displacement current have been included. They are completely equivalent to the two-fluid Rosenbluth-Simon equations which are discussed in the next section.

Only the terms $\rho[(\partial/\partial t + \underset{\sim}{u}_E \cdot \nabla)\underset{\sim}{u}_E - \underset{\sim}{g}]$ of Eq. (7.20) would have been obtained if CGL ordering had been assumed and the electrostatic force and displacement current ignored. All FGR effects are included in the term $(\Omega^{-1}\underset{\sim}{b} \times \nabla p_{\perp}^{+})^{(0)} \cdot \nabla \underset{\sim}{u}_E^{(1)}$. That these effects arise from the difference between the plasma and guiding center velocities is made explicit by the form of this term. Electrostatic effects produce the terms proportional to $(B/c)^2$. Thus, for the flute approximation, these effects simply add the magnetic mass density ($(B/c)^2$ in our units) to the plasma inertia; this result, previously obtained by Northrop,[31] holds for both FGR and CGL ordering.

7.3 The Rosenbluth-Simon Equations.

Rosenbluth and Simon[6] actually use a two-fluid formulation of the moment equations for the flute approximation and FGR ordering which consists of equations giving the density of each component through second order and Poisson's equation for $\underset{\sim}{E}$. They obtain their equation for the density from moments of the expansion of the Vlasov equation outlined in Chapter II.

Their equations for the density may also be obtained from the continuity equation (4.1) by using (4.3) to obtain $\underset{\sim}{u}_{\perp}$ through third order. For FGR ordering and the flute approximation, we obtain from Eq. (4.3), by use of the expression (7.10) and the relation (7.13),

$$\rho \underset{\sim}{u}_{\perp} = \rho \underset{\sim}{u}_E + \Omega^{-1} \underset{\sim}{b} \times \{\nabla \psi' + \rho(D\underset{\sim}{u}_E/Dt - g)\} + 0(\epsilon^4 \rho v_0) . \tag{7.21}$$

Substitution of this result into the continuity equation (4.1) gives

$$(\partial/\partial t + \underset{\sim}{u}_E \cdot \nabla)\rho + \nabla \cdot [\Omega^{-1} \underset{\sim}{b} \times (D\underset{\sim}{u}_E/Dt - \underset{\sim}{g})] = 0(\epsilon^5 \rho\Omega) . \qquad (7.22)$$

Except for notation, Eq. (7.22) is identical to Eq. (A.1) of reference 6. This equation determines ρ through order $\epsilon^2 mn_0$. Equation (7.22) and the second of Eqs. (7.12) for each plasma component, together with Poisson's equation, are the basic equations of Rosenbluth and Simon. The FGR effects again appear in the term $\underset{\sim}{u}_p^+ \cdot \nabla$ of D/Dt.

If FGR effects are neglected in the equation for the electron density, as is done by Rosenbluth and Simon, only the ion pressure and ion pressure equation (7.12) are needed. In this approximation the Rosenbluth-Simon equations are equivalent to the single-fluid equations (7.12) and (7.20), but perhaps are somewhat more complicated.

The main content of reference 6 is a discussion of variational methods for determining stability from the eigenvalue equation that results from linearizing the basic equations for the flute approximation and FGR ordering. In reference 6 this eigenvalue equation is obtained from the two-fluid formalism just described, but it is stated that the same equation can be obtained from single-fluid equations of the usual form if all proper contributions to the stress tensor are included. Derivation of the eigenvalue equation from the single-fluid equations in the form (7.12) and (7.20) is very simple and will be outlined here.

Perturbations of the one-dimensional equilibrium are conveniently described by the Langrangian variable[32] $\underset{\sim}{\xi}$, which gives the displacement of the guiding centers from their equilibrium trajectories. Thus, through first order in $\underset{\sim}{\xi}$,

$$\underset{\sim}{u}_E = \underset{\sim}{\overline{u}}_E + (\partial/\partial t + \underset{\sim}{\overline{u}}_E \cdot \nabla)\xi, \qquad (7.22)$$

where $\underset{\sim}{\overline{u}}_E$ is the equilibrium value of $\underset{\sim}{u}_E$. Following reference 6 we assume that equilibrium quantities vary in the x-direction only, that $\underset{\sim}{\overline{u}}_E$ is in the y-direction, and that $\underset{\sim}{b}$ is in the z-direction, and we make a Fourier transform in the variables y and t. All perturbed quantities are then of the form $\alpha(x) \exp i(\omega t + ky)$, where α is proportional to $\underset{\sim}{\xi}$.

From Eqs. (7.12) the perturbations of ρ and $p_\perp^+$ are

$$\hat{\rho} = -\xi_x \overline{\rho}' \qquad \text{and} \qquad \hat{p} = -\xi_x \overline{p}',$$

where $\overline{\rho}'$ and $\overline{p}'$ are the equilibrium gradients of ρ and $p_\perp^+$. The condition $\nabla \cdot \underset{\sim}{u}_E = 0$ becomes

$$\nabla \cdot (\partial/\partial t + \underset{\sim}{u}_E \cdot \nabla)\underset{\sim}{\xi} = (\partial/\partial t + \underset{\sim}{u}_E \cdot \nabla)\nabla \cdot \underset{\sim}{\xi} + (\nabla \underset{\sim}{u}_E) : \nabla \underset{\sim}{\xi} = 0$$

or

$$ik\xi_y = -[\overline{u}_E' k \xi_x / \omega_1 + \xi_x'], \qquad (7.24)$$

where

$$\omega_1 = \omega + k\bar{u}_E \qquad \text{and} \qquad \underset{\sim}{\xi}' = \partial\underset{\sim}{\xi}/\partial x .$$

By use of these results the linearized momentum equation (7.20) becomes

$$\underset{\sim}{b}\cdot\nabla\times[(T - \bar{\rho}'g)\xi_x \underset{\sim}{e}_x - (T/ik)\xi_x' \underset{\sim}{e}_y] = 0, \tag{7.25}$$

where, as in reference 6,

$$T \equiv \rho\omega_1^2[1 + B^2/\rho c^2 + k\bar{p}'/\rho\omega_1\Omega] \tag{7.26}$$

or

$$(T\xi_x')' - k^2(T - \bar{\rho}'g)\xi_x = 0. \tag{7.27}$$

Equation (7.27) is identical with the Rosenbluth-Simon eigenvalue equation (3.12) of reference 6.

The equations found by Rosenbluth and Simon from an expansion of the Vlasov equation are thus easily obtained from an m/e expansion of the familiar conservation equations of mass, momentum, and energy. The "finite gyro-radius" terms of this expansion, expressed in Eqs. (7.6) and (7.8), are easily found from the exact moment equations if the special assumptions of this section are made at the outset. This procedure provides an alternate derivation to that of reference 6. This alternate derivation emphasizes the relation of the Rosenbluth-Simon equations to equations describing more general configurations. In particular we have shown that closed moment equations are obtained for FGR ordering and finite β only for rather special cases.

CHAPTER VIII

Finite β Interchange Modes

8.1 Finite β Interchange Motion.

In this chapter we obtain, by use of the lowest-order Vlasov equation (3.14), closed hydromagnetic equations for certain interchange motions of a finite β plasma. Except for allowing for finite pressure, we make the same assumptions as in the previous chapter; namely $\underset{\sim}{b}$ constant, no parallel gradients, transverse motion, one-dimensional equilibrium, and FGR ordering. These assumptions are a generalization of the low-frequency flute approximation to finite β, and the equations obtained are very similar to those of the previous chapter. The equations of this chapter, however, are restricted to a class of interchange motions for which the gradient-B drift is always orthogonal to the gradient of $f^{(0)}$, and the appropriate hydromagnetic equations (given in Section 6.6) are closed even for finite β.

To obtain the equations for these modes we note that, by use of the result (6.37), the induction equation (6.32) may be written in the form

$$[\partial/\partial t + (\underset{\sim}{u}_E + \underset{\sim}{u}_R)\cdot\nabla] B^{(0)} = 0, \tag{8.1}$$

where

$$\underset{\sim}{u}_R \equiv [2\Omega(B^2 + 2\underset{\sim}{P}_\perp)]^{-1}\underset{\sim}{b} \times \nabla R_3^* + (\rho\Omega)^{-1}\underset{\sim}{b} \times \nabla p_\perp^+ .$$

The zero-order fourth-moments R_3^* required in this equation may be evaluated by use of the lowest-order Vlasov equation (3.14). If functions $f_0(\underset{\sim}{x}, \mu, t)$ and $f_1(\underset{\sim}{x}, \mu, t)$ are defined by the equations

$$[\partial/\partial t + (\underset{\sim}{u}_E + \underset{\sim}{u}_R)\cdot\nabla] f_0(\underset{\sim}{x}, \mu, t) = 0 \tag{8.2}$$

and

$$f^{(0)}(\underset{\sim}{x}, c_\perp, t) = f_0(\underset{\sim}{x}, \mu, t) + f_1(\underset{\sim}{x}, \mu, t),$$

the Vlasov equation (3.14) may be written in the form

$$(\underset{\sim}{u}_B - \underset{\sim}{u}_R)\cdot\nabla f_0 + [\partial/\partial t + (\underset{\sim}{u}_E + \underset{\sim}{u}_B)\cdot\nabla] f_1 = 0. \tag{8.3}$$

We now assume that the gradients of $B^{(0)}$ and $f^{(0)}$ are parallel at time $t = 0$, and choose $f_1(t = 0) = 0$. Then, since Eqs. (8.1) and (8.2) are of the same form, ∇f_0 remains parallel to $\nabla B^{(0)}$ throughout the motion, $\underset{\sim}{u}_B \cdot \nabla f_0 = 0$, and only $f_1^\pm$ contribute to the component of $\underset{\sim}{u}_R$ parallel to ∇f_0.

The ion and electron forms of Eq. (8.3) then admit the solution $f_1^+ = f_1^- = 0$. These modes, for which $f_1^\pm = 0$, are characterized by the equations

$$(\partial/\partial t + \underset{\sim}{u}_E\cdot\nabla) f^{(0)} = (\partial/\partial t + \underset{\sim}{u}_E\cdot\nabla) B^{(0)} = 0; \qquad (8.4)$$

they are thus simple interchange motions since both the guiding centers and magnetic field may be regarded as moving with the velocity $\underset{\sim}{u}_E^{(1)}$.

8.2 Hydromagnetic Equations for the Interchange Motions.

For the interchange motions discussed in the previous section the hydromagnetic equations (6.31), (6.32), (6.33), and (6.37) become

$$\underset{\sim}{b}\cdot\nabla\times\{\rho D\underset{\sim}{V}/Dt + \underset{\sim}{\lambda} + \nabla\cdot\underset{\sim}{\Pi}_2^{(2)} - \rho\underset{\sim}{g}\} = 0, \qquad (8.5)$$

$$\nabla\cdot\underset{\sim}{V}^{(1)} = 0, \qquad (8.6)$$

and $$D\rho^{(0)}/Dt = DB^{(0)}/Dt = Dp_\perp^{+(0)}/Dt = DR_3^*/Dt = 0, \qquad (8.7)$$

where $\underset{\sim}{\lambda}$ is given by Eq. (7.9) and

$$\underset{\sim}{\Pi}_2^{(2)} = (2\Omega)^{-1}[(\nabla\underset{\sim}{q}^*)^s - \underset{\sim}{I}_\perp\nabla\cdot\underset{\sim}{q}^*]. \qquad (8.8)$$

As is the case for low β, Eq. (8.6) takes a simpler form when written in terms of the average drift velocity $\underset{\sim}{u}_D$. For finite β,

$$\underset{\sim}{u}_D = \underset{\sim}{u}_E + \underset{\sim}{u}_B, \qquad \underset{\sim}{u}_B \equiv (p^+/\rho\Omega B)\underset{\sim}{b}\times\nabla B. \qquad (8.9)$$

To transform Eq. (8.5) from $\underset{\sim}{u}$ to $\underset{\sim}{u}_D$ note that Eqs. (7.13) and (7.14) hold for the finite β interchange motions, and, from Eqs. (8.6), (8.7), and (7.14):

$$\nabla \cdot \underset{\sim}{u}_D = \nabla \cdot \underset{\sim}{u}_E = 0;$$

$$D'\rho = D'B = D'p_\perp^+ = D'R_3^* = 0; \qquad (8.10)$$

with

$$D' \equiv \partial/\partial t + \underset{\sim}{u}_D \cdot \nabla;$$

and

$$\underset{\sim}{u}_B \cdot \nabla \underset{\sim}{u}_E = -(p_\perp^+/\rho\Omega B)\underset{\sim}{b} \times (\nabla \underset{\sim}{u}_E) \cdot \nabla B. \qquad (8.11)$$

By use of these results Eq. (8.5) may be written

$$\underset{\sim}{b} \cdot \nabla \times \{\rho[D' + (\underset{\sim}{u} - \underset{\sim}{u}_D) \cdot \nabla]\underset{\sim}{u}_D + \nabla \cdot \underset{\sim}{\Pi}_2^{(2)} - \rho\underset{\sim}{g}\} = 0, \qquad (8.12)$$

with

$$\underset{\sim}{u} - \underset{\sim}{u}_D = \underset{\sim}{u}_p - \underset{\sim}{u}_B = \rho^{-1}\underset{\sim}{b} \times \nabla(p_\perp^+/\Omega).$$

Since $\underset{\sim}{u}_D = \underset{\sim}{u}_E$ and $\underset{\sim}{u} - \underset{\sim}{u}_D = \underset{\sim}{u}_p$ for low β, when written in terms of the average drift velocity, Eqs. (8.10) and (8.12) are of the same form, except for $\nabla \cdot \underset{\sim}{\Pi}_2^{(2)}$, as the corresponding low β Eqs. (7.12) and (7.16).

The finite β effects are seen to be the terms $\underset{\sim}{u}_B \cdot \nabla$ of D' and $\nabla \cdot \underset{\sim}{\Pi}_2^{(2)}$. The former is just the change in $\underset{\sim}{u}_D$ produced by the gradient-B drift. Since ∇B and $\nabla p_\perp^+$ are in opposite directions (unless $\nabla p_\perp^-$ is larger than and opposite to $\nabla p_\perp^+$) the gradient-B drifts increase the stabilizing effect of the low β FGR effects.

By a calculation similar to the derivation of Eq. (7.9) we find

$$\nabla\cdot\underset{\approx}{\Pi}_2^{(2)} = (\Omega B)^{-1}[(\nabla B)\nabla\cdot\underset{\sim}{q}^* - (\nabla\underset{\sim}{q}^*)\cdot\nabla B], \tag{8.13}$$

where $\underset{\sim}{q}^*$, given by Eq. (6.26), represents effects of temperature gradients and deviations of the distribution of $v_\perp$ from Gaussian form. For a constant-temperature Maxwellian plasma, $\underset{\sim}{q}^*$ vanishes and the FGR stabilization of the interchange mode is enhanced by finite β. For more general distributions, however, the finite β contributions from $\nabla\cdot\underset{\approx}{\Pi}_2^{(2)}$ do not vanish and may be either stabilizing or destabilizing.

We emphasize that the finite β equations discussed in this section apply only to the class of interchange motions for which $f_1^+ = f_1^- = 0$.

