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Publication Date
2008-09-25

UCRL-16633

University of California

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Berkeley, California

Submitted to Science

UCRL-16633

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AEC Contract No. W-7405-eng-48

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January 1966

Abstract

The second-order nonlinear differential equation for the rotation of Mercury is shown to imply locked-in motion when the period is within the range

$$(2T/3)\left[1 - \lambda\cos\frac{2\pi t}{T} \pm \frac{2}{3}(21\lambda e/2)^{1/2}\right],$$

where e is the eccentricity and T the period of Mercury's orbit, the time t is measured from perihelion, and $\lambda = (B - A)/C$ measures the planet's distortion. For values near $2T/3$, the instantaneous period oscillates about $2T/3$ with period $(21\lambda e/2)^{-1/2}\,T$.

Radar (1) and visual (2) observations of the planet Mercury indicate a rotation period $T_r = 58.4 \pm 0.4$ days, close to 2/3 of the orbit period $T = 87.97$ days. Colombo (3) and Liu and O'Keefe (4) have surmised that a stable "locked-in" motion of this type can occur as a result of the inverse-cube term in the planetary potential (5,6) that arises for a body with unequal moments of inertia in the orbital plane. The existence of such a solution to the equations that govern the rotation of a rigid distorted planet has been demonstrated by Liu and O'Keefe by means of digital computations. In this report we present approximate <u>analytic</u> formulas that may afford further physical insight into the character of locked-in motion, that could facilitate the interpretation of observational data, and that indicate the dependence of the results upon the various parameters of the model. For simplicity, and for clarity in the exposition, the analysis outlined in this report is carried to no higher order than is required to exhibit the salient features of the phenomenon.

The differential equation for the orientation, θ, of the planet is given by Eq. 4 of the report by Liu and O'Keefe (4). In terms of the variable $\tau = 2\pi t/T$ it becomes, after insertion of the equation for the Keplerian orbit (7) of eccentricity e,

$$\frac{d^2\theta}{d\tau^2} + \frac{3}{2}\lambda\left[\frac{1 + e\cos f(\tau)}{1 - e^2}\right]^3 \sin 2[\theta - f(\tau)] = 0\ , \tag{1}$$

with the largest of the principal moments of inertia (C) taken perpendicular to the orbital plane, $\lambda \equiv (B - A)/C$ measuring the difference between the two smaller moments of inertia, and f denoting the true anomaly.

[Since damping effects have been ignored in this analysis, Eq. 1 is derivable from a simple Hamiltonian function, with periodic coefficients, in which $p = d\theta/d\tau$ is the canonical momentum conjugate to θ, and Liouville's theorem concerning the conservation of phase-space area applies to the variables θ, p.]

Substitution of the explicit variation of the true anomaly with time, as given by

$$f(\tau) = \tau + 2e \sin \tau \tag{2}$$

through the first-order term in e, converts Eq. 1 to the approximate form

$$\frac{d^2\theta}{d\tau^2} + \frac{3}{2}\lambda \left[(1 + 3\, e \cos \tau)\sin 2(\theta - \tau) - 4\, e \sin \tau \cos 2(\theta - \tau)\right] = 0 , \tag{3}$$

which forms the basis of the remainder of our analysis. [It is noted, from Eq. 2, that τ is to be regarded as measured from the time of perihelion passage, and θ is the angle made by the smallest of the moments of inertia (A) with the major axis of the orbit.] One expects that there may be periodic (locked-in) solutions to Eq. 1 or 3 that are stable, in the sense that neighboring solutions describe oscillatory motion about these periodic solutions.