8.3 Eigenvalue Equation for Stability.

The eigenvalue equation for the finite β interchange modes corresponding to the low β equation (7.27) is easily found. To find the contribution of $\nabla\cdot\underset{\approx}{\Pi}_2^{(2)}$ to the linearized form of Eq. (8.13) we note that

$$\underset{\sim}{q}^* = \underset{\sim}{\overline{q}} - \nabla(\underset{\sim}{\overline{q}}\xi_x), \tag{8.14}$$

where $\underset{\sim}{\overline{q}}$ is the equilibrium value of $\underset{\sim}{q}^*$. From this result a straightforward calculation leads to

$$\nabla\cdot\underset{\approx}{\Pi}_2^{(2)} = (\overline{q}\,\overline{B}''k^2/\Omega B)[\xi_x\underset{\sim}{e}_x - (\xi_x'\underset{\sim}{e}_y)/ik]. \tag{8.15}$$

Linearization of the remaining terms of Eq. (8.13) leads to the result

$$(T_\beta \xi_x')' - k^2[T_\beta - \overline{\rho}' g]\xi_x = 0, \tag{8.16}$$

where

$$T_\beta = \rho\omega_1^2[1 + (k/\rho\omega_1)(\overline{p}_\perp^+/\overline{\Omega})' - (\overline{q}\,\overline{B}'k^2/\rho\Omega B\omega_1^2)]. \tag{8.17}$$

Equations (8.16) and (8.17) are the generalization of the low β Rosenbluth-Simon eigenvalue equation (7.27) to the finite β interchange modes.

The eigenvalue equation (8.16) may be written in the form

$$\omega^2(O_1\xi_x) + \omega(O_2\xi_x) + O_3\xi_x = 0, \tag{8.18}$$

where

$$O_1\xi = (\rho\xi')' - k^2\rho\xi, \tag{8.19}$$

$$O_2\xi = (k\alpha\xi')' - k^3\alpha\xi, \tag{8.20}$$

and

$$\begin{aligned} O_3 = \{&[k^2\overline{u}_E(\alpha - \overline{\rho}\overline{u}_E) - (\overline{q}'\overline{B}'/\Omega B)]\xi'\}' \\ &- k^2[u_E(\alpha - \rho u_E) - \overline{q}'\overline{B}'/\Omega B - g\overline{\rho}']\}, \end{aligned} \tag{8.21}$$

with

$$\alpha \equiv (\overline{p}/\Omega)' + 2\rho\overline{u}_E = \rho(\overline{u}_D - 2\overline{u}_E)\cdot e_y.$$

The usual sufficient conditions for stability[32, 6] follow from Eq. (8.18): $-C > 0$ or, more generally,

$$B^2 > 4AC, \tag{8.22}$$

with

$$A \equiv \int d^3x \xi_x^* O_1 \xi_x, \qquad B \equiv \int d^3x \xi_x^* O_2 \xi_x, \quad \text{and} \quad C \equiv \int d^3x \xi_x^* O_3 \xi_x .$$

These conditions are not strong enough to be very useful in general. However, just as in the low β case of reference 6, if $\alpha = 0$ the condition (8.22) becomes necessary and sufficient for stability, analogous to the well-known variational principle[33] for static equilibria and CGL ordering.

For $\alpha = 0$, the condition (8.22) may be written

$$\int d^3x \{[\rho(\overline{u}_E)^2 - \overline{q}'\overline{B}'/\Omega B k^2][(\xi')^2 + k^2\xi^2] + \overline{\rho}' g \xi^2\} > 0. \tag{8.23}$$

This is a generalization of the low β condition (4.16) of reference 6. For low β the second term of the first factor of Eq. (8.23) vanishes, and thus, for $\alpha = 0$ and low β, the net effect of FGR corrections and mass motion is stabilizing. For finite β this conclusion holds only for uniform equilibrium temperature and Gaussian distribution of $v_\perp$; under more general conditions the finite β term from $\nabla \cdot \underset{\sim}{\Pi}_2^{(2)}$ may be either stabilizing or destabilizing.

For a simple example of finite β FGR effects we consider the gravitational interchange instability, and make the usual approximation[6,7,14] that $\underset{\sim}{\xi}$ is independent of x. The eigenvalue equation (8.16) then reduces to

$$[T_\beta - \bar{\rho}g]\,\xi_x = 0, \tag{8.24}$$

which has eigenvalues

$$\omega = \frac{1}{2}\left\{ -\frac{k}{\bar{\rho}}\left(\frac{\bar{p}}{\Omega}\right)' \pm \left[\left[\frac{k}{\bar{\rho}}\left(\frac{\bar{p}}{\Omega}\right)'\right]^2 + \frac{4\bar{\rho}'g}{\bar{\rho}} - \frac{4\bar{q}'\bar{B}'k^2}{\bar{\rho}\Omega\bar{B}}\right]\right\}. \tag{8.25}$$

The condition for the stability of the finite β gravitational interchange is thus

$$\bar{\rho}'g + (k^2/4\rho)[(\bar{p}/\Omega)']^2 + (k^2\bar{B}''/\Omega\bar{B})\{\Omega^{-1}[R'_\Delta + \bar{p}(\bar{p}/\rho)']\}' \geq 0, \tag{8.26}$$

where

$$R_\Delta \equiv \bar{R}_3/4 - \bar{p}^2/\bar{\rho} \geq -3p^2/4\bar{\rho}$$

represents deviations of the distribution from Gaussian form. This result may also be obtained from the condition (8.23).

For an isothermal, Maxwellian plasma with equal ion and electron temperatures, this stability condition becomes

$$\bar{\rho}'g + [(k/2\Omega)(1 + 2\bar{p}/\bar{B}^2)(\bar{p}')]^2 \geq 0, \tag{8.35}$$

the increase of the stabilizing low β FGR effect being due to the increase in the difference between the plasma and guiding center velocities produced by the ∇B drift. For more general finite β equilibria the FGR effects may be either more or less stabilizing, or even destabilizing.

We emphasize that the discussion of finite β effects in this chapter is limited to the special case of interchange modes for which the functions f_1^+ and f_1^- defined by Eqs. (8.2) vanish. Wave-particle resonances have also been neglected.

CHAPTER IX

FGR Corrections to the Firehose Instability

9.1 The Alfven Mode.

In this chapter we apply the equations of Chapter VI to find FGR corrections to the "Firehose" instability,[17] the instability of an Alfven wave propagating along a uniform magnetic field in a uniform, finite β plasma with sufficiently anisotropic pressure.

Yajima and Taniuti[18] have discussed this problem by use of the modified hydromagnetic equations of Roberts and Taylor.[7] They find these FGR effects to be stabilizing. Their calculation, however, omits the contributions of $\nabla \cdot \underset{\approx}{Q}_{\phi}^{(1)}$ to $\underset{\approx}{\Pi}^{(2)}$. These terms, which are not important for the interchange mode discussed by Roberts and Taylor, turn out to be the dominant FGR effects of the Alfven mode. When they are included, the FGR effects are found to destabilize the Alfven mode for most equilibrium distribution functions, but may have either sign depending on the choice of the equilibrium distribution.

Sato[19] has discussed this problem, in work concurrent with ours, from a solution of the Vlasov equation which assumes the equilibrium distributions to be of Gaussian form. He emphasizes the importance of the second-order terms, but incorrectly concludes that a hydromagnetic of all the FGR effects is not possible.

In the following we restrict the ion and electron equilibrium distributions only by the conditions spatial uniformity and vanishing electric field, and by the condition that the Alfven mode be near marginal stability. Our result, of which Sato's is a special case, exhibits the dependence of the FGR effects on the forms of the equilibrium distribution functions.

The Alfven wave propagating along a uniform magnetic field is characterized, in the linear approximation, by the fact that the plasma motion is transverse to the unperturbed field. This property depends only on the approximation of linearity, and is independent of the expansion in ϵ. This may be seen from the Vlasov equation in the form (3.2). If it is assumed that there are no transverse gradients, the ϕ-average of this equation, with $\underset{\sim}{C}_\perp = \underset{\sim}{\nu}_\perp = \underset{\sim}{v}_\perp - \underset{\sim}{u}_\perp$ and $C_\parallel = v_\parallel$, is

$$
\begin{aligned}
D^* h &+ \left[\underset{\sim}{u} \cdot D^* \underset{\sim}{b} + \frac{eE_\parallel}{m}\right] \frac{\partial h}{\partial C_\parallel} + \frac{C_\parallel}{C_\perp} (\underset{\sim}{u} \cdot D^* \underset{\sim}{b}) \frac{\partial h}{\partial C_\perp} \\
&= - \frac{1}{2\Pi} \oint d\phi \left\{ (D^* \underset{\sim}{b}) \cdot \underset{\sim}{C}_\perp \frac{\partial f_\phi}{\partial C_\parallel} + [\Omega(\underset{\sim}{u}_\perp - \underset{\sim}{u}_E) \times \underset{\sim}{b} \right. \\
&\quad - D^* \underset{\sim}{u}] \cdot \frac{\underset{\sim}{C}_\perp}{C_\perp} \frac{\partial f_\phi}{\partial C_\perp} + \left[\underset{\sim}{b} \cdot (D^* \underset{\sim}{u}) \times \underset{\sim}{C}_\perp \right. \\
&\quad \left.\left. + \frac{(\underset{\sim}{b} \times \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}) \cdot \underset{\sim}{C}_\perp C_\parallel}{|\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}| C_\perp^2} \right] \frac{\partial f_\phi}{\partial \phi} \right\},
\end{aligned}
\tag{9.1}
$$

where $D^* = \partial/\partial t + v_\parallel \underset{\sim}{b} \cdot \nabla$. When linearized about the equilibrium $h = \overline{h}(C_\perp, C_\parallel)$, $f_\phi = 0$, $\underset{\sim}{B} = \overline{\underset{\sim}{B}} =$ constant, and $\underset{\sim}{E} = 0$, Eq. (9.1) becomes

$$D^* \hat{h} + (e/m) E_\parallel \partial \overline{h} / \partial C_\parallel = 0, \tag{9.2}$$

where $\hat{h}$ is the perturbation of h.

Equation (9.2) for each component and Poisson's equation for $E_\parallel$ describe longitudinal electrostatic oscillations. The decoupling of these modes from the transverse waves for which $\hat{h}$ but not $\hat{f}_\phi$ vanishes is therefore independent of the expansion in ϵ.

For these transverse modes the perturbations of all ϕ-independent moments vanish. In particular

$$\rho_c = \hat{\rho}_t = V_\parallel = j_\parallel = \hat{p}_\perp = \hat{p}_\parallel = \hat{q}_\parallel^\perp = \hat{q}_\parallel^\parallel = 0, \tag{9.3}$$

the moment equations are very much simplified, and are closed to all orders in ϵ.

For frequencies of order $\epsilon\Omega$, $\epsilon \ll 1$, these modes are the ordinary hydromagnetic Alfven waves which propagate along the magnetic field. At higher frequencies they become cyclotron waves, and finally transverse electromagnetic waves.

The CGL hydromagnetic theory gives the well-known result

$$\omega^2 = \omega_0^2 \equiv (k^2/\rho)(B^2 + P_\perp - P_\parallel) \tag{9.4}$$

for the frequency of these Alfven waves. The "firehose" (Alfven wave) instability occurs when this frequency becomes imaginary, or when

$$P_{\parallel} > B^2 + P_{\perp} . \qquad (9.5)$$

It is produced by the centrifugal acceleration of the guiding centers moving along the curved field lines. Restoring forces are provided by the magnetic field energy and the total magnetic moment of the particles.

9.2 FGR Corrections.

The calculation leading to the result (9.4) is applicable if the wavelength is long compared to the gyro-radius, provided that $B^2 + P_{\perp} - P_{\parallel} \sim 1$; the small ϵ expansion is then valid and the motion characterized by CGL ordering.

Near the condition of marginal stability, however, FGR effects become important. To investigate these effects we assume

$$(B^2 + P_{\perp} - P_{\parallel})/B^2 \sim \epsilon^2 . \qquad (9.6)$$

The frequency ω_0 is then of order $\epsilon^2\Omega$, the motion characterized by FGR ordering, and the appropriate hydromagnetic equations are those of Section 6.5. For the Alfven mode these equations are simplified by the conditions (9.3) and (9.6).

Note that separation of the transverse modes and the conditions (9.3) are independent of the forms of the equilibrium distributions of the parallel ion and electron velocities. The only restrictions on these distributions are that $j_{\parallel}$ vanish and, for FGR ordering to apply, that ω_0 and the equilibrium parallel transport terms be small. We choose the coordinate system so that the parallel flow velocity vanishes.

By use of the conditions (9.3), the linearized single-fluid momentum equation becomes

$$\rho \partial \underset{\sim}{V} / \partial t = (B^2 + P_{\perp} - P_{\parallel}) \overline{\underset{\sim}{b}} \cdot \nabla \hat{\underset{\sim}{b}} - \overline{\underset{\sim}{b}} \cdot \nabla (\overline{\underset{\sim}{b}} \cdot \underset{\sim}{\Pi})^{(2)} , \qquad (9.7)$$

where all terms are evaluated through order $\epsilon^3 \rho \Omega v_0 \sim \rho \partial V^{(1)} / \partial t$, $\overline{\underset{\sim}{b}}$ and $\hat{\underset{\sim}{b}}$ are equilibrium and perturbed values of $\underset{\sim}{b}$, and

$$\begin{aligned} (\underset{\sim}{b} \cdot \underset{\sim}{\Pi})^{(2)} &= \Omega^{-1} \underset{\sim}{b} \times [(2p_{\parallel}^+ - p_{\perp}^+) \underset{\sim}{b} \cdot \nabla \underset{\sim}{V}] \\ &- \Omega^{-2} [(R_1^+ - 3R_2^+) - \rho^{-1} p_{\parallel}^+ (p_{\parallel}^+ - p_{\perp}^+) \\ &- \rho^{-1} (p_{\parallel}^+ - p_{\perp}^+)(B^2 + p_{\perp}^- - p_{\parallel}^-)] \underset{\sim}{b} \cdot \nabla \underset{\sim}{b} \\ &+ (2/\Omega)(q_{\parallel}^{\parallel} - q_{\parallel}^{\perp}) \underset{\sim}{b} \times (\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}) . \end{aligned} \qquad (9.8)$$

The result (9.8) is obtained from equations (5.8), (6.19), and the first-order terms of Maxwell's induction equation in the form (4.45). For the present case this last equation is

$$\partial \hat{\underset{\sim}{b}}^{(0)} / \partial t = \overline{\underset{\sim}{b}} \cdot \nabla \underset{\sim}{V}_{\perp}^{(1)} + (\rho \Omega)^{-1} (B^2 + p_{\perp}^{-} - p_{\parallel}^{-}) \overline{\underset{\sim}{b}} \cdot \nabla \hat{\underset{\sim}{b}}^{(0)} . \tag{9.9}$$

Equation (9.8) is easily derived directly from the recursion formula (4.14) if the special assumptions of this section are made at the outset.

The first term of Eq. (9.8) is just the usual collisionless viscosity; it is the only contribution to $\underset{\sim}{\Pi}$ included in the calculation of reference 18. The other terms arise from $\underset{\sim}{\Pi}_2^{(2)}$, from the FGR term of Eq. (9.9), and from the equilibrium heat flows.

Note that, because of the condition (9.6), only $\hat{\underset{\sim}{b}}^{(0)}$ is required in Eq. (9.7). Thus (9.7), (9.8), and (9.9) are closed hydromagnetic equations for the Alfven mode. If the FGR terms in Eqs. (9.8) and (9.9) are dropped, these equations reduce to the CGL form.

We now assume $\underset{\sim}{V}$ and $\hat{\underset{\sim}{b}}$ to be proportional to $\exp(i\omega t - ikx_1)$, and from Eqs. (9.7) through (9.9) obtain the dispersion equation

$$\begin{aligned} \omega^4 &- [2(\omega_0^2 - \omega_1^2) + \omega_2^2 + \omega_3^2]\omega^2 + 2(\omega_2 + \omega_3)\omega_q^2 \omega \\ &+ (\omega_0^2 - \omega_1^2 - \omega_2\omega_3)^2 - \omega_q^4 = 0, \end{aligned} \tag{9.10}$$

where

$$\begin{aligned} \omega_1^2 &= (k^4/\rho\Omega^2)[R_1 - 3R_2 - \rho^{-1}(p_{\parallel} - p_{\perp})(2p_{\parallel} - p_{\perp})], \\ \omega_2 &= (k^2/\rho\Omega)(p_{\parallel} - p_{\perp}), \\ \omega_3 &= (k^2/\rho\Omega)(2p_{\parallel} - p_{\perp}), \end{aligned} \tag{9.11}$$

and

$$\omega_q^2 = (2k^3/\rho\Omega)(q_{\parallel}^{\parallel} - q_{\parallel}^{\perp})$$

represent the FGR effects. All moments in Eqs. (9.11) refer to the ion distribution, the electron pressures having been eliminated from ω_1 and ω_2 by use of the condition (9.6) in the form

$$(B^2 + p_\perp^- - p_\parallel^-)^{(0)} = (p_\parallel^+ - p_\perp^+)^{(0)} . \qquad (9.12)$$

The frequency ω_1 represents the contributions from $\nabla \cdot \underset{\sim}{Q}_\phi$, and from the FGR corrections to the generalized Ohm's law, to $\underset{\sim}{b} \cdot \underset{\sim}{\Pi}^{(2)}$; ω_2 represents the FGR term of Eq. (9.9), ω_3 the ordinary collisionless viscosity, and ω_q the effect of the equilibrium heat flows. Note that ω_1, ω_2, ω_3 and ω_q are all of order $\epsilon^2\Omega$; hence all terms of the dispersion equation (9.10) are of order $(\epsilon^2\Omega)^4$.

The four roots of Eq. (9.10) are given by

$$\omega = \tfrac{1}{2}\{(\omega_2 + \omega_\nu) \pm [(\omega_2 - \omega_\nu)^2 + 4(\omega_0^2 - \omega_1^2 \pm \omega_q^2)]^{\frac{1}{2}}\}; \qquad (9.13)$$

these roots correspond to the two polarizations and two directions of propagation of the waves. In the CGL limit all four frequencies have the magnitude $|\omega_0|$. Degeneracy with respect to the direction of polarization is removed by the FGR effects because Ω_+ is no longer infinitely large, and the ion gyration defines a preferred polarization. Degeneracy with respect to the direction of propagation is removed by an anisotropic equilibrium heat flow.

An interesting special case arises when $\omega_0^2 - \omega_1^2 = \omega_q^2 = 0$. The frequencies (9.13) are then ω_2 and ω_ν. The frequency ω_2 corresponds to current waves produced by the FGR term of the generalized Ohm's law; for these waves the plasma velocity vanishes. The waves of frequency ω_ν propagate because of the collisionless viscosity, without perturbations of the magnetic field. These simple waves exist for this special case because the equilibrium distribution is such that small amplitude disortions of the magnetic field direction produce no stresses.

The condition for stability of all four modes is

$$\omega_0^2 + \tfrac{1}{4}(\omega_2 - \omega_3)^2 - \omega_1^2 - |\omega_q^2| \geq 0, \tag{9.14}$$

which may be compared to the CGL result $\omega_0^2 \geq 0$. Yajima and Taniuti[18] include only the terms represented by ω_2 and ω_3 in their calculation; the stability condition (9.14) reduces to their result if the terms in ω_1 and ω_q are omitted and $p^+_{\perp,\parallel}$ set equal to $\frac{1}{2}P_{\perp,\parallel}$, corresponding to their assumption of equal ion and electron temperatures. It is evident from (9.14) that these terms are always stabilizing.

By use of the relations (4.33) and (9.11), the stability condition may be written

$$\omega_0^2 + (k^2/\rho\Omega)^2\{[p_\perp^2 - (3/4)p_\parallel^2] + 3p_\parallel[p_\perp(R_2' - 1) - p_\parallel(R_1' - 1)]\}$$
$$- (2k^3/\rho\Omega)\,|q^\perp_\parallel - q^\parallel_\parallel| \geq 0. \tag{9.15}$$

Only the first of the FGR terms in (9.15) is present if the equilibrium distribution function is separable in $v_\perp$ and $v_\parallel$, and the distribution of $v_\perp$ is Gaussian; Sato's[19] result corresponds to this term. The additional FGR effects, produced by deviations of the distribution from this simple form, are given by the remaining terms of Eq. (9.15). The equilibrium heat flows are seen to be destabilizing. Other deviations of the distribution from separable, Gaussian form may be either stabilizing or destabilizing. The relation between FGR terms of the stability condition (9.15) and FGR modifications of the particle orbits will be discussed in the following section.

9.3 Particle Orbits.

The dispersion equation (9.10) may also be obtained from the particle orbits. This calculation illustrates the relation between FGR corrections to the hydromagnetic equations and the corresponding modifications of the orbits.

We assume fields of the forms

$$\underset{\sim}{B} = B\{\underset{\sim}{z} + [\hat{\underset{\sim}{b}}\exp(i\omega t + ikz) + cc]\}$$

and (9.16)

$$\underset{\sim}{E} = \hat{\underset{\sim}{E}}\exp(i\omega t + ikz) + cc.$$

Solution of the equation of motion for a particle moving in these fields, linearized in $\hat{b}$ and $\hat{E}$, gives the velocity transverse to the unperturbed magnetic field in the form

$$(\ddot{r})_\perp = v_\perp \alpha e^{i\Omega t} + v_D e^{i\theta} + cc, \qquad (9.17)$$

where

$$\alpha = |\alpha|(ie_x + e_y),$$

$$v_D = \frac{\Omega^2}{\Omega^2 - \omega'^2}\left\{\mp v_\parallel \hat{b} + \frac{\hat{E} \times z}{B} + \frac{i\omega' v_\parallel z \times \hat{b}}{\Omega} + \frac{i\omega' \hat{E}}{\Omega B}\right\}, \qquad (9.18)$$

$$\omega' = \omega + kv_\parallel, \qquad \theta = i\omega' t + kz_0,$$

and z_0 is the initial z-coordinate of the particle. The displacement of the particle in the z-direction is

$$r_z = z_0 + v_\parallel t - \frac{\Omega^3 v_\perp}{(\Omega^2 - \omega'^2)^2}\left\{\left[\frac{i\omega' b_y}{\Omega} - \frac{\Omega^2 + \omega'^2}{\Omega^2} b_x\right] \cos\phi \right.$$
$$\left. + \left[\frac{\omega' b_x}{\Omega} - i\left(1 + \frac{\omega'^2}{\Omega^2}\right) b_y\right] \sin\phi\right\} e^{i\theta} + cc, \qquad (9.19)$$

where ϕ is the azimuthal angle of the vector $\alpha \exp(i\Omega t) + cc$ with respect to the x-axis.

It is easily verified that this motion may be represented as a gyration with velocity $v_\perp$ at frequency Ω in a circular orbit normal to the vector

$$\underset{\sim}{n} = \underset{\sim}{z} + \frac{[1 + (\omega'/\Omega)^2]\hat{\underset{\sim}{b}} - 2i(\omega'/\Omega)(\hat{\underset{\sim}{b}} \times \underset{\sim}{z})}{[1 - (\omega'/\Omega)^2]^2} \tag{9.20}$$

about a guiding center which moves with the velocity $\underset{\sim}{v}_D + v_\parallel \underset{\sim}{z}$. In the limit $\Omega \to \infty$, $\underset{\sim}{n} = \underset{\sim}{b}$ and $\underset{\sim}{v}_D = \underset{\sim}{u}_E + v_\parallel \hat{\underset{\sim}{b}}$, in agreement with the zero-order guiding center velocity, and, in first order, $\underset{\sim}{v}_D$ agrees with the well-known first-order drifts.[22] Higher-order corrections to the particle orbits take two forms: the well-known zero- and first-order drifts are increased in magnitude by the resonance denominator of Eq. (9.18), and the normal to the plane of gyration deviates from the direction of the magnetic field.

Derivation of the results (9.17) through (9.20) assumed $\omega' \neq \Omega$; if $\omega' = \Omega$ no divergence occurs, but the time dependence of the orbit becomes secular.

The total current equals the sum over all particles of the guiding center current plus the curl of the magnetic moment $\underset{\sim}{\mu} = -(mv_\perp^2/2B)\underset{\sim}{n}$. Note that $\underset{\sim}{\mu}$ is parallel to $\underset{\sim}{b}$ only in lowest order. The current is easily evaluated from Eqs. (9.18) and (9.20) if resonances are ignored[19] $(\omega' \ll \Omega)$ and the denominators are expanded. The resulting expression for the current may be combined with Maxwell's equations to obtain the dispersion equation (9.10).

If $\omega \sim \epsilon\Omega$, so that CGL ordering applies, FGR effects are small and are easily understood from Eqs. (9.18) and (9.20). A first-order current is produced by the first-order guiding center drifts and the curl of the zero-order magnetic moment; this is the current of CGL theory. There are no first-order corrections to the CGL currents, but, because of the denominator in Eq. (9.18) and the electron-ion mass difference, there is a second-order current proportional to the zero-order guiding center velocity. Another contribution to the second-order current arises from the first-order difference between $\underset{\sim}{b}$ and $\underset{\sim}{n}$. These are orthogonal to the CGL current and cause the wave to become elliptically polarized, but do not change its frequency; they correspond to the collisionless viscosity and first-order corrections to the generalized Ohm's law of Section 5.5. This polarization may be considered as a superposition of a secondary Alfven wave on the original wave.

Third-order currents produce a frequency change of order $\epsilon^2\Omega$, corresponding to second-order corrections to CGL theory. They arise from the effect of the resonance denominators of Eqs. (9.18) and (9.20) on the first-order CGL currents, and from modifications of the CGL currents produced by the secondary Alfven wave. This latter effect may be regarded as resulting from the elliptical polarization of the secondary

Alfven wave, hence as a produce of first-order effects. The currents associated with the former effect are in the same direction as the CGL current; they therefore enhance both the stabilizing nature of $p_\perp$ and the destabilizing nature of $p_\parallel$.

If the frequency is low ($\omega_0 \sim \epsilon^2 \Omega$), so that FGR ordering applies, all the above effects become of the same order, but they still may be separated. The polarization effect is associated with FGR modifications of the electric field drift and magnetic moment of the ions which are of first order for CGL ordering. They therefore appear in the moment equations as the collisionless viscosity (and the terms of $\underset{\approx}{\Pi}$ proportional to $q_\parallel^{\perp,\parallel}$), corrections to the generalized Ohm's law, and the contribution to $\underset{\approx}{\Pi}^{(2)}$ associated with the difference between the plasma velocity and the electric field drift. Thus, from the discussion of Eq. (6.29), both the collisionless viscosity and the terms of $\underset{\approx}{\Pi}_2^{(2)}$ arising from $\nabla \cdot \underset{\approx}{P}$, as well as the corrections to Ohm's law, are associated with the polarization. In the stability condition (9.14) these effects appear as ω_2, ω_3, ω_q, and the parts of ω_1^2 not proportional to R_1 or R_2. Only the R_1 and R_2 terms of ω_1^2, from the contribution of $\nabla \cdot \underset{\approx}{R}^{(0)}$ to $\underset{\approx}{\Pi}^{(2)}$, arise from the enhancement of the CGL currents produced by the resonance denominators of Eq. (9.18) and (9.20).

The stability condition (9.15) may be written

$$\omega_0^2 + (k^2/\rho\Omega)^2\{[p_\perp - (3/2)p_\parallel]^2 + 3\rho R_2 - \rho R_1\}$$
$$- (2k^3/\rho\Omega)\,|q_\parallel^\perp - q_\parallel^\parallel| \geq 0. \tag{9.21}$$

In this form all polarization effects appear in the first and last FGR terms. Thus, in the absence of anisotropic equilibrium heat flows, these effects are stabilizing. The remaining terms exhibit the effect of the enhancement of the CGL currents by the resonance denominators of Eqs. (9.18) and (9.20).

CHAPTER X

Long-Term Equilibrium

10.1 Equilibrium Moment Equations

In this chapter we consider the moment equations that describe configurations which remain in equilibrium over the long times associated with FGR ordering. Since first-order velocities become important on the longer time scale, static "long-term" equilibria exist only in special cases. We will, however, assume that the zero-order plasma velocity and electric field vanish, corresponding to a static equilibrium on the CGL time scale.

Mirror confinement configurations of the "minimum-B"[34] type, for which the pressures are a function of the magnetic field strength only, are of particular interest. Taylor[35] has shown, by use of the CGL momentum equation, that equilibria of this type exist for low plasma pressures, and Northrop and Whiteman[20] have extended the discussion to finite pressures. Both these discussions give "short-term" equilibria. Northrop and Whiteman, however, have found this same class of finite-β equilibria (for the special case of vanishing electric field) by use of the general theory of adiabatic particle motion and some (but not all) of the conditions for long-term equilibrium.

In the following we indicate how this theory is related to the hydromagnetic description of long-term equilibria of the Taylor-Whiteman type, and extend the discussion to include the effects of first-order electric fields.

The equilibrium moment equations, and their CGL form for the static case, will be reviewed in the remainder of this section; extension to the longer time scale is discussed in Section 10. 2, and the special "long-term minimum-B" equilibria are considered in Section 10. 3.

The fundamental equilibrium equation is the condition of momentum balance

$$\rho \underset{\sim}{V} \cdot \nabla \underset{\sim}{V} = - \nabla \cdot \underset{\approx}{P}_t + \underset{\sim}{j} \times \underset{\sim}{B} . \tag{10. 1}$$

For equilibrium on the CGL time scale, only the zero-order terms of this condition are required; in order to distinguish between motion characterized by FGR ordering and "long-term" equilibrium, however, the first- and second-order terms must also be considered. The ϕ-dependent components of $\underset{\approx}{P}$ that then appear may be expressed in terms of ϕ-independent moment by the recursion relation (4. 15).

The condition (10. 1) provides a relation between some of the ϕ-independent moments of the distribution functions and the magnetic field. Additional constraints are provided by the equilibrium forms of the "equations of motion" (4. 1), (4. 16), (4. 17), (4. 29), (4. 30), (4. 32), ... for the ϕ-independent moments n, $p_\perp$, $p_\parallel$, $q_\parallel^\perp$, $q_\parallel^\parallel$, R_i, etc.