We consider, specifically, solutions for which $d\theta/dt \approx (3/2)(2\pi/T)$, and write

$$\theta = \frac{3}{2}\tau + \eta , \tag{4}$$

so that Eq. 3 becomes

$$\frac{d^2\eta}{d\tau^2} + \frac{3}{2}\lambda\left[(\cos\tau + \frac{7}{2}e - \frac{1}{2}e\cos 2\tau)\sin 2\eta + (\sin\tau - \frac{1}{2}e\sin 2\tau)\cos 2\eta\right] = 0. \tag{5}$$

When η is small, Eq. 5 may be linearized, to assume the form

$$\frac{d^2\eta}{d\tau^2} + \frac{3}{2}\lambda\,(2\cos\tau + 7e - e\cos 2\tau)\eta = -\frac{3}{2}\lambda(\sin\tau - \frac{1}{2}e\sin 2\tau)\ . \tag{6}$$

For $\lambda^2 \ll 1$, an approximate particular integral to the inhomogeneous Eq. 6 is readily obtained, and the solution to the corresponding linear homogeneous equation may be derived (8) by ignoring terms of average value zero in the coefficient of η . The solution thus includes a periodic motion, of period T , and a long-period oscillation of amplitude α_0 :

$$\eta = \frac{3}{2}\lambda(\sin\tau - \frac{1}{8}e\sin 2\tau) + \alpha_0 \sin\left[(\frac{21}{2}\lambda e)^{1/2}\tau + \alpha_1\right], \tag{7a}$$

or

$$\theta = \frac{3\pi t}{T} + \frac{3}{2}\lambda(\sin\frac{2\pi t}{T} - \frac{1}{8}e\sin\frac{4\pi t}{T}) + \alpha_0 \sin\left[(\frac{21}{2}\lambda e)^{1/2}\frac{2\pi t}{T} + \alpha_1\right] \tag{7b}$$

$$(\text{for } \alpha_0 \ll \pi)\ ,$$

where α_0 and α_1 are arbitrary constants.

If α_0 is not small, so that the slow excursions of η preclude linearization, a similar averaging of the coefficient of $\sin 2\eta$ in Eq. 5 suggests that these oscillations are essentially described by an equation of the form applicable to the motion of a physical pendulum:

$$\frac{d^2\eta}{d\tau^2} + \frac{21}{4}\lambda\, e \sin 2\eta = 0 , \tag{8}$$

for which one may write the first integral

$$\frac{1}{2}\left(\frac{d\eta}{dt}\right)^2 - \frac{21}{8}\lambda\, e \cos 2\eta = \text{const.} \tag{9}$$

With the excursions of η limited to $\pm \pi/2$ for oscillatory motion, the maximum value that $d\eta/d\tau$ can assume for locked-in motion (9) occurs when $\eta = 0$, and is

$$|d\eta/d\tau|_{max} = \left(\frac{21}{2}\lambda e\right)^{1/2} .$$

With inclusion of the contributions from the first terms on the right-hand side of Eq. 7b, therefore, the values of $d\theta/dt$ for locked-in motion are expected to lie between the limits

$$\left[\frac{d\theta}{dt}\right]_{max,min} = \frac{3\pi}{T}\left[1 + \lambda \cos \frac{2\pi t}{T} \pm \frac{2}{3}\left(\frac{21}{2}\lambda e\right)^{1/2}\right] , \tag{10}$$

where we have neglected the term proportional to λe .

The foregoing analysis serves to confirm that locked-in rotational motion with a period approximately 2/3 the period of revolution is dynamically possible. The form of the solution shown in Eq. 7b suggests, however, that observations of the rotation will indicate rates that vary during the course of a planetary year and that, in addition, slower variations of the rotational rate may occur with a period given by

$$T_{lib} = \left(\frac{21}{2} \lambda e \right)^{-1/2} T \qquad (11)$$

when the amplitude (α_0) of this libration is not large. An expression of the form given by Eq. 7b may be useful for the interpretation of data obtained by the sequential observation of surface features on the planet. More simply, the instantaneous period--as could be inferred from radar observations--would be (by differentiation of Eq. 7b when the term proportional to λe is neglected)

$$T_i = \frac{2\pi}{d\theta/dt} = \frac{2}{3} \left\{ 1 - \lambda \cos \frac{2\pi t}{T} - \frac{2}{3} \alpha_0 \left(\frac{21}{2} \lambda e \right)^{1/2} \cos \left[\left(\frac{21}{2} \lambda e \right)^{1/2} \frac{2\pi t}{T} + \alpha_1 \right] \right\} T \qquad (12)$$

for α_0 small, and, for any α_0 compatible with locked-in motion, would lie between the limits obtained from Eq. 10:

$$\left[T_i \right]_{max,min} = \frac{2}{3} \left[1 - \lambda \cos \frac{2\pi t}{T} \mp \frac{2}{3} \left(\frac{21}{2} \lambda e \right)^{1/2} \right] T . \qquad (13)$$