If the zero-order velocity vanishes, corresponding to a static CGL equilibrium, the zero-order terms of these equations are

$$- (\partial \underset{\sim}{V} / \partial t)_\perp = \nabla_\perp P_\perp + (P_\parallel - P_\perp)\, \underset{\sim}{b} \cdot \nabla \underset{\sim}{b} - \underset{\sim}{j}^{(1)} \times \underset{\sim}{B} = 0, \tag{10. 2}$$

$$- (\partial \underset{\sim}{V} / \partial t)_\parallel = \underset{\sim}{b} \cdot \nabla P_\parallel - (p_\parallel - P_\perp)\, \underset{\sim}{b} \cdot \nabla B/B = 0, \tag{10. 3}$$

$$- \partial \rho / \partial t = \nabla \cdot (\rho \underset{\sim}{u}) = 0, \tag{10. 4}$$

$$- B^{-2}\, \partial p_\perp / \partial t = \underset{\sim}{b} \cdot \nabla(q_\parallel^\perp / B^2) = 0, \tag{10. 5}$$

$$- (1/2B)\, \partial p_\parallel / \partial t = \underset{\sim}{b} \cdot \nabla(q_\parallel^\parallel / B) + (q_\parallel^\perp / B^2)\, \underset{\sim}{b} \cdot \nabla B = 0, \tag{10. 6}$$

$$- 2\, \partial q_\parallel^\parallel / \partial t = \underset{\sim}{b} \cdot \nabla R_1 - (R_1 - 3R_2)(\underset{\sim}{b} \cdot \nabla B)/B = 0, \tag{10. 7}$$

$$- \partial q_\parallel^\perp / \partial t = \underset{\sim}{b} \cdot \nabla R_2 - (R_2 - R_3/4)(\underset{\sim}{b} \cdot \nabla B)/B = 0, \tag{10. 8}$$

$$- B^{-1} \partial R_1 / \partial t = \underset{\sim}{b} \cdot \nabla(S_1/B) + (4S_2/B^2)\, \underset{\sim}{b} \cdot \nabla B = 0, \tag{10. 9}$$

$$- B^{-1} \partial R_2 / \partial t = \underset{\sim}{b} \cdot \nabla(S_2/B) + [(S_2 - S_3)/B^2]\, \underset{\sim}{b} \cdot \nabla B = 0, \tag{10. 10}$$

and
$$- (1/2B)\, \partial R_3 / \partial t = \underset{\sim}{b} \cdot \nabla(S_3/B^2) = 0, \tag{10. 11}$$

where all quantities except $\underset{\sim}{j}_\perp$ are of zero order. Equations (10. 2) and (10. 3) are the transverse and parallel components of (10. 1); Eqs. (10. 5), (10. 6), and (10. 9) through (10. 11) are the equilibrium forms of (5. 4), (5. 5), and (5. 18); and Eqs. (10. 7) and (10. 8) are obtained from (4. 29) and (4. 30) by use of the result (A. 33). The condition (10. 3) has been used to simplify Eqs. (10. 7) through (10. 11). Equations (10. 4) through (10. 11) for each component, plus (10. 3) and the parallel component of the generalized Ohm's law (which determines $\underset{\sim}{E}$), are moment of the zero-order Vlasov equation (3. 11). These equations determine those ϕ-independent moments that involve $\nu_{\parallel}$ (such as $u_{\parallel}$, $p_{\parallel}$, $q^{\perp}_{\parallel}$, $q^{\parallel}_{\parallel}$, R_1, and R_2) in terms of the magnetic field, the appropriate boundary conditions, and those that do not involve $\nu_{\parallel}$. The condition (10. 2) and transverse components of the generalized Ohm's law are moments of the recursion equation (3. 9) for $f_{\phi}^{(1)}$.

The zero-order equations (10. 1) through (10. 11) do not distinguish between "long-term" equilibrium and motion characterized by FGR ordering; the higher-order terms required to make this distinction are discussed in the following section.

10. 2 "Long-Term" Equilibrium

In order to describe "long-term" equilibrium ($\tau \sim \tau_g/\epsilon^2$) all terms of the momentum balance condition (10. 1) must be evaluated through second order. Since we assume that the zero-order plasma velocity vanishes, this equation becomes

$$\rho^{(0)}\underset{\sim}{V}^{(1)} \cdot \nabla \underset{\sim}{V}^{(1)} = -\nabla \cdot \underset{\approx}{P}_d - \nabla \cdot \underset{\approx}{\Pi}^{(1)} - \nabla \cdot \underset{\approx}{\Pi}^{(2)} + \underset{\sim}{j} \times \underset{\sim}{B} + 0(\epsilon^3 \rho v_o^2/L), \qquad (10.\ 12)$$

where $\underset{\approx}{P}_d = P_{\perp} \underset{\approx}{I}_{\perp} + P_{\parallel} \underset{\sim}{b}\underset{\sim}{b}$. $\underset{\approx}{\Pi}^{(1)}$ is given by Eqs. (5. 8) and (5. 7) or (5. 11), and vanishes unless there is zero-order heat flow along the field lines; $\underset{\approx}{\Pi}^{(2)}$ is discussed in Chapter VI. The solutions of Eq. (10. 12) are only slightly different from those of (10.1). These differences may, however, distinguish between stable and weakly unstable configurations, as well as between long-term equilibrium and slow, low frequency motion.

First-order corrections to Eqs. (10.4) through (10.11) and to the equations for the higher moments must also be considered. The first-order terms of the continuity equation (4.1) are, for each plasma component,

$$\nabla \cdot (\rho^{(0)} \underset{\sim}{u}^{(1)}) = \nabla \cdot \rho^{(0)} (\underset{\sim}{u}_E^{(1)} + \underset{\sim}{u}_p^{(1)} + u_{\parallel}^{(1)} \underset{\sim}{b}) = 0,$$

where

$$\underset{\sim}{u}_p = (\rho\Omega)^{-1} \underset{\sim}{b} \times \nabla p_{\perp} .$$

By use of the equilibrium condition $\nabla_{\perp} \cdot (B \underset{\sim}{u}_E) = 0$, this equation may be written

$$\underset{\sim}{u}_E^{(1)} \cdot [B\nabla(\rho/B) - \rho \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}] + \underset{\sim}{b} \cdot \nabla (u_{\parallel}^{(1)}/B = \psi_1 , \qquad (10.13)$$

where

$$\psi_1 = \Omega^{-1} \underset{\sim}{b} \cdot [(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b} + \nabla B/B) \times \nabla_{\perp} p_{\perp} + (\nabla \times \underset{\sim}{b})(\underset{\sim}{b} \cdot \nabla p_{\perp})]^{(0)}$$

represents "FGR" effects. The first-order terms of the pressure equations (6.12) and (6.13) give the relations

$$\underset{\sim}{u}_E^{(1)} \cdot [B^2\nabla(p_\perp/B^2) - p_\perp \underset{\sim}{b}\cdot\nabla\underset{\sim}{b}] - p_\perp \underset{\sim}{b}\cdot\nabla u_\parallel^{(1)} + B^2\underset{\sim}{b}\cdot\nabla(q_\parallel^\perp/B^2)^{(1)} = \psi_2 \tag{10.14}$$

and

$$\underset{\sim}{u}_E^{(1)} \cdot [B\nabla(p_\parallel/B) - 3p_\parallel \underset{\sim}{b}\cdot\nabla\underset{\sim}{b}] + p_\parallel \underset{\sim}{b}\cdot\nabla u_\parallel^{(1)} + B\underset{\sim}{b}\cdot\nabla(q_\parallel^\parallel/B)^{(1)} = \psi_3, \tag{10.15}$$

where

$$\psi_2 = (\rho\Omega)^{-1}(p_\parallel - 2p_\perp)\underset{\sim}{b}\cdot(\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}\times\nabla p_\perp) - \nabla\cdot[\Omega^{-1}\underset{\sim}{b}\times(\nabla R_3 - 4\underset{\sim}{b}\cdot\underset{\sim}{T})] - a_2$$

and

$$\psi_3 = (\rho\Omega)^{-1}p_\parallel \underset{\sim}{b}\cdot(\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}\times\nabla p_\perp) + \nabla\cdot\{\Omega^{-1}\underset{\sim}{b}\times[\nabla R_2 + (R_1 - 3R_2)\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}]\} + a_2,$$

with a_2 defined following Eq. (5.23).

The equations for the higher-order ϕ-independent moments give relations of a similar form.

The forms of the zero-order equations (10.4) through (10.11), and equations for the higher velocity moments as well, are thus modified in first-order by terms proportional to $\underset{\sim}{u}_E^{(1)}$ (which would have appeared in zero order if a zero-order electric field has been allowed), and by the "FGR" terms ψ_i. These modifications represent the effects of the first-order transverse drifts. The modified equations determine first-order corrections to moments that involve $v_\parallel$; they are discussed in more detail in the following section for the special case of mirror confinement and, in particular, "minimum-B" configurations of the Taylor-Whiteman type.

10.3 "Minimum-B" Configurations.

We now apply the equations outlined in the previous sections to mirror confinement systems, and, in particular, to configurations of the Taylor-Whiteman[20, 35] "minimum-B" type; in this way we extend the "short term" hydromagnetic equilibria of Taylor and Whiteman to times of order ϵ^{-2} times the gyro-period.

For mirror confinement systems the plasma density vanishes except over a finite portion of each field line; Eqs. (10.4) through (10.6), (10.9) through (10.11), and the zero-order equations for the higher ϕ-independent moments which are even in $v_{\parallel}$ may be integrated along the field lines to show that $u_{\parallel}$, $q_{\parallel}^{\perp}$, $q_{\parallel}^{\parallel}$, S_i, and higher moments odd in $v_{\parallel}$ vanish in zero order. Thus the zero-order distributions are even in $v_{\parallel}$, a result that may also be obtained from the zero-order Vlasov equation (3.11). For mirror confinement systems, then, both the assumption that $V_{\parallel}^{(0)}$ vanishes and the condition that $j_{\parallel}^{(0)}$ vanishes, necessary for adiabatic expansion to apply, are automatically satisfied.

Equations (10.3), (10.7), (10.8), and those for the higher moments odd in $v_{\parallel}$, together with the boundary conditions, determine $p_{\parallel}^{(0)}$, $R_1^{(0)}$, $R_2^{(0)}$, and the other ϕ-independent moments involving second and higher even powers of $v_{\parallel}$ in terms of the moments $\rho^{(0)}$, $p_{\perp}^{(0)}$, $R_3^{(0)}$, etc. which do not involve $v_{\parallel}$.

For "long-term" equilibrium Eqs. (10.13), (10.14), (10.15), and the first-order terms of the equations for the higher ϕ-independent moments must also be considered. When integrated along the field lines Eqs. (10.13), (10.14), and (10.15) yield

$$\int_L ds\, \underset{\sim}{u}_E^{(1)} \cdot [B\nabla(\rho/B) - \rho \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}] = \int_L ds\, \psi_1, \tag{10.16}$$

$$\int_L ds\, \underset{\sim}{u}_E^{(1)} \cdot [B^2\nabla(p_\perp/B^2) - p_\perp \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}] = \int_L ds\, \psi_2, \tag{10.17}$$

and

$$\int_L ds\, \underset{\sim}{u}_E^{(1)} \cdot [B\nabla(p_\parallel/B) - 3p_\parallel \underset{\sim}{b} \cdot \nabla \underset{\sim}{b}] = \int_L ds\, \psi_3, \tag{10.18}$$

where $\int_L ds$ indicates the integral along those portions of the field lines that lie within the plasma volume, and the mirror boundary conditions have been used. The equilibrium equations for the higher ϕ-independent moments give conditions on $\underset{\sim}{u}_E^{(1)}$ of a similar form. The "FGR" terms ψ_i of these equations do not vanish, in general. Except in special cases, therefore, $\underset{\sim}{u}_E^{(1)}$ does not vanish for "long-term" equilibrium. Once the boundary conditions are specified and $\underset{\sim}{u}_E^{(1)}$ is determined, the first-order velocity moments involving $v_\parallel$ may be found by integrating Eqs. (10.13), (10.14), (10.15), $\cdots$ along a field line from the boundary to a point in the plasma.

These first-order conditions have a simple physical interpretation: the FGR terms of Eqs. (10.13), (10.14), (10.15), $\cdots$ represent the first-order transverse gradient-B and curvature drifts. The conditions (10.16), (10.17), (10.18), $\cdots$ simply insure that there be no accumulation of guiding centers on any field line. Unless $\int_L ds\,\psi_i = 0$, these conditions require a nonzero $\underset{\sim}{E}^{(1)}$. The parallel transport terms are then determined so that the distribution along the field lines of guiding centers, thermal energy, etc. remain constant.

The equilibrium condition considered by Northrop and Whiteman in their discussion of the particle motion is equivalent to Eqs. (10.16), (10.17)$\cdots$. They assume that the magnetic field is self-consistent. Since they assume $\underset{\sim}{E}^{(1)} = 0$, they obtain only equilibria for which $\int_L ds\,\psi_i = 0$. We show in the following that the Taylor-Whiteman configurations are of this type, and consider the effect of first-order electric fields.

We first briefly review some properties of the low-β equilibria found by Taylor[35] and extended to finite-β by Northrop and Whiteman.[20] They assume that the plasma is confined in a finite volume by the mirror effect, and that the pressure components $P_\perp$ and $P_\parallel$ are functions of the magnetic field strength B only. Then the CGL equilibrium condition may be written

$$\nabla \times (\nu \underset{\sim}{B}) = 0, \qquad (10.19)$$

where

$$\nu = \nu(B) \equiv B^{-2}(P_\parallel - P_\perp) - 1,$$

and, if it is assumed that $\underset{\sim}{b} \cdot \nabla B = - B\nabla \cdot \underset{\sim}{b} \neq 0$, Eq. (10. 4) implies

$$P_{\parallel}' - B^{-1}(P_{\parallel} - P_{\perp}) = 0, \tag{10. 20}$$

where the prime indicates differentiation with respect to B. Whiteman obtained the simple form (10. 19) by showing that, for these equilibria, the first-order parallel current vanishes. Thus

$$j_{\parallel}^{(1)} = [(\nabla \times \underset{\sim}{B}) \cdot \underset{\sim}{b}]^{(0)} = [B(\nabla \times \underset{\sim}{b}) \cdot \underset{\sim}{b}]^{(0)} = 0. \tag{10. 21}$$

This property, which depends on the boundary conditions appropriate for mirror confinement, is easily recovered from (10. 19). Another important property of these equilibria, which follows from Eq. (10. 19) and the identity

$$(\nabla \times \underset{\sim}{b}) = - \underset{\sim}{b} \times (\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}),$$

is that the vectors $\nabla_{\perp} B$ and $\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}$ are colinear.

We consider now the "long-term" extension of these Taylor-Whiteman equilibria. Their characteristic feature is that $P_{\perp}$ and $P_{\parallel}$ are assumed to be functions of B only. The "long-term" equilibrium equations, however, require higher velocity moments in addition to $P_{\perp}$ and $P_{\parallel}$; we assume that all ϕ-independent moments are functions of B only. This assumption agrees with the distribution for the P(B) equilibria found by Northrop and Whiteman from the general theory of adiabatic particle motion. With this assumption, the result (10. 21), and the colinearity of $\nabla_\perp B$ and $\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}$, the FGR terms ψ_i of Eqs. (10. 13) through (10. 18) all vanish, since they are all of the form

$$\nabla \cdot [\, \alpha(B) \underset{\sim}{b} \times \nabla_\perp B] \equiv 0 .$$

This is also true of the FGR terms of the equations for the higher ϕ-independent moments. These equilibria have this simplifying property because the first-order gradient-B and curvature drifts are everywhere orthogonal to the gradient of the zero-order distribution functions (the finite-β interchange modes discussed in Chapter VIII have this same property).

Because of this result the conditions (10. 16) through (10. 18) are satisfied if $\underset{\sim}{E}_\perp^{(1)}$ vanishes. Since $E_\parallel^{(1)}$ vanishes if the ion and electron pressures and densities are equal (see Eq. (10. 22) below), "long-term" Taylor-Whiteman equilibria exist for which the electric field vanishes through first order; these equilibria correspond to those found by Northrop and Whiteman.

Note that if $\underset{\sim}{E}_\perp^{(1)}$ vanishes, Eqs. (10. 16), (10. 17), and (10. 18) reduce to their zero-order forms (10. 4), (10. 5), and (10. 6). For this case, in fact, the forms of Eqs. (10. 4) through (10. 11), and those for the higher moments as well, are all unchanged in first order. The distribution functions are therefore even in $v_\parallel$ in first order, as well as in zero order.

The condition that $\underset{\sim}{E}^{(1)}$ vanish is an unnecessary limitation of the theory. The parallel electric field is determined by the parallel component of the generalized Ohm's law

$$E_{\parallel}^{(1)} = (n_{-}e)^{-1}(\nabla \cdot \underset{\sim}{P}_{\Delta}) \cdot \underset{\sim}{b}, \tag{10.22}$$

where

$$\underset{\sim}{P}_{\Delta} = [(p_{\perp}^{-} - p_{\perp}^{-})\underset{\sim}{I}_{\perp} + (p_{\perp}^{+} - p_{\parallel}^{-})\underset{\sim}{b}\underset{\sim}{b}]^{(0)},$$

and the equilibrium condition (10.3) has been used. Thus $E_{\parallel}^{(1)}$ vanishes only if the ion and electron pressures are everywhere equal.

If it is assumed that all ϕ-independent moments are functions of B only, Eq. (10.22) yields

$$E_{\parallel}^{(1)} = -\underset{\sim}{b} \cdot \nabla \Phi_1(B), \tag{10.23}$$

where

$$\Phi_1(B) = -\int_{B_{min}}^{B} dB(n_{-}e)^{-1}[p_{\parallel}'^{+} - p_{\parallel}'^{-} - B^{-1}(p_{\parallel}^{+} - p_{\perp}^{+} - p_{\parallel}^{-} + p_{\perp}^{-})].$$

The total electrostatic potential thus has the form

$$\Phi = \Phi_1(B) + \Phi_2, \tag{10.24}$$

where Φ_2 is constant along every field line.

This potential yields

$$\underset{\sim}{u}_E^{(1)} = (\Phi_1'(B)/B^2)\underset{\sim}{b} \times \nabla B + (1/B)\underset{\sim}{b} \times \nabla\Phi_2 . \qquad (10.25)$$

The first term of $\underset{\sim}{u}_E^{(1)}$, arising from $\Phi_1(B)$, is orthogonal to $\nabla_\perp B$, hence does not enter the constraints (10.16), (10.17), (10.18), $\cdots$. Thus for any $\underset{\sim}{P}_\Delta$ there is always an electric field which satisfies the condition (10.22) and leaves the Taylor-Whiteman equilibria essentially unaltered.

The potential Φ_2 must satisfy the constraints (10.16), (10.17), (10.18), $\cdots$. (In the quasi-neutral approximation Poisson's equation need not be considered.) It is not obvious whether solutions to these conditions exist although it is easily verified that, except for very special circumstances, $\underset{\sim}{b} \times \nabla\Phi_2$ is not everywhere orthogonal to $\nabla_\perp B$. Thus a potential Φ_2, if it exists, would cause convection in the direction of $\nabla_\perp B$ and the first-order distributions would no longer be even in $v_\parallel$.

To summarize the results of this section, we have shown from the extended-equilibrium hydromagnetic equations that the Taylor-Whiteman equilibria are essentially unaltered by the electric field produced by unequal ion and electron pressures, and that for these configurations the first-order distribution functions are even in $v_\parallel$ unless a nonzero potential Φ_2 exists. The second-order corrections to Eq. (10.19) may be found by straightforward calculation from the results of Section 6.4. These corrections would be of importance only for a very detailed stability analysis.

ACKNOWLEDGMENTS

The author is indebted to Dr. T. G. Northrop for many valuable discussions and for guidance throughout the course of this work, and to S. Macmahon for continuing support and assistance in preparing the manuscript.

This work was supported by the U. S. Atomic Energy Commision.

APPENDIX A

Algebraic Details

A. 1 Velocity Moments of the Vlasov Equation.

In this appendix we outline the tensor manipulations used to obtain the results presented in Sections IV through VI, beginning with the derivation of the moment equations (4. 4) through (4. 7) from the Vlasov equation (3. 1).

Multiplication of the first term of Eq. (3. 1) by the mass m and N factors of $\underset{\sim}{\nu} = \underset{\sim}{v} - \underset{\sim}{u}$ $(N \geq 2)$, and integration over $\underset{\sim}{v}$ gives

$$\langle m\underset{\sim}{\nu}\cdots\underset{\sim}{\nu}\frac{\partial f}{\partial t}\rangle = \frac{\partial}{\partial t}\langle m\underset{\sim}{\nu}\cdots\underset{\sim}{\nu}f\rangle + \langle m\frac{\partial \underset{\sim}{u}}{\partial t}\underset{\sim}{\nu}\cdots\underset{\sim}{\nu}f\rangle + \langle m\underset{\sim}{\nu}\frac{\partial \underset{\sim}{u}}{\partial t}\cdots\underset{\sim}{\nu}f\rangle$$

$$+ \cdots + \langle m\underset{\sim}{\nu}\underset{\sim}{\nu}\cdots\frac{\partial \underset{\sim}{u}}{\partial t}f\rangle \qquad (A.1)$$

$$= \frac{\partial \underset{\sim}{M}_N}{\partial t} + \left[\frac{\partial \underset{\sim}{u}}{\partial t}\underset{\sim}{M}_{N-1}\right]^s ,$$

where $M_N = \langle m\underset{\sim}{\nu}\cdots\underset{\sim}{\nu}f\rangle$, with N factors of $\underset{\sim}{\nu}$, $N \geq 1$, is the N^{th} order velocity moment tensor and s indicates the symmetrizing operation introduced in Section 4. 1 and discussed in Section A. 3 below. Similarly, we obtain from the second term of Eq. (3. 1)

$$\langle m(\underset{\sim}{\nu}\cdots\underset{\sim}{\nu})(\underset{\sim}{u}+\underset{\sim}{\nu})\cdot\nabla f\rangle = \underset{\sim}{u}\cdot\nabla\underset{\sim}{M}_N + [\underset{\sim}{M}_{N-1}\underset{\sim}{u}\cdot\nabla\underset{\sim}{u}]^s$$

$$+ \nabla\cdot\underset{\sim}{M}_{N+1} + \underset{\sim}{M}_N\nabla\cdot\underset{\sim}{u} + [\underset{\sim}{M}_N\cdot\nabla\underset{\sim}{u}]^s . \qquad (A.2)$$

Finally, the remaining term of the Vlasov equation gives

$$\langle m(\underset{\sim}{\nu}\cdots\underset{\sim}{\nu})[(e/m)(\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B}) + \Omega\underset{\sim}{\nu}\times\underset{\sim}{B}]\cdot\nabla_{\underset{\sim}{v}} f\rangle$$

$$= -(e/m)(\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B})\langle m[\nabla_{\underset{\sim}{v}}(\underset{\sim}{\nu}\cdots\underset{\sim}{\nu})]f\rangle - \Omega\langle m[\nabla_{\underset{\sim}{\nu}}\cdot(\underset{\sim}{\nu}\times\underset{\sim}{b})(\underset{\sim}{\nu}\cdots\underset{\sim}{\nu})]f\rangle$$

$$= -[(e/m)(\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B})\underset{\sim}{M}_{N-1}]^s - \Omega[\underset{\sim}{M}_N\times\underset{\sim}{b}]^s . \qquad (A.3)$$

Combining terms yields

$$\frac{d\underset{\sim}{M}_N}{dt} + \underset{\sim}{M}_N\nabla\cdot\underset{\sim}{u} + \nabla\cdot\underset{\sim}{M}_{N+1} + [\underset{\sim}{M}_N\cdot\nabla\underset{\sim}{u}]^s$$

$$+ \rho^{-1}\left\{\underset{\sim}{M}_{N-1}\left[\frac{d\underset{\sim}{u}}{dt} - \frac{e}{m}(\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B})\right]\right\} = \Omega(\underset{\sim}{M}_N\times\underset{\sim}{b}) . \qquad (A.4)$$

Equation (4.7) is obtained by use of Eqs. (4.2) and (4.3) in the form

$$d\underset{\sim}{u}/dt - (e/m)[\underset{\sim}{E}+\underset{\sim}{u}\times\underset{\sim}{B}] = -\rho^{-1}\nabla\cdot\underset{\sim}{P}$$

to simplify the last term of the left-hand side of Eq. (A.4). Equations (4.4), (4.5), and (4.6) are obtained from (4.7) for n equals 2, 3, and 4, respectively.

A. 2 Tensor Notation.

Use of direct products of vectors and tensors is very convenient for our discussion. We adopt the convention that a dot or cross product between two tensors (or a vector and a tensor) operates on the vector factors adjacent to it. Thus, if $\underset{\sim}{R} = \underset{\sim}{a}_1\underset{\sim}{a}_2 \cdots \underset{\sim}{a}_N$ and $\underset{\sim}{R}' = \underset{\sim}{a}'_1\underset{\sim}{a}'_2 \cdots \underset{\sim}{a}'_M$, with $\underset{\sim}{a}_i$ and $\underset{\sim}{a}'_i$ arbitrary vectors, the $N + M - 2$ rank tensors

$$\underset{\sim}{R} \cdot \underset{\sim}{R}' = (\underset{\sim}{a}_N \cdot \underset{\sim}{a}'_1)(\underset{\sim}{a}_1 \cdots \underset{\sim}{a}_{N-1})(\underset{\sim}{a}'_2 \cdots \underset{\sim}{a}_M)$$

and

$$\underset{\sim}{R}' \cdot \underset{\sim}{R} = (\underset{\sim}{a}'_M \cdot \underset{\sim}{a}_1)(\underset{\sim}{a}'_1 \cdots \underset{\sim}{a}'_{M-1})(\underset{\sim}{a}_2 \cdots \underset{\sim}{a}_N) .$$

If there is more than one dot between two tensors, the scalar products are between successive pairs of vector factors, working out from the center. For example, the $N + M - 2S$ rank tensor

$$\underset{S \le \frac{1}{2}(N+M) \text{ dots}}{\underset{\sim}{R} \vdots \underset{\sim}{R}'} = (\underset{\sim}{a}_N \cdot \underset{\sim}{a}'_1)(\underset{\sim}{a}_{N-1} \cdot \underset{\sim}{a}'_2) \cdots (\underset{\sim}{a}_{N-S+1} \cdot \underset{\sim}{a}'_S) \times (\underset{\sim}{a}_1 \cdots \underset{\sim}{a}_{N-S})(\underset{\sim}{a}'_{S+1} \cdots \underset{\sim}{a}'_M) .$$

The cross product between two tensors is handled in the same way,

$$\underset{\sim}{R} \times \underset{\sim}{R}' = (\underset{\sim}{a}_1 \cdots \underset{\sim}{a}_{N-1})(\underset{\sim}{a}_N \times \underset{\sim}{a}'_1)(\underset{\sim}{a}'_2 \cdots \underset{\sim}{a}'_M)$$

being of rank $N + M - 1$.

The gradient operator behaves in the same way as a vector as far as its vector properties are concerned. The differentiation acts on all factors to its right, up to the first parenthesis.

A. 3 The Symmetry Operation $[\ ]^s$.

Because the velocity moment tensors $\mathbf{M}_N$ are completely symmetric, most of the tensor expressions of interest to us are also completely symmetric, or at least symmetric with respect to all but one vector factor. Expressions of this form may be symmetrized by summing over all cyclic permutations of their vector factors, this operation being indicated by the superscript s.

When manipulating expressions of this form care must be taken to include all terms. Thus, if $\mathbf{b}$ and $\mathbf{u}$ are arbitrary vectors and $\mathbf{T}_N$ an arbitrary symmetric tensor of rank N $(N \geq 1)$,

$$[\mathbf{b}\mathbf{T}_N]^s \cdot \mathbf{u} = \mathbf{u} \cdot [\mathbf{b}\mathbf{T}_N]^s = (\mathbf{u} \cdot \mathbf{b})\mathbf{T}_N + [(\mathbf{u} \cdot \mathbf{T}_N)\mathbf{b}]^s . \qquad (A.5)$$

Note that $[\mathbf{b}\mathbf{T}_N]^s$, and also $[\mathbf{b}\mathbf{T}_N]^s \cdot \mathbf{u}$, are sums of $N + 1$ terms, those of the latter expression being $(\mathbf{u} \cdot \mathbf{b})\mathbf{T}_N$ and the N terms of $[(\mathbf{u} \cdot \mathbf{T}_N)\mathbf{b}]^s$.

Care must be exercised when applying the gradient operator to these symmetrized expressions. For example,

$$\nabla \cdot [\mathbf{b}\mathbf{T}_N] = (\nabla \cdot \mathbf{b})\mathbf{T}_N + \mathbf{b} \cdot \nabla \mathbf{T}_N + [\mathbf{b}\nabla \cdot \mathbf{T}_N]^s + [\mathbf{T}_N \cdot \nabla \mathbf{b}]^s . \qquad (A.6)$$

The dot product in this expression is between ∇ and the left-hand vector factor of each of the N + 1 terms of $(\underset{\sim}{b}\underset{\sim}{T}_N)^s$, the first two terms arising from that term of $(\underset{\sim}{b}\underset{\sim}{T}_N)^s$ for which this factor is $\underset{\sim}{b}$.

An expression which is unsymmetric with respect to more than one vector may be symmetrized by repeated application of the operation s. This has been done in the representation (4.31) for $\underset{\sim}{R}$, which is discussed below.

A.4 The Tensors $\underset{\sim}{I}_\gamma$ and $\underset{\sim}{I}_\delta$.