For favorable values of α_0 a determination of λ may be feasible through observation of the slow libratory motion, with a period close to that expressed by Eq. 11, that is represented by the last term of Eq. 12. If, however, α_0 is very small--as could well result from the action of damping mechanisms--the term $-\frac{2}{3} \lambda \cos \frac{2\pi t}{T}$ in Eq. 12 will represent the larger contribution to the variation of the instantaneous period.

Substitution of the values $T = 87.97/365$ yr, $e = 0.2$, and $\lambda = 5 \times 10^{-5}$, as suggested by Liu and O'Keefe (4), into Eq. 11 leads to a libration period $T_{lib} = 23.5$ yr for small-amplitude variations, in substantial agreement with their computational results (4). Correspondingly, from the last term of Eq. 13, the maximum variation of the instantaneous period of rotation that could arise from this libratory motion would be approximately $\pm$ 0.40 day, in good agreement with recent computational results of Liu and O'Keefe (10). It is highly unlikely, of course, that such large variations are now actually occurring, because of the damping that would have resulted from tidal effects.

Although the detailed results presented in this report have been with reference to motion for which the rotation period is close to 2/3 the period of revolution, the existence of other stable modes of locked-in motion should not be overlooked. The possible range of variation for the rotational speed in general will be substantially smaller for the <u>higher-order</u> modes, for reasonable values of the parameter λ, and this feature will have significant implications concerning the magnitude of the damping present at times when the speed of planetary rotation may have been considerably greater than is now observed. Lower limits, which depend on λ, can be set to the rate of decrease of the rotational energy through the agency of damping if the rotational motion has passed through the higher-order modes during the past history of the planet. Similarly, an upper limit can be set on the amount of damping that will permit the rotation to remain locked in to the mode analyzed in this report. Work to be reported elsewhere indicates, moreover, that damping torques acting at present would shift the phase of the periodic solutions presented here, and this result suggests that information concerning the current magnitude of such torques may be inferred from more detailed observation of the rotational motion.

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References and Footnotes

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3. G. Colombo, ibid. 208, 575 (1965).

4. H. Liu and J. A. O'Keefe, Science 150, 1717 (1965).

5. S. F. Peale and T. Gold, Nature 206, 1240 (1965).

6. P. Goldreich, ibid. 208, 375 (1965).

7. This ignores the (small) effect of the nonuniform rotation (as computed in this paper) upon the orbital motion. Interestingly enough, it leads--most dramatically--to a perihelion motion which contains both a secular term and a term with period $(21\lambda e/2)^{-1/2}$ T. These terms are small--of the same order as the perihelion advance present when $A = B \neq C$ --even when compared with the general relativistic perihelion advance of 3.8 seconds of arc per century.

8. K. R. Symon et al., Phys. Rev. 103, 1837 (1956)--esp. p. 1858.

9. This result is seen to be $2/\pi$ times as great as the corresponding value that would have been inferred from use of $\alpha_0 = \pi/2$ in the solution given by Eq. 7 for the linearized problem. The period of these slow oscillations, moreover, will not be that suggested by the last term in Eq. 7a,b, but will approach infinity as the amplitude approaches $\pi/2$.

10. We are grateful for the opportunity to discuss the work of Liu and O'Keefe with these authors, and we appreciate their courtesy in making some of their recent computational results available to us. We thank Miss Penelope A. Collom for assistance with numerical checks of our analysis.

11. The work reported here was supported by the U. S. Atomic Energy Commission.