It is convenient to introduce the tensors

$$\underset{\sim}{I}_\perp = \underset{\sim}{e}_2\underset{\sim}{e}_2 + \underset{\sim}{e}_3\underset{\sim}{e}_3, \qquad \underset{\sim}{I}_\delta = \underset{\sim}{e}_2\underset{\sim}{e}_2 - \underset{\sim}{e}_3\underset{\sim}{e}_3,$$
$$\underset{\sim}{I}_\gamma = \underset{\sim}{e}_2\underset{\sim}{e}_3 + \underset{\sim}{e}_3\underset{\sim}{e}_2, \quad \text{and} \quad \underset{\sim}{I}_x = \underset{\sim}{e}_2\underset{\sim}{e}_3 - \underset{\sim}{e}_3\underset{\sim}{e}_2. \qquad (A.7)$$

The tensor $\underset{\sim}{I}_\perp$ is just the transverse projection operator. The rotation operator $\underset{\sim}{I}_x$ is related to the cross product by

$$\underset{\sim}{a}\cdot\underset{\sim}{I}_x = -\underset{\sim}{I}_x\cdot\underset{\sim}{a} = \underset{\sim}{b}\times\underset{\sim}{a} \quad \text{and} \quad \underset{\sim}{a}\underset{\sim}{c}:\underset{\sim}{I}_x = \underset{\sim}{b}\cdot(\underset{\sim}{a}\times\underset{\sim}{c}). \qquad (A.8)$$

The operators $\underset{\sim}{I}_\xi$ ($\xi = \perp, \gamma, \delta, x$) have the properties:

$$\underset{\sim}{I}_\xi:\underset{\sim}{I}_{\xi'} = 0, \qquad \xi' \neq \xi;$$
$$\underset{\sim}{I}_\perp:\underset{\sim}{I}_\perp = \underset{\sim}{I}_\gamma:\underset{\sim}{I}_\gamma = \underset{\sim}{I}_\delta:\underset{\sim}{I}_\delta = -\underset{\sim}{I}_x:\underset{\sim}{I}_x = 2; \qquad (A.9)$$

$$\underset{\sim}{I}_\gamma \cdot \underset{\sim}{I}_\gamma = \underset{\sim}{I}_\delta \cdot \underset{\sim}{I}_\delta = -\underset{\sim}{I}_x \cdot \underset{\sim}{I}_x = \underset{\sim}{I}_\perp;$$

$$\underset{\sim}{I}_\perp \cdot \underset{\sim}{I}_\xi = \underset{\sim}{I}_\xi; \quad \underset{\sim}{I}_\delta \cdot \underset{\sim}{I}_\gamma = -\underset{\sim}{I}_\gamma \cdot \underset{\sim}{I}_\delta = \underset{\sim}{I}_x; \tag{A.10}$$

$$\underset{\sim}{I}_\gamma \cdot \underset{\sim}{I}_x = \underset{\sim}{b} \times \underset{\sim}{I}_\gamma = -\underset{\sim}{I}_x \cdot \underset{\sim}{I}_\gamma = -\underset{\sim}{I}_\gamma \times \underset{\sim}{b} = -\underset{\sim}{I}_\delta,$$

$$\underset{\sim}{I}_\delta \cdot \underset{\sim}{I}_x = \underset{\sim}{b} \times \underset{\sim}{I}_\delta = -\underset{\sim}{I}_x \cdot \underset{\sim}{I}_\delta = -\underset{\sim}{I}_\delta \times \underset{\sim}{b} = +\underset{\sim}{I}_\gamma, \tag{A.11}$$

$$\underset{\sim}{b} \times \underset{\sim}{I}_x = \underset{\sim}{I}_x \times \underset{\sim}{b} = \underset{\sim}{I}_\perp;$$

$$\underset{\sim}{I}_\delta \cdot (\underset{\sim}{b} \times \underset{\sim}{a}) = (\underset{\sim}{I}_\delta \times \underset{\sim}{b}) \cdot \underset{\sim}{a} = -\underset{\sim}{I}_\gamma \cdot \underset{\sim}{a},$$

$$\underset{\sim}{I}_\gamma \cdot (\underset{\sim}{b} \times \underset{\sim}{a}) = (\underset{\sim}{I}_\gamma \times \underset{\sim}{b}) \cdot \underset{\sim}{a} = +\underset{\sim}{I}_\delta \cdot \underset{\sim}{a}. \tag{A.12}$$

These rather trivial identities simplify the calculations considerably. For example

$$(\underset{\sim}{I}_\gamma \underset{\sim}{q}) : \underset{\sim}{I}_\gamma = \underset{\sim}{q} \cdot (\underset{\sim}{I}_\gamma \cdot \underset{\sim}{I}_\gamma) = \underset{\sim}{q}_\perp = (\underset{\sim}{I}_\delta \underset{\sim}{q}) : \underset{\sim}{I}_\delta,$$

$$(\underset{\sim}{I}_\gamma \underset{\sim}{q}) : \underset{\sim}{I}_\delta = \underset{\sim}{q} \cdot (\underset{\sim}{I}_\delta \cdot \underset{\sim}{I}_\gamma) = \underset{\sim}{q} \cdot \underset{\sim}{I}_x = \underset{\sim}{b} \times \underset{\sim}{q}, \tag{A.13}$$

and

$$(\underset{\sim}{I}_\delta \underset{\sim}{q}) : \underset{\sim}{I}_\gamma = \underset{\sim}{q} \cdot (\underset{\sim}{I}_\gamma \cdot \underset{\sim}{I}_\delta) = -\underset{\sim}{q} \cdot \underset{\sim}{I}_x = -\underset{\sim}{b} \times \underset{\sim}{q}.$$

Because of the orthogonality relations (A. 9) it is clear that the transverse part of any second-rank tensor (i. e., the 22, 23, 32, and 33 components) may be written as a unique linear combination of the $\underset{\sim}{I}_\xi$. This representation is discussed in the following section.

A. 5 Representation of $\underset{\sim}{P}$, $\underset{\sim}{Q}$, and $\underset{\sim}{R}$.

Because of the orthogonality relations (A. 9) an arbitrary second-rank tensor $\underset{\sim}{T}$ can be written in the form

$$\underset{\sim}{T} = T_\perp \underset{\sim}{I}_\perp + T_\gamma \underset{\sim}{I}_\gamma + T_\delta \underset{\sim}{I}_\delta + T_x \underset{\sim}{I}_x + \underset{\sim}{b}(\underset{\sim}{b}\cdot\underset{\sim}{T})_\perp + (\underset{\sim}{T}\cdot\underset{\sim}{b})_\perp \underset{\sim}{b} + T_\parallel \underset{\sim}{b}\underset{\sim}{b}, \tag{A. 14}$$

where $T_\xi \equiv \frac{1}{2}\underset{\sim}{T} : \underset{\sim}{I}_\xi$ and $T_\parallel \equiv \underset{\sim}{T} : \underset{\sim}{b}\underset{\sim}{b}$.

By symmetrizing this expression we obtain

$$[\underset{\sim}{T}]^s = 2(\underset{\sim}{I}_\perp \underset{\sim}{I}_\perp + \underset{\sim}{I}_\gamma \underset{\sim}{I}_\gamma + \underset{\sim}{I}_\delta \underset{\sim}{I}_\delta) : \underset{\sim}{T} + \{\underset{\sim}{b}[\underset{\sim}{b}\cdot(\underset{\sim}{T})^s]_\perp\}^s + 2T_\parallel \underset{\sim}{b}\underset{\sim}{b} \tag{A. 15}$$

which reduces to the form (4. 9) and (4. 13) for a symmetric tensor $\underset{\sim}{P}$.

To show that Eq. (4. 21) is a representation of $\underset{\sim}{Q}$ consistent with the definitions (4. 19), and that the symmetry of $\underset{\sim}{Q}$ implies the condition (4. 20), we note that the definitions (4. 19) are recovered from (4. 22) by forming the contractions of this expression from the left with $\underset{\sim}{b}\underset{\sim}{b}$, $\underset{\sim}{I}_\perp$, $\underset{\sim}{I}_\gamma$, and $\underset{\sim}{I}_\delta$. Since the expression (4. 22) is clearly symmetric with respect to its first two factors, the necessary and sufficient condition that it be completely symmetric is just

$$(\underset{\sim}{I}_\perp \underset{\sim}{q}^\perp_\perp + \underset{\sim}{I}_\gamma \underset{\sim}{q}^\gamma_\perp + \underset{\sim}{I}_\delta \underset{\sim}{q}^\delta_\perp) : \underset{\sim}{I}_x = 0,$$

or

$$\underset{\sim}{b} \times \underset{\sim}{q}^\perp + \underset{\sim}{I}_\gamma \cdot \underset{\sim}{b} \times \underset{\sim}{q}^\gamma + \underset{\sim}{I}_\delta \cdot \underset{\sim}{b} \times \underset{\sim}{q}^\delta$$

$$= \underset{\sim}{b} \times (\underset{\sim}{q}^\perp_\perp - \underset{\sim}{I}_\gamma \cdot \underset{\sim}{q}^\gamma - \underset{\sim}{I}_\delta \cdot \underset{\sim}{q}^\delta) = 0. \tag{A. 16}$$

This last condition is identical with (4.20). We conclude, therefore, that this condition and the representation (4.22) follow from the symmetry of $\underset{\sim}{Q}$ and the definitions (4.19). These conclusions may also be verified by direct evaluation of the components of the various expressions.

We outline now the derivation of the representation (4.31) for the ϕ-independent part of the fourth velocity moment tensor $\underset{\sim}{R}$. This tensor may be written

$$\underset{\sim}{R} = \sum_{(h,i,j,k)} R_{hijk} (\underset{\sim}{e}_h \underset{\sim}{e}_i \underset{\sim}{e}_j \underset{\sim}{e}_k)^p ,$$

where the sum is carried out over the fifteen distinct sets (h, i, j, k), $1 \le h, i, j, k \le 3$; and the superscript p indicates a sum over all distinct permutations of the vectors $\underset{\sim}{e}_h, \underset{\sim}{e}_i, \underset{\sim}{e}_j$, and $\underset{\sim}{e}_k$.

We require a representation for only the ϕ-independent part of $\underset{\sim}{R}$. It is easily verified that there are ϕ-independent contributions only to the (1111), (1122), (1133), (2222), (2233), and (3333) terms of (A.17). The corresponding values of R_{hijk} are

$$R_{1111} = R_1 \equiv \langle m\nu_{\parallel}^4 f\rangle ,$$

$$R_{1122} = R_{1133} = \langle m\nu_{\parallel}^2 \nu_{\perp}^2 \cos^2\phi\rangle = R_2 \equiv (m/2)\langle \nu_{\parallel}^2 \nu_{\perp}^2 f\rangle ,$$

$$R_{2233} = (1/4) R_3 = (m/8)\langle \nu_{\perp}^4 f\rangle ,$$

and

$$R_{2222} = R_{3333} = (3/4) R_3 .$$

Equation (A. 17) thus takes the form

$$\underset{\sim}{R} - \underset{\sim}{R}_\phi = R_1 \underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b} + R_2 [\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{I}_\perp]^p$$

$$+ (R_3/4)[3(\underset{\sim}{e}_2\underset{\sim}{e}_2\underset{\sim}{e}_2\underset{\sim}{e}_2 + \underset{\sim}{e}_3\underset{\sim}{e}_3\underset{\sim}{e}_3\underset{\sim}{e}_3) + (\underset{\sim}{e}_2\underset{\sim}{e}_2\underset{\sim}{e}_3\underset{\sim}{e}_3)^p]. \quad (A. 18)$$

That this expression is equivalent to the representation (4. 31) may be seen by use of the identities

$$2[\underset{\sim}{e}_i\underset{\sim}{e}_i\underset{\sim}{e}_j\underset{\sim}{e}_j]^p = [\underset{\sim}{e}_i(\underset{\sim}{e}_i\underset{\sim}{e}_j\underset{\sim}{e}_j)^s]^s, \quad i \neq j,$$

and

$$12\underset{\sim}{e}_i\underset{\sim}{e}_i\underset{\sim}{e}_i\underset{\sim}{e}_i = [\underset{\sim}{e}_i(\underset{\sim}{e}_i\underset{\sim}{e}_i\underset{\sim}{e}_i)^s]^s.$$

These identities follow from the definitions of the operations p and s.

A. 6 Evaluation of $\underset{\approx}{\Pi}^{(1)}$.

By use of the identities (A. 11) we find from the representation (A. 15)

$$[\underset{\sim}{P} \times \underset{\sim}{b}]^s = P_{23}\underset{\sim}{I}_\delta - P_\delta \underset{\sim}{I}_\gamma + [\underset{\sim}{b}(\underset{\sim}{b}\cdot\underset{\sim}{P}) \times \underset{\sim}{b}]^s. \quad (A. 20)$$

The recursion relations (4. 14) and (4. 15) follow immediately from Eq. (4. 4) by use of this result and the orthogonality relations (A. 9). The second form of Eq. (4. 15) is obtained by the use of Eqs. (A. 10) and (A. 15), which give

$$[\underset{\sim}{I}_\gamma \underset{\sim}{I}_\delta - \underset{\sim}{I}_\delta \underset{\sim}{I}_\gamma] : \underset{\sim}{G} = \underset{\sim}{b} \times [\underset{\sim}{I}_\gamma \underset{\sim}{I}_\gamma + \underset{\sim}{I}_\delta \underset{\sim}{I}_\delta] : \underset{\sim}{G} = \underset{\sim}{b} \times [(\underset{\sim}{G})^s \cdot \underset{\sim}{I}_\perp - \underset{\sim}{I}_\perp \underset{\sim}{I}_\perp : \underset{\sim}{G}] .$$

The first-order terms of $\underset{\sim}{\Pi}$ are obtained by substituting $\underset{\sim}{G}^{(0)}$ into Eq. (4.14) or (4.15). Because $\underset{\sim}{\Pi}^{(0)}$ and $\underset{\sim}{Q}_\phi^{(0)}$ vanish,

$$\begin{aligned}
\underset{\sim}{G} = \{ & \underset{\sim}{b}[(p_\parallel - p_\perp) d\underset{\sim}{b}/dt + p_\parallel (\underset{\sim}{b} \cdot \nabla \underset{\sim}{u})_\perp + p_\perp (\nabla_\perp \underset{\sim}{u}) \cdot \underset{\sim}{b} + 2(q_\parallel^\parallel - q_\parallel^\perp) \underset{\sim}{b} \cdot \nabla \underset{\sim}{b} + \nabla_\perp q_\parallel^\perp] \}^s \\
& + p_\perp \{ [(\nabla_\perp \underset{\sim}{u}) \cdot \underset{\sim}{I}_\perp]^s - \underset{\sim}{I}_\perp \nabla_\perp \cdot \underset{\sim}{u} \} + q_\parallel^\perp [(\nabla_\perp \underset{\sim}{b})^s - \underset{\sim}{I}_\perp \nabla_\perp \cdot \underset{\sim}{b}] \\
& + [dp_\parallel /dt + p_\parallel (2\nabla \cdot \underset{\sim}{u} + \nabla_\parallel \cdot \underset{\sim}{u}) + 2\nabla \cdot (q_\parallel^\parallel \underset{\sim}{b}) - 2 q_\parallel^\perp \nabla \cdot \underset{\sim}{b}] \underset{\sim}{b}\underset{\sim}{b} \\
& + [dp_\perp /dt + p_\perp (\nabla \cdot \underset{\sim}{u} + \nabla_\perp \cdot \underset{\sim}{u}) + \underset{\sim}{b} \cdot \nabla (q_\parallel^\perp / B^2)] \underset{\sim}{I}_\perp .
\end{aligned} \qquad (A.22)$$

When substituted into (4.14) or (4.15) the first three terms of this expression give $\underset{\sim}{\Pi}^{(1)}$ immediately. The last two terms give the CGL pressure equations (5.4) and (5.5).

A.7 Evaluation of $\underset{\sim}{Q}_\phi^{(1)}$.

The recursion equations for $\underset{\sim}{q}_\perp^\parallel$, $\underset{\sim}{q}_\perp^\perp$, $\underset{\sim}{q}^\gamma$, and $\underset{\sim}{q}^\delta$ are found by use of the representation (4.22) to obtain, from Eq. (4.5),

$$\begin{aligned}
[\underset{\sim}{Q} \times \underset{\sim}{b}]^s = & \; 2[\underset{\sim}{b}\underset{\sim}{b}(\underset{\sim}{q}_\perp^\parallel \times \underset{\sim}{b})]^s - 2q_\parallel^\delta (\underset{\sim}{b}\underset{\sim}{I}_\gamma)^s + 2q_\parallel^\gamma (\underset{\sim}{b}\underset{\sim}{I}_\delta)^s \\
& + [\underset{\sim}{I}_\perp \underset{\sim}{q}_\perp^\perp \times \underset{\sim}{b}]^s + [\underset{\sim}{I}_\gamma (\underset{\sim}{q}_\perp^\gamma \times \underset{\sim}{b})]^s + [\underset{\sim}{I}_\delta (\underset{\sim}{q}_\perp^\delta \times \underset{\sim}{b})]^s \qquad (A.23) \\
= & \; \Omega^{-1} \underset{\sim}{H} .
\end{aligned}$$

By contraction with $\underset{\sim}{b}\underset{\sim}{b}$, $\underset{\sim}{b}\underset{\sim}{I}_{\gamma}$, and $\underset{\sim}{b}\underset{\sim}{I}_{\delta}$ this result immediately yields the recursion equations (4. 23), (4. 27), and (4. 28) for $\underset{\sim}{q}_{\perp}^{\parallel}$, $q_{\parallel}^{\gamma}$, and $q_{\parallel}^{\delta}$. Contraction of Eq. (A. 23) with $\underset{\sim}{I}_{\perp}$ gives the equation (4. 24) for $\underset{\sim}{q}_{\perp}^{\perp}$:

$$
\begin{aligned}
(2/\Omega)\underset{\sim}{b} \times \underset{\sim}{H} : \underset{\sim}{I}_{\perp} &= \tfrac{1}{2}\underset{\sim}{b} \times [\, 4\underset{\sim}{q}_{\perp}^{\perp} \times \underset{\sim}{b} + 2\underset{\sim}{I}_{\gamma} \cdot (\underset{\sim}{q}_{\perp}^{\gamma} \times \underset{\sim}{b}) + 2\underset{\sim}{I}_{\delta} \cdot (\underset{\sim}{q}_{\perp}^{\delta} \times \underset{\sim}{b})] \\
&= \tfrac{1}{2}\underset{\sim}{b} \times [\, 4\underset{\sim}{q}_{\perp}^{\perp} - 2\underset{\sim}{I}_{\gamma} \cdot \underset{\sim}{q}^{\gamma} - 2\underset{\sim}{I}_{\delta} \cdot \underset{\sim}{q}^{\delta}] \times \underset{\sim}{b} = \underset{\sim}{q}_{\perp}^{\perp}.
\end{aligned}
\tag{A. 24}
$$

To obtain the recursion equations (4. 25) and (4. 26) for $\underset{\sim}{q}_{\perp}^{\gamma}$ and $\underset{\sim}{q}_{\perp}^{\delta}$ we note that

$$
\begin{aligned}
\underset{\sim}{I}_{\gamma} : [\underset{\sim}{Q} \times \underset{\sim}{b}]^{s} &= -2\underset{\sim}{I}_{\delta} \cdot \underset{\sim}{q}_{\perp}^{\perp} + 4\underset{\sim}{q}_{\perp}^{\gamma} \times \underset{\sim}{b} - 2\underset{\sim}{q}_{\perp}^{\delta}, \\
\underset{\sim}{I}_{\delta} : [\underset{\sim}{Q} \times \underset{\sim}{b}]^{s} &= 2\underset{\sim}{I}_{\gamma} \cdot \underset{\sim}{q}_{\perp}^{\perp} + 4\underset{\sim}{q}_{\perp}^{\delta} \times \underset{\sim}{b} + 2\underset{\sim}{q}_{\perp}^{\gamma}.
\end{aligned}
\tag{A. 25}
$$

The solutions of these equations for $\underset{\sim}{q}_{\perp}^{\gamma}$ and $\underset{\sim}{q}_{\perp}^{\delta}$ are just Eqs. (4. 25) and (4. 26).

To find $\underset{\sim}{Q}_{\phi}^{(1)}$ we note that

$$
\begin{aligned}
\frac{d\underset{\sim}{Q}}{dt} &= 2\,\frac{dq_{\parallel}^{\parallel}}{dt}\,\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b} + 2(q_{\parallel}^{\parallel} - q_{\parallel}^{\perp})\left(\underset{\sim}{b}\underset{\sim}{b}\,\frac{d\underset{\sim}{b}}{dt}\right)^{s} + \frac{dq_{\parallel}^{\perp}}{dt}(\underset{\sim}{b}\underset{\sim}{I}_{\perp})^{s} + q_{\parallel}^{\perp}\left(\underset{\sim}{I}_{\perp}\,\frac{d\underset{\sim}{b}}{dt}\right)^{s}, \\
(\underset{\sim}{Q} \cdot \nabla\underset{\sim}{u})^{s} &= 2q_{\parallel}^{\parallel}[\underset{\sim}{b}\underset{\sim}{b}(\underset{\sim}{b} \cdot \nabla\underset{\sim}{u})]^{s} + q_{\parallel}^{\perp}[\underset{\sim}{b}(\nabla_{\perp}\underset{\sim}{u})^{s}]^{s},
\end{aligned}
\tag{A. 26}
$$

and

$$
(\underset{\sim}{P}\nabla \cdot \underset{\sim}{P})^{s} = p_{\perp}(\underset{\sim}{I}_{\perp}\nabla \cdot \underset{\sim}{P})^{s} + p_{\parallel}(\underset{\sim}{b}\underset{\sim}{b}\nabla \cdot \underset{\sim}{P})^{s},
$$

where all quantities are taken to zero order.

From this result and the expression for $\nabla\cdot R^{(0)}$ derived below we obtain

$$\underset{\sim}{H} = \{\underset{\sim}{b}\underset{\sim}{b}[\nabla_\perp R_2 - \rho^{-1} p_\parallel (\nabla\cdot\underset{\sim}{P})_\perp + (R_1 - 3R_2)\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}$$
$$+ 2q_\parallel^\parallel (\underset{\sim}{b}\cdot\nabla\underset{\sim}{u})_\perp + q_\parallel^\perp[(\nabla_\perp\underset{\sim}{u})\cdot\underset{\sim}{b}]^s + 2(q_\parallel^\parallel - q_\parallel^\perp) d\underset{\sim}{b}/dt]\}^s$$
$$+ \{\underset{\sim}{I}_\perp[\tfrac{1}{4}\nabla_\perp R_3 - \rho^{-1} p_\perp (\nabla\cdot\underset{\sim}{P})_\perp + (R_2 - R_3/4)\underset{\sim}{b}\cdot\nabla\underset{\sim}{b} + q_\parallel^\perp d\underset{\sim}{b}/dt]\}^s$$
$$+ \{\underset{\sim}{b}[(R_2 - R_3/4)(\nabla_\perp\underset{\sim}{b})^s + q_\parallel^\perp(\nabla_\perp\underset{\sim}{u})^s]\}^s, \qquad (A.27)$$

where terms proportional to $\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}$ and $(\underset{\sim}{b}\underset{\sim}{I}_\perp)^s$ have been dropped since they do not contribute to $\underset{\sim}{Q}_\phi$. (These terms give the zero-order terms of the "equations of motion" (4.29) and (4.30) for $q_\parallel^\perp$ and $q_\parallel^\parallel$.) The results (5.12) through (5.15) for the components of $\underset{\sim}{Q}_\phi^{(1)}$ follow immediately from this expression and the recursion relations (4.23) through (4.28).

A.8 Evaluation of $\nabla\cdot\underset{\sim}{R}^{(0)}$.

Calculation of the divergence of the ϕ-independent part of $\underset{\sim}{R}$ from the representation (4.31) requires evaluation of the expressions

$$\nabla\cdot\{R_j[\underset{\sim}{e}_i(\underset{\sim}{e}_i\underset{\sim}{I}_\perp)^s]^s\} = [\underset{\sim}{e}_i(\underset{\sim}{e}_i\underset{\sim}{I}_\perp)^s]^s\cdot\nabla R_j + R_j\nabla\cdot[\underset{\sim}{e}_i(\underset{\sim}{e}_i\underset{\sim}{I}_\perp)^s]^s . \qquad (A.28)$$

It is easily verified that

$$[\underset{\sim}{e}_i(\underset{\sim}{e}_i\underset{\sim}{I}_\perp)^s]^s\cdot\nabla R_j = 2[(\underset{\sim}{e}_i\underset{\sim}{I}_\perp)^s\underset{\sim}{e}_i\cdot\nabla R_j + (\underset{\sim}{e}_i\underset{\sim}{e}_i\nabla_\perp R)^s],$$

hence

$$[\underset{\sim}{b}(\underset{\sim}{b}\underset{\sim}{I}_{\perp})^{s}]^{s} \cdot \nabla R_{j} = 2[(\underset{\sim}{b}\underset{\sim}{I}_{\perp})^{s}\underset{\sim}{b} \cdot \nabla R_{j} + (\underset{\sim}{b}\underset{\sim}{b}\nabla_{\perp} R_{j})^{s}],$$

and

$$\sum_{i=2,3}[\underset{\sim}{e}_{i}(\underset{\sim}{e}_{i}\underset{\sim}{I}_{\perp})^{s}]^{s} \cdot \nabla R_{j} = 4(\underset{\sim}{I}_{\perp}\nabla_{\perp} R)^{s}. \qquad (A.29)$$

By use of (A. 6) and the identity

$$[\underset{\sim}{a}(\underset{\sim}{b}\underset{\sim}{c})^{s}]^{s} = [(\underset{\sim}{a}\underset{\sim}{b})^{s}\underset{\sim}{c}]^{s},$$

valid for arbitrary vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$, the second term of the expression (A. 28) becomes

$$\nabla \cdot [\underset{\sim}{e}_{i}(\underset{\sim}{e}_{i}\underset{\sim}{I}_{\perp})^{s}]^{s} = \tfrac{1}{2}\{(\underset{\sim}{e}_{i}\underset{\sim}{I}_{\perp})^{s}\nabla \cdot \underset{\sim}{e}_{i} + [\underset{\sim}{I}_{\perp}(\underset{\sim}{e}_{i} \cdot \nabla \underset{\sim}{e}_{i})]^{s} + 2[\underset{\sim}{e}_{i}(\underset{\sim}{e}_{i} \cdot \nabla \underset{\sim}{I}_{\perp})]^{s}$$
$$+ [\underset{\sim}{e}_{i}(\nabla_{\perp}\underset{\sim}{e}_{i})^{s}]^{s} + (\underset{\sim}{e}_{i}\underset{\sim}{e}_{i}\nabla \cdot \underset{\sim}{I}_{\perp})^{s}\}, \qquad (A.30)$$

with

$$\nabla \underset{\sim}{I}_{\perp} = -(\nabla \underset{\sim}{b})\underset{\sim}{b} - \underset{\sim}{b}\nabla \underset{\sim}{b}.$$

For i = 1, Eq. (A. 30) reduces to

$$\nabla \cdot [\underset{\sim}{b}(\underset{\sim}{b}\underset{\sim}{I}_{\perp})^{s}]^{s} = 2\{[(\underset{\sim}{b}\underset{\sim}{I}_{\perp})^{s} - 3\underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b}]\nabla \cdot \underset{\sim}{b} + [\underset{\sim}{I}_{\perp}(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b})]^{s}$$
$$+ [\underset{\sim}{b}(\nabla_{\perp}\underset{\sim}{b})^{s}]^{s} - 3[\underset{\sim}{b}\underset{\sim}{b}(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b})]^{s}\}. \qquad (A.31)$$

To evaluate the sum of (A. 30) for i = 2, 3, note that

$$\nabla \cdot \underset{\sim}{e}_i = \sum_{j=1}^{3} (\underset{\sim}{e}_j \cdot \nabla \underset{\sim}{e}_i) \cdot \underset{\sim}{e}_j = -(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}) \cdot \underset{\sim}{e}_i - \sum_{j=2,3} (\underset{\sim}{e}_j \cdot \nabla \underset{\sim}{e}_j) \cdot \underset{\sim}{e}_i ,$$

and

$$\nabla_\perp \underset{\sim}{e}_i = -(\nabla_\perp \underset{\sim}{b}) \cdot \underset{\sim}{e}_i \underset{\sim}{b} + (\nabla_\perp \underset{\sim}{e}_i) \cdot \underset{\sim}{I}_\perp .$$

By use of these results

$$\sum_{i=2,3} \nabla \cdot [\underset{\sim}{e}_i (e_i \underset{\sim}{I}_\perp)^s]^s = 4\{[(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b} + \underset{\sim}{b} \nabla \cdot \underset{\sim}{b}) \underset{\sim}{I}_\perp]^s - [\underset{\sim}{b}(\nabla_\perp \underset{\sim}{b})^s]^s\} . \qquad (A.32)$$

Combining Eqs. (4. 31), (A. 28), (A. 31), and (A. 32) yields, finally

$$\begin{aligned} \nabla \cdot (\underset{\sim}{R} - \underset{\sim}{R}_\phi) = & [\underset{\sim}{b} \cdot \nabla R_1 + (R_1 - 3R_2) \nabla \cdot \underset{\sim}{b}] \underset{\sim}{b}\underset{\sim}{b}\underset{\sim}{b} \\ & + [\underset{\sim}{b} \cdot \nabla R_2 + (R_2 - R_3/4) \nabla \cdot \underset{\sim}{b}] [\underset{\sim}{b} \underset{\sim}{I}_\perp]^s \\ & + (R_1 - 3R_2) [\underset{\sim}{b}\underset{\sim}{b}(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b})]^s + [\underset{\sim}{b}\underset{\sim}{b} \nabla_\perp R_2]^s \\ & + (R_2 - R_3/4) \{[I\, b \cdot \nabla b]^s + [b(\nabla\, b)^s]^s\} \\ & + (1/4) [\underset{\sim}{I}_\perp \nabla_\perp R_3]^s . \end{aligned} \qquad (A.33)$$

A. 9 Evaluation $\underset{\sim}{\Pi}_2^{(2)}$.

The expressions (6.19) and (6.20) for $\underset{\sim}{\Pi}_2^{(2)}$ are found by setting $\underset{\sim}{G}$ equal to $\underset{\sim}{Q}_\phi^{(1)}$ in the recursion formulae (4.14) and (4.15). From Eq. (5.16)

$$\begin{aligned}\nabla\cdot\underset{\sim}{Q}_\phi^{(1)} = {} & (\nabla\cdot\underset{\sim}{q}_\perp^{\parallel})\underset{\sim}{b}\underset{\sim}{b} + [\underset{\sim}{b}\underset{\sim}{q}_\perp^{\parallel}\cdot\nabla\underset{\sim}{b} + \underset{\sim}{q}_\perp^{\parallel}\nabla\cdot\underset{\sim}{b} \\ & + \underset{\sim}{b}\underset{\sim}{b}\cdot\nabla\underset{\sim}{q}_\perp^{\parallel} + \underset{\sim}{q}_\perp^{\parallel}\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}]^s + (\nabla\cdot\underset{\sim}{q}_\perp^{\perp})\underset{\sim}{I}_\perp \\ & + [\nabla_\perp\underset{\sim}{q}_\perp^{\perp} - \underset{\sim}{b}\underset{\sim}{q}_\perp^{\perp}\cdot\nabla\underset{\sim}{b} - \underset{\sim}{q}_\perp^{\perp}\underset{\sim}{b}\nabla\cdot\underset{\sim}{b} - \underset{\sim}{q}_\perp^{\perp}\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}]^s \\ & + \nabla\cdot\underset{\sim}{Q}_{\phi 1},\end{aligned}$$

where

$$\underset{\sim}{Q}_{\phi 1} \equiv \alpha[\underset{\sim}{b}\underset{\sim}{\chi}]$$

with

$$\alpha = (2\Omega)^{-1}(R_2 - R_3/4)$$

and

$$\underset{\sim}{\chi} = \underset{\sim}{b}\times[(\nabla_\perp\underset{\sim}{b})^s - \underset{\sim}{I}_\perp\nabla\cdot\underset{\sim}{b}].$$

By straightforward calculation we find

$$[\underset{\sim}{b}\cdot(\nabla\cdot\underset{\sim}{Q}_{\phi 1})]_\perp = \alpha[(\nabla\cdot\underset{\sim}{\chi})_\perp - (\underset{\sim}{b}\cdot\nabla\underset{\sim}{b})\cdot\underset{\sim}{\chi} + \underset{\sim}{\chi}\cdot\nabla\alpha/\alpha]$$

and

$$[\underset{\sim}{I}_{\perp} \cdot (\nabla \cdot \underset{\sim}{Q}_{\phi 1}) \cdot \underset{\sim}{I}_{\perp}]^{s} - \underset{\sim}{I}_{\perp}\underset{\sim}{I}_{\perp} : \nabla \cdot \underset{\sim}{Q}_{\phi 1}$$

$$= 2[\nabla \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \nabla a/a]\underset{\sim}{\chi} + (\underset{\sim}{\chi} \cdot \nabla \underset{\sim}{b})^{s}$$

$$+ 2a\underset{\sim}{b} \times \{[\nabla_{\perp}(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}) - (\nabla_{\perp}\underset{\sim}{b}) \cdot \nabla \underset{\sim}{b}]^{s}$$

$$- \underset{\sim}{I}_{\perp}[\nabla_{\perp} \cdot (\underset{\sim}{b} \cdot \nabla \underset{\sim}{b}) - (\nabla_{\perp}\underset{\sim}{b}) : \nabla_{\perp}\underset{\sim}{b}]\}$$

$$- 2a[(\underset{\sim}{b} \cdot \nabla \underset{\sim}{b})\underset{\sim}{b} \times (\underset{\sim}{b} \cdot \nabla \underset{\sim}{b})]^{s}.$$

Equation (6.19) follows immediately from Eqs. (4.14), (A.34), and (A.35). Equation (6.20) is obtained from Eqs. (4.15), (A.34), and (A.36) by use of the result

$$\underset{\sim}{b} \times (\underset{\sim}{\chi} \cdot \nabla \underset{\sim}{b})^{s} = -a\{[(\nabla_{\perp}\underset{\sim}{b})^{s} \cdot \nabla \underset{\sim}{b} - (\nabla_{\perp}\underset{\sim}{b})\nabla \cdot \underset{\sim}{b}]^{s}$$

$$- \underset{\sim}{I}_{\perp}[(\nabla_{\perp}\underset{\sim}{b}) : (\nabla_{\perp}\underset{\sim}{b})^{s} - (\nabla \cdot \underset{\sim}{b})^{2}]\}.$$

APPENDIX B

Northrop's Paradox

From Eq. (5. 9) the contibution of the collisionless viscosity to P_{13} is

$$P_{13}^{(1)} = \div \Omega^{-1}[\ (2P_{\parallel} - P_{\perp})(\underset{\sim}{b}\cdot\nabla\underset{\sim}{u})_2 + P_{\perp}(\underset{\sim}{e}_2\cdot\nabla\underset{\sim}{u})\cdot\underset{\sim}{b}]^{(0)}\ . \tag{B. 1}$$

For the special case B = constant, $P_{\parallel} = 0$, and $\underset{\sim}{u} = u(x_1)\underset{\sim}{e}_2$ at time $t = 0$, x_1 being the coordinate in the direction $\underset{\sim}{b}$, Eq. (B. 1) reduces to

$$P_{13}^{(1)}\ (0) = \Omega^{-1} P \quad \partial u / \partial x_1 \neq 0\ . \tag{B.2}$$

A simple distribution function consistent with the above configuration is

$$f(0) = ng(\nu_{\perp}^2)\,\delta\,(v_{\parallel}),\ \text{with} \int_0^{\infty} \nu_{\perp}\, g d\nu_{\perp} = 1. \tag{B. 3}$$

This distribution yields

$$P_{13}(0) \equiv 0. \tag{B. 4}$$

There is thus a contradiction, pointed out by Northrop,[21] between the result (B. 4) obtained from the simple distribution (B. 3), and also obtained by Kaufman[2] from an examination of the particle orbits, and the result (B. 1) obtained from the adiabatic expansion of the moment equations or

Vlasov equation.

This contradiction is resolved by the discussion of initial conditions given in Section 3. 3. The initial condition (3. 15), when applied to the present example, gives

$$f_\phi^{(1)}(0) = \frac{\sin\phi}{\Omega}\left[C_\perp\,\omega\frac{\partial f}{\partial C_\parallel} - C_\parallel\frac{\partial u}{\partial x_1}\frac{\partial f}{\partial C_\perp}\right]^{(0)} . \tag{B. 5}$$

This condition is obviously not satisfied by the ϕ-independent distribution function (B. 3); the small ϵ expansion and the result (B. 2) therefore do not apply for this choice of f(0).

The non-zero f_ϕ of Eq. (B. 5) is required to obtain a slowly varying f because of the $d\underset{\sim}{b}/dt$ associated with a non-zero $(\underset{\sim}{b}\cdot\nabla\underset{\sim}{E})_\perp$ by Maxwell's induction equation. This variation of $\underset{\sim}{B}$ amounts to a rotation of the magnetic field direction at an angular frequency of order $\epsilon\Omega$. After a time of order Ω^{-1}, therefore, the particles described by the distribution (B. 3) will have velocities in the direction $\underset{\sim}{b}(t)$ of order ϵv_0. These small parallel velocities, together with the velocity of gyration, produce a P_{13} which fluctuates at the gyro-frequency and therefore is not given by Eq. (B. 2).

The discrepancy between Eq. (B. 2) or (B. 1) and Kaufman's result is also due to the time variation of $\underset{\sim}{b}$; Kaufman ignores the induction effect and assumes a magnetic field constant in space and time.

In the following we find $P_{13}(t)$ from the particle orbits obtained in a rotating magnetic field. Equation (B. 5) will be recovered from the condition that $P_{13}(t)$ be slowly varying. In addition to illustrating the relation between the initial condition on $f_\phi(0)$ and the particle motion, this calculation will complete Kaufman's discussion of the collisionless viscosity by extending his derivation to include the effect of induction.

We consider a uniform plasma in a uniform rotating magnetic field. The initial direction of the magnetic field is $\underset{\sim}{b}(0) = \underset{\sim}{e}_1(0)$. The vectors $\underset{\sim}{e}_i(0)$ define a fixed Cartesian coordinate system; in the following the spatial coordinates x_i refer to this fixed system. Vector and tensor components, however, will be referred to the basis vectors $\underset{\sim}{e}_i(t)$ which rotate with the magnetic field. We assume that $\underset{\sim}{E}$ varies only in the direction $\underset{\sim}{b}$. Its initial value is then $\underset{\sim}{E}(0) = -B\omega x_1 \underset{\sim}{e}_3$ (where ω is the magnitude of $d\underset{\sim}{b}/dt$); hence to lowest order the plasma flows initially in the direction $\underset{\sim}{e}_3(0)$ with the velocity $u(x_1) = +\omega x_1$.

Kaufman's calculation of P_{13} assumed that $\underset{\sim}{b} \cdot \nabla \underset{\sim}{E}$ was small and neglected the time variation of $\underset{\sim}{b}$. In lowest order the contributions of $\underset{\sim}{b} \cdot \nabla\underset{\sim}{E}$ and $d\underset{\sim}{b}/dt$ to P_{13} will be additive; Kaufman's calculation, therefore, may be completed by neglecting the electric field and considering the particle motion in the rotating magnetic field. For this calculation the plasma may be considered to be initially at rest.

For $\omega t \ll 1$ the equation of motion of a particle moving in the rotating magnetic field is

$$\dot{\mathbf{v}} = \Omega \mathbf{v} \times [\mathbf{e}_1(0) + (\omega t)\mathbf{e}_2(0)] . \tag{B. 6}$$

Through first order in ωt and ω/Ω the general solution of this equation for $\mathbf{v}$ may be written

$$\begin{aligned} v_1 &= c_{\parallel} - (\omega/\Omega) c_{\perp} [\sin(\phi - \Omega t) - \sin\phi] , \\ v_2 &= c_{\perp} \cos(\phi - \Omega t) , \\ v_3 &= (\omega/\Omega) c_{\parallel} + c_{\perp} \sin(\phi - \Omega t) , \end{aligned} \tag{B. 7}$$

where the velocity components are referred to the rotating axes $\mathbf{e}_i(t)$, and the parameters $c_{\perp}$, $c_{\parallel}$, and ϕ specify the initial velocity.

This solution shows that the particles spiral around the rotating field lines, drift in the direction $\mathbf{e}_3$ with the $d\mathbf{b}/dt$ drift velocity, and also oscillate with frequency Ω about their mean velocity along the field lines. More complicated orbits would be found if $\mathbf{E}$ were included in the calculation; then there would be a steady acceleration along $\mathbf{B}$.[22]

The average velocity at time t is given by

$$\langle \mathbf{v}(t) \rangle = \int d^3c\, F(\mathbf{c})\, \mathbf{v}(\mathbf{c}, t) , \tag{B. 8}$$

where

$$\mathbf{c} = c_{\perp} \cos\phi\, \mathbf{e}_2(0) + c_{\perp} \sin\phi\, \mathbf{e}_3(0) + c_{\parallel} \mathbf{e}_1(0) ,$$

and $F(c)$ describes the initial distribution function. By use of Eqs. (B. 7) and (B. 8) the condition that $\langle \mathbf{v}(0) \rangle$ equals zero implies that

$$\langle c_{\parallel} \rangle = \langle c_{\perp} \cos\phi \rangle = \langle c_{\perp} \sin\phi \rangle = 0 \,, \tag{B. 9}$$

and that $\langle \underset{\sim}{v}(t) \rangle$ remains zero.

Let the contribution of $d\underset{\sim}{b}/dt$ to $P_{13}(t)$ be $P_{13}^{(3)}$. (Kaufman denotes the contributions of $\partial u_1 / \partial x_3$ and $\partial u_3 / \partial x_1$ by $P_{13}^{(1)}$ and $P_{13}^{(2)}$, respectively.) Then, by use of Eq. (B. 9)

$$\begin{aligned} P_{13}^{(3)}(t) &= m \langle v_1(t) v_3(t) \rangle \\ &= (\omega/\Omega) \left[p_{\parallel}(0) - p_{\perp}(0) \right] \\ &+ m \langle c_{\perp} c_{\parallel} \sin\phi + (\omega/\Omega) c_{\perp}^2 \sin^2\phi \rangle \cos\Omega t \\ &- m \langle c_{\perp} c_{\parallel} \cos\phi + (\omega/2\Omega) c_{\perp}^2 \sin 2\phi \rangle \sin\Omega t \\ &+ (\omega/2\Omega) m \left[\langle c_{\perp}^2 \cos 2\phi \rangle \cos 2\Omega t - \langle c_{\perp}^2 \sin 2\phi \rangle \sin 2\Omega t \right] . \end{aligned} \tag{B. 10}$$

The first term of this expression equals the last term of (B. 1). The remaining terms show that in general P_{13} oscillates rapidly about this value. If the initial distribution function is of the form (B. 3), Eq. (B. 10) reduces to

$$P_{13}^{(3)} = -(\omega/\Omega) p_{\perp}(0) \left[1 + \cos\Omega t \right] . \tag{B. 11}$$

The failure of (B. 1) and (B. 2) to apply in this case is reflected by the rapid oscillation of $P_{13}^{(3)}$.

If P_{13} is to be slowly varying, the initial distribution function must be such that the coefficients of the oscillating terms of Eq. (B. 10) vanish, at least through first order in ω/Ω. A simple way of satisfying these conditions is to assume

$$c_{\parallel} = s - (\omega/\Omega)\, c_{\perp} \sin\phi\,, \tag{B. 12}$$

where s is a variable uncorrelated with ϕ. It is evident that Eq. (B. 12) and (B. 9) are sufficient to cause the oscillating terms of (B. 10) to vanish. From Eqs. (B. 7) and (B. 12) it is seen that some of the particles must be moving initially along the magnetic field and that oscillations of v_1 are in phase with those of v_3.

Through first order in ω/Ω, $F(\underset{\sim}{c})$ may be expressed in terms of an arbitrary function $H(c_{\perp}, s)$ by the relation

$$\begin{aligned} F(c_{\perp}, c_{\parallel}, \phi) &= H[c_{\perp}, c_{\parallel} + (\omega/\Omega)\, c_{\perp} \sin\phi] \\ &= H(c_{\perp}, c_{\parallel}) + (\omega/\Omega)\, c_{\perp} \sin\phi\, \partial H/\partial c_{\parallel}\,. \end{aligned} \tag{B. 13}$$

Thus the ϕ-dependent part of the initial distribution $F(\underset{\sim}{c})$ is derivable from the ϕ-independent part when rapid fluctuations are absent, in agreement with the general result derived in Section 3. 3 .

Equation (B. 13) may be compared with Eq. (B. 5), obtained directly from the results of Section 3. 3 . The variable $\underset{\sim}{c}$ in Eq. (B.13) differs from the $\underset{\sim}{C}$ of (B. 5) only by the drift velocity $(\omega/\Omega)\, C_{\parallel}\, \underset{\sim}{e}_3$; the ϕ-dependence of the distribution is independent of this change in variables, and the ϕ-independent part is changed only in first order. Equation (B. 13) thus agrees with the first term of Eq. (B. 5) . The second term of Eq. (B. 5) would have been obtained if the effects of the electric field had been included in the calculation.

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